Algebraic Topology

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In preparing this lecture, I used the following references:

- Allen Hatcher, **Algebraic Topology**, Cambridge University Press
- Tammo tom Diek, **Algebraic Toplogy**, EMS Textbooks in Mathematics
- W. S. Massey, **Algebraic Topology: An Introduction**, Springer Graduate Texts
- W. S. Massey, **A Basic Course in Algebraic Topology**, Springer Graduate Texts
- Gerd Laures, Markus Szymik, **Grundkurs Topologie**, Spektrum Akademischer Verlag

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Contents

1 Topological spaces

1.1 Basic definitions

In this section we recall definitions and results from basic topology. The proofs can be found in topology textbooks and will be omitted here.

Definition 1.1.1:

1. A **topology** on a set X is a set $\mathcal{O}$ of subsets $U \subset X$ such that:

   T1: $X \in \mathcal{O}$, $\emptyset \in \mathcal{O}$

   T2: $U_i \in \mathcal{O}$ for all $i \in I$ implies $\bigcup_{i \in I} U_i \in \mathcal{O}$ for any index set I.

   T3: $U_1, ..., U_n \in \mathcal{O}$ implies $\cap_{i=1}^n U_i \in \mathcal{O}$.

   A subset $U \subset X$ is called **open** if $U \in \mathcal{O}$ and **closed** if $X \setminus U \in \mathcal{O}$. A **topological space** is a pair $(X, \mathcal{O})$ of a set X and a topology $\mathcal{O}$ on X.
2. If $\mathcal{O}_1$ and $\mathcal{O}_2$ are topologies on X, then $\mathcal{O}_2$ is called **coarser** than $\mathcal{O}_1$ and $\mathcal{O}_1$ **finer** than $\mathcal{O}_2$ if $\mathcal{O}_2 \subset \mathcal{O}_1$.
3. A subset $\mathcal{B} \subset \mathcal{O}$ is called a **basis** of $\mathcal{O}$ if every $U \in \mathcal{O}$ is a union of sets in $\mathcal{B}$. It is called a **subbasis** of $\mathcal{O}$ if the set of finite intersections of elements in $\mathcal{B}$ is a basis of $\mathcal{O}$.

Definition 1.1.2: Let $(X, \mathcal{O}_X)$ be a topological space and $A \subset X$.

1. The **interior** of A is the union of all open subsets of A: $A^\circ = \bigcup_{U \subset A, U \in \mathcal{O}_X} U$.
2. The **closure** of A the intersection of all closed sets containing A: $\overline{A} = \bigcap_{A \subset C, X \setminus C \in \mathcal{O}_X} C$.
3. The **boundary** of A is $\partial A = \overline{A} \cap \overline{X \setminus A}$.
4. A **neighbourhood** of $A \subset X$ is a set $B \subset X$ for which there exists an open set U with $A \subset U \subset B$.

Definition 1.1.3:

1. A topological space $(X, \mathcal{O}_X)$ is called **hausdorff** or **Hausdorff space** if for $p, q \in X$ with $p \neq q$ there are open neighbourhoods U_p of p and U_q of q with $U_p \cap U_q = \emptyset$.
2. A topological space $(X, \mathcal{O}_X)$ is called **connected** if $X = A \cup B$ with $A, B \in \mathcal{O}_X$ and $A \cap B = \emptyset$ implies $A = \emptyset$ or $B = \emptyset$.

Important examples of topologies are the discrete and the trivial topology on a set X, which are useful to construct counterexamples and to test hypotheses, and the topology induced by the distance function on a metric space.

Example 1.1.4:

1. For any set X the set $\mathcal{O}_{disc}$ of all subsets of X is a topology on X, the **discrete topology**. It is the finest possible topology on X and makes X a Hausdorff space.

2. For any set X, the set $\mathcal{O}_{triv} = \{X, \emptyset\}$ is a topology on X. It is the coarsest possible topology on X and called the **trivial topology** or **indiscrete topology**.
3. For any topological space $(X, \mathcal{O}_X)$, any subset $A \subset X$ becomes a topological space with the subspace topology $\mathcal{O}_{A,X} = \{A \cap U : \ U \in \mathcal{O}_X\}$. The subset A equipped with this topology is called a subspace of X.
4. Metric spaces (M, d) are Hausdorff spaces with the topology induced by the metric. A set $U \subset M$ is open with respect to this topology if and only if for all $x \in U$ there exists an $\epsilon > 0$ such that $U_\epsilon(x) = \{y \in M : \ d(x, y) < \epsilon\} \subset U$.
5. Normed vector spaces $(V, || \ ||)$ are Hausdorff spaces, since they are metric spaces with distance function $d : V \times V \to \mathbb{R}_0^+$, $d(x, y) = ||x - y||$. Two norms $|| \ ||_1$, $|| \ ||_2$ on a vector space V induce the same topology if and only if they are **equivalent**, i. e. there exist $c, c' \in \mathbb{R}^+$ such that
$$c||v||_1 \leq ||v||_2 \leq c'||v||_1 \qquad \forall v \in V.$$
If V is a finite-dimensional vector space, then all norms on V are equivalent and hence induce the same topology, the **standard topology** on V.

In the following, we will assume without stating this explicitly that metric spaces are equipped with the topology induced by their metric, that finite-dimensional vector spaces are equipped with their standard topology and that subsets of topological spaces are equipped with the subspace topology. In the following chapters, we will also often omit the topology and write X instead of $(X, \mathcal{O}_X)$.

When considering maps between topological spaces, it is natural to ask that these maps are not only maps between the underlying sets but compatible with the given topologies. This leads to the notion of a continuous map. Similarly, it is sensible to ask that an invertible map between topological spaces is not only continuous but also has a continuous inverse, i. e. is a homeomorphism.

Definition 1.1.5: A map $f : X \to Y$ between topological spaces $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ is called:

- **continuous** if $f^{-1}(V)$ is open for all open $V \subset Y$ or, equivalently, $f^{-1}(V)$ is closed for all closed $V \subset Y$.
- **open** (**closed**) if $f(U)$ is open (closed) for all open (closed) $U \subset X$.
- **homeomorphism** if it is bijective and continuous with a continuous inverse.

If $f : X \to Y$ is a homeomorphism, then $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are called **homeomorphic** and we write $(X, \mathcal{O}_X) \approx (Y, \mathcal{O}_Y)$. This is an equivalence relation. In the following, we denote by $C(X, Y)$ the set of continuous functions $f : X \to Y$ and by $\mathrm{Homeo}(X, Y)$ the set of all homeomorphisms $f : X \to Y$.

Example 1.1.6:

1. For any topological space $(X, \mathcal{O}_X)$ the identity map $\mathrm{id}_X : (X, \mathcal{O}_X) \to (X, \mathcal{O}_X)$ is a homeomorphism.
2. The composition $g \circ f : (X, \mathcal{O}_X) \to (Z, \mathcal{O}_Z)$ of two continuous maps (homeomorphisms) $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, $g : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ is continuous (a homeomorphism).

3. For any topological space $(Y, \mathcal{O}_Y)$ and any set X, all maps $f : (X, \mathcal{O}_{disc}) \mapsto (Y, \mathcal{O}_Y)$ and $g : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_{triv})$ are continuous.
4. Let $\mathcal{O}_1$, $\mathcal{O}_2$ be topologies on X. Then $\mathcal{O}_1$ is finer than $\mathcal{O}_2$ if and only if $\mathrm{id} : (X, \mathcal{O}_1) \to (X, \mathcal{O}_2)$ is continuous.
5. Let X be a set with at least two elements. Then the map $\mathrm{id}_X : (X, \mathcal{O}_{disc}) \to (X, \mathcal{O}_{triv})$ is bijective and continuous by 4., but not a homeomorphism, since 4. implies that its inverse $\mathrm{id}_X : (X, \mathcal{O}_{triv}) \to (X, \mathcal{O}_{disc})$ is not continuous.
6. For $n, m \in \mathbb{R}$ with $n \neq m$, $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$. We will prove this statement later in the lecture.

Continuous maps and homeomorphisms between topological spaces play a similar role as linear maps between vector spaces and vector space isomorphisms in linear algebra. In particular, one often does not distinguish topological spaces that are related by homeomorphisms and considers them topologically equivalent. We can also compare topological spaces *locally* by considering homeomorphisms between open neighbourhoods of points in these spaces. In particular, this allows us to introduce the notion of a topological manifold - a topological space that locally looks like $\mathbb{R}^n$.

Definition 1.1.7: An n-dimensional **topological manifold** is a Hausdorff space $(M, \mathcal{O})$ for which every point $m \in M$ has an open neighbourhood $U \subset M$ which is homeomorphic to an open neighbourhood $V \subset \mathbb{R}^n$.

Example 1.1.8:

1. Every open subset $U \subset \mathbb{R}^n$ is an n-dimensional topological manifold, since $\mathrm{id}_U : U \to U$ is a homeomorphism with the required properties.
2. The n-sphere $S^n = \{x \in \mathbb{R}^{n+1} : \ ||x|| = 1\}$ is an n-dimensional topological manifold. The **stereographic projection** maps

$$\phi_\pm : S^n \setminus \{(0, \ldots, 0, \pm 1)\} \to \mathbb{R}^n, \ (x_1, ..., x_{n+1}) \mapsto \left(\frac{x_1}{1 \mp x_{n+1}}, ..., \frac{x_n}{1 \mp x_{n+1}} \right) \qquad (1)$$

and the projection maps

$$\pi^k_\pm : \{x \in S^n : \pm x_k > 0\} \to U_1(0), \ (x_1, ..., x_{n+1}) \mapsto (x_1, ..., x_{k-1}, x_{k+1}, ..., x_{n+1}) \qquad (2)$$

are homeomorphisms onto open subsets of $\mathbb{R}^n$.
3. Consider the circle $C_R = \{(R\cos\phi, R\sin\phi, 0) : \phi \in [0, 2\pi]\}$ of radius $R > 0$ in $\mathbb{R}^3$. Then the **torus** with radii $0 < r < R$ in $\mathbb{R}^3$

$$T_{R,r} = \{p \in \mathbb{R}^3 : d(p, C_R) = r\}.$$

is a 2-dimensional topological manifold. Local homeomorphisms between open subsets in $\mathbb{R}^2$ and open subsets in $T_{R,r}$ are given by the maps

$$\begin{aligned} f_{\phi_0,\psi_0,\epsilon} : (\phi_0 - \epsilon, \phi_0 + \epsilon) \times (\psi_0 - \epsilon, \psi_0 + \epsilon) \to T \qquad & \phi_0, \psi_0 \in \mathbb{R}, 0 < \epsilon < \pi \\ (\phi, \psi) \mapsto ((R + r\cos\phi)\cos\psi, (R + r\cos\phi)\sin\psi, r\sin\phi) & \end{aligned}$$

which are homeomorphism onto their images.

4. The set $\{(x,0): x \in \mathbb{R}\} \cup \{(0,1)\} \subset \mathbb{R}^2$ is not a topological manifold. This follows from the fact that $\{(x,0) : x \in \mathbb{R}\} \approx R$ is a 1-dimensional manifold but the point $(0,1)$ does not have a neighbourhood homeomorphic to an open set in $\mathbb{R}$.

In many situations one wishes to derive conclusions about *global* properties of topological spaces or continuous functions between them from their *local* properties. This can be attempted by covering the topological space by neighbourhoods of points or, equivalently, open sets containing these points and to considering the relevant quantities on each of these open sets. When passing from the local to the global picture, it makes a crucial difference if one needs to consider infinitely many open sets of this type or just a finite number of them. Topological spaces for which a finite number of such open sets is sufficient are called *compact*.

