Hyperbolic Conservation Laws

Emil Wiedemann
Ulm University, Winter 2020/2021

Preface

These lecture notes are based on graduate-level courses I gave at Bonn in summer term 2016, and in Ulm in winter 2020/21. I have based most of these notes on Evans' books [**1**, **2**]. Students taking this course should have some acquaintance with functional analysis, but are not required to have taken a first course on partial differential equations. The course is intended for 14 weeks at 180 minutes per week, at a relatively slow pace. In any case, the second iteration in 2020/21 took place remotely, essentially as a reading class, due to the COVID-19 crisis, so that lecture times were largely fictitious.

Hyperbolic conservation laws form an important class that is often neglected in courses on partial differential equations (PDEs). A focus on elliptic and parabolic equations, however, often misleads students to think that PDEs always behave nicely and regularly, with classically established well-posedness theories. In the hyperbolic case, however, things are different: The ongoing lack of an existence and uniqueness theory for hyperbolic systems constitutes, as Peter Lax once said, a 'scientific scandal'. The lack of regularity, already exhibited in the scalar situation, is not just a shortcoming of current analysis, but is inherent in the models and physically visible, e.g., in the form of shock waves.

Conservation laws naturally appear in numerous fields of application, such as fluid dynamics, mathematical biology, collective behaviour, or traffic flow. This course, however, focuses on problems of mathematical analysis in terms of existence and uniqueness of entropy solutions, and an analysis of the Riemann problem. While the theory of scalar conservation laws can be considered reasonably complete, the corresponding problems for systems are wide open and constitute the subject of very active current research. This course can thus be the starting point for a Master's thesis.

I am deeply grateful to Raphael Wagner, who not only composed an exquisite set of example sheets, but also offered to typeset my (often sketchy and illegible) handwritten notes.

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CHAPTER 1

Introduction

A *conservation law* is a first-order PDE of the form

$$\partial_t u(x,t) + \operatorname{div} F(u(x,t)) = 0. \tag{1.1}$$

Here,

- the unknown is $u : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}^m$,
- $F : \mathbb{R}^m \to \mathbb{R}^{m\times n}$ is the given *flux function*,
- div is the divergence operator in x, i.e.

$$[\operatorname{div} F(u(x,t))]_i := \sum_{j=1}^{n} \partial_{x_j} [F(u(x,t))]_{ij}.$$

If $m > 1$, we say that (1.1) is a *system* of conservation laws.
If $m = 1$, then (1.1) is a *scalar* conservation law.
Hyperbolicity: An assumption on F only relevant for systems, to be discussed later.

Motivation:
Recall the *Gauß divergence theorem*:
Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be C^1 (continuously differentiable) and $\Omega \subset \mathbb{R}^n$ a bounded domain with smooth boundary $\partial\Omega$ and outer unit normal ν, then

$$\int_{\partial\Omega} \Phi(x)\cdot\nu(x)\,dS(x) = \int_{\Omega} \operatorname{div}\Phi(x)\,dx,$$

where dS means integration with respect to the surface measure on $\partial\Omega$ and dx indicates integration with respect to the n-dimensional Lebesgue measure.
Suppose now $u : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$ is the density of some quantity and $\Omega \subset \mathbb{R}^n$ a smooth and bounded domain. A conservation law then states that the rate of change of $\int_\Omega u(x,t)\,dx$ with time equals the flux across the boundary $-\int_{\partial\Omega} F(u(x,t))\cdot\nu(x)\,dS(x)$:

$$\frac{d}{dt}\int_\Omega u(x,t)\,dx = -\int_{\partial\Omega} F(u(x,t))\cdot\nu(x)\,dS(x) = -\int_\Omega \operatorname{div} F(u(x,t))\,dx.$$

As this is true for every domain Ω, we obtain

$$u(x,t) + \operatorname{div} F(u(x,t)) = 0,$$

i.e. (1.1) (for $m = 1$).

EXAMPLE 1.1. i) *Burgers' equation* (m=n=1)

$$\partial_t u + \partial_x \left(\frac{u^2}{2}\right) = 0.$$

ii) Consider the *wave equation*

$$\partial_{tt}w - \partial_{xx}w = 0.$$

Write $u_1 = \partial_x w, u_2 = \partial_t w$, then we have

$$\partial_t u_1 - \partial_x u_2 = 0 \text{ and } \partial_t u_2 - \partial_x u_1 = 0$$

or, writing $u = (u_1, u_2)$,

$$\partial_t u + \partial_x F(u) = 0 \text{ with } F(u) = -(u_2, u_1).$$

iii) Similarly, for the *nonlinear wave equation*

$$\partial_{tt}w - (p(\partial_x w))_x = 0 \, (p : \mathbb{R} \to \mathbb{R} \text{ given}),$$

we have the "p-system"

$$\partial_t u + \partial_x F(u) = 0 \text{ with } F(u_1, u_2) = (-u_2, -p(u_1)).$$

iv) The *isentropic Euler equations* of gas dynamics are

$$\partial_t \rho + \operatorname{div}(\rho v) = 0 \ \text{(conservation of mass)}$$
$$\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \nabla p(\rho) = 0 \ \text{(conservation of momentum)}$$

where $\rho \geq 0$ is the density, $v : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}^n$ the velocity, p a given pressure function (e.g. $p(\rho) = \rho^\gamma, \gamma > 1$) and $v \otimes v \in \mathbb{R}^{n \times n}$ with $(v \otimes v)_{ij} = v_i v_j$.
As long as $\rho > 0$, we can write

(1.2) $$u = (\rho, \rho v_1, ..., \rho v_n) \text{ and } \partial_t u + \operatorname{div} F(u) = 0,$$

where

$$\begin{cases} F_{1j}(u_1, ..., u_{n+1}) = u_{j+1} & : j = 1, ..., n \\ F_{ij}(u_1, ..., u_{n+1}) = \frac{u_{i+1}u_{j+1}}{u_1} + p(u_1)\delta_{ij} & : i > 1, j = 1, ..., n. \end{cases}$$

The vacuum state ($\rho = 0$) causes trouble.

Rough plan of the lecture:

- Scalar conservation laws:
  Characteristics, shocks, weak solutions and non-uniqueness, entropy conditions, uniqueness (Kružkov), existence via compensated compactness, Riemann problem
- Systems:
  Weak solutions, entropy, Riemann problem, existence of systems of two equations via compensated compactness, weak-strong uniqueness

Literature:

- Evans [**1**]: *Partial Differential Equations* – Chapters 3.4 and II
- Evans [**2**]: *Weak Convergence Methods for Nonlinear Partial Differential Equations* – Chapter 5

Exam: Oral exam at the end of the semester.

CHAPTER 2

Scalar Conservation Laws

2.1. The Method of Characteristics

2.1.1. Characteristic ODEs for general first order PDEs.

Let $\Omega \subset \mathbb{R}^n$ be a smooth domain and

$$\Phi : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}, (x, z, p) \mapsto \Phi(x, z, p)$$

be smooth. Consider the PDE

$$\Phi(x, u(x), Du(x)) = 0 \text{ on } \Omega \tag{2.1}$$

for the unknown $u : \mathbb{R}^n \to \mathbb{R}$.

Idea: Reduce (2.1) to a system of ODEs! To this end, let $x(s)$ be a curve in Ω ($s \in \mathbb{R}$). If u is a (smooth) solution of (2.1), set

$$\begin{aligned} &z(s) = u(x(s)) \text{ and} \\ &p(s) = Du(x(s)) = (\partial_{x_1} u(x(s)), ..., \partial_{x_n} u(x(s))). \end{aligned}$$

Differentiate this with respect to s:

$$\dot{p}_i(s) = \sum_{j=1}^{n} \partial_{x_j} \partial_{x_i} u(x(s)) \dot{x}_j(s).$$

On the other hand, differentiate (2.1) with respect to x_i:

$$\partial_{x_i} \Phi(x, u, Du) + \partial_z \Phi(x, u, Du) \partial_{x_i} u + \sum_{j=1}^{n} \partial_{p_j} \Phi(x, u, Du) \partial_{x_i} \partial_{x_j} u = 0.$$

Therefore, if

$$\dot{x}_j(s) = \partial_{p_j} \Phi(x(s), z(s), p(s)) = \partial_{p_j} \Phi(x(s), u(x(s)), Du(x(s))),$$

then

$$\dot{p}_i(s) = -\partial_{x_i} \Phi(x(s), z(s), p(s)) - \partial_z \Phi(x(s), z(s), p(s)) p_i(s).$$

Also differentiate $z(s) = u(x(s))$:

$$\dot{z}(s) = \sum_{j=1}^{n} \partial_{x_j} u(x(s)) \dot{x}_j(s) = \sum_{j=1}^{n} p_j(s) \partial_{p_j} \Phi(x(s), z(s), p(s)). \tag{2.2}$$

Hence we derived the *characteristic ODEs*

i) $\dot{x}(s) = D_p \Phi(x(s), z(s), p(s))$
ii) $\dot{z}(s) = D_p \Phi(x(s), z(s), p(s)) \cdot p(s)$
iii) $\dot{p}(s) = -D_x \Phi(x(s), z(s), p(s)) - D_z \Phi(x(s), z(s), p(s)) p(s)$.

We have shown:

THEOREM 2.1. *If u is a smooth solution of (2.1) and $x(s)$ solves i) with $z(s) = u(x(s)), p(s) = D_u(x(s))$, then z and p solve ii) and iii), respectively.*

A solution of i) is called a *characteristic curve* for the PDE (2.1).
Hence the strategy to solve (2.1) is:

- Solve the system of ODEs i) - iii) under appropriate *initial data* (s=0) related to a boundary condition $u = g$ on $\partial\Omega$ for (2.1);
- This gives values of u along each characteristic curve via ii)
- Hope that the characteristic curves cover all of Ω and are disjoint.

Picture for $\Omega = \{x = (x_1, ..., x_n) : x_n > 0\}$:

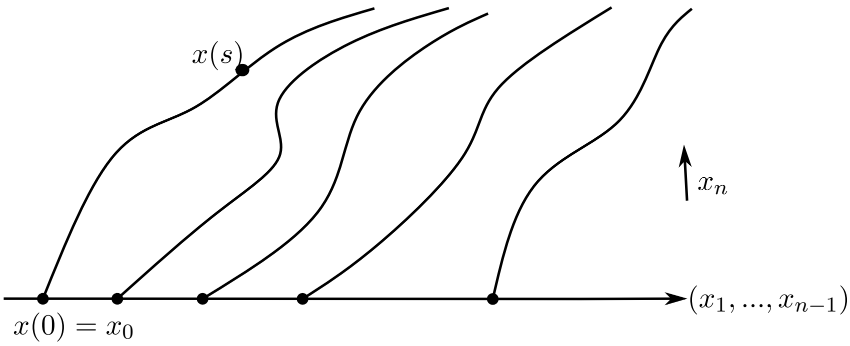


FIGURE 1. Characteristic curves

We will soon see that characteristics may intersect or not fill entire domains.

2.1.2. Application to scalar conservation laws.
Consider a scalar conservation law

$$\partial_t u + \operatorname{div} F(u) = 0 \, (u : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}, F : \mathbb{R} \to \mathbb{R}^n).$$

If u is a smooth solution, this is equivalent to

$$\partial_t u + F'(u) \cdot D_x u = 0 \text{ or } \Phi(y, u(y), D_y u(y)) = 0, \tag{2.3}$$

where $y = (x, t), D_y u = (D_x u, \partial_t u)$ and

$$\Phi(y, z, q) = q_{n+1} + F'(z) \cdot \underbrace{(q_1, ..., q_n)}_{=:p}.$$

Our domain is $\Omega = \mathbb{R}^n \times (0, \infty) \subset \mathbb{R}^{n+1}$. Note now that

$$\begin{aligned} D_y \Phi &= 0, \\ D_z \Phi &= F''(z) \cdot p, \\ D_q \Phi &= (F'(z), 1). \end{aligned}$$

Let $z(s) = u(y(s)), q(s) = D_y u(y(s))$. Hence i) becomes

$$\begin{cases} \dot{y}_i(s) = F_i'(z(s)) : i = 1, ..., n \\ \dot{y}_{n+1}(s) = 1 \end{cases} \tag{2.4}$$

and ii) is

$$\dot{z}(s) = F'(z(s)) \cdot p(s) + q_{n+1}(s) = 0$$

by virtue of the PDE (2.3). Do not need iii)! Let us impose in addition to (2.3) the initial condition

$$u(x, 0) = g(x),$$

and choose $y(0) = (x_0, 0) \in \partial\Omega$. Then first $\dot{y}_{n+1}(s) = 1$ and $y_{n+1}(0) = 0$ implies $y_{n+1}(s) = s$. Also, $\dot{z}(s) = 0$ together with $z(0) = u(y(0)) = u(x_0, 0) = g(x_0)$ yields $z(s) = g(x_0)$ for all $s \geq 0$, so that (2.4) gives

$$\dot{y}_i(s) = F_i'(g(x_0))$$

and so

$$y_i(s) = F_i'(g(x_0))s + (x_0)_i \, (i = 1, ..., n).$$

We have proved

THEOREM 2.2. *Suppose u is a smooth solution of*

$$\partial_t u + \operatorname{div} F(u) = 0 \text{ on } \mathbb{R}^n \times (0, \infty),$$
$$u(\cdot, 0) = g.$$

Then u is constant along the characteristics, which are given by $s \mapsto (F'(g(x_0))s + x_0, s)$.

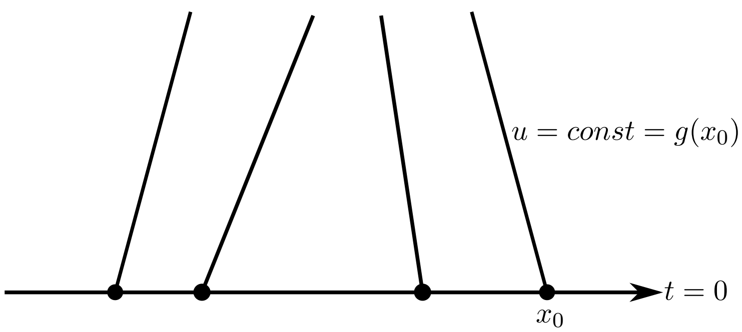


FIGURE 2. Linear characteristic curves

2.1.3. Formation of Shocks.

Consider the *Burgers' equation*

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) = 0 \text{ on } \mathbb{R} \times (0, \infty) \tag{2.5}$$

with smooth initial data $g : \mathbb{R} \to \mathbb{R}$ such that

$$g(x) = \begin{cases} 1 & : x \leq 0 \\ 0 & : x \geq 1 \\ \text{decreasing} & : 0 < x < 1 \end{cases}$$

By Theorem 2.2, the characteristic curves are given by

$$s \mapsto (F'(g(x_0))s + x_0, s) = (g(x_0)s + x_0, s)$$

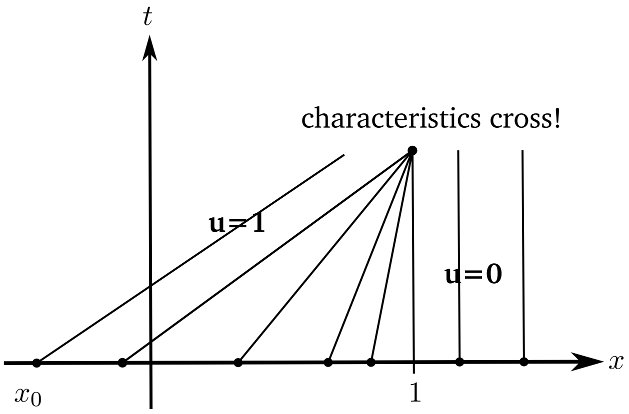


FIGURE 3. Crossing of characteristics

If u is a smooth solution, it is constant along the characteristics, so at the crossing point $u = 0$ and $u = 1 \to$ contradiction!
This shows:

There is smooth initial data such that (2.5) does not have a global-in-time smooth solution!

The crossing of characteristics and breakdown of a smooth solution is called a *shock formation*.

2.1.4. Rarefaction Waves.
Consider now (2.5) with (discontinuous) data

$$g(x) = \begin{cases} 0 & : x \leq 0 \\ 1 & : x > 0 \end{cases} \tag{2.6}$$

The notion of characteristic curves still makes sense and the solution is constant along

$$s \mapsto (g(x_0)s + x_0, s).$$

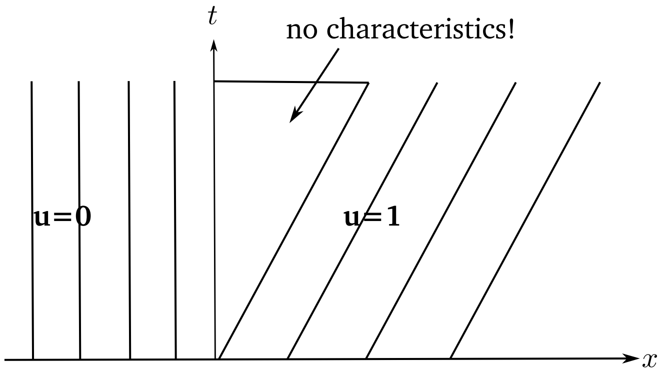


FIGURE 4. Non-determined area by characteristics

So the method of characterstics does not fully determine the solution.

2.2. Weak Solutions

Recall the shock example from 2.1.3. It seems that the solution becomes discontinuous after shock formation and "jumps" from 1 to 0. But then the PDE does not make sense anymore.

2.2.1. Definition of Weak Solution.

Consider a scalar conservation law in one dimension

$$\partial_t u + \partial_x F(u) = 0 \text{ in } \mathbb{R} \times (0, \infty)$$

$$u|_{t=0} = g$$

Assume for the moment that u is a smooth solution. Let $\varphi \in C_c^\infty(\mathbb{R} \times [0, \infty))$ (smooth with compact support), then multiply the PDE by φ and integrate in space and time:

$$\int_0^\infty \int_{-\infty}^\infty \partial_t u \varphi \, dx dt + \int_0^\infty \int_{-\infty}^\infty \partial_x F(u) \varphi \, dx dt = 0.$$

Integrate by parts:

$$-\int_0^\infty \int_{-\infty}^\infty \partial_t \varphi u \, dx dt - \int_{-\infty}^\infty \underbrace{u(x,0)}_{=g(x)} \varphi(x,0) \, dx - \int_0^\infty \int_{-\infty}^\infty F(u) \partial_x \varphi \, dx dt = 0.$$

But this makes sense even if only $u, F(u) \in L^1_{loc}(\mathbb{R} \times [0, \infty))$ and $g \in L^1_{loc}(\mathbb{R})$! (Recall $f \in L^1_{loc}(\Omega)$ if f is measurable and $\int_K |f| < \infty$ for every compact $K \subset \Omega$). Usually one is only interested in bounded solutions, hence we define:

Definition 2.3. Let $F : \mathbb{R} \to \mathbb{R}$ be continuous and $g \in L^\infty(\mathbb{R})$. A function $u \in L^\infty(\mathbb{R} \times (0, \infty))$ is a *weak solution* (or distributional solution, integral solution) of $\partial_t u + \partial_x F(u) = 0$ with initial datum g if

$$\int_0^\infty \int_{-\infty}^\infty \partial_t \varphi u \, dx dt + \int_0^\infty \int_{-\infty}^\infty \partial_x \varphi F(u) \, dx dt + \int_{-\infty}^\infty \varphi(x,0) g(x) \, dx = 0$$

for all $\varphi \in C_c^\infty(\mathbb{R} \times [0, \infty))$.

Without reference to an initial condition, u is simply called a weak solution if

$$\int_0^\infty \int_{-\infty}^\infty \partial_t \varphi u \, dx dt + \int_0^\infty \int_{-\infty}^\infty \partial_x \varphi F(u) \, dx dt = 0$$

for all $\varphi \in C_c^\infty(\mathbb{R} \times (0, \infty))$.

(Notice carefully the difference between $C_c^\infty(\mathbb{R} \times [0, \infty))$ and $C_c^\infty(\mathbb{R} \times (0, \infty))$: Functions from the latter space need to vanish at $t = 0$, while functions from the former don't.)

The following proposition is rather technical, but will be used several times in the sequel. We first need to introduce the space $C_w([0, \infty); L^\infty(\mathbb{R}))$: This is the space of functions $u : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ such that $u(\cdot, t) \in L^\infty(\mathbb{R})$ for every $t \geq 0$, and such that the map $[0, \infty) \to \mathbb{R}$,

$$t \mapsto \int_{\mathbb{R}} u(x,t) \psi(x) dx,$$

is continuous for every $\psi \in L^1(\mathbb{R})$. In this case one says that u is *weakly continuous* into L^∞.

PROPOSITION 2.4. *Let $u \in C_w([0,\infty); L^\infty(\mathbb{R}))$ be a weak solution of $\partial_t u + \partial_x F(u) = 0$. Let $g \in L^\infty(\mathbb{R})$. Then $u(\cdot, 0) = g$ if and only if u is a weak solution with initial datum g in the sense of Definition 2.3.*

PROOF. Suppose u is a weak solution such that $u \in C_w([0,\infty); L^\infty(\mathbb{R}))$ and $u(\cdot,0) = g$. Let $\varphi \in C_c^\infty(\mathbb{R} \times [0,\infty))$ (i.e. $\varphi(t=0)$ is not necessarily zero!). Let further $\chi_\varepsilon : \mathbb{R} \to \mathbb{R}$ be a 'cut-off' function, i.e.,

$$\chi(t) = \begin{cases} 0 & : 0 \le t \le \frac{1}{2} \\ 1 & : t \ge 1 \end{cases},$$

which is smooth and increasing in $\left(\frac{1}{2}, 1\right)$, and set $\chi_\varepsilon(t) = \chi\left(\frac{t}{\varepsilon}\right)$. Note $|\chi_\varepsilon'| \le \frac{C}{\varepsilon}$. Then $\chi(t)\varphi(x,t) \in C_c^\infty(\mathbb{R} \times (0,\infty))$ and hence, as u is a weak solution,

$$\int_0^\infty \int_{\mathbb{R}} \partial_t(\chi_\varepsilon \varphi) u + \partial_x(\chi_\varepsilon \varphi) u \, dx dt = 0.$$

Observe that

$$\lim_{\varepsilon \searrow 0} \iint \chi_\varepsilon(t) \partial_x \varphi u \, dx dt = \iint \partial_x \varphi u \, dx dt$$

by the dominated convergence theorem, as $\chi_\varepsilon \to 1$ a.e. and $|\partial_x \varphi u| \in L^1(\mathbb{R} \times [0,\infty))$ is a dominating function.

On the other hand,

$$\begin{aligned} \iint \partial_t(\chi_\varepsilon \varphi) u \, dx dt &= \iint \chi_\varepsilon \partial_t \varphi u \, dx dt + \iint \chi_\varepsilon'(t) \varphi u \, dx dt \\ &= \iint \chi_\varepsilon \partial_t \varphi u \, dx dt + \int_0^\infty \chi_\varepsilon'(t) \left(\int_{\mathbb{R}} \varphi u \, dx \right) dt. \end{aligned}$$

The first integral approaches $\iint \partial_t \varphi u \, dx dt$ as $(\varepsilon \to 0)$ by dominated convergence. For the second integral, note that

$$t \mapsto \int_{\mathbb{R}} \varphi(x,t) u(x,t) \, dx$$

is continuous on $[0,\infty)$ because $u \in C_w([0,\infty); L^\infty(\mathbb{R}))$ (exercise). In particular, since $u(\cdot, 0) = g$,

$$\lim_{t \searrow 0} \int \varphi u \, dx = \int \varphi(0) g \, dx.$$

But then, as χ_ε' is a nonnegative function of unit integral supported on $(0,\varepsilon)$,

$$\lim_{\varepsilon \searrow 0} \int \chi_\varepsilon'(t) \left(\int_{\mathbb{R}} \varphi u \, dx \right) dt = \int_{\mathbb{R}} \varphi(0) g(x) \, dx.$$

Putting everything together, we obtain for $(\varepsilon \searrow 0)$

$$\iint \partial_t \varphi u + \partial_x \varphi u \, dx dt + \int \varphi(x,0) g(x) \, dx = 0$$

hence u takes the initial datum in the sense of Definition 2.3.

Conversely, suppose u is a weak solution with initial data g in the sense of Definition 2.3. Then the same arguments as above yield, since

$$\lim_{t \searrow 0} \int \varphi u \, dx = \int \varphi(0) u(x,0) \, dx,$$

the equality

$$\iint \partial_t\varphi u + \partial_x\varphi u \, dxdt + \int \varphi(x,0)u(x,0)\, dx = 0,$$

but by assumption also

$$\iint \partial_t\varphi u + \partial_x\varphi u \, dxdt + \int \varphi(x,0)g(x)\, dx = 0.$$

Since this is true for all ϕ, we obtain $u(\cdot,0) = g$ as desired. □

EXAMPLE 2.5. Consider Burgers' equation with datum

$$g(x) = \begin{cases} 1 & : x \leq 0 \\ 0 & : x > 0 \end{cases}.$$

We claim that

$$u(x,t) = \begin{cases} 1 & : x \leq \frac{t}{2} \\ 0 & : x > \frac{t}{2} \end{cases}$$

is a weak solution.
Proof: Let $\varphi \in C_c^\infty(\mathbb{R} \times [0,\infty))$. Write

$$\Omega_1 = \left\{(x,t) : x < \frac{t}{2}\right\}, \Omega_2 = \left\{(x,t) : x > \frac{t}{2}\right\}.$$

Then

$$\begin{aligned}
&\int_0^\infty \int_{-\infty}^\infty \partial_t\varphi u + \partial_x\varphi \frac{u^2}{2}\, dxdt \\
&= \iint_{\Omega_1} \partial_t\varphi + \partial_x\varphi\frac{1}{2}\, dxdt + \iint_{\Omega_2} (\partial_t\varphi + \partial_x\varphi)0\, dxdt \\
&= \int_{\partial\Omega_1} \varphi\nu_t \, dS(x,t) + \int_{\partial\Omega_1} \frac{1}{2}\varphi\nu_x \, dS(x,t) \\
&= -\int_{\{x<0,t=0\}} \varphi \, dS(x,t) + \int_{\{x=\frac{t}{2}\}} \varphi\nu_t\, dS(x,t) + \int_{\{x=\frac{t}{2}\}} \frac{1}{2}\varphi\nu_x \, dS(x,t) \\
&= -\int_{-\infty}^\infty \varphi(x,0)g(x)\, dx + 0,
\end{aligned}$$

since $\nu_t = -\frac{1}{2}\nu_x$ on $\{x = \frac{t}{2}\}$.

2.2.2. Rankine-Hugoniot condition.

Let us generalize the previous example.

