All involutive solutions of the Yang-Baxter equation

Gandalf Lechner

joint work with Ulrich Pennig and Simon Wood

The Yang-Baxter equation (YBE)

V: finite-dimensional vector space, $R : V \otimes V \to V \otimes V$ linear.

The YBE is the algebraic equation

$$(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R)$$

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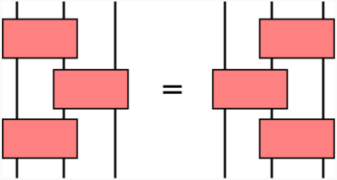
Graphical representation:



Everybody likes the YBE

The YBE appears in a remarkable number of fields:

- Statistical mechanics
- Quantum mechanics
- Integrable QFT
- Braid groups
- Knot theory
- Quantum groups
- Subfactors
- Quantum information theory
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Independent of the field of application, one is often interested in the **solutions** to the YBE.

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- Quantum groups [Drinfeld 86, Jimbo 86, ...] give many solutions of the YBE, but not a complete solution theory.
- **Here:** Consider solutions of YBE up to equivalence relation suggested by group theory and integrable AQFT [Alazzawi, GL 2017] .

The Yang-Baxter equation and the symmetric groups

Definition (for purpose of this talk)

V : finite-dim. Hilbert space. An **R-matrix** is a unitary $R \in \mathrm{End}(V \otimes V)$ that solves the YBE and satisfies $R^2 = 1$.

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Such involutive R's appear in

- elastic two-body S-matrices in integrable QFT
- symmetries of categories of vector spaces [Lyubashenko 1987]
- representations of Thompson's group $\mathcal{V}$ [Jones 2016]
- constructions of certain non-commutative spaces [Dubois-Violette, Landi 2017]
- ...

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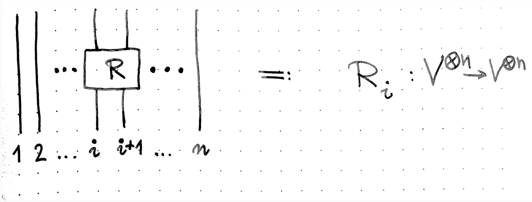
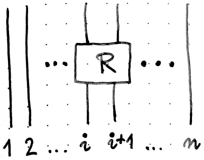
- $\mathcal{R}_0 :=$ set of all R-matrices (with any V)
- $S_n :=$ symmetric group of n letters. Generators σ_i, $i = 1, .., n-1$ satisfy

$$\sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i$$
$$\sigma_i\sigma_j = \sigma_j\sigma_i, \quad |i-j| > 1$$
$$\sigma_i^2 = e.$$

- **Any $R \in \mathcal{R}_0$ gives unitary rep. $\rho_R^{(n)}$ of S_n on $V^{\otimes n}$** via

$$\rho_R^{(n)}(\sigma_i) := \mathrm{id}_V^{\otimes(i-1)} \otimes R \otimes \mathrm{id}_V^{\otimes(n-i-1)}$$


R
1 2 … i i+1 … n
=: $R_i : V^{\otimes n} \to V^{\otimes n}$



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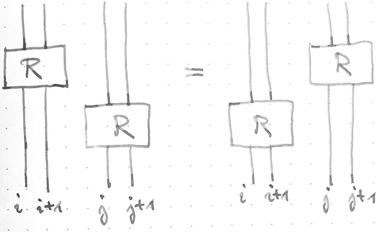
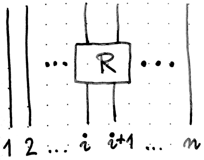


$$R_i R_j = R_j R_i \qquad |i-j|>1$$



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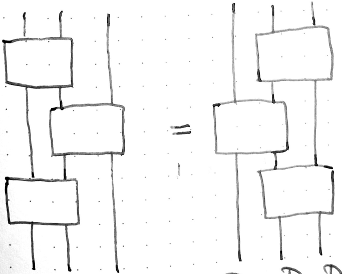
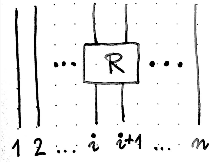


YBE

$R_i R_{i+1} R_i = R_{i+1} R_i R_{i+1}$



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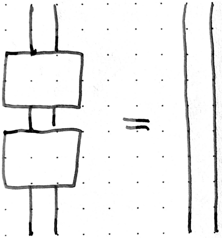

$$R^2 = 1$$

Equivalent R-matrices

Definition

$R, S \in \mathcal{R}_0$ are called equivalent, written

$$R \sim S,$$

if

$$\rho_R^{(n)} \cong \rho_S^{(n)} \qquad \text{for all } n \in \mathbb{N}.$$

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- **Example:** $R \in \mathcal{R}_0(V)$, $U : V \to V$ unitary. Then

$$R \sim (U \otimes U) R (U^{-1} \otimes U^{-1})$$

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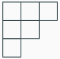
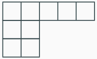
- **Aim:** Determine all R-matrices up to equivalence $\sim$.