Definition 1.1.9: Let $(X, \mathcal{O}_X)$ be a topological space. A subset $A \subset X$ is called **compact** if every open cover of A has a finite subcover

$$A \subset \bigcup_{i \in I} U_i \text{ with } U_i \in \mathcal{O}_X \ \forall i \in I \quad \Rightarrow \quad \exists \text{ finite } J \subset I \text{ with } A \subset \bigcup_{i \in J} U_i.$$

Remark 1.1.10:

1. A finite subset $A \subset X$ is compact with respect to any topology on X. A subset $A \subset X$ is compact with respect to $\mathcal{O}_{disc}$ if and only if A is finite, and every subset $A \subset X$ is compact with respect to $\mathcal{O}_{triv}$.
2. If $(X, \mathcal{O}_X)$ is a compact topological space and $A \subset X$ closed, then A is compact.
3. If $(X, \mathcal{O}_X)$ is a Hausdorff space and $A \subset X$ compact, then A is closed.
4. If $A \subset X$ is compact and $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ continuous, then $f(A)$ is compact.
5. If $(X, \mathcal{O}_X)$ is compact and $(Y, \mathcal{O}_Y)$ hausdorff, then every continuous map $f : X \to Y$ is closed. If additionally, f is injective (surjective, bijective), then f is an embedding (a quotient map, a homeomorphism).
6. Let (M, d) be a metric space and $A \subset M$. If A is compact, then A is bounded, i. e. for every $m \in M$ there exists a $C > 0$ such that $d(m, a) \leq C$ for all $a \in A$. The subset A is compact if and only if every sequence $(a_k)_{k \in \mathbb{N}}$, $a_k \in A$, has a convergent subsequence.
7. **Heine-Borel:** If $(V, || \ ||)$ is a finite dimensional vector space, then $A \subset V$ is compact if and only if A is closed and bounded. For infinite-dimensional normed vector spaces compact implies closed and bounded, but closed and bounded does not imply compact.

In the context of compact metric spaces, a particularly nice property of open covers is the existence of Lebesgue's number. It ensures that for any open cover, subsets of sufficiently small diameter are contained in one of the open sets of the cover.

Lemma 1.1.11: (Lebesgue's lemma)

Let (M, d) be a compact metric space and $(U_i)_{i \in I}$ an open cover of M. Then there exists a number $\lambda \in \mathbb{R}^+$, the **Lebesgue number** such that any $A \subset X$ with $\text{diam}(A) = \sup\{d(a,b) : a, b \in A\} < \lambda$ is contained in a set U_i with $i \in I$.

Proof:
Suppose there is no such number λ. Then for all $n \in \mathbb{N}$ there is a subset $A_n \subset M$ that satisfies $\mathrm{diam}(A_n) < 1/n$ and is not contained in any of the open sets U_i. Choose a sequence $(a_n)_{n\in\mathbb{N}}$, $a_n \in A_n$. Since M is compact, we can assume after choosing an appropriate subsequence that $a_n \to a \in A$ for $n \to \infty$ (Remark 1.1.10). Then there is a $k \in \mathbb{N}$ and an $\epsilon > 0$ such that $U_\epsilon(a) = \{m \in M : d(m,a) < \epsilon\} \subset U_k$. As the sequence $(a_n)_{n\in\mathbb{N}}$ converges, there exists an $N \in \mathbb{N}$ with $d(a_N, a) < \epsilon/2$ and $\mathrm{diam}(A_N) < \epsilon/2$. This implies for all $m \in A_N$

$$d(m,a) \leq d(m,a_N) + d(a_N,a) \leq \mathrm{diam}(A_n) + d(a_N,a) < \epsilon$$

and hence $A_N \subset U_k$. Contradiction. □

1.2 Basic constructions and examples

In this section, we recall the four basic constructions for topological spaces, namely subspaces, quotients, sums and products. These four basic constructions allow one to construct new topological spaces from given ones. As the ultimate goal is to relate these topological constructions to algebraic ones, it is essential to characterise them from a structural or abstract viewpoint. In the following, we will therefore emphasise *universal properties*, which characterise these constructions in terms of the existence and unqiueness of continuous maps with certain properties.

Definition 1.2.1:

1. Let $(X, \mathcal{O}_X)$ be a topological space and $A \subset X$ a subset. Then
$$\mathcal{O}_\subset = \{V \subset A : \exists U \in \mathcal{O}_X \,\text{with}\, U \cap A = V\}$$
is a topology on A, the **subspace topology** or **relative topology** on A.
2. An **embedding** is an injective, continuous map $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ that is a homeomorphism onto its image, i. e. for which the associated map $(X, \mathcal{O}_X) \to (f(X), \mathcal{O}_{f(X)\subset Y})$ is a homeomorphism.

Remark 1.2.2:

1. The subspace topology is the coarsest topology on A for which the inclusion map $i : A \to X$ is continuous.
2. For any subset $A \subset X$, the inclusion map $i : A \to X$, $a \mapsto a$ is an embedding with respect to the topology on X and the subspace topology on A.
3. Subspaces and embeddings are characterised uniquely by a **universal property**:
An injective map $i : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is an embedding if and only if for any topological space $(W, \mathcal{O}_W)$, a map $g : W \to X$ is continuous if and only if $i \circ g : W \to Y$ is continuous.

While Definition 1.2.1 is more concrete, the characterisation of subspaces and embeddings in this remark give a clear motivation. Subsets of topological spaces are canonically equipped with an inclusion map, and the topology on the subspace is chosen as coarse as possible, in such a way that this map becomes continuous. Similarly, an embedding is an injective map $i : X \to Y$ which is not only continuous but such that post-composition i does not cause information loss with respect to the topology. While for any two continuous maps $i : X \to Y$, $g : W \to X$, one has $i \circ g : W \to Y$ continuous, it is generally not true that the continuity of a map $i \circ g : W \to Y$ implies the continuity of $g : W \to Y$. A simple example is $X = \mathbb{R}$ with the standard topology and $Y = (\mathbb{R}, \mathcal{O}_{triv})$. Then $i = \mathrm{id} : \mathbb{R} \to \mathbb{R}$, $x \mapsto x$ is bijective and continuous, but not an embedding since for all maps $g : W \to X$, one has $i \circ g : W \to Y$ continuous - the map $i \circ g : W \to Y$ does not contain any information about the continuity of $g : W \to X$.

The second basic construction that allows one to obtain new topological spaces from given ones are topological quotients. While subspaces of topological spaces $(X, \mathcal{O}_X)$ are obtained from subsets $A \subset X$ and correspond to embeddings $i : A \to X$, quotient spaces are obtained from equivalence relations on X and correspond to identifications $\pi : X \to B$.

Definition 1.2.3:

1. Let $(X, \mathcal{O}_X)$ be a topological space and $f : X \to Y$ surjective. Then
$$\mathcal{O}_\sim = \{V \subset Y :\ f^{-1}(V) \in \mathcal{O}_X\}$$
is a topology on Y, the **quotient topology** on Y induced by f.
2. A surjective map $f : X \to Y$ between topological spaces $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ is called a **quotient map** or **identification** if $V \subset Y$ is open if and only if $f^{-1}(V) \subset X$ is open. In this case Y is called a **quotient space** of X.

Remark 1.2.4:

1. The quotient topology on Y is the finest topology on Y that makes $f : X \to Y$ continuous.
2. Any map $f : X \to Y$ defines an equivalence relation on X, namely $x \sim x'$ if $f(x) = f(x')$. If f is surjective there is a canonical bijection $X/\sim \xrightarrow{\sim} Y$, $[x] \mapsto f(x)$. Conversely, for any equivalence relation $\sim$ on X, the map $\pi : X \to X/\sim$, $x \mapsto [x]$ is surjective. The set $X/\sim$ with the quotient topology induced by $\pi : X \to X/\sim$ is called the **quotient space** of X with respect to $\sim$.
3. Quotient spaces and quotient maps can be characterised by their **universal properties**: For any continuous map $f : X \to Y$ that satisfies $f(x) = f(x')$ for $x, x' \in X$ with $x \sim x'$, there exists a unique continuous map $f_\sim : X/\sim \to Y$ with $f_\sim \circ \pi = f$, namely $f_\sim : [x] \mapsto f(x)$:
$$\begin{tikzcd} X/\sim \arrow[r, dashed, "\exists! f_\sim"] & Y \\ & X \arrow[ul, "\pi"] \arrow[u, "f"'] \end{tikzcd}$$
A surjective map $\pi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a quotient map if and only if for any topological space $(Z, \mathcal{O}_Z)$, a map $f_\sim : Y \to Z$ is continuous if and only if $f_\sim \circ \pi : X \to Z$ is continuous.
4. Quotients of topological spaces induce quotients of their subspaces. If $\sim$ is an equivalence relation on X and $A \subset X$ a subspace with inclusion map $i_A : A \to X$, then one obtains an equivalence relation $\sim_A$ on A by setting $a \sim_A a'$ if and only if $i_A(a) \sim i_A(a')$. The

universal property of quotient spaces then implies that there is a unique continuous map $i_{A/\sim}: A/\sim_A \to X/\sim$ such that the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[d, "\pi_A"'] & X \arrow[d, "\pi_X"] \\ A/\sim_A \arrow[r, "i_{A/\sim}"'] & X/\sim \end{tikzcd}$$

The map $i_\sim : A/\sim_A \to X/\sim$ is given by $i_\sim([a]) = [i_A(a)]$, and it is injective by definition.

5. Quotients of compact topological spaces are compact.

   For any open cover $X/\sim \subset \bigcup_{i\in I} U_i$, we obtain an open cover $X \subset \bigcup_{i\in I} \pi^{-1}(U_i)$, since $\pi^{-1}(U_i) \in \mathcal{O}_X$ by definition of the quotient topology. As X is compact, there is a finite $J \subset I$ with $X \subset \bigcup_{i\in J} \pi^{-1}(U_i)$, and $X/\sim \subset \bigcup_{i\in J} U_i$ is a finite subcover of $X/\sim$.

The advantage of the characterisation of topological quotients in Remark 1.2.4 is its clear motivation - the topology on the quotient is chosen as fine as possible such that the given surjective map $\pi : X \to Y$ is continuous. In analogy to the characterisation of an embedding, a quotient map is a map $\pi : X \to Y$ that is not only continuous but also such that pre-composition with π does not cause any information loss with respect to the topology. While $f \circ \pi : X \to Z$ is continuous for any two continuous maps $\pi : X \to Y$ and $f : Y \to Z$, the continuity of $f \circ \pi$ does in general not imply the continuity of $f : Y \to Z$. A simple example is $X = \mathbb{R}$ with the discrete topology, $Y = \mathbb{R}$ with the standard topology and $\pi = \mathrm{id} : X \to Y$. Then $f \circ \pi : X \to Z$ is continuous for any map $f : X \to Z$ and hence cannot contain any information about the continuity of f.

The characterisation of continuous functions $f_\sim : X/\sim \to Y$ in terms of functions $f : X \to Y$ that are constant on the equivalence classes is advantageous in practice since it can be much simpler. Moreover, in many situations it is not necessary to specify $f_\sim : X/\sim \to Y$ explicitly but sufficient to know that it exists and that it is unique.

Example 1.2.5:

1. **Circle:** Consider the interval $[0,1]$ with the equivalence relation $t \sim t$ for $t \in [0,1]$ and $0 \sim 1$. Then the quotient $[0,1]/\sim$ with the quotient topology is homeomorphic to S^1. A homeomorphism is induced by the map

$$\exp : [0,1] \to S^1 = \{z \in \mathbb{C} : |z| = 1\}, \quad t \mapsto e^{2\pi \mathrm{i} t} = \cos(2\pi t) + \mathrm{i}\sin(2\pi t).$$

   As $\exp : [0,1] \to S^1$ is continuous and $\exp(0) = \exp(1) = 1$, by the universal property of the quotient space there is a unique continuous map $\exp_\sim : [0,1]/\sim \to S^1$ with $\exp_\sim \circ \pi = \exp$, where $\pi : [0,1] \to [0,1]/\sim$, $t \mapsto [t]$ is the canonical surjection. As $e^{2\pi \mathrm{i} t} = e^{2\pi \mathrm{i} t'}$ implies $t - t' \in \mathbb{Z}$, the map $\exp_\sim : [0,1]/\sim \to S^1$ is bijective. As $[0,1]$ is compact, $[0,1]/\sim$ is compact by Remark 1.2.4, 5. As S^1 is hausdorff as a subspace of a Hausdorff space, Remark 1.1.10 then implies that $\exp_\sim : [0,1]/\sim \to S^1$ is a homeomorphism.

2. **Torus:** Consider $[0,1] \times [0,1]$ with the equivalence relation $(x,y) \sim (x,y)$, $(0,y) \sim (1,y)$ and $(x,0) \sim (x,1)$ for all $x, y \in [0,1]$. Then by a similar argument to the previous example, $[0,1]\times[0,1]/\sim$ with the quotient topology is homeomorphic to the torus $S^1 \times S^1$.