Suppose $u : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ is a weak solution of $\partial_t u + \partial_x F(u) = 0$, $\Omega \subset \mathbb{R} \times (0,\infty)$ a bounded domain with smooth boundary and γ a smooth curve that splits Ω in two subdomains Ω_l and Ω_r. Suppose u is smooth on Ω_l and Ω_r, but possibly discontinuous along γ. Let ν be the unit normal in direction of Ω_r, and

$$u_l(x,t) := \lim_{(y,s)\to(x,t),(y,s)\in\Omega_l} u(y,s)$$

and similarly

$$u_r(x,t) := \lim_{(y,s)\to(x,t),(y,s)\in\Omega_r} u(y,s)$$

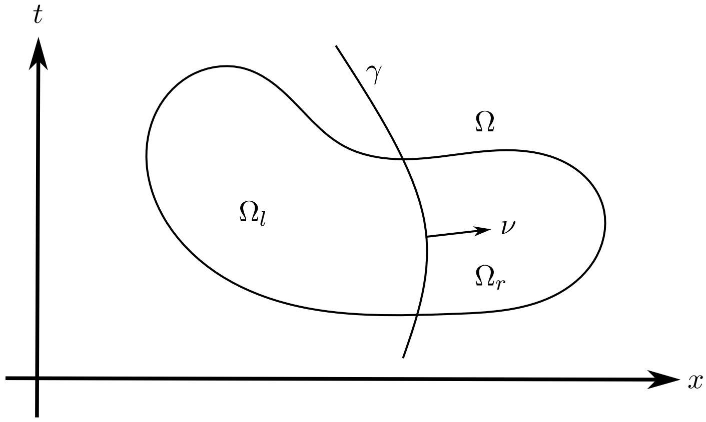


FIGURE 5

for $(x,t) \in \gamma$. Let $\varphi_l \in C_c^\infty(\Omega_l)$, then using Definition 2.3,

$$\iint_{\Omega_l} \partial_t \varphi_l u(y,s)\, dxdt + \iint_{\Omega_l} \partial_x \varphi_l F(u(x,t))\, dxdt = 0$$
$$\Leftrightarrow -\iint_{\Omega_l} \varphi_l (\partial_t u + \partial_x F(u))\, dxdt = 0.$$

Since this holds for every such φ_l, we have

$$\partial_t u + \partial_x F(u) = 0 \text{ in } \Omega_l$$

and analogously

$$\partial_t u + \partial_x F(u) = 0 \text{ in } \Omega_r.$$

Now let $\varphi \in C_c^\infty(\Omega)$. Since u is a weak solution,

$$\begin{aligned} 0 &= \iint_{\Omega} \partial_t \varphi u + \partial_x \varphi F(u)\, dxdt \\ &= \iint_{\Omega_l} \partial_t \varphi u + \partial_x \varphi F(u)\, dxdt + \iint_{\Omega_r} \partial_t \varphi u + \partial_x \varphi F(u)\, dxdt. \end{aligned}$$

But

$$\begin{aligned} &\iint_{\Omega_l} \partial_t \varphi u + \partial_x \varphi F(u)\, dxdt \\ &= -\iint_{\Omega_l} \varphi(\partial_t u + \partial_x F(u))\, dxdt + \int_\gamma \varphi(u_l \nu_t + F(u_l)\nu_x)\, dS \\ &= \int_\gamma \varphi(u_l \nu_t + F(u_l)\nu_x)\, dS \end{aligned}$$

and likewise

$$\iint_{\Omega_r} (\partial_t \varphi u + \partial_x F(u))\, dxdt = -\int_\gamma \varphi(u_r \nu_t + F(u_r)\nu_x)\, dS,$$

hence

$$\int_\gamma \varphi(u_l - u_r)\nu_t + \varphi(F(u_l) - F(u_r))\nu_x\, dS = 0.$$

As φ was arbitrary, we obtain

$$(u_l - u_r)\nu_t + (F(u_l) - F(u_r))\nu_x = 0$$

along γ. Suppose now γ is parametrized as

$$\gamma = \{(x,t) : x = s(t)\},$$

then

$$\nu = (\nu_x, \nu_t) = \frac{(1, -\dot{s})}{\sqrt{1+\dot{s}^2}}$$

so we get

$$F(u_l) - F(u_r) = \dot{s}(u_l - u_r) = \sigma(u_l - u_r) \quad (\sigma := \dot{s}). \tag{2.7}$$

Equation (2.7) is called the *Rankine-Hugoniot (jump) condition*.
We have thus shown that any weak solution of the type considered satisfies the Rankine-Hugoniot condition. The converse follows by reading the proof 'in reverse direction'. Therefore:

THEOREM 2.6. *If u is a smooth solution of $\partial_t u + \partial_x F(u) = 0$ on Ω_l and Ω_r but possibly discontinuous along γ, then it is a weak solution on Ω if and only if*

$$F(u_l) - F(u_r) = \dot{s}(u_l - u_r) \text{ along } \gamma,$$

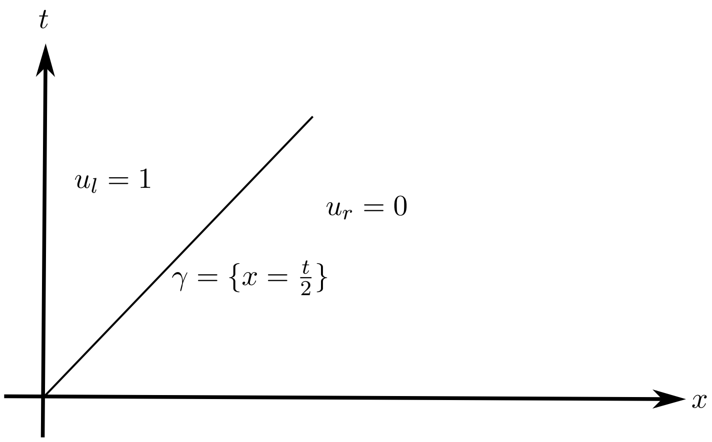
where $\gamma = \{(x,t) : x = s(t)\}$.

REMARK 2.7. Our choice of parametrization excludes curves with tangents parallel to the x-axis (Exercise).

EXAMPLE 2.8. Recall our solution to Burgers' equation:

$$\partial_t u + \partial_x \left(\frac{u^2}{2}\right) = 0 :$$

$$u(x,t) = \begin{cases} 1 & : x < \frac{t}{2} \\ 0 & : x > \frac{t}{2} \end{cases}$$



So $\dot{s}(t) \equiv \frac{1}{2}$. We have $F(u_l) - F(u_r) = \frac{1}{2}$ and $\dot{s}(u_l - u_r) = \frac{1}{2}$. So the Rankine-Hugoniot condition is indeed satisfied.

COROLLARY 2.9. *(of Theorem 2.6) If u is smooth on Ω_p and Ω_r and solves $\partial_t u + \partial_x F(u) = 0$ on Ω_p and Ω_r, and if u is continuous along γ, then u is a weak solution on Ω.*

EXAMPLE 2.10. Consider Burgers' equation with initial data

$$g(x) = \begin{cases} 0 & : x < 0 \\ 1 & : x > 0. \end{cases}$$

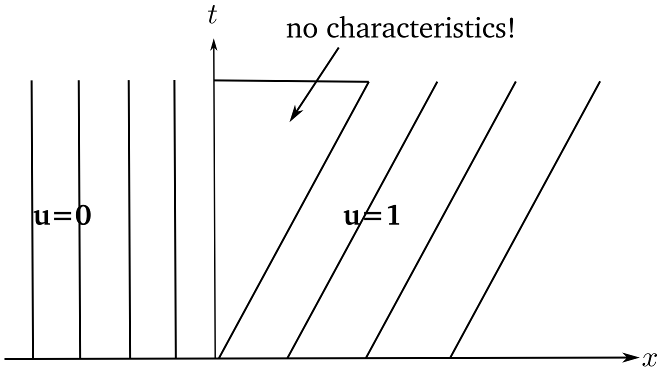


FIGURE 6. Non-determined area by characteristics

By the method of characteristics, if u is a solution, then $u = 0$ for $x < 0$ and $u = 1$ for $x > t$. What happens in between? Set

$$u_1(x,t) = \begin{cases} 0 & : x < \frac{t}{2} \\ 1 & : x > \frac{t}{2}. \end{cases}$$

Rankine-Hugoniot: $F(u_l) - F(u_r) = -\frac{1}{2}, \dot{s}(u_l - u_r) = -\frac{1}{2}$.
Hence u_1 is a weak solution, and it is easily seen that it is in $C_w([0,\infty); L^\infty(\mathbb{R}))$, so that indeed it is a weak solution of the initial value problem by Proposition 2.4. But consider also

$$u_2(x,t) = \begin{cases} 1 & : x > t \\ \frac{x}{t} & : 0 < x < t \\ 0 & : x < 0. \end{cases}$$

Note:

- u_2 is continuous in $\mathbb{R} \times (0, \infty)$;
- $u \equiv 0$ and $u \equiv 1$ are solutions on Ω_1, and Ω_3, respectively;
- $u(x,t) = \frac{x}{t}$ is a solution on Ω_2, since $\partial_t u + \partial_x \frac{u^2}{2} = -\frac{x}{t^2} + \frac{x}{t^2} = 0$.

Hence by Corollary 2.9, u_2 is also a weak solution of Burgers' equation with the same initial data (again cf. Proposition 2.4)! (u_2 is called a *rarefaction wave*.)
Disturbing discovery: Weak solutions are not necessarily unique! (given the same initial data).

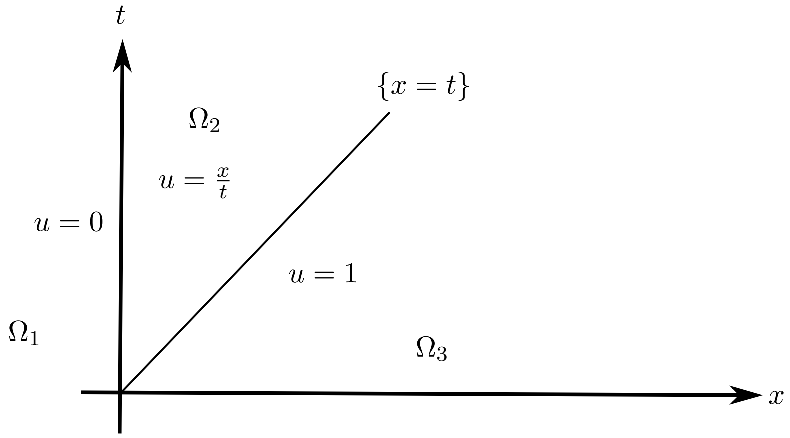


FIGURE 7. Alternative solution

2.3. Entropy Solutions

2.3.1. Definitions. Suppose u is a smooth solution of

$$\partial_t u + \partial_x F(u) = 0$$

and let $\eta : \mathbb{R} \to \mathbb{R}$ be convex and smooth. Multiply the PDE by $\eta'(u)$:

$$\eta'(u)\partial_t u + F'(u)\eta'(u)\partial_x u = 0.$$

Suppose $q : \mathbb{R} \to \mathbb{R}$ is such that $q'(z) = F'(z)\eta'(z)$ for all $z \in \mathbb{R}$. Then we get

$$0 = \eta'(u)\partial_t u + q'(u)\partial_x u = \partial_t \eta(u) + \partial_x q(u).$$

DEFINITION 2.11. Two functions $\eta, q : \mathbb{R} \to \mathbb{R}$ are called an *entropy/entropy-flux pair* for $\partial_t u + \partial_x F(u) = 0$ if

- η is convex
- $q'(z) = F'(z)\eta'(z)$ for all $z \in \mathbb{R}$.

In the scalar case, every smooth and convex function η is an entropy, because we can simply set

$$q(z) := \int_0^z F'(s)\eta'(s)\, ds.$$

We have seen: For smooth solutions $\partial_t \eta(u) + \partial_x q(u) = 0$ ("conservation of entropy"). What about discontinuous solutions?

EXAMPLE 2.12. Consider $\partial_t u + \partial_x \left(\frac{u^2}{2}\right) = 0, u(x,0) = g(x) = \begin{cases} 1 & : x < 0 \\ 0 & : x > 0. \end{cases}$

We have seen that $u(x,t) = \begin{cases} 1 & : x < \frac{t}{2} \\ 0 & : x > \frac{t}{2} \end{cases}$

is a weak solution. Set $\eta(z) = \frac{z^2}{2}$ and $q(z) = \int_0^z F'(s)\eta'(s)\, ds = \int_0^z s^2\, ds = \frac{z^3}{3}$.

Is it true that $\partial_t \eta(u) + \partial_x q(u) = 0$?

Problem: $u \notin C^1$ so we have to understand $\partial_t \eta(u) + \partial_x q(u)$ in a weak (distributional) sense.

DEFINITION 2.13. We say $\partial_t \eta(u) + \partial_x q'(u) \geq (\leq)\, 0$ in the sense of distributions if

$$\int_0^\infty \int_{\mathbb{R}} \partial_t \varphi \eta(u) + \partial_x \varphi q(u)\, dx dt \leq (\geq)\, 0$$

for all $\varphi \in C_c^\infty(\mathbb{R} \times (0,\infty)), \varphi \geq 0$.

So now let $\varphi \in C_c^\infty(\mathbb{R} \times (0,\infty))$ with $\varphi \geq 0$ and $\varphi > 0$ on a subset of $\{x = \frac{t}{2}\}$ of positive measure. Then,

$$\begin{aligned}
\int_0^\infty \int_{\mathbb{R}} \partial_t \varphi \frac{u^2}{2} + \partial_x \varphi \frac{u^3}{3}\, dx dt &= \iint_{\{x<\frac{t}{2}\}} \frac{1}{2}\partial_t \varphi + \frac{1}{3}\partial_x \varphi\, dx dt \\
&= \iint_{\{x=\frac{t}{2}\}} \frac{1}{2}\varphi \nu_t + \frac{1}{3}\varphi \nu_x\, dS \\
&= \iint_{\{x=\frac{t}{2}\}} -\frac{1}{4}\varphi \nu_x + \frac{1}{3}\varphi \nu_x\, dS \\
&= \iint_{\{x=\frac{t}{2}\}} \frac{1}{12}\nu_x \varphi\, dS \\
&> 0,
\end{aligned}$$

hence $\partial_t \eta(u) + \partial_x q(u) < 0$ ("entropy dissipation along the shock").

EXAMPLE 2.14. Recall the weak solution

$$u(x,t) = \begin{cases} 0 & : x < \frac{t}{2} \\ 1 & : x > \frac{t}{2} \end{cases}$$

for Burgers' equation with $g(x) = \begin{cases} 0 & : x < 0 \\ 1 & : x > 0. \end{cases}$ A computation similar to the previous example gives

$$\partial_t \frac{u^2}{2} + \partial_x \frac{u^3}{3} > 0 \text{ (weakly)}.$$

This phenomen can be described as 'entropy production along the shock'. But we also had the rarefaction wave solution

$$u(x,t) = \begin{cases} 0 & : x < 0 \\ \frac{x}{t} & : 0 < x < t \\ 1 & : x > t. \end{cases}$$

Since it is continuous and piecewise smooth, we have

$$\partial_t \eta(u) + \partial_x q(u) = 0 \text{ (exercise)}.$$

Sometimes the mathematical entropy is minus the physical entropy. The 2nd law of thermodynamics hence motivates:

DEFINITION 2.15. Let $g \in L^\infty(\mathbb{R})$. A function $u \in L^\infty(\mathbb{R} \times (0,\infty))$ is an *entropy solution* of

$$\begin{aligned} \partial_t u + \partial_x F(u) &= 0 \\ u(x,0) &= g(x) \end{aligned}$$

if

$$u - g \in C([0,\infty); L^1(\mathbb{R})), u(\cdot,t) \to g \text{ in } L^1$$

and

$$\int_0^\infty \int_{\mathbb{R}} \partial_t\varphi\eta(u) + \partial_x\varphi q(u)\, dxdt \geq 0$$

for all entropy / entropy-flux pairs η, q and all $\varphi \in C_c^\infty(\mathbb{R}\times(0,\infty)), \varphi \geq 0$.

Here, for reasons that will become apparent in the proof of the next theorem, we require $u-g$ to be in the function space $C([0,\infty); L^1(\mathbb{R}))$, which is defined as the space of functions $v : \mathbb{R}\times[0,\infty)$ such that $v(\cdot,t) \in L^1(\mathbb{R})$ for every $t \geq 0$, and such that

$$\lim_{s\to t}\int_{\mathbb{R}} |v(\cdot,t) - v(\cdot,s)|\, dx = 0$$

for all $t \geq 0$. In other words, the map $t \mapsto v(\cdot,t)$ is continuous from $[0,\infty)$ to $L^1(\mathbb{R})$ is continuous.

REMARK 2.16. Every entropy solution is a weak solution: Set $\eta = id$ and $q = F$. Since $u \in L^\infty(\mathbb{R}\times[0,\infty))$, $g \in L^\infty(\mathbb{R})$, and $u - g \in C([0,\infty); L^1(\mathbb{R}))$ imply $u \in C_w([0,\infty); L^\infty)$ (exercise), by Proposition 2.4 every entropy solution is even a weak solution with initial data g in the sense of Definition 2.3.

But not every weak solution is an entropy solution (cf. Example 2.14 with the 'non-physical shock').

2.3.2. Uniqueness of entropy solutions.

THEOREM 2.17. *(Kružkov 1970 [4])*
There exists at most one entropy solution of

$$\partial_t u + \partial_x F(u) = 0\, (F \in C^1)$$
$$u(x,0) = g(x)\, (g \in L^\infty).$$

Some preliminaries before we start the proof:
Dominated Convergence Theorem: Let $\Omega \subset \mathbb{R}^n$ be a domain. If $f_n \to f$ pointwise a.e. on Ω and if there exists $h \in L^1(\Omega)$ such that $|f_n(x)| \leq h(x)$ for every $n \in \mathbb{N}$ and almost every $x \in \Omega$, then $f \in L^1(\Omega)$ and

$$\lim_{n\to\infty}\int_\Omega f_n(x)\, dx = \int_\Omega f(x)\, dx. \tag{2.8}$$

PROPOSITION 2.18. *If $f \in L^1(\mathbb{R}^n)$ then*

$$\lim_{y\to 0} f(x-y) = f(x) \text{ in } L^1_{loc},$$

i.e.

$$\lim_{y\to 0}\int_K |f(x) - f(x-y)|\, dx = 0 \tag{2.9}$$

for every compact subset $K \subset \mathbb{R}^n$.

PROOF. Exercise. □

Mollification: Let $\eta \in C_c^\infty(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n}\eta\, dx = 1, \eta \geq 0$. Then η is called a standard mollifier. Set $\eta_\varepsilon(x) := \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}\right)$. We have:

PROPOSITION 2.19. *Let $f \in L^1_{loc}(\mathbb{R}^n)$ (i.e. $f \in L^1(K)$ for every compact subset $K \subset \mathbb{R}^n$) be continuous at $x_0 \in \mathbb{R}^n$. Then*

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} f(x + x_0)\eta_\varepsilon(x)\,dx = f(x_0)$$

PROOF. Let $\kappa > 0$ and choose $\varepsilon > 0$ so small that $|f(x+x_0) - f(x_0)| < \kappa$ on $\operatorname{supp} \eta_\varepsilon$. Then

$$\int_{\mathbb{R}^n} |f(x + x_0) - f(x_0)|\eta_\varepsilon(x)\,dx \le \kappa \int_{\mathbb{R}^n} \eta_\varepsilon(x)\,dx = \kappa$$

and the claim follows. □

PROOF. (Kružkov's theorem)
Step 1: Consider the entropies given by $\eta_k(z) = \beta_k(z - \alpha)$, where

- $\alpha \in \mathbb{R}$ is fixed
- $\beta_k : \mathbb{R} \to \mathbb{R}$ is smooth and convex and

$$\beta_k(z) \to |z| \text{ uniformly}$$
$$\beta_k'(z) \to \operatorname{sgn}(z) \text{ a.e.}$$
$$|\beta_k'| \text{ uniformly bounded.}$$

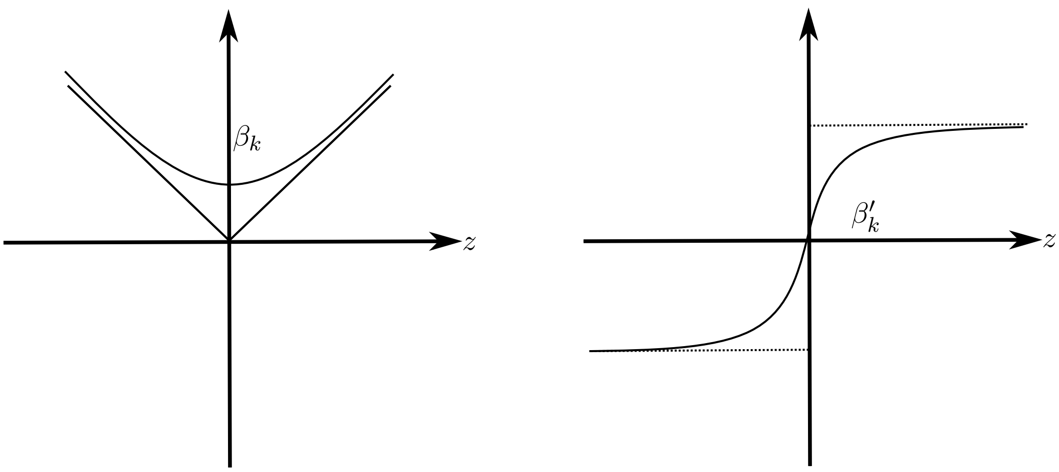


FIGURE 8

The entropy flux function is then given by

$$q_k(z) = \int_\alpha^z \beta_k'(\zeta - \alpha)F'(\zeta)\,d\zeta.$$

For fixed $z \in \mathbb{R}$, we have

$$q_k(z) \to \operatorname{sgn}(z - \alpha)(F(z) - F(\alpha))\,(k \to \infty).$$

Indeed this follows from dominated convergence, since

$$\beta_k'(\zeta - \alpha) \to \operatorname{sgn}(z - \alpha) \text{ a.e. } \zeta \in (\alpha, z)$$

and

$$|\beta_k'(\zeta - \alpha)F'(\zeta)| \le C|F'(\zeta)| \in L^1(\alpha, z).$$

Therefore

$$\begin{aligned}\lim_{k\to\infty} q_k(z) &= \lim_{k\to\infty}\int_\alpha^z \beta_k'(\zeta-\alpha)F'(\zeta)\,d\zeta\\ &= \int_\alpha^z \underbrace{\operatorname{sgn}(\zeta-\alpha)}_{\operatorname{sgn}(z-\alpha)} F'(\zeta)\,d\zeta\\ &= \operatorname{sgn}(z-\alpha)(F(z)-F(\alpha)).\end{aligned}$$

Let $\varphi \in C_c^\infty(\mathbb{R}\times(0,\infty)), \varphi \geq 0$. If u is an entropy solution, then for every k,

$$\iint \eta_k(u)\partial_t\varphi + q_k(u)\partial_x\varphi\,dxdt \geq 0.$$

Since $\eta_k(u)\to|\eta-\alpha|$ uniformly,

$$\iint \partial_t\varphi\eta_k(u)\,dxdt \to \iint \partial_t\varphi|u-\alpha|\,dxdt.$$

On the other hand,

$$q_k(u)\to \operatorname{sgn}(u-\alpha)(F(u)-F(\alpha))$$

for almost every x, t and

$$|q_k(u(x,t))| \leq C\int_\alpha^{u(x,t)} |F'(\zeta)|\,d\zeta \in L^\infty(\mathbb{R}\times(0,\infty))$$

as $u\in L^\infty(\mathbb{R}\times(0,\infty))$, hence by dominated convergence (take $C\int_\alpha^{\|u\|_{L^\infty}}|F'(\zeta)|\,d\zeta|\partial_x\varphi|$ as a dominating function)

$$\iint q_k(u)\partial_x\varphi\,dxdt \to \iint \operatorname{sgn}(u-\alpha)(F(u)-F(\alpha))\partial_x\varphi\,dxdt. \tag{2.10}$$

This shows

$$\int_0^\infty\int_{\mathbb{R}} |u-\alpha|\partial_t\varphi + \operatorname{sgn}(u-\alpha)(F(u)-F(\alpha))\partial_x\varphi\,dxdt \geq 0 \tag{2.11}$$

for all $\alpha\in\mathbb{R}, \varphi\in C_c^\infty(\mathbb{R}\times(0,\infty)), \varphi\geq 0$.
Step 2: Suppose $\tilde{u}$ is another entropy solution. As above,

$$\int_0^\infty\int_{\mathbb{R}} |\tilde{u}-\tilde{\alpha}|\partial_t\tilde{\varphi} + \operatorname{sgn}(\tilde{u}-\tilde{\alpha})(F(\tilde{u})-F(\tilde{\alpha}))\partial_y\tilde{\varphi}\,dyds \geq 0$$

for all $\tilde{\alpha}\in\mathbb{R}, \tilde{\varphi}\in C_c^\infty(\mathbb{R}\times(0,\infty)), \tilde{\varphi}\geq 0$.
Let now $\psi \in C_c^\infty(\underbrace{\mathbb{R}}_x\times\underbrace{\mathbb{R}}_y\times\underbrace{(0,\infty)}_t\times\underbrace{(0,\infty)}_s), \psi\geq 0$. Fix $(y,s)\in\mathbb{R}\times(0,\infty)$ and set in (2.11) $\alpha=\tilde{u}(y,s), \varphi(x,t)=\psi(x,y,t,s)$. Integrate this in y, s to get

$$\begin{aligned}\iiiint |u(x,t)-\tilde{u}(y,s)|\partial_t\psi + \operatorname{sgn}(u(x,t)-\tilde{u}(y,s))&\\ (F(u(x,t))-F(\tilde{u}(y,s)))\partial_x\psi\,dxdydsdt &\geq 0.\end{aligned}$$

Similarly, for fixed $(x,t) \in \mathbb{R} \times (0,\infty)$, setting in (2.3.2) $\tilde{\alpha} = u(x,t)$ and $\varphi(y,s) = \psi(x,y,t,s)$ yields

$$\iiiint |u(x,t) - \tilde{u}(y,s)|\partial_s\psi + \operatorname{sgn}(\tilde{u}(y,s) - u(x,t)) \\ (F(\tilde{u}(y,s)) - F(u(x,t)))\partial_y\psi \, dxdydsdt \geq 0.$$

Add the last two inequalities:

$$\iiiint |u(x,t) - \tilde{u}(y,s)|(\partial_t\psi + \partial_s\psi) \\ + \operatorname{sgn}(u(x,t) - \tilde{u}(y,s))(F(u(x,t)) - F(\tilde{u}(y,s)))(\partial_x\psi + \partial_y\psi) \geq 0. \tag{2.12}$$

Step 3: Choose ψ wisely. Let η be a standard mollifier on $\mathbb{R}$ ($\eta \in C_c^\infty(\mathbb{R}), \eta \geq 0, \int \eta = 1$) and set

$$\eta_\varepsilon(x) := \frac{1}{\varepsilon}\eta\left(\frac{x}{\varepsilon}\right). \tag{2.13}$$

Set $\psi(x,y,t,s) = \eta_\varepsilon\left(\frac{x-y}{2}\right)\eta_\varepsilon\left(\frac{t-s}{2}\right)\gamma\left(\frac{x+y}{2},\frac{t+s}{2}\right)$ for a $\gamma \in C_c^\infty(\mathbb{R}\times(0,\infty)), \gamma \geq 0$. Plug this into (2.12):

$$\iiiint \Bigg[|u(x,t) - \tilde{u}(y,s)|\partial_t\gamma\left(\frac{x+y}{2},\frac{t+s}{2}\right) \\ + \operatorname{sgn}(u(x,t) - \tilde{u}(y,s))\Bigg(F(u(x,t)) - F(\tilde{u}(y,s)\Bigg)\partial_x\gamma\left(\frac{x+y}{2},\frac{t+s}{2}\right)\Bigg] \\ \eta_\varepsilon\left(\frac{x+y}{2}\right)\eta_\varepsilon\left(\frac{t-s}{2}\right) dxdydtds \geq 0. \tag{2.14}$$

Change of variables:

$$\overline{x} = \frac{x+y}{2}, \overline{t} = \frac{t+s}{2}, \overline{y} = \frac{x-y}{2}, \overline{s} = \frac{t-s}{2}. \tag{2.15}$$

Write (2.14) as

$$\iint h(\overline{y},\overline{s})\eta_\varepsilon(\overline{y})\eta_\varepsilon(\overline{s}) \, d\overline{y}d\overline{s} \geq 0, \tag{2.16}$$

where

$$h(\overline{y},\overline{s}) = \iint |u(\overline{x}+\overline{y},\overline{t}+\overline{s}) - \tilde{u}(\overline{x}-\overline{y},\overline{t}-\overline{s})|\partial_t\gamma(\overline{x},\overline{t}) \\ + \operatorname{sgn}(u(\overline{x}+\overline{y};\overline{t}+\overline{s}) - \tilde{u}(\overline{x}-\overline{y},\overline{t}-\overline{s})) \\ \cdot (F(u(\overline{x}+\overline{y},\overline{t}+\overline{s})) - F(\tilde{u}(\overline{x}-\overline{y},\overline{t}-\overline{s})))\partial_x\gamma(\overline{x},\overline{t}) \, dxdt.$$

Next, by Proposition 2.18,

$$u(\overline{x}+\overline{y},\overline{t}+\overline{s}) \to u(\overline{x},\overline{t})$$

in $L^1_{loc}(\mathbb{R}\times(0,\infty))$ as $\overline{y},\overline{s} \to 0$ and

$$\tilde{u}(\overline{x}-\overline{y},\overline{t}-\overline{s}) \to \tilde{u}(\overline{x},\overline{t});$$

therefore this also holds for a Lipschitz function of $u(\overline{x}+\overline{y},\overline{t}+\overline{s})$ and $\tilde{u}(\overline{x}-\overline{y},\overline{t}-\overline{s})$:

Indeed, if $|\Phi(u,\tilde{u})-\Phi(v,\tilde{v})| \leq L(|u-v|+|\tilde{u}-\tilde{v}|)$ and $u_n \to u_0, \tilde{u_n} \to \tilde{u}_0$ in L^1_{loc}, then

$$\int_K |\Phi(u_n,\tilde{u}_n)-\Phi(u_0,\tilde{u}_0)|\,dxdt \leq L\int_K |u_n-u_0|+|\tilde{u}_n-\tilde{u}_0|\,dxdt \to 0.$$

Pick $\Phi(u,\tilde{u}) = |u-\tilde{u}|$ to get

$$|u(\overline{x}+\overline{y},\overline{t}+\overline{s})-\tilde{u}(\overline{x}-\overline{y},\overline{t}-\overline{s})| \to |u(\overline{x},\overline{t})-\tilde{u}(\overline{x},\overline{t})|$$

in L^1_{loc} as $\overline{y},\overline{s}\to 0$ and set $\Phi(u,v)=\operatorname{sgn}(u-v)(F(u)-F(v))$ to show

$$\operatorname{sgn}(u(\overline{x}+\overline{y},\overline{t}+\overline{s})-\tilde{u}(\overline{x}-\overline{y},\overline{t}-\overline{s}))(F(u(\overline{x}+\overline{y},\overline{t}+\overline{s})-F(\tilde{u}(\overline{x}-\overline{y},\overline{t}-\overline{s})))$$
$$\to \operatorname{sgn}(u(\overline{x},\overline{t})-\tilde{u}(\overline{x},\overline{t}))(F(u(\overline{x},\overline{t})-F(\tilde{u}(\overline{x},\overline{t})))$$

in L^1_{loc} as $\overline{y},\overline{s}\to 0$. This shows

$$\lim_{\overline{y},\overline{s}\to 0} h(\overline{y},\overline{s}) = \iint |u(\overline{x},\overline{t})-\tilde{u}(\overline{x},\overline{t})|\partial_t\gamma + \operatorname{sgn}(u(\overline{x},\overline{t})-\tilde{u}(\overline{x},\overline{t}))(F(u(\overline{x},\overline{t}))-F(\tilde{u}(\overline{x},\overline{t})))\partial_x\gamma\,d\overline{x}d\overline{t}$$

and in particular h is continuous at $(0,0)$. Hence (2.16) and Proposition 2.19 give (write now $x=\overline{x}, t=\overline{t}$)

$$\iint \underbrace{|u(x,t)-\tilde{u}(x,t)|}_{=:a(x,t)}\partial_t\gamma + \underbrace{\operatorname{sgn}(u(x,t)-\tilde{u}(x,t))(F(u(x,t))-F(\tilde{u}(x,t)))}_{=:b(x,t)}\partial_x\gamma\,dxdt \geq 0. \tag{2.17}$$

Step 4: Make again suitable choice of $\gamma(x,t)$: Let $0<t_1<t_2, r>0$, and

$$\gamma(x,t) = A(x)B(t),$$

where

- $A \in C_c^\infty(\mathbb{R},\mathbb{R})$,
- $A(x) = \begin{cases} 1 & : |x| \leq r \\ 0 & : |x| \geq r+1 \end{cases}$
- $|A'(x)| \leq 2$

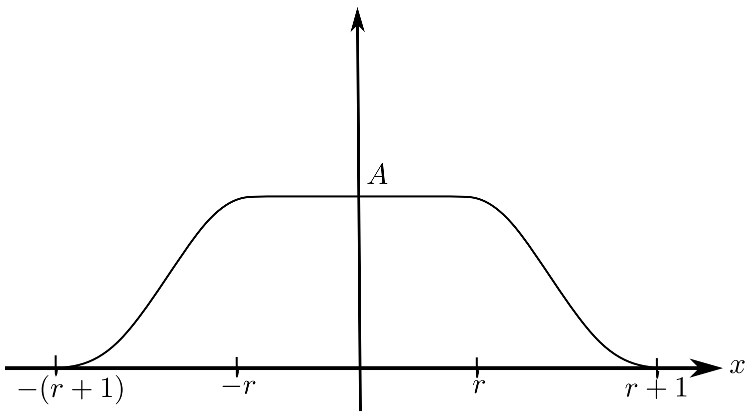


FIGURE 9

- $B \in C^{\infty}(\mathbb{R}, \mathbb{R})$
- $B(t) = \begin{cases} 0 & : 0 \le t \le t_1 \text{ or } t \ge t_2 + \delta \, (\delta < t_2 - t_1) \\ 1 & : t_1 + \delta \le t \le t_2 \end{cases}$
- $B(t)$ monotone on $(t_1, t_1 + \delta)$ and $(t_2, t_2 + \delta)$.

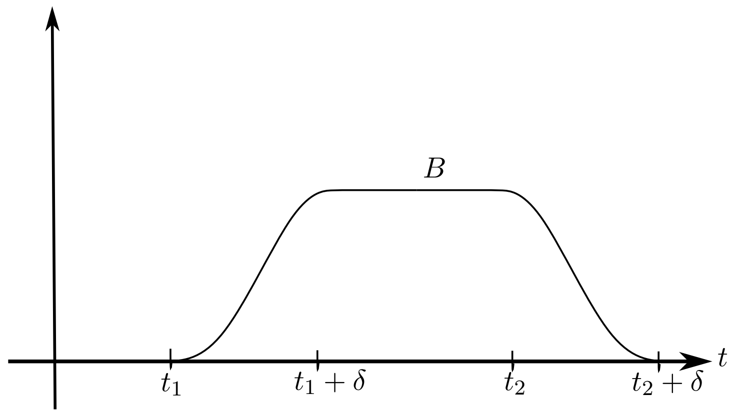


FIGURE 10

Then (2.17) becomes

$$\int_{t_1}^{t_1+\delta} \int a(x,t)A(x)B'(t)\,dxdt + \int_{t_2}^{t_2+\delta} a(x,t)A(x)B'(t)\,ddxdt$$
$$\ge -\int_{t_1}^{t_2+\delta} \int_{\{r \le |x| \le r+1\}} b(x,t)A'(x)B(t)\,dxdt.$$

As $(r \to \infty)$, the integral on the right hand side tends to 0: Indeed, since F is smooth (and in particular Lipschitz), we have

$$\begin{aligned} \sup_{t \in (t_1, t_2+\delta)} \int |b(x,t)|\,dx &\le \sup_t \int |F(u(x,t)) - F(\tilde{u}(x,t)))|\,dx \\ &\le L \sup_t \int |u(x,t) - g(x) - (\tilde{u}(x,t) - g(x))|\,dx \\ &\le L(\|u - g\|_{C_t L^1_x} + \|\tilde{u} - g\|_{C_t L^1_x}) \\ &< \infty. \end{aligned}$$

It follows

$$\int_{r \le |x| \le r+1} b(x,t)A'(x)B(t)\,dx \to 0 \, (r \to \infty)$$

for every t and

$$\left| \int_{r \le |x| \le r+1} b(x,t)A'(x)B(t)\,dx \right|$$

is bounded uniformly in t so that the time integral approaches 0 by the dominated convergence theorem.
Similarly, $(r \to \infty)$ on the left hand side yields

$$\int_{t_1}^{t_1+\delta} \int a(x,t)B'(t)\,dxdt + \int_{t_2}^{t_2+\delta} \int a(x,t)B'(t)\,dxdt \ge 0.$$

Now observe that $B'(t)$ is a standard mollification kernel on $(t_1, t_1+\delta)$ and likewise a negative standard mollification kernel on $(t_2, t_2+\delta)$, so that by Proposition 2.18, as $(\delta \to 0)$

$$\begin{aligned}&\int_{t_1}^{t_1+\delta}\int a(x,t)B'(t)\,dxdt + \int_{t_2}^{t_2+\delta}\int a(x,t)B'(t)\,dxdt\\ &\to \int a(x,t_1)\,dx - \int a(x,t_2)\,dx \geq 0.\end{aligned}$$

hence we have shown the L^1 *contraction property*

$$\int |u(x,t_2) - \tilde{u}(x,t_2)|\,dx \leq \int |u(x,t_1) - \tilde{u}(x,t_1)|\,dx$$

for $0 < t_1 \leq t_2$. Finally, if $u(\cdot,t) \to g, \tilde{u}(\cdot,t) \to g$ in L^1 as $(t \searrow 0)$, then it follows that

$$\int |u(x,t) - \tilde{u}(x,t)|\,dx = 0$$

for all $t > 0$, hence $u = \tilde{u}$ a.e. □

2.3.3. Riemann's Problem.

PROPOSITION 2.20. *Suppose that γ is a smooth curve splitting $\mathbb{R}\times(0,\infty)$ in two domains Ω_1, Ω_2 and $u \in L^\infty(\mathbb{R}\times(0,\infty))$ is a smooth solution of*

$$\partial_t u + \partial_x F(u) = 0 \text{ on } \Omega_1 \text{ and } \Omega_2$$

respectively. If $u \in C(\mathbb{R}\times(0,\infty))$, and if

$$u - g \in C([0,\infty); L^1(\mathbb{R}))$$

for some $g \in L^\infty(\mathbb{R})$, then u is an entropy solution of

$$\partial_t u + \partial_x F(u) = 0, u(x,0) = g(x).$$

PROOF. Let η, q be an entropy / entropy-flux pair and let $\varphi \in C_c^\infty(\mathbb{R}\times(0,\infty)), \varphi \geq 0$. Then

$$\begin{aligned}&\int_{\mathbb{R}\times(0,\infty)} \partial_t\varphi\eta(u) + \partial_x\varphi q(u)\,dxdt\\ &= \int_{\Omega_1} \partial_t\varphi\eta(u) + \partial_x\varphi q(u)\,dxdt + \int_{\Omega_2} \partial_t\varphi\eta(u) + \partial_x\varphi q(u)\,dxdt\\ &= -\int_{\Omega_1}\varphi\underbrace{(\partial_t\eta(u) + \partial_x q(u))}_{=0}\,dxdt - \int_{\Omega_2}\varphi\underbrace{(\partial_t\eta(u) + \partial_x q(u))}_{=0}\,dxdt\\ &\quad + \int_{\partial\Omega_1}\varphi\eta(u_1)\nu_t + \varphi q(u_1)\nu_x\,dS - \int_{\partial\Omega_2}\varphi\eta(u_2)\nu_t + \varphi q(u_2)\nu_x\,dS\\ &= 0,\end{aligned}$$

since $u_1 = u_2$ on γ by continuity (recall $u_1(x,t) = \lim_{y,s\in\Omega_1,(y,s)\to(x,t)}$ and similar for u_2). Note that on Ω_1 and Ω_2,

$$\partial_t\eta(u) + \partial_x q(u) = 0$$

since u is a smooth solution (cf. the computation in subsection 2.3.1). □

Consider

$$\partial_t u + \partial_x F(u) = 0$$
$$u(x,0) = g(x)$$

with special initial data

$$g(x) = \begin{cases} u_1 & : x < 0 \\ u_2 & : x > 0. \end{cases}$$

This is called *Riemann's problem*.

THEOREM 2.21. *Let F be smooth and strictly convex. The unique (entropy) solution of Riemann's problem is given by*

a) $u(x,t) = \begin{cases} u_1 & : \frac{x}{t} < \sigma \\ u_2 & : \frac{x}{t} > \sigma \end{cases}$

if $u_1 > u_2$, where $\sigma := \frac{F(u_1)-F(u_2)}{u_1-u_2}$.

b) $u(x,t) = \begin{cases} u_1 & : \frac{x}{t} < F'(u_1) \\ G\left(\frac{x}{t}\right) & : F'(u_1) < \frac{x}{t} < F'(u_2) \\ u_2 & : \frac{x}{t} > F'(u_2), \end{cases}$ *where $G = (F')^{-1}$ (well-defined by convexity of F!).*

PROOF. a) Let η, q be an entropy / entropy-flux pair and $\varphi \in C_c^\infty(\mathbb{R} \times (0,\infty)), \varphi \geq 0$. Then,

$$\begin{aligned}
&\iint_{\mathbb{R}\times(0,\infty)} \partial_t\varphi\eta(u) + \partial_x\varphi q(u)\,dxdt \\
&= \iint_{\{\frac{x}{t}<\sigma\}} \partial_t\varphi\eta(u_1) + \partial_x\varphi q(u_1)\,dxdt + \iint_{\{\frac{x}{t}>\sigma\}} \partial_t\varphi\eta(u_2) + \partial_x\varphi q(u_2)\,dxdt \\
&= -\iint_{\{\frac{x}{t}<\sigma\}} \varphi\underbrace{(\partial_t\eta(u_1) + \partial_x q(u_1))}_{=0}\,dxdt - \iint_{\{\frac{x}{t}>\sigma\}} \varphi\underbrace{(\partial_t\eta(u_2) + \partial_x q(u_2))}_{=0}\,dxdt \\
&\quad + \int_{\{\frac{x}{t}=\sigma\}} \eta(u_1)\varphi\nu_t + q(u_1)\varphi\nu_x - \eta(u_2)\varphi\nu_t - q(u_2)\varphi\nu_x\,dS \\
&= C\int_{\{\frac{x}{t}=\sigma\}} \nu_x\varphi[-\sigma(\eta(u_1)-\eta(u_2)) + q(u_1) - q(u_2)]\,dS.
\end{aligned}$$

Since $\nu_x\varphi \geq 0$, it suffices to show (assume $\sigma > 0$)

$$\begin{aligned}
&-\sigma(\eta(u_1)-\eta(u_2)) + (q(u_1)-q(u_2)) \geq 0 \text{ if } u_1 > u_2 \\
\Leftrightarrow\ &(F(u_1)-F(u_2))(\eta(u_1)-\eta(u_2)) \leq (q(u_1)-q(u_2))(u_1-u_2) \\
\Leftrightarrow\ &\int_{u_2}^{u_1} F'(z)\,dz \int_{u_2}^{u_1} \eta'(z)\,dz \leq \int_{u_2}^{u_1} q'(z)\,dz(u_1-u_2) \\
\Leftrightarrow\ &\int_{u_2}^{u_1} F'(z)\,dz \int_{u_2}^{u_1} \eta'(z)\,dz \leq (u_1-u_2)\int_{u_2}^{u_1} F'(z)\eta'(z)\,dz
\end{aligned}$$

and this is true (exercise). The case $\sigma < 0$ is similar. Note also that $u - g \in C([0,\infty); L^1(\mathbb{R}))$, as can be seen by direct computation.

b) We use Proposition 2.20: Clearly $u - g \in C_t L^1_x$, and u is continuous on $\mathbb{R} \times (0, \infty)$. Hence u is the entropy solution if it is a solution on $\{F'(u_1) \leq \frac{x}{t} \leq F'(u_2)\}$. Check this: Recall $G = (F')^{-1}$.

$$\begin{aligned}\partial_t G\left(\frac{x}{t}\right) + \partial_x F\left(G\left(\frac{x}{t}\right)\right) &= G'\left(\frac{x}{t}\right)\left(-\frac{x}{t^2}\right) + F'\left(G\left(\frac{x}{t}\right)\right) G'\left(\frac{x}{t}\right)\frac{1}{t} \\ &= G'\left(\frac{x}{t}\right)\left[-\frac{x}{t^2} + \frac{x}{t}\frac{1}{t}\right] \\ &= 0.\end{aligned}$$

□

2.4. Compensated Compactness

2.4.1. Viscosity Approximation. Let again F be smooth and $g \in L^\infty(\mathbb{R})$. *Goal:* Show existence of an entropy solution of

$$\begin{aligned}\partial_t u + \partial_x F(u) = 0 \text{ on } \mathbb{R} \times (0, \infty) \\ u(x, 0) = g(x).\end{aligned}$$

Idea: Approximation by a better equation! Consider for $\varepsilon > 0$ the parabolic PDE

$$\begin{aligned}\partial_t u_\varepsilon + \partial_x F(u_\varepsilon) &= \varepsilon \partial_{xx} u_\varepsilon \\ u_\varepsilon(x, 0) &= g(x),\end{aligned} \tag{2.18}$$

for which existence of smooth solutions can be shown more easily (cf. heat equation ...) Let us assume for the moment that u_ε is smooth,

$$\sup_{\varepsilon > 0} \|u_\varepsilon\|_{L^\infty(\mathbb{R} \times (0,\infty))} < \infty \text{ and } u_\varepsilon - g \to u - g \text{ in } C_t L^1_x$$

for some $u \in L^\infty(\mathbb{R} \times (0, \infty))$ (this is in practice very difficult to show). We claim that then u is the entropy solution. To see this, let η, q be an entropy/entropy-flux pair. Multiply (2.18) by $\eta'(u_\varepsilon)$ to obtain

$$\begin{aligned}0 &= \eta'(u_\varepsilon)\partial_t u_\varepsilon + \eta'(u_\varepsilon)F'(u_\varepsilon)\partial_x u_\varepsilon - \varepsilon \partial_{xx} u_\varepsilon \eta'(u_\varepsilon) \\ &= \partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon) - \varepsilon \partial_{xx} \eta(u_\varepsilon) + \varepsilon \underbrace{\eta''(u_\varepsilon)(\partial_x u_\varepsilon)^2}_{\geq 0 \, (\eta \text{ convex})}\end{aligned}$$

so that $\partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon) \leq \varepsilon \partial_{xx} \eta(u_\varepsilon)$. Let now $\varphi \in C^\infty_c(\mathbb{R} \times (0, \infty)), \varphi \geq 0$. Then

$$\iint \varphi \partial_t \eta(u_\varepsilon) + \varphi \partial_x q(u_\varepsilon) \, dxdt \leq \varepsilon \iint \partial_{xx} \eta(u_\varepsilon) \varphi \, dxdt$$

so

$$\iint \partial_t \varphi \eta(u_\varepsilon) + \partial_x \varphi q(u_\varepsilon) \, dxdt \geq -\varepsilon \iint \partial_{xx} \varphi \eta(u_\varepsilon) \, dxdt.$$

Since, on the support of φ, $u_\varepsilon \to u$ in $C_t L^1_x$, we conclude

$$\iint \partial_t \varphi \eta(u_\varepsilon) + \partial_x \varphi q(u_\varepsilon) \, dxdt \to \iint \partial_t \varphi \eta(u) + \partial_x \varphi \eta(u) \, dxdt$$

and

$$\left| \varepsilon \iint \partial_{xx} \varphi \eta(u_\varepsilon) \, dxdt \right| \leq \varepsilon \sup_{|z| \leq \|u_\varepsilon\|_{L^\infty}} |\eta(z)| \int |\partial_{xx} \varphi| \, dx \to 0$$

hence

$$\iint \partial_t\varphi\eta(u) + \partial_x\varphi q(u)\,dxdt \geq 0$$

so u is the entropy solution. This gives another motivation for our definition of entropy solution.
Problem: "Strong" convergence like $u_\varepsilon \to u$ in $C_tL^1_x$ is very hard to prove! If we only have an L^∞-bound $\|u_\varepsilon\|_{L^\infty} \leq C$, the best we can hope for is *weak convergence*.

2.4.2. Weak convergence. Let $(X, \|\cdot\|_X)$ be a normed space and X' its dual space, i.e. the space of bounded linear functionals on X. For $x' \in X'$, write $\langle x, x'\rangle$ instead of $x'(x)$. The dual space is itself a normed (even Banach!) space with norm given by

$$\|x'\|_{X'} := \sup\{|\langle x, x'\rangle| : \|x\|_X \leq 1\}.$$

DEFINITION 2.22. a) We say $x_n \rightharpoonup x$ weakly in X if

$$\lim_{n\to\infty}\langle x_n, x'\rangle = \langle x, x'\rangle$$

for all $x' \in X'$.
b) We say $x'_n \overset{*}{\rightharpoonup} x'$ weakly-* in X' if

$$\lim_{n\to\infty}\langle x, x'_n\rangle = \langle x, x'\rangle$$

for every $x \in X$.

Weakly convergent sequences are always bounded.

THEOREM 2.23. *(Banach-Alaoglu) Let X be a separable normed space and $(x'_n) \subset X'$ a bounded sequence, i.e.*

$$\sup_{n\in\mathbb{N}}\ \sup_{x\in X, \|x\|_X\leq 1} |\langle x, x'_n\rangle| < \infty,$$

then there is a subsequence $(x'_{n_k})_{k\in\mathbb{N}}$ that converges weakly-.*

EXAMPLE 2.24.
- Let $1 < p < \infty$, then $L^p(\Omega)$ is a separable normed space with norm

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p\,dx\right)^{\frac{1}{p}}, \tag{2.19}$$

  and its dual is $L^q(\Omega)$ for $\frac{1}{p} + \frac{1}{q} = 1$. The duality pairing is

$$\langle f, g\rangle = \int_\Omega f(x)g(x)\,dx.$$

  Hence $f_n \rightharpoonup f$ weakly means

$$\int f_n g\,dx \to \int fg\,dx$$

  for all $g \in L^q(\Omega)$. On the other hand, $(L^q(\Omega))' = L^p(\Omega)$ so $(L^p(\Omega))'' = L^p(\Omega)$ (we say $L^p(\Omega)$ is *reflexive*) and hence the weak and the weak-* topology on $L^p(\Omega)$ are the same.

- $L^1(\Omega)$ is also a separable normed space and its dual is $L^\infty(\Omega)$, but $L^1(\Omega)$ is not reflexive because $(L^\infty(\Omega))' \supsetneq L^1(\Omega)$. In $L^\infty(\Omega)$ one mostly uses the weak-* topology: $f_n \overset{*}{\rightharpoonup} f$ in $L^\infty(\Omega)$ if
$$\int f_n g\,dx \to \int f g\,dx$$
for all $g \in L^1(\Omega)$.
- Let $\Omega \subset \mathbb{R}^d$ be a domain and $C_0(\Omega)$ the space of continuous functions $\Omega \to \mathbb{R}$ vanishing at the boundary. Then the dual of $C_0(\Omega)$ can be identified, by the Riesz-Markov representation theorem, with the space of finite signed measures, i.e., the set of $\mu = \mu^+ - \mu^-$ where μ^+ and μ^- are finite measures supported on two disjoint measurable sets Ω^+ and Ω^-, respectively. The dual pairing is given by
$$\langle f, \mu\rangle = \int_\Omega f d\mu := \int_{\Omega^+} f d\mu^+ - \int_{\Omega^-} f d\mu^-.$$
- The difference between weak and strong convergence is most apparent in the presence of *oscillations*: Set $\Omega = (0,1), f_n(x) = \sin(nx)$. It is easy to see
$$f_n \overset{*}{\rightharpoonup} 0$$
(i.e. for all $g \in L^1(0,1) : \int g(x)\sin(nx)\,dx \to 0$, "Riemann-Lebesgue-Lemma"), but f_n does not converge strongly in $L^\infty(0,1)$ (i.e. uniformly).

Weak convergence and nonlinearities
Recall we want to pass to the limit in the equation
$$\partial_t u_\varepsilon + \partial_x F(u_\varepsilon) = \varepsilon \partial_{xx} u_\varepsilon,$$
and we know $u_\varepsilon \overset{*}{\rightharpoonup}$ in L^∞. In order to pass to the limit, we need $F(u_\varepsilon) \overset{*}{\rightharpoonup} F(u)$. But this is not true!
Example: $f_n(x) = \sin(nx)$ on $(0,1)$ and $f_n \overset{*}{\rightharpoonup} 0$, but $f_n^2 = \sin^2(nx) \overset{*}{\rightharpoonup} \frac{1}{2} \neq 0^2$.
So weak convergence does not commute with nonlinear functions:
$$F(\text{w*-}\lim u_\varepsilon) \neq \text{w*-}\lim F(u_\varepsilon)!$$
Our goal is to exclude such oscillatory effects.

PROPOSITION 2.25. *Let X be a normed space and X' its dual, and let either $x_n \to x$ strongly in X and $x_n' \rightharpoonup x'$ weakly in X', or $x_n \rightharpoonup x$ weakly in X and $x_n' \to x'$ strongly in X'. Then,*
$$\langle x_n, x_n'\rangle \to \langle x, x'\rangle.$$

The proof is left as an exercise.

2.4.3. Sobolev Spaces. Let $\Omega \subset \mathbb{R}^n$ be a domain. A function $u \in L^1_{loc}(\Omega)$ is weakly differentiable if there exists $v \in L^1_{loc}(\Omega)$ such that
$$\int_\Omega u D\varphi\,dx = -\int_\Omega \varphi v\,dx$$

for all $\varphi \in C_c^\infty(\Omega)$, and we write $v = Du$.
If $k \in \mathbb{N}$ and $1 \le p \le \infty$, then

$$W^{k,p}(\Omega) = \{u : \Omega \to \mathbb{R} : \partial_\alpha u \in L^p(\Omega) \text{ for all multiindices } \alpha \text{ with } |\alpha| \le k\}.$$

Similarly for $W^{k,p}_{loc}(\Omega)$. The norm is given by

$$\|u\|_{W^{k,p}(\Omega)} = \sum_{|\alpha|\le k} \|\partial_\alpha\|_{L^p}.$$

The space $W_0^{1,p}(\Omega)$ is the closure of $C_c^\infty(\Omega)$ under the $W^{1,p}$-norm.
The space $W^{-1,p}(\Omega)$ is the dual of $W_0^{1,p'}(\Omega)(1 < p < \infty, \frac{1}{p} + \frac{1}{p'} = 1)$, so the norm is

$$\|f\|_{W^{-1,p}(\Omega)} = \sup_{\|u\|_{W_0^{1,p'}(\Omega)} \le 1} \left| \int_\Omega f u \, dx. \right|$$

Note $W^{-1,q}_{loc} \subset W^{-2,r}_{loc}$ whenever $q \ge r$.

Rellich Compactness Theorem: If Ω is bounded and smooth then

- $W^{1,p}(\Omega) \Subset L^q(\Omega)$ for $1 \le q < p*$, where $p* = \frac{np}{n-p}$.
- $W^{1,p}(\Omega) \Subset C(\overline{\Omega})$ if $p > n$.

Here: $X \Subset Y$ means compact embedding, i.e. if $(x_n)_{n\in\mathbb{N}}$ is bounded in X then there exists a subsequence (x_{n_k}) s.t. $x_{n_k} \to x$ in Y ("(x_n) is precompact in Y").

THEOREM 2.26. *Let $\Omega \subset \mathbb{R}^n$ be bounded and smooth and (μ_n) be bounded in $\mathcal{M}(\Omega)$. Then (μ_n) is precompact in $W^{-1,q}(\Omega)$ for each $1 \le q < \frac{n}{n-1}$.*

PROOF. By Banach-Alaoglu there is a subsequence (still called (μ_n)) such that $\mu_n \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega)$. Consider $W_0^{1,q'}(\Omega)$ with $\frac{1}{q} + \frac{1}{q'} = 1$. Since $1 \le q < \frac{n}{n-1}$, $q' > n$, so that

$$W_0^{1,q'}(\Omega) \Subset C_0(\overline{\Omega}).$$

In particular $B := \{\varphi \in W_0^{1,q'}(\Omega) : \|\varphi\|_{W_0^{1,q'}} \le 1\}$ is compact in $C_0(\overline{\Omega})$ hence for every $\delta > 0$ there are finitely many functions $\{\varphi_i\}_{i=1,\dots,N_\delta} \subset C_0(\overline{\Omega})$ such that

$$\min_{1\le i\le N_\delta} \|\varphi - \varphi_i\|_\infty < \delta$$

for all $\varphi \in B$. Therefore for $\varphi \in B$,

$$\begin{aligned} \left| \int_\Omega \varphi \, d\mu_n - \int_\Omega \varphi \, d\mu \right| &\le \int_\Omega |\varphi - \varphi_i| \, d\mu_n + \left| \int_\Omega \varphi_i \, d\mu_n - \int_\Omega \varphi_i \, d\mu \right| + \int_\Omega |\varphi_i - \varphi| \, d\mu \\ &\le 2\delta \sup_n |\mu_n|(\Omega) + \left| \int_\Omega \varphi_i \, d\mu_n - \int_\Omega \varphi \, d\mu \right| \end{aligned}$$

for some $1 \le i \le N_\delta$. Let now $\varepsilon > 0$ and choose $\delta > 0$ such that

$$2\delta \sup_n |\mu_n|(\Omega) < \frac{\varepsilon}{2}.$$

Since $\mu_n \overset{*}{\rightharpoonup} \mu$, for every $1 \le i \le N_\delta$ there exists N_i such that

$$\left| \int_\Omega \varphi_i \, d\mu_n - \int_\Omega \varphi \, d\mu \right| < \frac{\varepsilon}{2}$$

for all $n \ge N_i$. Therefore, if $n > \max_{1 \le i \le N_\delta} N_i$,

$$\left| \int_\Omega \varphi \, d\mu_n - \int_\Omega \varphi \, d\mu \right| < \varepsilon$$

for all $\varphi \in B$ hence $\mu_n \to \mu$ in $W^{-1,q}(\Omega)$. □

THEOREM 2.27. *Let $1 < p < \infty, \Omega$ bounded smooth domain, and $f \in W^{k,p}(\Omega)$ for some $k \ge -1$ (in particular $f \in W^{-1,p}(\Omega)$ is allowed!). Then there exists a unique solution in the sense of distributions of*

$$-\Delta u = f, u \in W^{k+2,p}(\Omega) \cap W_0^{1,p}(\Omega). \tag{2.20}$$

Moreover, (f_n) is precompact in $W^{k,p}(\Omega)$ if and only if (u_n) is precompact in $W^{k+2,p}(\Omega)$.