R-matrices and Young diagrams

- Recall Young diagrams = integer partitions

 $= 3 + 2 + 1,$  $= 1 + 1 + 1 + 1,$  $= 5 + 2 + 2 \quad \ldots$

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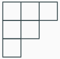
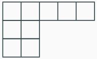
 $= 3 + 2 + 1,$  $= 1 + 1 + 1 + 1,$  $= 5 + 2 + 2 \quad \ldots$

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Theorem I: Structure of $\mathcal{R}_0/\sim$

- Equivalence classes of R-matrices are in $1 : 1$ correspondence with pairs of Young diagrams:

$$\mathcal{R}_0/\sim \;\cong\; (\mathbb{Y} \times \mathbb{Y}) \setminus \{(\varnothing, \varnothing)\}$$

- Dimension = total number of boxes in the two diagrams.
- To each pair (Y, Y') of diagrams, an explicit normal form R-matrix $R_{Y,Y'} \in \mathcal{R}_0$ can be constructed.

Partial traces of R-matrices

- Recall the partial trace $\mathrm{ptr} : \mathrm{End}(V \otimes V) \to \mathrm{End}(V)$ defined by

$$\mathrm{ptr}(A \otimes B) := \mathrm{Tr}(A) \cdot B\,.$$

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$R, S \in \mathcal{R}_0$.

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Example: $R = \left(\ydiagram{4,2,2}\,,\ \ydiagram{3,3}\right)$

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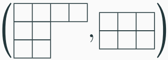
Example: $R =$ 

eigenvalue	multiplicity
$+4$	$4{\times}1$
$+2$	$2{\times}2$
-3	$3{\times}2$

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- Normalized trace on tensor products ($d = \dim V$):

$$\tau = \frac{\mathrm{Tr}_V}{d} \otimes \frac{\mathrm{Tr}_V}{d} \otimes \frac{\mathrm{Tr}_V}{d} \otimes \ldots$$

- Given $R \in \mathcal{R}_0$,

$$\boldsymbol{\chi_R := \tau \circ \rho_R} : S_\infty \longrightarrow \mathbb{C}$$

is a (normalized) character of S_∞ ("Yang-Baxter character").

Special property of Yang-Baxter characters: χ_R "factorizes": For $\sigma, \sigma' \in S_\infty$ with disjoint supports,

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Theorem [Thoma 1964]

- A character χ of S_∞ is extremal if and only if it factorizes.
- $\mathbb{T}$:= all real sequences $\{\alpha_i\}_i$, $\{\beta_i\}_i$ such that
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  Extremal characters are in $1:1$ correspondence with $\mathbb{T}$ via

$$\chi(n\text{-cycle}) = \sum_i \alpha_i^n + (-1)^{n+1} \sum_i \beta_i^n, \qquad n \geq 2.$$

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- Each R defines a point $(\alpha, \beta) \in \mathbb{T}$. The α_i, β_i are the good parameters to characterize R.

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Theorem III: Yang-Baxter characters of S_∞

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Example: $B, F \in \mathbb{N}_0$

$$\alpha_1 = \ldots = \alpha_B = \beta_1 = \ldots = \beta_F = \frac{1}{B+F}$$

This YB-character appears in superselection theory [Doplicher, Haag, Roberts 1971]

Yang-Baxter subfactors

$R \in \mathcal{R}_0$

- Normalized trace τ = state on the *-algebra generated by all R_i, $i \in \mathbb{N}$.
- In GNS representation:

$$\mathcal{M}_R := \pi_\tau(\rho_R(S_\infty))'' = \{\pi_\tau(R_i) \,:\, i \in \mathbb{N}\}''$$

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- $\mathcal{N}_R' \cap \mathcal{M}_R = \mathbb{C}$ if and only if $R \in \{\pm 1, \pm F\}$ [Gohm-Köstler 2010, Yamashita 2012]

The τ-preserving conditional expectation E of $\mathcal{N}_R' \cap \mathcal{M}_R \subset \mathcal{M}_R$ can be computed.

