Perturbative Analysis of Products of Random Matrices

Störungstheoretische Analyse von Produkten von Zufallsmatrizen

Der Naturwissenschaftlichen Fakultät
der Friedrich-Alexander-Universität
Erlangen-Nürnberg

zur

Erlangung des Doktorgrades Dr. rer. nat.

vorgelegt von

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Als Dissertation genehmigt

von der Naturwissenschaftlichen Fakultät

der Friedrich-Alexander-Universität Erlangen-Nürnberg

Tag der mündlichen Prüfung: 10.05.2021

Vorsitzender des Promotionsorgans: Prof. Dr. Wolfgang Achtziger

Gutachter: Prof. Dr. Hermann Schulz-Baldes
Prof. Dr. Andreas Knauf
Prof. Dr. Peter Müller

Acknowledgements

First and foremost, I am deeply grateful to my supervisor, *Prof. Dr. Hermann Schulz-Baldes*, for his guidance and his support during our close collaboration in the past four years.

Moreover, I appreciate discussions and conversations with my current and former colleagues in the *Mathematical Physics* group, *Dr. Chris Bourne*, *Joris De Moor*, *Nora Doll*, *Dr. Stefan Fleischer*, *Dr. Julian Großmann*, *Prof. Dr. Andreas Knauf*, *Dr. Nikolay Martynchuk*, *Manuel Quaschner*, *Dr. Johannes Singer*, *Tom Stoiber* and *Dr. Daniele Toniolo*, with some current and former members of the *Stochastics and Dynamical Systems* group, *Prof. Dr. Andrej Depperschmidt*, *Dr. Max Grieshammer* and *Prof. Dr. Gerhard Keller*, and with all other colleagues not mentioned here. Furthermore, I thank *Prof. Dr. Peter Fiebig*, *Dr. Hubertus Grillmeier*, *Prof. Dr. Günther Grün*, *Dr. Ksenija Kitanov* and *Dr. Thomas Singer* for the collaboration in teaching.

I also gratefully acknowledge the support of the *Deutsche Forschungsgemeinschaft* and the *Studienstiftung des deutschen Volkes*.

Finally, I am deeply indebted to my family and my friends for their permanent support.

Contents

Zusammenfassung

Das *Anderson-Modell*, welches zur quantenmechanischen Beschreibung ungeordneter Festkörper entwickelt wurde [7], wird in Abschnitt 1.1 definiert. Daraufhin wird erklärt, welche *Transporteigenschaften* im Anderson-Modell von der Fachwelt der Physiker größtenteils erwartet werden (siehe Abschnitt 1.2). Je nach Dimension, Energie und Stärke der Unordnung werden entweder *Anderson-Lokalisierung* oder *Anderson-Delokalisierung* vermutet. Bestimmte mathematische Begriffe ermöglichen es, einige der erwarteteten Transporteigenschaften präzise zu formulieren (siehe Abschnitt 1.3). Das Phänomen der Anderson-Lokalisierung wurde bereits weitgehend mathematisch verstanden, die Anderson-Delokalisierung stellt die Fachwelt der Mathematiker allerdings bislang noch vor große Herausforderungen [94]. Ein Überblick über die bedeutendsten Ergebnisse zur Anderson-Lokalisierung ist in Abschnitt 1.4 zu finden. Im Jahr 2000 formulierte Barry Simon drei Vermutungen und listete damit die wesentlichen offenen Fragen zur Anderson-Delokalisierung auf [93] (siehe Abschnitt 1.5). Bisherige Herangehensweisen an diese offenen Fragen sind in Abschnitt 1.6 zusammengefasst. Bei einer der von Simon formulierten Vermutungen geht es um die *dynamische Delokalisierung* im Anderson-Modell. Diese Vermutung steht im Mittelpunkt eines Langzeitprojekts des Verfassers und dessen Betreuers.

Eine Wiederholung relevanter Definitionen, Begriffe und wohlbekannter Tatsachen erfolgt in Abschnitt 2.

Das Langzeitprojekt zur dynamischen Delokalisierung basiert auf der Idee, das Anderson-Modell mit Näherungen endlichen Volumens zu betrachten, innerhalb derer formale Lösungen der zugehörigen Eigenwertgleichung mithilfe von *Transfermatrizen* entlang einer bestimmten Raumrichtung entwickelt werden können (siehe Abschnitt 3.1). Genaueres hierzu findet sich in Abschnitt 3.2. Die Kernaufgabe besteht darin, *delokalisierte Zustände* zu finden. Im Gegensatz zu *lokalisierten Zuständen* handelt es sich bei delokalisierten Zuständen um formale Lösungen, die trotz der Unordnung nahezu gleichmäßig räumlich ausgebreitet sind. Bei der Suche nach delokalisierten Zuständen ist die Betrachtung des *quasi-eindimensionalen Grenzfalls* hilfreich (siehe Abschnitt 3.3). Dies erfordert die Analyse einer assoziierten *Zufallsdynamik* auf der *komplexen Graßmann-Mannigfaltigkeit*, die durch eine *hyperbolische Matrix* induziert wird, welche wiederum mit einer *zufälligen Störung* versehen ist (siehe Abschnitt 3.4).

Um sich an diese diffizile Herausforderung heranzutasten, wurde zunächst eine vergleichbare Zufallsdynamik auf dem reell-projektiven Raum untersucht (siehe Abschnitt 4.1). Hierbei entstand die Publikation [31]. In Abschnitt 4.2 wird ein bestimmter Beweis aus [31] skizziert und diskutiert, warum die skizzierte Beweisführung das Interesse an durch hyperbolische Zufallsmatrizen induzierte Zufallsdynamiken auf einem mit einer semipermeablen Barriere ausgestatteten Kreis wecken kann. Die Analyse von derartigen Zufallsdynamiken mit Methoden der *Erneuerungstheorie* ist ein wesentlicher Bestandteil der Publikation [29], in der *Pseudo-Lücken* der *integrierten Zustandsdichte* in zufälligen *Hopping-Modellen* bewiesen werden (siehe Abschnitt 4.3). In Abschnitt 4.4 wird genauer erörtert, wie delokalisierte Zustände tatsächlich mit dynamischer Delokalisierung zusammenhängen. Hierbei wird ersichtlich, dass die Durchführung der Strategie des Langzeitprojekts auf dem geplanten Wege die Benutzung von Transfermatrizen bei Energien mit kleinem, nicht-verschwindenden Imaginärteil erfordert. Als Spezialfall gehören 2×2 Transfermatrizen bei Energien mit kleinem, nicht-verschwindenden Imaginärteil und nicht zu großem Realteil zu einer bestimmten Klasse von zweiparametrigen, zufälligen Störungen von elliptischen Zufallsmatrizen, deren assoziierte *Möbiusdynamik* im Jahr 2009 von Barthel untersucht wurde [12] (siehe Abschnitt 4.5). Das Preprint [30] enthält eine Analyse des *Furstenberg-Maßes* dieser Möbiusdynamik, welche unter anderem eine Verallgemeinerung einer approximativen Formel für den *Lyapunov-Exponenten* der induzierenden Zufallsmatrizen von Barthel hervorbringt (siehe Abschnitt 4.6).

Erklärungen zu Beiträgen in Koautorenschaft zu [31, 29, 30] sind in Abschnitt 5 zu finden.

Abstract

The *Anderson model*, which was constructed to explain quantum mechanical effects of disorder [7], is defined in Section 1.1. Thereafter we summarize the *transport properties* that the physics community expects to occur in the Anderson model (see Section 1.2). Depending on the dimension, the energy and the strength of disorder, either *Anderson localization* or *Anderson delocalization* is expected. Certain mathematical notions allow to formulate some of the expected transport properties precisely (see Section 1.3). The phenomenon of Anderson localization is widely understood mathematically, but Anderson delocalization is still a major challenge for the mathematics community [94]. An overview of the most important results on Anderson localization is given in Section 1.4. In 2000, Barry Simon catalogued the central open questions on Anderson delocalization by formulating three conjectures [93] (see Section 1.5). Some former attempts to address these open questions are summarized in Section 1.6. One of the conjectures formulated by Simon is about *dynamical delocalization* in the Anderson model. This conjecture is in the focus of a long-term project of the author and his supervisor.

Relevant definitions, notions and well-known facts are recapitulated in Section 2.

The long-term project on the dynamical delocalization is based on the idea to work with finite volume approximations of the Anderson model, in which formal solutions of the corresponding eigenvalue equation can be produced by the *transfer matrix technique* in a certain direction of space (see Section 3.1). For details, see Section 3.2. The core task is to find *extended states*. In contrast to *localized states*, extended states are formal solutions that are somewhat evenly spread out in space despite the disorder. When looking for extended states, it is insightful to consider the *quasi-one-dimensional limit* (see Section 3.3). This requires the analysis of an associated *random dynamics* on the *complex Grassmannian* that is induced by *randomly perturbed hyperbolic matrices* (see Section 3.4).

To approach this difficult challenge, a comparable random dynamics on the real projective space was studied primarily (see Section 4.1). These studies gave rise to the publication [31]. In Section 4.2, we sketch a certain proof in [31] and discuss why the latter may awake the interest in random dynamical systems that are induced by hyperbolic random matrices and defined on a circle endowed with a semipermeable barrier. The analysis of such random dynamical systems by means of *renewal theory* is a major part of the publication [29], in which *pseudo-gaps* of the *integrated density of states* in *random hopping models* are proven (see Section 4.3). We then explain in greater detail, how extended states are actually related to dynamical delocalization (see Section 4.4). At this, it becomes clear that the implementation of the long-term project in the planned manner requires the use of transfer matrices at energies whose imaginary part is small but non-zero. As a special case, 2×2 transfer matrices at energies whose imaginary part is small but non-zero and whose real part is not too large belong to a certain class of two parameter perturbations of elliptic random matrices, whose associated *Möbius dynamics* was

studied in 2009 by Barthel [12] (see Section 4.5). The preprint [30] contains an analysis of the *Furstenberg measure* of this Möbius dynamics, which yields among others a generalization of Barthel's approximate formula for the *Lyapunov exponent* of the inducing random matrices (see Section 4.6).

Section 5 contains statements on the authors' contributions to the publications [31, 29, 30].

1 Introduction

This thesis presents the achievements within a long-term project that the author (Florian Dorsch) and his supervisor (Prof. Dr. Hermann Schulz-Baldes) have been working on in close collaboration since 2016. The introduction summarizes the scientific context, the long-range objective and the long-range strategy (originating from the supervisor) that motivated them to focus on the concrete scientific works exhibited below up to this point. The collaboration on the long-range objective is ongoing and other partial projects are being under way.

This introduction is a revised version of the first three sections of a research proposal [28] written by the author at the beginning of his doctoral studies for the purpose of a scholarship application.

1.1 The Anderson Model

The *Anderson model* was constructed in 1958 by the physicist P.W. Anderson [7] to explain quantum mechanical effects of disorder. It was intended to be the simplest model that still represents real physical disordered systems to a suitable extent [7]. The (d-dimensional) Anderson model is given by a random discrete Schrödinger operator

$$(H_\omega \psi)(n) = (\Delta\psi)(n) + \lambda\, V_\omega(n)\,\psi(n)\,, \qquad \psi \in \ell^2(\mathbb{Z}^d)\,, \qquad n \in \mathbb{Z}^d \tag{1}$$

(cf. *e.g.* [94]). Here, Δ denotes the (negative) *discrete Laplacian*[1], defined by

$$(\Delta\psi)(n) = -\sum_{\substack{m\in\mathbb{Z}^d \\ |n-m|=1}} \psi(m)\,, \qquad \psi \in \ell^2(\mathbb{Z}^d)\,, \qquad n \in \mathbb{Z}^d\,, \tag{2}$$

and $\{V_\omega(n)\}_{n\in\mathbb{Z}^d}$ is a sequence of independent and identically distributed and compactly supported random variables (see Section 2.1), which is called the *random potential*. Each $V_\omega(n)$ assigns a random energy to a site of the lattice $\mathbb{Z}^d$. The strength of these random energies is controlled by a non-negative *disorder parameter* λ also called the *coupling constant* (see *e.g.* [61]).

In the following Section 1.2, we summarize the transport properties that the physics community expects to occur in the Anderson model.

1.2 Physicists' Expectations

The belief in a strong dependence of the transport properties on the dimension, the coupling constant and the energy was expressed by Anderson already [7]. While a mathematical description follows in the subsequent sections, we now recall some expectations of the physics community.

At sufficiently large disorder λ or extreme energies, H_ω is expected to be dynamically localized with exponentially localized eigenstates [5]. This is called *Anderson localization* [9] or *strong localization* [99]. Moreover, at arbitrary disorder and at all energies, Anderson localization is expected in the one-dimensional case and (in a possibly weaker form) also in the two-dimensional

[1] The choice of the sign of Δ in (2) varies in the literature.

case [94]. In higher dimensions $d \geq 3$, however, and for small disorder λ and away from extreme energies, H_ω is expected to exhibit diffusive transport [5] with extended eigenstates [94]. This is called *Anderson delocalization*[2] or *weak localization* [71]. The boundary between the two regimes is called *mobility edge* [5]. The expectations for higher dimensions $d \geq 3$ are summarized in Figure 1. As will be outlined below, some of these expectations have reached the status of mathematical results (see Section 1.4), others remain open (see Section 1.5).

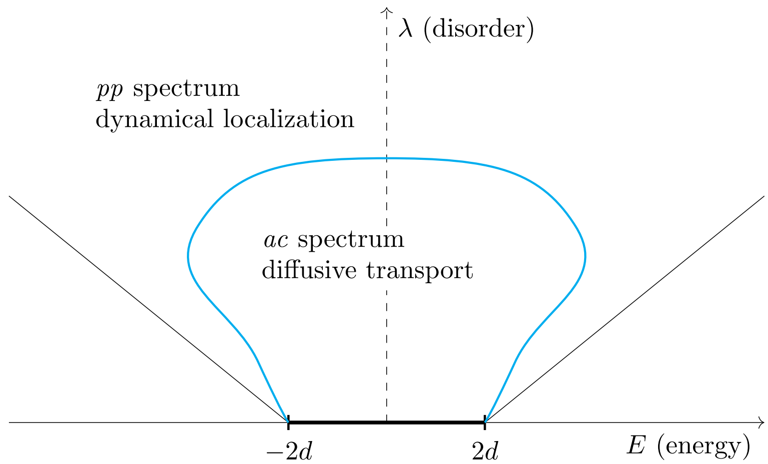

Figure 1: Expected phase diagram of the Anderson model in dimensions $d \geq 3$ (see *e.g.* [5]).

In the next Section 1.3, we recall a basic mathematical result on the spectrum of the Anderson model and introduce time-averaged moments of the position operator in order to define the notions of *spectral* and *dynamical localization*. These notions allow to consider transport properties in the mathematical language. Thereafter, we present the most important mathematical results on the strong localization regime in Section 1.4. The definitions, facts and techniques on the Anderson model stated in Sections 1.3 and 1.4 can be found in the original works from the 1970's, 1980's and 1990's [62, 77, 54, 52, 70, 4, 44, 43, 17] and also in a number of more recent reviews [20, 94, 67, 5, 61, 68]. Some of the former results were achieved by means of the *transfer matrix technique*, which we briefly sketch in the course of Section 1.4, since this technique is crucial for our long-term project. We refrain from quoting all the respective necessary assumptions on the random potential in this introduction, since it is supposed to provide rather an overview than a pedantic catalogue of results. For a typical example, the reader may think of the random variables $V_\omega(n)$ being uniformly distributed on some interval, which is the case that was originally discussed by Anderson.

1.3 Spectral and Dynamical Localization

For the purpose of a mathematical formulation of the transport properties, we recall a basic theorem on the spectrum of the Anderson model and introduce the *time-averaged q-th moment*.

[2]This term is used rather rarely in the literature, *e.g.*, by Huang et al, who sudied quasi-periodic lattices [60].

Theorem 1. *The spectrum $\sigma(H_\omega)$ of H_ω is almost surely given by*

$$\Sigma := [-2d, 2d] + \lambda\,\mathrm{supp}(V_\omega(1))\,.$$

Moreover, Σ can be decomposed almost surely into absolutely continuous, singular continuous and pure point part, namely more precisely, there exist $\Sigma_{\mathrm{ac}}, \Sigma_{\mathrm{sc}}, \Sigma_{\mathrm{pp}} \subset \Sigma$, for which one has

$$\sigma_{\mathrm{ac}}(H_\omega) = \Sigma_{\mathrm{ac}}\,, \qquad \sigma_{\mathrm{sc}}(H_\omega) = \Sigma_{\mathrm{sc}}\,, \qquad \sigma_{\mathrm{pp}}(H_\omega) = \Sigma_{\mathrm{pp}} \tag{3}$$

almost surely.

Definition 1. For $q > 0$, we define the *time-averaged q-th moment of the position operator* of a state initially localized at the origin and evolved with states energetically in a measurable set $I \subset \mathbb{R}$ as

$$M_q(T, I) = \mathbf{E} \int_0^\infty \frac{\mathrm{d}t}{T}\, e^{-2\frac{t}{T}} \sum_{j\in\mathbb{Z}^d} |j|^q \left| \left\langle \delta_{j,\cdot} \middle| \exp[-itH_\omega]\chi_I(H_\omega)\delta_{0,\cdot} \right\rangle \right|^2 ,$$

where $\mathbf{E}$ denotes the average, χ_I the indicator function of I and $\delta_{\cdot,\cdot}$ the Kronecker symbol.

The almost sure decomposition (3) and the quantity of time-averaged q-th moment allows to introduce two notions of localization: the *spectral localization* and the *dynamical localization*.

Definition 2. Let $I \subset \mathbb{R}$ be an open interval. We say that H_ω exhibits *spectral localization* in I if $I \subset \Sigma_{\mathrm{pp}}$. Furthermore, we say that H_ω exhibits *dynamical localization* in I if $M_q(T, I)$ is uniformly bounded in T for all $q > 0$.

As for dynamical localization, there are actually various technical notions used in the literature. For sake of concreteness, we chose one of them.

Remark 1. By variants of the so-called RAGE theorem, dynamical localization in some open interval $I \subset \mathbb{R}$ implies spectral localization in I. The converse is false, see *e.g.* [81].

1.4 Mathematical Results on the Strong Localization Regime

In the one-dimensional case, dynamical localization with exponentially decaying eigenstates has been proven at arbitrary disorder and energies, and this also leads to spectral localization in all Σ. This was also extended to strips, as will be described shortly. Furthermore, dynamical and hence spectral localization was proven in any dimension at large disorder and, additionally, at arbitrary disorder near the band edges of the almost-sure spectrum (see Figure 1).

1.4.1 Localization in One Dimension and on a Strip

In 1973, Ishii [62] demonstrated the absence of absolutely continuous spectrum in the one-dimensional case for all energies and at any strength of disorder. In the continuum version of the model, a full proof of spectral and dynamical localization was first given in 1977 by Goldsheid, Molchanov and Pastur [54]. For the discrete model described above, the first proof of these facts is due to Kunz and Souillard [70] in 1980.

The techniques used for the proof in the one-dimensional case is relevant for our long-term project, so we describe them in more detail. The solution $\psi \in \ell^2(\mathbb{Z})$ of the eigenvalue equation

$$H_\omega \psi = E\psi\,, \quad E \in \mathbb{R}\,, \tag{4}$$

is determined by its values at two suceeding points $\Psi(n) := (\psi(n+1), \psi(n))^\intercal$, $n \in \mathbb{Z}$, via

$$\Psi(n) = \mathcal{T}_\omega^E(n)\Psi(n-1)\,, \quad n \in \mathbb{Z}\,, \tag{5}$$

where the (random) 2×2-matrices

$$\mathcal{T}_\omega^E(n) := \begin{pmatrix} \lambda V_\omega(n) - E & -1 \\ 1 & 0 \end{pmatrix}\,, \quad n \in \mathbb{Z}\,, \tag{6}$$

belong to the symplectic group $\mathrm{SP}(2, \mathbb{R})$. The matrices $\mathcal{T}_\omega^E(n)$ are called *transfer matrices*. By iteration, one obtains

$$\Psi(n) = \left(\prod_{l=m+1}^{n} \mathcal{T}_\omega^E(l) \right) \Psi(m)\,, \quad n, m \in \mathbb{Z}\,. \tag{7}$$

A way to show localization in one dimension, therefore, is to apply the theory of products of random matrices (see [15], Part A) on the products of the random transfer matrices as in the bracket in (7) (see [15], Part B, Chapter II & III). In the 1960's substantial results on products of random matrices were achieved by Furstenberg and Kesten [46], Furstenberg [45] and Osseledec [75]. Later on, building on these methods, a lot of efforts were made to address the localization on a strip, *i.e.*, on the Anderson model on $\ell^2(\mathbb{Z} \times \{1, \ldots, N\}) \cong \ell^2(\mathbb{Z}) \otimes \mathbb{C}^N$, *e.g.*, by Goldsheid and Margulis [53]. Many results obtained for the one-dimensional Schrödinger operator turned out to be adaptable to that more general case (see [15], Part B, Chapter IV).

1.4.2 Localization in Higher Dimensions

In 1983, Fröhlich and Spencer [44] initially developed the method of multiscale analysis, whose essential ingredient is the exponential decay of the resolvent kernel in the distance of its entries. Two years later, Fröhlich, Martinelli, Scoppola and Spencer [43] proved spectral localization in arbitrary dimension for large disorder or near the band tails by means of this method. For the case of large disorder, the result in [43] was refined by Carmona, Klein and Martinelli [17] in 1987. The basic ideas in [43] were adapted in 1989 by von Dreifus and Klein [32], who implemented them in a technically simpler way. In 1993, Aizenman and Molchanov [4] introduced the new fractional moments method, which includes the use of low moments of the resolvent kernel and provides a mathematically less involved way to prove spectral localization in higher dimensions. In 1998, Germinet and De Biévre [48] proved dynamical localization in higher dimensions using multiscale analysis, but in a weaker sense than defined above. In 2001, the result in [48] was strengthened by Damanik and Stollmann [21], who proved the uniform boundedness of the q-th moment for subcritical $q < q_0$. Shortly after, Germinet and Klein [50] extended the result in [21] to arbitrary $q > 0$, whereby they actually proved dynamical localization in the sense defined above. Also the fractional moments method was used to address dynamical localization in higher dimensions [2, 80] and for continuum models [3]. Localization results for continuum models on the case of Bernoulli random potentials were obtained by Bourgain and Kenig [16] and on the case of Poisson random potentials by Germinet, Hislop and Klein [49].

1.5 Simon's List of Open Problems

A list of open problems on the Anderson model has been put forward in 2000 by Simon [93]. It is just a mathematical formulation of the physicists' expectations.

In the two-dimensional case, spectral localization is expected in the whole spectrum:

Conjecture 1 (Localization in two dimensions)**.** If $d = 2$, one has $\Sigma = \Sigma_{\mathrm{pp}}$ for all $\lambda > 0$.

Proving localization in two dimensions is likely the most difficult problem on the list. Indeed, dimension two is critical and thus very fragile. For instance, adding a magnetic field leads to the quantum Hall effect and divergence of the localization length, *i.e.*, the rate of exponential decay of the eigenfunctions, at certain energies [69]. Even in the two-dimensional Anderson model in the strict sense described above, the localization length is expected to behave like $e^{1/\lambda}$ in the coupling constant [1], which indicates that any perturbative approach may run into difficulties.

Our focus is rather on the (expected) weak localization regime in the higher-dimensional case $d \geq 3$ about which Simon formulated the following two open problems:

Conjecture 2 (Extended states)**.** For $d \geq 3$, there exist $\lambda > 0$ such that Σ_{ac} is non-empty.

Conjecture 3 (Quantum diffusion)**.** For $d \geq 3$, there exist $\lambda > 0$ such that $M_2(T, \mathbb{R}) \sim T$ as $T \to \infty$.

The main purpose of the long-term project the author and his supervisor are working on is to prove the following rather modest version of Conjecture 3:

Conjecture 4. For some $d \geq 3$, there exist $\lambda > 0$ and $q > 0$ such that one has

$$\liminf_{T\to\infty} \frac{\log[M_q(T, \mathbb{R})]}{\log(T)} > 0\,. \tag{8}$$

Remark 2. The positivity of the left side of (8) for some $q > 0$ implies that all $q > 0$ fulfil

$$\liminf_{T\to\infty} \frac{\log[M_q(T, \mathbb{R})]}{\log(T)} \geq \frac{1}{2}\left[\frac{q}{d} - 11\right]$$

if H_ω satisfies a Wegner estimate in $\mathbb{R}$ (see [50], Theorem 2.10).

Before we describe our strategy for a proof of Conjecture 4 in more detail in Section 3, we recollect some attempts by the mathematical physics community which aimed at a better understanding of the weak localization regime.

1.6 Contributions to the Weak Localization Regime

1.6.1 Wegner's N-Orbital Model

The Wegner N-orbital model modifies the Anderson model by supposing that over every lattice site there are N atomic levels. All these levels are connected by a full random matrix, for example, taken from the Gaussian orthogonal ensemble. Wegner argued that there should be diffusion in this model, at least in the limit $N \to \infty$ [101]. A rigorous analysis based on free probability was given by Neu and Speicher [74]. This led to closed formulas for the 2-point and 4-point Green functions. This was then indeed sufficient to deduce diffusion [91].

1.6.2 Universality in Full Random Matrices

Beginning in the mid 2000's, substantial progress was made by Erdös, Ramirez, Schlein, Tao, Vu, Yau and Yin (see [39, 38, 41, 40, 97, 98, 37, 35, 42]) and others, who studied the eigenvalue statistics of (full) Wigner matrices. It was proven that under certain assumptions the local spectral statistics in the bulk are universal, *i.e.*, it coincides with that of the associated Gaussian ensemble (GOE or GUE) in the thermodynamic limit. Finite volume approximations of the Anderson model, however, give rise to ensembles of very sparsely filled random matrices. Numerous numerical studies were performed earlier [34], which supported the conjecture that the eigenvalue statistics of the Anderson model is the same as the one of full matrices. Nevertheless, so far no one succeeded in proving these numerical results. Stated in a different way, it has not been possible to show why full random matrices are relevant for the Anderson model.

1.6.3 Random Band Matrices

A way to improve our understanding of the weak localization regime is to change the model and study random band matrices. In 2009, Schenker [85] extended the fractional moment method by Aizenman and Molchanov (see above) to prove localization of operators associated to random band matrices in a regime of subcritical width, *i.e.*, provided the band width is sufficiently small relative to the system size. On the other hand, Erdös et al [36] proved spectral delocalization (in the sense of GOE statistics of the eigenvalue process) in a regime in which the band width increases sufficiently fast relative to the system size. Further substantial contributions were made by Disertori, Pinson and Spencer [26], who proved that the expectation of the density of states of Gaussian band matrices coincides with Wigner's semicircle law up to an error depending on the band width. Again it is not clear how to connect these results to the Anderson model.

