Partially hyperbolic random dynamics on Grassmannians

Hermann Schulz-Baldes
FAU Erlangen-Nürnberg

joint work with Joris De Moor and Florian Dorsch
`arXiv preprint arXiv:2206.03444`

Abel Conference, Cergy
June 2022

Overview

Anderson model on a strip of width L

Random discrete Schrödinger operator on $\mathbb{Z} \times \{1, \dots, \mathsf{L}\}$ for $\mathsf{L} \in \mathbb{N}$

Hamiltonian $H = \Delta_{\mathbb{Z}} + \Delta_{\mathsf{L}} + \lambda V$ with weak coupling $\lambda > 0$

Discrete Laplacian $\Delta_{\mathsf{L}} = -(S + S^*)$ with $S : \mathbb{C}^{\mathsf{L}} \to \mathbb{C}^{\mathsf{L}}$ cyclic shift

For centered i.i.d. random variables $\omega_{n,j} \in [-1, 1]$

$$V = \sum_{n \in \mathbb{Z}} V_n \qquad , \qquad V_n = \sum_{j=1}^{\mathsf{L}} \omega_{n,j} \, |n, j\rangle\langle n, j|$$

Study $H\psi = E\psi$ for $E \in \mathbb{R}$ via random transfer matrices:

$$\mathcal{T}_n^E = \begin{pmatrix} E - (\Delta_L + \lambda V_n) & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & -\lambda V_n \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} E - \Delta_L & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$$

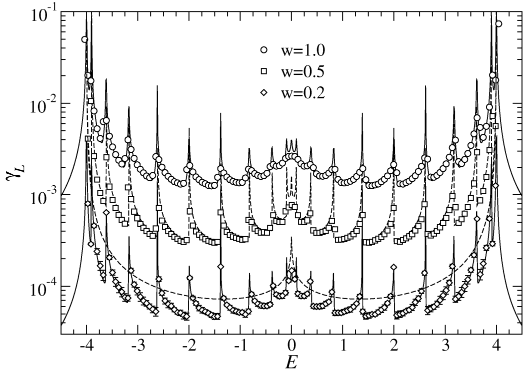
Abel, Lacroix, Spies (1990): Anderson loc. (with Margulis, Goldsheid)

Perturbation theory for Lyapunov exponents (2004 with Römer)

Perturbation theory for Lyapunov exponents

Set $2\cos(k_l) = E - 2\cos(\frac{2\pi l}{\mathrm{L}})$ for $l = 1, \ldots, \mathrm{L}$, then for $p = 1, \ldots, \mathrm{L}$:

$$\gamma_p \;=\; \frac{\lambda^2}{4\mathrm{L}} \left(\frac{1}{\mathrm{L}_e} \sum_{l \text{ elliptic}} \frac{1}{|\sin(k_l)|} \right)^2 \left(\mathrm{L} - p + \frac{1}{2} \right) \;+\; \mathcal{O}_{\mathrm{L}}(\lambda^3)$$



Problem: bad control on separation of elliptic/hyperbolic channels

From Anderson to toy model

Rewrite:

$$\mathcal{T}_n^E \;=\; \exp\left[\lambda \begin{pmatrix} 0 & -V_n \\ 0 & 0 \end{pmatrix}\right] \begin{pmatrix} E - \Delta_L & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \;\in\; \mathbb{R}^{2L\times 2L} \;\text{ symplectic}$$

Fourier transform $\mathcal{F} : \mathbb{C}^L \to \mathbb{C}^L$ extended to $\mathcal{F} = \mathcal{F} \oplus \mathcal{F}$ gives:

$$\mathcal{F}\,\mathcal{T}_n^E\,\mathcal{F}^* \;=\; e^{\lambda \mathcal{P}_n} \mathcal{R}$$

with block-diag. $\mathcal{R}$ with 2×2 blocks (elliptic/hyperbolic open/closed):

$$\mathcal{P}_n \;=\; \begin{pmatrix} 0 & -\mathcal{F} V_n \mathcal{F}^* \\ 0 & 0 \end{pmatrix} \quad , \quad \mathcal{R} \;=\; \begin{pmatrix} E - \mathcal{F}\Delta_L\mathcal{F}^* & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$$

Simplifications:

suppress: symplectic structure, energy dependence E and $\mathrm{Ker}(\mathcal{P}_n)$

choose: $\mathcal{R} > 0$ diagonal (phases absorbed)