3. **Möbius strip:** Consider $[0,1]\times[0,1]$ with the equivalence relation $(x,y)\sim(x,y)$, $(x,0)\sim(1-x,1)$ for all $x,y\in[0,1]$. Then the quotient space $[0,1]\times[0,1]/\sim$ with the quotient topology is called the **Möbius strip**.
4. **Real projective space:** Consider $\mathbb{R}^{n+1}\setminus\{0\}$ with the equivalence relation $x\sim_{\mathbb{R}} y$ if there is a $\lambda\in\mathbb{R}^\times$ with $y=\lambda x$. Then the quotient space $\mathbb{RP}^n=(\mathbb{R}^{n+1}\setminus\{0\})/\sim_{\mathbb{R}}$ with the quotient topology is called **real projective space**. Points in $\mathbb{RP}^n$ correspond to lines through the origin in $\mathbb{R}^{n+1}$. $\mathbb{RP}^n$ is a topological manifold of dimension n.
5. **Complex projective space:** Consider $\mathbb{C}^{n+1}\setminus\{0\}$ with the equivalence relation $x\sim_{\mathbb{C}} y$ if there is a $\lambda\in\mathbb{C}^\times$ with $y=\lambda x$. Then $\mathbb{CP}^n=(\mathbb{C}^{n+1}\setminus\{0\})/\sim_{\mathbb{C}}$ with the quotient topology is called **complex projective space**. Points in $\mathbb{CP}^n$ correspond to planes through the origin in $\mathbb{C}^{n+1}\approx\mathbb{R}^{2n+2}$. $\mathbb{CP}^n$ is a topological manifold of dimension $2n$.

Another important example of quotient spaces are topological spaces that are obtained by collapsing a given nonempty subspace to a point. In this case, the relevant equivalent relation identifies all points that are contained in the subspace.

Example 1.2.6: If $\emptyset\neq A\subset X$ is a subspace of a topological space X, then the inclusion map $\iota: A\to X$, $a\mapsto a$ defines an equivalence relation on X, namely $x\sim x'$ if $x=x'$ or $x,x'\in\iota(A)$. The associated quotient space $X/\sim$ with the quotient topology is called the topological space obtained by **collapsing** $A\subset X$. The universal property of quotient spaces implies that for any topological space $(Y,\mathcal{O}_Y)$ continuous functions $f_\sim: X/\sim\,\to Y$ are in bijection with continuous functions $f: X\to Y$, that are constant on A.

Two other essential constructions that allow one to construct new topological spaces from a given family of topological space are topological sums and products. In both cases, it is necessary to first consider the underlying constructions for sets. Given a family of sets $(X_j)_{j\in J}$, we consider their **disjoint union**

$$\amalg_{j\in J}X_j=\{(x_j,j):\ j\in J,\ x_j\in X_j\},$$

and the canonical inclusions $i_k: X_k\to\amalg_{j\in J}X_j$, $x\mapsto(x,k)$. For a family of maps $f_j: X_j\to Y_j$, we have the associated **sum map**

$$\amalg_{j\in J}f_j:\amalg_{j\in J}X_j\to\amalg_{j\in J}Y_j,\quad (x,j)\mapsto(f_j(x),j)$$

and for a family of maps $f_j: X_j\to Y$, a map

$$\langle f_j\rangle_{j\in J}:\amalg_{j\in J}X_j\to Y,\quad (x,j)\mapsto(f_j(x)).$$

Requiring that all canonical inclusions are continuous yields a topology on the disjoint union, the sum topology. This topology is characterised by a universal property, which allows one to define continuous maps from topological sums to other topological spaces uniquely by specifying their behaviour on each component.

Definition 1.2.7: Let $(X_j,\mathcal{O}_j)_{j\in J}$ be a family of topological spaces. Then

$$\mathcal{O}_{\amalg}=\{U\subset\amalg_{j\in J}X_j: U\cap i_k(X_k)\in i_k(\mathcal{O}_k)\}$$

is a topology on $\amalg_{j\in J}X_j$, and $(\amalg_{j\in J}X_j,\mathcal{O}_{\amalg})$ is called the **topological sum** of $(X_j,\mathcal{O}_j)$. For $J=\{1,...,n\}$, we also write $X_1+\ldots+X_n$ instead of $(\amalg_{j\in J}X_j,\mathcal{O}_{\amalg})$ and $f_1+\ldots+f_n$ or $(f_1,...,f_n)$ instead of $\amalg_{j\in J}f_j$.

Remark 1.2.8:

1. The sum topology is the finest topology on $\amalg_{j\in J} X_j$ for which all inclusion maps $i_k : X_k \to \amalg_{j\in J} X_j$ are continuous.
2. The sum topology is characterised by a **universal property**:
   For any family of continuous maps $f_j : X_j \to Y$ there exists a unique continuous map $f : \amalg_{j\in J} X_j \to Y$ with $f \circ i_j = f_j$ for all $j \in J$, namely $f = \langle f_j \rangle_{j\in J}$:
   $$\begin{tikzcd} Y & \amalg_{j\in J} X_j \arrow[l, dashed, "\exists! f"'] \\ & X_j \arrow[ul, "f_j"] \arrow[u, "i_j"'] \end{tikzcd} \qquad (3)$$
3. The sum $\amalg_{j\in J} f_j : \amalg_{j\in J} X_j \to \amalg_{j\in J} Y_j$ of continuous maps $f_j : X_j \to Y_j$ is continuous.
4. Sums of topological spaces are associative up to homeomorphism:
   $(X_1 + X_2) + X_3 \approx X_1 + (X_2 + X_3)$ for all triples of topological spaces $(X_i, \mathcal{O}_i)$, $i = 1, 2, 3$.

The other standard construction for a family of sets $(X_j)_{j\in J}$ is their **product set** or **Cartesian product** $\Pi_{j\in J} X_j = \{(x_j)_{j\in J},\, x_j \in X_j\}$. Instead of inclusion maps, one then has **projection maps** $\mathrm{pr}_k : \Pi_{j\in J} X_j \to X_k$, $(x_j)_{j\in J} \mapsto x_k$ on the kth factor. For each family of maps $f_j : X_j \to Y_j$, there is a **product map**

$$\Pi_{j\in J} f_j : \Pi_{j\in J} X_j \to \Pi_{j\in J} Y_j, \quad (x_j)_{j\in J} \mapsto (f_j(x_j))_{j\in J}$$

and for each family of maps $f_j : W \to X_j$ a map

$$(f_j)_{j\in J} : W \to \Pi_{j\in J} X_j, \quad w \mapsto (f_j(w))_{j\in J}.$$

It is then natural to equip the cartesian product $\Pi_{j\in J} X_j$ with the coarsest topology that makes all projection maps $\mathrm{pr}_k : \Pi_{j\in J} X_j \to X_k$ continuous. The resulting topological space has a universal property that characterises continuous maps $f : W \to \Pi_{j\in J} X_j$ in terms of the associated maps $\mathrm{pr}_k \circ f : W \to X_k$.

Definition 1.2.9: Let $(X_j, \mathcal{O}_j)_{j\in J}$ be a family of topological spaces. Then

$$\mathcal{O}_\Pi = \{\text{arbitrary unions of finite intersections of sets } \mathrm{pr}_j^{-1}(U_j), U_j \in \mathcal{O}_j\}$$

is a topology on $\Pi_{j\in J} X_j$, the **product topology** and $(\Pi_{j\in J} X_j, \mathcal{O}_\Pi)$ is called the **product** of the topological spaces $(X_j, \mathcal{O}_j)$. For $J = \{1, ..., n\}$, we also write $X_1 \times \ldots \times X_n$ instead of $(\Pi_{j\in J} X_j, \mathcal{O}_\Pi)$ and $f_1 \times \ldots \times f_n$ instead of $\Pi_{j\in J} f_j$.

Remark 1.2.10:

1. The product topology is the coarsest topology on the set $\Pi_{j\in J} X_j$ that makes all projection maps $\mathrm{pr}_k : \Pi_{j\in J} X_j \to X_k$ continuous.

2. The product topology is characterised by a **universal property**:
   For a family of continuous maps $f_j : W \to X_j$ there exists a unique continuous map $f : W \to \Pi_{j\in J} X_j$ with $\mathrm{pr}_j \circ f = f_j$ for all $j \in J$, namely $f = (f_j)_{j\in J}$:

$$\begin{tikzcd} W \arrow[r, dashed, "\exists! f"] \arrow[dr, "f_j"'] & \Pi_{j\in J} X_j \arrow[d, "\mathrm{pr}_j"] \\ & X_j \end{tikzcd} \tag{4}$$

3. The product $\Pi_{j\in J} f_j : \Pi_{j\in J} X_j \to \Pi_{j\in J} Y_j$ of continuous maps $f_j : X_j \to Y_j$ is continuous.
4. Products of topological spaces are associative up to homeomorphism:
   $(X_1 \times X_2) \times X_3 \approx X_1 \times (X_2 \times X_3)$ for all triples of topological spaces $(X_i, \mathcal{O}_i)$, $i = 1, 2, 3$.

Note that topological sums and products are based on the same input data, a family $(X_j)_{j\in J}$ of topological spaces, and that the diagrams (3) and (4) that characterise their universal properties are related by a reversal of arrows. One says that these constructions are dual to each other. While finite topological sums and products

Example 1.2.11:

1. $\phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n+m}$, $(x, y) \mapsto (x, y)$ is homeomorphism between $\mathbb{R}^n \times \mathbb{R}^m$ with the product topology and $\mathbb{R}^{n+m}$ with the standard topology.
2. For any $0 < r < R$, the torus $T_{R,r} \subset \mathbb{R}^3$ from example 1.1.8 is homeomorphic to the product $S^1 \times S^1$ via $f : S^1 \times S^1 \to T_{R,r}$, $((x, y), (u, v)) \mapsto (Rx + ru, Ry + ru, rv)$.

To conclude our discussion of the general properties of topological sums and products, we recall that desirable properties of topological spaces such as compactness and the hausdorff property are compatible with sums and products in the following sense.

Remark 1.2.12: Let $(X_j, \mathcal{O}_j)_{j\in J}$ be a family of topological spaces.

1. If $(X_j, \mathcal{O}_j)$ is hausdorff for all $j \in J$, then $(\amalg_{j\in J} X_j, \mathcal{O}_\amalg)$ is hausdorff.
2. If $(X_j, \mathcal{O}_j)$ is compact for all $j \in J$, then $(\Pi_{j\in J} X_j, \mathcal{O}_\Pi)$ is compact (**Tychonoff**).
3. If J is finite, then $(\amalg_{j\in J} X_j, \mathcal{O}_\amalg)$ and $(\Pi_{j\in J} X_j, \mathcal{O}_\Pi)$ are compact (hausdorff) if and only if $(X_j, \mathcal{O}_j)$ is compact (hausdorff) for all $j \in J$

1.3 Pullbacks and pushouts, attaching and CW-complexes

In the following, we consider topological spaces, which are obtained by combining the four basic constructions - subspaces, quotients, sums and products. Two fundamental constructions which cover a large class of relevant examples are pullbacks and pushouts. They are obtained by combining topological products with subspaces and topological sums with quotients, respectively. We will see later that these constructions have algebraic counterparts and can be formulated more generally in any category.

Definition 1.3.1: Let $(W, \mathcal{O}_W)$, $(Y, \mathcal{O}_Y)$, $(X_i, \mathcal{O}_{X_i})$, $i = 1, 2$, be topological spaces.

1. For a pair of continuous maps $f_i : X_i \to Y$, the set
$$X_1 \times_Y X_2 = \{(x_1, x_2) \in X_1 \times X_2 :\ f_1(x_1) = f_2(x_2)\} \subset X_1 \times X_2$$
becomes a topological space when equipped with the subspace topology induced by the product topology on $X_1 \times X_2$. It is called the **pullback** or **fibre product** of X_1 and X_2.
2. For a pair of continuous maps $g_i : W \to X_i$, $i_1 \circ g_1(w) \sim i_2 \circ g_2(w)$ for all $w \in W$ defines an equivalence relation on $X_1 + X_2$. The quotient space $X_1 + X_2/\sim$ with the quotient topology induced by the sum topology on $X_1 + X_2$ is called the **pushout** of X_1 and X_2 and denoted $X_1 +_W X_2$.

As combinations of topological products with subspaces and of topological sums with quotients, pullback and pushout inherit universal properties, which arise from the universal properties of the underlying constructions.

