Summer school on analysis and applied mathematics

Minimizing movement schemes, thresholding scheme for mean curvature flow

Felix Otto

Max Planck Institute for Mathematics in the Sciences,
Leipzig

with Selim Esedoğlu CPAM '15

with Tim Laux CalcVar '16, '20, Adv. Studies Pure Math '20

course based on math $>$ arXiv:1910.11442
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A hybrid topic

geometric analysis:
Flow of a surface by its mean curvature

materials science: growth of grains in polycrystals

analysis on metric spaces:
De Giorgi's tools for gradient flows

scientific computing: Osher's thresholding scheme

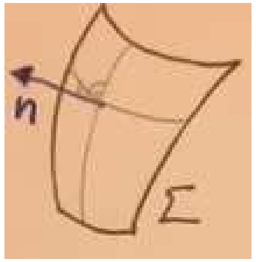
Geometric analysis: Mean curvature flow

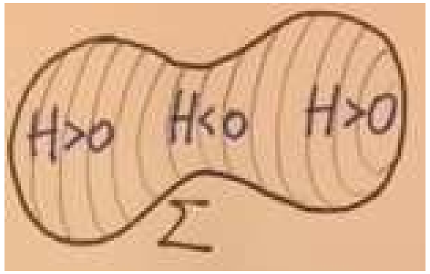
What is the flow of a surface Σ by its mean curvature?

Mean curvature H

=sum of principle curvatures

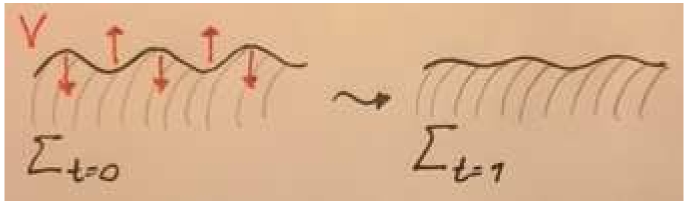
$H := \kappa_1 + \kappa_2$



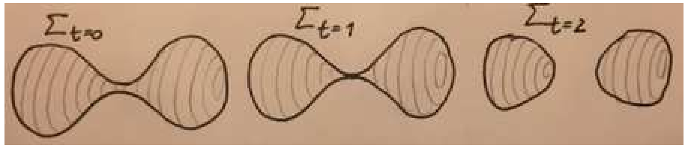


Normal velocity $V = -H$.

Typically has smoothing effect ...



... but singularities occur



"geometric heat flow" (extrinsic vs. intrinsic, cf. Ricci flow)

Materials science: interfacial energy

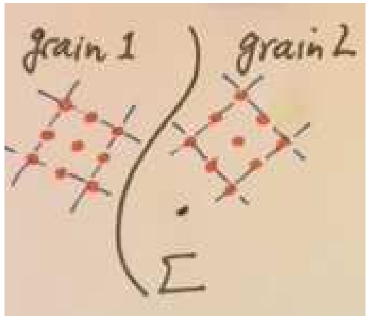
Mean curvature H = first variation of surface area $\Longrightarrow$

Flow by mean curvature (MCF) reduces surface area:

$$\frac{d}{dt}(\text{surface area of } \Sigma) = -\int_\Sigma V^2 = -\int_\Sigma H^2$$

Polycrystals made of single-crystal grains

lattice misorientation
leads to interfacial energy
between grains

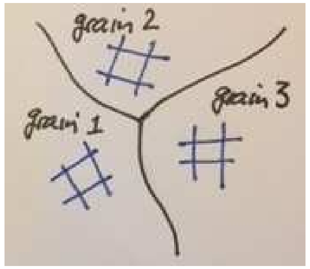


Aging of polycrystals via diffusion-less phase transition

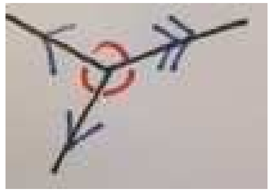
reduction of interfacial energy by MCF (Mullins)

Materials science: grain growth

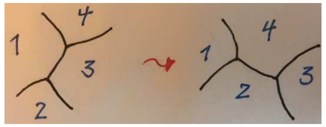
triple junctions

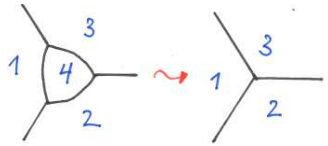


Herring: local balance of surface tensions
→ angle condition

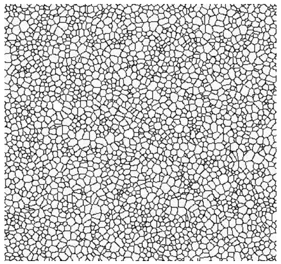


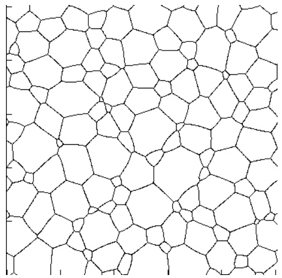
generic singularity:
exchange of neighbors,
vanishing of grains





coarsening of grain configuration
= grain growth
(Kinderlehrer et. al.)