PROOF. 'Standard' elliptic theory (see e.g. [**3**][Thm. 9.15]) □

LEMMA 2.28. *(Interpolation of L^p-spaces)*
Let $\Omega \subset \mathbb{R}^n$ be measurable and let $1 \le p \le q < \infty$. For $\theta \in [0,1]$ set $\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}$ (note such r are from $[p,q]$). Then if $f \in L^p(\Omega) \cap L^q(\Omega)$, then $f \in L^r(\Omega)$ and

$$\|f\|_{L^r} \le \|f\|_{L^p}^{1-\theta} \|f\|_{L^q}^{\theta}.$$

PROOF.

$$\begin{aligned} \|f\|_{L^r}^r &= \int |f|^{(1-\theta)r} |f|^{\theta r} \\ &\le \| |f|^{(1-\theta)r} \|_{L^{\frac{p}{(1-\theta)r}}} \| |f|^{\theta r} \|_{L^{\frac{q}{\theta r}}} \\ &= \|f\|_{L^p}^{(1-\theta)r} \|f\|_{L^q}^{\theta r}. \end{aligned}$$

□

COROLLARY 2.29. *Let $\Omega \subset \mathbb{R}^n$ be a domain. Suppose $(u_n)_{n\in\mathbb{N}}$ is precompact in $L^p(\Omega)$ and bounded in $L^q(\Omega)$ for $1 \le p < q < \infty$. Then $(u_n)_{n\in\mathbb{N}}$ is precompact in $L^r(\Omega)$ for any $p \le r < q$.*

PROOF. Let $(u_n)_{n\in\mathbb{N}}$ converge to u in $L^p(\Omega)$ and let $\theta \in (0,1)$ be such that $\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}$. Then by Lemma 2.28,

$$\|u_{n_k} - u\|_{L^r} \le \underbrace{\|u_{n_k} - u\|_{L^p}^{1-\theta}}_{\to 0} \underbrace{\|u_{n_k} - u\|_{L^q}^{\theta}}_{\text{bdd.}} \to 0.$$

□

COROLLARY 2.30. *Let $\Omega \subset \mathbb{R}^n$ bounded and smooth. Suppose $(f_n)_{n\in\mathbb{N}}$ is bounded in $W^{-1,p}(\Omega)$ for a $p > 2$. Let $f_n = g_n + h_n$, where*

- *g_n is precompact in $W^{-1,2}(\Omega)$*
- *h_n is bounded in $\mathcal{M}(\Omega)$.*

Then (f_n) is precompact in $W^{-1,2}(\Omega)$.

PROOF. By Theorem 2.27, there exists a unique $v_n \in W_0^{1,2}(\Omega)$ such that

$$-\Delta v_n = g_n.$$

Moreover, by Theorem 2.26 there is a unique $w_n \in W_0^{1,q}(\Omega)$ for any $1 \le q < \frac{n}{n-1}$ such that

$$-\Delta w_n = h_n,$$

and

- (v_n) is precompact in $W_0^{1,2}(\Omega)$,
- (w_n) is precompact in $W_0^{1,q}(\Omega)$.

Set $u_n := v_n + w_n$, then (since $q \le 2$) (u_n) is precompact in $W_0^{1,q}(\Omega)$, and u_n is the unique solution in $W_0^{1,q}(\Omega)$ of

$$-\Delta u_n = g_n + h_n = f_n.$$

By Theorem 2.27 and the assumption on (f_n), the sequence (u_n) is bounded in $W_0^{1,p}(\Omega)$. It follows from Corollary 2.29 that (u_n) is precompact in $W_0^{1,2}(\Omega)$ and hence $(f_n)_{n\in\mathbb{N}}$ is precompact in $W^{-1,2}(\Omega)$. □

2.4.4. Div-Curl-Lemma. Given a vector field $v = (v_1, ..., v_d) \in L^2(\Omega; \mathbb{R}^d)$ ($\Omega \subset \mathbb{R}^d$ bounded smooth domain), then $\operatorname{div} v \in W^{-1,2}(\Omega)$ is defined by duality as

$$\langle \varphi, \operatorname{div} v \rangle = -\int \nabla\varphi \cdot v \, dx$$

for $\varphi \in W_0^{1,2}(\Omega)$. Also we define $(\operatorname{curl} v)_{ij} = \partial_j v_i - \partial_i v_j$ in the sense of distributions, i.e.

$$\langle \varphi, \operatorname{curl} v \rangle_{ij} = \int \partial_i \varphi v_j - \partial_j \varphi v_i \, dx,$$

for $\varphi \in W_0^{1,2}(\Omega)$.
The Laplacian of v is interpreted row-wise, i.e.

$$\Delta v = (\Delta v_1, ..., \Delta v_d).$$

LEMMA 2.31. *(Div-Curl-Lemma – Murat 1978 [**5**])*
Let Ω bounded and smooth and let $(v_n)_{n\in\mathbb{N}}, (w_n)_{n\in\mathbb{N}}$ be two bounded sequences in $L^2(\Omega; \mathbb{R}^d)$ such that

i) $(\operatorname{div} v_n)$ *is precompact in* $W^{-1,2}(\Omega)$
ii) $(\operatorname{curl} w_n)$ *is precompact in* $W^{-1,2}(\Omega)$.

If $v_n \rightharpoonup v$ and $w_n \rightharpoonup w$ weakly in L^2, then $v_n \cdot w_n \rightharpoonup v \cdot w$ weakly in L^1.

PROOF. Consider $u_n \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$ solving

$$-\Delta u_n = w_n \text{ in } \Omega.$$

As w_n is bounded in L^2, u_n is bounded in $W^{2,2}(\Omega)$. Set $z_n = -\operatorname{div} u_n, y_n = w_n - Dz_n$. Then

- z_n is bounded in $W^{1,2}$ and

- $y_n^i = w_n^i - \partial_i z_n$
  $= -\Delta u_n^i + \sum_j \partial_i \partial_j u_n^j$
  $= \sum_j \partial_j (\partial_i u_n^j - \partial_j u_n^i)$
  $= -(\operatorname{div} \operatorname{curl} u_n)_i.$

But by ii) $(\operatorname{curl} u_n)$ is precompact in $W^{1,2}_{loc}$, hence (y_n) is precompact in L^2_{loc} (Exercise). Hence, after passing to subsequences if necessary,

- $z_n \rightharpoonup z$ in $W^{1,2}(\Omega)$
- $y_n \to y$ in $L^2_{loc}(\Omega)$, and $z = -\operatorname{div} u, y = w - Dz$, where $u \in W^{1,2}_0 \cap W^{2,2}$ solves $-\Delta u = w$.

Next let $\varphi \in C_c^\infty(\Omega)$, then

$$\int v_n \cdot w_n \varphi \, dx = \int v_n (y_n + Dz_n) \varphi \, dx.$$

As $v_n \rightharpoonup v$ weakly in L^2 (up to a subsequence) and $y_n \varphi \to y \varphi$ strongly in L^2, by Proposition 2.25

$$\int v_n \cdot y_n \varphi \, dx \to \int v \cdot y \varphi \, dx.$$

Moreover,

$$\int v_n \cdot Dz_k \varphi \, dx$$
$$= - \int \underbrace{\operatorname{div} v_n}_{\text{strongly in } W^{-1,2}} \underbrace{z_n}_{\text{weakly in } W^{1,2}} \varphi \, dx - \int \underbrace{v_n}_{\text{weakly in } L^2} \cdot D\varphi \underbrace{z_n}_{\text{strongly in } L^2 \text{ (Rellich)}} dx$$
$$\to - \int \operatorname{div} v z \varphi \, dx - \int v \cdot D\varphi z \, dx$$
$$= \int v \cdot Dz \varphi \, dx$$

hence

$$\int v_n \cdot w_n \varphi \, dx \to \int v(y + Dz) \, dx = \int v \cdot w \varphi \, dx.$$

Extension to $\varphi \in L^\infty$ follows by approximation, considering that $(v_n \cdot w_n)$ is bounded in L^∞. □

2.4.5. Young measures. Weak convergence $u_n \rightharpoonup u$ does not go well with nonlinearities. Take a different viewpoint: Identify $u_n(x)$ with $x \mapsto \delta_{u_n(x)}$, so $\delta_{u_n(x)}$ is a probability measure parametrized by the domain. Hope that

$$\delta_{u_n(x)} \rightharpoonup \delta_{u(x)}$$

in some sense.

THEOREM 2.32. *(Fundamental Theorem of Young measures)*
Let $\Omega \subset \mathbb{R}^n$ be a domain and $u_n \in L^\infty(\Omega; \mathbb{R}^m)$ be bounded. Then there exists

a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ *and, for a.e.* $x \in \Omega$, *a probability measure* ν_x, *such that*

$$F(u_{n_k}) \overset{*}{\rightharpoonup} \int_{\mathbb{R}^m} F(z)\, d\nu_x(z) =: \langle F, \nu_x\rangle$$

in $L^\infty(\Omega)$ *for every* $F \in C(\mathbb{R}^m)$. *The family* $(\nu_x)_{x\in\Omega}$ *is the Young measure generated by* (u_{n_k}).

To prove this, we need

LEMMA 2.33. *(Disintegration)*
Let μ *be a finite nonnegative Radon measure on* $\mathbb{R}^{n\times m}$. *Let* $\sigma = \pi_{\mathbb{R}^n}(\mu)$ *be the projection of* μ *into* $\mathbb{R}^n$, *i.e.*

$$\sigma(E) = \mu(E \times \mathbb{R}^m)$$

for $E \subset \mathbb{R}^n$ *Borel measurable. Then for* σ*-a.e.* $x \in \mathbb{R}^n$ *there is a probability measure* ν_x *on* $\mathbb{R}^m$ *such that*

$$x \mapsto \int_{\mathbb{R}^m} f(x,z)\, d\nu_x(z) \tag{2.21}$$

is σ*-measurable and*

$$\int_{\mathbb{R}^{m\times n}} f(x,y)\, d\mu(x,y) = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} f(x,z)\, d\nu_x(z) \right) d\sigma(x)$$

for every bounded continuous function f.

PROOF. (of Theorem 2.32)
First assume Ω is bounded. Define for each $n \in \mathbb{N}$ a measure by

$$\mu_n(E) = \int_\Omega \mathbb{1}_E(x, u_n(x))\, dx$$

for any $E \subset \Omega\times\mathbb{R}^m$ Borel measurable. Then $\sup_n \mu_n(\Omega\times\mathbb{R}^m) = \mathcal{L}^n(\Omega) < \infty$, so by Banach-Alaoglu there is a subsequence (μ_{n_k}) such that $\mu_{n_k} \overset{*}{\rightharpoonup} \mu \geq 0$. Let σ be the projection of μ onto Ω, i.e. $\sigma(E) = \mu(E \times \mathbb{R}^m)$. Then, on the one hand, if $V \subset \Omega$ is open,

$$\sigma(V) = \mu(V \times \mathbb{R}^m) \leq \liminf_k \mu_{n_k}(V \times \mathbb{R}^m) = \mathcal{L}^n(V)$$

by lower semicontinuity under weak convergence, so that $\sigma \leq \mathcal{L}^n|_\Omega$.
On the other hand, let $K \subset \Omega$ be compact. Since (u_{n_k}) is bounded in L^∞, there exists $R > 0$ (specifically $R > \sup_k \|u_{n_k}\|_{L^\infty}$) such that $\operatorname{supp}\mu, \operatorname{supp}\mu_{n_k} \subset \Omega \times B(0,R)$. Hence

$$\begin{aligned}\sigma(K) = \mu(K \times \mathbb{R}^m) &= \mu(K \times \overline{B(0,R)}) \\ &\geq \limsup_k \mu_{n_k}(K \times \overline{B(0,R)}) = \mathcal{L}^n(K)\end{aligned}$$

hence $\sigma \geq \mathcal{L}^n|_\Omega$ so in total we have shown $\sigma = \mathcal{L}^n|_\Omega$.
Next, by Lemma 2.33, for a.e. $x \in \Omega$ there is a probability measure ν_x such that

$$\int_{\mathbb{R}^{n\times m}} f(x,z)\, d\mu(x,z) = \int_\Omega \left(\int_{\mathbb{R}^m} f(x,z)\, d\nu_x(z) \right) dx$$

for f bounded and continuous. Let $f(x,z) = \varphi(x)F(z)\,(\varphi \in C_c(\Omega), F \in C_c(\mathbb{R}^m))$. Hence

$$\lim_{k\to\infty}\int_\Omega \varphi(x)F(u_{n_k}(x))\,dx = \lim_{k\to\infty}\int_{\mathbb{R}^{n\times m}} f(x,z)\,d\mu_{n_k}(x,z) = \int_{\mathbb{R}^{n\times m}} f(x,z)\,d\mu(x,z) = \int_\Omega \varphi(x)\left(\int_{\mathbb{R}^m} F(z)\,d\nu_x(z)\right)dx.$$

But by approximation this is also true for $\varphi \in L^1(\Omega)$ so

$$F(u_{n_k}) \overset{*}{\rightharpoonup} \int_{\mathbb{R}^n} F(z)\,d\nu_x(z)$$

in L^∞ for $F \in C_c(\mathbb{R}^m)$. Finally, as (u_{n_k}) are bounded in $L^\infty, F \in C(\mathbb{R}^m)$ can be altered to have compact support so the convergence is even true for $F \in C(\mathbb{R}^m)$.

Now let Ω be a general domain (not necessarily bounded) and exhaust it by bounded domains Ω_j, so that

$$\Omega_j \subset \Omega_{j+1} \text{ and } \bigcup_{j\in\mathbb{N}} \Omega_j = \Omega.$$

For each $j \in \mathbb{N}$, let ν_x^j be the Young measure generated by a subsequence $(u_{n_{k,j}})_{k\in\mathbb{N}}$. We assume $(n_{k,j})_{k\in\mathbb{N}} \supset (n_{k,j+1})_{k\in\mathbb{N}}$ for all $j \in \mathbb{N}$. Then clearly $\nu_x^j = \nu_x^{j+1}$ for a.e. every $x \in \Omega_j$ (test weak-* convergence with $\varphi \in C_c(\Omega_j)$). Hence for $x \in \Omega_j$ we define $\nu_x = \nu_x^j$ and $\{\nu_x\}_{x\in\Omega}$ is generated by the diagonal sequence $(u_{n_{k,k}})_{k\in\mathbb{N}}$: Indeed for $\varphi \in C_c(\Omega)$ and $F \in C(\mathbb{R}^m)$ there exists $j \in \mathbb{N}$ such that

$$\operatorname{supp}\varphi \in \Omega_j.$$

Since $(u_{n_{k,k}})_{k\geq j}$ is a subsequence of $(u_{n_{k,j}})_{k\in\mathbb{N}}$ by our construction, we obtain

$$\int_\Omega \varphi(x)F(u_{n_{k,k}}(x))\,dx = \int_{\Omega_j} \varphi(x)F(u_{n_{k,k}}(x))\,dx \to \int_{\Omega_j}\varphi(x)\int_{\mathbb{R}^m} F(z)\,d\nu_x^j(z)\,dx = \int_\Omega \varphi(x)\int_{\mathbb{R}^m} F(z)\,d\nu_x(z)\,dx.$$

The statement for $\varphi \in L^1(\Omega)$ follows by approximation. □

2.4.6. Application to Scalar Conservation Laws. So consider $\partial_t u_\varepsilon + \partial_x F(u_\varepsilon) = \varepsilon\partial_{xx}u_\varepsilon$ and assume $\|u_\varepsilon\|_{L^\infty(\mathbb{R}\times(0,\infty))}$ is bounded in $\varepsilon > 0$ and u_ε is smooth. From the L^∞ bound we get (up to a subsequence)

$$u_\varepsilon \overset{*}{\rightharpoonup} u$$

in $L^\infty(\mathbb{R}\times(0,\infty))$.

THEOREM 2.34. *Let F be strictly convex. Then u is an entropy solution of $\partial_t u + \partial_x F(u) = 0$.*

REMARK 2.35. Ignore initial data.

PROOF. Up to a subsequence, $(u_\varepsilon)_{\varepsilon>0}$ generates a Young measure $(\nu_{x,t})$ by Theorem 2.32, so that

$$u(x,t) = \int_{\mathbb{R}} z \, d\nu_{x,t}(z)$$

for a.e. x, t and

$$F(u_\varepsilon) \overset{*}{\rightharpoonup} \int_{\mathbb{R}} F(z) \, d\nu_{x,t}(z) =: \langle F, \nu_{x,t} \rangle \tag{2.22}$$

in L^∞. Let now η, q be an entropy / entropy-flux pair. Then

$$\eta(u_\varepsilon) \overset{*}{\rightharpoonup} \langle \eta, \nu_{x,t} \rangle = \int \eta(z) \, d\nu_{x,t}(z)$$

and

$$q(u_\varepsilon) \overset{*}{\rightharpoonup} \langle q, \nu_{x,t} \rangle$$

in L^∞. As u_ε are smooth, we can multiply the parabolic PDE by $\eta'(u_\varepsilon)$ to obtain, as before,

$$\partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon) = \varepsilon \partial_{xx} \eta(u_\varepsilon) - \varepsilon \eta''(u_\varepsilon)(\partial_x u_\varepsilon)^2.$$

We want to apply the div-curl-lemma to

$$v_\varepsilon = (F(u_\varepsilon), u_\varepsilon), w_\varepsilon = (\eta(u_\varepsilon), -q(u_\varepsilon)).$$

Note that div and curl are taken in the variables (x,t)! Hence

$$\operatorname{div}_{x,t} v_\varepsilon = \partial_t u_\varepsilon + \partial_x F(u_\varepsilon) = \varepsilon \partial_{xx} u_\varepsilon$$

and

$$\operatorname{curl}_{x,t} w_\varepsilon = \partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon) = \varepsilon \partial_{xx} \eta(u_\varepsilon) - \varepsilon \eta''(u_\varepsilon)(\partial_x u_\varepsilon)^2$$

and we need to show that both are precompact in $W^{-1,2}_{loc}(\mathbb{R} \times (0,\infty))$. For this, multiply the parabolic PDE with u_ε and integrate in x:

$$\begin{aligned} 0 &= \int \partial_t u_\varepsilon u_\varepsilon \, dx + \int u_\varepsilon \partial_x F(u_\varepsilon) \, dx - \varepsilon \int u_\varepsilon \partial_{xx} u_\varepsilon \, dx \\ &= \frac{1}{2} \frac{d}{dt} \int u_\varepsilon^2 \, dx + \varepsilon \int (\partial_x u_\varepsilon)^2 \, dx. \end{aligned}$$

Note $\int u_\varepsilon \partial_x F(u_\varepsilon) \, dx = 0$ since $u_\varepsilon \partial_x F(u_\varepsilon) = \partial_x Q(u_\varepsilon)$, where Q is the entropy flux corresponding to $H(z) = \frac{z^2}{2}$.
Integration in the time thus yields

$$\frac{1}{2} \int_{\mathbb{R}} u_\varepsilon^2 \, dx + \varepsilon \int_0^t \int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 \, dx ds = \frac{1}{2} \int_{\mathbb{R}} u_0^2 \, dx \text{ (indep. of } \varepsilon).$$

This means $\sqrt{\varepsilon} \partial_x u_\varepsilon \in L^2(\mathbb{R} \times (0,\infty))$ is bounded and so $\sqrt{\varepsilon} \partial_{xx} u_\varepsilon$ is bounded in $W^{-1,2}(\mathbb{R} \times (0,\infty))$ and hence $\varepsilon \partial_{xx} u_\varepsilon \to 0$ in $W^{-1,2}(\mathbb{R} \times (0,\infty))$. Next consider $\varepsilon \partial_{xx} \eta(u_\varepsilon) - \varepsilon \eta''(u_\varepsilon)(\partial_x u_\varepsilon)^2$. We have that $\sqrt{\varepsilon} \partial_x u_\varepsilon$ is bounded in $L^2(\mathbb{R} \times (0,\infty))$ hence so is $\sqrt{\varepsilon} \eta'(u_\varepsilon) \partial_x u_\varepsilon = \sqrt{\varepsilon} \partial_x \eta(u_\varepsilon)$. Therefore $\varepsilon \partial_x \eta(u_\varepsilon)$ is precompact in L^2 so $\varepsilon \partial_{xx} \eta(u_\varepsilon)$ is precompact in $W^{-1,2}$. Also, $\varepsilon (\partial_x u_\varepsilon)^2$ is in $L^1(\mathbb{R} \times (0,\infty))$ so $\varepsilon \eta''(u_\varepsilon)(\partial_x u_\varepsilon)^2$ is bounded in $\mathcal{M}(\mathbb{R} \times (0,\infty))$.

Moreover, $\partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon)$ is bounded in $W^{-1,\infty}$ and so Corollary 2.30 applies and yields

$$\varepsilon \partial_{xx} \eta(u_\varepsilon) - \varepsilon \eta''(u_\varepsilon)(\partial_x u_\varepsilon)^2$$

is precompact in $W^{-1,2}_{loc}(\mathbb{R} \times (0,\infty))$. Therefore

$$F(u_\varepsilon)\eta(u_\varepsilon) - u_\varepsilon q(u_\varepsilon) \rightharpoonup \langle F, \nu\rangle \langle \eta, \nu\rangle - u\langle q, \nu\rangle.$$

On the other hand,

$$F(u_\varepsilon)\eta(u_\varepsilon) - u_\varepsilon q(u_\varepsilon) \overset{*}{\rightharpoonup} \int F(z)\eta(z) - zq(z)\, d\nu(z)$$

hence

$$\int F(z)\eta(z) - zq(z)\, d\nu(z) = \langle F, \nu\rangle \langle \eta, \nu\rangle - u\langle q, \nu\rangle$$

for a.e. x, t or equivalently

$$\int (F(z) - \langle F, \nu\rangle)\eta(z) + (u - z)q(z)\, d\nu(z) = 0$$

a.e. Now set (cf. Kruzkhov!) $\eta(z) = |z - u(x,t)|$. Note as long as we keep x, t fixed this is a valid choice of entropy! Correspondingly, we obtain

$$q(z) = \operatorname{sgn}(z - u(x,t))(F(z) - F(u(x,t))).$$

Using this entropy / entropy-flux pair, we get

$$\int (F(z) - \langle F, \nu\rangle)|u - z| - |u - z|(F(z) - F(u))\, d\nu = 0$$

$$\Leftrightarrow (F(u) - \langle F, \nu\rangle) \int |z - u(x,t)|\, d\nu_{x,t} = 0.$$

Hence for a.e. x, t, $F(u) = \langle F, \nu\rangle$ or $\nu_{x,t} = \delta_{u(x,t)}$. But since F is strictly convex, $F(u) = \langle F, \nu\rangle$ is equivalent to $\nu_{x,t} = \delta_{u(x,t)}$ (by Jensen's inequality) hence $\langle \eta, \nu\rangle = \eta(u)$ and $q(u_\varepsilon) \rightharpoonup q(u)$ and from

$$\partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon) = \varepsilon \partial_{xx} \eta(u_\varepsilon) - \varepsilon \eta''(u_\varepsilon)(\partial_x u_\varepsilon)^2$$

we have for any $\varphi \in C_c^\infty(\mathbb{R} \times (0,\infty)), \varphi \geq 0:$

$$\iint \varphi(\partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon))\, dxdt = -\int \partial_t \varphi \eta(u_\varepsilon) + \partial_x \varphi q(u_\varepsilon)\, dxdt$$

$$\to -\int \partial_t \varphi \eta(u) + \partial_x \varphi q(u)\, dxdt,$$

$$\varepsilon \int \varphi \partial_{xx} \eta(u_\varepsilon)\, dxdt = -\varepsilon \underbrace{\int \partial_{xx} \varphi \eta(u_\varepsilon)\, dxdt}_{\to \int \partial_{xx}\varphi\eta(u) < \infty} \to 0$$

and $\varepsilon \eta''(u_\varepsilon)(\partial_x u_\varepsilon)^2 \geq 0$ hence

$$\int \partial_t \varphi \eta(u) + \partial_x \varphi q(u)\, dxdt \geq 0$$

and u is an entropy solution. □

CHAPTER 3

Hyperbolic Systems of Conservation Laws

Consider systems in one space dimension:

$$\partial_t u + \partial_x F(u) = 0 \text{ in } \mathbb{R} \times (0, \infty),$$

where

$$u : \mathbb{R} \times (0, \infty) \to \mathbb{R}^m \text{ and } F : \mathbb{R}^m \to \mathbb{R}^m$$

3.1. Basics

3.1.1. Rankine-Hugoniot condition. We say $u \in L^\infty(\mathbb{R} \times (0, \infty))$ is a *weak solution* with initial datum $g \in L^\infty(\mathbb{R})$ if for all $\varphi \in C_c^\infty(\mathbb{R} \times [0, \infty); \mathbb{R}^m)$,

$$\int_0^\infty \int_{\mathbb{R}} \partial_t \varphi \cdot u + \partial_x \varphi \cdot F(u)\, dxdt + \int_{\mathbb{R}} \varphi(x, 0) \cdot g(x)\, dx = 0.$$

Without reference to an initial condition, u is simply called a weak solution if

$$\int_0^\infty \int_{\mathbb{R}} \partial_t \varphi \cdot u + \partial_x \varphi \cdot F(u)\, dxdt = 0.$$

for all $\varphi \in C_c^\infty(\mathbb{R} \times (0, \infty); \mathbb{R}^m)$.

Let u be a weak solution on $\Omega \subset \mathbb{R} \times (0, \infty)$ (i.e. we test only with $\varphi \in C_c^\infty(\Omega)$) which is smooth on either side of a smooth curve:

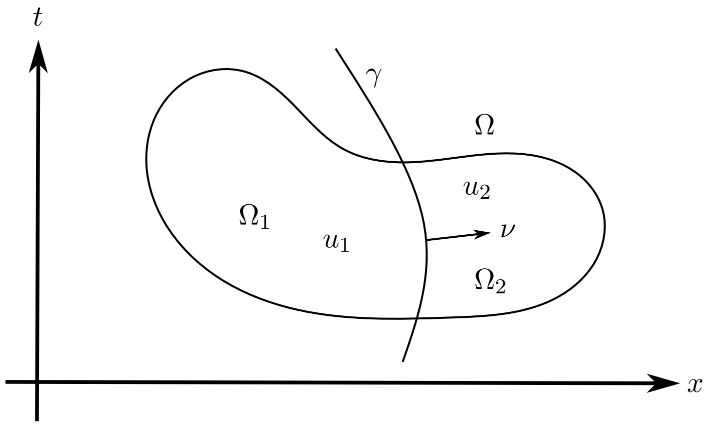


FIGURE 1

Let $\varphi \in C_c^\infty(\Omega_1;\mathbb{R}^m)$, then

$$0=\iint_{\Omega_1}\partial_t\varphi\cdot u+\partial_x\varphi\cdot F(u)\,dtdt=-\iint_{\Omega_1}\varphi\cdot(\partial_t u+\partial_x F(u))\,dxdt$$

so that $\partial_t u+\partial_x F(u)=0$ on Ω_1 and similarly on Ω_2. Now let $\varphi\in C_c^\infty(\Omega;\mathbb{R}^m)$. Then

$$\begin{aligned}0&=\int_0^\infty\int_\Omega\partial_t\varphi\cdot u+\partial_x\varphi\cdot F(u)\,dxdt\\&=\iint_{\Omega_1}\partial_t\varphi\cdot u+\partial_x\varphi\cdot F(u)\,dxdt+\iint_{\Omega_2}\partial_t\varphi\cdot u+\partial_x\varphi\cdot F(u)\,dxdt\\&=-\iint_{\Omega_1}\underbrace{(\partial_t u+\partial_x F(u))}_{=0}\cdot\varphi\,dxdt-\iint_{\Omega_2}\underbrace{(\partial_t u+\partial_x F(u))}_{=0}\cdot\varphi\,dxdt\\&\quad+\int_\gamma(\varphi\cdot u_1)\nu_t+(F(u_1)\cdot\varphi)\nu_x\,dS\\&\quad-\int_\gamma(\varphi\cdot u_2)\nu_t+(F(u_2)\cdot\varphi)\nu_x\,dS\end{aligned}$$

hence we obtain

$$(F(u_1)-F(u_2))\nu_x+(u_1-u_2)\nu_t=0$$

along γ.
Suppose $\gamma=\{(x,t):x=s(t)\}$, then for all t, $\nu=(\nu_x,\nu_t)=(1,-\dot{s})$ and we obtain the *Rankine-Hugoniot condition*

$$\boxed{F(u_1)-F(u_2)=\dot{s}(t)(u_1-u_2)\text{ along }\gamma}$$

3.1.2. Hyperbolicity.

For motivation, consider particular smooth solutions of the conservation law:
Travelling waves: They have the form

$$u(x,t)=v(x-\sigma t)\text{ (cf. shocks!)}$$

where $v:\mathbb{R}\to\mathbb{R}^m$ is the *profile* and σ is the *speed*. Plugging this into $\partial_t u+DF(u)\partial_x u=0$, we find

$$0=-\sigma v'(x-\sigma t)+DF(v(x-\sigma t))v'(x-\sigma t),$$

i.e. σ is an eigenvalue of $DF(v)$ with eigenvector v'. Hence if we want to find m linearly independent travelling waves, we need to assume

DEFINITION 3.1. If for every $z\in\mathbb{R}^m$ the eigenvalues of $DF(z)\in\mathbb{R}^{m\times m}$ are real and distinct, then the system of conservation laws is called *strictly hyperbolic*.