Lemma 1.3.2: Let $(W, \mathcal{O}_W)$, $(Y, \mathcal{O}_Y)$ and $(X_i, \mathcal{O}_{X_i})$, $i = 1, 2$, be topological spaces.

1. **Universal property of the pullback:** Let $X_1 \times_Y X_2$ be the pullback of X_1 and X_2 with continuous functions $f_i : X_i \to Y$ and $\pi_i = \mathrm{pr}_i|_{X_1 \times_Y X_2} : X_1 \times_Y X_2 \to X_i$. Then the inner rectangle in the following diagram commutes, and for any two continuous functions $g_i : W \to X_i$ for which the outer quadrilateral commutes, there is a unique continuous function $g : W \to X_1 \times_Y X_2$ such that the two triangles commute
$$\begin{tikzcd} W \arrow[dr, dashed, "\exists! g"'] \arrow[drr, bend left, "g_2"] \arrow[ddr, bend right, "g_1"'] & & \\ & X_1 \times_Y X_2 \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] & X_2 \arrow[d, "f_2"] \\ & X_1 \arrow[r, "f_1"'] & Y. \end{tikzcd} \tag{5}$$

2. **Universal property of the pushout:** Let $X_1 +_W X_2$ be the pushout of X_1 and X_2 with continuous functions $g_i : W \to X_i$ and $\iota_i = \pi \circ i_i : X_i \to X_1 +_W X_2$. Then the inner rectangle in the following diagram commutes, and for any two continuous functions $f_i : X_i \to Y$ for which the outer quadrilateral commutes, there is a unique continuous function $f : X_1 +_W X_2 \to Y$ such that the two triangles commute
$$\begin{tikzcd} Y & & \\ & X_1 +_W X_2 \arrow[ul, dashed, "\exists! f"] & X_2 \arrow[l, "\iota_2"'] \arrow[ull, bend right, "f_2"'] \\ & X_1 \arrow[u, "\iota_1"] \arrow[uul, bend left, "f_1"] & W. \arrow[l, "g_1"] \arrow[u, "g_2"'] \end{tikzcd} \tag{6}$$

Proof:
1. The inner rectangle in (5) commutes by definition, since we have $f_1 \circ \mathrm{pr}_1(x_1, x_2) = f_1(x_1) = f_2(x_2) = f_2 \circ \mathrm{pr}_2(x_1, x_2)$ for all $(x_1, x_2) \in X_1 \times_Y X_2$. The existence and uniqueness of the continuous function $g : W \to X_1 \times_Y X_2$ follows from the universal property of the product topology. The latter states that for $g_i : W \to X_i$ there exists a unique map $g : W \to X_1 \times X_2$

with $\mathrm{pr}_i \circ g = g_i$. It follows that $f_2 \circ \mathrm{pr}_2 \circ g = f_2 \circ g_2 = f_1 \circ g_1 = f_1 \circ \mathrm{pr}_1 \circ g$ and hence $g(W) \subset X_1 \times_Y X_2$. By definition, one then has $\pi_i \circ g = \mathrm{pr}_i|_{X_1 \times_Y X_2} \circ g = \mathrm{pr}_i \circ g = g_i$ for $i = 1, 2$.

2. The inner rectangle in diagram (11) commutes, as the equivalence relation on $X_1 + X_2$ implies $\iota_1 \circ g_1(w) = \pi \circ i_1 \circ g_1(w) = \pi \circ i_2 \circ g_2(w) = \iota_2 \circ g_2(w)$ for all $w \in W$. The existence and uniqueness of the map f follows from the universal property of the sum and the quotient topology. By definition of the sum topology, there exists a unique continuous map $f' : X_1 + X_2 \to Y$ with $f' \circ i_j = f_j$. The map f' is constant on the equivalence classes in $X_1 + X_2$ since $f' \circ i_1 \circ g_1(w) = f_1 \circ g_1(w) = f_2 \circ g_2(w) = f' \circ i_2 \circ g_2(w)$ for all $w \in W$. By the universal property of the quotient topology, there is a unique continuous map $f : X_1 +_W X_2 \to Y$ with $f \circ \pi = f'$, and we have $f \circ \iota_2 \circ g_2 = f \circ \pi \circ i_2 \circ g_2 = f' \circ i_2 \circ g_2 = f' \circ i_1 \circ g_1 = f \circ \pi \circ i_1 \circ g_1 = f \circ \iota_1 \circ g_1$. □

From their universal properties it is evident that pullback and pushout are dual constructions - the diagrams (5) and (11) which characterise their universal properties are related to each other by a reversal of arrows, the former makes use of topological sums and the latter of topological products. While the pullback may seem more intuitive at first sight - it involves a subspace of a topological product, while the pushout involves quotients, pushouts also have a clear geometrical interpretation, namely in the attaching or gluing of topological spaces. For this, one considers two topological spaces $X_1 = X$, $X_2 = Y$, a subspace $A \subset X_1$ and a continuous function $f : A \to Y$. The maps $g_i : A \to X_i$ are then given by $g_1 = i_A : A \to X$ and $g_2 = f$.

Definition 1.3.3: Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces, $A \subset X$ and $f : A \to Y$ continuous. Then the pushout $X +_A Y$ of X and Y by $g_1 = i_A : A \to X$ and $g_2 = f : A \to Y$ is denoted $X \cup_f Y$ and called the topological space obtained by **attaching** or **gluing** X to Y.

1. If $X = \amalg_{i \in I} D^n$ is a topological sum of n-discs $D^n = \{x \in \mathbb{R}^n : ||x|| \leq 1\}$, $A = \amalg_{i \in I} \partial D^n = \amalg_{i \in I} S^{n-1}$ and the attaching map $f = \langle f_i \rangle_{i \in I} : \amalg_{i \in I} S^{n-1} \to Y$ is given by continuous maps $f_i : S^{n-1} \to Y$ via the universal property of the topological sum, one speaks of **attaching n-cells** to Y with the **attaching maps** $f_i : S^{n-1} \to Y$. The map $\iota_j = \pi \circ i_j : D^n \to X \cup_f Y$ is called the **characteristic map** of the jth n-cell D^n.
2. If X is a topological sum $\amalg_{i \in I} X_i$, $A = \amalg_{i \in I} \{p_i\}$ with $p_i \in X_i$ and $Y = \{p\}$ then $X \cup_f Y$ is called the **wedge sum** of the topological spaces X_i and and denoted $\vee_{i \in I} X_i$.

Note that attaching is not symmetric with respect to the topological spaces X and Y. As it is given by the equivalence relation $i_X(a) \sim i_Y \circ f(a)$ on $X + Y$, it does not identify the inclusions of different points in Y, and consequently, the topological space Y can be embedded in $X \cup_f Y$. However, this equivalence relation identifies points $a, a' \in A \subset X$ with $f(a) = f(a')$. For this reason, in general there is no embedding of X into $X \cup_f Y$ unless $f : A \to Y$ is injective - one speaks of *attaching X to Y*. As attaching is a special case of the pushout construction, it follows immediately that it can be characterised by a universal property. Moreover, one can show that the order in which different topological spaces are attached to a given topological space Y does not matter- the resulting topological spaces are homeomorphic.

Lemma 1.3.4: Let $(X_i, \mathcal{O}_{X_i})$, $(Y, \mathcal{O}_Y)$ be topological spaces, $A_i \subset X_i$ subspaces with inclusion maps $j_i : A_i \to X_i$ and $f_i : A_i \to Y$ continuous functions. Then one has:

1. **Universal property of attaching:** The map $\iota_Y : Y \to Y \cup_{f_1} X_1$ is an embedding, and for any two continuous maps $g_{X_1} : X_1 \to Z$, $g_Y : Y \to Z$ with $g_Y \circ f_1 = g_{X_1} \circ j_1$ there is

a unique map $g : X_1 \cup_{f_1} Y \to Z$ for which the following diagram commutes

$$
\begin{tikzcd}
Z & & \\
& X_1 \cup_{f_1} Y \arrow[ul, dashed, "\exists ! g"'] & Y \arrow[l, "\iota_Y"'] \arrow[ull, bend right, "g_Y"'] \\
& X_1 \arrow[u, "\iota_{X_1}"] \arrow[uul, bend left, "g_{X_1}"] & A_1. \arrow[l, "j_1"] \arrow[u, "f_1"']
\end{tikzcd}
\tag{7}
$$

2. **Commutativity of attaching:** For topological spaces $(Y, \mathcal{O}_Y)$, $(X_i, \mathcal{O}_{X_i})$, $i = 1, 2$, subspaces $A_i \subset X_i$ and continuous functions $f_i : A_i \to Y$ one has

$$X_1 \cup_{\tilde{f}_1} (X_2 \cup_{f_2} Y) \approx X_2 \cup_{\tilde{f}_2} (X_1 \cup_{f_1} Y) \approx (X_1 + X_2) \cup_{\langle f_1, f_2 \rangle} Y$$

with $\tilde{f}_1 = \iota_Y \circ f_1 : X_1 \to X_2 \cup_{f_2} Y$, $\tilde{f}_2 = \iota_Y \circ f_2 : X_2 \to X_1 \cup_{f_1} Y$ and $\langle f_1, f_2 \rangle : A_1 + A_2 \to Y$ defined by the universal property of the sum topology.

Proof. The first claim is obvious, and for the second claim we prove the identity

$$X_2 \cup_{\iota_Y \circ f_2} (X_1 \cup_{f_1} Y) \approx (X_1 + X_2) \cup_{f_1 + f_2} Y.$$

To simplify notation, we set $V := (X_1 + X_2) \cup_{f_1 + f_2} Y$, $W := X_1 \cup_{f_1} Y$ and $Z = X_2 \cup_{\iota_Y \circ f_2} (X_1 \cup_{f_1} Y)$. Using the universal property for the attaching of X_1 to Y with f_1, the universal property for the attaching of X_2 to W with $\iota_Y \circ f_2$ and the universal properties of the sums $A_1 + A_2$ and $X_1 + X_2$, we obtain the following diagram, in which the triangles $A_1 Y (A_1 + A_2)$, $Y A_2 (A_1 + A_2)$ and $Z(X_1 + X_2) X_2$, the quadrilaterals $A_1 (A_1 + A_2)(X_1 + X_2) X_1$, $A_2 X_2 (X_1 + X_2)(A_1 + A_2)$, $Z(X_1 + X_2) X_1 W$ and $X_1 W Y A_1$ and the pentagon $Z X_2 A_2 Y W$ commute.

$$
\begin{tikzcd}
& & W \arrow[dll, "\iota"'] & X_1 \arrow[l, "\iota_{X_1}"'] \arrow[dddr, bend left, "i_{1,12}"] & \\
Z & & Y \arrow[u, "\iota_Y"] & A_1 \arrow[u, "j_1"'] \arrow[l, "f_1"'] \arrow[d, "j_{1,12}"] & \\
& X_2 \arrow[ul, "\iota_{X_2}"] & A_2 \arrow[l, "j_2"'] \arrow[u, "f_2"'] \arrow[r, "j_{2,12}"'] & A_1 + A_2 \arrow[ul, "f_1 + f_2"] \arrow[dr, "\langle j_1, j_2 \rangle"'] & \\
& & & & X_1 + X_2 \arrow[uullll, bend left, "\iota_{Z,12}"] \arrow[from=ulll, bend right, "i_{2,12}"']
\end{tikzcd}
\tag{8}
$$

Denote now by $\iota_{Y,12} : Y \to V$ and $\iota_{X_1 + X_2} : X_1 + X_2 \to V$ the canonical maps for the attaching of $X_1 + X_2$ to Y with $f_1 + f_2$. Then the universal property of attaching implies $\iota_{X_1+X_2} \circ \langle j_1, j_2 \rangle = \iota_{Y,12} \circ (f_1 + f_2)$. From this and the commutativity of $A_1(A_1 + A_2)(X_1 + X_2)X_1$, we obtain

$$\iota_{X_1+X_2} \circ i_{1,12} \circ j_1 = \iota_{X_1+X_2} \circ \langle j_1, j_2 \rangle \circ j_{1,12} = \iota_{Y,12} \circ (f_1 + f_2) \circ j_{1,12} = \iota_{Y,12} \circ f_1.$$