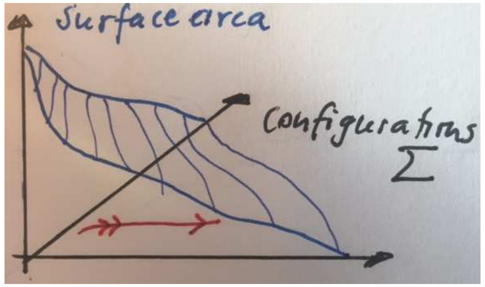




Analysis on metric spaces: gradient flows

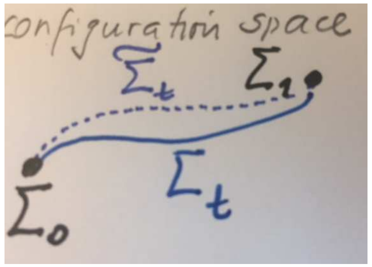
MCF is a *gradient flow*: $V = -H$ can be interpreted as $\frac{d\Sigma}{dt} = -\text{grad}_{|\Sigma}(\text{surface area})$

Gradient flow
= steepest descent
in energy landscape



Geometry of configuration space matters,
"in the large" described by *induced distance*

$d^2(\Sigma_0, \Sigma_1)$
$:= \inf\{\int_0^1 \int_\Sigma V^2 dt\}$



... degenerates
(Michor
& Mumford '06)

Gradient flows: natural discretization in time

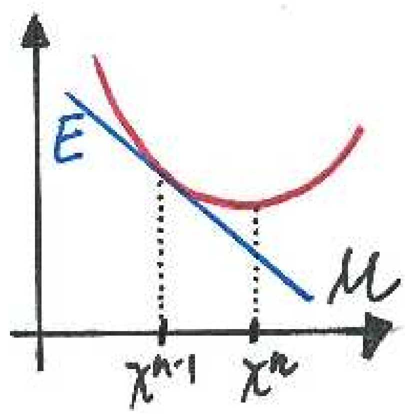
$(\mathcal{M}, d)$ metric space, E function on $\mathcal{M}$

(typical elements of $\mathcal{M}$ denoted by χ)

Natural time discretization with time step size $h > 0$:

χ^n minimizes $\frac{1}{2h} d^2(\chi, \chi^{n-1}) + E(\chi)$ among all $\chi \in \mathcal{M}$.

= De Giorgi's minimizing movements scheme



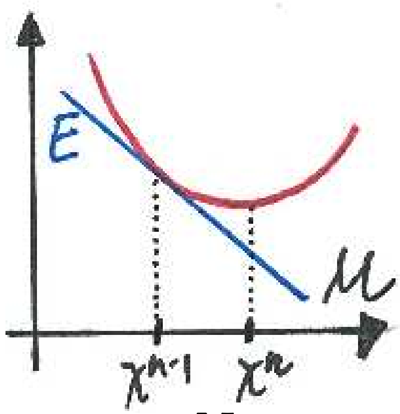
If $(\mathcal{M}, d)$ Euclidean then
minimizing movements = implicit Euler for $\frac{d\chi}{dt} = -\mathrm{grad}_{|\chi} E$.

Passage to limit in minimizing movements scheme

Natural time discretization with time step size $h > 0$:

χ^n minimizes $\quad \frac{1}{2h}d^2(\chi,\chi^{n-1}) + E(\chi) \quad$ among all $\chi \in \mathcal{M}$.

= De Giorgi's minimizing movements scheme



Easy a priori estimate $E(\chi^N) + \sum_{n=1}^{N} \frac{1}{2h} d^2(\chi^n,\chi^{n-1}) \le E(\chi^0)$

misses dissipation relation $E(\chi(T)) + \int_0^T g_\chi(\frac{d\chi}{dt},\frac{d\chi}{dt})dt \le E(\chi(0))$

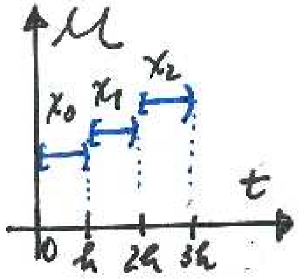
by a factor $\frac{1}{2}$. Way out:

De Giorgi's "variational interpolation", "metric slope".

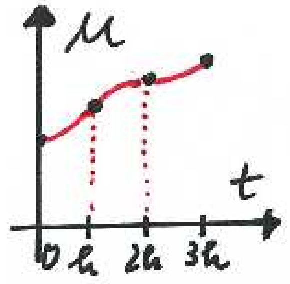
De Giorgi's tools

Two interpolations of $\{\chi^n\}_{n\in\mathbb{N}}$

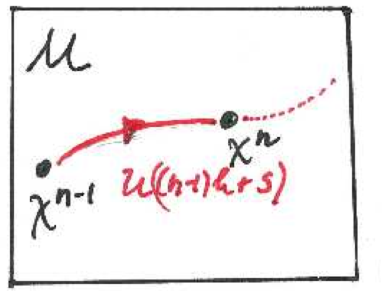
piecewise constant χ^h



"variational" u^h



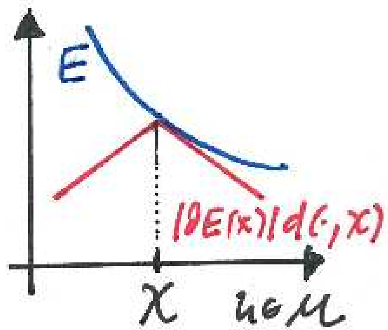
$u^h((n\text{-}1)h + s)$ minimizes
$\frac{1}{2s} d^2(u, \chi^{n-1}) + E(u)$
among all $u \in \mathcal{M}$



"Metric slope" $|\partial E(\chi)|$

$$:= \limsup_{d(u,\chi)\to 0} \frac{(E(\chi)-E(u))_+}{d(u,\chi)}$$

maximal local downwards slope



De Giorgi's tools designed to provide a path ...

Obtain

$$E(\chi^N)+\int_0^{Nh}\frac{1}{2h^2}d^2(\chi^h(t+h),\chi^h(t))dt+\int_0^{Nh}\frac{1}{2}|\partial E(u^h(t))|^2dt$$
$$\leq E(\chi^0).$$

Similar to limit:

$$E(\chi(T))+\int_0^T\frac{1}{2}g_\chi(\frac{d\chi}{dt},\frac{d\chi}{dt})dt+\int_0^T\frac{1}{2}g_\chi(\mathrm{grad}E_{|\chi},\mathrm{grad}E_{|\chi})dt$$
$$\leq E(\chi^0),$$

(formally) equivalent to $\frac{d\chi}{dt}=-\mathrm{grad}E_{|\chi}$.