Simplified toy model

$$\mathcal{T}_n \;=\; e^{\lambda \mathcal{P}_n}\, \mathcal{R} \;\in\; \mathrm{Gl}(\mathsf{L}, \mathbb{C})$$

with diagonal partially hyperbolic

$$\mathcal{R} \;=\; \mathrm{diag}\big(\kappa_{\mathsf{L}}, \ldots, \kappa_1\big) \qquad\qquad \kappa_1 \geqslant \cdots \geqslant \kappa_{\mathsf{L}} > 0$$

and i.i.d. matrices $\mathcal{P}_n$ with further Hypothesis 1-5 (later)

Example: $\mathcal{R} = \mathcal{F}(\Delta_L + s)\mathcal{F}^* > 0$ for some $s \in (2, \infty)$ so that

$$\mathcal{R} \;=\; s\,\mathbf{1} \;-\; 2\,\mathrm{diag}\big(1, \cos(\tfrac{2\pi}{\mathsf{L}}), \cos(\tfrac{2\pi}{\mathsf{L}}), \cos(\tfrac{2\pi 2}{\mathsf{L}}), \ldots, \cos(\tfrac{\pi(\mathsf{L}-1)}{\mathsf{L}})\big)$$

and

$$\mathcal{P}_n \;=\; \mathcal{F}\Big(\sum_{j=1}^{\mathsf{L}} \omega_{n,j} |j\rangle\langle j|\Big)\mathcal{F}^* \;\in\; \mathbb{C}^{\mathsf{L}\times\mathsf{L}}$$

random Toeplitz matrix. This will satisfy Hypothesis 1-5 !

Markov process on unit vectors (random dynamics)

Using group action $\circ$ of $\mathrm{Gl}(\mathsf{L},\mathbb{C})$ on $\mathbb{S}_{\mathbb{C}}^{\mathsf{L}-1} = \left\{ v \in \mathbb{C}^{\mathsf{L}} : \|v\| = 1 \right\}$

$$\mathcal{T} \circ v \;=\; \frac{\mathcal{T} v}{\|\mathcal{T} v\|}$$

one gets Markov chain on compact state space $\mathbb{S}_{\mathbb{C}}^{\mathsf{L}-1}$

$$v_n \;=\; \mathcal{T}_n \circ v_{n-1} \;=\; (e^{\lambda \mathcal{P}_n} \mathcal{R}) \circ v_{n-1} \;=\; e^{\lambda \mathcal{P}_n} \circ (\mathcal{R} \circ v_{n-1})$$

Furstenberg measure

Suppose strong irreducibility and contractibility

Then $\exists$ unique invariant measure μ_λ on $\mathbb{S}_{\mathbb{C}}^{\mathsf{L}-1}$

$$\mathbb{E} \int \mu_\lambda(dv)\, f(\mathcal{T} \circ v) \;=\; \int \mu_\lambda(dv)\, f(v) \qquad , \qquad f \in C(\mathbb{S}_{\mathbb{C}}^{\mathsf{L}-1})$$

Under suitable coupling assumptions: $\mathrm{supp}(\mu_\lambda) = \mathbb{S}_{\mathbb{C}}^{\mathsf{L}-1}$

Aim: More information on μ_λ for λ small

Unperturbed deterministic dynamics for $\lambda = 0$

Assume $\mathcal{R} = \mathrm{diag}(\kappa_L, \dots, \kappa_1)$ with strict inequalities $\kappa_1 > \kappa_2 > \cdots > \kappa_L$

$$\mathcal{R}^N \circ v_0 \xrightarrow{N\to\infty} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \text{ for a.e. } v_0, \qquad \text{but: } \mathcal{R}^N \circ \begin{pmatrix} v_0^{(1)} \\ \vdots \\ v_0^{(J-1)} \\ v_0^{(J)} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \xrightarrow{N\to\infty} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

stable fixed point: $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ unstable fixed points: $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}$

Random perturbation: escape from unstable fixed points

but for $\lambda \geqslant 1 - \frac{\kappa_{l+1}^2}{\kappa_l^2}$ possibly arrive at all unstable fixed points