1.6.4 Stochastic Differential Equations

Bachmann and De Roeck [10] conducted further research on the Anderson model on a strip in the context of the DMPK-Theory of disordered quantum wires. Its principal postulate is that the random transfer matrix as a function of the wire length satisfies an Ito stochastic differential equation motivated by the "Maximum Entropy" Assumption. Exploiting two symmetries of the transfer matrix (see [10], eq. 9), one obtains a diagonal matrix which fully describes the eigenvalue statistics via the so-called DMPK equation (see [10], eq. 24). Almost at the same time and independently, Valko and Virag [100] obtained very similar results when also studying stochastic differential equations related to the Anderson model. The results in [10] and [100] are applicable to cases in which the unperturbed transfer matrices are (conjugated to) unitary matrices. Later on, Sadel and Virag [84] extended the results in [10] and [100] to cases in which the unperturbed transfer matrices have eigenvalues with moduli different from 1. In fact, the moduli of the eigenvalues of the unperturbed transfer matrices will play an important role below, namely under the heading of so-called elliptic and hyperbolic channels, which will be introduced later.

2 Preliminaries

We now recollect basic definitions, introduce notations for later use and recall well-known facts. This section does *not* contain any yet unknown scientific result.

The symbol $\mathbb{K}$ will always denote either the real line $\mathbb{R}$ or the complex plane $\mathbb{C}$. Moreover, we will assume that L and q are positive integers for which one has $q \leq \mathsf{L}$.

2.1 Random Matrices[3] and Random Operators[4]

Definition 3. Let $(\Sigma, \mathscr{A}, \mathbb{P})$ be a probability space and $(X, \mathcal{X})$ be a measurable space. Then, we call an $(\mathscr{A}, \mathcal{X})$-measurable map $\Sigma \to X\,,\ \sigma \mapsto Y_\sigma$ an *X-valued random variable* over $(\Sigma, \mathscr{A}, \mathbb{P})$.

If X is a locally compact Hausdorff space[5] and if $\mathbb{P}$ is a regular Borel measure[6], we call the complement $\mathrm{supp}(Y_\sigma) = X \setminus X_0$ of the largest open set X_0 satisfying $\mathbb{P}(Y_\sigma \in X_0) = 0$ the *support*[7] of Y_σ. If $\mathrm{supp}(Y_\sigma)$ is compact, we call Y_σ *compactly supported.*

A sequence $Y_{\sigma_1}, Y_{\sigma_2}, \dots$ of X-valued random variables $\Omega \to X\,,\ (\sigma_1, \sigma_2, \dots) \mapsto Y_{\sigma_n}$ over the infinite product space $(\Omega, \mathcal{A}, \mathbf{P}) = \bigotimes_{n \in \mathbb{N}} (\Sigma, \mathscr{A}, \mathbb{P})$ is called *independent and identically distributed (i.i.d.).* We write $Y_\omega(n) = Y_{\sigma_n}$, where the stochastic label $\omega = (\sigma_1, \sigma_2, \dots)$ lies in Ω.

We denote the expectation w.r.t. $\mathbb{P}$ and $\mathbf{P}$ by $\mathbb{E}$ and $\mathbf{E}$, respectively.

Definition 4. We call a $\mathbb{K}^{\mathsf{L} \times \mathsf{L}}$-valued random variable $\mathcal{T}_\sigma$ a (*real* or *complex*) $\mathsf{L} \times \mathsf{L}$ *random matrix.* We call $\mathcal{T}_\sigma$ *invertible* if it is $\mathrm{GL}(\mathsf{L}, \mathbb{K})$-valued and — for even $\mathsf{L} \equiv 2L$ — we call $\mathcal{T}_\sigma$ *symplectic* if it is $\mathrm{SP}(2L, \mathbb{R})$-valued. Here, $\mathrm{SP}(2L, \mathbb{R})$ denotes the *symplectic group* defined by

$$\mathrm{SP}(2L, \mathbb{R}) = \left\{ \mathcal{T} \in \mathbb{R}^{2L \times 2L} : \mathcal{T}^\intercal \mathcal{J} \mathcal{T} = \mathcal{J} \right\}\,, \qquad \mathcal{J} = \begin{pmatrix} \mathbf{0} & -\mathbf{1}_L \\ \mathbf{1}_L & \mathbf{0} \end{pmatrix}.$$

We endow the set $\mathcal{L}(\mathscr{H})$ of self-adjoint bounded operators on a Hilbert space $\mathscr{H}$ with the σ-field of Borel sets with respect to the weak operator topology (see *e.g.* [78]). We call an $\mathcal{L}(\mathscr{H})$-valued random variable over a probability space $(\Sigma, \mathscr{A}, \mathbb{P})$ a *random operator* on $\mathscr{H}$ over $(\Sigma, \mathscr{A}, \mathbb{P})$.

We are now prepared to formulate the precise definition of the Anderson model.

Definition 5. Let $(\Omega, \mathcal{A}, \mathbf{P}) = \bigotimes_{n \in \mathbb{Z}^d} (\Sigma, \mathscr{A}, \mathbb{P})$ be the infinite product space as in Definition 3 and denote the stochastic label by $\omega = \{\sigma_n\}_{n \in \mathbb{Z}^d} \in \Omega$, where $\sigma_n \in \Sigma$. Then, a sequence $V_{\sigma_n} \equiv V_\omega(n)$, $n \in \mathbb{Z}^d$ of i.i.d., real-valued and compactly supported random variables defines the Anderson model as the random operator H_ω by equation (1) (cf. [94], Section 2.2).

Definition 6. A family $\{S_x\}_{x \in I}$ of measure preserving transformations $S_x : \Sigma \to \Sigma$ on a probability space $(\Sigma, \mathscr{A}, \mathbb{P})$ is called *ergodic* if all $A \in \mathscr{A}$ satisfy the implication

$$S_x^{-1}(A) = A \qquad \forall\, x \in I \qquad \Longrightarrow \qquad \mathbb{P}(A) \in \{0, 1\}\,.$$

[3] The definitions of (sequence of i.i.d.) random variable(s) and random matrix can *e.g.* be found in §1.1 of [96].
[4] For the definitions of random operator, ergodicity and (integrated) density of states, see §3.1 and §3.4 of [5].
[5] For the definitions of Hausdorff and local compactness, see *e.g.* Ch. 5, Sec. 13 and Ch. 6, Sec. 18 of [102].
[6] For the definition of regular measure, see Chapter II, Section 1 of [76].
[7] For the definition of the support of a measure, see *e.g.* [57]. It can be readily transferred to random variables.

Definition 7. We call a random operator H_σ on $\ell^2(\mathbb{Z}^d)$ over $(\Sigma, \mathscr{A}, \mathbb{P})$ *ergodic*[8] if there exists an ergodic group $\{S_x\}_{x\in\mathbb{Z}^d}$ of measure preserving transformations $S_x : \Sigma \to \Sigma$ such that

$$H_{S_x(\sigma)} = U_{x,\sigma} H_\sigma U_{x,\sigma}^* \qquad \text{and} \qquad (U_{x,\sigma}\psi)(x') = \exp\left[\imath\phi_{x,\sigma}(x')\right]\, \Psi(x'-x) \qquad \forall\, \psi \in \ell^2(\mathbb{Z}^d)$$

hold for some unitary operator $U_{x,\sigma}$ and some $\phi_{x,\sigma}(x') \in \mathbb{R}$ for all $x, x' \in \mathbb{Z}^d$ and all $\sigma \in \Sigma$.

Remark 3. By setting $S_x : \Omega \to \Omega\,, \{\sigma_n\}_{n\in\mathbb{Z}^d} \mapsto \{\sigma_{n-x}\}_{n\in\mathbb{Z}^d}$ and $(U_x\psi)(x') = \psi(x'-x)$, one shows that the Anderson model is an ergodic operator (see [5], §3.1).

An important quantity for ergodic operators H_σ defined on $\ell^2(\mathbb{Z}^d)$ is a probability measure called *density of states*, which measures, in a sense, the distribution of the eigenvalues of H_σ; its cumulative distribution function is called *integrated density of states*, which then measures the fraction of eigenvalues of H_σ below a certain energy (see *e.g.* [20], Section 9.2).

Definition 8. For an ergodic operator H_σ on $\ell^2(\mathbb{Z}^d)$ over $(\Sigma, \mathscr{A}, \mathbb{P})$, the *density of states* $\mathfrak{n}$ is defined as the (unique and existent) probability measure on $\mathbb{R}$ satisfying

$$\exists\ \Sigma_0 \in \mathscr{A} : \quad \mathbb{P}(\Sigma_0) = 1 \quad \wedge \quad \lim_{L\to\infty} \frac{\operatorname{tr}\left[f\left(\Lambda_L^d H_\sigma \Lambda_L^d\right)\right]}{L^d} = \int_{\mathbb{R}} \mathrm{d}\mathfrak{n}(x)\, f(x) \quad \forall\ (\sigma, f) \in \Sigma_0 \times C_0(\mathbb{R})\,,$$

where Λ_L^d denotes the projection onto $\ell^2([0,L)^d \cap \mathbb{Z}^d)$ given by $(\Lambda_L^d\psi)(n) = \chi_{[0,L)^d}(n)\,\psi(n)$. The *integrated density of states* $\mathcal{N}$ is defined as the cumulative distribution function of $\mathfrak{n}$, *i.e.*,

$$\mathcal{N} : \mathbb{R} \longrightarrow [0,1]\,, \quad E \longmapsto \mathfrak{n}((-\infty, E])\,.$$

2.2 The Stiefel Manifold, the Grassmannian and the Exterior Power[9]

Definition 9. For a vector space V over $\mathbb{K}$, we define an equivalence relation $\sim$ on $V \setminus \{0\}$ by

$$v \sim v' \ :\Longleftrightarrow\ \mathbb{K}v = \mathbb{K}v'\,.$$

The set $(V\setminus\{0\})/\!\sim$ of equivalence classes is called the *projective space over* V (*cf. e.g.* [47], Chapter 5) and can be identified by

$$\mathscr{P}(V) = \{\mathbb{K}v : v \in V \setminus \{0\}\}\ .$$

If V is endowed with a norm $\|\cdot\|_V$, we call the set of unit vectors

$$\mathscr{S}(V) = \{v \in V : \|v\|_V = 1\}$$

the *unit sphere of* V (see *e.g.* [23], Chapter II, §1).

[8] In §3.1 of [5], the term *standard ergodic* operator is used instead.

[9] In this section, the author used Chapter IV.2 of [18] and Chapter 8 of [27] to recapitulate the definitions and facts related to the exterior power unless otherwise stated.

Definition 10. The *non-compact Stiefel manifold* $\hat{\mathbb{F}}_{\mathsf{L},q}(\mathbb{K})$ is defined by

$$\hat{\mathbb{F}}_{\mathsf{L},q}(\mathbb{K}) = \left\{\Psi \in \mathbb{K}^{\mathsf{L}\times q} :\ \mathrm{rank}(\Psi) = q\right\}$$

and consists of all *non-singular q-frames*, which are the $\mathsf{L} \times q$ matrices whose column vectors are linearly independent; the *compact Stiefel manifold* $\mathbb{F}_{\mathsf{L},q}(\mathbb{K})$ is defined by

$$\mathbb{F}_{\mathsf{L},q}(\mathbb{K}) = \left\{\Phi \in \mathbb{K}^{\mathsf{L}\times q} :\ \Phi^*\Phi = \mathbf{1}\right\}$$

and consists of all *orthonormal q-frames*, which are the $\mathsf{L} \times q$ matrices whose column vectors are orthonormal (see [63], Section 2).

Remark 4. Given some non-singular q-frame $\Psi \in \hat{\mathbb{F}}_{\mathsf{L},q}(\mathbb{K})$, the Gram-Schmidt procedure (applied to its column vectors from the left to the right) allows to compute the *thin QR factorization*, which yields a unique couple (Φ, S) of an orthonormal q-frame $\Phi \in \mathbb{F}_{\mathsf{L},q}(\mathbb{K})$ and an upper triangular $q \times q$ matrix with positive diagonal entries S for which one has

$$\Psi = \Phi S$$

(see [55], Sections 5.2.6 & 5.2.7). This defines a projection

$$\varpi : \hat{\mathbb{F}}_{\mathsf{L},q}(\mathbb{K}) \longrightarrow \mathbb{F}_{\mathsf{L},q}(\mathbb{K})\,,\ \ \Psi \longmapsto \Phi \tag{9}$$

from the non-compact Stiefel manifold onto the compact one.

Definition 11. We identify the *Grassmannian* $\mathbb{G}_{\mathsf{L},q}(\mathbb{K})$ of q-dimensional linear subspaces of $\mathbb{K}^{\mathsf{L}}$ as

$$\mathbb{G}_{\mathsf{L},q}(\mathbb{K}) = \{\Phi\Phi^* :\ \Phi \in \mathbb{F}_{\mathsf{L},q}(\mathbb{K})\}\ ,$$

which is the set of all $\mathbb{K}^{\mathsf{L}\times\mathsf{L}}$-valued orthogonal projections with rank q (*cf.* [33], Section 2.5).

Remark 5. The projection

$$\Pi : \mathbb{F}_{\mathsf{L},q}(\mathbb{K}) \to \mathbb{G}_{\mathsf{L},q}(\mathbb{K})\,,\ \ \Phi \mapsto \Phi\Phi^*$$

maps an orthogonal q-frame to the orthogonal projection onto its range (see [33], Section 2.5).

Remark 6. We endow $\mathbb{F}_{\mathsf{L},q}(\mathbb{K})$ and $\mathbb{G}_{\mathsf{L},q}(\mathbb{K})$ with the Frobenius norm $\|\cdot\|_{\mathrm{F}}$ (see [55], Section 2.3.1).

Definition 12. For $v_1, \ldots, v_q \in \mathbb{K}^{\mathsf{L}}$, the alternating q-linear functional

$$\begin{aligned} v_1 \wedge \cdots \wedge v_q : \left[\,\left(\mathbb{K}^{\mathsf{L}}\right)^*\,\right]^{\times q} &\longrightarrow \mathbb{C}\,, \\ (f_1, \ldots, f_q) &\longmapsto \det\left((f_j(v_k))_{j,k}\right)\ , \end{aligned}$$

where $(\cdot)^*$ denotes the dual, is called a *decomposable q-vector*, for which we also write

$$\Lambda\Phi = v_1 \wedge \cdots \wedge v_q\,, \tag{10}$$

where $\Phi = (v_1, \ldots, v_q)$ is the $\mathsf{L} \times q$ matrix whose columns are $v_1, \ldots, v_q$. The linear span

$$\Lambda^q\mathbb{K}^{\mathsf{L}} = \mathrm{span}\left(\Lambda_0^q\mathbb{K}^{\mathsf{L}}\right)\ , \qquad\qquad \Lambda_0^q\mathbb{K}^{\mathsf{L}} = \left\{\Lambda\Phi : \Phi \in \mathbb{K}^{\mathsf{L}\times q}\right\}$$

of the decomposable q-vectors is called *q-th exterior power*.

Remark 7. The identity (10) defines the map

$$\Lambda : \mathbb{K}^{\mathsf{L}\times q} \longrightarrow \Lambda_0^q \mathbb{K}^{\mathsf{L}}\,,\ (v_1,\ldots,v_q) \longmapsto v_1 \wedge \cdots \wedge v_q\,.$$

Remark 8. The bi- or sesquilinear extension of

$$\begin{aligned}\langle\cdot,\cdot\rangle_\wedge : \Lambda_0^q\mathbb{K}^{\mathsf{L}} \times \Lambda_0^q\mathbb{K}^{\mathsf{L}} &\longrightarrow \mathbb{K}\,,\\ (\Lambda\Phi_1,\Lambda\Phi_2) &\longmapsto \det\left(\Phi_1^*\Phi_2\right)\end{aligned}$$

to $\Lambda^q\mathbb{K}^{\mathsf{L}}$ is an inner product, which then also induces a norm $\|\cdot\|_\wedge = \sqrt{\langle\cdot,\cdot\rangle_\wedge}$ on $\Lambda^q\mathbb{K}^{\mathsf{L}}$.

Remark 9. Two decomposable q-vectors $v_1\wedge\cdots\wedge v_q$ and $w_1\wedge\cdots\wedge w_q$ are proportional if and only if the sets of vectors $\{v_1,\ldots,v_q\}$ and $\{w_1,\ldots,w_q\}$ span the same linear subspace of $\mathbb{K}^{\mathsf{L}}$. Therefore the Grassmannian $\mathbb{G}_{\mathsf{L},q}(\mathbb{K})$ can be embedded into the projective space of the exterior power $\Lambda^q\mathbb{K}^{\mathsf{L}}$ by the *Plücker embedding*, which is defined with the aid of the compact Stiefel manifold $\mathbb{F}_{\mathsf{L},q}(\mathbb{K})$ as

$$\mathscr{J} : \mathbb{G}_{\mathsf{L},q}(\mathbb{K}) \longrightarrow \mathscr{P}\left(\Lambda^q\mathbb{K}^{\mathsf{L}}\right)\,,\quad \Pi(\Phi)\longmapsto \mathbb{K}\,\Lambda\Phi\,,\qquad \Phi\in\mathbb{F}_{\mathsf{L},q}(\mathbb{K})\,.$$

In our representation, the Plücker embedding maps the orthogonal projection onto the range of some $\Phi\in\mathbb{F}_{\mathsf{L},q}(\mathbb{K})$ to the (one-dimensional) linear span of the decomposable q-vectors $v_1\wedge\cdots\wedge v_q$ for which the set $\{v_1,\ldots,v_q\}$ spans the range of Φ. In particular, the image of $\mathscr{J}$ is given by

$$\mathscr{J}\left(\mathbb{G}_{\mathsf{L},q}(\mathbb{K})\right) = \left\{\mathbb{K}u : u\in\Lambda_0^q\mathbb{K}^{\mathsf{L}}\right\}\,. \tag{11}$$

The set $\Lambda_0^p\mathbb{K}^{\mathsf{L}}$ is *not* a linear subspace of $\Lambda^p\mathbb{K}^{\mathsf{L}}$. Due to (11), however, we call the image of $\mathscr{J}$ the *projective space over* $\Lambda_0^p\mathbb{K}^{\mathsf{L}}$ and define — in abuse of the above notation — the expressions

$$\mathscr{P}\left(\Lambda_0^q\mathbb{K}^{\mathsf{L}}\right) := \mathscr{J}\left(\mathbb{G}_{\mathsf{L},q}(\mathbb{K})\right)\,,\qquad \mathscr{S}\left(\Lambda_0^q\mathbb{K}^{\mathsf{L}}\right) := \left\{u\in\Lambda_0^q\mathbb{K}^{\mathsf{L}} : \|u\|_\wedge = 1\right\}\,.$$

An overview is given by the following commutative diagram[10]:

$$\begin{tikzcd} \mathbb{F}_{\mathsf{L},q}(\mathbb{K}) \arrow[r, "\Pi"] \arrow[d, "\Lambda"'] & \mathbb{G}_{\mathsf{L},q}(\mathbb{K}) \arrow[d, "\mathscr{J}"] \\ \mathscr{S}\left(\Lambda_0^q\mathbb{K}^{\mathsf{L}}\right) \arrow[r, "{[\cdot]_\sim}"] & \mathscr{P}\left(\Lambda_0^q\mathbb{K}^{\mathsf{L}}\right) \end{tikzcd} \tag{12}$$

2.3 Random Dynamical Systems Generated by Sequences of i.i.d. Invertible Random Matrices

In Section 2.3.1, we define an action of the group $\mathrm{GL}(\mathsf{L},\mathbb{K})$ on the spaces occurring in (12).

[10]The author could not find the diagram in this form in the literature. However, it is well-known and easy to verify that this diagram commutes. This is *not* a new discovery of the author.

2.3.1 The Action of the General Linear Group[11]

Definition 13. The compact Stiefel manifold $\mathbb{F}_{\mathsf{L},q}(\mathbb{K})$ is acted on by $\mathrm{GL}(\mathsf{L},\mathbb{K})$ via

$$\mathrm{GL}(\mathsf{L},\mathbb{K}) \times \mathbb{F}_{\mathsf{L},q}(\mathbb{K}) \longrightarrow \mathbb{F}_{\mathsf{L},q}(\mathbb{K})\,, \qquad (\mathcal{T},\Phi) \longmapsto \mathcal{T}\cdot\Phi := \varpi(\mathcal{T}\Phi)\,, \tag{13}$$

where ϖ was defined by (9) in the previous section (see Section 2 of [82]).

Remark 10. If $q' \in \mathbb{N}$ is such that $q+q' \leq \mathsf{L}$ and $\Phi \in \mathbb{F}_{\mathsf{L},q}(\mathbb{K})$ and $\Phi' \in \mathbb{F}_{\mathsf{L},q'}(\mathbb{K})$ are such that $(\Phi,\Phi') \in \mathbb{F}_{\mathsf{L},q+q'}(\mathbb{K})$, then one has

$$\mathcal{T}\cdot(\Phi,\Phi') = (\mathcal{T}\cdot\Phi,\Phi'') \in \mathbb{F}_{\mathsf{L},q+q'}(\mathbb{K}) \tag{14}$$

for some $\Phi'' \in \mathbb{F}_{\mathsf{L},q'}(\mathbb{K})$ which is typically different from $\mathcal{T}\cdot\Phi'$.

Definition 14. The action (13) induces an action of $\mathrm{GL}(\mathsf{L},\mathbb{K})$ on the Grassmannian $\mathbb{G}_{\mathsf{L},q}(\mathbb{K})$ via

$$\mathrm{GL}(\mathsf{L},\mathbb{K}) \times \mathbb{G}_{\mathsf{L},q}(\mathbb{K}) \longrightarrow \mathbb{G}_{\mathsf{L},q}(\mathbb{K})\,, \qquad (\mathcal{T},\Pi(\Phi)) \longmapsto \mathcal{T}\bullet\Pi(\Phi) := \Pi(\mathcal{T}\cdot\Phi) \tag{15}$$

(see Section 2 of [90]).

To define an action of $\mathrm{GL}(\mathsf{L},\mathbb{K})$ on $\mathscr{P}\left(\Lambda_0^q\mathbb{K}^{\mathsf{L}}\right)$ and $\mathscr{S}\left(\Lambda_0^q\mathbb{K}^{\mathsf{L}}\right)$, we first recall how an invertible matrix $\mathcal{T} \in \mathrm{GL}(\mathsf{L},\mathbb{K})$ generates a linear operator $\Lambda^q\mathcal{T}$ on the exterior power $\Lambda^q\mathbb{K}^{\mathsf{L}}$.

Definition 15. For $\mathcal{T} \in \mathrm{GL}(\mathsf{L},\mathbb{K})$, we define the operator $\Lambda^q\mathcal{T}$ on $\Lambda^q\mathbb{K}^{\mathsf{L}}$ as the linear extension of

$$\Lambda_0^q\mathbb{K}^{\mathsf{L}} \longrightarrow \Lambda_0^q\mathbb{K}^{\mathsf{L}}\,, \qquad v_1\wedge\cdots\wedge v_q \longmapsto \mathcal{T}v_1\wedge\cdots\wedge\mathcal{T}v_q$$

to $\Lambda^q\mathbb{K}^{\mathsf{L}}$ (see [18], Chapter IV.2).

Remark 11. The operator $\Lambda^q\mathcal{T}$ leaves the set of decomposable q-vectors $\Lambda_0^q\mathbb{K}^{\mathsf{L}}$ invariant and its operator norm is realized on the decomposable q-vectors already, *i.e.*,

$$\|\Lambda^q\mathcal{T}\|_\Lambda := \sup\left\{\|\Lambda^q\mathcal{T}u\|_\wedge :\ u \in \mathscr{S}\left(\Lambda^q\mathbb{K}^{\mathsf{L}}\right)\right\} = \sup\left\{\|\Lambda^q\mathcal{T}u\|_\wedge :\ u \in \mathscr{S}\left(\Lambda_0^q\mathbb{K}^{\mathsf{L}}\right)\right\}$$

(see [19]).

The set of all such maps $\Lambda^q\mathcal{T}$ then induces an action of $\mathrm{GL}(\mathsf{L},\mathbb{K})$ on $\mathscr{P}\left(\Lambda_0^q\mathbb{K}^{\mathsf{L}}\right)$ and $\mathscr{S}\left(\Lambda_0^q\mathbb{K}^{\mathsf{L}}\right)$.

[11] The actions $\cdot$ and $\star$ defined in (13) and (17) were defined for a special case in the references provided here. The generalization to the situation herein is straightforward and can presumably be found elsewhere in the literature. This also applies to the action $\bullet$ defined in (15), where even the representation is different in the reference. As for the action $*$ defined in (16), the author did not find this very form in the literature but it is easily reconstructable after reading Sections 5.4 & 9.1 of the named reference.

Definition 16. The projective space $\mathscr{P}\left(\Lambda_0^q\mathbb{K}^{\mathsf{L}}\right)$ over $\Lambda_0^q\mathbb{K}^{\mathsf{L}}$ is acted on by $\mathrm{GL}(\mathsf{L},\mathbb{K})$ via

$$\begin{aligned} \mathrm{GL}(\mathsf{L},\mathbb{K}) \times \mathscr{P}\left(\Lambda_0^q\mathbb{K}^{\mathsf{L}}\right) &\longrightarrow \mathscr{P}\left(\Lambda_0^q\mathbb{K}^{\mathsf{L}}\right) , \\ (\mathcal{T}, \mathbb{K}u) &\longmapsto \mathcal{T} * (\mathbb{K}u) := \mathbb{K}\ \Lambda^q \mathcal{T} u \,, \qquad u \in \Lambda_0^q\mathbb{K}^{\mathsf{L}} \setminus \{0\} \end{aligned} \tag{16}$$

(*cf.* [27]) and the set of decomposable unit q-vectors $\mathscr{S}\left(\Lambda_0^q\mathbb{K}^{\mathsf{L}}\right)$ is acted on by $\mathrm{GL}(\mathsf{L},\mathbb{K})$ via

$$\begin{aligned} \mathrm{GL}(\mathsf{L},\mathbb{K}) \times \mathscr{S}\left(\Lambda_0^q\mathbb{K}^{\mathsf{L}}\right) &\longrightarrow \mathscr{S}\left(\Lambda_0^q\mathbb{K}^{\mathsf{L}}\right) , \\ (\mathcal{T}, u) &\longmapsto \mathcal{T} \star u := \Lambda^p \mathcal{T} u \left\| \Lambda^p \mathcal{T} u \right\|_{\wedge}^{-1} \end{aligned} \tag{17}$$

(see Section 3.2 of [88]).