Sandier-Serfaty '04 ... Liero-Mielke-Peletier-Renger '17

... to a (soft) convergence result

The thresholding scheme

Merriman & Bence & Osher '92:

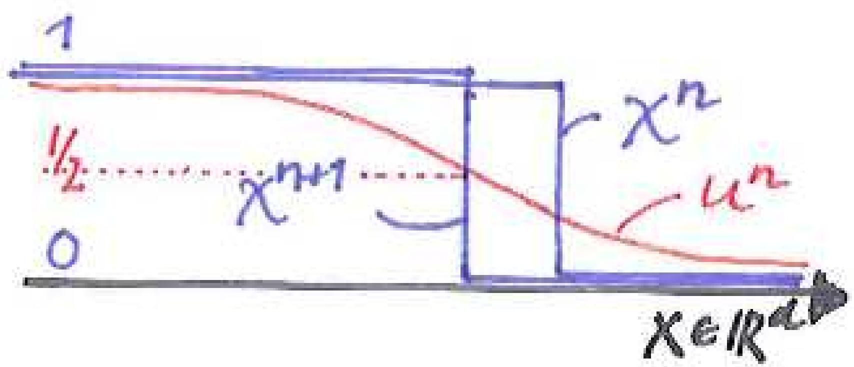
Computational scheme for flow by mean curvature (MCF)

Here just time discretization; time-step size h; $\chi \in \{0, 1\}$

$$\chi^{n-1} \overset{\text{convolution}}{\rightsquigarrow} u^n := G_h * \chi^{n-1} \overset{\text{thresholding}}{\rightsquigarrow} \chi^n := I(u^n \geq \tfrac{1}{2})$$

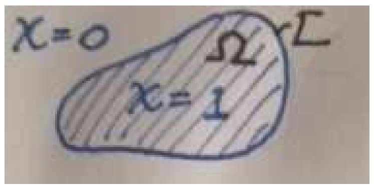
G_h heat kernel at time h

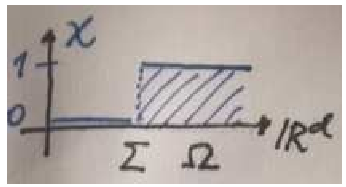
$=$ Gaussian of variance h



description of Σ in terms

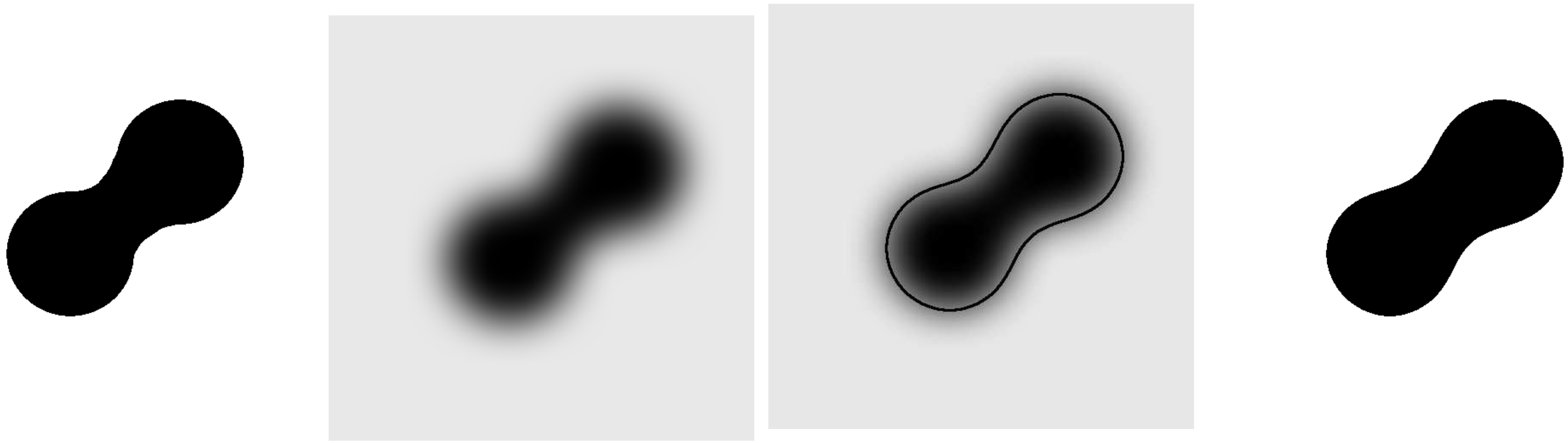
of characteristic function χ





Easy to implement

$$\chi^{n-1} \overset{\text{convolution}}{\rightsquigarrow} u^n := G_h * \chi^{n-1} \overset{\text{thresholding}}{\rightsquigarrow} \chi^n := I(u^n \geq \tfrac{1}{2})$$



χ^{n-1} u^n $\{u^n = \frac{1}{2}\}$ χ^n

Low complexity: Fast Fourier Transform for convolution

Connects to more general level set methods,
efficient thanks to fast marching algorithm (Sethian)

Convergence in the two-phase case

$$\chi^{n-1} \overset{\text{convolution}}{\rightsquigarrow} u^n := G_h * \chi^{n-1} \overset{\text{thresholding}}{\rightsquigarrow} \chi^n := I(u^n \geq \tfrac{1}{2})$$

Thresholding satisfies comparison principle:

$$\chi^{n-1} \leq \tilde{\chi}^{n-1} \Longrightarrow u^n \leq \tilde{u}^n \Longrightarrow \chi^n \leq \tilde{\chi}^n$$