The universal property for attaching X_1 to Y with f_1 then implies that there is a unique continuous map $\iota' : W \to V$ with $\iota' \circ \iota_{X_1} = \iota_{X_1+X_2} \circ i_{1,12}$ and $\iota' \circ \iota_Y = \iota_{Y,12}$. We obtain the diagram

$$
\begin{tikzcd}
 & & V & & \\
 & & W \arrow[u, "\iota'"] \arrow[dll, "\iota"'] & X_1 \arrow[l, "\iota_{X_1}"'] \arrow[dddr, "i_{1,12}"] & \\
Z & & Y \arrow[u, "\iota_Y"] & A_1 \arrow[u, "j_1"] \arrow[l, "f_1"'] \arrow[d, "j_{1,12}"] & \\
 & X_2 \arrow[ul, "\iota_{X_2}"] \arrow[drrr, "i_{2,12}"'] & A_2 \arrow[u, "f_2"] \arrow[l, "j_2"'] \arrow[r, "j_{2,12}"'] & A_1 + A_2 \arrow[ul, "f_1+f_2"'] \arrow[dr, "\langle j_1, j_2\rangle"'] & \\
 & & & & X_1 + X_2 \arrow[uuuull, bend right, "\iota_{X_1+X_2}"'] \arrow[ullll, bend left, "\iota_{Z,12}"]
\end{tikzcd}
\tag{9}
$$

in which the new quadrilateral $VWX_1(X_1+X_2)$ and pentagon $VWY(A_1+A_2)(X_1+X_2)$ also commute. From this diagram, we obtain

$$\iota' \circ \iota_Y \circ f_2 = \iota' \circ \iota_Y \circ (f_1 + f_2) \circ j_{2,12} = \iota_{X_1+X_2} \circ \langle j_1, j_2\rangle \circ j_{2,12} = \iota_{X_1+X_2} \circ i_{2,12} \circ j_2,$$

and by the universal property of $X_2 \cup_{\tilde{f}_2} (X_1 \cup_{f_1} Y)$ there is a unique continuous map $\iota'' : Z \to V$ with $\iota'' \circ \iota = \iota'$ and $\iota'' \circ \iota_{X_2} = \iota_{X_1+X_2} \circ i_{2,12}$. As we have

$$
\begin{aligned}
\iota_{Z,12} \circ \langle j_1, j_2\rangle \circ j_{2,12} &= \iota_{Z,12} \circ i_{2,12} \circ j_2 = \iota_{X_2} \circ j_2 = \iota \circ \iota_Y \circ f_2 = \iota \circ \iota_Y \circ (f_1 + f_2) \circ j_{2,12} \\
\iota_{Z,12} \circ \langle j_1, j_2\rangle \circ j_{1,12} &= \iota_{Z,12} \circ i_{1,12} \circ j_1 = \iota \circ \iota_{X_1} \circ j_1 = \iota \circ \iota_Y \circ f_1 = \iota \circ \iota_Y \circ (f_1 + f_2) \circ j_{1,12},
\end{aligned}
$$

the universal property of the sum $A_1 + A_2$ implies $\iota_{Z,12} \circ \langle j_1, j_2\rangle = \iota \circ \iota_Y \circ (f_1 + f_2)$, i. e. the quadrilateral $Z(X_1+X_2)(A_1+A_2)YW$ commutes. With the universal property of attaching $X_1 + X_2$ to Y with $f_1 + f_2$ it then follows that there is a unique continuous map $r : V \to Z$ with $r \circ \iota_{X_1+X_2} = \iota_{Z,12}$ and $r \circ \iota'' \circ \iota \circ \iota_Y = r \circ \iota' \circ \iota_Y = \iota \circ \iota_Y$.

$$
\begin{tikzcd}
 & & V \arrow[ddll, bend right, "r"'] & & \\
 & & W \arrow[u, "\iota'"] \arrow[dll, "\iota"'] & X_1 \arrow[l, "\iota_{X_1}"'] \arrow[dddr, "i_{1,12}"] & \\
Z \arrow[uurr, "\iota''"] & & Y \arrow[u, "\iota_Y"] & A_1 \arrow[u, "j_1"] \arrow[l, "f_1"'] \arrow[d, "j_{1,12}"] & \\
 & X_2 \arrow[ul, "\iota_{X_2}"] \arrow[drrr, "i_{2,12}"'] & A_2 \arrow[u, "f_2"] \arrow[l, "j_2"'] \arrow[r, "j_{2,12}"'] & A_1 + A_2 \arrow[ul, "f_1+f_2"'] \arrow[dr, "\langle j_1, j_2\rangle"'] & \\
 & & & & X_1 + X_2 \arrow[uuuull, bend right, "\iota_{X_1+X_2}"'] \arrow[ullll, bend left, "\iota_{Z,12}"]
\end{tikzcd}
\tag{10}
$$

The universal property of attaching X_2 to $X_1 \cup_{f_1} Y$ with $\iota_Y \circ f_2$ guarantees that $r \circ \iota'' = \mathrm{id}_Z$ if $r \circ \iota'' \circ \iota = \iota$ and $r \circ \iota'' \circ \iota_{X_2} = \iota_{X_2}$. The second identity follows directly, since $r \circ \iota'' \circ \iota_{X_2} = r \circ \iota_{X_1+X_2} \circ i_{2,12} = \iota_{Z,12} \circ i_{2,12} = \iota_{X_2}$. The first follows with the universal property of attaching X_1 to Y with f_1 from the identities $r \circ \iota'' \circ \iota \circ \iota_Y = \iota \circ \iota_Y$ and $r \circ \iota'' \circ \iota \circ \iota_{X_1} = r \circ \iota_{X_1+X_2} \circ i_{1,12} = \iota_{Z,12} \circ i_{1,12} = \iota \circ \iota_{X_1}$. This shows that $r \circ \iota'' = \mathrm{id}_Z$.

To show that $\iota'' \circ r = \mathrm{id}_V$, by the universal property of attaching $X_1 + X_2$ to Y with $f_1 + f_2$, it is sufficient to show that $\iota'' \circ r \circ \iota_{X_1+X_2} = \iota_{X_1+X_2}$ and $\iota'' \circ r \circ \iota' \circ \iota_Y = \iota' \circ \iota_Y$. The last identity follows since $\iota'' \circ r \circ \iota' \circ \iota_Y = \iota'' \circ r \circ \iota'' \circ \iota \circ \iota_Y = \iota'' \circ \iota \circ \iota_Y = \iota' \circ \iota_Y$. The first identity follows from the universal property of $X_1 + X_2$ since $\iota'' \circ r \circ \iota_{X_1+X_2} \circ \iota_{2,12} = \iota'' \circ \iota_{Z,12} \circ i_{2,12} = \iota'' \circ \iota_{X_2} = \iota_{X_1+X_2} \circ \iota_{2,12}$ and $\iota'' \circ r \circ \iota_{X_1+X_2} \circ \iota_{1,12} = \iota'' \circ \iota_{Z,12} \circ i_{1,12} = \iota'' \circ \iota \circ \iota_{X_1} = \iota_{X_1+X_2} \circ i_{1,12}$. This shows that $\iota'' \circ r = \mathrm{id}_V$, and hence $r : V \to Z$ and $\iota'' = r^{-1} : Z \to V$ are homeomorphisms.

□

Example 1.3.5:

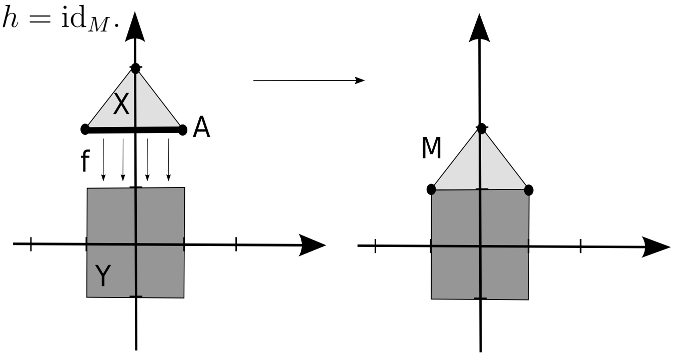
1. Let $X = \mathrm{conv}(\{p_1, p_2, p_3\}) \subset \mathbb{R}^2$ the triangle with corners $p_1 = (-1, 2)$, $p_2 = (1, 2)$, $p_3 = (0, 3)$, consider the subspace $A = [p_1, p_2] = [-1, 1] \times \{2\}$ and the topological space $Y = [-1, 1] \times [-1, 1]$. Then the topological space obtained by gluing of X to Y with the attaching map $f : A \to Y$, $(x, 2) \mapsto (x, 1)$ is homeomorphic to

$$M = Y \cup \mathrm{conv}(\{(-1, 1), (1, 1), (0, 2)\})$$

This follows because the continuous maps $g_Y = i_Y : Y \to M$, $(x, y) \mapsto (x, y)$ and $g_X : X \to M$, $(x, y) \mapsto (x, y - 1)$ satisfy the condition $g_X \circ i_A = g_Y \circ f$. By Lemma 1.3.4, this implies the existence of a unique continuous map $g : X \cup_f Y \to M$ with $g \circ \pi \circ i_X = g_X$ and $g \circ \pi \circ i_Y = g_Y$. To show that g is a homeomorphism, consider the map

$$h : M \to X \cup_f Y, \quad z \mapsto \begin{cases} [z] & z \in M \cap Y \\ [z + (0, 1)] & z \in M \setminus Y. \end{cases}$$

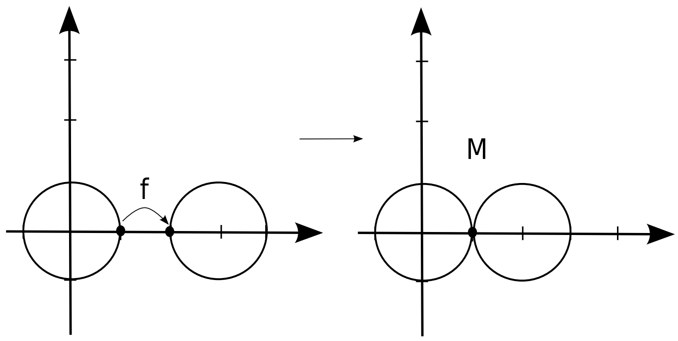
It is continuous, since $[z] = [z + (0, 1)]$ in $X \cup_f Y$ for all $z \in [(-1, 1), (1, 1)]$, and we have $h \circ g = \mathrm{id}_{X \cup_f Y}$, $g \circ h = \mathrm{id}_M$.



Attaching a space X to Y with attaching map $f : A \to Y$

2. If one chooses instead $A = X$ and $f : X \to Y$, $(x, y) \mapsto (x, 1)$, then $\pi \circ i_Y : Y \to X \cup_f Y$ is a homeomorphism, but the corresponding map $\pi \circ i_X : X \to X \cup_f Y$ is not an embedding.
3. For a wedge sum $X \vee Y$, both the maps $\pi \circ i_X : X \to X \vee Y$, $\pi \circ i_Y : Y \to X \vee Y$ are embeddings.
4. The wedge sum of $X = \{x \in \mathbb{R}^2 : ||x|| = 1\}$ and $Y = \{x \in \mathbb{R}^2 : ||x - (3, 0)|| = 1\}$ with respect to the map $f : \{(1, 0)\} \to \{(2, 0)\}$ is homeomorphic to

$$M = \{x \in \mathbb{R}^2 : ||x|| = 1\} \cup \{x \in \mathbb{R}^2 : ||x - (2, 0)|| = 1\}.$$

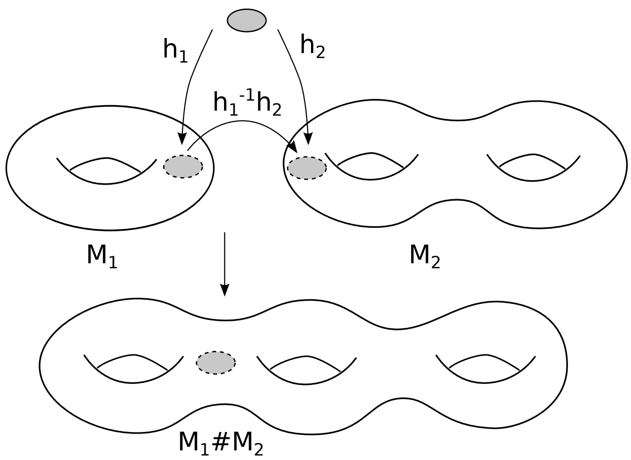


The wedge sum of two circles

Example 1.3.6: To show that a Hausdorff space $(Z, \mathcal{O}_Z)$ is obtained by attaching an n-cell to a compact topological space $(Y, \mathcal{O}_Y)$ with an attaching map $f : S^{n-1} \to Y$, it is sufficient to find injective continuous functions $g_1 : D^n \to Z$, $g_2 : Y \to Z$ with $g_1|_{S^{n-1}} = g_2 \circ f$, $Z \subset g_1(D^n) \cup g_2(Y)$ and $g_1(x) = g_2(y)$ if and only if $x \in S^{n-1}$ and $y = f(x)$.