Towards a quantitative description

Local hyperbolicity of $\mathcal{R} = \mathrm{diag}(\kappa_{\mathsf{L}}, \ldots, \kappa_1)$ measured by relative gaps

$$\eta(\mathsf{I},\mathsf{J}) \;=\; 1 - \frac{\kappa_{\mathsf{J}}^2}{\kappa_{\mathsf{I}}^2} \in [0,1] \qquad , \qquad \mathsf{I} \leqslant \mathsf{J}$$

For $\mathsf{L}_a, \mathsf{L}_b, \mathsf{L}_c \in \mathbb{N}$ with $\mathsf{L}_a + \mathsf{L}_b + \mathsf{L}_c = \mathsf{L}$, subdivide

$$v \;=\; \begin{pmatrix} a(v) \\ b(v) \\ c(v) \end{pmatrix} \in \mathbb{S}_{\mathbb{C}}^{\mathsf{L}-1}$$

in which $a(v)$, $b(v)$ and $c(v)$ are of lengths L_a, L_b and L_c such that

Hypothesis 1: Macroscopic relative gap for $\mathcal{R}$

$\boldsymbol{\eta} = \eta(\mathsf{L}_c, \mathsf{L}_b + \mathsf{L}_c)$ satisfies $\boldsymbol{\eta} > 0$

Aim: quantitative bound on $\|a(v)\|$ for $\lambda \gg$ local relative gaps $\eta(\mathsf{I}, \mathsf{I}+1)$

Hypothesis

Perturbation $e^{\lambda\mathcal{P}} = \mathbf{1} + \lambda\mathcal{P} + \mathcal{O}(\lambda^2)$ leaves unstable fixed points

Hypothesis 2: Coupling assumption on $\mathcal{P}$

Random matrices $\mathcal{P}$ centered and $\|P\| \leqslant 1$ for $P \in \mathrm{supp}(\mathcal{P})$

$$\beta \;=\; \inf\left\{\mathbb{E}\,\|c(\mathcal{P}v)\|^2 : \quad v \in \mathbb{S}_{\mathbb{C}}^{L-1}\,, \quad c(v) = 0\right\}$$

satisfies $\beta > 0$

Hypothesis 3: Small coupling constant

$\lambda \leqslant C\beta^{\frac{8}{3}}\boldsymbol{\eta}^{-\frac{1}{3}}$ for some constant C

Hypothesis 4: Dominated microscopic gaps (λ intermediate)

$$\forall\, l \in \{L_c, \ldots, L_b + L_c\} : \quad \eta(l, l+1) < 2^4\,\lambda$$

Main result on dynamics of vectors

Dynamics restricted to equator $a(v) = 0$, up to errors (rare excursions)

Theorem

Under Hypotheses 1-4, all $v_0 \in \mathbb{S}_{\mathbb{C}}^{\mathrm{L}-1}$ and $n \geqslant T_0 = C\beta^{-1}\lambda^{-2}$ obey

$$\mathbb{E}\,\|a(v_n)\|^2 \;\leqslant\; 10\,\boldsymbol{\eta}^{-1}\,\lambda^2$$

In terms of Furstenberg measure

$$\int \mu_\lambda(dv)\,\|a(v)\|^2 \;\leqslant\; 10\,\boldsymbol{\eta}^{-1}\,\lambda^2$$

Allows to deduce bound on largest Lyapunov exponent (later)

Scaling of upper bound optimal, also equilibration time T_0 optimal

Flexibility of choice of $\mathrm{L}_a, \mathrm{L}_b, \mathrm{L}_c$ (this has influence on $\boldsymbol{\eta}$)

Result applies directly to toy model with random Toeplitz matrices

Generalization to bound on dynamics on Grassmannian later

Very rough idea of proof

Basic hyperbolicity estimates on action $\mathcal{R}\circ$:

$$\|a\,(\mathcal{R}\circ v)\,\|^2 \;\leqslant\; \|a(v)\|^2\left[1 \,-\, \eta\,\|c(v)\|^2\right]$$

$$\|c\,(\mathcal{R}\circ v)\,\|^2 \;\geqslant\; \|c(v)\|^2\left[1 \,+\, \eta\,\|a(v)\|^2\right]$$

Here splitting $b(v) = (b_\uparrow(v), b_\downarrow(v))$ and setting $u(v) = (a(v), b_\uparrow(v))$,

$$\|a\,(\mathcal{R}\circ v)\,\|^2 \;\leqslant\; \|a(v)\|^2\left[1 \,-\, \frac{\eta}{2}\,(1 \,-\, \|u(v)\|^2)\right]$$