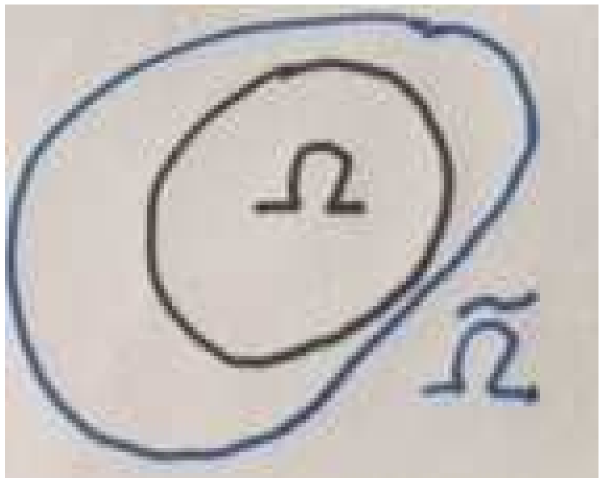
Evans '93,

Barles & Georgelin '95,

Ishii & Pires & Souganidis '99:

convergence to MCF

in sense of viscosity solution (Evans-Spruck)



Straightforward extension to multi-phase version

N phases, eg $\chi = \{\chi_i\}_{i=1,\cdots,N}$ with $\sum_{i=1}^N \chi_i = 1$

$\chi^{n-1} \rightsquigarrow u^n, u_i^n := G_h * \chi_i^{n-1} \rightsquigarrow \chi^n, \chi_i^n := I(u_i^n \geq u_j^n \; \forall j)$

Application to grain growth:
eg. Elsey & Esedoḡlu & Smereka '11 ($d = 3$ and $N \geq 100{,}000$)

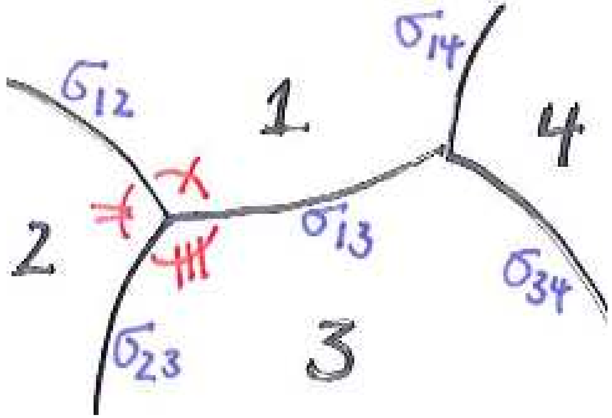
Long-time existence of multi-phase MCF:
Kim & Tonegawa via Brakke's notion '15,

Strong solutions past singularities for $d = 2$ (networks):
Mantegazza&Novaga&Tortorelli '04, Ilmanen&Neves&Schulze '18,

Weak-strong uniqueness (d=2): Fischer&Hensel&Laux&Simon '20

Two tasks

1) Generalization to $\binom{N}{2}$
surface tensions σ_{ij}
(Esedoḡlu & O. '15),
and mobilities (Esedoḡlu & Salvador '18)
interfacial energy depends
on misorientation of grains



2) (conditional) convergence (Laux & O. '16, '20, '20)

Both based on **minimizing movement interpretation**
of thresholding (Esedoḡlu & O. CPAM'15)

2-phase thresholding ...

Thresholding $\chi^n = I(G_h{*}\chi^{n-1} > \frac{1}{2})$ minimizes

$$\underbrace{\frac{1}{\sqrt{h}}\int (v\text{-}\chi^{n-1})\, G_h{*}(v\text{-}\chi^{n-1})}_{\text{distance}^2 \text{ of } v \text{ to } \chi^{n-1}} + \underbrace{\frac{1}{\sqrt{h}}\int (1\text{-}v)\, G_h{*}v}_{\text{energy of } v}$$

among all functions $v \in [0,1]$. Why? Just linear algebra:

$$\overset{G_h \text{ symm.}}{=} \frac{1}{\sqrt{h}}\int v\,(1 - 2G_h{*}\chi^{n-1}) + \frac{1}{\sqrt{h}}\int \chi^{n-1}\, G_h{*}\chi^{n-1}.$$

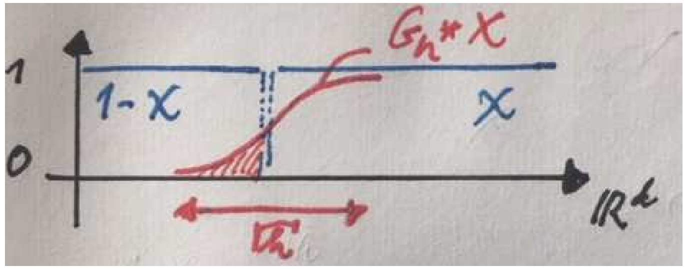
... interpreted as minimizing movements (EO'15)

Link of minimizing movements interpretation ...

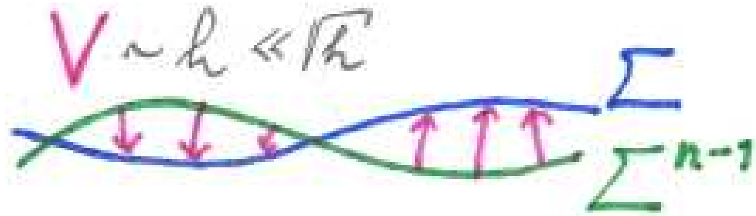
Recall : Thresholding χ^n minimizes among all v

$$\underbrace{\frac{1}{\sqrt{h}}\int (v\text{-}\chi^{n-1})\, G_h*(v\text{-}\chi^{n-1})}_{\text{distance}^2 \text{ of } v \text{ to } \chi^{n-1}} + \underbrace{\frac{1}{\sqrt{h}}\int (1\text{-}v)\, G_h*v}_{\text{energy of } v}$$

$$\frac{1}{\sqrt{h}}\int (1\text{-}\chi)\, G_h*\chi \;\approx\; c_0 \,\text{surface area of}\, \Sigma$$



$$\frac{1}{\sqrt{h}}\int (\chi\text{-}\chi^{n-1})\, G_h*(\chi\text{-}\chi^{n-1}) \;\approx\; \frac{c_0}{h}\int_\Sigma V^2$$