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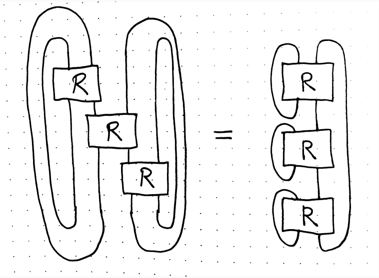

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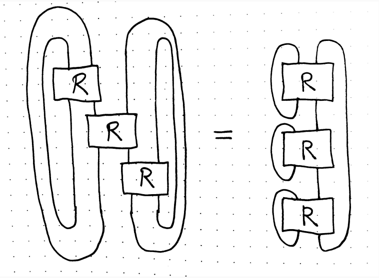

- With these results, one can prove Thm. II and Thm. III

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- **Plan:** Build R-matrix from simple blocks by "direct sum"

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$$X \boxplus Y = X \oplus Y \oplus F \quad \text{on}$$

$$(V \oplus W) \otimes (V \oplus W) = (V \otimes V) \oplus (W \otimes W) \oplus ((V \otimes W) \oplus (W \otimes V)).$$

[Lyubashenko 87, Gurevich 91, Hietarinta 93]

Setting: V, W Hilbert spaces, $X \in \mathrm{End}(V \otimes V)$, $Y \in \mathrm{End}(W \otimes W)$. Define

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Proposition

- $\boxplus$ is commutative and associative.
- $\boxplus$ preserves the YBE: $R, S \in \mathcal{R}_0 \Rightarrow R \boxplus S \in \mathcal{R}_0$.
- $\mathrm{ptr}(R \boxplus S) = \mathrm{ptr}\, R \oplus \mathrm{ptr}\, S$.

Let $d_1^+, \ldots, d_n^+, d_1^-, \ldots, d_m^- \in \mathbb{N}$. **Normal form R-matrix**

$$N := 1_{d_1^+} \boxplus \ldots \boxplus 1_{d_n^+} \boxplus (-1_{d_1^-}) \boxplus \ldots \boxplus (-1_{d_m^-}).$$

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- Let $d := d_1^+ + \dots + d_n^+ + d_1^- + \dots + d_m^-$. Then χ_N has Thoma parameters

$$\alpha_i = \frac{d_i^+}{d}, \qquad \beta_j = \frac{d_j^-}{d}.$$

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- This leads to the proof of Thm. I.

Example: $\mathcal{R}_0(\mathbb{C}^2)/\sim$ has 5 elements:

$\begin{pmatrix} \pm & & & \\ & \pm & & \\ & & \pm & \\ & & & \pm \end{pmatrix}$	$\begin{pmatrix} \pm & & & \\ & & \pm & \\ & \pm & & \\ & & & \pm \end{pmatrix}$	$\begin{pmatrix} + & & & \\ & & + & \\ & + & & \\ & & & - \end{pmatrix}$
$(\square\square, \varnothing)$, $(\varnothing, \square\square)$	$(\begin{smallmatrix}\square\\\square\end{smallmatrix}, \varnothing)$, $(\varnothing, \begin{smallmatrix}\square\\\square\end{smallmatrix})$	$(\square, \square)$
$\alpha_1 = 1$, $\beta_1 = 1$	$\alpha_1 = \alpha_2 = \frac{1}{2}$, $\beta_1 = \beta_2 = \frac{1}{2}$	$\alpha_1 = \beta_1 = \frac{1}{2}$

Repitition

Involutive R-matrices are governed by the following rules:

(~) There is a natural equivalence in terms of S_∞-representations.

(I) $\mathcal{R}_0/\sim \,\cong (\mathbb{Y} \times \mathbb{Y}) \backslash \{(\varnothing, \varnothing)\}$.

(II) $R \sim S \iff \operatorname{ptr} R \cong \operatorname{ptr} S$.

(III) Thoma parameters (α, β) of Yang-Baxter characters are characterized by:

(1) only finitely many α_i, β_i are non-zero,
(2) $\sum_i (\alpha_i + \beta_i) = 1$, and
(3) all α_i, β_i are rational.

(N) In each equivalence class, one can construct an explicit representative by using $\boxplus$.

Outlook

The following generalizations are on our agenda:

- Introduce a **spectral parameter** $\longrightarrow$ QFT!
- **Drop the assumption** $R^2 = 1 \longrightarrow$ braid groups!