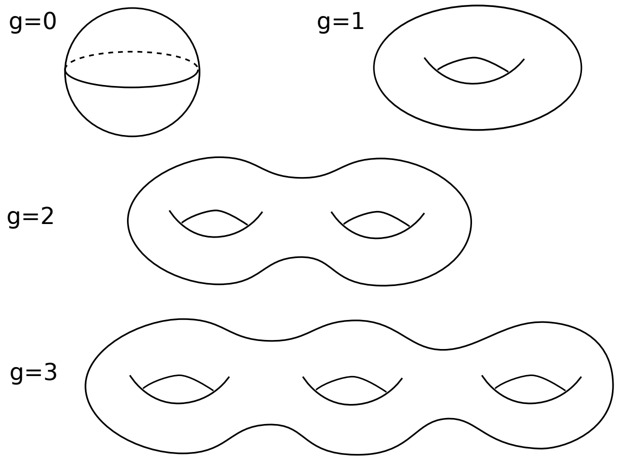
As $g_1|_{S^{n-1}} = g_2 \circ f$, the universal property of the attaching guarantees the existence of a unique continuous function $g : D^n \cup_f Y \to Z$ with $g([x]) = g_1(x)$ and $g([y]) = g_2(y)$ for all $x \in D^n$ and $y \in Y$. As $Z \subset g_1(D^n) \cup g_2(Y)$, g is surjective and as g_1, g_2 are injective with $g_1|_{S^{n-1}} = g_2 \circ f$, the map g is bijective since $g([x]) = g_1(x) = g([y]) = g_2(y)$ implies $x \in S^{n-1}$, $y = f(x)$ and hence $[x] = [y]$. As Z is hausdorff and Y compact, it then follows directly from Remark 1.1.10 that $g : D^n \cup_f Y \to Z$ is a homeomorphism.

An important example of topological spaces that can be constructed by attaching or gluing are topological manifolds. Every point $p \in M$ has an open neighbourhood $V \subset M$ which is homeomorphic to an open subset $U \subset \mathbb{R}^n$, and without restriction of generality, we can assume that it contains the unit disc $D^n \subset \mathbb{R}^n$. This yields an embedding $h : D^n \to M$ and allows one to remove an n-disc from M and glue it to another manifold N along the resulting circle. The resulting construction is called connected sum of topological manifolds.



Connected sum of two manifolds M_1 and M_2

Definition 1.3.7: Let M_1, M_2 be connected n-dimensional topological manifolds and $h_i : D^n \to M_i$ embeddings. Then the **connected sum** $M_1 \# M_2$ of M_1 and M_2 is the gluing of $M_1 \setminus h_1(\mathring{D}^n)$ to $M_2 \setminus h_2(\mathring{D}^n)$ with the homeomorphism $h_2 \circ h_1^{-1}|_{S^{n-1}} : h_1(S^{n-1}) \to h_2(S^{n-1})$. The connected sum $T^{\#g} = T \# T \# \ldots \# T$ of $g \in \mathbb{N}$ tori $T = S^1 \times S^1$ is called a **surface of genus** g. For $g = 0$, one sets $T^{\#0} = S^2$.



Surfaces of genus g

Remark 1.3.8:

1. The connected sum $M_1 \# M_2$ is a connected topological manifold of dimension n and depends on the choice of the embeddings $h_i : D^n \to M_i$ only up to homeomorphisms.
2. One can show that every oriented compact connected topological manifold of dimension 2 is homeomorphic to a surface of genus $g \in \mathbb{N}_0$.

Many topological spaces can be built up from simpler ones by successively attaching n-cells of different dimensions. A collection of n-cells for a given $n \in \mathbb{N}$ can be attached to a topological space Y either one after another or simultaneously. One one hand, the commutativity of attaching construction (see Lemma 1.3.4) guarantees that the result depends on the order only up to homeomorphisms and hence the resulting topological spaces are equivalent. A Hausdorff space that can be realised in this way by successively attaching finitely many or infinitely many n-cells to a discrete set is called a CW-complex.

Definition 1.3.9: Let $(X, \mathcal{O}_X)$ be a Hausdorff space. A **CW-decomposition** or **cell decomposition** of X is a sequence of subspaces $X^0 \subset \ldots \subset X$ such that

(CW1) X^0 is discrete and $X = \bigcup_{n \geq 0} X^n$.

(CW2) For each $n \in \mathbb{N}$, X^n is obtained from X^{n-1} by attaching n-cells.

(CW3) A subset $C \subset X$ is closed with respect to $\mathcal{O}_X$ if and only if it is closed with respect to $\mathcal{O}_{X^n}$ for all $n \geq 0$: X is equipped with the **weak topology**.

A Hausdorff space X with a CW-decomposition is called a **CW-complex** or **cell complex**. The space X^n is called the **n-skeleton** of X and the sequence $X^0 \subset X^1 \subset \ldots \subset X$ the **skeleton filtration**. If $n = \inf\{m \in \mathbb{N}_0 : X = X^m\} \in \mathbb{N}_0$, then n is called the **cellular dimension** of X. A CW-complex of cellular dimension one is called a **graph**. A CW-complex X is called **finite** if X^0 is finite and X is obtained by attaching finitely many cells, i. e. X^0 is finite, X is of cellular dimension n for an $n \in \mathbb{N}_0$ and X^k is obtained from X^{k-1} by attaching finitely many k-cells for all $1 \leq k \leq n$.

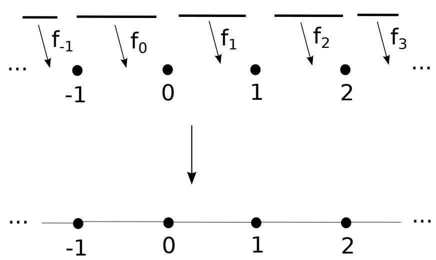
Remark 1.3.10:

1. The axiom (CW3) states that X is equipped with the **weak topology** induced by the topologies of the n-skeleta X^n. This is the letter W in *CW-complex*. This condition is irrelevant for finite CW-complexes but necessary in the infinite case.
2. As X^n is obtained from X^{n-1} by attaching n-cells, it is a quotient of a direct sum $(\amalg_{j\in J_n} D^n) \amalg X^{n-1}$ and there is a canonical surjection $\pi_n : (\amalg_{j\in J_n} D^n) \amalg X^{n-1} \to X^n$. By definition, $A \subset X^n$ is open (closed) if and only if $\pi^{-1}(A) \cap X^{n-1}$ and $\pi^{-1}(A) \cap D^n$ are open (closed) for all n-cells in $\amalg_{j\in J_n} D^n$. As by (CW3) a subset $A \subset X$ is open (closed) if and only if $A \cap X^n$ is open (closed) for all $n \in \mathbb{N}_0$ and X^0 is discrete by (CW1), it follows inductively that $A \subset X$ is open (closed) if and only if its preimage $\iota_j^{-1}(A) \subset D^k$ under the characteristic map $\iota_j : D^k \to X$ is open (closed) for all $j \in J_k$, $k \in \mathbb{N}_0$.
3. If $A \subset X$ is compact, then the sets $\iota_j(\mathring{D}^k) \cap A$ for $j \in J_k$, $k \in \mathbb{N}_0$ form an open covering of A, which has a finite sub covering. This implies that $A \cap \iota_j(\mathring{D}^k) \neq \emptyset$ for only finitely many $k \in \mathbb{N}_0$ and $j \in J_k$. As $\iota_j(\partial D^k)$ is a compact subset of X^{k-1}, it follows that $\iota_j(\partial D^k) \cap \iota_i(\mathring{D}^m) \neq 0$ for only finitely many m-cells with $m < k$. Inductively, we obtain a finite subcomplex of $(X, \cup_{n\geq 0} X^n)$ which contains A. This is called the **closure finiteness condition** and corresponds to the C in *CW-complex*.

In the following we will mainly work with finite CW-complexes, but it is important to also consider infinite CW-complexes for completeness. It can be shown that every every topological manifold is *homotopy equivalent* to a CW-complex and every topological space is *weakly homotopy equivalent* to a CW-complex. This implies that for many purposes, it is sufficient to consider CW-complexes instead of more general topological spaces. However, these results require that one admits infinite CW-complexes.

Example 1.3.11:

1. $\mathbb{R}$ has the structure of a CW-complex with skeleton filtration $X^0 = \mathbb{Z}$, $X^1 = \mathbb{R}$. The 1-skeleton X^1 is obtained from X^0 by attaching $\amalg_{n\in\mathbb{Z}} D^1$ with the attaching maps $f_n : \{-1, 1\} \to \mathbb{Z}$, $f_n(-1) = n - 1$, $f_n(1) = n$ for $n \in \mathbb{Z}$.

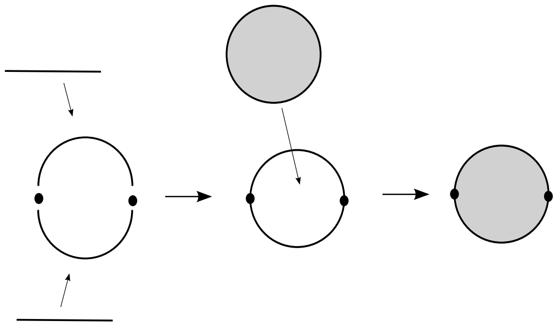


CW structure of $\mathbb{R}$

2. The closed disc $D^2 = \{x \in \mathbb{R}^2 : \ ||x|| \leq 1\}$ has the structure of a CW-complex with skeleton filtration

$$X^0 = \{(1,0)\}, \quad X^1 = S^1, \quad X^2 = D^2.$$

X^1 is obtained from X^0 by attaching a 1-cell with the attaching map $f_1 : S^0 = \{1, -1\} \to X^0$, $f_1(1) = f_1(-1) = (1,0)$, and X^2 is obtained from X^1 by attaching a 2-cell with $f_2 = \mathrm{id}_{S^1} : S^1 \to X^1$.



CW structure of the disc D^2

To show that $X^1 \approx D^1 \cup_{f_1} X^0$, we recall Example 1.3.6 and consider the continuous map $F_1 : D^1 \to X^1$, $x \mapsto (-\cos(\pi x), -\sin(\pi x))$ and the inclusion $i_1 : X^0 \to X^1$. As we have $F_1|_{S^0} = i_1 \circ f_1$, the universal property of attaching yields a unique continuous map $F_1' : D^1 \cup_{f_1} X^0 \to X^1$ with $F_1'([x]) = F_1(x)$ for all $x \in D^1$ and $F_1'((1,0)) = (1,0)$. As $X^1 \subset F_1(D^1)$ and $F_1(x) = f_1(1,0)$ implies $x \in S^1$ and $(1,0) = f_1(x)$, this map is bijective. As X^1 is hausdorff and X^0 compact, this implies that $F_1' : D^1 \cup_{f_1} X^0 \to X^1$ is a homeomorphism.

Similarly, we show that $X^2 \approx D^2 \cup_f X^1$, by considering the continuous map $F_2 = \mathrm{id}_{D^2} : D^2 \to X^2$ together with the canonical inclusion $i_2 : X^1 \to X^2$. Again, $F_2|_{S^1} = i_2 \circ f_2$, $X^2 \subset F_2(D^2)$ and $F_2(x) = f_2(y)$ implies $x = y \in S^1$. As X^2 is hausdorff and X^1 compact, this implies the existence of a homeomorphism $F_2' : D^2 \cup_{f_2} X^1 \to X^2$.