... to mean curvature flow

Multiphase thresholding as minimizing movement (EO'15)

a) $E_h(\chi) := \sum_{i\neq j} \frac{1}{\sqrt{h}}\int \chi_i\, G_h * \chi_j$
Γ-converges to $c_0 \sum_{i\neq j} \frac{1}{2}\int |\nabla\chi_i| + |\nabla\chi_j| - |\nabla(\chi_i+\chi_j)|$
$= c_0 \sum_{i\neq j}$ area of interface between phase i and phase j

$= c_0$ total interfacial energy

b) $-E_h(\chi-\chi') = \sum_i \frac{1}{\sqrt{h}}\int (\chi_i-\chi_i')\, G_h * (\chi_i-\chi_i')$
$= \sum_i \frac{1}{\sqrt{h}}\int |G_{\frac{h}{2}} * (\chi_i-\chi_i')|^2$ is a distance^2 of χ and χ'

c) thresholding means that χ^n minimizes
$2E_h(\chi;\chi^{n-1}) = -E_h(\chi-\chi^{n-1}) + E_h(\chi) + const$,
which is of form $\frac{1}{2h}$distance$^2(\chi,\chi^{n-1})$ + energy(χ)

Scheme preserves comparison and *gradient flow structure*

Natural generalization to $\{\sigma_{ij}\}$ (EO'15)

a) $E_h(\chi) := \sum_{i,j} \sigma_{ij} \frac{1}{\sqrt{h}} \int \chi_i G_h * \chi_j$
Γ-converges to $c_0 \sum_{i,j} \sigma_{ij} \frac{1}{2} \int |\nabla\chi_i| + |\nabla\chi_j| - |\nabla(\chi_i + \chi_j)|$
$= c_0$ total interfacial energy (eg Ambrosio&Braides'90)
provided $\{\sigma_{ij}\}$ satisfies triangle inequality

New element in proof: monotonicity $E_{k^2h}(\chi) \leq E_h(\chi)$

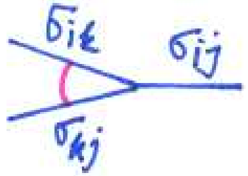
b) $-E_h(\chi - \chi')$ is a distance^2 of χ and χ'
provided $\{\sigma_{ij}\}$ negative semi-definite on $\delta\chi$ with $\sum_i \delta\chi_i = 0$.

c) χ^n minimizes $-E_h(\chi - \chi^{n-1}) + E_h(\chi)$ turns into
$\chi^{n-1} \rightsquigarrow u_i^n := \sum_j \sigma_{ij} G_h * \chi_j^{n-1} \rightsquigarrow \chi_i^n := I(u_i^n \leq u_j^n \ \forall j)$

Get right thresholding scheme by reverse engineering, retaining the complexity.

Assumptions on surface tensions $\{\sigma_{ij} = \sigma_{ji}\}$ ...

triangle inequality: $\sigma_{ik} + \sigma_{kj} \geq \sigma_{ij}$,

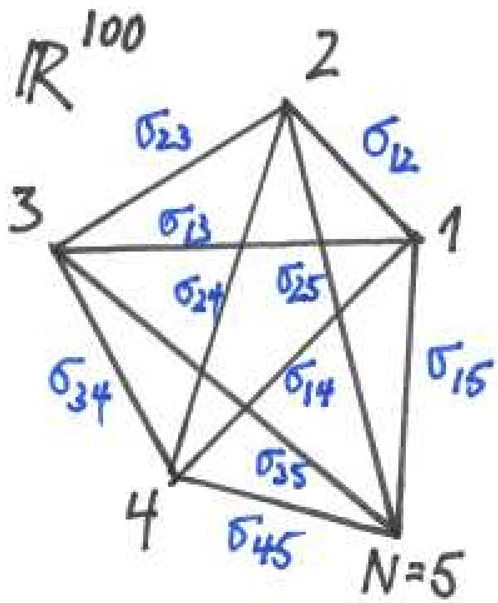


negative semi-definite: $\sum_{ij} \delta\chi_i \, \sigma_{ij} \, \delta\chi_j \leq 0$ for $\sum_i \delta\chi_i = 0$

$\{\sigma_{ij}\}$ ℓ^∞-embeddable
$\Longleftrightarrow$ triangle inequality

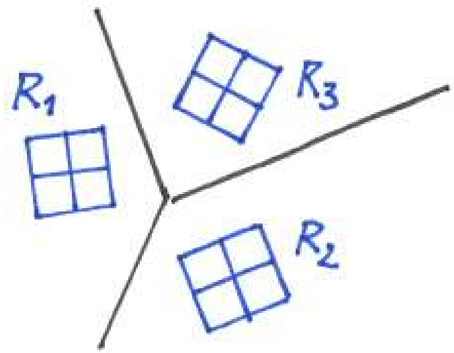
$\{\sqrt{\sigma_{ij}}\}$ ℓ^2-embeddable
$\Longleftrightarrow$ negative semi-definite

$\Longleftarrow$ { $N \leq 4 \Longrightarrow$
$\{\sigma_{ij}\}$ ℓ^1-embeddable
$\Longrightarrow$ easy treatment



... relate to embeddability

Assumptions on $\{\sigma_{ij}\}$ (triangle inequ. +negative def.) ...