We will now always assume strict hyperbolicity. For $z\in\mathbb{R}^m$, write

$$\lambda_1(z)<\lambda_2(z)<...<\lambda_m(z)$$

for the eigenvalues of $DF(z)$ and $r_k(z)\in\mathbb{R}^m$ the corresponding eigenvector $(k=1,...,m)$, so that

$$DF(z)r_k(z)=\lambda_k(z)r_k(z).$$

for each z, $\{r_k(z)\}_{k=1,...,m}$ form a basis of $\mathbb{R}^m$. Next, since $DF(z)$ and $DF(z)^t$ have the same spectrum, there exists a basis $\{l_k(z)\}_{k=1,...,m}$ of eigenvectors of $DF(z)^t$:

$$DF(z)^t l_k(z) = \lambda_k(z) l_k(z)$$

for $k = 1, ..., m$. This is sometimes written as

$$l_k(z) DF(z) = \lambda_k(z) l_k(z),$$

so that $l_k(z)$ are called the *left (row-)eigenvectors* of $DF(z)$ and $r_k(z)$ are the *right eigenvectors*. Note that

$$\begin{aligned} \lambda_k(z)(l_l(z) r_k(z)) &= l_l(z)(DF(z) r_k(z)) \\ &= (l_l(z) DF(z)) r_k(z) \\ &= \lambda_l(z)(l_l(z) r_k(z)), \end{aligned}$$

hence if $l \neq k$ (and therefore $\lambda_k(z) \neq \lambda_l(z)$) it follows that $l_l(z) \perp r_k(z)$.

THEOREM 3.2. *(Smooth dependence of λ_k, r_k, l_k on z)*
Assume F is smooth and strictly hyperbolic. Then

i) *λ_k depend smoothly on z for $k = 1, ..., m$.*
ii) *r_k and l_k can be chosen smooth in z and such that $|r_k(z)| = |l_k(z)| = 1$ for all z, k.*

PROOF. Fix $k \in \{1, ..., m\}$ and $z_0 \in \mathbb{R}^m$. Then

$$\lambda_1(z_0) < ... < \lambda_m(z_0)$$

and $r_k(z_0)$ can be chosen such that

$$\begin{aligned} DF(z_0) r_k(z_0) &= \lambda_k(z_0) r_k(z_0), \\ |r_k(z_0)| &= 1. \end{aligned}$$

Without loss of generality assume

$$r_k(z_0) = e_m = (0, ..., 0, 1).$$

We want to use the Implicit Function Theorem for $\Phi : \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^{m+1}$,

$$\Phi(r, \lambda, z) = (DF(z) r - \lambda r, |r|^2 - 1).$$

Note that $\Phi(r_k(z_0), \lambda_k(z_0), z_0) = 0$. Hence if we can show that

$$\det \left. \frac{\partial \Phi(r, \lambda, z)}{\partial (r, \lambda)} \right|_{(r_k(z_0), \lambda_k(z_0), z_0)} \neq 0,$$

then the desired result follows in a neighbourhood of z_0. Note

$$\frac{\partial \Phi(r, \lambda, z)}{\partial (r, \lambda)} = \begin{pmatrix} DF(z) - \lambda I_m & -r \\ 2r & 0 \end{pmatrix}$$

hence (recall $r_k(z_0) = e_m$)

$$\left. \frac{\partial \Phi(r, \lambda, z)}{\partial (r, \lambda)} \right|_{(r_k(z_0), \lambda_k(z_0), z_0)} = \begin{pmatrix} & & & 0 \\ & & & 0 \\ & DF(z_0) - \lambda_k(z_0) I_m & & \vdots \\ & & & -1 \\ 0 & \dots & 2 & 0 \end{pmatrix}.$$

We define

$$B_\varepsilon := \det[DF(z_0) - (\lambda_k(z_0) + \varepsilon)I].$$

As $r_k(z_0) = e_m$, we have

$$B_\varepsilon e_m = -\varepsilon e_m.$$

Therefore

$$\begin{pmatrix} & & & 0 \\ & & & 0 \\ & B_\varepsilon & & \vdots \\ & & & -1 \\ 0 & \dots & 2 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} & & & 0 \\ & & & 0 \\ & I_m & & \vdots \\ & & & (-\varepsilon)^{-1} \\ 0 & \dots & 0 & 1 \end{pmatrix}}_{\det=1} = \begin{pmatrix} & & & 0 \\ & & & 0 \\ & B_\varepsilon & & \vdots \\ & & & 0 \\ 0 & \dots & 2 & 2(-\varepsilon)^{-1} \end{pmatrix}$$

hence

$$\begin{aligned} \det \begin{pmatrix} & & & 0 \\ & & & 0 \\ & B_\varepsilon & & \vdots \\ & & & -1 \\ 0 & \dots & 2 & 0 \end{pmatrix} &= 2(-\varepsilon)^{-1} \det B_\varepsilon \\ &= 2(-\varepsilon)^{-1}[\Pi_{j\neq k}(\lambda_j(z_0) - (\lambda_k(z_0) + \varepsilon))](-\varepsilon) \\ &= 2\Pi_{j\neq k}(\lambda_j(z_0) - \lambda_k(z_0) - \varepsilon). \end{aligned}$$

But the LHS converges (as $\varepsilon \to 0$) to

$$\det \begin{pmatrix} & & & 0 \\ & & & 0 \\ & DF(z_0) - \lambda_k(z_0)I_m & & \vdots \\ & & & -1 \\ 0 & \dots & 2 & 0 \end{pmatrix}$$

and the RHS converges to

$$2\Pi_{j\neq k}(\lambda_j(z_0) - \lambda_k(z_0)) \neq 0 \text{ (hyperbolicity!)}.$$

This shows the assertion in a neighbourhood of $z_0 \in \mathbb{R}^m$. Finally, let

$$R = \sup\{r > 0 : \lambda_k(z), r_k(z) \text{ are well-defined and smooth on } B(z_0, r)\}.$$

If $R = \infty$ we are done. If $R < \infty$, cover $\partial B(z_0, R)$ with finitely many open balls, to which λ_k, r_k can be smoothly extended by the same arguments as above (choose centres $z_1, z_2, \dots$).

This contradicts the choice of R. The left eigenvectors l_k can be treated similarly. □

EXAMPLE 3.3. i) Recall the "p-system" (nonlinear wave equation)

$$\partial_t u + \partial_x F(u) = 0,$$

where $F(u_1, u_2) = (-u_2, -p(u_1))$. Hence

$$Df(u_1, u_2) = \begin{pmatrix} 0 & -1 \\ -p'(u_1) & 0 \end{pmatrix}.$$

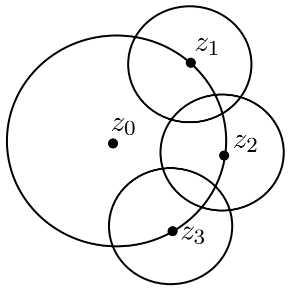


FIGURE 2

The eigenvalues are the solutions of

$$\det \begin{pmatrix} \lambda & 1 \\ p'(u_1) & \lambda \end{pmatrix} = 0 \Leftrightarrow \lambda = \pm\sqrt{p'(u_1)}.$$

Hence the $p - system$ is strictly hyperbolic iff $p' > 0$.

ii) Recall the barotropic Euler equations (here in one space dimension):

$$\partial_t m + \partial_x \left(\frac{m^2}{\rho} + p(\rho) \right) = 0$$
$$\partial_t \rho + \partial_x m = 0.$$

Hence $F(m, \rho) = \left(\frac{m^2}{\rho} + p(\rho), m \right)$ and

$$DF(m, \rho) = \begin{pmatrix} 2\frac{m}{\rho} & -\frac{m^2}{\rho^2} + p'(\rho) \\ 1 & 0 \end{pmatrix}.$$

We obtain the eigenvalues:

$$\left(2\frac{m}{\rho} - \lambda \right)(-\lambda) + \frac{m^2}{\rho^2} - p'(\rho) = 0$$
$$\Leftrightarrow \lambda^2 - 2\frac{m}{\rho}\lambda + \frac{m^2}{\rho^2} - p'(\rho) = 0.$$

This has two distinct real solutions iff

$$4\frac{m^2}{\rho^2} - 4\frac{m^2}{\rho^2} + p'(\rho) = p'(\rho) > 0$$

hence we require $\rho > 0, p'(\rho) > 0$.

3.2. Riemann's Problem

Consider a system of conservation laws with initial data

$$g(x) = \begin{cases} u_l & : x < 0 \\ u_r & : x > 0 \end{cases}$$

for vectors $u_l, u_r \in \mathbb{R}^m$.

3.2.1. Simple waves. A *simple wave solution* is a solution of the form $u(x,t) = v(w(x,t))$ where $v : \mathbb{R} \to \mathbb{R}^m$ and $w : \mathbb{R} \times (0,\infty) \to \mathbb{R}$ (for $w(x,t) = x - \sigma t$ we get the travelling waves). Plug this into the conservation law to get

$$v'(w(x,t))\partial_t w + DF(v(w(x,t)))v'(w(x,t))\partial_x w = 0.$$

If $v'(s) = r_k(v(s))$ then this becomes

$$v'(w(x,t))\partial_t w + \lambda_k(v(w(x,t)))v'(w(x,t))\partial_x w = 0,$$

so that v, w will solve the system if

$$\partial_t w + \lambda_k(v(w(x,t)))\partial_x w = 0$$
$$v'(s) = r_k(v(s)).$$

Hence the strategy: First solve the ODE $v'(s) = r_k(v(s))$ and then solve the scalar conservation law

$$\partial_t w + \lambda_k(v(w(x,t)))\partial_x w = 0.$$

In this case, the solution $u(x,t) = v(w(x,t))$ is called a *k-simple wave*.
Let's turn to the ODE first.

DEFINITION 3.4. Let $z_0 \in \mathbb{R}^m$. Denote by $R_k(z_0)$ the path of the solution of

$$v'(s) = r_k(v(s))$$

which passes through z_0, i.e.

$$R_k(z_0) = \{v(s) : s \in \mathbb{R}, v'(s) = r_k(v(s)), v(0) = z_0\}.$$

Then $R_k(z_0)$ is called the *k-th rarefaction curve* through z_0.

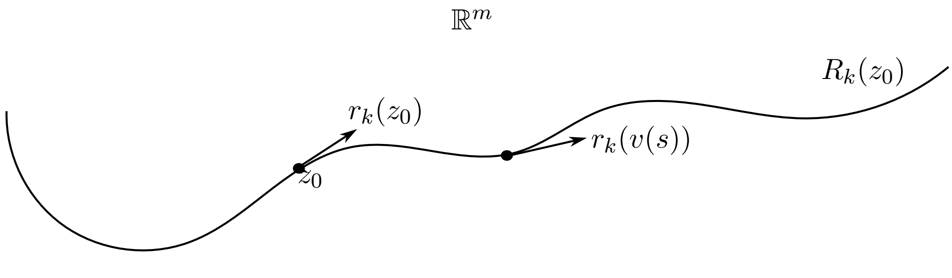


FIGURE 3. k-th rarefaction curve

Next consider the scalar law $\partial_t w + \lambda_k(v(w))\partial_x w$. This takes the form $\partial_t w + \partial_x F_k(w) = 0$ upon setting

$$F_k(s) = \int_0^s \lambda_k(v(t))\,dt.$$

In order to solve the scalar Riemann problem, we need F_k to be strictly convex (or concave), see Theorem 2.34. Check the 2nd derivative:

$$F_k'(s) = \lambda_k(v(s)),$$
$$F_k''(s) = D\lambda_k(v(s)) \cdot v'(s) = D\lambda_k(v(s)) \cdot r_k(v(s))$$

hence F_k is

- strictly convex if $\forall z \in \mathbb{R}^m : D\lambda_k(z) \cdot r_k(z) > 0$,
- strictly concave if $\forall z \in \mathbb{R}^m : D\lambda_k(z) \cdot r_k(z) < 0$

and affine if $D\lambda_k(z) \cdot r_k(z) = 0$ for all $z \in \mathbb{R}^m$.

DEFINITION 3.5. The pair $(\lambda_k(z), r_k(z))$ is *genuinely nonlinear* if for all $z \in \mathbb{R}^m$

$$D\lambda_k(z) \cdot r_k(z) \neq 0 \,(\text{it follows } D\lambda_k \cdot r_k > (<)\, 0).$$

It is *linearly degenerate* if for all $z \in \mathbb{R}^m$

$$D\lambda_k(z) \cdot r_k(z) = 0.$$

If (λ_k, r_k) is genuinely nonlinear, we write

$$R_k^+(z_0) = \{z \in R_k(z_0) : \lambda_k(z) > \lambda_k(z_0)\}$$
$$R_k^-(z_0) = \{z \in R_k(z_0) : \lambda_k(z) < \lambda_k(z_0)\},$$

so that

$$R_k(z_0) = R_k^+(z_0) \dot\cup \{z_0\} \dot\cup R_k^-(z_0).$$

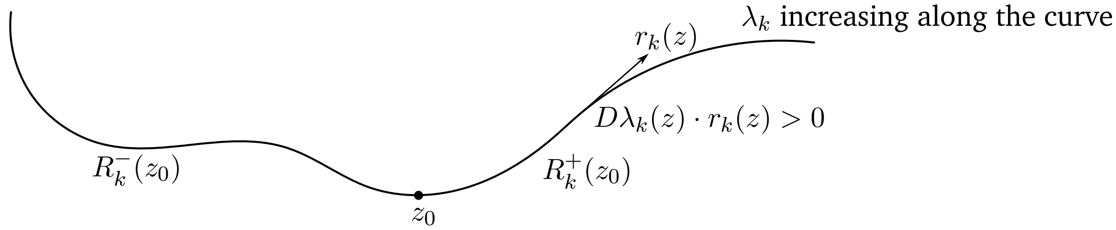


FIGURE 4

3.2.2. Rarefaction Waves.

THEOREM 3.6. *Suppose for some $k \in \{1, ..., m\}$ the pair (λ_k, r_k) is genuinely nonlinear and $u_r \in R_k^+(u_l)$.*
Then there is a k-simple wave which is a weak solution of the Riemann problem.

PROOF. Let v be the solution of

$$v'(s) = r_k(v(s)), v(0) = u_l \in \mathbb{R}^m.$$

By assumption $u_r \in R_k^+(u_l)$ there exists $w_r \in \mathbb{R}$ (wlog: $w_r > 0$) such that $v(w_r) = u_r$.
Next, consider the scalar Riemann problem

$$\partial_t w + \partial_x F_k(w) = 0$$
$$w(x,0) = g(x) = \begin{cases} 0 & : x < 0 \\ w_r & : x > 0. \end{cases}$$

By genuine nonlinearity, F_k is strictly convex because $\lambda_k(u_r) > \lambda_k(u_l)$, so that the solution is given by (Theorem 2.34)

$$w(x,t) = \begin{cases} 0 & : \frac{x}{t} < F_k'(0) \\ G_k\left(\frac{x}{t}\right) & : F_k'(0) < \frac{x}{t} < F_k'(w_r) \\ w_r & : \frac{x}{t} > F_k'(w_r), \end{cases}$$

where $G_k = (F_k')^{-1}$. Hence

$$u(x,t) = v(w(x,t))$$

is a weak solution of the systems. The case $w_r < 0$ is similar. □

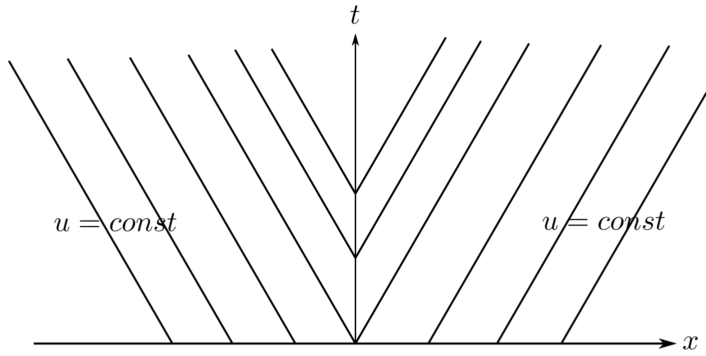


FIGURE 5

REMARK 3.7. This is also an entropy solution, as it is continuous (cf. Proposition 2.20).

3.2.3. Shock waves. Recall the Rankine-Hugoniot condition

$$F(u_1) - F(u_2) = \dot{s}(u_1 - u_2),$$

i.e. in particular $F(u_1) - F(u_2)$ has to be parallel to $u_1 - u_2$ if u_1, u_2 are the two values on either side of a shock.

DEFINITION 3.8. Fix $z_0 \in \mathbb{R}^m$, then the *shock set* is defined as

$$S(z_0) = \{z \in \mathbb{R}^m : F(z) - F(z_0) = \sigma(z - z_0) \text{ for some scalar } \sigma = \sigma(z, z_0)\}.$$

THEOREM 3.9. *(Structure of the shock set)*
Fix $z_0 \in \mathbb{R}^m$. There is a neighbourhood of z_0 such that $S(z_0) = \bigcup_{k=1}^m S_k(z_0)$ for smooth curves $S_k(z_0)$ with the following properties:

i) $S_k(z_0)$ *passes through* z_0 *with tangent* $r_k(z_0)$,
ii) $\lim_{z \to z_0, z \in S_k(z_0)} \sigma(z, z_0) = \lambda_k(z_0)$
iii) $\sigma(z, z_0) = \frac{\lambda_k(z) + \lambda_k(z_0)}{2} + \mathcal{O}(|z - z_0|^2)$ *as* $(z \to z_0)$ *with* $z \in S_k(z_0)$.

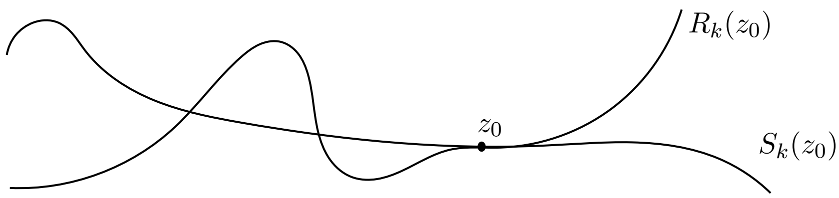


FIGURE 6. contact between $R_k(z_0)$ and $S_k(z_0)$

PROOF. Set $B(z) := \int_0^1 DF(z_0 + t(z - z_0))\, dt$, so that $B(z)(z - z_0) = F(z) - F(z_0)$. We have $z \in S(z_0)$ if and only if

$$(B(z) - \sigma I_m)(z - z_0) = 0$$

for some scalar $\sigma(z, z_0)$.
Since $B(z_0) = DF(z_0)$, the characteristic polynomial

$$\lambda \mapsto \det(\lambda I_m - B(z_0))$$

has m distinct real solutions (hyperbolicity) and so the same is true for

$$\lambda \mapsto \det(\lambda I_m - B(z))$$

for z in some neighbourhood of z_0. Moreover, in this neighbourhood there exist smooth functions $\hat{\lambda}_1(z) < ... < \hat{\lambda}_m(z)$ and unit vectors $\hat{r}_k(z), \hat{l}_k(z)$ $(k = 1, ..., m)$ such that

$$B(z)\hat{r}_k(z) = \hat{\lambda}_k(z)\hat{r}_k(z)$$
$$\hat{l}_k(z)B(z) = \hat{\lambda}_k(z)\hat{l}_k(z)$$

and $\hat{\lambda}_k(z_0) = \lambda_k(z_0), \hat{r}_k(z_0) = r_k(z_0), \hat{\lambda}_k(z_0) = \lambda_k(z_0)$. Also $\{\hat{r}_k(z)\}_{k=1,\ldots,m}$ and $\{\hat{l}_k(z)\}_{k=1,\ldots,m}$ are bases of $\mathbb{R}^m$ with

$$\hat{l}_l(z) \cdot \hat{r}_k(z) = 0$$

whenever $k \neq l$. Indeed all this follows from Theorem 3.2, where the gradient property of DF was never used.
Recall $z \in S(z_0)$ if and only if

$$(B(z) - \sigma I_m)(z - z_0) = 0$$

for some σ. This will be true if $\sigma = \hat{\lambda}_k(z)$ for some $k \in \{1, ..., m\}$ and if $z - z_0$ is parallel to $\hat{r}_k(z)$, which is equivalent by orthogonality to

$$\hat{l}_l(z) \cdot (z - z_0) = 0$$

for all $l \neq k$. We want to use the Implicit Function Theorem to find a curve $\varphi_k(t)$ such that $\varphi_k(0) = z_0$ and

$$\hat{l}_l(\varphi_k(t)) \cdot (\varphi_k(t) - z_0) = 0$$

for all t in a neighbourhood of 0.
Set $\Phi_k : \mathbb{R}^m \to \mathbb{R}^{m-1}$,

$$\Phi_k(z) = (\hat{l}_1(z)\cdot(z-z_0), ..., \hat{l}_{k-1}(z)\cdot(z-z_0), \hat{l}_{k+1}(z)\cdot(z-z_0), ..., \hat{l}_m(z)\cdot(z-z_0)).$$

Clearly $\Phi_k(z_0) = 0$ and

$$D\Phi_k(z_0) = \begin{pmatrix} \hat{l}_1(z_0) \\ \vdots \\ \hat{l}_{k-1}(z_0) \\ \hat{l}_{k+1}(z_0) \\ \vdots \\ \hat{l}_m(z_0) \end{pmatrix} \in \mathbb{R}^{(m-1)\times m}.$$

By hyperbolicity, $D\Phi_k(z_0)$ has maximal rank, so that by the Implicit Function Theorem there exists indeed a smooth curve $\varphi_k : \mathbb{R} \to \mathbb{R}^m$ such that $\varphi_k(0) =$

z_0 and $\Phi_k(\varphi_k(t)) = 0$ for all t near 0. The path of φ_k is then defined to be $S_k(z_0)$.
Next, without loss of generality, we may assume $|\dot{\varphi}_k| = 1$. By construction we have

$$(\varphi_k(t) - z_0) \cdot \hat{l}_l = 0$$

for all $l \neq k$, so that

$$\varphi_k(t) = z_0 + \mu(t)\hat{r}_k(\varphi_k(t))$$

for smooth μ with $\mu(0) = 0$. Differentiate the equality w.r.t. t, we find

$$\dot{\varphi}_k(t) = \dot{\mu}(t)\hat{r}_k(\varphi_k(t)) + \mu(t)\frac{d}{dt}(\hat{r}_k\varphi_k(t))$$

hence

$$\dot{\varphi}_k(0) = \dot{\mu}(0)\hat{r}_k(z_0)$$

and, due to $|\dot{\varphi}_k| = 1 = |\hat{r}_k|$,

$$\dot{\varphi}_k(0) = \hat{r}_k(z_0).$$

This shows i).
Near $t = 0$

$$F(\varphi_k(t)) - F(z_0) = \sigma(\varphi_k(t), z_0)(\varphi_k(t) - z_0)$$

for some smooth $\sigma : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$. Differentiate this with respect to t

$$DF(\varphi_k(t))\dot{\varphi}_k(t) = \frac{d}{dt}\sigma(\varphi_k(t), z_0)(\varphi_k(t) - z_0) + \sigma(\varphi_k(t), z_0)\dot{\varphi}_k(t).$$

Set $t = 0$

$$DF(z_0) \cdot \underbrace{\dot{\varphi}_k(0)}_{=r_k(z_0)} = \sigma(z_0, z_0) \underbrace{\dot{\varphi}_k(0)}_{=r_k(z_0)}$$

whence $\sigma(z_0, z_0) = \lambda_k(z_0)$, thus proving ii).
For iii), set $\sigma(t) := \sigma(\varphi_k(t), z_0), \lambda_k(t) := \lambda_k(\varphi_k(t)), r_k(t) := r_k(\varphi_k(t))$.
Differentiate $F(\varphi_k(t)) - F(z_0) = \sigma(t)(\varphi_k(t) - z_0)$ twice:

$$\begin{aligned}&(D^2F(\varphi_k(t))\dot{\varphi}_k(t))\dot{\varphi}_k(t) + DF(\varphi_k(t))\ddot{\varphi}_k(t)\\ &= \ddot{\sigma}(t)(\varphi_k(t) - z_0) + 2\dot{\sigma}(t)\dot{\varphi}_k(t) + \sigma(t)\ddot{\varphi}_k(t)\end{aligned}$$

and set $t = 0$:

$$D^2F(z_0)r_k(z_0))r_k(z_0) + DF(z_0)\ddot{\varphi}_k(0) = 2\dot{\sigma}(0)r_k(z_0) + \lambda_k(z_0)\ddot{\varphi}_k(0)$$

or

$$(2\dot{\sigma}(0)I - D^2F(z_0)r_k(z_0))r_k(z_0) = (DF(z_0) - \lambda_k(z_0)I)\ddot{\varphi}_k(0). \tag{3.1}$$

Next, set $\psi_k(t)$ to be the solution of

$$\dot{\psi}_k(t) = r_k(\psi_k(t)), \psi_k(0) = z_0, |\dot{\psi}| = 1$$

(i.e. the unit speed parametrization of $R_k(z_0)$). Then $DF(\psi_k(t))r_k(t) = \lambda_k(t)r_k(t)$. Take the time derivative:

$$(D^2F(\psi_k(t))r_k(t))r_k(t) + DF(\psi_k(t))\dot{r}_k(t) = \dot{\lambda}_k(t)r_k(t) + \lambda_k(t)\dot{r}_k(t)$$

and set $t = 0$:

$$(D^2F(z_0)r_k(z_0))r_k(z_0) + DF(z_0)\dot{r}_k(0) = \dot{\lambda}_k(0)r_k(z_0) + \lambda_k(z_0)\dot{r}_k(0)$$

or

$$(D^2F(z_0)r_k(z_0) - \dot{\lambda}_k(0)I)r_k(z_0) = -(DF(z_0) - \lambda_k(z_0)I)\dot{r}_k(0). \tag{3.2}$$

Add (3.1) and (3.2):

$$(2\dot{\sigma}(0) - \dot{\lambda}_k(0))r_k(z_0)I = (DF(z_0) - \lambda_k(z_0)I)(\ddot{\varphi}_k(0) - \dot{r}_k(0)).$$

Multiply ("from the left") with $l_k(z_0)$:

$$l_k(z_0)r_k(z_0)(2\dot{\sigma}(0) - \dot{\lambda}_k(0)) = l_k(z_0)(DF(z_0) - \lambda_k(z_0)I)(\ddot{\varphi}_k(0) - \dot{r}_k(0)) = 0.$$

But $l_k(z_0) \cdot r_k(z_0) \neq 0$ since $l_l(z_0) \perp r_k(z_0)$ for all $l \neq k$ and $(l_k)_k, (r_k)_k$ form a basis of $\mathbb{R}^m$. Therefore

$$2\dot{\sigma}(0) - \dot{\lambda}_k(0) = 0.$$

But by Taylor's Theorem,

$$\begin{aligned} 2\sigma(t) &= 2\sigma(0) + 2\dot{\sigma}(0)t + \mathcal{O}(t^2) \\ &= \underbrace{2\sigma(0)}_{=2\lambda_k(z_0)} + \underbrace{\dot{\lambda}_k(0)t}_{=\lambda_k(t)-\lambda_k(0)+\mathcal{O}(t^2)} \\ &= \lambda_k(z_0) + \lambda_k(t) + \mathcal{O}(t^2) \end{aligned}$$

hence

$$\sigma(\underbrace{\varphi_k(t)}_{z}, z_0) = \sigma(t) = \frac{1}{2}(\lambda_k(z_0) + \lambda_k(\psi_k(t))) + \mathcal{O}(t^2).$$

The result now follows from

$$|\lambda_k(\psi_k(t)) - \lambda_k(\varphi_k(t))| = \mathcal{O}(t^2),$$

which also follows from Taylor's theorem as both functions agree up to order one by i). □

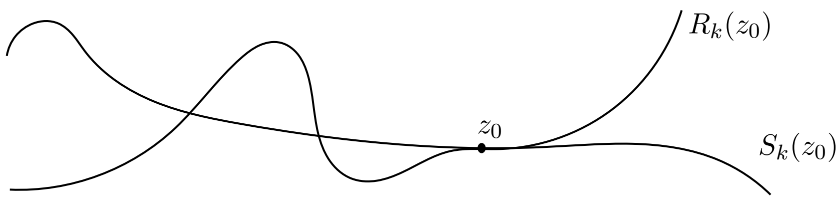


FIGURE 7

If (λ_k, r_k) is linearly degenerate, then $R_k(z_0)$ and $S_k(z_0)$ actually agree:

THEOREM 3.10. *Suppose for some $k \in \{1, ..., m\}$ that (λ_k, r_k) is linearly degenerate, then for all $z_0 \in \mathbb{R}^m$*

$$R_k(z_0) = S_k(z_0)$$

and

$$\sigma(z, z_0) = \lambda_k(z) = \lambda_k(z_0)$$

for all $z \in S_k(z_0)$.