All these actions are closely related to each other by the following commuting diagrams[12]:

$$\begin{tikzcd} \mathbb{F}_{\mathsf{L},q}(\mathbb{K}) \arrow[r, "\mathcal{T}\cdot"] \arrow[d, "\Lambda"'] & \mathbb{F}_{\mathsf{L},q}(\mathbb{K}) \arrow[d, "\Lambda"] \\ \mathscr{S}\left(\Lambda_0^q\mathbb{K}^{\mathsf{L}}\right) \arrow[r, "\mathcal{T}\star"] & \mathscr{S}\left(\Lambda_0^q\mathbb{K}^{\mathsf{L}}\right) \end{tikzcd} \qquad \begin{tikzcd} \mathbb{G}_{\mathsf{L},q}(\mathbb{K}) \arrow[r, "\mathcal{T}\bullet"] \arrow[d, "\mathscr{J}"'] & \mathbb{G}_{\mathsf{L},q}(\mathbb{K}) \arrow[d, "\mathscr{J}"] \\ \mathscr{P}\left(\Lambda_0^q\mathbb{K}^{\mathsf{L}}\right) \arrow[r, "\mathcal{T}*"] & \mathscr{P}\left(\Lambda_0^q\mathbb{K}^{\mathsf{L}}\right) \end{tikzcd} \tag{18}$$

2.3.2 The Lyapunov Exponents

We let $\{\mathcal{T}_\omega(n)\}_{n\in\mathbb{N}}$ be a sequence of i.i.d. invertible real $\mathsf{L}\times\mathsf{L}$ random matrices (throughout Section 2.3.2). The following Hypothesis holds *e.g.* in case of compactly supported $\mathcal{T}_\omega(n)$.

Hypothesis 1. The averages $\mathbf{E}\max\{\log\|\mathcal{T}_\omega(1)\|, 0\}$ and $\mathbf{E}\max\{\log\|\mathcal{T}_\omega(1)^{-1}\|, 0\}$ are finite.

Theorem 2. *Under Hypothesis 1, the numbers $\gamma_1, \ldots, \gamma_{\mathsf{L}}$ specified by the equations*

$$\sum_{l=1}^{q} \gamma_l = \lim_{N\to\infty} \frac{1}{N} \mathbf{E} \log \left\| \Lambda^q \left(\mathcal{T}_\omega(N) \ldots \mathcal{T}_\omega(1) \right) \right\|_{\Lambda} , \qquad q = 1, \ldots, \mathsf{L} , \tag{19}$$

are well-defined and finite. Moreover, $l \mapsto \gamma_l$ is non-increasing (see [15], Part A, Section III.5).

Definition 17. The numbers $\gamma_1, \ldots, \gamma_L$ specified by the equations (19) under Hypothesis 1 are called *Lyapunov exponents associated to* $\{\mathcal{T}_\omega(n)\}_{n\in\mathbb{N}}$ (see [15], Part A, Section III.5).

The following two hypotheses hold in many situations:

Hypothesis 2. The semigroup T generated by $\mathrm{supp}\left(\mathcal{T}_\omega(1)\right)$ is q-strongly irreducible, *i.e.*, for any finite union F of proper linear subspaces of $\Lambda^q\mathbb{R}^{\mathsf{L}}$, there is some $\mathcal{T} \in T$ such that $\Lambda^q\mathcal{T}F \not\subset F$.

[12] As with (12), the author could not find these diagrams in the depicted forms in the literature. Likewise, the diagrams are well-known and it is easy to verify that they commute. This is *not* a new discovery of the author.

Hypothesis 3. The semigroup T generated by $\operatorname{supp}(\mathcal{T}_\omega(1))$ is q-contracting, *i.e.*, there exists a sequence $\{\mathcal{T}_n\}_{n\in\mathbb{N}} \subset T$ for which $\Lambda^q\mathcal{T}_n \|\Lambda^q\mathcal{T}_n\|^{-1}$ converges to an operator of rank one.

Proposition 1. *Under Hypotheses 1, 2 and 3, all $u \in \Lambda^q\mathbb{R}^{\mathsf{L}} \setminus \{0\}$ satisfy*

$$\sum_{l=1}^{q} \gamma_l = \lim_{N\to\infty} \frac{1}{N} \log \|\Lambda^q (\mathcal{T}_\omega(N) \dots \mathcal{T}_\omega(1))\, u\|_\wedge \,, \qquad q = 1, \dots, \mathsf{L} \tag{20}$$

for $\mathbf{P}$*-almost every* ω (see [15], Part A Section IV.1, Theorem 1.2.iii) *so that one has in particular*

$$\sum_{l=1}^{q} \gamma_l = \lim_{N\to\infty} \frac{1}{N} \mathbf{E} \log \|\Lambda^q (\mathcal{T}_\omega(N) \dots \mathcal{T}_\omega(1))\, u\|_\wedge \,, \qquad q = 1, \dots, \mathsf{L}\,. \tag{21}$$

Another interesting property of the Lyapunov exponents is provided by the *Osseledec theorem*:

Theorem 3. *Assume Hypothesis 1. For* $\mathbf{P}$*-almost all* ω *there is decreasing sequence of subspaces*

$$\mathbb{R}^{\mathsf{L}} = V_\omega^1 \supset V_\omega^2 \supset \dots \supset V_\omega^{\mathsf{L}} \supset V_\omega^{\mathsf{L}+1} = \{0\}$$

for which all $l = 1, \dots, \mathsf{L}$ *satisfy*

$$\lim_{N\to\infty} \frac{1}{N} \log \|\mathcal{T}_\omega(N) \dots \mathcal{T}_\omega(1)\, v\| = \gamma_l \qquad \forall\, v \in V_\omega^l \setminus V_\omega^{l+1}$$

(*cf.* [18], Section IV.2).

A plenty of special properties in the case of symplectic $\mathcal{T}_\omega(n)$ is discussed in Section 2.3.5.

2.3.3 Random Dynamical Systems

In Section 2.3.1, we defined an action of the group of invertible matrices $\mathrm{GL}(\mathsf{L}, \mathbb{K})$ on the spaces $\mathbb{F}_{\mathsf{L},q}(\mathbb{K})$, $\mathbb{G}_{\mathsf{L},q}(\mathbb{K})$, $\mathscr{S}\left(\Lambda_0^q\mathbb{K}^{\mathsf{L}}\right)$ and $\mathscr{P}\left(\Lambda_0^q\mathbb{K}^{\mathsf{L}}\right)$, respectively. Now a sequence $\{\mathcal{T}_\omega(n)\}_{n\in\mathbb{N}}$ of i.i.d. invertible (real or complex) $\mathsf{L}\times\mathsf{L}$ random matrices can be used to define *random dynamical systems* on these spaces, namely as

$$\Phi_\omega(n) = \mathcal{T}_\omega(n) \cdot \Phi_\omega(n-1) \qquad \forall\, n \in \mathbb{N}\,, \qquad \text{where} \quad \Phi_\omega(0) \equiv \Phi(0) \in \mathbb{F}_{\mathsf{L},q}(\mathbb{K})\,, \tag{22}$$

on the compact Stiefel manifold, as

$$Q_\omega(n) = \mathcal{T}_\omega(n) \bullet Q_\omega(n-1) \qquad \forall\, n \in \mathbb{N}\,, \qquad \text{where} \quad Q_\omega(0) \equiv Q(0) \in \mathbb{G}_{\mathsf{L},q}(\mathbb{K})\,, \tag{23}$$

on the Grassmannian, as

$$u_\omega(n) = \mathcal{T}_\omega(n) \star u_\omega(n-1) \qquad \forall\, n \in \mathbb{N}\,, \qquad \text{where} \quad u_\omega(0) \equiv u(0) \in \mathscr{S}\left(\Lambda_0^q\mathbb{K}^{\mathsf{L}}\right)\,, \tag{24}$$

on the decomposable unit q-vectors and

$$\mathtt{u}_\omega(n) = \mathcal{T}_\omega(n) * \mathtt{u}_\omega(n-1) \qquad \forall\, n \in \mathbb{N}\,, \qquad\qquad \text{where} \quad \mathtt{u}_\omega(0) \equiv \mathtt{u}(0) \in \mathscr{P}\left(\Lambda_0^p \mathbb{K}^{\mathsf{L}}\right)\,, \tag{25}$$

on the projective space over the decomposable q-vectors[13]. All these random dynamical systems are closely related to each other via the commuting diagrams in (12) and (18).

Remark 12. A telescoping argument[14] allows to rewrite the identities (20) or (21) for the Lyapunov exponents (under Hypotheses 1, 2 and 3) in terms of the random dynamical system (24). Indeed, all $u(0) \in \mathscr{S}\left(\Lambda_0^p \mathbb{R}\right)$ satisfy

$$\sum_{l=1}^{q} \gamma_l = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \log \left\| \Lambda^q \mathcal{T}_\omega(n+1)\, u_\omega(n) \right\|_\wedge\,, \qquad\qquad q = 1, \ldots, \mathsf{L} \tag{26}$$

for $\mathbf{P}$-almost every ω and one has

$$\sum_{l=1}^{q} \gamma_l = \lim_{N\to\infty} \frac{1}{N} \mathbf{E} \sum_{n=0}^{N-1} \log \left\| \Lambda^q \mathcal{T}_\omega(n+1)\, u_\omega(n) \right\|_\wedge\,, \qquad\qquad q = 1, \ldots, \mathsf{L}\,. \tag{27}$$

The dynamics (23) or (25) can also be used to rewrite (26) and (27), namely by choosing representatives of $\mathtt{u}_\omega(n)$ in $\mathscr{S}\left(\Lambda_0^q \mathbb{R}^{\mathsf{L}}\right)$. Clearly, the choice has no impact on the norms in (26) and (27).

Remark 13. As the first diagram in (18) commutes, (26) implies that all $\Phi(0) \in \mathbb{F}_{\mathsf{L},q}(\mathbb{R})$ obey

$$\sum_{l=1}^{q} \gamma_l = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \log \left\| \Lambda^q \mathcal{T}_\omega(n+1)\, \Lambda\Phi_\omega(n) \right\|_\wedge\,, \qquad\qquad q = 1, \ldots, \mathsf{L} \tag{28}$$

for $\mathbf{P}$-almost every ω (whenever Hypotheses 1, 2 and 3 are fulfilled) so that one has in particular

$$\sum_{l=1}^{q} \gamma_l = \lim_{N\to\infty} \frac{1}{N} \mathbf{E} \sum_{n=0}^{N-1} \log \left\| \Lambda^q \mathcal{T}_\omega(n+1)\, \Lambda\Phi_\omega(n) \right\|_\wedge\,, \qquad\qquad q = 1, \ldots, \mathsf{L}\,. \tag{29}$$

Henceforth, we will mainly work with the random dynamical system (22) on the compact Stiefel manifold, since it is the most informative. The other systems (23), (24) and (25) can readily be obtained from (22) with the aid of the projections Π, Λ and $[\cdot]_\sim \circ \Lambda$ (see (12)).

[13]These types of random dynamical systems can be found in the literature, see *e.g.* [19] for the Grassmannian or Section 3.3 of [88] for a special case of (24).

[14]The standard telescoping argument to rewrite (20) and (21) can *e.g.* be found in the introduction of [15] for the case $q = 1$. The derivation works also if $q > 1$, where the result remains valid (see Section 4.2 of [88]).

2.3.4 Invariant Measures and the Furstenberg Measure

We let $\mathcal{T}_\sigma$ be an invertible $\mathsf{L} \times \mathsf{L}$ random matrix and suppose that $\{\mathcal{T}_\omega(n)\}_{n\in\mathbb{N}}$ is a sequence of independent copies of $\mathcal{T}_\sigma$ (throughout Section 2.3.4).

Definition 18. A probability measure μ on $\mathbb{F}_{\mathsf{L},q}(\mathbb{K})$ for which all indicator functions $f = \chi_A$ of Borel sets $A \subset \mathbb{F}_{\mathsf{L},q}(\mathbb{K})$ satisfy

$$\int_{\mathbb{F}_{\mathsf{L},q}(\mathbb{K})} \mathrm{d}\mu(\Phi)\ f(\Phi) \ = \ \int_{\mathbb{F}_{\mathsf{L},q}(\mathbb{K})} \mathrm{d}\mu(\Phi)\ \mathbb{E}\, f(\mathcal{T}_\sigma \cdot \Phi) \tag{30}$$

is called $(\mathcal{T}_\sigma\cdot)$-*invariant*[15]. In an analogous way, we define the notions of $(\mathcal{T}_\sigma\bullet)$-, $(\mathcal{T}_\sigma\star)$- and $(\mathcal{T}_\sigma *)$-*invariant* probability measure on the spaces $\mathbb{G}_{\mathsf{L},q}(\mathbb{K})$, $\mathscr{S}(\Lambda_0^q\mathbb{K}^{\mathsf{L}})$ and $\mathscr{P}(\Lambda_0^q\mathbb{K}^{\mathsf{L}})$, respectively.

Remark 14. Let ϱ_0 be a probability measure on $\mathbb{F}_{\mathsf{L},q}(\mathbb{K})$ and consider the sequence

$$(\xi_N)_{N\in\mathbb{N}}\,, \qquad \text{where} \qquad \xi_N \ = \ \frac{1}{N}\sum_{n=1}^{N} \varrho_n\,, \qquad\qquad \text{with} \qquad \varrho_n \ = \ \left((\mathcal{T}_\sigma\cdot)_*\right)^{(n)}(\varrho_0)\,,$$

where $\left((\mathcal{T}_\sigma\cdot)_*\right)^{(n)}(\varrho_0)$ is the n-th iterate of the image of ϱ_0 under $\mathcal{T}_\sigma\cdot$ (see *e.g.* [14], Chapter 3.6). Every weak limit point of $(\xi_N)_{N\in\mathbb{N}}$ is a $(\mathcal{T}_\sigma\cdot)$-invariant probability measure and, as $\mathbb{F}_{\mathsf{L},q}(\mathbb{K})$ is compact, at least one such weak limit point exists (see [15], Part A, Chapter I.3, Lemma 3.5).

If ϱ_0 is the Dirac measure (see *e.g.* [14], Section 1.3) at some $\Phi(0) \in \mathbb{F}_{\mathsf{L},q}(\mathbb{K})$, then ϱ_n is the probability measure according to which the random dynamics $\Phi_\omega(n)$ given by (22) is distributed.

Remark 15. We call a *unique* $(\mathcal{T}_\sigma\bullet)$- or $(\mathcal{T}_\sigma *)$-invariant measure the *Furstenberg measure*[16].

Proposition 2. *Suppose that $\mathcal{T}_\sigma$ is $\mathrm{GL}(\mathsf{L},\mathbb{R})$-valued and that $\{\mathcal{T}_\omega(n)\}_{n\in\mathbb{N}}$ fulfils Hypotheses 1, 2 and 3. Then there is a unique $(\mathcal{T}_\sigma *)$-invariant Furstenberg measure μ_{F} on $\mathscr{P}\left(\Lambda_0^q\mathbb{R}^{\mathsf{L}}\right)$ and any corresponding probability measure $\tilde{\mu}_{\mathrm{F}}$ on $\mathscr{S}\left(\Lambda_0^q\mathbb{R}^{\mathsf{L}}\right)$ satisfying $\tilde{\mu}_{\mathrm{F}} \circ ([\cdot]_\sim)^{-1} = \mu_{\mathrm{F}}$ is linked to the associated Lyapunov exponents via*

$$\sum_{l=1}^{q} \gamma_l = \int_{\mathscr{S}\left(\Lambda_0^q\mathbb{R}^{\mathsf{L}}\right)} \mathrm{d}\tilde{\mu}_{\mathrm{F}}(u)\ \mathbb{E}\,\log\left\|\Lambda^q\mathcal{T}_\sigma\, u\right\|_\wedge$$

(see [15], Part A, Section IV.1, Theorem 1.2.ii).

[15] The notion of invariant measure is common in the literature. However, the assumptions on the function f in (30) are varying. For example, the definition in [15], Part A, Section III.2 requires the validity of (30) for all f being indicator functions of Borel sets, whereas there are simply made no assumptions on f in [88], Section 3.3.

[16] This wording is repeatedly used in the literature, see *e.g.* [58].

Remark 16. In the special case $q = 1$, where $\mathscr{P}(\Lambda_0^1\mathbb{R}^{\mathsf{L}}) = \mathbb{R}\mathsf{P}^{\mathsf{L}-1}$, and again under Hypotheses 1, 2 and 3, the distribution of the dynamics $\mathtt{u}_\omega(n)$ given by (25) converges to μ_{F} even uniformly in the starting point $\mathtt{u}(0)$, namely one has

$$\sup_{\mathtt{u}(0)\in\mathbb{R}\mathsf{P}^{\mathsf{L}-1}} \left| \mathbf{E}\, f(\mathtt{u}_\omega(n)) - \int_{\mathbb{R}\mathsf{P}^{\mathsf{L}-1}} \mathrm{d}\mu_{\mathrm{F}}(\mathtt{u})\, f(\mathtt{u}) \right| \to 0 \qquad \text{as} \qquad n \to \infty$$

for all continuous functions $f : \mathbb{R}\mathsf{P}^{\mathsf{L}-1} \to \mathbb{R}$ (see [15], Part A, Chapter III.4, Theorem 4.3).

2.3.5 The Case of Symplectic Matrices[17]

Random dynamical systems that are generated by i.i.d. *symplectic* matrices in the way described in Section 2.3.3 are of particular importance for the long-term project of the author and his supervisor. In the following, we set $\mathsf{L} \equiv 2L$ and $q \leq L$ and recapitulate relevant related facts.

We define certain subspaces of $\mathbb{F}_{2L,q}(\mathbb{R})$ and $\mathbb{G}_{2L,q}(\mathbb{R})$ that play an important role in the context of random dynamical systems generated by symplectic random matrices.

Definition 19. We call a q-frame $\Phi \in \mathbb{F}_{2L,q}(\mathbb{R})$ *isotropic* if it satisfies $\Phi^*\mathcal{J}\Phi = 0$. The set of all isotropic q-frames is called the *isotropic Stiefel manifold* $\mathbb{IF}_{2L,q}(\mathbb{R})$ and the set of all isotropic L-frames is called the *Lagrangian Stiefel manifold* $\mathbb{LF}_L(\mathbb{R}) = \mathbb{IF}_{2L,L}(\mathbb{R})$. Accordingly, we define the *isotropic Grassmannian* by $\mathbb{IG}_{2L,q}(\mathbb{R}) = \{\Pi(\Phi) : \Phi \in \mathbb{IF}_{2L,q}(\mathbb{R})\}$ and the *Lagrangian Grassmannian* by $\mathbb{LG}_L(\mathbb{R}) = \mathbb{IG}_{2L,L}(\mathbb{R})$.

Remark 17. The sets $\mathbb{IF}_{2L,q}(\mathbb{R})$ and $\mathbb{IG}_{2L,q}(\mathbb{R})$ are invariant under the action of $\mathrm{SP}(2L,\mathbb{R})$.

The symmetry of the eigenvalues of symplectic matrices is recognizable in the Lyapunov exponents of sequences of i.i.d. symplectic random matrices that fulfil Hypothesis 1:

Proposition 3. *Let $\{\mathcal{T}_\omega(n)\}_{n\in\mathbb{N}}$ be a sequence of i.i.d. symplectic $2L \times 2L$ random matrices that fulfils Hypothesis 1. Then, the associated Lyapunov exponents exhibit the symmetry*

$$\gamma_l = -\gamma_{2L-l+1}\,, \qquad l = 1, \dots, L\,. \tag{31}$$

One must consider that Hypothesis 2 is always violated for $q = 2$ if the $\mathcal{T}_\omega(n)$ are symplectic. Nevertheless, it is convenient to replace it by the following alternative Hypothesis:

Hypothesis 4. The semigroup T generated by $\mathrm{supp}\,(\mathcal{T}_\omega(1))$ is $\mathbb{IF}_{2L,q}(\mathbb{R})$-strongly irreducible, *i.e.*, for any finite union F of proper linear subspaces of $\mathrm{span}\,(\Lambda\,(\mathbb{IF}_{2L,q}(\mathbb{R})))$, there is some $\mathcal{T} \in T$ such that $\Lambda^q\mathcal{T}F \neq F$.

Under Hypothesis 4 (together with Hypotheses 1 and 3), the identities (28) and (29) remain true for all isotropic q-frames $\Phi(0) \in \mathbb{IF}_{2L,q}(\mathbb{R})$. Similarly, Proposition 2 can be adapted:

[17] The author used [88], [90] and [15], Part A, Section IV.3 to recapitulate the facts stated in this section.

Proposition 4. *Let $\mathcal{T}_\sigma$ be a symplectic $2L \times 2L$ random matrix. Moreover, let $\{\mathcal{T}_\omega(n)\}_{n\in\mathbb{N}}$ be a sequence of independent copies of $\mathcal{T}_\sigma$ that fulfils Hypotheses 1, 3 and 4. Then there is a unique $(\mathcal{T}_\sigma *)$-invariant Furstenberg measure μ_{F} on the subset $([\cdot]_\sim \circ \Lambda)(\mathbb{IF}_{2L,q}(\mathbb{R}))$ of $\mathscr{P}\left(\Lambda_0^q \mathbb{R}^{2L}\right)$ and any corresponding probability measure $\tilde{\mu}_{\mathrm{F}}$ on the subset $\Lambda(\mathbb{IF}_{2L,q}(\mathbb{R}))$ of $\mathscr{S}\left(\Lambda_0^q \mathbb{R}^{2L}\right)$ satisfying $\tilde{\mu}_{\mathrm{F}} \circ ([\cdot]_\sim)^{-1} = \mu_{\mathrm{F}}$ is linked to the associated Lyapunov exponents via*

$$\sum_{l=1}^{q} \gamma_l = \int_{\Lambda\left(\mathbb{IF}_{2L,q}(\mathbb{R})\right)} \mathrm{d}\tilde{\mu}_{\mathrm{F}}(u)\ \mathbb{E}\ \log \|\Lambda^q \mathcal{T}_\sigma\, u\|_\wedge\ .$$

2.3.6 Symplectic Channels

The values of the random matrices the author and his supervisor deal with in their long-term project are not only symplectic but even belong to the subclass of matrices of the form

$$\mathcal{T} = \begin{pmatrix} A & -\mathbf{1}_L \\ \mathbf{1}_L & 0 \end{pmatrix}, \qquad\qquad A = A^\intercal \in \mathbb{R}^{L\times L}. \tag{32}$$

These matrices are decomposable into L two-dimensional eigenspaces of eigenvalue pairs called *symplectic channels* as *e.g.* considered in [88], Section 2.3.

Proposition 5 follows by mimicking the proof of Proposition 11.1 in [92] in this special case.

Proposition 5. *Let $\mathcal{T}$ be of the form (32). Then, there is an $\mathcal{S} \in \mathrm{SP}(2L, \mathbb{R})$ such that one has*

$$\mathcal{S}^{-1}\mathcal{T}\mathcal{S} = \mathscr{U}\left(S(d_1) \oplus \cdots \oplus S(d_L)\right)\mathscr{U}^{-1}, \qquad\qquad S(d) = \begin{pmatrix} d & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SP}(2, \mathbb{R}), \tag{33}$$

where $d_1, \ldots, d_L \in \mathbb{R}$ are such that $|d_1| \geq \cdots \geq |d_L|$ and $\mathscr{U}$ is an isomorphism given by

$$\mathscr{U} : \bigoplus_{l=1}^{L} \mathbb{C}^{2\times 2} \to \mathscr{U}\left(\bigoplus_{l=1}^{L} \mathbb{C}^{2\times 2}\right) \subset \mathbb{C}^{2L\times 2L}, \quad \bigoplus_{l=1}^{L} \begin{pmatrix} \alpha_l & \beta_l \\ \gamma_l & \delta_l \end{pmatrix} \mapsto \begin{pmatrix} \alpha_1 & & & \beta_1 & & \\ & \ddots & & & \ddots & \\ & & \alpha_L & & & \beta_L \\ \gamma_1 & & & \delta_1 & & \\ & \ddots & & & \ddots & \\ & & \gamma_L & & & \delta_L \end{pmatrix}.$$

Remark 18. An eigenvalue pair $\{\kappa_l, \kappa_l^{-1}\}$ of some $S(d_l)$ is given by $\left\{\frac{1}{2}\left[d_l \pm \sqrt{d_l^2 - 4}\,\right]\right\}$. We set

$$\kappa_l = \frac{1}{2}\left[d_l + \mathrm{sgn}(d_l)\sqrt{d_l^2 - 4}\right]$$

without loss of generality, which implies $|\kappa_l| \geq 1$ and the sequence of inequalities

$$|\kappa_1| \geq \cdots \geq |\kappa_L| \geq |\kappa_L^{-1}| \geq \cdots \geq |\kappa_1^{-1}| > 0\,. \tag{34}$$

Definition 20. Based on $|d_l|$, we subdivide the matrices $S(d_l)$ into three types[18]:

- If $|d_l| > 2$, the matrix $S(d_l)$ is called *hyperbolic* and κ_l and κ_l^{-1} lie on the real line.
- If $|d_l| = 2$, the matrix $S(d_l)$ is called *parabolic* and κ_l and κ_l^{-1} are either $+1$ or -1.
- If $|d_l| < 2$, the matrix $S(d_l)$ is called *elliptic* and κ_l and κ_l^{-1} lie on the complex unit circle.

We call the associated eigenspaces *symplectic channels* and denote them by $\mathscr{C}_l$. A symplectic channel $\mathscr{C}_l$ is then spanned by the l-th and the $(L+l)$-th canonical unit vector and is also called *hyperbolic*, *parabolic* and *elliptic* for $|d_l| > 2$, $|d_l| = 2$ and $|d_l| < 2$, respectively.

Proposition 6 describes a phenomenon called *symplectic blocking* in [88].

Proposition 6. *The intersection of a symplectic channel $\mathscr{C}_l$ and the range of some $\Phi \in \mathbb{IF}_{2L,q}(\mathbb{K})$ or $Q \in \mathbb{IG}_{2L,q}(\mathbb{K})$ is at most one-dimensional. In particular, the intersection of $\mathscr{C}_l$ and the range of some $\Phi \in \mathbb{LF}_L(\mathbb{K})$ or $Q \in \mathbb{LG}_L(\mathbb{K})$ is precisely one-dimensional.*

The matrices $S(d_l)$ can be brought into a canonical form by a symplectic conjugation.