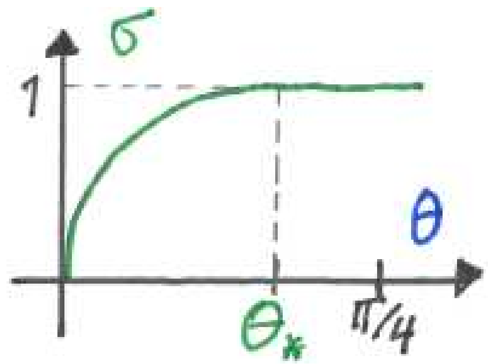


Symmetry: $\angle(R_i R_j{}^{-1}) \in [0, \pi]$

$$\theta_{ij} = \min_{R \in \text{octahedral group}} \angle(R R_i R_j{}^{-1})$$

Read-Shockley (dislocations $\rightsquigarrow$ grain boundaries [Lauteri&Luckhaus '16])

$$\sigma_{ij} = \begin{cases} \frac{\theta_{ij}}{\theta_*}(1 - \log \frac{\theta_{ij}}{\theta_*}) & \theta_{ij} \leq \theta_* \\ 1 & \theta_{ij} \geq \theta_* \end{cases}$$



... satisfied for common grain boundary model

Convergence of multi-phase thresholding

Holds for any number of phases N provided

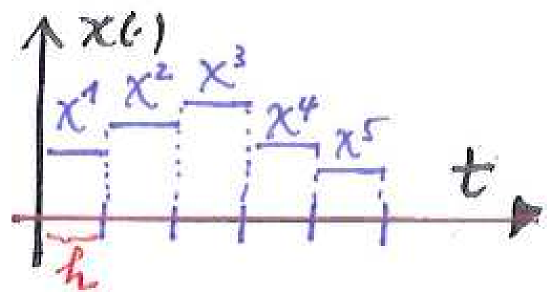
$\{\sigma_{ij}\}_{i,j=1,\cdots,N}$ negative definite & strict triangle inequality

State here for $N = 2$ where $E_h(\chi) = \frac{1}{\sqrt{h}} \int_{[0,1)^d} (1-\chi) G_h * \chi$

χ^0 initial data with $\{E_h(\chi^0)\}_{h \downarrow 0}$ bounded

i. e. $\int_{[0,1)^d} |\nabla \chi^0| < \infty$,

χ_h piecewise constant interpolation of $\{\chi^n\}_n$



Have 3 *conditional* convergence results:

to BV solution, Brakke solution, De Giorgi-type solution

Robust notions ("BV") for flow past singularities

Let χ be the characteristic function of set Ω

Robust notion of area of boundary and surface measure

$$\int |\nabla\chi| := \sup\left\{\int_{[0,1)^d} \chi\nabla\cdot\xi \,\middle|\, \xi\in C^\infty([0,1)^d,\mathbb{R}^d),\ \sup|\xi|\le 1\right\}.$$

χ = function of **B**ounded **V**ariation, Ω = Caccioppoli set

Robust notion of (inner) normal $\nu\in L^\infty(|\nabla\chi|)$:

$\nabla\chi = \nu|\nabla\chi|$ (polar factorization of a measure)

Robust notion of normal velocity $V\in L^2(|\nabla\chi|dt)$

$\partial_t\chi = V|\nabla\chi|$ in distributional sense, that is,

$\int_0^1\int \partial_t\zeta\chi + \zeta V|\nabla\chi|dt = 0$ for all $\zeta\in C_0^\infty((0,1)\times[0,1)^d)$

Robust notion of mean curvature $H\in L^2(|\nabla\chi|)$

$$\int H\xi\cdot\nabla\chi = \int(\nabla\cdot\xi - \nu\cdot D\xi\nu)|\nabla\chi|$$

for all $\xi\in C^\infty([0,1)^d,\mathbb{R}^d)$

Convergence to BV solution (LO'16 CalcVar)

Theorem 1. Suppose $\chi_h \to \chi$ in $L^1((0,1)\times[0,1)^d)$ and

$$\int_0^1 E_h(\chi_h(t))dt \to c_0 \int_0^1 \int |\nabla\chi| dt.$$

Then there exists $V \in L^2(|\nabla\chi|dt)$ such that

for all $\zeta \in C_0^\infty((0,1)\times[0,1)^d)$

$$\int_0^1 \int \partial_t\zeta\chi + \zeta V|\nabla\chi|dt = 0 \text{ (normal velocity } = V\text{)}$$

and for all $\xi \in C^\infty([0,1]\times[0,1)^d, \mathbb{R}^d)$

$$\int_0^1 \int (\nabla\cdot\xi - \nu\cdot D\xi\nu + 2V\nu\cdot\xi)|\nabla\chi|dt = 0 \text{ (mean curv. } = -2V\text{)}$$

here $\int := \int_{[0,1)^d}$

A *conditional* convergence result

Suppose $\chi_h \to \chi$ in $L^1((0,1)\times[0,1)^d)$ and

$$\int_0^1 E_h(\chi_h(t))dt \to c_0 \int_0^1 \int |\nabla\chi| dt.$$

Then $\exists\, V \in L^2(|\nabla\chi| dt)$ s. t. $\forall\, \zeta \in C_0^\infty((0,1)\times[0,1)^d)$, $\xi \in C^\infty([0,1]\times[0,1)^d, \mathbb{R}^d)$

$$\int_0^1 \int \partial_t\zeta\chi + \zeta V |\nabla\chi| dt = 0$$

$$\int_0^1 \int (\nabla\cdot\xi - \nu\cdot D\xi\nu + 2V\nu\cdot\xi)|\nabla\chi| dt = 0$$

Same assumption and notion of solution as in Luckhaus & Sturzenhecker '95 on

minimizing movement scheme for MCF introduced by Almgren & Taylor & Wang '93,

but more robust proof (no minimal surface regularity theory)

Satisfied in mean-convex case:

Laux & De Philippis '20 for ATW, Fuchs & Laux for thresholding

The scheme of Almgren&Taylor&Wang

Recall general structure of minimizing movements scheme:

χ^n minimizes $\quad \frac{1}{2h} d^2(\chi, \chi^{n-1}) + E(\chi)$.

Almgren-Taylor-Wang scheme: Ω^n minimizes

$\frac{1}{2h}\int_{\Omega \triangle \Omega^{n-1}} \mathrm{dist}(\cdot, \Omega^{n-1}) +$ surface area of $\partial\Omega$.