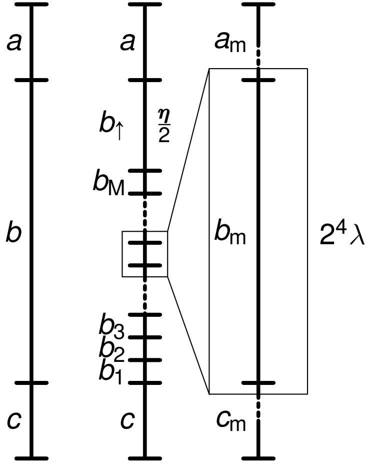
As $\mathcal{P}$ centered

$$\begin{aligned}\mathbb{E}\,\|a\left(e^{\lambda\mathcal{P}}\mathcal{R}\circ v\right)\|^2 \;&\leqslant\; \mathbb{E}\,\|a(v)\|^2\left[1 \,-\, \frac{\eta}{2}\,(1-\|u(v)\|^2)\right] \,+\, C\lambda^2 \\ &\leqslant\; \mathbb{E}\,\|a(v)\|^2\left[1 \,-\, \frac{\eta}{2}(1-\delta)\right] \,+\, C\lambda^2 \,+\, \mathbb{P}\Big(\|u(v))\|^2 > \delta\Big)\end{aligned}$$

for some $\delta < 1$. Iteration possible provided

$$\mathbb{P}\Big(\|u(v))\|^2 > \delta\Big) \;\leqslant\; C\,\lambda^2$$

For bound on $\mathbb{P}(\|u(v))\|^2 > \delta)$, split $b_\downarrow(v)$ in M parts with gaps $\geqslant 2^4\lambda$



Moving up requires passage by each b_m (mountain ridge)

This M-dimensional ridge has a length $M = \mathcal{O}(\lambda^{-1})$ (by Hypothesis 4)

Move from b_m to b_{m+1} needs 2^4 $\mathcal{P}$-kicks upwards (large deviations)

$$\begin{pmatrix} \vdots \\ v_{m+1} \\ v_m \\ \vdots \\ v_1 \end{pmatrix}$$

In compact phase space $\mathbb{S}_{\mathbb{C}}^{L-1}$, large regions have strong hyperbolicity

Between unstable fixed points, λ dominates local hyperbolicity,

but it's a long way (high dimension!)

At same time, there is diffusion into *c*-part by Hypothesis 2

Dynamics on Grassmanian

Grassmanian manifold of q-dimensional subspaces with $\mathrm{q} \leqslant \mathrm{L}_c$

$$\mathbb{G}_{\mathrm{L,q}} \;=\; \left\{ Q \in \mathbb{C}^{\mathrm{L}\times\mathrm{L}} : \quad Q = Q^2 = Q^* \,, \quad \mathrm{Tr}(Q) = \mathrm{q} \right\}$$

Action $\cdot : \mathrm{GL}(\mathrm{L},\mathbb{C}) \times \mathbb{G}_{\mathrm{L,q}} \to \mathbb{G}_{\mathrm{L,q}}$ defined by:

$$\mathcal{T} \cdot Q \;=\; \mathcal{T} Q \mathcal{T}^* (\mathcal{T} Q \mathcal{T}^*)^{-2} \mathcal{T} Q \mathcal{T}^*$$

This is a **group action**:

$$\forall\, \mathcal{T}_1, \mathcal{T}_2 \in \mathrm{GL}(\mathrm{L},\mathbb{C}), \;\; Q \in \mathbb{G}_{\mathrm{L,q}} : \quad \mathcal{T}_2 \cdot (\mathcal{T}_1 \cdot Q) \;=\; (\mathcal{T}_2 \mathcal{T}_1) \cdot Q$$

Random dynamical system on $\mathbb{G}_{\mathrm{L,q}}$

$$Q_n \;=\; \mathcal{T}_n \cdot Q_{n-1} \;=\; (e^{\lambda \mathcal{P}_n} \mathcal{R}) \cdot Q_{n-1} \qquad , \qquad Q_0 \in \mathbb{G}_{\mathrm{L,q}}$$

Remark: Decomposable vector dynamics in $\Lambda^{\mathrm{q}}\mathbb{C}^{\mathrm{L}}$ yields $\mathcal{O}(\boldsymbol{\eta}^{-1}\mathrm{q}^2\lambda^2)$