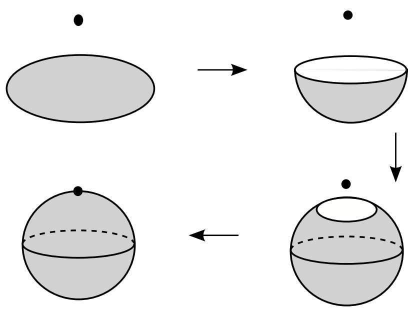
3. For $n \geq 1$, the n-sphere S^n has the structure of a CW-complex with skeleton filtration

$$X^0 = X^1 = \ldots = X^{n-1} = \{(1, 0, ..., 0)\}, \qquad X^n = S^n.$$

X^n is obtained by attaching an n-cell to X^{n-1} with the attaching map $f : S^{n-1} \to X^{n-1}$, $x \mapsto (0, ..., 0, 1)$. Again, one verifies easily with Example 1.3.6 that the inclusion map $i : (1, 0, ..., 0) \to (1, 0, ..., 0)$ and the map

$$F : D^n \to X, \quad (x_1, ..., x_n) \mapsto (2x_1\sqrt{1 - ||x||^2}, ..., 2x_n\sqrt{1 - ||x||^2}, 2||x||^2 - 1)$$

induce a homeomorphism $F' : D^n \cup_f X^{n-1} \to S^n$.



CW structure of the sphere S^2

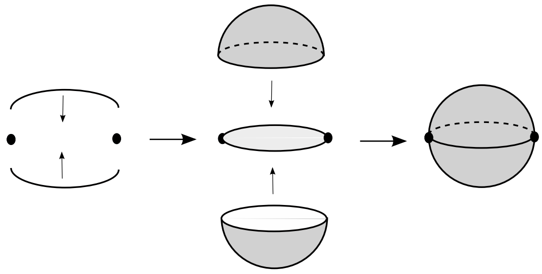
4. For $n \geq 1$, the n-sphere S^n has the structure of a CW-complex with skeleton filtration

$$X^0 = S^0, \quad X^1 = S^1, \quad X^2 = S^2, \quad \ldots, \quad X^n = S^n.$$

X^k is obtained from X^{k-1} by attaching two k-cells with attaching maps $f_k = \mathrm{id}_{S^{k-1}} : S^{k-1} \to S^{k-1}$. Together with the inclusions $i : X^{k-1} \to X^k$, $x \mapsto x$ the maps

$$F_k^{\pm} : D^k \to D^k \cup_{f_\pm^k} S^{k-1}, \quad x \mapsto (x, \pm\sqrt{1 - ||x||^2})$$

then induce a homeomorphism $F^k : D^k \cup_{\langle F_k^+, F_k^- \rangle} X^{k-1} \to S^k$.



CW structure of the sphere S^2

5. Real projective space $\mathbb{R}\mathrm{P}^n$ has the structure of a CW-complex with a skeleton filtration

$$X^0 = \{1\}, \quad X^1 = \mathbb{R}\mathrm{P}^1, \quad X^2 = \mathbb{R}\mathrm{P}^2, \quad \ldots, \quad X^n = \mathbb{R}\mathrm{P}^n,$$

where X^k is obtained from X^{k-1} by attaching a single k-cell with the attaching map $f_k : S^{k-1} \to \mathbb{R}\mathrm{P}^{k-1}$, $x \mapsto [x]$. This follows from the fact that $\mathbb{R}\mathrm{P}^k$ is homeomorphic to a quotient $S^k/\sim$ with the equivalence relation $x \sim \pm x$ on S^k that identifies antipodal points. By considering the inclusion $i_{k-1} : \mathbb{R}\mathrm{P}^{k-1} \to \mathbb{R}\mathrm{P}^k$, $[(x_1, ..., x_k)] \mapsto [(x_1, ..., x_k, 0)]$ and the continuous map $F_k : D^k \to \mathbb{R}\mathrm{P}^k$, $x \mapsto [(x, \sqrt{1 - ||x||^2})]$ one obtains a homeomorphism $D^k \cup_{f_k} \mathbb{R}\mathrm{P}^{k-1} \to \mathbb{R}\mathrm{P}^k$.

6. Complex projective space has the structure of a CW-complex with skeleton filtration

$$X^0 = X^1 = \{1\}, \quad X^2 = X^3 = \mathbb{C}\mathrm{P}^2, \quad \ldots, \quad X^{2n-2} = X^{2n-2} = \mathbb{C}\mathrm{P}^{n-1}, \quad X^{2n} = \mathbb{C}\mathrm{P}^n,$$

where $X^{2k} = \mathbb{C}\mathrm{P}^k$ is obtained from $X^{2k-1} = \mathbb{C}\mathrm{P}^{k-1}$ by attaching a $2k$-cell with the attaching map $f_k : S^{2k-1} \to \mathbb{C}\mathrm{P}^{k-1}, (x_1, x_2, ..., x_{2k-1}, x_{2k}) \mapsto [(x_1 + \mathrm{i}x_2, ..., x_{2k-1} + \mathrm{i}x_{2k})]$. This follows from the fact that $\mathbb{C}\mathrm{P}^k$ is homeomorphic to the quotient $S^{2k-1}/\sim$, where $x \sim y$ if there is a $\lambda \in S^1 \subset \mathbb{C}$ with $(x_1 + \mathrm{i}x_2, ..., x_{2k-1} + \mathrm{i}x_{2k}) = \lambda(y_1 + \mathrm{i}y_2, ..., y_{2k-1} + \mathrm{i}y_{2k})$. The proof is analogous to the last example.

1.4 Topological groups and homogeneous spaces

The product topology and quotient topology are essential in the description of algebraic objects such as groups, vector spaces and algebras that also have the structure of topological spaces. In this case, it is natural to require that all algebraic operations are compatible with the topology. Since for instance the group multiplication is a map $\cdot : G \times G \to G$, a sensible compatibility condition requires the product topology on $G \times G$. By considering quotients of a topological group G with respect to a subgroup $H \subset G$, one then obtains the so-called *homogeneous spaces*. Many important examples such as $\mathbb{R}^n$, the n-spheres S^n, complex projective space $\mathbb{C}\mathrm{P}^n$ and affine spaces can be realised this way and are equipped with additional geometrical structures induced by the underlying groups such as metrics. The description of geometry in terms of symmetry groups was the central goal of Klein's famous *Erlangen programme*, in which homogeneous spaces played a central role.

Definition 1.4.1: A **topological group** is a topological space $(G, \mathcal{O}_G)$ with a group structure such that the multiplication map $\cdot : G \times G \to G$, $(g, h) \to g \cdot h$ is continuous with respect to the product topology on $G \times G$ and $\mathcal{O}_G$ and the inversion map $\mathrm{inv} : G \to G$, $g \mapsto g^{-1}$ is continuous with respect to $\mathcal{O}_G$.

Example 1.4.2: Any matrix group $G \subset \mathrm{Mat}(n, \mathbb{C})$ is a topological group.

The set $\mathrm{Mat}(n, \mathbb{C}) \approx\approx \mathbb{R}^{2n^2}$ is equipped with the standard topology of $\mathbb{R}^{2n^2}$ and any matrix group $G \subset \mathrm{Mat}(n, \mathbb{C})$ becomes a topological space when equipped with the subspace topology. As the multiplication map $\cdot : \mathrm{Mat}(n, \mathbb{C}) \times \mathrm{Mat}(n, \mathbb{C}) \to \mathrm{Mat}(n, \mathbb{C})$ is linear, it is continuous with respect to the product topology on $\mathrm{Mat}(n, \mathbb{C}) \times \mathrm{Mat}(n, \mathbb{C})$ and the standard topology on $\mathrm{Mat}(n, \mathbb{C})$. Similarly, the inversion map $\mathrm{inv} : \{M \in \mathrm{Mat}(n, \mathbb{C}) : \det M \neq 0\} \to \{M \in \mathrm{Mat}(n, \mathbb{C}) : \det M \neq 0\}$, $M \mapsto M^{-1}$ is continuous, since it is a rational function. By the universal property of subspaces, the restrictions of these maps to $G \times G$ and G are continuous.

Example 1.4.3: Any group G becomes a topological group when equipped with the discrete topology. This is due to the fact that associated product topology on $G \times G$ coincides with the discrete topology on $G \times G$. This implies that any map $G \times G \to X$ and $G \to X$ in a topological space X becomes continuous, in particular the multiplication $\cdot : G \times G \to G$ and the inversion $\mathrm{inv} : G \to G$. A topological group $(G, \mathcal{O}_{disc})$ is called a **discrete group**.

Definition 1.4.4: Let $(G, \mathcal{O}_G)$ be a topological group and $(X, \mathcal{O}_X)$ a topological space.

1. A **(left) action** of G on X is a continuous map $\rhd : G \times X \to X$, $(g, x) \mapsto g \rhd x$ with
$$(g \cdot h) \rhd x = g \rhd (h \rhd x), \qquad e \rhd x = x \quad \forall g, h \in G, x \in X.$$
A topological space X with a left action $\rhd : G \times X \to X$ of G is called a **G-space**.
2. A G-space is called a **homogeneous space** for G if the action of G on X is **transitive**, i. e. for any $x, x' \in X$ there is a $g \in G$ with $x' = g \rhd x$.
3. A left action of G on X defines an equivalence relation on X, namely $x \sim x'$ if there exists a $g \in G$ with $x' = g \rhd x$. The associated equivalence classes $G \rhd x = \{g \rhd x : \; g \in G\}$ are called the G-**orbits** of $x \in X$, and the quotient space $X/G = X/\sim$ with the quotient topology is called the the **orbit space**.
4. If $(X, \rhd_X), (Y, \rhd_Y)$ are both G-spaces for a given topological group G, then a continuous map $f : X \to Y$ is called G-**equivariant** or **morphism of G-spaces** if $f(g \rhd_X x) = g \rhd_Y f(x)$ for all $g \in G$, $x \in X$.

Remark 1.4.5: Analogously, one defines a **right action** of a topological group $(G, \mathcal{O}_G)$ on a topological space $(X, \mathcal{O}_X)$ as a continuous map $\lhd : X \times G \to X$ with

$$x \lhd (g \cdot h) = (x \lhd g) \lhd h, \quad x \lhd e = x \qquad \forall g, h \in G, x \in X.$$

If $\rhd : G \times X \to X$ is a left action of G on X, then $\lhd : X \times G \to X$, $(x, g) \mapsto g^{-1} \rhd x$ is a right action of G on X. If $\lhd : X \times G \to X$ is a right action of G on X, then $\rhd : G \times X \to X$, $(g, x) \mapsto x \lhd g^{-1}$ is a left action of G on X.

The universal property of the quotient space implies that for any topological space $(Z, \mathcal{O}_Z)$, continuous maps $f_\sim : X/G \to Z$ are in bijection with continuous maps $f : X \to Z$ that are constant on each orbit $G \rhd x$. It follows that any G-equivariant continuous map $g : X \to Y$ induces a continuous map $g_\sim : X/G \to Y/G$, since the associated map $\pi \circ g : X \to Y/G$, $x \mapsto G \rhd_Y f(x)$ is constant on each G-orbit.

Example 1.4.6: Let $(G, \mathcal{O}_G)$ be a topological group and $H \subset G$ a subgroup, equipped with the subspace topology. Then by definition $\lhd : G \times H \to G$, $(g, h) \mapsto g \cdot h$ is a continuous right action of H on G. The associated orbit space is is the space of right cosets G/H with the quotient topology. For every $g \in G$, the left translation $l_g : G \to G$, $u \mapsto g \cdot u$ is continuous and H-equivariant and hence induces a unique continuous map $l'_g : G/H \to G/H$, $kH \mapsto (gk)H$. It follows that the map $\rhd : G \times G/H \to G/H$, $(g, g'H) \mapsto (gg')H$ is a transitive left action of G on G/H. Hence $(G/H, \rhd)$ is a homogeneous space for G. One can show that the homogeneous space G/H is hausdorff if and only if $H \subset G$ is a closed subgroup and that the group structure on G induces a topological group structure on G/H if and only if $H \subset G$ is **normal**, i. e. $gHg^{-1} = H$ for all $g \in G$.

One can show that any hausdorff homogeneous space $(X, \rhd)$ for a compact topological group G is of the form G/H where $H \subset G$ is a closed subgroup. They key idea is to choose a point $x \in X$ and to consider the elements of G that act trivially on this point.

Lemma 1.4.7: Let $(G, \mathcal{O}_G)$ be a compact topological group and $(X, \mathcal{O}_X)$ a Hausdorff space that is a homogeneous space for G. Denote by $G_x = \{g \in G : g \rhd x = x\}$ the **stabiliser subgroup** of a point $x \in X$. Then there is a G-equivariant homeomorphism $\phi : X \to G/G_x$.