Lemma 7. *For each $S(d_l)$, there is some $M(d_l) \in \mathrm{SP}(2,\mathbb{R})$ such that $M(d_l)^{-1}S(d_l)M(d_l)$ reads*

$$R_h(\kappa_l) = \begin{pmatrix} \kappa_l^{-1} & \\ & \kappa_l \end{pmatrix} \quad \text{or} \quad R_p(\kappa_l) = \operatorname{sgn}(\kappa_l) \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \quad \text{or} \quad R_e(\kappa_l) = \begin{pmatrix} \cos(\eta_l) & \sin(\eta_l) \\ -\sin(\eta_l) & \cos(\eta_l) \end{pmatrix} ,$$

where $\eta_l \in [0,\pi)$ is such that $\kappa_l = e^{-\imath\eta_l}$, depending on whether $S(d_l)$ is hyperbolic or parabolic or elliptic (see [24], Section 11.4 and [88], Section 2.1).

Remark 19. In the elliptic case, a diagonalization is possible by means of the *Cayley transform*, albeit at the expense of leaving $\mathrm{SP}(2,\mathbb{R})$, viz.,

$$\widehat{R}_e(\kappa_l) = CR_e(\kappa_l)C^{-1} = \begin{pmatrix} \kappa_l^{-1} & \\ & \kappa_l \end{pmatrix} , \qquad C = \sqrt{\frac{-\imath}{2}} \begin{pmatrix} 1 & -\imath \\ 1 & \imath \end{pmatrix} ,$$

where $\widehat{R}_e(\kappa_l)$ lies in the *generalized Lorentz group* $\mathrm{SU}(1,1) = C\ \mathrm{SP}(2,\mathbb{R})C^{-1}$ given by

$$\mathrm{SU}(1,1) = \left\{ T \in \mathbb{C}^{2\times 2} : T^*GT = G\,,\ \det(T) = 1 \right\} , \qquad G = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

(see *e.g.* [89], Section 2).

To summarize Section 2.3.6, we formulate Corollary 8 and a subsequent remark[19].

[18] One has $\mathrm{SP}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})$ (see *e.g.* [51], Example 4.8.1). The subdivision of $\mathrm{SL}(2,\mathbb{R})$ into hyperbolic, parabolic and elliptic matrices can *e.g.* be found in [24] as the beginning of Section 11.4 combined with Exercise 11.2.

[19] The author could not find Corollary 8 or Remark 20 in this form in the literature. The statements therein are easily verifiable and possibly findable in the literature. The author does *not* claim their novelty.

Corollary 8. *Each $\mathcal{T}$ of the form (32) can be decomposed as*

$$\widetilde{\mathcal{T}} = \mathcal{M}^{-1}\mathcal{T}\mathcal{M} = \mathscr{U}\left(\bigoplus_{l=1}^{L_h} R_h(\kappa_l) \oplus \bigoplus_{l=L_h+1}^{L_h+L_p} R_p(\kappa_l) \oplus \bigoplus_{l=L-L_e+1}^{L} R_e(\kappa_l)\right)\mathscr{U}^{-1}, \tag{35}$$

where $\mathcal{M} = \mathcal{S}\mathscr{U}\left(\bigoplus_{l=1}^{L} M(d_l)\right)\mathscr{U}^{-1}$ lies in $\mathrm{SP}(2L,\mathbb{R})$ and $\mathcal{S}$ and $\mathscr{U}$ are as in Proposition 5. Here, L_h, L_p and L_e denotes the number of hyperbolic, parabolic and elliptic channels, respectively. Further, if $\mathcal{T}$ has no parabolic channels, i.e., if $L_p = 0$, a diagonalization of $\mathcal{T}$ is given by

$$\widehat{\mathcal{T}} = \mathcal{W}_{L_h}\widetilde{\mathcal{T}}\mathcal{W}_{L_h}^{-1} = \mathcal{V}\mathscr{U}\left(\bigoplus_{l=1}^{L}\begin{pmatrix}\kappa_l^{-1} & \\ & \kappa_l\end{pmatrix}\right)\mathscr{U}^{-1}\mathcal{V} = \mathrm{diag}\big(\kappa_1^{-1},\dots,\kappa_L^{-1},\kappa_L,\dots,\kappa_1\big) \tag{36}$$

with $\mathcal{W}_{L_h} = (\mathcal{C}_{L_h}\mathcal{V})^{-1}$ and $\mathcal{C}_{L_h} = \mathscr{U}\left(\bigoplus_{l=1}^{L_h}\mathbf{1}_2 \oplus \bigoplus_{l=L_h+1}^{L} C\right)\mathscr{U}^{-1}$, where the permutation

$$\mathcal{V} = \mathrm{diag}\left(\mathbf{1}_L, \begin{pmatrix} & & 1\\ & \iddots & \\ 1 & & \end{pmatrix}\right)$$

makes the diagonal entries non-decreasing in modulus according to inequality (34).

Remark 20. Given some $\mathcal{T} \in \mathrm{SP}(2L,\mathbb{R})$, the matrix $\widetilde{\mathcal{T}}$ obtained by (35) lies again in $\mathrm{SP}(2L,\mathbb{R})$ but $\widehat{\mathcal{T}}$ obtained by (36) lies in $\mathcal{W}_{L_h}\mathrm{SP}(2L,\mathbb{R})\mathcal{W}_{L_h}^{-1}$, which is a different subgroup of $\mathrm{GL}(2L,\mathbb{C})$. The above statements on random dynamical systems generated by i.i.d. symplectic matrices can be readily transposed. Similarly to Remark 17, the sets $\mathcal{W}_{L_h}\mathbb{IF}_{2L,q}(\mathbb{R}) \subset \mathbb{F}_{2L,q}(\mathbb{C})$ and $\mathcal{W}_{L_h}\mathbb{IG}_{2L,q}(\mathbb{R})\mathcal{W}_{L_h}^{-1} \subset \mathbb{G}_{2L,q}(\mathbb{C})$ are invariant under the action of $\mathcal{W}_{L_h}\mathrm{SP}(2L,\mathbb{R})\mathcal{W}_{L_h}^{-1}$. A transformed symplectic channel $\mathcal{W}_{L_h}\mathscr{C}_l$ is spanned by the l-th and the $(2L-l)$-th canonical unit vector. Similarly to Proposition 6, the intersection of a transformed channel $\mathcal{W}_{L_h}\mathscr{C}_l$ and the range of some $\Phi \in \mathcal{W}_{L_h}\mathbb{IF}_{2L,q}(\mathbb{R})$ or $Q \in \mathcal{W}_{L_h}\mathbb{IG}_{2L,q}(\mathbb{R})\mathcal{W}_{L_h}^{-1}$ is at most one-dimensional. In particular, the intersection of $\mathcal{W}_{L_h}\mathscr{C}_l$ and the range of some $\Phi \in \mathcal{W}_{L_h}\mathbb{LF}_L(\mathbb{K})$ or $Q \in \mathcal{W}_{L_h}\mathbb{LG}_L(\mathbb{K})\mathcal{W}_{L_h}^{-1}$ is precisely one-dimensional.

Remark 21. We call a diagonal matrix whose diagonal entries are non-zero and non-decreasing in modulus a *hyperbolic matrix*[20].

2.4 The Möbius Transformation and the Random Möbius Dynamics

In Sections 2.2 and 2.3, we introduced the spaces $\mathbb{F}_{\mathsf{L},q}(\mathbb{K})$, $\mathbb{G}_{\mathsf{L},q}(\mathbb{K})$, $\mathscr{S}\left(\Lambda_0^q\mathbb{K}^{\mathsf{L}}\right)$ and $\mathscr{P}\left(\Lambda_0^q\mathbb{K}^{\mathsf{L}}\right)$ and defined an action of the general linear group $\mathrm{GL}(\mathsf{L},\mathbb{K})$ on each of these spaces, on which a sequence of i.i.d. invertible $\mathsf{L}\times\mathsf{L}$ random matrices then induces a random dynamical system.

In the case $q=1$, $\mathsf{L}=2$ and $\mathbb{K}=\mathbb{C}$, the projective space over $\Lambda_0^1\mathbb{C}^2 = \mathbb{C}^2$ is given by the *complex projective line* $\mathbb{C}\mathsf{P}^1$, which can be identified with $\overline{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$ (see [79], Section 17.1).

[20]This is not common in the literature, where the term "hyperbolic matrix" is usually used for a square matrix whose eigenvalues lie off the imaginary axis, see, *e.g.*, [66], Chapter 3, Section 5 or [11], §2.5.3.

The latter is called the *Riemann sphere* (see *e.g.* [65], Chapter 1). The unit sphere of $\mathbb{C}^2$ is given by $\mathbb{S}^1_{\mathbb{C}} = \{u \in \mathbb{C}^2 : \|u\| = 1\}$. For $u \in \mathbb{S}^1_{\mathbb{C}}$, a point $\mathbb{C}u \in \mathbb{CP}^1$ is identified with $\pi(u)$, where

$$\pi(u) = \begin{cases} ab^{-1}\,, & b \neq 0\,, \\ \infty\,, & b = 0\,, \end{cases} \qquad \text{where} \qquad u = \begin{pmatrix} a \\ b \end{pmatrix}$$

(see [79], Section 17.1). The action (16) is then converted into the *Möbius transformation*

$$\begin{aligned} &\mathrm{GL}(2,\mathbb{C}) \times \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}\,, \\ &\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \right) \longmapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \diamond z := \begin{cases} \infty\,, & z = -dc^{-1}\,, \\ ac^{-1}\,, & z = \infty\,, \\ \frac{az+b}{cz+d}\,, & \text{otherwise}\,, \end{cases} \end{aligned} \tag{37}$$

i.e., all $\mathcal{T} \in \mathrm{GL}(2,\mathbb{C})$ and all $u \in \mathbb{S}^1_{\mathbb{C}}$ satisfy the identity

$$\mathcal{T} \diamond \pi(u) = \pi(\mathcal{T} \star u) \tag{38}$$

(see [79], Section 17.3).

Equipped with the Möbius transformation (37) and the relation (38), we now transfer some notions and facts discussed in Section 2.3 to the case of invertible complex 2×2 random matrices. For this, we let $\mathcal{T}_\sigma$ be an invertible complex 2×2 random matrix and suppose that $\{\mathcal{T}_\omega(n)\}_{n\in\mathbb{N}}$ is a sequence of independent copies of $\mathcal{T}_\sigma$ in the following.

Definition 21. A probability measure μ on $\overline{\mathbb{C}}$ is called $(\mathcal{T}_\sigma\,\diamond)$-*invariant* if it satisfies

$$\int_{\overline{\mathbb{C}}} \mathrm{d}\mu(z)\; f(z) = \int_{\overline{\mathbb{C}}} \mathrm{d}\mu(z)\; \mathbb{E}\, f(\mathcal{T}_\sigma \diamond z)$$

for all indicator functions $f = \chi_A$ of Borel sets $A \subset \overline{\mathbb{C}}$.

Remark 22. We call a unique $(\mathcal{T}_\sigma\,\diamond)$-invariant measure the *Furstenberg measure* (cf. Remark 15).

We introduce the upper Lyapunov exponent γ_1 also for sequences of i.i.d. invertible complex 2×2 random matrices.

Definition 22. Suppose that $\{\mathcal{T}_\omega(n)\}_{n\in\mathbb{N}}$ satisfies Hypothesis 1. Then the number

$$\gamma_1 = \lim_{N\to\infty} \frac{1}{N} \mathbf{E}\, \log \|\mathcal{T}_\omega(N) \dots \mathcal{T}_\omega(1)\|$$

is (well-defined and finite and) called the *upper Lyapunov exponent*[21].

[21] The definition of the upper Lyapunov exponent works for invertible *complex* 2×2 random matrices just as for *real* ones. For this, Definition 2.1 in [15], Part A, Section I.2 may be translated using natural realifications $\mathbb{C}^2 \to \mathbb{R}^4$ and $\mathrm{GL}(2,\mathbb{C}) \to \mathrm{GL}(4,\mathbb{R})$. *E.g.*, for the case of $\mathrm{SL}(\mathbb{C})/\{\pm\}$-valued random matrices, see [25], Section 4.

The random dynamics (25) on $\mathbb{C}\mathsf{P}^1$ can be converted into a random dynamics on $\overline{\mathbb{C}}$.

Definition 23. The sequence $\{\mathcal{T}_\omega(n)\}_{n\in\mathbb{N}}$ induces a *random Möbius dynamics* by

$$z_\omega(n) = \mathcal{T}_\omega(n) \diamond z_\omega(n-1) \qquad \forall\, n \in \mathbb{N}\,, \qquad\qquad \text{where} \quad z_\omega(0) \equiv z(0) \in \overline{\mathbb{C}}\,. \tag{39}$$

The following two Hypotheses are similar to Hypotheses 2 and 3.

Hypothesis 5. For any finite set $F \subset \overline{\mathbb{C}}$, one has $\operatorname{supp}(\mathcal{T}_\omega(1)) \diamond F \not\subset F$.

Hypothesis 6. The semigroup generated by $\operatorname{supp}(\mathcal{T}_\omega(1))$ is not relatively compact.

If the values of $\mathcal{T}_\omega(1)$ lie in

$$\mathrm{SL}(2,\mathbb{C}) = \left\{T \in \mathbb{C}^{2\times 2} : \det(T) = 1\right\}\,,$$

then the Hypotheses 5 and 6 imply the uniqueness of the $(\mathcal{T}_\omega(1)\,\diamond)$-invariant probability measure.

Proposition 9. *Suppose that the $\mathcal{T}_\omega(n)$ are* $\mathrm{SL}(2,\mathbb{C})$*-valued and satisfy Hypotheses 1, 5 and 6. Then there is a unique $(\mathcal{T}_\omega(1)\,\diamond)$-invariant Furstenberg measure μ_{F} on $\overline{\mathbb{C}}$* (see [13], Proposition 4.7)*. Moreover, any corresponding probability measure $\tilde{\mu}_{\mathrm{F}}$ on $\mathbb{S}^1_{\mathbb{C}}$ satisfying $\tilde{\mu}_{\mathrm{F}} \circ \pi^{-1} = \mu_{\mathrm{F}}$ is linked to the upper Lyapunov exponent associated to $\{\mathcal{T}_\omega(n)\}_{n\in\mathbb{N}}$ via*

$$\gamma_1 = \int_{\mathbb{S}^1_{\mathbb{C}}} \mathrm{d}\tilde{\mu}_{\mathrm{F}}(u)\ \mathbf{E}\,\log \|\mathcal{T}_\omega(1)\,u\| \tag{40}$$

(cf. [13], Theorem 4.28).

Such random Möbius dynamics have *e.g.* been studied by Ambroladze and Wallin [6], who considered a situation, where the support of the $\mathcal{T}_\omega(n)$ is such that the actions $\mathcal{T}_\omega(n)\diamond$ map the complex upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \Im\mathfrak{m}(z) > 0\}$ onto itself. Thus the orbit $\{z_\omega(n)\}_{n\in\mathbb{N}}$ lies in $\mathbb{H}$ whenever $z(0)$ does. Ambroladze and Wallin provide a certain assumption under which this orbit is proven to tend to $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ almost surely (see [6], Theorem 1).

Now since the Möbius transformation $C\diamond$ under the Cayley transform (see Remark 19) maps $\mathbb{H}$ bijectively onto the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ (see [15], Part A, Section II.7), a conjugation of the $\mathcal{T}_\omega(n)$ by C then yields a random Möbius dynamics on $\mathbb{D}$ because the $C\mathcal{T}_\omega(n)C^{-1}$ then leave $\mathbb{D}$ invariant.

For example, the Möbius transformation under the semigroup of *sub-Lorentzian matrices*

$$\mathrm{SU}_{\leq}(1,1) = \left\{T \in \mathbb{C}^{2\times 2} : T^*GT \leq G\,,\ \det(T) = 1\right\}\,, \qquad G = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\,,$$

leaves $\mathbb{D}$ invariant (see [30], eq. 3). This is the case dealt with in Section 4.6.

2.5 A Few Basics of Renewal Theory[22]

As relevant in Section 4.3, we recall two basic definitions and a basic theorem of renewal theory.

Definition 24. For a sequence $\{X_\omega(n)\}_{n\in\mathbb{N}}$ of i.i.d. $[0,\infty)$-valued random variables, we define

$$N_\omega : [0,\infty) \longrightarrow \mathbb{N}_0\,, \quad t \longmapsto \max\left\{ N \in \mathbb{N}_0 : \sum_{n=1}^{N} X_\omega(n) \le t \right\}$$

and call the $X_\omega(n)$ *interarrival times* specifying the *renewal process* $\{N_\omega(t)\}_{t\ge 0}$.

A renewal process $\{N_\omega(t)\}_{t\ge 0}$ can be interpreted as the (random) number of occurrences of some event in the time interval $[0,t]$. We call such an occurrence a *renewal*. After the n-th renewal, the system starts *renewed* and it takes the (random) interarrival time $X_\omega(n+1)$ until the re-occurrence of the event, *i.e.*, until the $(n+1)$-th renewal. This (random) interarrival time is equal in each such cycle.

Definition 25. Associated to a renewal process $\{N_\omega(t)\}_{t\ge 0}$, the *renewal function* is defined by

$$m : [0,\infty) \longrightarrow [0,\infty]\,, \quad t \longmapsto \mathbf{E}\, N_\omega(t)\,.$$

The *elementary renewal theorem* links the renewal function to the average interarrival time:

Theorem 4. *If the interarrival time* $X_\omega(1)$ *obeys* $\mathbf{P}(X_\omega(1)=0)=0$ *and* $\mathbf{E}\, X_\omega(1) < \infty$, *one has*

$$\lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mathbf{E}\, X_\omega(1)}\,.$$

[22] The author used Sections 10.1 and 10.2 of [56] to recall the definitions and the theorem stated in this section.

3 The Long-Term Project of Anderson Delocalization

In this section, we describe the strategy with which the author and his supervisor pursue a proof of Conjecture 4 stated above. The idea of this strategy is due to the supervisor and was explained to the author in the course of many discussions. While the steps outlined below have yet to be elaborated, the author and his supervisor have made progress in understanding some of the expected phenomena, namely by the rigorous analysis of special cases of the problems emerging in the plan described below.

This section contains (revised) parts of the author's research proposal [28] (mentioned above).

3.1 Intuitive Description of the General Strategy

The strategy of our approach is to produce formal solutions of the Schrödinger equation of the Anderson model in a finite volume, which will be chosen to be a d-dimensional cube. These formal solutions will be produced by the *transfer matrix technique*[23] in one particular chosen direction of space. They become true eigenstates of the Schrödinger equation if and only if both boundary conditions on the $(d-1)$-dimensional (finite) sides are satisfied. We plan to single out those of the solutions which do not grow substantially under the application of the transfer matrices. They produce so-called *extended states*, where, in contrast, exponentially growing (or decaying) solutions lead to *localized states*[24]. Of course, the transfer matrices are random and therefore also the set of solutions produced is random. Thus one has to show that with sufficiently large probability there are at least many "almost extended states". The more detailed description below will indicate where we plan to look for such states. Once they are at our disposal, we aim at achieving our main goal by using the fact that the *Green matrices*, which are directly related to the moments $M_q(T, \mathbb{R})$, can be described by the same transfer matrix techniques[25].

3.2 Producing the Formal Solutions by Transfer Matrices

As explained in Section 1.4.1 for the one-dimensional case, solutions of the Schrödinger equation can be obtained via transfer matrices by an iterative expansion of some initially given values that are next to one another. As for the higher-dimensional case, we consider the Anderson model on a rod of the width K and the length N, *i.e.*, on the Hilbert space $\ell^2(\{-N, \ldots, N\}, \mathbb{C}^L)$, where we choose $L = K^{d-1}$ to actually deal with a d-dimensional cube in the case $K = 2N+1$. The corresponding Hamiltonian can then be written as

$$(H_{L,N,\omega}\psi)(n) = -\psi(n+1) - \psi(n-1) + \Delta_L\psi(n) + \lambda V_\omega(n)\psi(n)\,, \qquad n = -N, \ldots, N\,, \tag{41}$$

with $\psi \equiv \{\psi(n)\}_{n=-N}^{N}$, where the $\psi(n)$ are $\mathbb{C}^L$-valued[26]. In the end, we will be mainly interested in the case $K = 2N+1$ but also the quasi-one-dimensional limit $N \to \infty$ is interesting in itself and will be considered below. In equation (41), Δ_L is the transverse discrete Laplacian with

[23] This technique was described in Section 1.4.1 for the one-dimensional Anderson model.

[24] For the notions "extended" and "localized states" (mentioned in Section 1.2 already), see, *e.g.*, Ch. 1 of [5].

[25] For this, we refer to [21] and [64] but especially to the discussion in Section 4.4.

[26] The definition (41) of $H_{L,N,\omega}$ and the eq. (42) can be found in Section 1 of [88] for the case $d = 2$ and $N = \infty$.

adequate boundary conditions acting on each $\psi(n)$ separately and the $V_\omega(n)$ are i.i.d. diagonal random matrices, whose L (real-valued) diagonal entries are i.i.d., respectively, and give rise to a random potential. On top of that, one has to impose longitudinal boundary conditions on $H_{L,N,\omega}$, namely define the yet undetermined values $\psi(-N-1)$ and $\psi(N+1)$ in terms of $\psi(N)$ and $\psi(-N)$ (cf. [90], Section 4). The Schrödinger equation $H_{L,N,\omega}\psi = E\psi$ can be rewritten by using *transfer matrices* $\mathcal{T}^E_\omega(n)$ as

$$\begin{pmatrix} \psi(n+1) \\ \psi(n) \end{pmatrix} = \mathcal{T}^E_\omega(n) \begin{pmatrix} \psi(n) \\ \psi(n-1) \end{pmatrix}, \qquad \mathcal{T}^E_\omega(n) = \begin{pmatrix} \Delta_L + \lambda V_\omega(n) - E\mathbf{1}_L & -\mathbf{1}_L \\ \mathbf{1}_L & 0 \end{pmatrix}. \tag{42}$$

For $E \in \mathbb{R}$, the $\mathcal{T}^E_\omega(n)$ are symplectic and even of the form (32) discussed in Section 2.3.6.

Now given some boundary values $\psi(-N-1)$ and $\psi(-N)$, the rewritten eigenvalue equation (42) then produces a formal eigenfunction of the Schrödinger equation $H_{L,N,\omega}\psi = E\psi$, which is not necessarily a proper eigenvector of $H_{L,N,\omega}$ because the obtained values $\psi(N)$ and $\psi(N+1)$ may violate the imposed longitudinal boundary conditions.

3.3 The Quasi-One-Dimensional Limit

To better explain the way we the plan to extract the "almost extended states", it is convenient to first consider the quasi-one-dimensional limit, *i.e.*, to look at a rod of a fixed width $K < \infty$ and a divergent length $N \to \infty$. This is *not* the system we are eventually interested in. However, it will be argued below why we expect the asymptotic phenomena in the quasi-one-dimensional limit to occur already at finite length to an extend that may be sufficient for our purposes.

We start in the middle of the rod and focus here only on the expansion of solutions of (41) in the positive direction. The negative direction can be dealt with in the same manner due to

$$\begin{pmatrix} \psi(n-1) \\ \psi(n) \end{pmatrix} = \mathcal{T}^E_\omega(n) \begin{pmatrix} \psi(n) \\ \psi(n+1) \end{pmatrix}.$$

For the expansion of solutions in the positive direction, one uses the transfer matrices $\mathcal{T}^E_\omega(n)$ for $n \in \mathbb{N}$. For $E \in \mathbb{R}$, these specify a sequence of i.i.d. symplectic $2L \times 2L$ random matrices. Associated to this sequence are Lyapunov exponents $\gamma^E_1, \ldots, \gamma^E_{2L}$ as defined and discussed in Section 2.3.2. According to the Osseledec theorem (Theorem 3), they determine all possible values of the average (exponential) growth of a solution. More precisely, for each Lyapunov exponent γ^E_l, there is some initial condition, *i.e.*, some values of $\psi(0)$ and $\psi(1)$, for which the average growth of the logarithm of the modulus of the solution is equal to γ^E_l. Thus a solution is an "almost extended state", *i.e.*, it hardly increases or decreases in modulus, if the corresponding Lyapunov exponent is close to zero.

Now according to Proposition 3, the symplecticity of the $\mathcal{T}^E_\omega(n)$ implies the symmetry

$$\gamma^E_l = -\gamma^E_{2L-l+1} \qquad l = 1, \ldots, L,$$

so that we merely focus on the non-negative exponents $\gamma^E_1, \ldots, \gamma^E_L$ for the moment. They obey

$$\gamma^E_1 \geq \gamma^E_2 \geq \ldots \gamma^E_{L-1} \geq \gamma^E_L \geq 0, \tag{43}$$

since the map $l \mapsto \gamma_l^E$ is non-increasing according to Theorem 2. In fact, the Lyapunov exponents associated to the transfer matrices for the Anderson model on a rod were proven to be mutually distinct and non-zero (see [53], §7) so that (43) is strengthened to

$$\gamma_1^E > \gamma_2^E > \ldots \gamma_{L-1}^E > \gamma_L^E > 0\,. \tag{44}$$

Of course, our particular focus is now on the small Lyapunov exponents. A quantitative estimate on the smallest positive Lyapunov exponent γ_L^E for the Anderson model on a strip, *i.e.*, $d = 2$, was proven in a perturbative regime of small disorder λ for energies E in the spectrum of the unperturbed Hamiltonian (see [88], Theorem 1.(i)). It shows that γ_L^E is bounded from below by a term of the order λ^2/L, up to errors of the order $\mathcal{O}(\lambda^3)$. In fact, under a supplementary assumption called the *random phase property*, the lower Lyapunov exponents were proven to be equidistant, namely one has $\gamma_l^E - \gamma_{l+1}^E \sim \lambda^2/L$ for $1 \ll l < L$ (see [82], Theorem 1). This result can be generalized to the Anderson model on a rod, *i.e.*, to arbitrary $d \geq 2$ (see [82], Section 10).