Quantity of interest and modified Hypothesis 2

For same splitting $\mathsf{L} = \mathsf{L}_a + \mathsf{L}_b + \mathsf{L}_c$ set

$$\hat{P}_a = \mathrm{diag}(\mathbf{1}_{\mathsf{L}_a}, 0, 0) , \quad \hat{P}_b = \mathrm{diag}(0, \mathbf{1}_{\mathsf{L}_b}, 0) , \quad \hat{P}_c = \mathrm{diag}(0, 0, \mathbf{1}_{\mathsf{L}_c})$$

Then introduce $\mathsf{d} : \mathbb{G}_{\mathsf{L},\mathsf{q}} \longrightarrow [0, \mathsf{q}]$ by

$$d(Q) = \mathrm{Tr}\big(\hat{P}_a Q \hat{P}_a\big)$$

Hilbert-Schmidt weight of Q in a-part. **Not** a metric!

For $\mathsf{q} = 1$ and $Q = vv^*$, as above $d(Q) = \|a(v)\|^2$

Modified Hypothesis 2: Coupling assumption on $\mathcal{P}$

Random matrices $\mathcal{P}$ centered and $\|P\| \leqslant 1$ for $P \in \mathrm{supp}(\mathcal{P})$

$$\beta_{\mathsf{q}} = \inf \left\{ \mathbb{E} \, \|c(W\mathcal{P}v)\|^2 : \quad v \in \mathbb{S}^{\mathsf{L}-1}_{\mathbb{C}} , \quad c(v) = 0 , \quad W \in \mathbb{G}_{\mathsf{L},\mathsf{L}-\mathsf{q}+1} \right\}$$

satisfies $\beta_{\mathsf{q}} > 0$

Main result on Grassmanian

Fact: $q \mapsto \beta_q$ is nonincreasing

Example: For random Toeplitz model, $\beta_q > 0$ for all $q \leqslant L_c$

Hypothesis 3 and 4 unchanged (small $\lambda \gg \eta(l, l+1)$ microgaps)

Hypothesis 5: Condition on the dimension q

$q \leqslant \min\{L_c,\ C\lambda^{-\frac{1}{5}}\}$ for some constant C

Theorem

Under Hypotheses 1-5, all $Q_0 \in \mathbb{G}_{L,q}$ and $n \geqslant T_0 = C\beta^{-1}q^2\lambda^{-2}$ obey

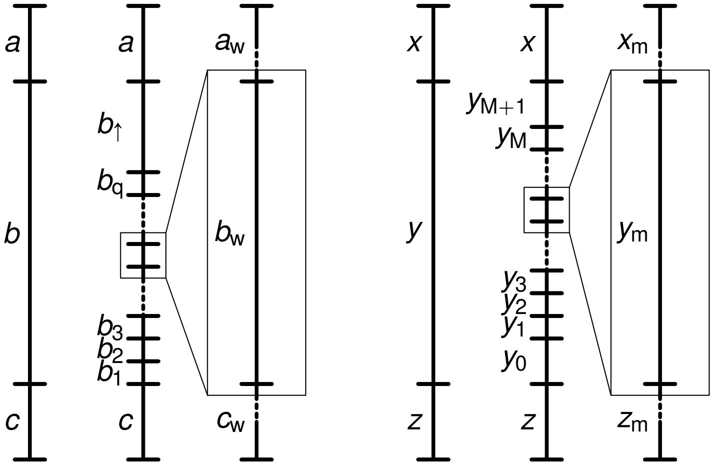
$$\mathbb{E}\, d(Q_n) \leqslant 10\,\eta^{-1} q\, \lambda^2$$

Again implies bound on Furstenberg measure

Scaling of bound optimal, but equilibration time by factor q^2 too large

Reason: iterative proof over dimension w from 1 to q

Two splittings: first in dimension $w = 1, \ldots, q$, then same as $q = 1$



Add iteratively $v \in \mathbb{S}_{\mathbb{C}}^{L-1}$ to $W \in \mathbb{G}_{L,w}$ with $Wv = \mathbf{0}$:

$$\mathcal{T} \cdot (W + vv^*) \;=\; \mathcal{T} \cdot W + \left[((\mathcal{T} \cdot W)^{\perp}\mathcal{T}) \circ v\right]\left[((\mathcal{T} \cdot W)^{\perp}\mathcal{T}) \circ v\right]^*$$

