# Singular half-sided modular inclusions and deformation quantization


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joint work with Charley Scotford


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# Overview of the talk

- **Main topic:** Certain (half-sided modular, HSM) inclusions $\mathcal{N} \subset \mathcal{M}$ (subfactors of type $\text{III}_1$) of von Neumann algebras (equivalently formulated as **Borchers triples**)
- Closely related to chiral conformal quantum field theories on the real line $\mathbb{R}$ (or the circle $S^1 \cong \mathbb{R} \cup \{\infty\}$)
- This relation will guide us to the question whether a HSM $\mathcal{N} \subset \mathcal{M}$ is **singular** (trivial relative commutant, $\mathcal{N}' \cap \mathcal{M} = \mathbb{C}$) or not.
- Will construct examples of singular half-sided inclusions by a **deformation procedure** (warped convolution, Rieffel deformation).

# Geometric preliminaries

- Symmetries of real line $\mathbb{R}$:
  - translations $x \mapsto x + a$, dilations $x \mapsto e^t \cdot x$, reflection $x \mapsto -x$
  - generate affine group **Aff** of $\mathbb{R}$
- In *conformal* theories, have also conformal (Möbius) transformations

$$x \mapsto \frac{ax+b}{cx+d}, \qquad x \in \mathbb{R} \cup \{\infty\} \cong S^1$$

generate Möbius group **Möb** = $\mathsf{PSL}(2, \mathbb{R})$

✎

# Conformal nets

**Idea:** Consider unitary rep $U$ of Möb on Hilbert space $\mathcal{H}$ and model quantum fields localized in interval $I$ by an algebra of operators $\mathcal{A}(I) \subset \mathcal{B}(\mathcal{H})$.

**Definition (or wishlist:)**

- For every $I \in \mathcal{I}$, there is a (von Neumann) algebra $\mathcal{A}(I) \subset \mathcal{B}(\mathcal{H})$
- $I_1 \subset I_2 \Longrightarrow \mathcal{A}(I_1) \subset \mathcal{A}(I_2)$
- Locality: If $I_1$ and $I_2$ are disjoint, then $\mathcal{A}(I_1)$ and $\mathcal{A}(I_2)$ commute.
- Covariance: $U(g)\mathcal{A}(I)U(g)^{-1} = \mathcal{A}(gI)$
- Positivity: The translations $U(x) = e^{iPx}$ have positive generator $P > 0$
- Vacuum I: There is a unique (up to a scalar) vector $\Omega \in \mathcal{H}$ that is invariant under $U$.
- Vacuum II: $\Omega$ is cyclic for all $\mathcal{A}(I)$

Such a structure is called a **conformal net**.

**Although well motivated, it is a complicated definition. For a classification, one would like to connect it to simpler data.**

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Let us start from a simpler situation (on the real line):

- only **one** algebra, the one corresponding to the half line $\mathbb{R}_+$
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A (one-dimensional) **Borchers triple** $(\mathcal{M}, T, \Omega)$ consists of a von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ and a unitary representation $T$ of $\mathbb{R}$ on $\mathcal{H}$ s.t.

- $T$ has positive generator. The subspace of $T$-invariant vectors is $\mathbb{C}\Omega$.
- $T(x)\mathcal{M}T(-x) \subset \mathcal{M}$ for $x \geq 0$.
- $\Omega$ is cyclic and separating for $\mathcal{M}$.

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- Gives rise to **half-sided modular inclusion** (Borchers, Wiesbrock)

$$\mathcal{N} \subset \mathcal{M}, \qquad \mathcal{N} := T(1)\mathcal{M}T(-1),$$
$$\Delta_{\mathcal{M}}^{it}\mathcal{N}\Delta_{\mathcal{M}}^{-it} \subset \mathcal{N} \qquad t \leq 0$$

- To construct a map (Borchers triples) → (conformal nets), one needs modular theory.

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- Define interval algebras by

$$\mathcal{A}(a,b) := T(a)\mathcal{M}T(-a) \cap T(b)\mathcal{M}'T(-b).$$

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- **If $\Omega$ is cyclic for the interval algebras $\mathcal{A}(a,b)$ ("standard" situation)**, use modular unitaries $\Delta_{\mathcal{A}(a,b)}^{it}$ to generate representation of Möbius group.

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**Theorem ([Longo,Guido,Wiesbrock 98])**

- *In the standard situation, this construction yields a conformal net on $S^1$.*
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Unfortunately, the "standard" situation is not really standard ..