In fact, we aim at controlling a certain fraction of the lower Lyapunov exponents associated to the transfer matrices $\mathcal{T}_\omega^E(n)$ of the actual Anderson model on a rod. Our approach to this is described in Section 3.4 below. Of course, the mere control of the Lyapunov exponents defined in the quasi-one-dimensional limit $N \to \infty$ is not enough to make a conclusion on the behaviour of the formal eigenfunctions up to a finite length $N = (K-1)/2$. However, a formal eigenfunction ψ corresponding to an exponent γ_l^E exhibits the asymptotic behaviour

$$|\psi(N)| \sim \exp\left[\gamma_l^E N\right] \qquad \text{as} \quad N \to \infty\,,$$

which indicates that this eigenfunction may not start to significantly decrease (or significantly increase) on sample sizes K of the order $\mathcal{O}\left((\gamma_l^E)^{-1}\right)$. This intuitive point may be made precise by following the argument for the limit $N \to \infty$ only up to a finite length $N = (K-1)/2$. In [88], *e.g.*, the finite length N contributes with terms of the order $\mathcal{O}(N^{-1})$ to the error estimates. Therefore, if the L_{ae} lowest Lyapunov exponents $\gamma_L^E, \gamma_{L-1}^E, \ldots, \gamma_{L-L_{\mathrm{ae}}+1}^E$ are of the order $\mathcal{O}(K^{-1})$, then the corresponding L_{ae} formal eigenfunctions may be "almost extended" within a cube of side length K. The number L_{ae} clearly depends on L and thus, in turn, on K. However, it is reasonable to expect L_{ae} to be proportional to L, since this is, *e.g.*, the case for $\lambda = 0$ and $d = 2$ (see [88], Section 2).

3.4 Random Perturbations of Hyperbolic Dynamics

The starting point for the analysis sketched in Section 3.3 is the decomposition[27] of the transfer matrices $\mathcal{T}_\omega^E(n)$ given by (42) into the deterministic parts and the random perturbations, viz.,

$$\mathcal{T}_\omega^E(n) = \left[\mathbf{1} + \lambda\,\mathcal{P}_\omega(n)\right]\,\mathcal{T}^E = e^{\lambda\mathcal{P}_\omega(n)}\,\mathcal{T}^E\,.$$

Here, $\mathcal{T}^E$ is the (deterministic) transfer matrix at $\lambda = 0$ and $\mathcal{P}_\omega(n)$ are i.i.d. random matrices,

$$\mathcal{T}^E = \begin{pmatrix} \Delta_L - E\mathbf{1}_L & -\mathbf{1}_L \\ \mathbf{1}_L & 0 \end{pmatrix}, \qquad\qquad \mathcal{P}_\omega(n) = \begin{pmatrix} 0 & V_\omega(n) \\ 0 & 0 \end{pmatrix}.$$

[27] A very similar decomposition can *e.g.* be found in Section 2.2 of [88].

Again, for $E \in \mathbb{R}$, the $\mathcal{T}^E_\omega(n)$ are of the form (32) and so is $\mathcal{T}^E$. Furthermore, the $\mathcal{P}_\omega(n)$ lie in the Lie algebra $\mathfrak{sp}(2L,\mathbb{R})$, which is the set of *Hamiltonian matrices* (see [73], Section 3.6), viz.,

$$\mathfrak{sp}(2L,\mathbb{R}) = \left\{ P \in \mathbb{R}^{2L\times 2L} : P^\intercal \mathcal{J} + \mathcal{J} P = 0 \right\} .$$

As discussed in Section 2.3.6, the form (32) allows to decompose the matrix $\mathcal{T}^E$ into L two-dimensional symplectic channels, namely there is a symplectic conjugation transforming it into the form $\widetilde{\mathcal{T}}^E = (\mathcal{M}^E)^{-1}\mathcal{T}^E\mathcal{M}^E$ as given by the first statement (35) of Corollary 8. The explicit transformation was performed in [88] for a strip and can be generalized to a rod (see [82], Section 10). For sake of simplicity, we assume henceforth that $\mathcal{T}^E$ has no parabolic channels, which is indeed the generic case; in general, however, the fulfilment of this assumption depends on L and E (see [88], Section 2.1). This allows to even diagonalize $\mathcal{T}^E$ and obtain the form

$$\widehat{\mathcal{T}}^E = \mathcal{W}^E_{L_h}\widetilde{\mathcal{T}}^E(\mathcal{W}^E_{L_h})^{-1} = \operatorname{diag}\big((\kappa^E_1)^{-1},\ldots,(\kappa^E_L)^{-1},\kappa^E_L,\ldots,\kappa^E_1\big) \tag{45}$$

as in the second statement (36) of Corollary 8, where L_h is the number of hyperbolic channels of $\mathcal{T}^E$. From now on, we work with the representation (45) and hence we adapt the random perturbations adequately by defining the random matrices

$$\widehat{\mathcal{T}}^E_\omega(n) = \Big[\mathbf{1} + \lambda\widehat{\mathcal{P}}^E_\omega(n)\Big]\,\widehat{\mathcal{T}}^E\,, \qquad\qquad \widehat{\mathcal{P}}^E_\omega(n) = \mathcal{W}^E_{L_h}(\mathcal{M}^E)^{-1}\mathcal{P}_\omega(n)\mathcal{M}^E(\mathcal{W}^E_{L_h})^{-1}\,.$$

For the remainder of the present Section 3.4, we fix L and pick an $E \in \mathbb{R}$ for which the matrix $\widehat{\mathcal{T}}^E$ has relevant fractions of both hyperbolic and elliptic channels.

Now the Lyapunov exponents in the unperturbed case $\lambda = 0$ can readily be read off as the logarithms of the diagonal entries of $\widehat{\mathcal{T}}^E$, viz., $\gamma^E_l = \log(\kappa^E_l)$ for $l = 1,\ldots,L$. Again, we only focus on the exponents $\gamma^E_1,\ldots,\gamma^E_L$. Associated to the L_e elliptic channels, there are L_e vanishing Lyapunov exponents $\gamma^E_{L_h+1} = \cdots = \gamma^E_L = 0$. An initial condition corresponding to one of them then constitutes a formal eigenfunction whose modulus remains constant throughout the sample and even in the quasi-one-dimensional limit, *i.e.*, an extended state.

The task is now to control the deviation from this behaviour in the presence of some tiny disorder $\lambda > 0$, *i.e.*, to analyze how many initial conditions still constitute at least "almost extended states" (see Section 3.3) despite the random perturbation. As a quantitative statement in the quasi-one-dimensional limit $N \to \infty$, we pursue a suitable upper bound for the average of the L_e lowest Lyapunov exponents $\gamma^E_L,\ldots,\gamma^E_{L-L_e+1}$. A reasonable upper bound would be

$$\frac{1}{L_e}\sum_{l=L-L_e+1}^{L}\gamma^E_l \le \log(\kappa^E_{p+1}) + \mathtt{C}_p\,\lambda^2 \tag{46}$$

for some p that is such that $L_h > p \gg 1$ so that $0 < \log(\kappa_{p+1}) \ll 1$ and some constant $\mathtt{C}_p < \infty$ that does *not* depend on L or L_e but is somehow increasing in p. If the right side of (46) becomes small for some suitable p, such an upper bound would then indicate that there may be at least L_{ae} small Lyapunov exponents for some number $L_{\mathrm{ae}} < L_e$, which can then be considered as the number of "almost elliptic channels". A suitable p may then be regarded as the number of "strongly hyperbolic channels". Again, the link to the finite system may be made precise by following an argument for the limit $N \to \infty$ only up to a finite length $N = (K-1)/2$.

In fact, the argument requires the study of the associated random dynamical system given by

$$Q^E_\omega(n) = \widehat{\mathcal{T}}^E_\omega(n) \bullet Q^E_\omega(n-1) \tag{47}$$

for all $n \in \mathbb{N}$, where $Q^E_\omega(0) \equiv Q(0)$ lies in $\mathcal{W}^E_{L_h} \mathbb{IG}_{2L,L_h}(\mathbb{R})(\mathcal{W}^E_{L_h})^{-1}$ and hence so do all $Q^E_\omega(n)$.

Lemma 10. *If the $V_\omega(n)$ are centered and such that $\{\mathcal{T}^E_\omega(n)\}_{n\in\mathbb{N}}$ fulfils Hypotheses 1, 2 and 4,*

$$\frac{1}{L_e} \sum_{l=L-L_e+1}^{L} \gamma^E_l \leq \log(\kappa^E_{p+1}) + \frac{1}{2}\frac{(\kappa^E_1)^2}{(\kappa^E_{p+1})^2}\left[\frac{1}{L_e}\varlimsup_{N\to\infty}\frac{1}{N}\,\mathbf{E}\sum_{n=0}^{N-1}\mathrm{tr}\left(\mathfrak{P}_p\left[\mathbf{1}-Q^E_\omega(n)\right]\mathfrak{P}_p\right)+\lambda^2\|\widehat{\mathcal{P}}^E\|^2\right], \tag{48}$$

where $\|\widehat{\mathcal{P}}^E\| := \operatorname{ess\,sup}_\omega \|\widehat{\mathcal{P}}^E_\omega(1)\|$ and $\mathfrak{P}_p = \mathrm{diag}\,(\mathbf{0}_{2L-p}, \mathbf{1}_p)$, holds for all $p = 1, \ldots, L_h$.

Proof. Let $\Phi(0) \in \mathcal{W}^E_{L_h}\mathbb{IF}_{2L,L_h}(\mathbb{R})$ such that $\Pi(\Phi(0)) = Q(0)$ and pick some $\Psi(0) \in \mathbb{F}_{2L,L_e}(\mathbb{C})$ such that $(\Phi(0), \Psi(0)) \in \mathcal{W}^E_{L_h}\mathbb{LF}_L(\mathbb{R})$. We then define

$$(\Phi^E_\omega(n), \Psi^E_\omega(n)) = \widehat{\mathcal{T}}^E_\omega(n)\cdot(\Phi^E_\omega(n-1), \Psi^E_\omega(n-1)) \tag{49}$$

for all $n \in \mathbb{N}$ with $(\Phi^E_\omega(0), \Psi^E_\omega(0)) \equiv (\Phi(0), \Psi(0))$. In view of Remark 10, one then has

$$\Phi^E_\omega(n) = \widehat{\mathcal{T}}^E_\omega(n)\cdot\Phi^E_\omega(n-1) \qquad \forall\, n \in \mathbb{N}\,, \tag{50}$$

but the $\Psi^E_\omega(n)$ do not fulfil a relation similar to (50). It follows from the identity (29) that

$$\sum_{l=L-L_e+1}^{L} \gamma^E_l = \tfrac{1}{2}\lim_{N\to\infty}\tfrac{1}{N}\mathbf{E}\sum_{n=0}^{N-1}\left[\log\left(\left\|\Lambda^L\widehat{\mathcal{T}}^E_\omega(n)\Lambda(\Phi^E_\omega(n),\Psi^E_\omega(n))\right\|^2_\wedge\right)-\log\left(\left\|\Lambda^{L_h}\widehat{\mathcal{T}}^E_\omega(n)\Lambda\Phi^E_\omega(n)\right\|^2_\wedge\right)\right]. \tag{51}$$

Now all $\mathcal{Y} \in \mathrm{GL}(2L,\mathbb{C})$ and all $(\Phi,\Psi) \in \mathbb{F}_{2L,L_h}(\mathbb{C})\times\mathbb{F}_{2L,L_e}(\mathbb{C})$ for which $(\Phi,\Psi) \in \mathbb{F}_{2L,L}(\mathbb{C})$ obey

$$\begin{aligned}
&\log\left(\left\|\Lambda^L\mathcal{Y}\Lambda(\Phi,\Psi)\right\|^2_\wedge\right) - \log\left(\left\|\Lambda^{L_h}\mathcal{Y}\Lambda\Phi\right\|^2_\wedge\right)\\
&= \log\left[\det\left(\Phi^*\mathcal{Y}^*\mathcal{Y}\Phi\right)^{-1/2}\,\det\left((\Phi,\Psi)^*\mathcal{Y}^*\mathcal{Y}(\Phi,\Psi)\right)\,\det\left(\Phi^*\mathcal{Y}^*\mathcal{Y}\Phi\right)^{-1/2}\right]\\
&= \log\left[\det\left(\begin{pmatrix}(\Phi^*\mathcal{Y}^*\mathcal{Y}\Phi)^{-1/2} & \\ & \mathbf{1}_{L_e}\end{pmatrix}\begin{pmatrix}\Phi^*\mathcal{Y}^*\mathcal{Y}\Phi & \Phi^*\mathcal{Y}^*\mathcal{Y}\Psi\\ \Psi^*\mathcal{Y}^*\mathcal{Y}\Phi & \Psi^*\mathcal{Y}^*\mathcal{Y}\Psi\end{pmatrix}\begin{pmatrix}(\Phi^*\mathcal{Y}^*\mathcal{Y}\Phi)^{-1/2} & \\ & \mathbf{1}_{L_e}\end{pmatrix}\right)\right]\\
&= \log\left[\det\begin{pmatrix}\mathbf{1}_{L_h} & (\Phi^*\mathcal{Y}^*\mathcal{Y}\Phi)^{-1/2}\,\Phi^*\mathcal{Y}^*\mathcal{Y}\Psi\\ \Psi^*\mathcal{Y}^*\mathcal{Y}\Phi\,(\Phi^*\mathcal{Y}^*\mathcal{Y}\Phi)^{-1/2} & \Psi^*\mathcal{Y}^*\mathcal{Y}\Psi\end{pmatrix}\right]\\
&= \log\left[\det\left(\Psi^*\mathcal{Y}^*\mathcal{Y}\Psi - \Psi^*\mathcal{Y}^*\mathcal{Y}\Phi\,(\Phi^*\mathcal{Y}^*\mathcal{Y}\Phi)^{-1}\,\Phi^*\mathcal{Y}^*\mathcal{Y}\Psi\right)\right]\\
&= \mathrm{tr}\left[\log\left(\Psi^*\mathcal{Y}^*\mathcal{Y}\Psi - \Psi^*\mathcal{Y}^*\mathcal{Y}\Phi\,(\Phi^*\mathcal{Y}^*\mathcal{Y}\Phi)^{-1}\,\Phi^*\mathcal{Y}^*\mathcal{Y}\Psi\right)\right]\\
&\leq \mathrm{tr}\left[\log\left(\Psi^*\mathcal{Y}^*\mathcal{Y}\Psi\right)\right]\\
&= L_e\,\mathrm{tr}\left[L_e^{-1}\,\log\left(\Psi^*\mathcal{Y}^*\mathcal{Y}\Psi\right)\right]\\
&\leq L_e\,\log\left[L_e^{-1}\,\mathrm{tr}\left(\Psi^*\mathcal{Y}^*\mathcal{Y}\Psi\right)\right]\,,
\end{aligned}$$

where we applied the concavity of the logarithm to the mean of the eigenvalues of $\Psi^*\mathcal{Y}^*\mathcal{Y}\Psi$. Thus,

$$\begin{aligned}
&\mathbf{E}\left[\log\left(\left\|\Lambda^L\widehat{\mathcal{T}}^E_\omega(n)\Lambda(\Phi^E_\omega(n),\Psi^E_\omega(n))\right\|^2_\wedge\right)-\log\left(\left\|\Lambda^{L_h}\widehat{\mathcal{T}}^E_\omega(n)\Lambda\Phi^E_\omega(n)\right\|^2_\wedge\right)\right]\\
&\leq L_e\,\mathbf{E}\,\log\left[L_e^{-1}\operatorname{tr}\left(\Psi^E_\omega(n)^*(\widehat{\mathcal{T}}^E_\omega(n+1))^*\widehat{\mathcal{T}}^E_\omega(n+1)\Psi^E_\omega(n)\right)\right]\\
&\leq L_e\,\log\left[L_e^{-1}\,\mathbf{E}\operatorname{tr}\left(\Psi^E_\omega(n)^*(\widehat{\mathcal{T}}^E_\omega(n+1))^*\widehat{\mathcal{T}}^E_\omega(n+1)\Psi^E_\omega(n)\right)\right]\,,
\end{aligned}\tag{52}$$

where we used the concavity of the logarithm to apply Jensen's inequality. Moreover, since $\widehat{\mathcal{P}}_\omega(n)$ is centered and due to the inequality $(\widehat{\mathcal{T}}^E)^*\widehat{\mathcal{T}}^E\leq(\kappa^E_{p+1})^2\,[\mathbf{1}-\mathfrak{P}_p]+(\kappa^E_1)^2\,\mathfrak{P}_p$, one has

$$\begin{aligned}
&\mathbf{E}\operatorname{tr}\left(\Psi^E_\omega(n)^*(\widehat{\mathcal{T}}^E_\omega(n+1))^*\widehat{\mathcal{T}}^E_\omega(n+1)\Psi^E_\omega(n)\right)\\
&=\mathbf{E}\operatorname{tr}\left(\Psi^E_\omega(n)^*(\widehat{\mathcal{T}}^E)^*\widehat{\mathcal{T}}^E\Psi^E_\omega(n)\right)+\lambda^2\,\mathbf{E}\operatorname{tr}\left(\Psi^E_\omega(n)^*(\widehat{\mathcal{T}}^E)^*\widehat{\mathcal{P}}^E_\omega(n+1)^*\widehat{\mathcal{P}}^E_\omega(n+1)\widehat{\mathcal{T}}^E\Psi^E_\omega(n)\right)\\
&\leq L_e\,(\kappa^E_{p+1})^2+(\kappa^E_1)^2\,\mathbf{E}\operatorname{tr}\left(\Psi^E_\omega(n)^*\mathfrak{P}_p\Psi^E_\omega(n)\right)+L_e\,\lambda^2\,(\kappa^E_1)^2\,\mathbf{E}\left\|\widehat{\mathcal{P}}^E_\omega(n+1)\right\|^2\\
&=L_e\,(\kappa^E_{p+1})^2\,\left[1+(\kappa^E_1)^2(\kappa^E_{p+1})^{-2}\,L_e^{-1}\,\left[\mathbf{E}\operatorname{tr}\left(\Psi^E_\omega(n)^*\mathfrak{P}_p\Psi^E_\omega(n)\right)+L_e\,\lambda^2\,\mathbf{E}\left\|\widehat{\mathcal{P}}^E_\omega(n+1)\right\|^2\right]\right]\\
&\leq L_e\,(\kappa^E_{p+1})^2\,\exp\left[(\kappa^E_1)^2(\kappa^E_{p+1})^{-2}\,L_e^{-1}\,\left[\mathbf{E}\operatorname{tr}\left(\mathfrak{P}_p\Psi^E_\omega(n)\Psi^E_\omega(n)^*\mathfrak{P}_p\right)+L_e\,\lambda^2\,\|\widehat{\mathcal{P}}^E\|^2\right]\right]\,.
\end{aligned}\tag{53}$$

Combining (52), (53) and $\Psi^E_\omega(n)\Psi^E_\omega(n)^*\leq\mathbf{1}-\Phi^E_\omega(n)\Phi^E_\omega(n)^*=\mathbf{1}-Q^E_\omega(n)$ yields

$$\begin{aligned}
&\mathbf{E}\left[\log\left(\left\|\Lambda^L\widehat{\mathcal{T}}^E_\omega(n)\Lambda(\Phi^E_\omega(n),\Psi^E_\omega(n))\right\|^2_\wedge\right)-\log\left(\left\|\Lambda^{L_h}\widehat{\mathcal{T}}^E_\omega(n)\Lambda\Phi^E_\omega(n)\right\|^2_\wedge\right)\right]\\
&\leq 2\,L_e\,\log(\kappa^E_{p+1})+\frac{(\kappa^E_1)^2}{(\kappa^E_{p+1})^2}\,\left[\mathbf{E}\operatorname{tr}\left(\mathfrak{P}_p\left[\mathbf{1}-Q^E_\omega(n)\right]\mathfrak{P}_p\right)+L_e\,\lambda^2\|\widehat{\mathcal{P}}^E\|^2\right]\,,
\end{aligned}$$

which implies (48) in view of (51). $\square$

Now in order to apply the upper bound in (48), we need to control the first summand in the square bracket of its right side. This term can be rewritten as

$$\frac{1}{L_e}\,\overline{\lim_{N\to\infty}}\,\frac{1}{N}\,\mathbf{E}\sum_{n=0}^{N-1}\mathtt{d}^2(\mathfrak{P}_p|Q^E_\omega(n))\,,\tag{54}$$

where we define the *distance to the expanding directions of the strongly hyperbolic channels*[28] by

$$\mathtt{d}(\mathfrak{P}_p|\cdot):\mathbb{G}_{2L,L_h}(\mathbb{C})\longrightarrow[0,p]\,,\qquad Q\longmapsto\left[\operatorname{tr}\left(\mathfrak{P}_p\left[\mathbf{1}-Q\right]\mathfrak{P}_p\right)\right]^{\frac{1}{2}}\,.$$

If we had $L_h=p$, the map $\mathtt{d}(\mathfrak{P}_p|\cdot)$ would be a proper distance obeying $\mathtt{d}(\mathfrak{P}_p|Q)=2^{-\frac{1}{2}}\|\mathfrak{P}_p-Q\|_{\mathrm{F}}$.

[28]The author is not aware of any work in the literature that has defined or considered this quantity.

We now discuss on an intuitive level why the distance to the expanding directions of the strongly hyperbolic channels is part of the upper bound in (48) and why this quantity can be expected to become typically small as the random dynamical system (50) or (47) evolves.

In the unperturbed case, $i.e.$, at $\lambda = 0$, all "typical" initial points $\Phi(0) \in \mathcal{W}^E_{L_h}\mathbb{IF}_{2L,L_h}(\mathbb{R})$ obey

$$\Phi^E_\omega(n) \longrightarrow \widehat{\Phi} \qquad \text{as} \quad n \longrightarrow \infty\,, \qquad\qquad \text{where} \qquad \widehat{\Phi} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix} \in \mathbb{F}_{2L,L_h}(\mathbb{R})\,,$$

$i.e.$, the dynamics converges to an L_h-frame whose range is the sum of all eigenspaces of the hyperbolic channels associated with the eigenvalues whose moduli are larger than 1. In other words, the dynamics converges to a frame spanning the "increasing directions" of $\widehat{\mathcal{T}}^E$. The trivial reason for this convergence behaviour is the strict inequality $|\kappa^E_{L_h}| > |\kappa^E_{L_h+1}|$ meaning that the lowermost L_h entries are *more attractive* than the $2L - L_h$ remaining entries. If we now pair $\Phi(0)$ with some "typical" $\Psi(0) \in \mathbb{F}_{2L,L_e}(\mathbb{C})$ satisfying $(\Phi(0), \Psi(0)) \in \mathcal{W}^E_{L_h}\mathbb{IF}_{2L,L}(\mathbb{R})$ (as in the proof of Lemma 10), the resulting extended dynamics (49) satisfies

$$(\Phi^E_\omega(n), \Psi^E_\omega(n)) \longrightarrow (\widehat{\Phi}, \widehat{\Psi}) \qquad \text{as} \quad n \longrightarrow \infty\,,$$

where the range of $\widehat{\Psi}$ is orthogonal to the range of $\widehat{\Phi}$, $i.e.$, it is orthogonal to the "increasing directions" of $\widehat{\mathcal{T}}^E$. An a priori consequence of the latter fact is that the L_e-dimensional range of $\widehat{\Psi}$ can consist of "modulus-preserving directions" (eigenspaces of elliptic channels) and "decreasing directions" (eigenspaces of hyperbolic channels associated with the eigenvalues whose moduli are smaller than 1). Now one way to exclude that the range of $\widehat{\Psi}$ contains "decreasing directions" is to exploit the symplectic blocking (described in Section 2.3.5). It avoids the range $\widehat{\Psi}$ from containing "decreasing directions", since the corresponding "increasing directions" belonging to the same respective (hyperbolic) channels are already occupied in the range of $\widehat{\Phi}$. Therefore, the range of $\widehat{\Psi}$ definitely contains only "modulus-preserving directions", which then correspond to the vanishing Lyapunov exponents and thus the extended states in the unperturbed case.

Now let us turn on a small perturbation so that we have $0 < \lambda \ll 1$. If we choose λ tiny with respect to the relative differences in the spectral gaps of $|\widehat{\mathcal{T}}^E|$, then the perturbation hardly affects the qualitative behaviour of the dynamics, which is then quasi-deterministic. However, when eventually working with the d-dimensional Anderson model on a cube of size K (as described in Section 3.2), we do not only need to take the limit $(K-1)/2 = N \to \infty$ in the longitudinal direction but also the limit $K^{d-1} = L \to \infty$ in the transversal direction. Now, as $L \to \infty$, the size of the $\widehat{\mathcal{T}}^E_\omega(n)$ increases and the spectrum of $|\widehat{\mathcal{T}}^E|$ becomes finer and finer (see [88] for the case $d = 2$). In particular, the strict inequality $|\kappa^E_{L_h}| > |\kappa^E_{L_h+1}|$ gets closer and closer to an equality while the coupling constant λ is supposed to remain constant. To circumvent this, we subdivide the hyperbolic channels into p strongly hyperbolic ones and $L_h - p$ weakly hyperbolic ones, where we require the fraction pL_h^{-1} to be constant as L varies so that the coupling constant λ can at least be chosen tiny with respect to the gap $|\kappa_p^E| > |\kappa_{L_h+1}^E|$. The behaviour of the random dynamics (50) can then be expected to qualitatively differ from the one in the unperturbed case. The range of the random L_h-frame $\Phi_\omega^E(n)$ does then (for sufficiently large n) certainly not contain all L_h "increasing directions" of $\widehat{\mathcal{T}}^E$ but it presumably at least almost contains the $L_h - p$ "most increasing directions" typically, *i.e.*, the distance

$$\mathtt{d}(\mathfrak{P}_p|Q_\omega^E(n)), \qquad \text{where} \quad Q_\omega^E(n) = \Pi(\Phi_\omega^E(n)), \tag{55}$$

is presumably typically small. In other words, while the lowermost L_h entries are more attractive than the $2L - L_h$ remaining ones in the unperturbed case, we now only expect the lowermost p entries to be *typically more attractive* than the $2L - L_h$ uppermost ones. The range of the random L_e-frame $\Psi_\omega^E(n)$ in the extended random dynamical system (49) is then (for sufficiently large n) presumably typically almost orthogonal to the "most increasing directions" of $\widehat{\mathcal{T}}^E$ and, due to the symplectic blocking, almost orthogonal to its "most decreasing directions", too. The range of $\Psi_\omega^E(n)$, therefore, presumably contains (for sufficiently large n) typically roughly only at most "weakly increasing" or "weakly decreasing directions" of $\widehat{\mathcal{T}}^E$, which then correspond to small Lyapunov exponents and thus "almost extended states".