Application to Lyapunov exponents

$\gamma_1, \dots, \gamma_L \geqslant 0$ associated to $(\mathcal{T}_n)_{n\in\mathbb{N}}$ for $q \in \{1, \dots, L\}$

$$\sum_{w=1}^{q} \gamma_w \;=\; \lim_{N\to\infty} \frac{1}{N}\,\mathbb{E}\,\log\,\left\|\Lambda^q(\mathcal{T}_N\cdots\mathcal{T}_1)\right\|_{\Lambda^q\mathbb{C}^L}$$

$$=\; \frac{1}{2}\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\mathbb{E}\,\log\,\det\left(\Phi_n^*\mathcal{T}_{n+1}^*\mathcal{T}_{n+1}\Phi_n\right)$$

where $Q_n = \Phi_n\Phi_n^*$ expressed in terms of a normalized q-frame

$$(\Phi_n)^*\Phi_n \;=\; \mathbf{1}_q \qquad , \qquad \Phi_n \in \mathbb{C}^{L,q}$$

Set of q-frames forms a $U(q)$-cover of $\mathbb{G}_{L,q}$

Lemma

For $Q = \Phi\Phi^* \in \mathbb{G}_{L,q}$ and centered $\mathcal{P}$ with $\|\mathcal{P}\| \leqslant 1$:

$$\mathbb{E}\log\,\det\left(\Phi^*(e^{\lambda\mathcal{P}}\mathcal{R})^*e^{\lambda\mathcal{P}}\mathcal{R}\Phi\right) \;\geqslant\; 2\,q\,\log(\kappa_{L_b+L_c}) \\ -\,2\,\mathbb{E}\,d(Q)\,\log\frac{\kappa_{L_b+L_c}}{\kappa_L} - 3\,q\,\lambda^2$$

Theorem

Under Hypotheses 1-5,

$$\frac{1}{\mathsf{q}} \sum_{\mathsf{w}=1}^{\mathsf{q}} \gamma_{\mathsf{w}} \geqslant \log(\kappa_{\mathsf{L}_b+\mathsf{L}_c}) - \left[\tfrac{3}{2} + 10\,\boldsymbol{\eta}^{-1} \log \tfrac{\kappa_{\mathsf{L}_b+\mathsf{L}_c}}{\kappa_{\mathsf{L}}} \right] \lambda^2$$

As $(\mathcal{T} \cdot Q)^{\perp} = (\mathcal{T}^{-1})^{*} \cdot Q^{\perp}$, Lyapunov of $(\mathcal{T}^{-1})^{*}$ are $\gamma'_{\mathsf{w}} = -\gamma_{\mathsf{L}-\mathsf{w}+1}$

Corollary

If Hypotheses 1-5 hold for $\mathcal{R}^{-1}$ and distribution of $\mathcal{P}^{*}$,

$$\frac{1}{\mathsf{q}} \sum_{\mathsf{w}=\mathsf{L}-\mathsf{q}+1}^{\mathsf{L}} \gamma_{\mathsf{w}} \leqslant \log(\kappa_{\mathsf{L}_b}) - \left[\tfrac{3}{2} + 10\,\boldsymbol{\eta}^{-1} \log \tfrac{\kappa_1}{\kappa_{\mathsf{L}_c}} \right] \lambda^2$$

Conclusion

- Random dynamics on $\mathbb{G}_{\mathsf{L},\mathsf{q}}$, induced by $\mathcal{T}_n = e^{\lambda \mathcal{P}_n} \mathcal{R}$
- $\mathcal{R}$ has local hyperbolicity dominated by λ
- Coupling assumptions on $\mathcal{P}_n$
- Result: upper bound on expectation to be in upper *a*-part

$$\mathbb{E}\, \mathsf{d}(Q) \; = \; \mathbb{E}\, \mathrm{Tr}(\hat{P}_a Q \hat{P}_a) \; = \; \mathcal{O}(\boldsymbol{\eta}^{-1} \mathsf{q}\, \lambda^2)$$

or in terms fo Furstenberg measure

$$\int \mu_\lambda(dQ)\, \mathsf{d}(Q) \; = \; \mathcal{O}(\boldsymbol{\eta}^{-1} \mathsf{q}\, \lambda^2)$$

- Control of the perturbation: Ladder construction
- Application: Bounds on Lyapunov exponents

?