PROOF. Let again v be the solution of

$$\dot{v}(s) = r_k(v(s)), v(0) = z_0,$$

then by linear degeneracy $s \mapsto \lambda_k(v(s))$ is constant ($D\lambda_k \cdot r_k = 0$), so that

$$\begin{aligned} F(v(s)) - F(z_0) &= \int_0^s DF(v(t))\dot{v}(t)\,dt \\ &= \int_0^s DF(v(t))r_k(v(t))\,dt \\ &= \int_0^s \lambda_k(v(t))r_k(v(t))\,dt \\ &= \lambda_k(z_0)\int_0^s \dot{v}(t)\,dt \\ &= \lambda_k(z_0)(v(s) - z_0). \end{aligned}$$

□

Let's use all this to solve Riemann's Problem. Let (λ_k, r_k) be linearly degenerate and suppose

$$u_r \in S_k(u_l).$$

Then set

$$u(x,t) = \begin{cases} u_l & : x < \sigma t \\ u_r & : x > \sigma t \end{cases}$$

for $\sigma = \sigma(u_r, u_l) = \lambda_k(u_l) = \lambda_k(u_r)$ (cf. Theorem 3.10).
In light of the Rankine-Hugoniot condition, this is a weak solution.
Interpret this in terms of characteristics: This is a travelling wave solution so, as we saw before, $\partial_x u$ is an eigenvector of $DF(u)$, so that

$$0 = \partial_t u + DF(u)\partial_x u = \partial_t u + \lambda_k(u)\partial_x u = \frac{d}{dt}u(x(t),t)$$

for

$$\dot{x}(t) = \lambda_k(\underbrace{u(x(t),t)}_{g(x_0)}).$$

So the characteristics are given by

$$x(t) = \begin{cases} x_0 + \lambda_k(u_l)t & : x_0 < 0 \\ x_0 + \lambda_k(u_r)t & : x_0 > 0 \end{cases}$$

but since $\lambda_k(u_l) = \lambda_k(u_r) = \sigma$, the characteristics are parallel to the discontinuity! cf. in the scalar case the transport equation

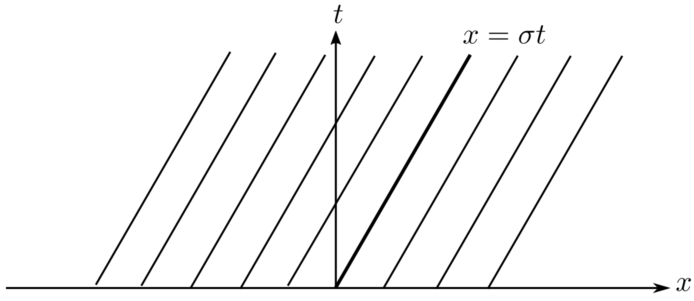


FIGURE 8

$$\partial_t u + \sigma \partial_x u = 0.$$

Next, assume (λ_k, r_k) is genuinely nonlinear and

$$u_r \in S_k(u_l).$$

Again,

$$u(x,t) = \begin{cases} u_l & : x < \sigma t \\ u_r & : x > \sigma t \end{cases}$$

is a weak solution by the Rankine-Hugoniot condition if $\sigma = \sigma(u_r, u_l)$.

Case 1: $\lambda_k(u_r) < \lambda_k(u_l)$. If u_r is sufficiently close to u_l, then by Theorem 3.9 we get

$$\lambda_k(u_r) < \sigma(u_r, u_l) < \lambda_k(u_l).$$

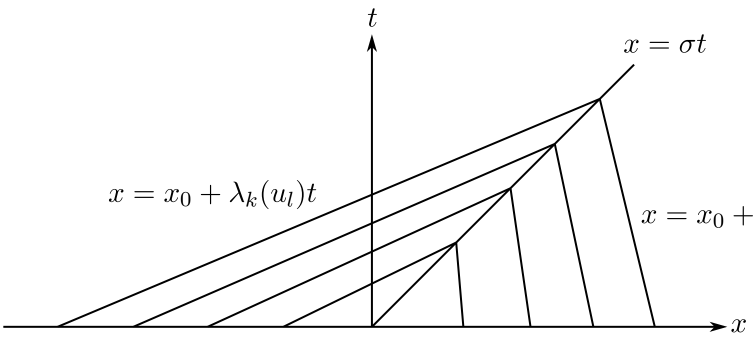


FIGURE 9. Shock formation (cf. Burgers' equation) - entropy solution

Case 2: $\lambda_k(u_r) > \lambda_k(u_l)$ so by Theorem 3.9 iii)

$$\lambda_k(u_l) < \sigma(u_r, u_l) < \lambda_k(u_r).$$

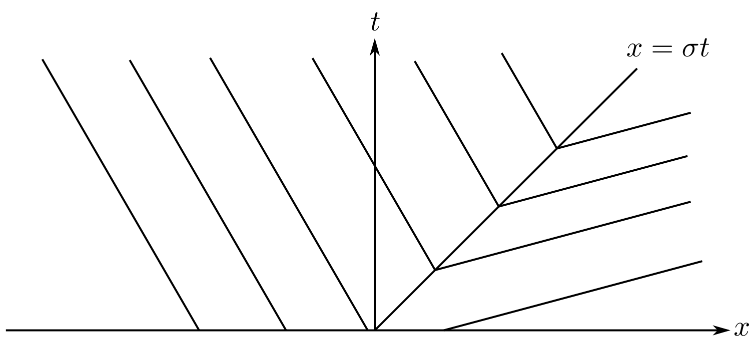


FIGURE 10. "non-physical shock" (cf. Burgers' equation) - not an entropy solution

We haven't yet defined entropy solutions for systems, but for the Riemann problem this definition is reasonable:

DEFINITION 3.11. Assume (λ_k, r_k) is genuinely nonlinear. We say (u_l, u_r) is *admissible* if

- $u_r \in S_k(u_l)$ and
- $\lambda_k(u_r) < \sigma(u_r, u_l) < \lambda_k(u_l)$ ("Lax entropy condition").

If (u_l, u_r) is admissible, then the corresponding solution is called a *k-shock wave*.

DEFINITION 3.12. Assume (λ_k, r_k) is genuinely nonlinear. Set

$$S_k^+(z_0) = \{z \in S_k(z_0) : \lambda_k(z_0) < \sigma(z, z_0) < \lambda_k(z)\}$$

and

$$S_k^-(z_0) = \{z \in S_k(z_0) : \lambda_k(z) < \sigma(z, z_0) < \lambda_k(z_0)\}.$$

Note that (since $D\lambda_k \cdot r_k > (<)0$) in a neighbourhood of z_0,

$$S_k(z_0) = S_k^+(z_0) \cup \{z_0\} \cup S_k^-(z_0).$$

Observe: (u_l, u_r) is admissible if and only if $u_r \in S_k^-(u_l)$.

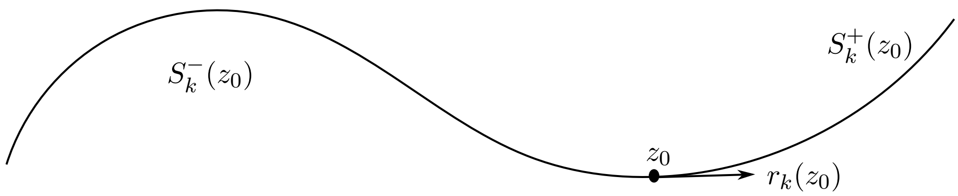


FIGURE 11

3.2.4. Local solution of Riemann's Problem.

DEFINITION 3.13. i) If (λ_k, r_k) is genuinely nonlinear, write

$$T_k(z_0) := R_k^+(z_0) \cup \{z_0\} \cup S_k^-(z_0).$$

ii) If (λ_k, r_k) is linearly degenerate, set

$$T_k(z_0) := R_k(z_0) = S_k(z_0).$$

By Theorem 3.9, $T_k(z_0)$ is of regularity C^1. So $u_r \in T_k(u_l)$ means that u_l and u_r can be joined by

- a k-rarefaction wave,
- a k-shockwave or
- a k-contact discontinuity.

THEOREM 3.14. *(Local solution of Riemann's Problem)*
Assume for each $k \in \{1, ..., m\}$ that (λ_k, r_k) is either genuinely nonlinear or linearly degenerate. Let $u_l \in \mathbb{R}^m$, then for each u_r sufficiently close to u_l there is a weak solution of Riemann's problem.

PROOF. For each $k \in \{1, ..., m\}$ let τ_k be a parameter measuring arclength on the curve T_k:
If $z, \tilde{z} \in T_k(z_0)$ then

$$\tau_k(\tilde{z}) - \tau_k(z) = \text{distance between } \tilde{z} \text{ and } z \text{ along } T_k(z_0).$$

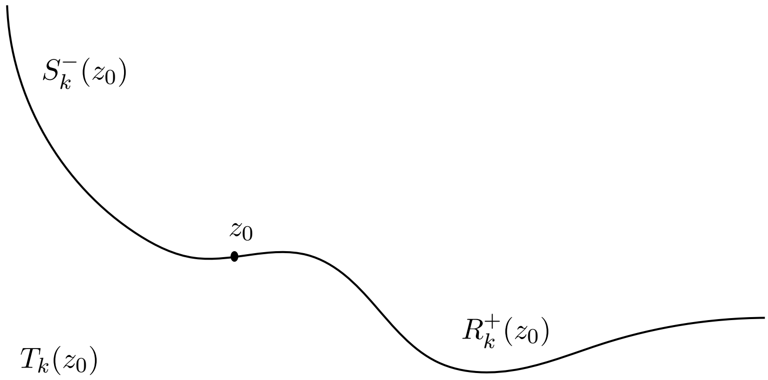


FIGURE 12

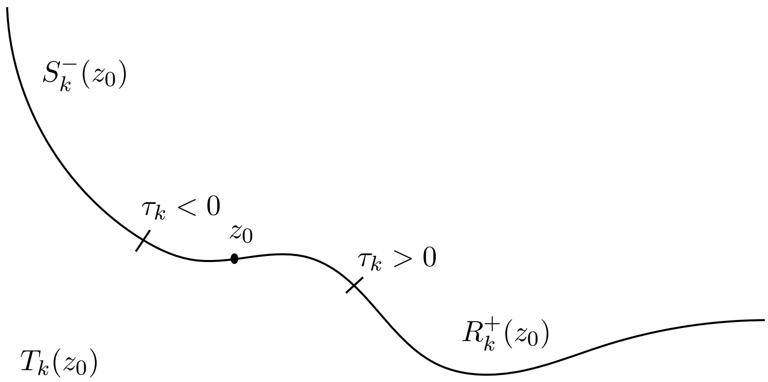


FIGURE 13

τ_k should **increase** in direction of $R_k^+(z_0)$. Set $u_l = z_0$. Want to connect u_l to u_r along curves T_k. To this end choose "intermediate states" in the following way:

$$\begin{aligned}
&z_1 \in T_1(z_0), \tau_1(z_1) - \tau_1(z_0) = t_1,\\
&z_2 \in T_2(z_1), \tau_2(z_2) - \tau_2(z_1) = t_2,\\
&\vdots\\
&z_m \in T_m(z_{m-1}), \tau_m(z_m) - \tau_m(z_{m-1}) = t_m.
\end{aligned}$$

This is well-defined for sufficiently small $t = (t_1, ..., t_m)$. Set $z = z_m$ and write $\Phi(t) = z$. Since T_k are C^1, also Φ is C^1 in a neighbourhood of 0.
We want to apply the Inverse Function Theorem, hence need to show $D\Phi(0)$ is nonsingular. Note that

$$\Phi(0, ..., t_k, ..., 0) - \Phi(0, ..., 0) = t_k r_k(z_0) + \mathcal{O}(t_k) \text{ as } (t_k \to 0)$$

since $T_k(z_0)$ is C^1. Hence

$$\frac{\partial \Phi}{\partial t_k}(0) = r_m(z_0)$$

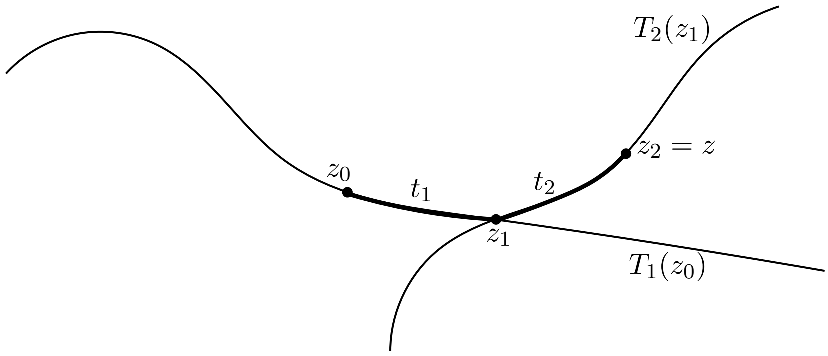


FIGURE 14

and therefore

$$D\Phi(0) = (r_1(z_0)|...|r_k(z_0)).$$

Since $\{r_k(z_0)\}_k$ forms a basis, $D\Phi(0)$ is indeed nonsingular, and hence in a neighbourhood of z_0 there is a unique $t = (t_1, .., t_m)$ s.t. $\Phi(t) = z$.
In particular, if u_l and u_r are sufficiently close, then there exists a unique t such that $\Phi(t) = u_r$.
Recall: if $z_k \in R_k^+(z_{k-1})$ then the corresponding rarefaction wave is

$$\begin{cases} z_{k-1} & : \frac{x}{t} < \lambda_k(z_{k-1}) \\ G_k\left(\frac{x}{t}\right) & : \lambda_k(z_{k-1}) < \frac{x}{t} < \lambda_k(z_k), G_k := (F_k')^{-1} \\ z_k & : \lambda_k(z_k) < \frac{x}{t}. \end{cases}$$

If $z_k \in S_k^-(z_{k-1})$ then the shock is

$$\begin{cases} z_{k-1} & : \frac{x}{t} < \sigma(z_k, z_{k-1}) \\ z_k & : \sigma(z_k, z_{k-1}) < \frac{x}{t}, \end{cases}$$

and similarly if (λ_k, r_k) is linearly degenerate. Note that $\lambda_1(z_0) < \lambda_2(z_0) < ... < \lambda_m(z_0)$, and by Theorem 3.9 iii)

$$\lambda_k(z_k) < \sigma(z_k, z_{k-1}) < \lambda_k(z_{k-1})$$

whenever $z_k \in S_k^-(z_{k-1})$ and therefore the rarefaction, shocks and/or contact discontinuities do not intersect.
A (Lax-)entropy solution is therefore given by "glueing" the various part together. □

EXAMPLE 3.15. i) Consider $m = 2, z_1 \in S_1^-(u_l), u_r \in S_2^-(z_1)$.
ii) Consider $m = 2, z_1 \in R_1^+(u_l), u_r \in S_2^-(z_1)$

REMARK 3.16. The solution of Riemann's problem can be used to prove existence of entropy solutions if $TV(g)$ (total variation of initial data) is small. Idea: Approximate g by piecewise constant data and solve Riemann's problem, then pass to the limit ("front-tracking").

3.3. Riemann Invariants (m=2)

We specialise to a system of two conservation laws, $m = 2$.

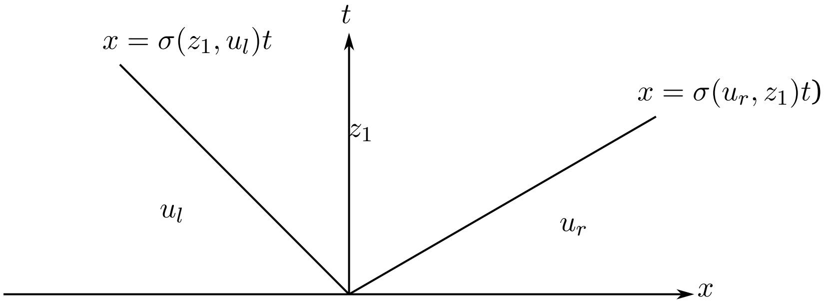


FIGURE 15. Example 3.15 i)

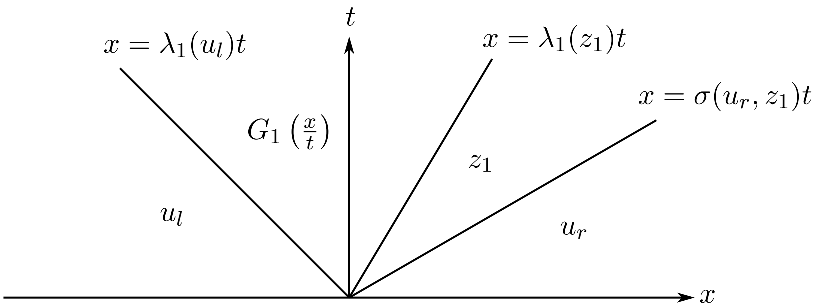


FIGURE 16. Example 3.15 ii)

DEFINITION 3.17. A function $w^i : \mathbb{R}^2 \to \mathbb{R}$ is an i-th *Riemann invariant* if it is constant along the rarefaction curve $R_i(z_0)$ for all $z_0 \in \mathbb{R}^2$.

Idea: Transform variables in phase space:

$$w(z) = (w^1(z_1, z_2), w^2(z_1, z_2)).$$

THEOREM 3.18. *Suppose* $u = (u^1, u^2)$ *is a smooth solution of*

$$\partial_t u + \partial_x F(u) = 0.$$

Set $v(x,t) = w(u(x,t))$. *Then*

$$\begin{aligned} \partial_t v^1 + \lambda_2(u)\partial_x v^1 &= 0 \\ \partial_t v^2 + \lambda_1(u)\partial_x v^2 &= 0 \end{aligned}$$

in $\mathbb{R} \times (0, \infty)$.

PROOF. Let $i \neq j, i, j \in \{1, 2\}$. Then

$$\begin{aligned} \partial_t v^i + \lambda_j(u)\partial_x v^i &= Dw^i(u) \cdot \partial_t u + \lambda_j(u) Dw^i(u) \cdot \partial_x(u) \\ &= Dw^i(u) \cdot (-\partial_x F(u) + \lambda_j(u) \cdot \partial_x(u)) \\ &= \underbrace{Dw^i(u)}_{||\, l_j(u)} \cdot (-DF(u) + \lambda_j(u) I)\partial_x u \\ &= 0. \end{aligned}$$

Indeed, w^i constant along R_k means $Dw^i(u) \cdot r_i(u) = 0$, hence $Dw^i(u) \,||\, l_j$. □

REMARK 3.19. The assumption $D\lambda_i(z)\cdot r_i(z) \neq 0$ of genuine nonlinearity can be rephrased in terms of Riemann invariants as (if Dw is non-singular)

$$\frac{\partial \lambda_i}{\partial w^j} \neq 0 \, (i \neq j).$$

Indeed: If $\frac{\partial \lambda_i}{\partial w^j} = 0$, then

$$0 = \frac{\partial \lambda_i}{\partial w^j} = \sum_{k=1}^{2} \frac{\partial \lambda_i}{\partial z_k} \frac{\partial z_k}{\partial w^j}.$$

But

$$\sum_{k=1}^{2} \frac{\partial w^i}{\partial z_k} \frac{\partial z_k}{\partial w^j} = \delta_{ij} = 0 \, (i \neq j),$$

it follows that $D\lambda_i \,||\, Dw^i \,||\, l_j$, hence $D\lambda_i \cdot r_i = 0$. Hence we have shown

$$D\lambda_i \cdot r_i = 0 \text{ if and only if } \frac{\partial \lambda_i}{\partial w^j} \neq 0.$$

EXAMPLE 3.20. (1-D isentropic Euler equations)
Consider

$$\begin{aligned} \partial_t \rho + \partial_x(\rho v) &= 0 \\ \partial_t(\rho v) + \partial_x(\rho v^2 + p(\rho)) &= 0 \end{aligned}$$

where $p = p(\rho)$ is smooth and $p'(\rho) > 0$ (this guarantees strict hyperbolicity).
Set $(u_1, u_2) = (\rho, \rho v)$ and recall that we get

$$\partial_t u + \partial_x F(u) = 0$$

for $F = (F_1, F_2) = (z_2, \frac{z_2^2}{z_1} + p(z_1))$ whenever $z_1 > 0$. Hence

$$DF = \begin{pmatrix} 0 & 1 \\ -\left(\frac{z_2}{z_1}\right)^2 + p'(z_1) & \frac{2z_2}{z_1} \end{pmatrix}.$$

We compute the eigenvalues

$$\lambda_1 = \frac{z_2}{z_1} - \sqrt{p'(z_1)}, \lambda_2 = \frac{z_2}{z_1} + \sqrt{p'(z_1)}.$$

Setting $v = \frac{z_2}{z_1}$ and $\sigma = \sqrt{p'(z_1)}$, this becomes

$$\lambda_{1/2} = v \mp \sigma.$$

Consider the characteristic ODE

i) $\dot{x}_1(t) = v(x_1(t), t) + \sigma(x_1(t), t) = \lambda_2(x_1(t), t)$
ii) $\dot{x}_2(t) = v(x_2(t), t) - \sigma(x_2(t), t) = \lambda_1(x_2(t), t)$,

where $\sigma(x, t) = \sqrt{p(\rho(x, t))}, t \geq 0$.
If w^i are Riemann invariants then $w^i(u)$ is constant along the trajectory of

i) and ii), respectively. Indeed,

$$\begin{aligned}
&\frac{d}{dt}w^1(u(x_1(t),t)) \\
&= Dw^1(u(x_1(t),t)) \cdot (\partial_t u(x_1(t),t) + \partial_x u(x_1(t),t)\dot{x}_1(t)) \\
&= Dw^1(u(x_1(t),t)) \cdot (-DF(u)\partial_x u + \lambda_2(u)\partial_x u) \\
&= 0
\end{aligned}$$

as before. Similarly for ii).
Next, write Euler as

iii) $\partial_t \rho + \rho \partial_x v + \partial_x \rho v = 0$,
iv) $\partial_t \rho v + \rho \partial_t v + \partial_x \rho v^2 + 2\rho v \partial_x v + \partial_x p(\rho) = 0$.

Multiplying iii) with $\sigma^2 = p'(\rho)$ gives

v) $\partial_t p(\rho) + \sigma^2 \rho \partial_x v + v \partial_x p(\rho) = 0$.

Multiplying iii) with v and subtracting from iv) yields

$$\rho \partial_t v + \rho v \partial_x v + \partial_x p(\rho) = 0.$$

Multiply this by σ and and add/subtract v):

$$\begin{aligned}
&\partial_t p(\rho) + (v+\sigma)\partial_x p(\rho) + \rho\sigma(\partial_t v + (v+\sigma)\partial_x v) = 0 \\
&\partial_t p(\rho) + (v-\sigma)\partial_x p(\rho) - \rho\sigma(\partial_t v + (v-\sigma)\partial_x v) = 0.
\end{aligned}$$

Recalling our ODE, this can be written as

$$\begin{aligned}
&\frac{d}{dt}(p\circ\rho)(x_1(t),t) + \rho(x_1(t),t)\sigma(x_1(t),t)\frac{d}{dt}v(x_1(t),t) = 0 \\
&\frac{d}{dt}(p\circ\rho)(x_2(t),t) - \rho(x_2(t),t)\sigma(x_2(t),t)\frac{d}{dt}v(x_2(t),t) = 0.
\end{aligned}$$

Keeping in mind $\frac{d(p\circ\rho)}{dt} = \sigma^2 \frac{d\rho}{dt}$, we get

$$\frac{\sigma}{\rho}\frac{d\rho}{dt} \pm \frac{dv}{dt} = 0$$

along the trajectories $(x_i(t),t)$. We have seen that $w^1(\rho, v)$ is constant along $(x_1(t),t)$, i.e.

$$\begin{aligned}
0 &= \frac{d}{dt}(w^1(\rho(x_1(t),t)), v(x_1(t),t)) \\
&= \frac{dw^1}{d\rho}\frac{d}{dt}\rho(x_1(t),t) + \frac{\partial w^1}{dv}\frac{d}{dt}v(x_1(t),t).
\end{aligned}$$

This is the case if

$$\frac{\partial w^1}{\partial \rho} = \frac{\sigma(\rho)}{\rho}, \frac{dw^1}{dv} = 1.$$

Similarly

$$\frac{\partial w^2}{\partial \rho} = \frac{\sigma(\rho)}{\rho}, \frac{dw^2}{\partial v} = -1.$$

Hence the Riemann invariants are

$$w^1 = \int_1^\rho \frac{\sigma(s)}{s}\, ds + v, w^2 = \int_1^\rho \frac{\sigma(s)}{s}\, ds - v.$$

THEOREM 3.21. *Consider the IVP*

$$\partial_t u + \partial_x F(u) = 0$$
$$u(x,0) = g(x)$$

(two equations) with $g \in C_c^\infty(\mathbb{R})$. Suppose further

$$\frac{\partial \lambda_i}{\partial w^j} \geq c > 0$$

for some constant c, $i,j \in \{1,2\}, i \neq j$ ("strong genuine nonlinearity").
If $\partial_x v^1$ or $\partial_x v^2 < 0$ somewhere on $\mathbb{R} \times \{t=0\}$, then u blows up in finite time (i.e. there is no smooth solution u for all times $t \geq 0$.)