A minimizing movement scheme “avant la lettre”.

Recall Michor & Mumford ’06: canonical d degenerates.

As opposed to thresholding: an academic scheme
(however Chambolle & Novaga ’07).

Recall minimizing movements interpr. of thresholding:

χ^n minimizes $\quad -E_h(\chi - \chi^{n-1}) + E_h(\chi)$.

Gradient flow comes with energy (in)equality

$H :=$ mean curvature, $V =$ normal velocity

Seek energy inequality $\int (2V)^2 |\nabla \chi| = \int H^2 |\nabla \chi| \leq -2 \frac{d}{dt} \int |\nabla \chi|$

Build-in into both notions of solutions of

De Giorgi $\frac{1}{2} \int H^2 |\nabla \chi| + \frac{1}{2} \int (2V)^2 |\nabla \chi| \leq -2 \frac{d}{dt} \int |\nabla \chi|$

Brakke $\int (\zeta H^2 + \nu \cdot \nabla \zeta H) |\nabla \chi| \leq -2 \frac{d}{dt} \int \zeta |\nabla \chi|$ for $\zeta \geq 0$

use De Giorgi's ideas to establish Brakke's

Convergence to Brakke-type sol. (LO'20 CalcVar)

Theorem 2. Suppose $\chi_h \to \chi$ in $L^1((0,1)\times[0,1)^d)$ and

$$\int_0^1 E_h(\chi_h(t))dt \to c_0\int_0^1\int|\nabla\chi|dt.$$

Then there exists $H \in L^2(|\nabla\chi|dt)$ such that

for all $\xi \in C^\infty((0,1)\times[0,1)^d,\mathbb{R}^d)$

$$\int_0^1\int(\nabla\cdot\xi-\nu\cdot D\xi\nu-\nu\cdot\xi H)|\nabla\chi|dt = 0 \text{ (mean curv. } = H)$$

and for all $\zeta \in C^\infty((0,1)\times[0,1)^d,[0,\infty))$

$$\int_0^1\int(-2\partial_t\zeta+\zeta H^2+\nu\cdot\nabla\zeta H)|\nabla\chi|dt\leq 0$$

(2normal velocity $= -H$)

Contains correct inequality $2\frac{d}{dt}\int|\nabla\chi|\leq-\int H^2|\nabla\chi|$

"Brakke-type" because
Brakke's inequality is expressed in BV-framework instead of varifold-framework

Recall: convergence by De Giorgi's tools

Obtain from variational interpolation and metric slope

$$E_h(\chi^N)+\int_0^{Nh}\frac{1}{2h^2}d^2(\chi^h(t+h),\chi^h(t))dt+\int_0^{Nh}\frac{1}{2}|\partial E(u^h(t))|^2dt$$
$$\leq E_h(\chi^0).$$

Similar to characterization of $\frac{d\chi}{dt}=-\text{grad}_{|\chi}E$ by inequality:

$$E(\chi(T))+\int_0^T\frac{1}{2}g_\chi\left(\frac{d\chi}{dt},\frac{d\chi}{dt}\right)dt+\int_0^T\frac{1}{2}g_\chi(\text{grad}E_{|\chi},\text{grad}E_{|\chi})dt$$
$$\leq E(\chi^0),$$

which takes the form (after division by c_0):

$$\int|\nabla\chi(T)|+\int_0^T\int(V^2+(\tfrac{H}{2})^2)|\nabla\chi|dt\leq\int|\nabla\chi^0|.$$

Convergence to De Giorgi-type solution (LO'20 Proc.,Laux & Lelmi '22 Calc. Var.)

Theorem 3. Suppose $\chi_h \to \chi$ in $L^1((0,1)\times[0,1)^d)$ and

$$\int_0^1 E_h(\chi_h(t))dt \to c_0 \int_0^1 \int |\nabla\chi| dt.$$

Then $\exists\ V \in L^2(|\nabla\chi|dt)$ s. t. $\forall\ \zeta \in C_0^\infty((0,1)\times[0,1)^d)$

$$\int_0^1 \int \partial_t\zeta\chi + \zeta V|\nabla\chi|dt = 0 \quad \text{(normal velocity} = V\text{)}$$

and $\exists\ H \in L^2(|\nabla\chi|dt)$ s. t. $\forall\ \xi \in C^\infty((0,1)\times[0,1)^d,\mathbb{R}^d)$

$$\int_0^1 \int (\nabla\cdot\xi - \nu\cdot D\xi\nu - \nu\cdot\xi H)|\nabla\chi|dt = 0 \quad \text{(mean curv.} = H\text{)}$$

with the property that for all $T \in (0,1)$

$$\int |\nabla\chi(T)| + \int_0^T \int (V^2 + (\tfrac{H}{2})^2)|\nabla\chi|dt \le \int |\nabla\chi^0|.$$

$$E(\chi(T)) + \int_0^T \tfrac{1}{2} g_\chi(\tfrac{d\chi}{dt},\tfrac{d\chi}{dt})dt + \int_0^T \tfrac{1}{2} g_\chi(\text{grad}E_{|\chi},\text{grad}E_{|\chi})dt \le E(\chi^0)$$

Lower semi-continuity in metric term, sketch of proof

Goal: $c_0\int_0^T\int V^2|\nabla\chi| \le \liminf_{h\downarrow 0} \sum_{0<nh<T} \frac{1}{2h} d_h^2(\chi_h^n, \chi_h^{n-1})$

$$= \lim_{h\downarrow 0} \sqrt{h}\int_0^T\int |G_{\frac{h}{2}} * \frac{\chi_h(t+h)-\chi_h(t)}{h}|^2 := \int_0^T\int d\mu.$$

Convergence assumption yields
"convergence of normals down to scale $\sqrt{h}$", i. e.
$\frac{1}{\sqrt{h}}\Big(\chi_h(\cdot+\sqrt{h}\hat{z}) - \chi_h\Big)_+ \rightharpoonup \Big(\hat{z}\cdot\nabla\chi\Big)_+$

Good time scale $\tau := \alpha\sqrt{h}$ with $\alpha \in (0,\infty)$ to be chosen later.