We are thus led to analyze how (55) evolves in order to obtain a suitable upper bound for the expression (54) by merely using an assumption of the type

$$\lambda \ll \frac{|\kappa_p^E| - |\kappa_{L_h+1}^E|}{|\kappa_{L_h+1}^E|}\,.$$

Due to prior results (see Section 3.3), we expect (54) to be of the order $\mathcal{O}(\lambda^2)$ as indicated in (46).

This makes the starting point of the journey from the first to the second to the third paper of the author and his supervisor. This journey is summarized in Section 4.

4 The Journey from the 1st to the 2nd to the 3rd Paper

To approach the challenge described in Section 3.4, the author and his supervisor primarily investigated a concrete toy model in [31] (see Section 4.1), which is as simple as possible but still contains the essence of the task. This makes the first of the three research papers this thesis is based on. In Section 4.2, we outline how the author and his supervisor were led to conduct the research from which the second paper arose. The second paper is then described in Section 4.3. We then discuss the connection between the moment $M_q(T, \mathbb{R})$ and the Green matrices and how the desired lower bound for the former (Conjecture 4) may be achieved by dealing with transfer matrices at *complex* energies (see Section 4.4). In Section 4.5, we consider 2×2 transfer matrices at complex energies and recall some perturbative results on the Lyapunov exponent of the latter. A generalization of these results is part of the third paper summarized in Section 4.6.

4.1 A Toy Model

We consider a random dynamical system on an $(\mathsf{L}-1)$-dimensional sphere $\mathbb{S}^{\mathsf{L}-1}$, $\mathsf{L} \geq 3$, given by

$$u_\omega(n) = \mathcal{T}_\omega(n) \star u_\omega(n-1) \qquad \forall\, n \in \mathbb{N}\,, \qquad\qquad \text{where} \quad u_\omega(0) \equiv u(0) \in \mathbb{S}^{\mathsf{L}-1}\,, \tag{56}$$

which is defined as in (24) with $q = 1$ and $\mathbb{K} = \mathbb{R}$. Here, the $\mathcal{T}_\omega(n)$ are of the form

$$\mathcal{T}_\omega(n) \;=\; \mathcal{R}\left[\mathbf{1} \;+\; \lambda\, r_\omega(n)\, U_\omega(n)\right]\,, \tag{57}$$

where λ lies in $(0, 1)$ and $\mathcal{R}$ is a deterministic and hyperbolic matrix, for which we write

$$\mathcal{R} = \operatorname{diag}(\kappa_{\mathsf{L}}, \ldots, \kappa_1)\,, \qquad\qquad \text{where} \qquad \kappa_1 \geq \cdots \geq \kappa_{\mathsf{L}} > 0\,. \tag{58}$$

Moreover, $\{r_\omega(n)\}_{n\in\mathbb{N}}$ and $\{U_\omega(n)\}_{n\in\mathbb{N}}$ are assumed to be sequences of i.i.d. random variables taking on values in the interval $[0, 1]$ and in the orthogonal group $\mathrm{O}(\mathsf{L})$, respectively. Further, we suppose that $r_\omega(1) \not\equiv 0$ and that $U_\omega(1)$ is distributed according to the Haar measure on $\mathrm{O}(\mathsf{L})$.

Theorem 5. *There is a unique $(\mathcal{T}_\omega(1)*)$-invariant Furstenberg measure μ_{F} on $\mathbb{R}\mathsf{P}^{\mathsf{L}-1}$* (see [15], Part A, Theorem III.4.3) *and any $(\mathcal{T}_\omega(1)\star)$-invariant probability measure $\tilde{\mu}_{\mathrm{F}}$ on $\mathbb{S}^{\mathsf{L}-1}$ is absolutely continuous with respect to the surface measure on $\mathbb{S}^{\mathsf{L}-1}$* (see [31], Theorem 1.1).

As mentioned above for the actual transfer matrices in the Anderson model, choosing λ tiny with respect to the relative differences in the spectral gaps of the unperturbed hyperbolic matrix results in a quasi-deterministic behaviour. This makes the system more easily comprehensible but, as for our actual long-term strategy, results within this case are inapplicable. The local expansion rates $\delta\mathcal{R}_i = \kappa_i \kappa_{i+1}^{-1} - 1$ allow to distinguish the latter case from the more difficult one.

Theorem 6. *Let $\tilde{\mu}_{\mathrm{F}}$ be a $(\mathcal{T}_\omega(1)\star)$-invariant probability measure. If $\lambda < 2^{-4} \min\{\delta\mathcal{R}_i, \frac{1}{2}\}$ for some $i = 1, \ldots, \mathsf{L}-1$, then the support of $\tilde{\mu}_{\mathrm{F}}$ is a strict subset of $\mathbb{S}^{\mathsf{L}-1}$; if $\lambda > \delta\mathcal{R}_i$ for all $i = 1, \ldots, \mathsf{L}-1$ and $1 \in \operatorname{supp}(r_\omega(1))$, then the support of $\tilde{\mu}_{\mathrm{F}}$ equals $\mathbb{S}^{\mathsf{L}-1}$* (see [31], Theorem 1.1).

The system is only non-trivial if λ is large enough such that the effect of the hyperbolic structure of the unperturbed matrix can be overcome on a local level. As discussed above for the actual transfer matrices in the Anderson model, we are interested in how this hyperbolic structure at least dominates on a global level. For this, we split each vector $v = (v_1, \ldots, v_{\mathsf{L}})^{\mathsf{T}} \in \mathbb{R}^{\mathsf{L}}$ into its upper part $\mathfrak{a}(v) \in \mathbb{R}^{\mathsf{L}_\mathfrak{a}}$, middle part $\mathfrak{b}(v) \in \mathbb{R}^{\mathsf{L}_\mathfrak{b}}$ and lower part $\mathfrak{c}(v) \in \mathbb{R}^{\mathsf{L}_\mathfrak{c}}$ via

$$\mathfrak{a}(v) \;=\; (v_1, \ldots, v_{\mathsf{L}_\mathfrak{a}})^{\mathsf{T}} \;, \qquad \mathfrak{b}(v) \;=\; (v_{\mathsf{L}_\mathfrak{a}+1}, \ldots, v_{\mathsf{L}_\mathfrak{a}+\mathsf{L}_\mathfrak{b}})^{\mathsf{T}} \;, \qquad \mathfrak{c}(v) \;=\; (v_{\mathsf{L}_\mathfrak{a}+\mathsf{L}_\mathfrak{b}+1}, \ldots, v_{\mathsf{L}})^{\mathsf{T}} \;,$$

where $(\mathsf{L}_\mathfrak{a}, \mathsf{L}_\mathfrak{b}, \mathsf{L}_\mathfrak{c}) \in \mathbb{N}^{\times 3}$ is such that $\mathsf{L}_\mathfrak{a} + \mathsf{L}_\mathfrak{b} + \mathsf{L}_\mathfrak{c} = \mathsf{L}$. Associated to such a partition, we define the *macroscopic gap* $\mathfrak{g} = \mathfrak{g}\,(\mathcal{R}, \mathsf{L}_\mathfrak{b}, \mathsf{L}_\mathfrak{c}) \in [0,1]$ between the upper and lower part by

$$\mathfrak{g} \;=\; \min\left\{1\,,\, \kappa_{\mathsf{L}_\mathfrak{c}}^2 \kappa_{\mathsf{L}_\mathfrak{b}+\mathsf{L}_\mathfrak{c}+1}^{-2} \;-\; 1\right\} \;.$$

If $\mathfrak{g} > 0$, the entries of the upper part $\mathfrak{a}$ can be seen as the repulsive entries of the hyperbolic action $\mathcal{R}\star$. The deviation of the random path $\{u_\omega(n)\}_{n\in\mathbb{N}}$ from the attractive part of the phase space can therefore be measured as the norm of the upper part $\|\mathfrak{a}(u_\omega(n))\|$. Our main result provides a quantitative bound on the expectation value of $\|\mathfrak{a}(u_\omega(N))\|^2$ for sufficiently large N.

Theorem 7. *Assume* $(\mathsf{L}_\mathfrak{a}, \mathsf{L}_\mathfrak{b}) \neq (1,1)$ *and* $\mathfrak{g} > 0$. *Then, for all* $\lambda \leq \frac{1}{4}$, *there exist* $N_0 \in \mathbb{N}$ *such that*

$$\mathbf{E}\,\|\mathfrak{a}(u_\omega(N))\|^2 \;\leq\; 2\left(\frac{\mathsf{L}}{\mathsf{L}_\mathfrak{a}+\mathsf{L}_\mathfrak{b}}\right)^{\frac{\mathsf{L}_\mathfrak{a}+\mathsf{L}_\mathfrak{b}-2}{\mathsf{L}_\mathfrak{c}+2}} \left(\frac{6}{\mathfrak{g}}\,\frac{\mathsf{L}_\mathfrak{a}}{\mathsf{L}_\mathfrak{c}}\,\lambda^2\right)^{\frac{\mathsf{L}_\mathfrak{c}}{2+\mathsf{L}_\mathfrak{c}}} \tag{59}$$

is satisfied for all $N \geq N_0$ *and all* $u(0) \in \mathbb{S}^{\mathsf{L}-1}$ (see [31], Theorem 1.2).

The upper bound in (59) is an intensive quantity. In the thermodynamic limit $\mathsf{L} \to \infty$, therefore, the upper bound does not explode provided that $\mathsf{L}_\mathfrak{a}$, $\mathsf{L}_\mathfrak{b}$ and $\mathsf{L}_\mathfrak{c}$ are of the same order. Moreover, in the limit $\mathsf{L}_\mathfrak{c} \to \infty$, the upper bound is approximately of the expected order $\mathcal{O}(\lambda^2)$.

Corollary 11. *Assume* $(\mathsf{L}_\mathfrak{a}, \mathsf{L}_\mathfrak{b}) \neq (1,1)$ *and* $\mathfrak{g} > 0$ *and* $\lambda \leq \frac{1}{4}$. *Then, the inequality*

$$\int_{\mathbb{S}^{\mathsf{L}-1}} \mathrm{d}\tilde{\mu}_{\mathrm{F}}(u)\; \|\mathfrak{a}(u)\|^2 \;\leq\; 2\left(\frac{\mathsf{L}}{\mathsf{L}_\mathfrak{a}+\mathsf{L}_\mathfrak{b}}\right)^{\frac{\mathsf{L}_\mathfrak{a}+\mathsf{L}_\mathfrak{b}-2}{\mathsf{L}_\mathfrak{c}+2}} \left(\frac{6}{\mathfrak{g}}\,\frac{\mathsf{L}_\mathfrak{a}}{\mathsf{L}_\mathfrak{c}}\,\lambda^2\right)^{\frac{\mathsf{L}_\mathfrak{c}}{2+\mathsf{L}_\mathfrak{c}}} \tag{60}$$

holds for any $(\mathcal{T}_\omega(1)\star)$*-invariant probability measure* $\tilde{\mu}_{\mathrm{F}}$ *on* $\mathbb{S}^{\mathsf{L}-1}$ (see [31], Corollary 1.3).

Now the upper bound (60) allows, in turn, to deduce a lower bound for the top Lyapunov exponent γ_1 associated to the sequence defined by (57), namely by using Proposition 2. Such a lower bound for the top Lyapunov exponent is obtained in Corollary 12 and is just the opposite of the desired upper bound (46) for the lower Lyapunov exponents but it demonstrates, in principle, that the deviation of a Lyapunov exponent at positive λ is controllable with the aid of a macroscopic gap. Corollary 12 was not formulated in [31] and therefore we prove it below.

Corollary 12. *Assume* $(\mathsf{L}_\mathfrak{a}, \mathsf{L}_\mathfrak{b}) \neq (1,1)$ *and* $\mathfrak{g} > 0$ *and* $\lambda \leq \frac{1}{4}$. *Then, one has*

$$\gamma_1 \geq \log(\kappa_{\mathsf{L}-\mathsf{L}_\mathfrak{a}}) - 2\,\frac{\kappa_{\mathsf{L}-\mathsf{L}_\mathfrak{a}}^2}{\kappa_{\mathsf{L}}^2}\left(\frac{\mathsf{L}}{\mathsf{L}_\mathfrak{a}+\mathsf{L}_\mathfrak{b}}\right)^{\frac{\mathsf{L}_\mathfrak{a}+\mathsf{L}_\mathfrak{b}-2}{\mathsf{L}_\mathfrak{c}+2}} \left(\frac{6}{\mathfrak{g}}\,\frac{\mathsf{L}_\mathfrak{a}}{\mathsf{L}_\mathfrak{c}}\,\lambda^2\right)^{\frac{\mathsf{L}_\mathfrak{c}}{2+\mathsf{L}_\mathfrak{c}}} \;. \tag{61}$$

Proof. The sequence $\{\mathcal{T}_\omega(n)\}_{n\in\mathbb{N}}$ defined by (57) clearly satisfies Hypotheses 1, 2 and 3. Thus

$$\gamma_1 = \int_{\mathbb{S}^{\mathsf{L}-1}} \mathrm{d}\tilde{\mu}_{\mathrm{F}}(u)\,\mathbf{E}\,\log\|\mathcal{T}_\omega(1)\,u\| = \frac{1}{2}\int_{\mathbb{S}^{\mathsf{L}-1}} \mathrm{d}\tilde{\mu}_{\mathrm{F}}(u)\,\mathbf{E}\,\log\|\mathcal{R}(\mathbf{1}+\lambda\,r_\omega(1)\,U_\omega(1))\,u\|^2 \qquad (62)$$

holds for any $(\mathcal{T}_\omega(1)\star)$-invariant probability measure $\tilde{\mu}_{\mathrm{F}}$ on $\mathbb{S}^{\mathsf{L}-1}$ by Proposition 2. This implies

$$\begin{aligned}\gamma_1 &\geq \log(\kappa_{\mathsf{L}-\mathsf{L}_\mathfrak{a}}) \\ &+ \frac{1}{2}\int_{\mathbb{S}^{\mathsf{L}-1}} \mathrm{d}\tilde{\mu}_{\mathrm{F}}(u)\,\mathbf{E}\left[\log\|(\mathbf{1}+\lambda\,r_\omega(1)\,U_\omega(1))\,u\|^2 - \frac{\kappa^2_{\mathsf{L}-\mathsf{L}_\mathfrak{a}}}{\kappa^2_{\mathsf{L}}}\|\mathfrak{a}((\mathbf{1}+\lambda\,r_\omega(1)\,U_\omega(1))\star u)\|^2\right]\end{aligned} \qquad (63)$$

because all $\mathcal{Y}\in\mathrm{GL}(\mathsf{L},\mathbb{R})$ and all $u\in\mathbb{S}^{\mathsf{L}-1}$ satisfy the inequality

$$\begin{aligned}\|\mathcal{R}\mathcal{Y}u\|^2 &\geq \kappa^2_{\mathsf{L}}\,\|\mathfrak{a}(\mathcal{Y}u)\|^2 + \kappa^2_{\mathsf{L}-\mathsf{L}_\mathfrak{a}}\,\left[\|\mathfrak{b}(\mathcal{Y}u)\|^2 + \|\mathfrak{c}(\mathcal{Y}u)\|^2\right] \\ &= \kappa^2_{\mathsf{L}-\mathsf{L}_\mathfrak{a}}\,\|\mathcal{Y}u\|^2\,\left[1-(1-\kappa^2_{\mathsf{L}}\kappa^{-2}_{\mathsf{L}-\mathsf{L}_\mathfrak{a}})\|\mathfrak{a}(\mathcal{Y}\star u)\|^2\right] \\ &\geq \kappa^2_{\mathsf{L}-\mathsf{L}_\mathfrak{a}}\,\|\mathcal{Y}u\|^2\,\exp\left[-\kappa_{\mathsf{L}}^{-2}\kappa^2_{\mathsf{L}-\mathsf{L}_\mathfrak{a}}\|\mathfrak{a}(\mathcal{Y}\star u)\|^2\right]\,,\end{aligned}$$

where we used that $1-x\geq e^{-cx}$ holds for all $c\geq 1$ and $x\in[0,1-c^{-1}]$ with $c=\kappa_{\mathsf{L}}^{-2}\kappa^2_{\mathsf{L}-\mathsf{L}_\mathfrak{a}}$ and $x=(1-\kappa^2_{\mathsf{L}}\kappa^{-2}_{\mathsf{L}-\mathsf{L}_\mathfrak{a}})\|\mathfrak{a}(\mathcal{Y}\star u)\|^2$. Next, we show that $\mathbf{E}\log\|(\mathbf{1}+\lambda\,r_\omega(1)\,U_\omega(1))\,u\|^2$ is positive. For this, we use the bound $\log(1+w)\geq w-\frac{w^2}{2}+\frac{w^3}{3}-\frac{w^4}{2}$ for $w\geq-\frac{1}{2}$ with $w=\lambda\mathtt{r}\,[2\mathtt{S}+\lambda\mathtt{r}]$, where $\mathtt{r}=r_\omega(1)$ and $\mathtt{S}=\langle u,U_\omega(1)u\rangle$. Now the average of the second line on the right side of

$$\begin{aligned}w-\frac{w^2}{2}+\frac{w^3}{3}-\frac{w^4}{2} &= \lambda^2\mathtt{r}^2\left[1-\frac{\lambda^2\mathtt{r}^2}{2}+\frac{\lambda^4\mathtt{r}^4}{3}-\frac{\lambda^6\mathtt{r}^6}{2}-2\left[1+2(2\mathtt{S}^2-1)\lambda^2\mathtt{r}^2+6\lambda^4\mathtt{r}^4\right]\mathtt{S}^2\right] \\ &\quad + 2\lambda\mathtt{r}\left[1-\lambda^2\mathtt{r}^2+\lambda^4\mathtt{r}^4-2\lambda^6\mathtt{r}^6\right]\mathtt{S}+8\lambda^3\mathtt{r}^3\left[3^{-1}-2\lambda^2\mathtt{r}^2\right]\mathtt{S}^3\end{aligned}$$

vanishes because all terms of odd order in the entries of $U_\omega(1)$ are centered. Furthermore, one has $2\mathtt{S}^2-1\leq 1$, as $|\mathtt{S}|\leq 1$, and $\mathbf{E}\,\mathtt{S}^2=\mathsf{L}^{-1}\leq\frac{1}{3}$ (see [31], p. 16, antepenultimate eq.). Therefore,

$$\begin{aligned}\mathbf{E}\log\|(\mathbf{1}+\lambda\,r_\omega(1)\,U_\omega(1))\,u\|^2 &\geq \lambda^2\,\mathbf{E}\,\mathtt{r}^2\left[1-\frac{\lambda^2\mathtt{r}^2}{2}+\frac{\lambda^4\mathtt{r}^4}{3}-\frac{\lambda^6\mathtt{r}^6}{2}-2\left[1+2\lambda^2\mathtt{r}^2+6\lambda^4\mathtt{r}^4\right]\frac{1}{3}\right] \\ &= 6^{-1}\,\lambda^2\,\mathbf{E}\,\mathtt{r}^2\left[2-11\,\lambda^2\mathtt{r}^2-22\,\lambda^4\mathtt{r}^4-3\,\lambda^6\mathtt{r}^6\right]\end{aligned}$$

is indeed positive because $\lambda|\mathtt{r}|\leq\frac{1}{4}$. Hence and due to Lemma 2.9 in [31], inequality (63) implies

$$\gamma_1\geq\log(\kappa_{\mathsf{L}-\mathsf{L}_\mathfrak{a}})-\frac{1}{2}\frac{\kappa^2_{\mathsf{L}-\mathsf{L}_\mathfrak{a}}}{\kappa^2_{\mathsf{L}}}\left(\int_{\mathbb{S}^{\mathsf{L}-1}}\mathrm{d}\tilde{\mu}_{\mathrm{F}}(u)\,\|\mathfrak{a}(u)\|^2+3\,\lambda^2\frac{\mathsf{L}_\mathfrak{a}}{\mathsf{L}}\right)\,. \qquad (64)$$

Combining the inequality (64) with the statement (60) of Corollary 11 yields

$$\gamma_1\geq\log(\kappa_{\mathsf{L}-\mathsf{L}_\mathfrak{a}})-\frac{\kappa^2_{\mathsf{L}-\mathsf{L}_\mathfrak{a}}}{\kappa^2_{\mathsf{L}}}\left[\left(\frac{\mathsf{L}}{\mathsf{L}_\mathfrak{a}+\mathsf{L}_\mathfrak{b}}\right)^{\frac{\mathsf{L}_\mathfrak{a}+\mathsf{L}_\mathfrak{b}-2}{\mathsf{L}_\mathfrak{c}+2}}\left(\frac{6}{\mathfrak{g}}\frac{\mathsf{L}_\mathfrak{a}}{\mathsf{L}_\mathfrak{c}}\lambda^2\right)^{\frac{\mathsf{L}_\mathfrak{c}}{2+\mathsf{L}_\mathfrak{c}}}+\frac{3}{2}\frac{\mathsf{L}_\mathfrak{a}}{\mathsf{L}}\lambda^2\right]\,,$$

which, in turn, implies (61) due to

$$\frac{3}{2}\frac{\mathsf{L}_\mathfrak{a}}{\mathsf{L}}\lambda^2=\left(\frac{3}{2}\frac{\mathsf{L}_\mathfrak{a}}{\mathsf{L}}\lambda^2\right)^{\frac{\mathsf{L}_\mathfrak{c}}{2+\mathsf{L}_\mathfrak{c}}}\left(\frac{3}{2}\frac{\mathsf{L}_\mathfrak{a}}{\mathsf{L}}\lambda^2\right)^{\frac{2}{2+\mathsf{L}_\mathfrak{c}}}\leq\left(\frac{3}{2}\frac{\mathsf{L}_\mathfrak{a}}{\mathsf{L}}\lambda^2\right)^{\frac{\mathsf{L}_\mathfrak{c}}{2+\mathsf{L}_\mathfrak{c}}}\leq\left(\frac{\mathsf{L}}{\mathsf{L}_\mathfrak{a}+\mathsf{L}_\mathfrak{b}}\right)^{\frac{\mathsf{L}_\mathfrak{a}+\mathsf{L}_\mathfrak{b}-2}{\mathsf{L}_\mathfrak{c}+2}}\left(\frac{6}{\mathfrak{g}}\frac{\mathsf{L}_\mathfrak{a}}{\mathsf{L}_\mathfrak{c}}\lambda^2\right)^{\frac{\mathsf{L}_\mathfrak{c}}{2+\mathsf{L}_\mathfrak{c}}}. \qquad \square$$

The publication [31] contains further discussion and further results on this toy model and the proofs of the results stated above. The notation in [31] is slightly different.

4.2 An Excursion to Sisyphus & Renewal Processes

When we studied the dynamics $\{u_\omega(n)\}_{n\in\mathbb{N}}$ induced by randomly perturbed hyperbolic matrices

$$\mathcal{T}_\omega(n) \;=\; \mathcal{R}\,[\mathbf{1} \,+\, \lambda\, r_\omega(n)\, U_\omega(n)]\;, \qquad \mathcal{R} = \mathrm{diag}(\kappa_{\mathsf{L}}, \ldots, \kappa_1)\,, \qquad \kappa_1 \geq \cdots \geq \kappa_{\mathsf{L}} > 0$$

described in Section 4.1, we were particularly interested in the case of medium-sized strengths of perturbation, where the coupling constant λ exceeds the local expansion rates $\delta\mathcal{R}_i = \kappa_i\kappa_{i+1}^{-1} - 1$ only barely. In that case, the repulsive (upper) entries of $u_\omega(N)$ are typically only weakly occupied for large N according to Theorem 7. Nonetheless, it is possible that these repulsive entries are even fully occupied. In line with this, a $(\mathcal{T}_\omega(1)\star)$-invariant measure is supported by the entire phase space according to Theorem 6. To prove the latter, it was demonstrated that two arbitrary points in the phase space are always connected by finite paths:

Lemma 13. *Suppose that $\lambda > \max_{i=1,\ldots,\mathsf{L}-1}\delta\mathcal{R}_i$ and that $1 \in \mathrm{supp}(r_\omega(1))$. Then, for every couple $v, w \in \mathbb{S}^{\mathsf{L}-1}$, there exist $N \in \mathbb{N}$ and $\{s_n\}_{n=1}^N \subset \mathrm{supp}(r_\omega(1))$ and $\{\mathscr{U}_n\}_{n=1}^N \subset \mathrm{O}(\mathsf{L})$ such that*

$$w = \prod_{n=1}^{N} \mathcal{R}\,[\mathbf{1} \,+\, \lambda\, s_n\, \mathscr{U}_n] \star v$$

(see [31], Lemma 2.7).

For the details of the proof of Lemma 13, the reader may look at the actual publication. However, a very rough sketch of this proof for $\mathsf{L} = 3$ is already very insightful. The core of the proof is to construct a path from the most attractive to the most repulsive point in the phase space:

$$\begin{pmatrix}0\\0\\1\end{pmatrix} \longrightarrow \cdots \longrightarrow \boxed{\textbf{?}} \longrightarrow \cdots \longrightarrow \begin{pmatrix}1\\0\\0\end{pmatrix} \tag{65}$$

The perturbation can overcome the hyperbolicity locally, which allows to construct the paths

$$\begin{pmatrix}0\\0\\1\end{pmatrix} \longrightarrow \cdots \longrightarrow \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix}0\\1\\0\end{pmatrix} \longrightarrow \cdots \longrightarrow \begin{pmatrix}1\\0\\0\end{pmatrix}$$

by gradually shifting the "vector mass" from the lower entry into the middle one or from the middle entry into the upper one. These paths are unlikely but possible and can be combined to fill the gap in (65). A direct jump of the "vector mass" from the lower entry to the upper one without the occupation of the middle one is even more unlikely, since it requires the perturbation to overcome the hyperbolicity globally. Now for large $\mathsf{L} \gg 3$ and suitable values of $\kappa_1, \ldots, \kappa_{\mathsf{L}}$, the global hyperbolicity dominates over the perturbation so that such a direct jump becomes then actually impossible. In that case, one can still rely on the stepwise construction

$$\begin{pmatrix}0\\0\\\vdots\\0\\\vdots\\0\\1\end{pmatrix} \longrightarrow \cdots \longrightarrow \begin{pmatrix}0\\0\\\vdots\\0\\\vdots\\1\\0\end{pmatrix} \longrightarrow \cdots\cdots\cdots \longrightarrow \begin{pmatrix}0\\0\\\vdots\\1\\\vdots\\0\\0\end{pmatrix} \longrightarrow \cdots\cdots\cdots \longrightarrow \begin{pmatrix}0\\1\\\vdots\\0\\\vdots\\0\\0\end{pmatrix} \longrightarrow \cdots \longrightarrow \begin{pmatrix}1\\0\\\vdots\\0\\\vdots\\0\\0\end{pmatrix}. \tag{66}$$

Once the dynamics has arrived at the most repulsive point on the right of (66), it can reach any other less repulsive point with the aid of an adequate kick by the perturbation and the subsequent iterate hyperbolic action (again, for details, see [31]). More generally, a path from some point to some more attractive point is rather likely and can happen quickly.