PROOF. Suppose u is a smooth solution. Set $a = \partial_x v^1, b = \partial_x v^2$, where $v = w(u)$ solves

$$\begin{aligned} \partial_t v^1 + \lambda_2(u)\partial_x v^1 &= 0 \\ \partial_t v^2 + \lambda_1(u)\partial_x v^2 &= 0. \end{aligned} \tag{3.3}$$

Note: v^1 is constant along the curve $(x_1(s), s)$, where

$$\dot{x}_1(s) = \lambda_2(u(x_1(s), s)), x_1(0) = x_0$$

and v^2 is constant along the curve $(x_2(s), s)$,

$$\dot{x}_2(s) = \lambda_1(u(x_2(s), s)), x_2(0) = x_0.$$

Since u is smooth, the characteristics cover $\mathbb{R}^2$ and in particular v is **bounded**. Differentiate (3.3) with respect to x:

$$\partial_t a + \lambda_2 \partial_x a + \frac{\partial \lambda_2}{\partial w_1} a^2 + \frac{\partial \lambda_2}{\partial w_2} ab = 0$$

and write the second equation of (3.3) as

$$\partial_t v^2 + \lambda_2 \partial_x v^2 = (\lambda_2 - \lambda_1) b.$$

Combining these gives

$$\partial_t a + \lambda_2 \partial_x a + \frac{\partial \lambda_2}{\partial w_1} a^2 + \left(\frac{1}{\lambda_2 - \lambda_1} \frac{\partial \lambda_2}{\partial w_2} (\partial_t v^2 + \lambda_2 \partial_x v^2) \right) a = 0. \tag{3.4}$$

Next, set

$$\xi(t) = \exp\left(\int_0^t \frac{1}{\lambda_2 - \lambda_1} \frac{\partial \lambda_2}{\partial w_2} (\partial_t v^2 + \lambda_2 \partial_x v^2)(x_1(s), s) \, ds \right).$$

Set also

$$\gamma(\mu) = \int_0^\mu \left(\frac{1}{\lambda_2 - \lambda_1} \frac{\partial \lambda_2}{\partial w_2} \right) (v_0^1, v^2) \, dv^2,$$

where $v_0^1 := v^1(x_0, 0)$. Then,

$$\begin{aligned} &\frac{d}{ds} \gamma(v^2(x_1(s), s)) \\ &= \frac{1}{\lambda_2 - \lambda_1} \frac{\partial \lambda_2}{\partial w_2} (v_0^1, v^2(x_1(s), s)) \cdot (\partial_t v^2 + \lambda_2 \partial_x v^2) \\ &= \frac{1}{\lambda_2 - \lambda_1} \frac{\partial \lambda_2}{\partial w_2} (v^1, v^2)(\partial_t v^2 + \lambda_2 \partial_x v^2)(x_1(s), s), \end{aligned}$$

where we used that v^1 is constant along $(x_1(s), s)$. Hence

$$\begin{aligned}\xi(t) &= \exp\left(\int_0^t \frac{d}{ds}\gamma(v^2(x_1(s), s))\, ds\right)\\ &= \exp(\gamma(v^2(x_1(t), t)) - \gamma(v^2(x_0, 0))).\end{aligned}$$

Since v^2 is bounded, it follows

$$0 < m \le \xi < M < \infty$$

for all times $t \ge 0$. Next, set $\alpha(t) = a(x_1(t), t)$ and compute

$$\begin{aligned}\frac{d}{dt}(\xi\alpha)^{-1} &= -\frac{1}{(\xi\alpha)^2}\frac{d}{dt}(\xi\alpha)\\ &= -\frac{1}{(\xi\alpha)^2}\left(\xi\alpha\frac{1}{\lambda_2 - \lambda_1}\frac{\partial\lambda_2}{\partial w_2}(\partial_t v^2 + \lambda_2 v^2) + \xi(\partial_t a + \lambda_2\partial_x a)\right)\\ &\underbrace{=}_{(3.4)} -\frac{1}{\xi\alpha^2}\left(-\frac{\partial\lambda_2}{\partial w_1}\alpha^2\right)\\ &= \frac{1}{\xi}\frac{\partial\lambda_2}{\partial w_1},\end{aligned}$$

so that

$$(\xi\alpha)^{-1}(t) = (\xi\alpha)^{-1}(0) + \int_0^t \frac{1}{\xi(s)}\frac{\partial\lambda_2}{\partial w_1}(s)\, ds.$$

Note $\xi(0) = 1$, so

$$\alpha(t) = \xi^{-1}(t)\left(\alpha(0)^{-1} + \int_0^t \frac{1}{\xi(s)}\frac{\partial\lambda_2}{\partial w_1}(s)\, ds\right)^{-1}$$

and

$$\alpha(t) = \alpha(0)\xi^{-1}(t)\left(1 + \alpha(0)\int_0^t \frac{1}{\xi(s)}\frac{\partial\lambda_2}{\partial w_1}(s)\, ds\right)^{-1}.$$

Since $\frac{\partial\lambda_2}{\partial w_1} \ge c > 0$ and $\xi \ge m > 0$, this is well-defined for all t if $\alpha(0) \ge 0$. But if $\alpha(0) < 0$, there exists a finite time T such that

$$\lim_{t\nearrow T}\alpha(t) = \infty,$$

so that the solution blows up at time T. But $\alpha(0) = \partial_x v^1(x_0, 0)$ hence $\partial_x v^1(x_0, 0) < 0$ for some x_0 is sufficient for blow-up. A similar argument works for v^2. □

3.4. More on entropy conditions

3.4.1. The Lax condition and viscosity. Recall Lax's entropy condition for shock waves:

$$\lambda_k(u_r) < \sigma(u_r, u_l) < \lambda_k(u_l),$$

if (λ_k, r_k) genuinely nonlinear. Find more general conditions.
Recall the viscosity limit: A reasonable solution should arise as the limit $\varepsilon \searrow 0$ of

$$\partial_t u_\varepsilon + \partial_x F(u_\varepsilon) = \varepsilon\partial_{xx}u_\varepsilon. \tag{3.5}$$

Consider solutions of the special form

$$u_\varepsilon(x,t) = v\left(\frac{x-\sigma t}{\varepsilon}\right) \text{ (consistent with scaling!)}$$

and seek the profile v and speed σ. Insert this into (3.5):

$$-\sigma v'\left(\frac{x-\sigma t}{\varepsilon}\right)\frac{1}{\varepsilon} + DF(v)v'\left(\frac{x-\sigma t}{\varepsilon}\right)\frac{1}{\varepsilon} = \frac{1}{\varepsilon}v''\left(\frac{x-\sigma t}{\varepsilon}\right),$$

i.e.

$$-\sigma v' + DF(v)v' = v''.$$

Suppose $u_l, u_r \in \mathbb{R}^m$ are given and

$$\lim_{s\to-\infty} v(s) = u_l,\ \lim_{s\to\infty} v(s) = u_r,\ \lim_{s\to\pm\infty} \dot v = 0,$$

then

$$\lim_{\varepsilon\searrow 0} u_\varepsilon(x,t) = \begin{cases} u_l & : x < \sigma t \\ u_r & : x > \sigma t. \end{cases}$$

so the viscosity limit is a shock wave connecting u_l and u_r. Try to solve

$$-\sigma\dot v + DF(v)\dot v = \ddot v.$$

Integrate to obtain

$$\dot v F(v) - \sigma v + c,$$

$c \in \mathbb{R}^m$ constant. Taking the limits $s \to \pm\infty$ we see

$$F(u_l) - \sigma u_l + c = 0,$$
$$F(u_r) - \sigma u_r + c = 0,$$

hence

$$F(u_l) - F(u_r) = \sigma(u_l - u_r) \text{ (Rankine-Hugoniot).}$$

Determining the constant via $s \to -\infty$, in particular, gives $c = \sigma u_l - F(u_l)$, hence our ODE reads

$$\dot v = F(v) - F(u_l) - \sigma(v - u_l).$$

From Rankine-Hugoniot we see moreover that, for given u_l, and u_r close by, necessarily $u_r \in S_k(u_l)$ for some k and $\sigma = \sigma(u_r, u_l)$. Even more, we have:

THEOREM 3.22. *If there is a travelling wave solution of (3.5) connecting u_l to u_r sufficiently close, then $u_r \in S_k^-(u_l)$.*

PROOF. We have already seen $u_r \in S_k(u_l)$ for some k and $\sigma = \sigma(u_r, u_l)$. Set

$$G(z) := F(z) - F(u_l) - \sigma(z - u_l),$$

so that our ODE becomes

$$\dot v = G(v).$$

Moreover, by Rankine-Hugoniot we have

$$G(u_l) = G(u_r) = 0,$$

as well as

$$DG(z) = DF(z) - \sigma I$$

and so

$$DG(u_l) = DF(u_l) - \sigma I.$$

Hence the eigenvalues of $DG(u_l)$ are $\{\lambda_k(u_l) - \sigma\}_{k=1}^m$ with left and right eigenvectors $\{l_k(u_l)\}, \{r_k(u_l)\}$.
But as $u_r \in S_k(u_l)$ and $|u_r - u_l| \ll 1$, we have by Theorem 3.9iii)

$$\sigma = \frac{\lambda_k(u_r) + \lambda_k(u_l)}{2} + \mathcal{O}(|u_l - u_r|^2)$$

and thus

$$\lambda_k(u_l) - \sigma = \frac{\lambda_k(u_l) - \lambda_k(u_r)}{2} + \mathcal{O}(|u_r - u_l|^2).$$

We argue that $\lambda_k(u_l) - \sigma$ has to be strictly positive. Indeed, u_l is an equilibrium point of $\dot{v} = G(v)$, as $G(u_l) = 0$. If u_r is close to u_l. then $u_r - u_l$ is almost parallel to $r_k(u_l)$, since $u_r \in S_k(u_l)$. But if a trajectory of $\dot{v} = G(v)$ leaves the equilibrium at u_l in the direction r_k, then by standard ODE theory (linearised stability!) the eigenvalue $\lambda_k(u_l) - \sigma$ of $DG(u_l)$ corresponding to $r_k(u_l)$ has to be positive.
But if u_r is sufficiently close to u_l, $\lambda_k(u_l) - \sigma > 0$ implies $\lambda_k(u_l) > \lambda_k(u_r)$, hence $u_r \in S_k^-(u_l)$. □

REMARK 3.23. i) The converse statement holds (if (λ_k, r_k) is genuinely nonlinear): If $u_r \in S_k^-(u_l)$ for some k, there exists a travelling wave solution of (3.5).

ii) This justifies Lax's entropy condition: A shock wave arises from a viscosity limit if and only if Lax's entropy condition is satisfied.

3.4.2. Liu's Condition: An Example.

If (λ_k, r_k) is not genuinely nonlinear, then $u_r \in S_k^-(u_l)$ does not make sense.

DEFINITION 3.24. Let $u_r \in S_k(u_l)$ for some k. Then (u_r, u_l) satisfies (Tai-Ping) *Liu's entropy condition* if

$$\sigma(z, u_l) > \sigma(u_r, u_l)$$

for each z on the curve $S_k(u_l)$ between u_l and u_r.

THEOREM 3.25. *If (λ_k, r_k) is genuinely nonlinear, $u_r \in S_k^-(u_l)$, and u_r close to u_l, then Liu's entropy condition is equivalent to Lax' entropy condition.*

PROOF. *Liu ⇒ Lax:* Let $z = u_l$, then by Theorem 3.9 ii), $\sigma(u_l, u_l) = \lambda_k(u_l)$, and by Theorem 3.9 iii), $\sigma(u_l, u_r)$ is strictly between $\lambda_k(u_l)$ and $\lambda_k(u_r)$.
Liu's condition implies $\lambda_k(u_l) > \sigma(u_r, u_l)$ and it follows

$$\lambda_k(u_l) > \sigma(u_r, u_l) > \lambda_k(u_r),$$

i.e. Lax' condition.
Lax ⇒ Liu: Let $\lambda_k(u_r) < \sigma(u_r, u_l) < \lambda_k(u_l)$ and $z \in S_k^-(u_l)$ between u_r and u_l. Then by genuine nonlinearity and Theorem 3.9 iii)

$$\sigma(z, u_l) = \frac{\lambda_k(z) + \lambda_k(u_l)}{2} + \mathcal{O}(|z - u_l|^2)$$

is strictly decreasing along $S_k(u_l)^-$, hence

$$\sigma(z, u_l) > \sigma(u_r, u_l),$$

i.e. Liu's condition. □

THEOREM 3.26. *(without proof)*
If u_r is sufficiently close to u_l, then there exists a travelling wave solution to (3.5) if and only if the Liu entropy condition is satisfied.

EXAMPLE 3.27. (p-system) Recall

$$\begin{aligned}\partial_t u_1 - \partial_x u_2 &= 0\\ \partial_t u_2 - \partial_x p(u_1) &= 0,\end{aligned}$$

which is strictly hyperbolic if and only if $p' > 0$.
Two conceivable ways to add viscosity: (3.5) would yield

$$\begin{aligned}\partial_t u_1^\varepsilon - \partial_x u_2^\varepsilon &= \varepsilon \partial_{xx} u_1\\ \partial_t u_2^\varepsilon - \partial_x p(u_1^\varepsilon) &= \varepsilon \partial_{xx} u_2\end{aligned}$$

"artifical viscosity" - no physical meaning. Or

$$\begin{aligned}\partial_t u_1^\varepsilon - \partial_x u_2^\varepsilon &= 0\\ \partial_t u_2^\varepsilon - \partial_x p(u_1^\varepsilon) &= \varepsilon \partial_{xx} u_2\end{aligned} \tag{3.6}$$

"physical viscosity". Let's go for (3.6).
Assume $u^\varepsilon = v\left(\frac{x-\sigma t}{\varepsilon}\right)$ is a smooth solution with

$$\lim_{s\to-\infty} v(s) = u_l, \lim_{s\to\infty} v(s) = u_r, \lim_{s\to\pm\infty} \dot{v}(s) = 0.$$

From (3.6) we get

$$\begin{aligned}-\sigma\dot{v}_1 - \dot{v}_2 &= 0\\ -\sigma\dot{v}_2 - p(\dot{v_1}) &= \ddot{v}_2\end{aligned}$$

Integrating gives

$$\begin{aligned}\sigma v_1 + v_2 = \sigma u_l^1 + u_l^2 &= \sigma u_r^1 + u_r^2 \text{ and}\\ \dot{v}^2 &= \sigma(u_l^2 - v^2) + p(u_l^1) - p(v^1)\\ &= \sigma(u_r^2 - v^2) + p(u_r^1) - p(v^1).\end{aligned} \tag{3.7}$$

It follows that

$$\begin{aligned}\sigma u_l^1 + u_l^2 &= \sigma u_r^1 + u_r^2 \text{ and}\\ \sigma u_l^2 + p(u_l^1) &= \sigma u_r^2 + p(u_r^1).\end{aligned}$$

Solve for σ :

$$\sigma^2(u_r^1 - u_l^1) = \sigma(u_l^2 - u_r^2) = p(u_r^1) - p(u_l^1),$$

hence

$$\sigma^2 = \frac{p(u_r^1) - p(u_l^1)}{u_r^1 - u_l^1} > 0,$$

since $p' > 0$. Take $\sigma > 0$. Then Liu's condition becomes

$$\frac{p(z_1) - p(u_l^1)}{z_1 - u_l'} > \frac{p(u_r^1) - p(u_l^1)}{u_r' - u_l'}$$

for every $z \in S_k(u_l)$ between u_l and u_r.
Consider again (3.7). Eliminate v^2:

$$\dot{v}^1 = -\frac{\dot{v}_2}{\sigma} = \frac{p(v^1) - p(u_l^1)}{\sigma} - \underbrace{(u_l^2 - v^2)}_{\sigma(v^1 - u_l^1)} =: g(v^1).$$

But $g(u_l^1) = 0$ and also $g(u_r^1) = 0$ by our formula for σ.
Suppose now $u_r^1 > u_l^1$.

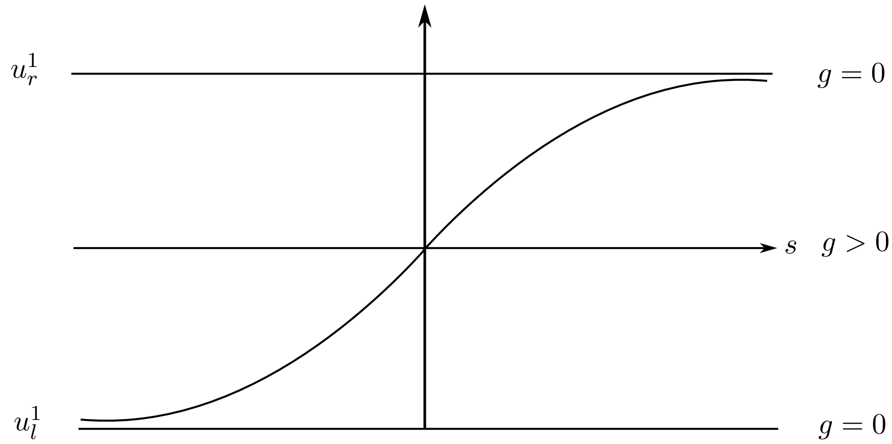


FIGURE 17

In order for $\dot{v} = g(v)$ to have a solution connecting u_l^1 and u_r^2, we need $g(z^1) > 0$ for z^1 between u_l^1 and u_r^2, i.e.

$$p(z^1) - p(u_l^1) > \sigma^2(z^1 - u_l^1) = \frac{p(u_r^1) - p(u_l^1)}{u_r^1 - u_l^1}(z^1 - u_l^1),$$

i.e. Liu's condition.
But note that, for u_l and u_r sufficiently close, $u_r \in S_k(u_l)$ implies that

$$u_r^1 > z^1 > u_l^1$$

for every $z \in S_k(u_l)$ between u_l and u_r (since $r_k(u_l)$ is not parallel to $(0,1)$), hence Liu's condition is equivalent to the existence of the travelling wave.
If $u_r^1 < u_l^1$ then a similar argument works (taking $\sigma < 0$).

3.4.3. Entropy / Entropy-flux pairs.

DEFINITION 3.28. Two smooth functions $\eta, q : \mathbb{R}^m \to \mathbb{R}$ are called an entropy / entropy-flux pair if η is convex and

$$D\eta DF = Dq.$$

This can be motivated as in the scalar case: If $\partial_t u + \partial_x F(u) = 0$ for smooth u, then multiplication with $D\eta(u)$ gives

$$\partial_t \eta(u) + \underbrace{D\eta(u) DF(u)}_{Dq(u)} \partial_x u = 0$$

so $\partial_t \eta(u) + \partial_x q(u) = 0$.

DEFINITION 3.29. A function $u \in L^\infty(\mathbb{R} \times (0, \infty))$ is an *entropy solution* of

$$\partial_t u + \partial_x F(u) = 0$$

with initial data $g \in L^\infty$ if

$$\partial_t \eta(u) + \partial_x q(u) \leq 0$$

in the sense of distributions, i.e.

$$\int_0^\infty \int_{-\infty}^\infty \partial_t \varphi \eta(u) + \partial_x \varphi q(u)\, dx dt + \int_{-\infty}^\infty \varphi(x, 0) g(x)\, dx \geq 0$$

for all $\varphi \in C_c^\infty(\mathbb{R} \times [0, \infty)), \varphi \geq 0$, and all entropy/entropy-flux pairs. Without reference to an initial condition, u is an entropy solution if

$$\int_0^\infty \int_{-\infty}^\infty \partial_t \varphi \eta(u) + \partial_x \varphi q(u)\, dx dt \geq 0$$

for all $\varphi \in C_c^\infty(\mathbb{R} \times (0, \infty)), \varphi \geq 0$, and all entropy/entropy-flux pairs.

THEOREM 3.30. *Suppose* (u_ε) *is a sequence of smooth solutions of*

$$\partial_t u_\varepsilon + \partial_x F(u_\varepsilon) = \varepsilon \partial_{xx} u_\varepsilon,$$

bounded in L^∞*, and such that*

$$u_\varepsilon \to u$$

pointwise a.e. Then u *is an entropy solution of*

$$\partial_t u + \partial_x F(u) = 0.$$

PROOF. We have

$$\begin{aligned} \partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon) &= \varepsilon D\eta(u_\varepsilon) \partial_{xx} u_\varepsilon \\ &= \varepsilon \partial_{xx} \eta(u_\varepsilon) - \underbrace{\varepsilon (D^2 \eta(u_\varepsilon)(\partial_x u_\varepsilon) \partial_x u_\varepsilon)}_{\geq 0}. \end{aligned}$$

Let now $\varphi \in C_c^\infty(\mathbb{R} \times (0, \infty)), \varphi \geq 0$. Then

$$\iint \partial_t \varphi \eta(u_\varepsilon) + \partial_x \varphi q(u_\varepsilon)\, dx dt = -\varepsilon \iint \partial_{xx} \varphi \eta(u_\varepsilon)\, dx dt + \varepsilon \iint \varphi D^2 \eta(u_\varepsilon) \partial_x u_\varepsilon \partial_x u_\varepsilon\, dx dt.$$

But by dominated convergence, since

$$\varepsilon \partial_{xx} \varphi \eta(u_\varepsilon) \to 0$$

a.e. and a dominating function is given by

$$\|\varphi\|_{C^2} \chi_{\operatorname{supp} \varphi} \sup_{x, \varepsilon} |\eta(u_\varepsilon)|,$$

we conclude

$$\iint \partial_t \varphi \eta(u) + \partial_x \varphi q(u)\, dx dt \geq 0.$$

□

COROLLARY 3.31. *A solution of the form*

$$u(x,t) = \begin{cases} u_l & : x < \sigma t \\ u_r & : x > \sigma t \end{cases}$$

that satisfies Lax's or Liu's entropy condition is also an entropy solution in this sense.

REMARK 3.32. The converse is not true in general: Consider the linearly degenerate "system"

$$\partial_t u_1 + \partial_x u_1 = 0$$
$$\partial_t u_2 + 2\partial_x u_2 = 0$$

and $g(x) = (g_1(x), g_2(x))$ for

$$g_1(x) = 0, g_2(x) = \begin{cases} u_l & : x < 0 \\ u_r & : x > 0, \end{cases}$$

so $(0, u_r) \in S_2(0, u_l)$. Then the solution (contact discontinuity) is given by

$$u_1(x,t) = 0, u_2(x,t) = \begin{cases} 0 & : x < 2t \\ u_r & : x > 2t \end{cases}$$

but $\sigma(z, u_l)$ is constant in z, so that Liu's condition is not satisfied. On the other hand it is not difficult to see that $\partial_t \eta(u) + \partial_x q(u) = 0$ for every entropy/entropy-flux pair (exercise).

Warning: Unlike in the scalar case, it may be difficult to find any entropy / entropy-flux pair at all! In particular for the choice $\eta = |\cdot|$ there may not be a corresponding entropy flux (so that the proof of uniqueness and compensated compactness are not transferable to systems).

EXAMPLE 3.33. Consider again the p-system, for which

$$DF(z) = \begin{pmatrix} 0 & -1 \\ -p'(z_1) & 0 \end{pmatrix}.$$

We claim that

$$\eta(z) = \frac{z_2^2}{2} + \int_0^{z_1} p(w)\, dw$$

is an entropy with corresponding flux

$$q(z) = -p(z_1)z_2.$$

Indeed, η is convex since $p' > 0$ and

$$Dq(z) = \begin{pmatrix} -p'(z_1)z_2 \\ -p(z_1) \end{pmatrix}$$

whereas

$$D\eta(z)DF(z) = (p(z_1), z_2) \begin{pmatrix} 0 & -1 \\ -p'(z_1) & 0 \end{pmatrix} = (-p'(z_1)z_2, -p(z_1)).$$

EXAMPLE 3.34. For the isentropic Euler equations, an entropy is given by

$$\eta(\rho, v) = \frac{1}{2}\rho v^2 + P(\rho),$$

where

$$P(\rho) = \rho \int_1^\rho \frac{\rho(r)}{r^2}\, dr$$

is the *pressure potential*. The corresponding flux ist

$$q(\rho, v) = \left(\frac{1}{2}\rho v^2 + p(\rho) + P(\rho)\right) v.$$

Physically, η is interpreted as the *energy*.

We will use the following entropies for systems of two equations:

LEMMA 3.35. *(Lax)*
Let $m = 2$ and $w = (w_1, w_2)$ be a Riemann invariant for

$$\partial_t u + \partial_x F(u) = 0 \,(Dw \neq 0).$$

There exists, for each $k \in \mathbb{Z}$ with $|k|$ sufficiently large, entropy /entropy-flux pairs (η^k, q^k) of the asymptotic forms

$$\eta^k(w) = e^{kw_1}\left(A_0(w) \pm \frac{1}{k}A_1(w) + \mathcal{O}\left(\frac{1}{k^2}\right)\right)$$
$$q^k(w) = e^{kw_1}\left(B_0(w) \pm \frac{1}{k}B_1(w) + \mathcal{O}\left(\frac{1}{k^2}\right)\right).$$

such that $A_0 > 0$ and A_0, A_1, B_0, B_1 are smooth and independent of k.

3.5. Compensated Compactness for Systems of Two Equations

Consider the system

$$\partial_t u_\varepsilon + \partial_x F(u_\varepsilon) = \varepsilon \partial_{xx} u_\varepsilon$$

and assume u_ε is smooth in $\mathbb{R} \times (0, \infty)$ and satisfies

$$\sup_\varepsilon \|u_\varepsilon\|_{L^\infty} < \infty$$

as well as $\sup_\varepsilon \|\sqrt{\varepsilon}\partial_x u_\varepsilon\|_{L^2} < \infty$ (satisfied e.g. for Euler).