Consider increment $\delta\chi := \chi_h(t+\tau) - \chi_h(t) \in \{-1,0,1\}$;

have $|\delta\chi| = \delta\chi\, G_h{*}\delta\chi \;+\; \delta\chi(\delta\chi - G_h{*}\delta\chi)$.

Lower semi-continuity in metric term, sketch of proof

Recall: time scale $\tau := \alpha\sqrt{h}$, increment $\delta\chi := \chi_h(t+\tau) - \chi_h(t)$

splitting $\quad |\delta\chi| = \delta\chi\, G_h{*}\delta\chi \; + \; \delta\chi(\delta\chi - G_h{*}\delta\chi)$.

Recall consequence of convergence assumption

$$\frac{1}{\sqrt{h}}\Big(\chi_h(\cdot{+}\sqrt{h}\hat{z}) - \chi_h\Big)_+ \rightharpoonup \Big(\hat{z}\cdot\nabla\chi\Big)_+.$$

Hence if we further split $\quad \delta\chi = \delta\chi_+ - \delta\chi_-$ we have "orthogonality" $\frac{1}{\sqrt{h}}\int \delta\chi_+\, G_h{*}\delta\chi_- \to 0$.

Allows to replace $\quad \delta\chi(\delta\chi - G_h{*}\delta\chi) \rightsquigarrow$

$$\delta\chi_+(\delta\chi_+ - G_h{*}\delta\chi_+) + \delta\chi_-(\delta\chi_- - G_h{*}\delta\chi_-)$$
$$= \delta\chi_+\, G_h{*}(1\text{-}\delta\chi_+) + \delta\chi_-\, G_h{*}(1\text{-}\delta\chi_-).$$

Lower semi-continuity in metric term, sketch of proof

Recall we still need to control

$$\frac{1}{\tau}\int\Big(\delta\chi_{+}\,G_h*(1\text{-}\delta\chi_{+}) + \delta\chi_{-}G_h*(1\text{-}\delta\chi_{-})\Big)$$

where $\delta\chi = \chi_h(t+\tau) - \chi_h(t)$, $\tau = \alpha\sqrt{h}$.

For any normal $\nu_0 \in S^{d-1}$ to be chosen later

$$\int\Big(\delta\chi_{+}\,G_h*(1\text{-}\delta\chi_{+}) + \delta\chi_{-}G_h*(1\text{-}\delta\chi_{-})\Big)$$
$$= \int dx \int_{z\cdot\nu_0>0} dz G_h(z)(|\delta\chi_{+}(x+z)\text{-}\delta\chi_{+}(x)|+|\delta\chi_{-}(x+z)\text{-}\delta\chi_{-}(x)$$

Discrete mixed derivative in time τ and space z;
use 2 pointwise estimates ("time like", "space like"):

$$|\delta\chi_{+}(x+z)\text{-}\delta\chi_{+}(x)| + |\delta\chi_{-}(x+z)\text{-}\delta\chi_{-}(x)|$$
$$\leq \begin{cases} |\chi_h(t+\tau,x+z)\text{-}\chi_h(t,x+z)| + |\chi_h(t+\tau,x)\text{-}\chi_h(t,x)| \\ |\chi_h(t+\tau,x+z)\text{-}\chi_h(t+\tau,x)| + |\chi_h(t,x+z)\text{-}\chi_h(t,x)| \end{cases}$$

sketch of proof, end

Recall we still need to control

$$\frac{1}{\tau}\int dx \int_{z\cdot\nu_0>0} dz G_h(z)(|\delta\chi_+(x+z)\text{-}\delta\chi_+(x)| + |\delta\chi_-(x+z)\text{-}\delta\chi_-(x)|).$$

Use $|\delta\chi_+(x+z)\text{-}\delta\chi_+(x)| + |\delta\chi_-(x+z)\text{-}\delta\chi_-(x)|$

$$\leq \begin{cases} |\chi_h(t+\tau,x+z)\text{-}\chi_h(t,x+z)| + |\chi_h(t+\tau,x)\text{-}\chi_h(t,x)| & \text{for } z\cdot\nu_0 > \tau V_0 \\ |\chi_h(t+\tau,x+z)\text{-}\chi_h(t+\tau,x)| + |\chi_h(t,x+z)\text{-}\chi_h(t,x)| & \text{for } z\cdot\nu_0 < \tau V_0 \end{cases}$$

with $V_0 \in (0,\infty)$ to be chosen.

Convergence assumption yields $|\partial_t\chi|$

$$\leq \alpha\mu + 2\int_{\hat{z}\cdot\nu_0>\alpha V_0} G_1(\hat{z})d\hat{z}|\partial_t\chi| + \frac{2}{\alpha}\int_{0<\hat{z}\cdot\nu_0<\alpha V_0} G_1(\hat{z})|\hat{z}\cdot\nabla\chi|d\hat{z}.$$

Localize in good point x on boundary and choose

$\nu_0 = \nu(x)$, $V_0 := |V(x)|$, divide by $\alpha \downarrow 0$.

Recover $c_0 V^2 \leq \frac{d\mu}{d|\nabla\chi|}$ with desired $c_0 := \int(\hat{z}_1)_+ G_1(\hat{z})d\hat{z} = \frac{1}{\sqrt{2\pi}}$.

Summary

geometric analysis:

Flow of a surface by its mean curvature

materials science: growth of grains in polycrystals

analysis on metric spaces:

De Giorgi's tools for gradient flows

scientific computing: Osher's thresholding scheme

stable second-order versions (Zaitzeff&Esedoḡlu&Garikipati)

co-dimension two (Laux&Yip)