Heavily simplified, one can also think of (66) as a person, say the mythological Greek king Sisyphus (see [59], Book XI), ascending a vertical ladder in the stormy weather of the underworld. The gravitational field is supposed to be exceedingly strong in the underworld so that each step requires a long period of time in which the storm is coincidentally beneficial to the ascension. At any position, however, Sisyphus may lose his grip and fall down onto the bottom or at least onto some inferior step, where he may try to ascend again. The higher the level, the more laterally tumultuous the tempest is. When Sisyphus reaches the upper steps of the ladder, he is in greatest danger of falling. This is the situation in which he deviates from the attractive part of the phase space. One way of controlling the deviation from the attractive part of the phase space in the actual dynamics $\{u_\omega(n)\}_{n\in\mathbb{N}}$ was elaborated in Theorem 7. Another criterion to judge Sisyphus' volatile success is the fraction of times at which he stands on one of the upper steps or at least somewhere among the upper steps.

As a further simplification, we may assume that Sisyphus can lose his grip only after having reached the top of the ladder but that he falls down then at once onto the bottom coercively. After a fall, Sisyphus starts *renewed* and tries to ascend again. This is outlined in Figure 2.

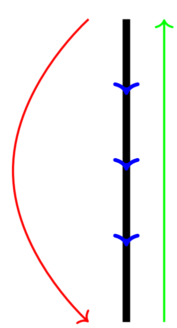

Figure 2: Sisyphus tries to ascend (green arrow) the ladder (black bar) against the force of gravity (blue arrows) and falls back (red arrow) onto the bottom once he has reached the top.

As a suitable measure for Sisyphus' success in the model depicted in Figure 2, one may ask for the number of finished climbs within a certain time. Now this (random) number can in fact be understood as a renewal process as discussed in Section 2.5. Here, the renewal corresponds to a fall of Sisyphus and the (random) durations Sisyphus needs to accomplish the ascensions correspond to the interarrival times. The expected number of ascensions achieved within a certain time is then in fact the renewal function.

Now the dynamics on the line sketched in Figure 2 can be translated into a dynamics on a one-dimensional sphere endowed with a semipermeable barrier, which is depicted in Figure 3.

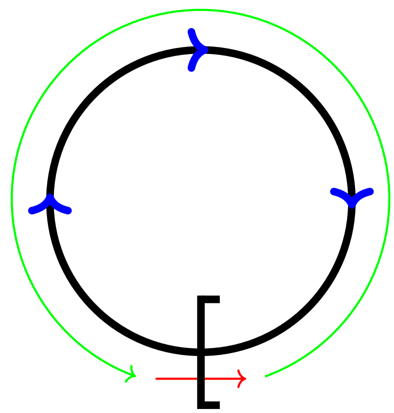

Figure 3: Sisyphus tries to move counterclockwise (green arrow) on the sphere (black circle) against a clockwise drift (blue arrows) and passes through (red arrow) a semipermeable barrier (black square bracket) after each lap.

Admittedly, the model depicted in Figure 3 is only limitedly related to the original problem presented above. Nevertheless, the described considerations made the author and his supervisor become interested in random dynamical systems on a circle endowed with a semipermeable barrier and a drift against the direction of permeability, in fact, especially when these systems are induced by perturbed hyperbolic matrices and are investigable by means of renewal theory.

As chance would have it, Prof. Dr. Günter Stolz (University of Alabama at Birmingham, USA) brang a certain open question on the *random dimer hopping model* (see Section 4.3) to their attention[29] in the spring of 2019. It turned out that a similar question was answerable if one approximates a certain random dynamical system on a circle induced by certain perturbed hyperbolic matrices by a model similar to the one depicted in Figure 3 and solves a certain problem within this approximation by means of renewal theory. That is why the author and his supervisor realized the project corresponding to [29] summarized in the following Section 4.3.

[29] Thank you again, Günter.

4.3 Pseudo-Gaps for Random Hopping Models

The *Su-Schrieffer-Heeger (SSH) model* describes electrons hopping on a (finite) one-dimensional lattice with staggered hopping amplitudes occuring naturally in many solid state systems as *e.g.* polyacetylene; the infinite version of the SSH Hamiltonian acts on $\ell^2(\mathbb{Z})$ and is given by

$$(H_{\mathrm{SSH}}\psi)(n) = -t(n+1)\,\psi(n+1) - t(n)\,\psi(n-1)\,, \qquad \psi \in \ell^2(\mathbb{Z})\,, \tag{67}$$

(see [8], Chapter 1). Here, the hopping amplitudes $t(n)$ are non-negative and staggered, *i.e.*,

$$t(n) \geq 0 \qquad \text{and} \qquad t(n) = t(n+2) \qquad \text{and} \qquad t(n) \neq t(n+1) \qquad \forall\, n \in \mathbb{Z}\,.$$

We call the random version of (67) over $(\Sigma, \mathscr{A}, \mathbb{P})$ the *random dimer hopping model*. It is given by

$$(H_\sigma\psi)(n) = -t_\sigma(n+1)\,\psi(n+1) - t_\sigma(n)\,\psi(n-1)\,, \qquad \psi \in \ell^2(\mathbb{Z})\,, \tag{68}$$

where $\{t_\sigma(n)\}_{n\in\mathbb{Z}}$ is a sequence of independent non-negative random variables called *hopping terms*[30] such that, for all $n \in \mathbb{Z}$, the distributions of $t_\sigma(n)$ and $t_\sigma(n+2)$ are equal, whereas the distributions of $t_\sigma(n)$ and $t_\sigma(n+1)$ are different. Hypothesis 7 strengthens the latter assumption.

Hypothesis 7. The $t_\sigma(n)$ have compact support in $(0,\infty)$ and obey $\mathbb{E}\,\log(t_\sigma(1)) > \mathbb{E}\,\log(t_\sigma(2))$.

Moreover, Hypothesis 8 excludes a deterministic order between $t_\sigma(1)$ and $t_\sigma(2)$.

Hypothesis 8. The event $t_\sigma(1) < t_\sigma(2)$ occurs with a non-zero probability.

Before we explain the connection to the system discussed in Section 4.2 (see Figure 3), we formulate our main result on the integrated density of states (see Section 2.1) of random dimer hopping models satisfying Hypotheses 7 and 8. As discussed in Section 2.1, the integrated density of states is defined for ergodic operators. In fact, the random operator H_σ defined by (68) is *not* ergodic, since the distributions of the even and odd hopping terms are different. However, a certain redefinition of the underlying probability space yields an associated random operator $\widehat{H}_{\widehat{\sigma}}$ on $\ell^2(\mathbb{Z})$ over another probability space $(\widehat{\Sigma}, \widehat{\mathscr{A}}, \widehat{\mathbb{P}})$ which is then indeed ergodic. This redefinition was performed in Section 4.1 of [64] for the more general *random polymer model* and carried over to Section 3.1 of [29]. In some sense, it equalizes the even and odd sites of the lattice but preserves the dimer structure and the distributions of the hopping terms of the model (for details, see [29]). For the sake of a more accessible approach, this subtlety was skipped in the introduction of [29], where the following statement was formulated.

Theorem 8. *If H_σ, given by (68), fulfils Hypotheses 7 and 8, there is a unique $\nu > 0$ such that*

$$\mathbb{E}\left(\frac{t_\sigma(1)}{t_\sigma(2)}\right)^{\nu} = 1$$

is satisfied. Moreover, the integrated density of states $\mathcal{N}$ of the ergodic operator $\widehat{H}_{\widehat{\sigma}}$ associated to H_σ is Hölder continuous at zero with any Hölder exponent smaller than ν, i.e., for all $\delta > 0$, there exists some $C_\delta \in (0,\infty)$ such that all $E \in \mathbb{R}$ obey the inequality

$$|\mathcal{N}(E) - \mathcal{N}(0)| \leq C_\delta\,|E|^{\nu-\delta} \tag{69}$$

(see [29], Theorem 1).

[30] The (non-Hermitian) SSH model with random hopping terms (and imaginary potentials) was studied in [72].

Remark 23. Theorem 8 can be generalized to the random polymer model (see [29], Theorem 8).

Remark 24. If the number ν is large, the density of states $\mathfrak{n}$ has a pseudo-gap at zero, *i.e.*, the eigenvalues of H_σ are typically away from zero as in the numerical example depicted in Figure 4.

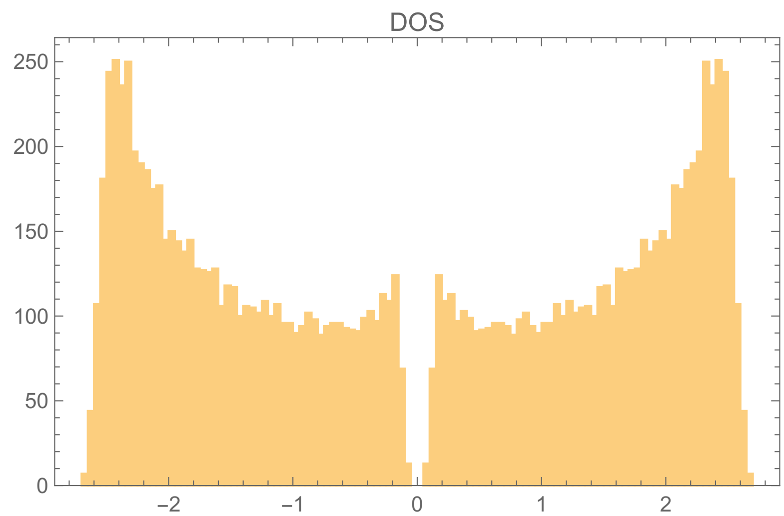

Figure 4: Eigenvalue histogram[31] of a random realization of a finite volume approximation of H_σ of length 6500, where the even and odd hopping terms are as in eq. 6 of [29] with $c_{\text{ev}} = 1.3$, $c_{\text{od}} = 1.1$, $\lambda_{\text{ev}} = 0.4$ and $\lambda_{\text{od}} = 0$ resulting in $\nu \approx 10.79$.

We now elaborate on the connection to the system discussed in Section 4.2 (see Figure 3). To prove Theorem 8, we rewrite the eigenvalue equation $H_\sigma\psi = E\psi$ in terms of the transfer matrices

$$T_\sigma^E(n) = \frac{1}{t_\sigma(n)} \begin{pmatrix} -E & -t_\sigma(n)^2 \\ 1 & 0 \end{pmatrix}, \qquad n \in \mathbb{Z},$$

as

$$\Xi_\sigma^E(n) = T_\sigma^E(n)\, \Xi_\sigma^E(n-1), \qquad \text{where} \qquad \Xi_\sigma^E(n) = (t_\sigma(n+1)\,\psi(n+1), \psi(n))^\intercal .$$

The transfer matrices $T_\sigma^E(n)$ as well as the vectors $\Xi_\sigma(n)$ are similar to the $\mathcal{T}_\omega^E(n)$ and $\Psi(n)$ given in Section 1.4.1. Under the above assumptions on the hopping terms $t_\sigma(n)$, the $T_\sigma^E(n)$ are independent of each other. However, in contrast to the $\mathcal{T}_\omega^E(n)$, the $T_\sigma^E(n)$ are not distributed identically due to the dimer structure. This is different in case of the two-step transfer matrices

$$\mathsf{T}_\sigma^E(n) = T_\sigma^E(2n)\, T_\sigma^E(2n-1) = \frac{1}{t_\sigma(2n-1)\, t_\sigma(2n)} \begin{pmatrix} E^2 - t_\sigma(2n)^2 & E\, t_\sigma(2n-1)^2 \\ -E & -t_\sigma(2n-1)^2 \end{pmatrix}, \qquad n \in \mathbb{Z},$$

which are then i.i.d. and can be used to rewrite the eigenvalue equation $H_\sigma\psi = E\psi$ as

$$\Upsilon_\sigma^E(n) = \mathsf{T}_\sigma^E(n)\, \Upsilon_\sigma^E(n-1), \qquad \text{where} \qquad \Upsilon_\omega^E(n) = (t_\sigma(2n+1)\,\psi(2n+1), \psi(2n))^\intercal .$$

[31] The histogram was computed by the author by using a Mathematica script written by his supervisor.

Now the random dynamics $\{u^E_\sigma(n)\}_{n\in\mathbb{N}}$ on $\mathbb{S}^1$ induced by $\{\mathsf{T}^E_\sigma(n)\}_{n\in\mathbb{N}}$ (as in Section 2.3) fulfils

$$u^E_\sigma(n) = \Upsilon^E_\sigma(n)\,\|\Upsilon^E_\sigma(n)\|^{-1} \qquad \text{for} \qquad u^E_\sigma(0) = \Upsilon^E_\sigma(0)\,\|\Upsilon^E_\sigma(0)\|^{-1}\,. \tag{70}$$

We choose the initial conditions $(\psi_\sigma(1), \psi_\sigma(0)) = (1, 0)$ so that $u_\sigma(0) = (1, 0)^\intercal$ and lift the $u^E_\sigma(n)$ to the real line by introducing the *Prüfer variables* $\theta^E_\sigma(n) \in \mathbb{R}$ (see [64], Section 3) defined by

$$u^E_\sigma(n) = \begin{pmatrix} \cos\theta^E_\sigma(n) \\ \sin\theta^E_\sigma(n) \end{pmatrix} \qquad \text{and} \qquad -\frac{\pi}{2} < \theta^E_\sigma(n) - \theta^E_\omega(n-1) < \frac{3\pi}{2}\,,$$

where one has $\theta^E_\sigma(0) = 0$ due to $u^E_\sigma(0) = (1, 0)^\intercal$. Now it follows from the oscillation theorem that

$$\left|\frac{1}{\pi}\theta^E_\sigma(N) - \operatorname{tr}\left[\chi_{(-\infty,E]}\left(\Lambda^1_{2N} H_\sigma \Lambda^1_{2N}\right)\right]\right| \le \frac{1}{2}\,,$$

where Λ^1_{2N} is the projection onto $\ell^2([0, 2N) \cap \mathbb{Z})$ (see [64], Section 3.2), which, in turn, implies

$$\frac{1}{2N}\operatorname{tr}\left[\chi_{(0,E]}\left(\Lambda^1_{2N} H_\sigma \Lambda^1_{2N}\right)\right] = \frac{1}{2\pi N}\left[\theta^E_\sigma(N) - \theta^0_\sigma(N)\right] + \mathcal{O}(N^{-1})\,, \tag{71}$$

where we assumed $E > 0$ without loss of generality. The left side of (71) would converge to

$$\mathcal{N}(E) - \mathcal{N}(0) \tag{72}$$

as $N \to \infty$ with probability 1 if H_σ were replaced by the associated ergodic operator $\widehat{H}_{\widehat{\sigma}}$ (see Section 2.1). In fact, Fatou's lemma (see [14], Chapter 2.8) implies that this limit is not affected if the averages are taken for all N before. An upper bound on (72) is then obtained as an upper bound for the limit of the average of the analogue of the right side of (71) with adapted modified Prüfer variables (see [29] for details). For sake of simplicity, we refrain from embarking on this modification here and rather focus on an upper bound for the limit superior of the average of the right side of (71) itself, since this is enough to explain the core of the proof.

First, we write the two-step transfer matrices as perturbations of hyperbolic matrices, viz.,

$$\mathsf{T}^E_\sigma(n) = -\left[\mathbf{1} + E\begin{pmatrix} 0 & -1 \\ t_\sigma(2n)^{-2} & 0 \end{pmatrix} + \mathcal{O}(E^2)\right]\begin{pmatrix} \kappa_\sigma(n) & 0 \\ 0 & \kappa_\sigma(n)^{-1} \end{pmatrix}\,, \qquad \kappa_\sigma(n) = \frac{t_\sigma(2n)}{t_\sigma(2n-1)}\,, \tag{73}$$

where both the perturbation (first factor) and the hyperbolic matrix (second factor) are random. This is not entirely consistent with the notion of hyperbolic matrix as introduced in Remark 21, since the diagonal entries of the second factor (73) are random and not necessarily non-decreasing. Now since $\mathsf{T}^0_\sigma(n) \star u_\sigma(0) = u_\sigma(0)$, one has $\theta^0_\sigma(n) = 0$ for all $n \in \mathbb{N}$. Therefore,

$$\limsup_{N\to\infty} \frac{1}{2\pi N}\,\mathbb{E}\,\theta^E_\sigma(N) \tag{74}$$

is the quantity we aim at an upper bound for. Note that an increase of the Prüfer variable by 2π corresponds to a counterclockwise rotation of the dynamics (70) on $\mathbb{S}^1$. Thus we aim at controlling the average number of counterclockwise rotations per time step.

Now the random number $\kappa_\sigma(n)$ is typically smaller than 1 (see Hypothesis 7) and therefore the bottom right entry of the hyperbolic matrix in (73) is typically the dominant one. Roughly speaking, its action presses a vector in $\mathbb{S}^1$ typically to the lower entry. More precisely, $(0,1)^\intercal$ is an attractive fixed point and $(1,0)^\intercal$ is a repulsive one. Moreover, the first order term of the perturbation generates a rotation, which is counterclockwise due to $E>0$. For small $E \ll 1$, the contribution of this rotation is only dominant close to the two fixed points, where the hyperbolic part is almost ineffective. In the bulk, however, the hyperbolic action dominates and produces a random drift away from $(1,0)^\intercal$ and towards $(0,1)^\intercal$. Henceforth, we view the dynamics on the projective space $\mathbb{R}\mathsf{P}^1$, on which a counterclockwise rotation corresponds to an increase of the Prüfer variable by π (see Figure 5).

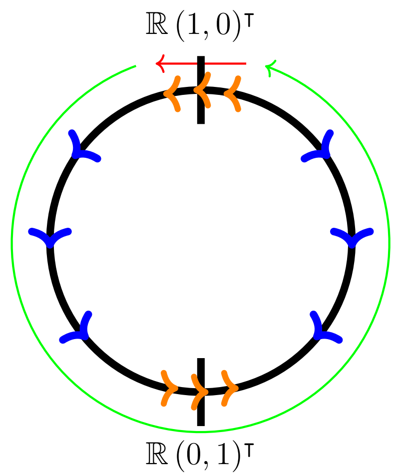

Figure 5: A counterclockwise rotation (green arrow) on the projective space (black circle) starts and ends by passing through $\mathbb{R}\,(1,0)^\intercal$ (red arrow). The random drift (blue arrows) is counterclockwise on the left side (clockwise on the right side), where it is beneficial to (counterproductive for) such a rotation. Close to the fixed points (black bars), the drift is almost ineffective. The perturbation (orange arrows) dominates there and causes a small counterclockwise rotation.[32]

Whereas a motion against the random drift produced by the hyperbolic action is possible, albeit unlikely, the counterclockwise motion close to the fixed points $\mathbb{R}\,(1,0)^\intercal$ and $\mathbb{R}\,(0,1)^\intercal$ caused by the locally dominating perturbation is certain. Thus there are neighborhoods of the fixed points that can only be passed in the counterclockwise direction, *i.e.*, they are semipermeable. In particular, these neighborhoods avoid full clockwise rotations. Now the random drift on the left side of the circle (in Figure 5) benefits the counterclockwise rotation. In contrast, the random drift on the right side counteracts the counterclockwise rotation. Hence the latter is more profitable when it is used to control the number of counterclockwise rotations.

It is therefore reasonable to collapse the bulk of the left side and the two semipermeable neighborhoods of the fixed points to a single semipermeable barrier. By doing so, one indeed

[32] Figure 5 corresponds to the left image of Figure 1 in [29].

obtains the renewal process of the type depicted in Figure 3. Since parts of the original route need not be covered any more, a rotation is speeded up in this renewal process. Thus the order in the bound is correct but the clockwise random drift, which is the main hurdle, is kept.

This explains the connection to the system discussed in Section 4.2. Having got up to this point already, it is not a big deal to sketch the rest of the proof. Now in the simplified model (Figure 3), the completion of a counterclockwise rotation is a renewal. Therefore the average number of counterclockwise rotations equates to the renewal function (see Section 2.5). According to Theorem 4, the asymptotic average number of renewals per time step then equals to the inverse of the expected time for one renewal. In conclusion, the inverse of the expected time for one rotation in the simplified model is an upper bound for (74). Finally, an adequate lower bound on the expected time for one rotation is achieved by a large deviation estimate.

A detailed proof of the more general version of Theorem 8, namely for the case of arbitrary random polymer models, further results and further discussion are contained in the publication [29]. The notation in [29] is slightly different.

4.4 From Extended States to Delocalization[33]

So far, we discussed how the transfer matrices $\mathcal{T}_\omega^E(n)$ can be used to produce formal solutions of the Schrödinger equation $H_{L,N,\omega}\psi = E\psi$ (see Section 3.2) and why small associated Lyapunov exponents indicate "almost extended states" (see Section 3.3). We also explained how an upper bound for the average of the lower Lyapunov exponents can be obtained by analyzing an associated random dynamical system (see Section 3.4). In this section, we suggest how the above may be used to actually obtain a lower bound for the time-averaged q-th moment of the position operator $M_q(T,\mathbb{R})$ of a state initially localized at the origin and evolved with states of all energies (see Definition 1). Such a lower bound was proven for the random polymer model by Jitomirskaya et al under certain assumptions (see [64]). Their proof relies on the identity

$$\int_0^\infty \mathrm{d}t\ e^{-2\frac{t}{T}}\ \left|\left\langle \delta_{m,\cdot}\right| \exp[-itH_\omega]\,\delta_{n,\cdot}\right\rangle\Big|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} dE\ \left|\left\langle \delta_{m,\cdot}\right|\left(H_\omega - E - \frac{\imath}{T}\right)^{-1}\delta_{n,\cdot}\right\rangle\Big|^2 \tag{75}$$

for all $T>0$ and $n,m\in\mathbb{Z}^d$ (see [22], Section 1), which we also use in the following.

As hinted at in Section 3.2, we plan to exploit the longitudinal spreading of wave packets

[33]The idea described in this section is due to the author's supervisor and was mentioned in [28] already.

within a cube for the purpose of a lower bound for $M_q(T,\mathbb{R})$. For this, we start with the estimate

$$\begin{aligned}
&M_q(T,\mathbb{R}) \\
&= \mathbf{E}\int_0^\infty \frac{\mathrm{d}t}{T}\, e^{-2\frac{t}{T}} \sum_{j\in\mathbb{Z}^d} |j|^q \left|\left\langle \delta_{j,\cdot}\middle| \exp[-itH_\omega]\,\delta_{0,\cdot}\right\rangle\right|^2 \\
&\geq \mathbf{E}\int_0^\infty \frac{\mathrm{d}t}{T}\, e^{-2\frac{t}{T}} \sum_{n=-N}^{N}\sum_{k\in\mathbb{Z}^{d-1}} n^q \left|\left\langle \delta_{(n,k),\cdot}\middle| \exp[-itH_\omega]\,\delta_{(0,0),\cdot}\right\rangle\right|^2 \\
&= (2N+1)^{1-d} \sum_{\substack{x\in\mathbb{Z}^{d-1}\\ |x|_1\leq N}} \mathbf{E}\int_0^\infty \frac{\mathrm{d}t}{T}\, e^{-2\frac{t}{T}} \sum_{n=-N}^{N}\sum_{k\in\mathbb{Z}^{d-1}} n^q \left|\left\langle \delta_{(n,k+x),\cdot}\middle| \exp[-itH_\omega]\,\delta_{(0,x),\cdot}\right\rangle\right|^2 \\
&\geq (2N+1)^{1-d} \sum_{\substack{x\in\mathbb{Z}^{d-1}\\ |x|_1\leq N}} \mathbf{E}\int_0^\infty \frac{\mathrm{d}t}{T}\, e^{-2\frac{t}{T}} \sum_{n=-N}^{N}\sum_{\substack{k\in\mathbb{Z}^{d-1}\\ |k|_1\leq N}} n^q \left|\left\langle \delta_{(n,k),\cdot}\middle| \exp[-itH_\omega]\,\delta_{(0,x),\cdot}\right\rangle\right|^2 \\
&= \frac{(2N+1)^{1-d}}{2\pi\, T} \sum_{\substack{x\in\mathbb{Z}^{d-1}\\ |x|_1\leq N}} \mathbf{E}\int_{-\infty}^\infty \mathrm{d}E \sum_{n=-N}^{N}\sum_{\substack{k\in\mathbb{Z}^{d-1}\\ |k|_1\leq N}} n^q \left|\left\langle \delta_{(n,k),\cdot}\middle| \left(H_\omega - E - \frac{\imath}{T}\right)^{-1} \delta_{(0,x),\cdot}\right\rangle\right|^2 \\
&= \frac{(2N+1)^{1-d}}{2\pi\, T} \sum_{\substack{x\in\mathbb{Z}^{d-1}\\ |x|_1\leq N}} \mathbf{E}\int_{-\infty}^\infty \mathrm{d}E \sum_{n=-N}^{N}\sum_{\substack{k\in\mathbb{Z}^{d-1}\\ |k|_1\leq N}} n^q \left|\left\langle \delta_{(n,k),\cdot}\middle|\mathbf{\Lambda}_N^d \left(H_\omega - E - \frac{\imath}{T}\right)^{-1} \mathbf{\Lambda}_N^d\,\delta_{(0,x),\cdot}\right\rangle\right|^2 ,
\end{aligned}
\tag{76}$$

where the ergodicity of H_ω (see Remark 3) and (75) were used in the third and fifth step, respectively. Here, $(\mathbf{\Lambda}_N^d\psi)(n) = \chi_{[-N,N]^d}(n)\,\psi(n)$ is the projection onto $\ell^2([-N,N]^d\cap\mathbb{Z}^d)$.