THEOREM 3.36. *Under the stated assumptions, and if (λ_1, r_1) and (λ_2, r_2) are genuinely nonlinear, there is a subsequence (u_{ε_k}) which converges pointwise to an entropy solution*

$$\partial_t u + \partial_x F(u) = 0.$$

PROOF. By Theorem 2.32, there exists a subsequence (u_{ε_k}) generating a Young measure (ν_{x_t}), i.e.

$$f(u_{\varepsilon_k}) \overset{*}{\rightharpoonup} \int_{\mathbb{R}} f(z)\, d\nu_{x,t}(z) \text{ in } L^\infty(\mathbb{R} \times (0, \infty))$$

for every $f \in C(\mathbb{R}^2)$.
Let $(\eta_1, q_1), (\eta_2, q_2)$ be two entropy / entropy-flux pairs, and set

$$v_k := (q_2(u_{\varepsilon_k}), \eta_2(u_{\varepsilon_k})),$$
$$w_k := (\eta_1(u_{\varepsilon_k}), -q_1(u_{\varepsilon_k})).$$

Setting $f = \eta_1 q_2 - \eta_2 q_1$ in Theorem 2.32, we get

$$v_k \cdot w_k \overset{*}{\rightharpoonup} \int (\eta_1 q_2 - \eta_2 q_1)(z)\, d\nu_{x,t}(z) =: \langle \nu_{x,t}, \eta_1 q_2 - \eta_2 q_1 \rangle.$$

Next, apply the div-curl-lemma to v_k, w_k. To this end, note that

$$\begin{aligned} \operatorname{div} v_k &= \partial_t \eta_2(u_{\varepsilon_k}) + \partial_x q_2(u_{\varepsilon_k})) \\ \operatorname{curl} w_k &= -(\partial_t \eta_1(u_{\varepsilon_k}) + \partial_x q_1(u_{\varepsilon_k})). \end{aligned}$$

But multiply $\partial_t u_{\varepsilon_k} + \partial_x F(u_{\varepsilon_k}) = \varepsilon \partial_{xx} u_{\varepsilon_k}$ by $D\eta_1(u_{\varepsilon_k})$ to find

$$\begin{aligned} \partial_t \eta_1(u_{\varepsilon_k}) + \partial_x q_1(u_{\varepsilon_k}) &= D\eta_1(u_{\varepsilon_k}) \partial_{xx} u_{\varepsilon_k} \\ &= \varepsilon \partial_{xx} \eta_1(u_{\varepsilon_k}) - \varepsilon (D^2 \eta_1(u_{\varepsilon_k}) \partial_x u_{\varepsilon_k}) \cdot \partial_x u_{\varepsilon_k}. \end{aligned}$$

We need to show that this precompact in $W^{-1,2}_{loc}$. To this end note that $\varepsilon \eta_1'(u_{\varepsilon_k}) \partial_x u_{\varepsilon_k}$ is precompact in L^2, because it equals $\sqrt{\varepsilon} \underbrace{\sqrt{\varepsilon} \partial_x u_{\varepsilon_k}}_{\text{bounded in } L^2} \underbrace{\eta_1'(u_{\varepsilon_k})}_{\text{bounded in } L^\infty}$.
Since this quantity equals $\varepsilon D_x \eta_1'(u_{\varepsilon_k})$, it follows that $\varepsilon \partial_{xx} \eta_1(u_{\varepsilon_k})$ is precompact in $W^{-1,2}(\mathbb{R} \times (0, \infty))$.
Moreover, $\underbrace{\varepsilon |\partial_x u_{\varepsilon_k}|^2}_{\text{bounded in } L^1} \underbrace{|D^2 \eta_1(u_{\varepsilon_k})|}_{\text{bounded in } L^\infty}$ is bounded in $L^1(\mathbb{R} \times (0, \infty))$, hence in $\mathcal{M}(\mathbb{R} \times (0, \infty))$, so Corollary 2.30 applies and gives the derived precompactness in $W^{-1,2}_{loc}(\mathbb{R} \times (0, \infty))$. We conclude

$$\begin{aligned} v_k \cdot w_k &\overset{*}{\rightharpoonup} \text{w*-lim } v_k \cdot \text{w*-lim } w_k \\ &= \langle \nu, \eta_1 \rangle \langle \nu, q_2 \rangle - \langle \nu, \eta_2 \rangle \langle \nu, q_1 \rangle \end{aligned}$$

and hence

$$\langle \nu, \eta_1 q_2 - \eta_2 q_1 \rangle = \langle \nu, \eta_1 \rangle \langle \nu, q_2 \rangle - \langle \nu, \eta_2 \rangle \langle \nu, q_1 \rangle$$

almost everywhere. We want to show $\nu_{x,t}$ is a Dirac measure a.e. First, prove the following

LEMMA 3.37. *If η, q is an entropy/entropy-flux pair, viewed as a function of $w = (w_1, w_2)$, then*

$$\frac{\partial q}{\partial w_1} = \lambda_2 \frac{\partial \eta}{\partial w_1}, \frac{\partial q}{\partial w_2} = \lambda_1 \frac{\partial \eta}{\partial w_2}.$$

PROOF. By definition, $D\eta(z) DF(z) = Dq(z)$, i.e.

$$DF^t(z) D\eta(z) = Dq(z).$$

Consider the diffeomorphism $z = z(w)$, then by the chain rule

$$D_w q(w) = Dz(w)^t D_z q(z)$$

and similarly

$$D_w \eta(w) = Dz(w)^t D_z \eta(z),$$

so that

$$D_w q = Dz^t D_z q = Dz^t D_z F^t D_z \eta = Dz^t D_z F^t (Dz^t)^{-1} D_w \eta.$$

But $Dz(w) = Dw(z(w))^{-1}$, hence

$$D_w q(w) = Dw^{-t}(z) D_z F^t(z) Dw^t(z) D_w \eta.$$

But recall that $Dw_1||l_2$ and $Dw_2||l_1$, hence

$$\begin{aligned} Dw^{-t}DF^tDw^t &= Dw^{-t}DF^t(\underbrace{Dw_1}_{||l_2}|\underbrace{Dw_2}_{||l_1}) \\ &= Dw^{-t}(\lambda_2 Dw_1|\lambda_1 Dw_2) \\ &= Dw^{-t}Dw^t\begin{pmatrix}\lambda_2 & 0 \\ 0 & \lambda_1\end{pmatrix} \\ &= \begin{pmatrix}\lambda_2 & 0 \\ 0 & \lambda_1\end{pmatrix} \end{aligned}$$

and the claim follows. □

Continue the proof of Theorem 3.36.
Recall

$$\eta^k(w) = e^{kw_1}\left(A_0(w) \pm \frac{1}{k}A_1(w) + \mathcal{O}\left(\frac{1}{k^2}\right)\right),\ A_0 > 0$$
$$q^k(w) = e^{kw_1}\left(B_0(w) \pm \frac{1}{k}B_1(w) + \mathcal{O}\left(\frac{1}{k^2}\right)\right)$$

from Lemma 3.35. Insert this into the just derived lemma and match coefficients of k^1 and k^0 which yields

$$\frac{\partial \eta^k}{\partial w_1} = ke^{kw_1}\left(A_0 + \frac{1}{k}A_1 + \mathcal{O}\left(\frac{1}{k^2}\right)\right) + e^{kw_1}\left(\frac{\partial A_0}{\partial w_1} + \frac{1}{k}\frac{\partial A_1}{\partial w_1} + \mathcal{O}\left(\frac{1}{k^2}\right)\right)$$
$$\frac{\partial \eta^k}{\partial w_2} = e^{kw_1}\left(\frac{\partial A_0}{\partial w_2} + \frac{1}{k}\frac{\partial A_1}{\partial w_2} + \mathcal{O}\left(\frac{1}{k^2}\right)\right)$$
$$\frac{\partial q^k}{\partial w_1} = ke^{kw_1}\left(B_0 + \frac{1}{k}B_1 + \mathcal{O}\left(\frac{1}{k^2}\right)\right) + e^{kw_1}\left(\frac{\partial B_0}{\partial w_1} + \frac{1}{k}\frac{\partial B_1}{\partial w_1} + \mathcal{O}\left(\frac{1}{k^2}\right)\right)$$
$$\frac{\partial q^k}{\partial w_2} = e^{kw_1}\left(\frac{\partial B_0}{\partial w_2} + \frac{1}{k}\frac{\partial B_1}{\partial w_2} + \mathcal{O}\left(\frac{1}{k^2}\right)\right)$$

hence Lemma 3.37 gives

$$\begin{aligned} B_0 &= \lambda_2 A_0, \\ \frac{\partial B_0}{\partial w_2} &= \lambda_1 \frac{\partial A_0}{\partial w_2}, \\ B_1 + \frac{\partial B_0}{\partial w_1} &= \lambda_2\left(A_1 + \frac{\partial A_0}{\partial w_1}\right). \end{aligned}$$

Hence we have

$$B_1 - \lambda_2 A_1 = \lambda_2\frac{\partial A_0}{\partial w_1} - \frac{\partial B_0}{\partial w_1} = \frac{\partial}{\partial w_1}(\lambda_2 A_0 - B_0) - \frac{\partial \lambda_2}{\partial w_1}A_0 = -\frac{\partial \lambda_2}{\partial w_1}A_0.$$

Next, fix x, t and define $R := \{w \in \mathbb{R}^2 : w_i^- \leq w_i \leq w_i^+\ (i = 1, 2)\}$ as the smallest rectangle in $\mathbb{R}^2$ containing $\operatorname{supp}\nu_{x,t}$. We want to show that R is a point.
For contradiction, suppose $w_1^- < w_1^+$ ($w_2^- < w_2^+$ is similar). Since $A_0 > 0$, for large $|k|$, $\eta^k = e^{kw_1}\left(A_0 + \mathcal{O}\left(\frac{1}{k}\right)\right) > 0$, so that the measure defined by

$$\mu_k(E) = \frac{1}{\langle \nu_{x,t}, \eta^k\rangle}\int_E \eta^k(w)\, d\nu_{x,t}(w)$$

is a probability measure supported in $\mathbb{R}$. Hence, up to subsequences,

$$(\mu_k)_{k>0} \overset{*}{\rightharpoonup} \mu^+, (\mu_k)_{k<0} \overset{*}{\rightharpoonup} \mu^- \; (|k| \to \infty)$$

for probability measures μ^+, μ^- with support in R.
Claim:

$$\operatorname{supp} \mu^+ \subset R \cap \langle w_1 = w_1^+ \rangle$$
$$\operatorname{supp} \mu^- \subset R \cap \langle w_1 = w_1^- \rangle$$

Proof of the claim: Consider only $+$. Let $\varphi \in C(R)$ such that $\varphi = 0$ near

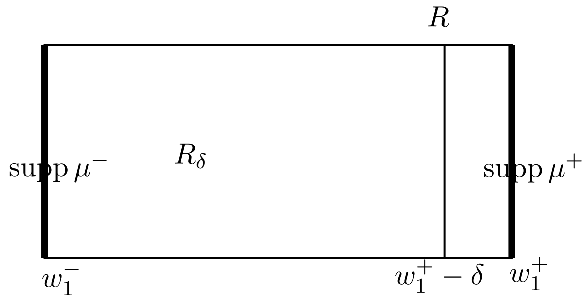


FIGURE 18

$\langle w = w_1^+ \rangle$. Hence, there exists $\delta > 0$ such that $\varphi(w_1, w_2) = 0$ if $w_1 > w_1^+ - \delta$ and let $R_\delta = [w_1^-, w_1^+ - \delta] \times [w_2^-, w_2^+.]$. Then

$$\begin{aligned}
\left| \int_R \varphi(w)\, d\mu^+(w) \right| &= \left| \lim_{k\to\infty} \int \varphi(w) \eta^k(w)\, d\nu_{x,t}(w) \frac{1}{\langle \nu_{x,t}, \eta^k \rangle} \right| \\
&\le \|\varphi\|_\infty \left| \limsup_{k\to\infty} \frac{\int_{R_\delta} \eta^k(w)\, d\nu_{x,t}(w)}{\int_R \eta^k(w)\, d\nu_{x,t}(w)} \right| \\
&= \|\varphi\|_\infty \left| \limsup_{k\to\infty} \frac{\int_{R_\delta} e^{kw_1} A_0(w)\, d\nu_{x,t}(w)}{\int_R e^{kw_1} A_0(w)\, d\nu_{x,t}(w)} \right| \\
&\le \|\varphi\|_\infty C \limsup_{k\to\infty} \frac{\int_{R_\delta} e^{kw_1}\, d\nu_{x,t}(w)}{\int_R e^{kw_1}\, d\nu_{x,t}(w)} \; (0 < m < A_0 < M) \\
&\le \|\varphi\|_\infty C \limsup_{k\to\infty} \frac{e^{k(w_1^+ - \delta)} \nu_{x,t}(R_\delta)}{e^{k(w_1^+ - \delta/2)} \nu_{x,t}(R \setminus R_{\delta/2})} \\
&= 0.
\end{aligned}$$

This proves the claim. The argument for μ^- is similar.
Let us define

$$\lambda_2^+ = \int_{R \cap \{w_1 = w_1^+\}} \lambda_2\, d\mu^+, \lambda_2^- = \int_{R \cap \{w_1 = w_1^-\}} \lambda_2\, d\mu^-.$$

Let now (η, q) be any entropy/entropy-flux pair and use $(\eta_1, q_1) = (\eta_k, q_k)$ and $(\eta_2, q_2) = (\eta, q)$ above:

$$\langle \nu_{x,t}, \eta_k q - \eta q_k \rangle = \langle \nu_{x,t}, \eta_k \rangle \langle \nu_{x,t}, q \rangle - \langle \nu_{x,t}, \eta \rangle \langle \nu_{x,t}, q_k \rangle. \tag{3.8}$$

Recalling the definition of μ_k, we obtain

$$\begin{aligned}\langle \nu_{x,t}, q_k\rangle \frac{1}{\langle \nu_{x,t}, \eta_k\rangle} &= \frac{1}{\langle \nu_{x,t}, \eta_k\rangle} \int e^{k_1 w} \lambda_2 A_0(w)\, d\nu_{x,t}(w) + \mathcal{O}\left(\frac{1}{k}\right) \\ &= \int \lambda_2(w)\, d\mu_k + \mathcal{O}\left(\frac{1}{k}\right).\end{aligned}$$

and similarly

$$\begin{aligned}\langle \nu_{x,t}, \eta_k q - \eta q_k\rangle \frac{1}{\langle \nu_{x,t}, \eta_k\rangle} &= \frac{1}{\langle \nu_{x,t}, \eta_k\rangle} \int e^{kw_1} A_0 q - \eta e^{kw_1} A_0 \lambda_2\, d\nu + \mathcal{O}\left(\frac{1}{k}\right) \\ &= \int q - \lambda_2 \eta\, d\mu_k + \mathcal{O}\left(\frac{1}{k}\right).\end{aligned}$$

Hence dividing (3.8) by $\langle \nu, \eta_k\rangle$ and letting $k \to \pm\infty$ we obtain

$$\int_{R\cap\{w_1 = w_1^\pm\}} q - \lambda_2 \eta\, d\mu^\pm = \langle \nu, q\rangle - \langle \nu, \eta\rangle \lambda_2^\pm.$$

Next, insert (η^k, q^k) and (η^{-k}, q^{-k}) in (3.8) to get

$$\langle \nu, \eta_k q_{-k} - \eta_{-k} q_k\rangle = \langle \nu, \eta_k\rangle\langle \nu, q_{-k}\rangle - \langle \nu, \eta_{-k}\rangle\langle \nu, q_k\rangle$$

and so

$$\frac{\langle \nu, \eta_k q_{-k} - \eta_{-k} q_k\rangle}{\langle \nu, \eta_k\rangle\langle \nu, \eta_{-k}\rangle} = \frac{\langle \nu, q_{-k}\rangle}{\langle \nu, \eta_{-k}\rangle} - \frac{\langle \nu, q_k\rangle}{\langle \nu, \eta_k\rangle}.$$

Let $k \to \pm\infty$: The RHS converges to $\lambda_2^- - \lambda_2^+$ as above. For the LHS, the numerator is of order $\frac{1}{k}$ (because: $\eta_k q_{-k} - \eta_{-k} q_k \sim A_0 B_0 - A_0 B_0 + \frac{1}{k}(A_1 B_0 - A_0 B_1 - A_0 B_1 + A_1 B_0) + \mathcal{O}\left(\frac{1}{k^2}\right)$) whereas the denominator is of order $e^{k(w_1^+ - w_1^-)} \to \infty$. Hence in the limit $|k| \to \infty$ we get

$$\lambda_2^- - \lambda_2^+ = 0.$$

By (3.5), this also implies

$$\int_{R\cap\{w_1 = w_1^+\}} q - \eta\lambda_2\, d\mu^+ = \int_{R\cap\{w_1 = w_1^-\}} q - \eta\lambda_2\, d\mu^- \tag{3.9}$$

for any entropy/entropy-flux pair (η, q). Set

$$(\eta, q) = (\eta_k, q_k)$$

and expand (3.9) up to order $\frac{1}{k}$:

$$\begin{aligned}&e^{kw_1^+} \int_{R\cap\{w_1 = w_1^+\}} \frac{B_1 - \lambda_2 A_1}{k} + \mathcal{O}\left(\frac{1}{k^2}\right) d\mu^+ \\ &= e^{kw_1^-} \int_{R\cap\{w_1 = w_1^-\}} \frac{B_1 - \lambda_2 A_1}{k} + \mathcal{O}\left(\frac{1}{k^2}\right) d\mu^-.\end{aligned}$$

Since $w_1^+ \neq w_1^-$, it follows that

$$\int_{R\cap\{w_1 = w_1^\pm\}} B_1 - \lambda_2 A_1\, d\mu^\pm = 0.$$

But we recall that we had derived

$$B_1 - \lambda_2 A_1 = -\frac{\partial \lambda_2}{\partial w_1} A_0,$$

so that

$$\int_{R\cap\{w_1=w_1^\pm\}} \frac{\partial \lambda_2}{\partial w_1} A_0\, d\mu^\pm = 0.$$

But $A_0 > 0$ and $\frac{\partial\lambda_2}{\partial w_1}$ does not change sign by genuine nonlinearity, therefore we get a contradiction to $w_1^+ > w_1^-$.
A similar argument yields $w_2^+ = w_2^-$ hence the support of $\nu_{x,t}$ is $u(x,t)$ for a.e. x, t, and pointwise convergence to an entropy solution then follows easily. □

REMARK 3.38. The following argument justifies the assumption

$$\varepsilon \int_0^T \int_{\mathbb{R}} |\partial_x u|^2\, dxdt < \infty :$$

If (η, q) is an entropy / entropy-flux pair, then

$$\partial_t \eta(u) + \partial_x q(u) = \varepsilon \partial_{xx}\eta(u) - \varepsilon(\partial_x u, D^2\eta(u)\partial_x u)$$

hence integration in x and t gives

$$\eta(u(T)) + \varepsilon \int_0^T \int_{-\infty}^{\infty} (\partial_x u, D^2\eta(u)\partial_x u)\, dxdt \le \eta(u(0))$$

(if u is smooth, sufficiently decaying).
Hence if η is *strongly convex* in the sense that

$$(\xi, D^2\eta\xi) \ge m|\xi|^2$$

for some $m > 0$ and all $\xi \in \mathbb{R}^2$ then we obtain a uniform $L^2_{x,t}$-bound for $\sqrt{\varepsilon}\partial_x u$, as desired.

REMARK 3.39. Different subsequences might converge to different entropy solutions.

3.6. Weak-strong uniqueness

For general systems of conservation laws, there is no known uniqueness result for entropy solutions ("scientific scandal", P. Lax).
The second best type of result is *weak-strong uniqueness*:

THEOREM 3.40. *(C. Dafermos)*
Suppose the hyperbolic system of m equations

$$\partial_t u + \partial_x F(u) = 0$$

has a strongly convex entropy, $D^2\eta \ge cId_m$ for some $c > 0$.
Suppose there is a solution $\overline{u} \in C^1(\mathbb{R}\times[0,T)) \cap L^\infty$ with $\overline{u}(x,0) = g(x)$. Then every entropy solution u with initial data g coincides with $\overline{u}$, i.e.

$$u = \overline{u} \text{ on } \mathbb{R}\times[0,T).$$

PROOF. Define the functions

$$h(x,t) = \eta(u) - \eta(\overline{u}) - D\eta(\overline{u})\cdot(u-\overline{u}) \text{ "relative entropy"}$$
$$Y(x,t) = q(u) - q(\overline{u}) - D\eta(\overline{u})\cdot(F(u)-F(\overline{u}))$$
$$Z(x,t) = D^2\eta(\overline{u})(F(u) - F(\overline{u}) - DF(\overline{u})\cdot(u-\overline{u})).$$

Since $u, \overline{u} \in L^\infty$ and η, q, F are smooth, by Taylor's theorem there is a constant $C > 0$ such that

$$|q(u) - q(\overline{u}) - Dq(\overline{u}) \cdot (u - \overline{u})| \leq C|u - \overline{u}|^2,$$
$$|F(u) - F(\overline{u}) - DF(\overline{u})(u - \overline{u})| \leq C|u - \overline{u}|^2.$$

Hence,

$$\begin{aligned}
&|Y(x,t)| \\
&= |q(u) - q(\overline{u}) - D\eta(\overline{u}) \cdot (F(u) - F(\overline{u}))| \\
&\leq |q(u) - q(\overline{u}) - Dq(\overline{u}) \cdot (u - \overline{u}) \\
&\qquad + Dq(\overline{u}) \cdot (u - \overline{u}) - D\eta(\overline{u}) \cdot (F(u) - F(\overline{u}))| \\
&\leq C|u - \overline{u}|^2 + |Dq(\overline{u})(u - \overline{u}) - D\eta(\overline{u})(F(u) - F(\overline{u}) \\
&\qquad - DF(\overline{u})(u - \overline{u})) - D\eta(\overline{u})DF(\overline{u}) \cdot (u - \overline{u})| \\
&\leq C|u - \overline{u}|^2,
\end{aligned}$$

where we let the value of the constant C increase from time to time. On the other hand, strong convexity of η implies

$$\begin{aligned}
|h(x,t)| &= |\eta(u) - \eta(\overline{u}) - D\eta(\overline{u}) \cdot (u - \overline{u})| \\
&= \frac{1}{2}|(D^2\eta(\overline{u}) \cdot (u - \overline{u})) \cdot (u - \overline{u})| + \mathcal{O}(|u - \overline{u}|^3) \text{ for } |u - \overline{u}| \text{ small} \\
&\geq c|u - \overline{u}|^2
\end{aligned}$$

(for $|u - \overline{u}|$ small this follows from $\mathcal{O}(|u - \overline{u}|^3) \ll |u - \overline{u}|^2$, for $|u - \overline{u}|$ large it follows from $u, \overline{u} \in L^\infty$ and $u \neq \overline{u}$ implies $h \neq 0$).
It follows that

$$|Y(x,t)| \leq C|h(x,t)|.$$

Similarly,

$$|Z(x,t)| \leq |D^2\eta(\overline{u})||F(u) - F(\overline{u}) - DF(\overline{u})(u - \overline{u})| \leq C|u - \overline{u}|^2$$

and therefore

$$|Z(x,t)| \leq C|h(x,t)|.$$

Since $\overline{u}$ is a C^1-smooth solution, the usual computation gives

$$\partial_t \eta(\overline{u}) + \partial_x q(\overline{u}) = 0,$$

and by assumption

$$\partial_t \eta(u) + \partial_x q(u) \leq 0$$

in the sense of distributions.
So if $\varphi \in C_c^\infty(\mathbb{R} \times [0,T))$, then

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} \partial_t \varphi h + \partial_x \varphi Y \, dx dt \\
&= \iint \partial_t \varphi(\eta(u) - \eta(\overline{u}) - D\eta(\overline{u}) \cdot (u - \overline{u})) \\
&\qquad \partial_x \varphi(q(u) - q(\overline{u}) - D\eta(\overline{u}) \cdot (F(u) - F(\overline{u}))) \, dx dt \qquad (3.10) \\
&\geq \iint -\partial_t \varphi D\eta(\overline{u}) \cdot (u - \overline{u}) - \partial_x \varphi D\eta(\overline{u}) \cdot (F(u) - F(\overline{u}))
\end{aligned}$$

(boundary terms at $t = 0$ cancel since $u(0) = \overline{u}(0) = g$).
Moreover, use $\varphi D\eta(\overline{u}) \in C^1$ as a test function for

$$\partial_t u + \partial_x F(u) = 0$$
$$\partial_t \overline{u} + \partial_x F(\overline{u}) = 0$$

to obtain

$$\iint \partial_t(\varphi D\eta(\overline{u})) \cdot (\overline{u} - u) + \partial_x(\varphi D\eta(\overline{u})) \cdot (F(\overline{u}) - F(u)) = 0.$$

Moreover observe that

$$\begin{aligned} &\partial_t(D\eta(\overline{u})) \cdot (\overline{u} - u) + \partial_x D\eta(\overline{u}) \cdot (F(\overline{u}) - F(u)) \\ &= D^2\eta(\overline{u}) \cdot (\overline{u} - u) \cdot \partial_t \overline{u} + D^2\eta(\overline{u}) \cdot (F(\overline{u}) - F(u)) \cdot \partial_x \overline{u} \\ &= -D^2\eta(\overline{u}) \cdot \partial_x F(\overline{u}) \cdot (\overline{u} - u) + D^2\eta(\overline{u}) \cdot (F(\overline{u}) - F(u)) \cdot \partial_x \overline{u} \\ &= -Z(x,t) \cdot \partial_x \overline{u}. \end{aligned}$$

Recall (3.10) to conclude

$$\iint \partial_t \varphi h + \partial_x \varphi Y \, dxdt \geq \iint \varphi Z(x,t) \cdot \partial_x \overline{u} \, dxdt. \tag{3.11}$$

Next choose a particular test function

$$\varphi(x,t) = w(t) X(x,t),$$

where, for some $\tau < T, R > 0, \varepsilon$ small,

$$w(t) = \begin{cases} 1 & : 0 \leq t \leq \tau - \varepsilon \\ 0 & : t \geq \tau \end{cases}$$

and w is linearly decreasing on $[\tau - \varepsilon, \tau]$.

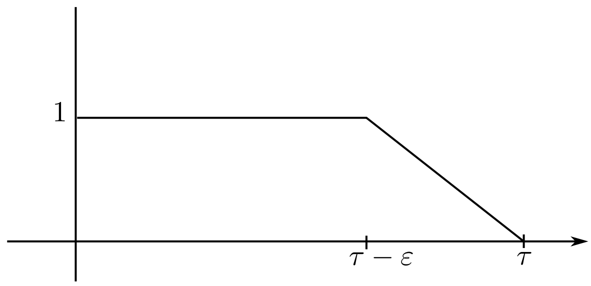


FIGURE 19

The function X is given by

$$X(x,t) = \begin{cases} 1 & : |x| \leq R + C(\tau - t) \\ 1 - \frac{1}{\varepsilon}(|x| - R - C(\tau - t)) & : 0 \leq |x| - (R + C(\tau - t)) \leq \varepsilon \\ 0 & : \text{ otherwise}, \end{cases}$$

where C satisfies $|Y| \leq C|h|$.
Observe that

- $0 \leq \varphi \leq 1$
- $\varphi = 0$ if $t \geq \tau$ or $|x| \geq \varepsilon + R + C(\tau - t)$
- $\partial_t \varphi = -\frac{1}{\varepsilon}$ on $(-R, R) \times (\tau - \varepsilon, \tau)$

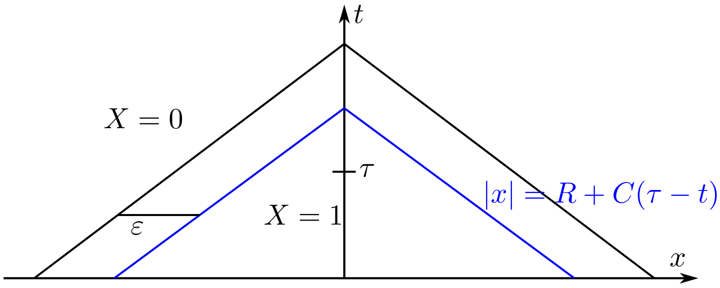


FIGURE 20

- $|\partial_x\varphi| \le -\frac{1}{\varepsilon}\partial_t\varphi$.

Therefore,

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_{\tau-\varepsilon}^{C}\int_{|x|\le R+C(\tau-t)} h(x,t)\,dxdt \\
&= -\int_{\tau-\varepsilon}^{C}\int_{|x|\le R+C(\tau-t)} \underbrace{\partial_t\varphi}_{=-1/\varepsilon \text{ on } (\tau-\varepsilon,\tau)} h + \underbrace{\partial_x\varphi}_{=0 \text{ on } |x|\le R+c(\tau-t)} Y\,dxdt \\
&\le -\int_0^\tau\int_{|x|\le\varepsilon+R+C(\tau-t)} \partial_t\varphi h + \partial_x\varphi Y\,dxdt \\
&\le_{3.11} -\int_0^\tau\int_{|x|\le\varepsilon+R+C(\tau-t)} \varphi Z(x,t)\cdot\partial_x\overline{u}(x,t)\,dxdt \\
&\le \int_0^\varepsilon\int_{|x|\le\varepsilon+R+C(\tau-t)} |Z||\partial_x\overline{u}|\,dxdt.
\end{aligned}$$

But since $\partial_x\overline{u}\in L^\infty$ and $|Z|\le C'h$, we get

$$\frac{1}{\varepsilon}\int_{\tau-\varepsilon}^{\tau}\int_{|x|\le R+C(\tau-t)} h(x,t)\,dxdt \le C'\int_0^\tau\int_{|x|\le\varepsilon+R+C(\tau-t)} h(x,t)\,dxdt,$$

for every $\tau<T$ and $R>0$. In particular we may replace R by $R+C(s-t)$ to get ($\tau<s<T$)

$$\frac{1}{\varepsilon}\int_{\tau-\varepsilon}^{\tau}\int_{|x|\le R+C(s-t)} h(x,t)\,dxdt \le C'\int_0^\tau\int_{|x|\le\varepsilon+R+C(s-t)} h(x,t)\,dxdt.$$

By the Lebesgue differentiation Theorem, the limit $\varepsilon\searrow 0$, gives for $a.e.$ $\tau<s$

$$\underbrace{\int_{|x|\le R+C(\tau-t)} h(x,\tau)\,dx}_{=:g(\tau)} \le C'\int_0^t\underbrace{\int_{|x|\le R+C(s-t)} h(x,t)\,dx\,dt}_{=g(t)}$$

so that

$$g(\tau)\le\int_0^\tau g(t)\,dt$$

a.e. $\tau<T$. It follows from Gronwall's lemma that $g=0$, a.e., hence by $h\ge 0$ and the arbitrary choices of R and τ, $h\equiv 0$ a.e.
It follows that $u=\overline{u}$ for almost every x,t. □

REMARK 3.41. i) The theorem (with similar proof) is valid also for $x \in \mathbb{R}^d$.

ii) In the absence of smooth solutions, solutions need not be unique:
For isentropic Euler and $m \geq 2$, there exist $\rho_0, v_0 \in L^\infty$ such that there are infinitely many entropy solutions with this data (De Lellis - Székelyhidi 2010).

Bibliography

[1] L. C. EVANS. Partial Differential Equations (= Graduate Studies in Mathematics **19).** *American Mathematical Society, Providence, RI,* 2nd Ed. 2010.

[2] L. C. EVANS. Weak Convergence Methods for Nonlinear Partial Differential Equations *CBMS Regional Conference Series in Mathematics 74, American Mathematical Society, Providence, RI, 1990,* 1990.

[3] D. GILBARG and N. S. TRUDINGER. Elliptic partial differential equations of second order. Reprint of the 1998 edition. Classics in Mathematics. *Springer-Verlag, Berlin,* 2001.

[4] S. N. KRUŽKOV. First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.)* **81 (123)** (1970), 228–255.

[5] F. MURAT. Compacité par compensation. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **5** (1978), no. 3, 489–507