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*This space is independent of $I$ and invariant under the net $\mathcal{A}$.*

Three cases:

1. $\mathcal{H}_{\text{loc}} = \mathcal{H}$. (**standard situation**)
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2. $\mathbb{C}\Omega \subsetneq \mathcal{H}_{\text{loc}} \subsetneq \mathcal{H}$.
   - Here the construction works as in (1) after restriction to $\mathcal{H}_{\text{loc}}$.
3. $\mathcal{H}_{\text{loc}} = \mathbb{C}\Omega$.
   - This is the case of a **singular Borchers triple**.
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**Does case (3) occur?**

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**Does case (3) occur?** Answer from 2019: Yes. **[Longo, Tanimoto, Ueda 19]** have free probability construction to get an example of (3).

# The algebra at infinity

Let $(\mathcal{M}, T, \Omega)$ be a BT, write $\alpha_x = \mathrm{Ad}T(x)$, $\sigma_t = \mathrm{Ad}\Delta^{it}$, $\mathcal{N} = \alpha_1(\mathcal{M})$.

**The algebra at infinity:**

$$\mathscr{X} := \bigcap_{t\in\mathbb{R}} \sigma_t(\mathcal{N} \vee J\mathcal{N}J) = \bigcap_{I\in\mathcal{I}} \mathcal{A}(I)'.$$

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Remarks/Lemmas:

- The larger $\mathscr{X}$, the smaller $\mathcal{H}_{\mathrm{loc}}$.
- $\mathscr{X} = \mathbb{C}1 \iff \mathcal{H}_{\mathrm{loc}} = \mathcal{H}$ (standard case).
- $\mathscr{X} = \mathcal{B}(\mathcal{H}) \iff \mathcal{H}_{\mathrm{loc}} = \mathbb{C}\Omega$ (singular case)

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**How to construct elements in $\mathscr{X}$**

Let $A \in \mathcal{M}$, $B' \in \mathcal{M}'$, and let $L$ be a weak limit point of $\sigma_t(\alpha_1(A)\alpha_{-1}(B'))$ as $t \to -\infty$. Then $L \in \mathscr{X}$. ✎

# Warping

**Plan:** Find a BT such that $\sigma_t(\alpha_1(A)\alpha_{-1}(B')) \to P_\Omega$ weakly as $t \to -\infty$.

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- Fix a deformation parameter $Q$, an antisymmetric $(2 \times 2)$-matrix.
- Deform smooth operators $A \in \mathcal{B}(\mathcal{H})$ on smooth vects $\Psi \in \mathcal{H}$ according to

$$A_Q \Psi := (2\pi)^{-2} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} e^{-ipx} T(Qp) A T(-Qp) \cdot T(x) \Psi \, dp \, dx,$$

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**Facts [Buchholz,GL,Summers 2011]:**

- $A_Q$ extends to a bounded operator. $A \mapsto A_Q$ is a faithful representation of the Rieffel-deformed $C^*$-algebra $(\mathcal{C}_Q, \times_Q, \|\cdot\|_Q)$.
- Let $\mathcal{M}_Q := \{A_Q : A \in \mathcal{M} \text{ smooth}\}''$. If $\kappa \geq 0$, then also $(\mathcal{M}_Q, T, \Omega)$ is a Borchers triple.

**Theorem ([GL/Scotford 2021] Deforming the free field leads to singular BTs)**

*Consider the free field triple $(\mathcal{M}, T, \Omega)$. Then there exist operators $A$ affiliated to $\mathcal{M}$ and $B'$ affiliated to $\mathcal{M}'$ such that for $\kappa > 0$*

$$\operatorname*{w-lim}_{t\to-\infty} \Delta^{it}\alpha_1(A_Q)\alpha_{-1}(B'_{-Q})\Delta^{-it} = P_\Omega.$$

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- Here $A, B$ can be chosen as free quantum field operators.
- The proof relies on a Riemann-Lebesgue type argument

$$\int dp_1 \cdots dp_n \, dq' \, \overline{\Phi_n(p)} \Psi_n(p) f^+(q') \overline{g^+(q')} \prod_{l=1}^{n} e^{i(p_l, Q\Lambda_t q')} \longrightarrow 0 \quad \text{as } t \to -\infty$$

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**Conjecture**

Take any 2d Borchers triple with $\mathcal{H}_{\text{loc}} = \mathcal{H}$, then deform it with deformation parameter $\kappa > 0$. Then $(\mathcal{H}_{\text{loc}})_Q = \mathbb{C}\Omega$ (singular case).

**Outlook/Open Questions:**

- This result hints at $\mathcal{H}_{\text{loc}}$ being unstable under deformations.
- What are the intrinsic properties of this example that distinguish it from case (1) and (2)?
- Are further deformation results possible? ($\to$ quantum group symmetry)