One is now tempted to naively replace H_ω by the Hamiltonian $H_{L,N,\omega}$ defined by (41). However, H_ω and $H_{L,N,\omega}$ act on different Hilbert spaces and not even the left and the right side of

$$\mathbf{\Lambda}_N^d\,(H_\omega - z)^{-1}\,\mathbf{\Lambda}_N^d \neq \left(\mathbf{\Lambda}_N^d\,H_\omega\,\mathbf{\Lambda}_N^d - z\right)^{-1}, \qquad \text{where} \qquad z = E + \imath T^{-1}, \tag{77}$$

are equal. But whatever the difference in (77) is in detail, the bound (76) suggests the study of

$$\frac{1}{L}\sum_{x=1}^{L} \mathbf{E}\int_{-\infty}^\infty \mathrm{d}E \sum_{n=-N}^{N}\sum_{k=1}^{L} n^q \left|\left\langle \Phi_{n,k}\middle|\,(H_{L,N,\omega}-z)^{-1}\,\Phi_{0,x}\right\rangle\right|^2 , \tag{78}$$

where $\Phi_{n,k}\in\ell^2(\{-N,\dots,N\},\mathbb{C}^L)$ is such that $\Phi_{n,k}(m) = \delta_{n,m}e_k$, where e_k is the k-th canonical unit vector in $\mathbb{C}^L$. Now (78) can be rewritten[34] in terms of the Green matrices

$$\mathtt{G}_\omega^z(n) = \left(\left((H_{L,N,\omega}-z)^{-1}\,\Phi_{0,1}\right)(n),\dots,\left((H_{L,N,\omega}-z)^{-1}\,\Phi_{0,L}\right)(n)\right)$$

[34]Such a rewriting was performed in Section 6 of [64] for the case $L=1$ and $N=\infty$.

(see [87]) as the left side of

$$\begin{aligned} &\frac{1}{L}\,\mathbf{E}\int_{-\infty}^{\infty}\mathrm{d}E\sum_{n=-N}^{N} n^q\,\mathrm{tr}\left[\mathtt{G}^z_\omega(n)^*\mathtt{G}^z_\omega(n)\right] \\ &\quad\geq \frac{1}{2}\frac{1}{L}\,\mathbf{E}\int_{-\infty}^{\infty}\mathrm{d}E\sum_{\substack{n=-N+1\\ n\neq 0}}^{N-1} n^q\,\mathrm{tr}\left[\begin{pmatrix}\mathtt{G}^z_\omega(n+\mathrm{sgn}(n))\\ \mathtt{G}^z_\omega(n)\end{pmatrix}^*\begin{pmatrix}\mathtt{G}^z_\omega(n+\mathrm{sgn}(n))\\ \mathtt{G}^z_\omega(n)\end{pmatrix}\right]. \end{aligned} \tag{79}$$

Now inserting $\psi=(H_{L,N,\omega}-z)^{-1}\Phi_{0,l}$ into (41) for $l=1,\dots,L$ yields

$$\Phi_{0,l}(n) = -\sum_{\pm}\left((H_{L,N,\omega}-z)^{-1}\Phi_{0,l}\right)(n\pm 1)+\left[\Delta_L+\lambda V_\omega(n)-z\mathbf{1}_L\right]\left((H_{L,N,\omega}-z)^{-1}\Phi_{0,l}\right)(n)\,.$$

Since $\Phi_{0,l}(n)=\delta_{0,n}\,e_l$, this implies that all $n\in\{-N+1,\dots,N-1\}\setminus\{0\}$ satisfy

$$\begin{pmatrix}\mathtt{G}^z_\omega(n+1)\\ \mathtt{G}^z_\omega(n)\end{pmatrix}=\mathcal{T}^z_\omega(n)\begin{pmatrix}\mathtt{G}^z_\omega(n)\\ \mathtt{G}^z_\omega(n-1)\end{pmatrix}\qquad\text{and}\qquad\begin{pmatrix}\mathtt{G}^z_\omega(n-1)\\ \mathtt{G}^z_\omega(n)\end{pmatrix}=\mathcal{T}^z_\omega(n)\begin{pmatrix}\mathtt{G}^z_\omega(n)\\ \mathtt{G}^z_\omega(n+1)\end{pmatrix}, \tag{80}$$

i.e., the sequence of Green matrices can be expanded using the transfer matrices as well as the formal solutions of the Schrödinger equation. Now due to (80), the right side of (79) equals

$$\begin{aligned} \frac{1}{2}\frac{1}{L}\,\mathbf{E}\int_{-\infty}^{\infty}\mathrm{d}E\sum_{n=1}^{N-1} n^q\,\mathrm{tr}\Bigg[&\begin{pmatrix}\mathtt{G}^z_\omega(1)\\ \mathtt{G}^z_\omega(0)\end{pmatrix}^*\mathcal{T}^z_\omega(1)^*\dots\mathcal{T}^z_\omega(n)^*\mathcal{T}^z_\omega(n)\dots\mathcal{T}^z_\omega(1)\begin{pmatrix}\mathtt{G}^z_\omega(1)\\ \mathtt{G}^z_\omega(0)\end{pmatrix} \\ &+\begin{pmatrix}\mathtt{G}^z_\omega(-1)\\ \mathtt{G}^z_\omega(0)\end{pmatrix}^*\mathcal{T}^z_\omega(-1)^*\dots\mathcal{T}^z_\omega(-n)^*\mathcal{T}^z_\omega(-n)\dots\mathcal{T}^z_\omega(-1)\begin{pmatrix}\mathtt{G}^z_\omega(-1)\\ \mathtt{G}^z_\omega(0)\end{pmatrix}\Bigg]. \end{aligned} \tag{81}$$

We refrain from giving any further details here, since they are not relevant for the sequel in this thesis. However, the key message, which already (81) hints at to some extent, is that the quantity $M_q(T,\mathbb{R})$ is indeed linked to the random dynamics (47) induced by the (transformed) transfer matrices discussed in Section 3.4 *but only almost*. In contrast to (47), the transfer matrices have to be considered at energies z which have a non-zero imaginary part T^{-1}. Nevertheless, we are interested in $M_q(T,\mathbb{R})$ in the range of large times $T\gg 1$ so that it seems to be sufficient to deal with small imaginary parts of the energy z. In this regime, the transfer matrices are only "approximately symplectic" and, in particular, the structure of the symplectic channels discussed in Section 2.3.6 is only approximately present there.

Now to understand how a small imaginary part of z affects the growth of the Green matrices $\mathtt{G}^z_\omega(n)$ expanded by the transfer matrices $\mathcal{T}^z_\omega(n)$, it is natural to analyze the Lyapunov exponents associated to $\{\mathcal{T}^z_\omega(n)\}_{n\in\mathbb{N}}$ in a perturbative manner. The case $L=1$ is discussed in Section 4.5.

4.5 The Two Parameter Perturbation of 2×2 Transfer Matrices

If $L=1$, the transfer matrices $\mathcal{T}^z_\omega(n)$ are 2×2 random matrices. For $z=E+\imath T^{-1}$, one has

$$\mathcal{T}^z_\omega(n)=\begin{pmatrix}\lambda V_\omega(n)-E+\imath\delta & -1\\ 1 & 0\end{pmatrix}=S(-E)\,\exp\left[-\left(\lambda\,V_\omega(n)+\imath\,\delta\right)\begin{pmatrix}0&0\\1&0\end{pmatrix}\right], \tag{82}$$

where $\delta = -T^{-1}$ and $S(-E)$ is the 2×2 symplectic matrix given by (33). The cases of hyperbolic, parabolic and elliptic $S(-E)$ were discussed in Section 2.3.6. In the elliptic case, where $|E| < 2$, the eigenvalues of $S(-E)$ lie on the complex unit circle and one has

$$M(-E)^{-1}\mathcal{T}^z_\omega(n)M(E) = \begin{pmatrix} \cos(k) & -\sin(k) \\ \sin(k) & \cos(k) \end{pmatrix} \exp\left[-\frac{\imath\,\delta + \lambda\, V_\omega(n)}{\sin(k)} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right] \tag{83}$$

for some $M(-E) \in \mathrm{SP}(2,\mathbb{R})$, where $k = \arccos(-E/2)$ (cf. Lemma 7 and see Section 9 of [86] for the case $\delta = 0$). We are now interested in the perturbative regime of a small parameter δ and, of course, a small coupling constant λ.

In 2004, Schrader et al [86] conducted a perturbative analysis of the upper Lyapunov exponent $\gamma_1 = \gamma_1(\lambda)$ associated to the sequence

$$\begin{pmatrix} \cos(\eta_\omega(n)) & \sin(\eta_\omega(n)) \\ -\sin(\eta_\omega(n)) & \cos(\eta_\omega(n)) \end{pmatrix} \exp\left[\lambda\, \widehat{P_\omega}(n) + \lambda^2\, \widehat{P'_\omega}(n) + \mathcal{O}(\lambda^3)\right], \tag{84}$$

where $\eta_\omega(n)$ are i.i.d. real-valued random variables and $\widehat{P_\omega}(n)$ and $\widehat{P'_\omega}(n)$ are i.i.d. random matrices with values in the Lie algebra $\mathfrak{sp}(2,\mathbb{R})$ of $\mathrm{SP}(2,\mathbb{R})$. This generalizes (83) at $\delta = 0$, where

$$\eta_\omega(n) = -k\,, \qquad \widehat{P_\omega}(n) = -\frac{V_\omega(n)}{\sin(k)} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \widehat{P'_\omega}(n) = 0\,. \tag{85}$$

Schrader et al proved under certain assumptions that $\gamma_1(\lambda)$ obeys the asymptotic approximation

$$\gamma_1(\lambda) = \mathcal{D}\,\lambda^2 + \mathcal{O}(\lambda^3)\,, \tag{86}$$

where $\mathcal{D}$ is a non-negative constant defined in Section 4.6 below (see [86], Theorem 1).

In 2009, Barthel [12] studied a generalization of (84) that also covers cases such as (83), namely

$$\begin{pmatrix} \cos(\eta_\omega(n)) & \sin(\eta_\omega(n)) \\ -\sin(\eta_\omega(n)) & \cos(\eta_\omega(n)) \end{pmatrix} \exp\left[\lambda\, \widehat{P_\omega}(n) + \lambda^2\, \widehat{P'_\omega}(n) + \imath\,\delta\, \widehat{Q_\omega}(n) + \mathcal{O}(\lambda^3, \lambda\,\delta, \delta^2)\right], \tag{87}$$

where also the $\widehat{Q_\omega}(n)$ are i.i.d. random matrices with values in $\mathfrak{sp}(2,\mathbb{R})$. In case of (83), one has

$$\widehat{Q_\omega}(n) = -\frac{1}{\sin(k)} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{88}$$

Barthel proved under certain assumptions that $\gamma_1 = \gamma_1(\lambda, \delta)$ obeys the asymptotic approximation

$$\gamma_1(\lambda,\delta) = \mathcal{D}\,\lambda^2 + \mathcal{O}(\lambda^3, \delta) \tag{89}$$

(see [12], Proposition 5.1), which then generalizes the result (86) of Schrader et al to non-zero δ.

The author and his supervisor, in turn, generalized the result of Barthel by proving

$$\gamma_1(\lambda,\delta) = \mathcal{C}\,\delta + \mathcal{D}\,\lambda^2 + \mathcal{O}(\lambda^3, \lambda\,\delta, \delta^2)\,, \tag{90}$$

where $\mathcal{C}$ is a non-negative constant defined in Section 4.6 below (see [30], Theorem 4).

This is one of the results of their third paper [30], which is summarized in the next Section 4.6.

4.6 Random Möbius Dynamics on the Unit Disc and Perturbation Theory for Lyapunov Exponents

As well as Barthel [12], the author and his supervisor worked with the representation that is obtained by diagonalizing the unperturbed matrix in (87) by means of the Cayley transform (see Remark 19). In this representation, the matrices (87) are of the form

$$\mathcal{T}_\omega(n) = \begin{pmatrix} e^{\imath\eta_\omega(n))} & 0 \\ 0 & e^{-\imath\eta_\omega(n))} \end{pmatrix} \exp\left[\lambda\, P_\omega(n) + \lambda^2\, P'_\omega(n) + \imath\,\delta\, Q_\omega(n) + \mathcal{O}(\lambda^3, \lambda\,\delta, \delta^2)\right], \qquad (91)$$

where $P_\omega(n) = C\,\widehat{P_\omega}(n)\,C^{-1}$ and $P'_\omega(n) = C\,\widehat{P'_\omega}(n)\,C^{-1}$ and $Q_\omega(n) = C\,\widehat{Q_\omega}(n)\,C^{-1}$.

Here, we assume $\mathbf{E}\,e^{2\imath\eta_\omega(1)} \neq 1 \neq \mathbf{E}\,e^{4\imath\eta_\omega(1)}$ and suppose that the error term $\mathcal{O}(\lambda^3, \lambda\,\delta, \delta^2)$ is bounded in norm by $C(\lambda^3 + \lambda\,\delta + \delta^2)$ for a uniform constant $C < \infty$. Moreover, we assume that the $P_\omega(n)$ and $P'_\omega(n)$ and $Q_\omega(n)$ are compactly supported and that their values lie in the Lie algebra of $\mathrm{SU}(1,1)$, which is given by

$$\mathfrak{su}(1,1) = \left\{P \in \mathbb{C}^{2\times 2} : P^*G + GP = 0\,,\ \mathrm{tr}(P) = 0\right\}\,, \qquad G = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

(see *e.g.* [95], Section A.10). A further *key assumption* is that the values of the random matrices $\mathcal{T}_\omega(n)$ lie in the semigroup of sub-Lorentzian matrices $\mathrm{SU}_\leq(1,1)$ so that the unit disc $\mathbb{D}$ is left invariant under the Möbius transformation $\mathcal{T}_\omega(n)\,\diamond$ (see Section 2.4). Therefore the orbit of the random Möbius dynamics (39) lies in $\mathbb{D}$ whenever $z(0)$ does. Moreover, we suppose that $\mathcal{T}_\omega(1)$ fulfils Hypotheses 5 and 6. Since $\mathrm{SU}_\leq(1,1) \subset \mathrm{SL}(2,\mathbb{C})$, Proposition 9 then implies the existence of a unique $(\mathcal{T}_\omega(1)\,\diamond)$-invariant Furstenberg measure $\mu = \mu_{\lambda,\delta}$. The support of $\mu_{\lambda,\delta}$ is then clearly contained in the closure $\overline{\mathbb{D}}$ of the unit disc.

If the random matrices $\mathcal{T}_\omega(n)$ stem from the transfer matrices in the one-dimensional Anderson model with compactly supported potentials $V_\omega(n)$ and complex energy $z = E - \imath\delta$ (see Section 4.5), where $E \in (-2,2)\setminus\{0\}$ and $\delta \geq 0$, all these assumptions are satisfied by $\eta_\omega(n)$, $P_\omega(n)$, $P'_\omega(n)$ and $Q_\omega(n)$, which are given by (85) and (88) — up to conjugation by C (see [30]).

We now introduce the constants $\mathcal{C}$ and $\mathcal{D}$ mentioned in Section 4.5 by

$$\mathcal{C} = -\frac{\imath}{2}\,\mathbf{E}\,\mathrm{tr}\left[G\,P_\omega(1)\right] \qquad \text{and} \qquad \mathcal{D} = \frac{1}{2}\mathbf{E}\left(|\beta_\omega(1)|^2\right) + \mathfrak{Re}\left(\frac{\mathbf{E}(\beta_\omega(1))\,\mathbf{E}\left(e^{2\imath\eta_\omega(1)}\,\overline{\beta_\omega(1)}\right)}{1 - \mathbf{E}\left(e^{2\imath\eta_\omega(1)}\right)}\right)\,,$$

where $\beta_\omega(n) = \begin{pmatrix}0\\1\end{pmatrix}^\intercal P_\omega(n)\begin{pmatrix}1\\0\end{pmatrix}$. These are non-negative (see [30], Remark 7 & [86], Proposition 1).

While the result (90) on the upper Lyapunov exponent $\gamma_1(\lambda,\delta)$ was stated in Section 4.5 already, we have not sketched yet how it was obtained. In fact, to reduce the proof to the essential, one has to approximate the radial distribution of the Furstenberg measure $\mu_{\lambda,\delta}$ on the unit disc for small λ and δ. For this, the relative size of λ^2 and δ turns out to be crucial. If both $\mathcal{C}$ and $\mathcal{D}$ are positive, it is convenient to measure this relative size in terms of the quantity

$$\alpha = 2\,\frac{\mathcal{C}}{\mathcal{D}}\,\frac{\delta}{\lambda^2}\,,$$

which allows to formulate the main result.

Theorem 9. *The radial distribution of* $\mu_{\lambda,\delta}$ *is approximated in a weak sense by the radial density*

$$\varrho_\alpha(s) = \frac{\alpha}{(1-s)^2}\,\exp\left[-\frac{\alpha\, s}{1-s}\right] \tag{92}$$

if $\mathcal{C} > 0$ *and* $\mathcal{D} > 0$*, namely more precisely, all* $h \in C^2([0,1])$ *satisfy*

$$\int_{\overline{\mathbb{D}}} \mathrm{d}\mu_{\lambda,\delta}(x)\; h(|x|^2) \;=\; \int_0^1 \mathrm{d}s\;\, \varrho_\alpha(s)\, h(s) \;+\; \mathcal{O}(\lambda, \lambda^{-1}\delta) \tag{93}$$

(see [30], Theorem 1).

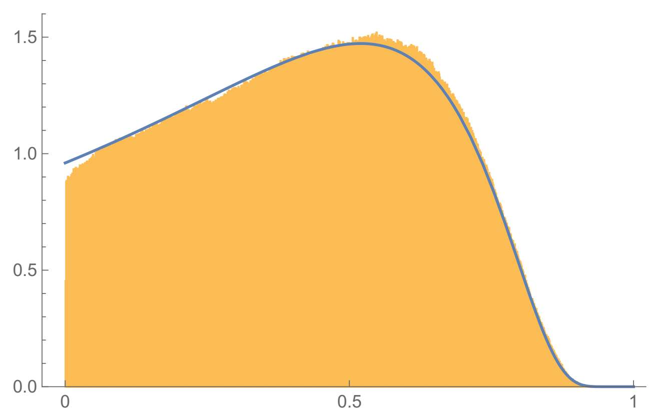

Figure 6: Approximate radial density ϱ_α (blue) given by (92) and numerical histogram obtained after $2 \cdot 10^7$ iterations (yellow)[35]. The numerics were done with the transfer matrices (83) conjugated by C, where the real part of the energy is given by $E = -2\,\cos(2)$ and the random potentials $V_\omega(n)$ are distributed uniformly on $[-1, 1]$. Therefore one has $\mathcal{C} = [2\,\sin(2)]^{-1}$ and $\mathcal{D} = [24\,\sin(2)^2]^{-1}$. The chosen parameters are $\lambda = 0.05$ and $\delta = 1.1 \cdot 10^{-4}$.

Remark 25. The approximation (93) is only profitable for small λ and δ for which $\delta = o(\lambda)$.

Remark 26. Inserting a smooth approximation h of $\chi_{[0,s]}$ into (93) yields formally

$$\mu_{\lambda,\delta}\left(\{x \in \overline{\mathbb{D}} : |x|^2 \leq s\}\right) \approx 1 - \exp\left[\frac{-\alpha\, s}{1-s}\right]\,. \tag{94}$$

This is, however, up to an uncontrolled error term depending on h. Further, (94) implies formally

$$\mu_{\lambda,\delta} \longrightarrow \delta_1 \quad \text{as} \quad \alpha \longrightarrow 0 \qquad\qquad \text{and} \qquad\qquad \mu_{\lambda,\delta} \longrightarrow \delta_0 \quad \text{as} \quad \alpha \longrightarrow \infty\,, \tag{95}$$

[35] The figure was created by the author, who advanced a Mathematica script of his supervisor for that purpose.

again up to uncontrolled errors (see [30], Remark 2). The following two *precise* results (see [30], Theorem 1) are more modest but still describe the (approximate) phenomenon indicated by (95):

$$\mathcal{C} > 0 \qquad \Longrightarrow \qquad \int_{\overline{\mathbb{D}}} \mathrm{d}\mu_{\lambda,\delta}(x)\ |x|^2 \ =\ \mathcal{O}(\lambda, \delta, \lambda^2\,\delta^{-1}) \tag{96}$$

and

$$\mathcal{D} > 0 \qquad \Longrightarrow \qquad \int_{\overline{\mathbb{D}}} \mathrm{d}\mu_{\lambda,\delta}(x)\ |x|^2 \ = 1 + \ \mathcal{O}(\lambda^{\frac{1}{2}}, \delta^{\frac{1}{2}}\lambda^{-1})\,. \tag{97}$$

On the one hand, (96) states that the mass of $\mu_{\lambda,\delta}$ is mainly concentrated in the center of the disc if $\lambda = o(\delta^{\frac{1}{2}})$. On the other hand, (97) states that the mass of $\mu_{\lambda,\delta}$ is mainly concentrated close to the edge of the disc if $\delta = o(\lambda^2)$. The two cases are illustrated by numerics in Figure 7.

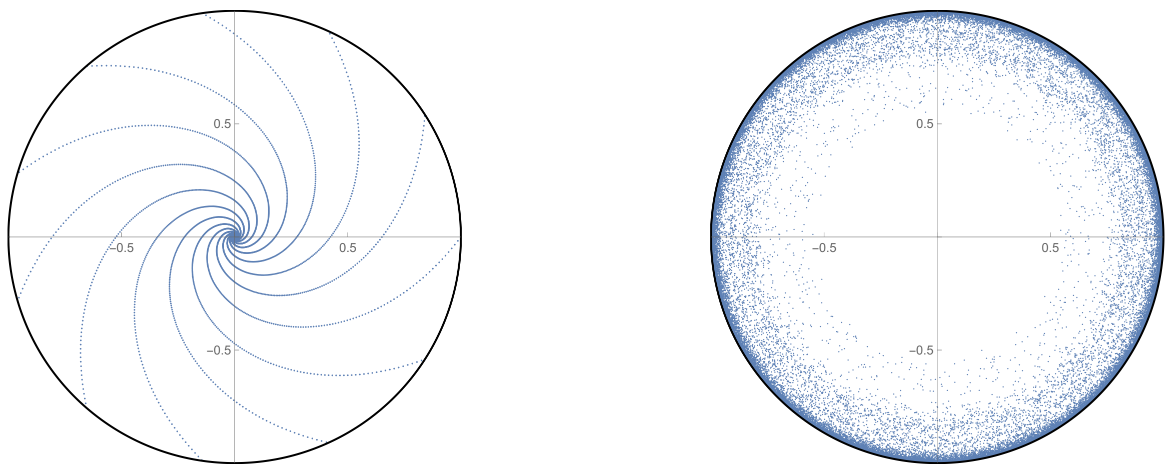

Figure 7: Plots of the orbit $\{z_\omega(n)\}_{n=1}^N$ of the random Möbius dynamics (39) on the unit disc with $z(0) = 1$ induced by the same random matrices as in Figure 6, but with $\lambda = 10^{-4}$ and $\delta = 1.1 \cdot 10^{-3}$ on the left and $\lambda = 0.1$ and $\delta = 9 \cdot 10^{-6}$ on the right[36]. The number of iterations is $N = 5 \cdot 10^3$ on the left and $N = 5 \cdot 10^4$ on the right. The bulk of the spiral is merely due to the thermalization and does not occur if $z(0) = 0$.

The proofs of Theorem 9 and the asymptotic approximation (90) of the upper Lyapunov exponent as well as further results and further discussion on the random Möbius dynamics generated by the random matrices (91) are contained in the preprint [30]. The notation in [30] is slightly different.

[36] The plots were created by the author, who advanced a Mathematica script of his supervisor for that purpose.

5 Publications

In this section, we list the three publications [31, 29, 30] on which this thesis is based and furnish each of the publications with a brief statement on the authors' contributions. More detailed statements on the authors' contributions were submitted separately. Within these statements, the labels of equations and sections refer to the labels in the respective works and *not* to the ones in this thesis.

5.1 Random Perturbations of Hyperbolic Dynamics

Reference [31]:

F. Dorsch and H. Schulz-Baldes. *Random perturbations of hyperbolic dynamics.* Electronic Journal of Probability **24**, no. 89, 1-23 (2019).

Digital Object Identifier (DOI): 10.1214/19-EJP340

Authors' Contributions to Reference [31]:

Prof. Dr. Hermann Schulz-Baldes (HSB) came up with the idea of investigating a toy model in the real projective space. The concrete set-up originated in discussions with Florian Dorsch (FD) and was modified during the review process. HSB and FD worked in close collaboration and had many discussions with Prof. Dr. Andreas Knauf, who gave helpful and constructive comments. Section 1 was mainly written by HSB. Sections 2 and 3 were mainly written and elaborated by FD under the supervision of HSB.

5.2 Pseudo-Gaps for Random Hopping Models

Reference [29]:

F. Dorsch and H. Schulz-Baldes. *Pseudo-gaps for random hopping models.* Journal of Physics A: Mathematical and Theoretical **53**, 185201 (2020).

Digital Object Identifier (DOI): 10.1088/1751-8121/ab5e8c

Authors' Contributions to Reference [29]:

Prof. Dr. Hermann Schulz-Baldes (HSB) reported his scientific exchange with Prof. Dr. Günter Stolz (GS) on the random dimer hopping model to Florian Dorsch (FD). As a result, HSB and FD began to do research on a certain question GS had posed to HSB. In the course of this, HSB and FD actually discovered the phenomenon of pseudo-gaps occuring in the model under certain assumptions, on which they then wrote this article. HSB and FD worked in close collaboration. Sections 1, 2 and 3 were mainly written by HSB, while Section 4 was mainly written by FD.

5.3 Random Möbius Dynamics on the Unit Disc and Perturbation Theory for Lyapunov Exponents

Reference [30]:

F. Dorsch and H. Schulz-Baldes, *Random Möbius dynamics on the unit disc and perturbation theory for Lyapunov exponents*, `arXiv:2008.02174v2` (2021).

To appear in *Discrete and Continuous Dynamical Systems, Series B.*

Digital Object Identifier (DOI): 10.3934/dcdsb.2021076

Authors' Contributions to Reference [30]:

Prior to the collaboration with Florian Dorsch (FD), Prof. Dr. Hermann Schulz-Baldes (HSB) had written a draft with the purpose to analyze the asymptotic approximation of the upper Lyapunov exponent for the special cases $\epsilon = o(\delta^{\frac{1}{2}})$ and $\delta = o(\epsilon^2)$, respectively. The sketched proof therein had contained a flaw. FD came up with the idea of applying the oscillatory phase argument to the second order to fix the flaw and proved formula (17). Based on Appendix B of [83], HSB explained the method to derive the approximate radial density to FD, who then elaborated the details of this derivation. HSB mainly wrote Section 1, in which the numerics were done by FD. Sections 2, 3 and 4 were mainly written by FD. The beginning of Section 5 (pages 28 & 29) was mainly written by HSB and the rest of Section 5 was mainly written by FD.

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