Proof:
Choose a point $x \in X$ and let $G_x = \{g \in G : g \rhd x = x\}$ be its stabiliser. Clearly, $G_x \subset G$ is a subgroup, since $e \rhd x = x$ and $g \rhd x = x$, $g' \rhd x = x$ implies $(gg') \rhd x = g \rhd (g' \rhd x) = g \rhd x = x$. This shows that G/G_x with the quotient topology is a topological space.

The map $\rho_x : G \to X$, $g \mapsto g \rhd x$ is constant on the cosets gG_x and continuous since $\rhd : G \times X \to X$ is continuous. By the universal property of the quotient space, the map $\phi : G/G_x \to X$, $gG_x \mapsto g \rhd x$ is continuous, and it follows directly from the definition that ϕ is G-equivariant. Moreover, ϕ is surjective since G acts transitively on X, and ϕ is injective since $\rho_x(g) = \rho_x(g')$ implies $(g^{-1}g') \rhd x = g^{-1} \rhd (g' \rhd x) = g^{-1} \rhd (g \rhd x) = x \Rightarrow g^{-1}g' \in G_x \Rightarrow gG_x = g'G_x$. As X is hausdorff and G/G_x is compact as a quotient of the compact topological space G, it follows that $\phi : G/G_x \to G$ is a homeomorphism (see Remark 1.1.10). $\square$

Example 1.4.8: S^{n-1} is homeomorphic to the homogeneous space $\mathrm{O}(n,\mathbb{R})/\mathrm{O}(n-1,\mathbb{R})$, where

$$\mathrm{O}(n,\mathbb{R}) = \{A \in \mathrm{GL}(n,\mathbb{R}) : A^T = A^{-1}\}$$

is the real orthogonal group. For this, consider the group action

$$\rhd : \mathrm{O}(n,\mathbb{R}) \times S^{n-1} \to S^{n-1}, \quad (A,x) \mapsto A \cdot x.$$

The stabiliser subgroup of the point $e_1 = (1,0,...,0) \in S^{n-1}$ is the matrix group

$$H = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} : B \in \mathrm{O}(n-1,\mathbb{R}) \right\} \cong \mathrm{O}(n-1,\mathbb{R}).$$

The group $O(n,\mathbb{R})$ is compact since it can be identified with the closed and bounded subset

$$\mathrm{O}(n,\mathbb{R}) = \{A = (a_1,...,a_n) \in \mathbb{R}^n \times \ldots \times \mathbb{R}^n : ||a_i||_{\mathbb{R}^n} = 1, \langle a_i, a_j\rangle_{\mathbb{R}^n} = 0 \text{ for } i \neq j\} \subset \mathbb{R}^{n^2}.$$

As S^{n-1} is hausdorff, Lemma 1.4.7 implies that $\phi : \mathrm{O}(n,\mathbb{R})/\mathrm{O}(n-1,\mathbb{R}) \to S^{n-1}$, $[A] \mapsto Ae_1$ is a homeomorphism.

1.5 Exercises for Chapter 1

Exercise 1: Show that real projective space $\mathbb{RP}^n$ and complex projective space $\mathbb{CP}^n$ are topological manifolds of dimension n and $2n$, respectively.

Exercise 2:
(a) Show that the topological space obtained from $\mathbb{R}$ by collapsing the subspace $(0,1) \subset \mathbb{R}$ is not hausdorff.
(b) Show that the topological space obtained from $\mathbb{R}$ by collapsing the subspace $[0,1] \subset \mathbb{R}$ is homeomorphic to $\mathbb{R}$.

Exercise 3: Show that the quotient $\mathbb{R}^2/\sim$ with $(x_1,x_2) \sim (y_1,y_2)$ if $x_1 - y_1, x_2 - y_2 \in \mathbb{Z}$ is homeomorphic to the torus $T = S^1 \times S^1$.

Exercise 4: Consider the Möbius strip $M = [0,1] \times [0,1]/\sim$ with the equivalence relation given by $(x,0) \sim (1-x,1)$ for all $x \in [0,1]$. Determine the boundary ∂M and an embedding $i : S^1 \to \partial M$.

Exercise 5: Let X, Y be topological spaces.
(a) Consider the pullback $X \times_{\{p\}} Y$ with respect to the point space $\{p\}$ and formulate its universal property.
(b) Consider the pushout $X +_\emptyset Y$ with respect to the empty space $\emptyset$ and formulate its universal property.
(c) Show that the pullback $X \times_Y Y$ of topological spaces X, Y with $\mathrm{id}_Y : Y \to Y$ and a continuous map $f : X \to Y$ is homeomorphic to the graph of f.
(d) Show that for any continuous map $f : X \to Y$ continuous map $g : W \to \mathrm{graph}(f)$ correspond bijectively to pairs of continuous maps $g_X : W \to X$, $g_Y : W \to Y$ that satisfy the condition $f \circ g_X = g_Y$.

Exercise 6: Let X, Y be topological spaces, $A \subset X$ a subspace and $X \cup_f Y$ the topological space obtained by attaching X to Y with the attaching map $f : A \to Y$. Prove that the map $\iota_X : X \to X \cup_f Y$, $x \mapsto [x]$ is an embedding if and only if f is injective.

Exercise 7: Consider the n-discs $D^n = \{x \in \mathbb{R}^n : ||x|| \leq 1\}$ and the n-spheres $S^n = \{x \in \mathbb{R}^{n+1} : ||x|| = 1\}$ for $n \geq 0$.

(a) Show that the topological space obtained by collapsing $\partial D^n = S^{n-1} = \{x \in \mathbb{R}^n : ||x|| = 1\}$ in D^n is homeomorphic to S^n.
(b) Show that attaching an n-cell to an $(n-1)$-sphere with the attaching map $\mathrm{id} : S^{n-1} \to S^{n-1}$ yields a topological space homeomorphic to D^n.
(c) Show that attaching an n-cell to the n-disc with the attaching map $i : S^{n-1} \to D^n$, $x \mapsto x$ yields a topological space homeomorphic to S^n.

Exercise 8:

(a) Show that real projective space $\mathbb{RP}^n$ is obtained by attaching an n-cell to $\mathbb{RP}^{n-1}$.
(b) Show that complex projective space $\mathbb{CP}^n$ is obtained by attaching a $2n$-cell to $\mathbb{CP}^{n-1}$.

Hint: $\mathbb{RP}^n$ and $\mathbb{CP}^n$ can be realised as quotients of, respectively, the n-sphere S^n and the $(2n+1)$-sphere S^{2n+1}.

Exercise 9: Show that the topological space $S^n \times S^m$, $n, m \in \mathbb{N}$, can be obtained by attaching an $(n+m)$-cell to the wedge sum $S^n \vee S^m$.

Exercise 10: Consider the torus $T = [0,1] \times [0,1]/\sim$ where $\sim$ is the equivalence relation on $[0,1] \times [0,1]$ given by $(x,0) \sim (x,1)$, $(0,y) \sim (1,y)$ for all $x, y \in [0,1]$. Show that the torus has the structure of a CW-complex with one 0-cell, two 1-cells and one 2-cell.

Exercise 11: Consider the **Klein bottle** $K = [0,1] \times [0,1]/\sim$ where $\sim$ is the equivalence relation on $[0,1] \times [0,1]$ given by $(x,0) \sim (x,1)$, $(0,y) \sim (1,1-y)$ for all $x, y \in [0,1]$. Show that the Klein bottle has the structure of a CW-complex with one 0-cell, two 1-cells and one 2-cell.

Exercise 12: Let X, Y be topological spaces, $A \subset X$ a subspace and $f : A \to Y$ continuous. Prove that the topological space obtained by first attaching X to Y with $f : A \to Y$ and then collapsing the subspace $i_X(X) \subset X \cup_f Y$ is homeomorphic to the topological space obtained by collapsing the subspace $f(A) \subset Y$ in Y.

Exercise 13: Show that complex projective space $\mathbb{CP}^n$ is homeomorphic to $\mathrm{U}(n+1)/\mathrm{U}(1) \times \mathrm{U}(n)$, where $U(n) = \{A \in \mathrm{Mat}(n, \mathbb{C}) : A^\dagger := \overline{A^T} = A^{-1}\}$ is the unitary group.

Exercise 14: Let Y, X_i with $i = 1, 2$ be topological spaces, $A_i \subset X_i$ subspaces and $f_i : A_i \to Y$ continuous maps. Denote by $X_i \cup_{f_i} Y$ the topological space obtained by attaching X_i to Y with attaching map $f : A_i \to Y$, by $i_{X_i} : X_i \to X_i + Y$ and $i_{Y,i} : Y \to X_i + Y$ the inclusion maps for the topological sums and by $\pi_i : X_i + Y \to X_i \cup_{f_i} Y$ the canonical surjections for the quotient. Show that

$$X_1 \cup_{\tilde{f}_1} (X_2 \cup_f Y) \approx X_2 \cup_{\tilde{f}_2} (X_1 \cup_{f_1} Y) \approx (X_1 + X_2) \cup_{\langle f_1, f_2 \rangle} Y$$

where $\tilde{f}_1 = \pi_2 \circ i_{Y,2} \circ f_1 : X_1 \to X_2 \cup_{f_2} Y$, $\tilde{f}_2 = \pi_1 \circ i_{Y,1} \circ f_2 : X_2 \to X_1 \cup_{f_1} Y$ and $\langle f_1, f_2 \rangle : A_1 + A_2 \to Y$ is the map defined by the universal property of the sum topology.

Exercise 15: Let W, X, Y, Z be topological spaces, $f : W \to X$, $g : W \to Y$, $g : X \to Z$ and $k : Y \to Z$ continuous functions. Then the diagram

$$\begin{tikzcd}
Z & Y \arrow[l, "k"'] \\
X \arrow[u, "h"] & W \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}$$

is called a *pushout square* if it commutes and for all continuous maps $\phi_X : X \to U$, $\phi_Y : Y \to U$ with $\phi_X \circ f = \phi_Y \circ g$ there is a unique continuous map $\phi : Z \to U$ such that the following diagram commutes

$$\begin{tikzcd}
U & & \\
& Z \arrow[ul, dashed, "\exists!\phi"'] & Y \arrow[l, "k"'] \arrow[llu, bend right, "\phi_Y"'] \\
& X \arrow[u, "h"] \arrow[luu, bend left, "\phi_X"] & W. \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd} \tag{11}$$

Suppose that the following diagram of continuous maps and topological spaces commutes

$$\begin{tikzcd}
Z & Y \arrow[l, "k"'] & U \arrow[l, "l"'] \\
X \arrow[u, "h"] & W \arrow[l, "f"] \arrow[u, "g"'] & V. \arrow[l, "m"] \arrow[u, "n"']
\end{tikzcd}$$

(a) Suppose the subdiagram $UVWY$ is a pushout square. Show that in that case $UVXZ$ is a pushout square if and only if $YWXZ$ is a pushout square.

(b) Give an example in which $UVXZ$ and $YWXZ$ are pushout squares but $UVWY$ is not.

2 Algebraic background

2.1 Categories and functors

As algebraic topology aims to describe and ultimately classify topological spaces in terms of algebraic structures, it is necessary to determine which mathematical structures can be used to express such a characterisation. Since they must encompass both, topological and algebraic structures and should be based on as few assumptions as possible, it is clear that these mathematical structures need to be rather abstract.

It is also clear that they should involve an assignment of an algebraic structure such as a group, an algebra or a module to each topological space but that such an assignment is not sufficient in itself. As one does not want to distinguish topological spaces that are homeomorphic, one needs to incorporate the notion of homeomorphisms or, more generally, continuous maps in the picture. Continuous maps and homeomorphisms must correspond, respectively, to structure preserving maps in the algebraic setting (group, algebra and module homomorphisms) and structure preserving maps with a structure preserving inverse (group, algebra and module isomorphisms).

This forces one to first develop a mathematical language which describes mathematical structures and structure preserving maps between them and is sufficiently general to be applied to the topological as well as the algebraic context. The minimum requirements for this to make sense are a notion of composition for structure preserving maps (the composite of two structure preserving maps is structure preserving) that should be associative and a notion of identity maps. By imposing only these requirements, one obtains the notion of a category.

Definition 2.1.1: A **category** $\mathcal{C}$ consists of:

- a class $\mathrm{Ob}\,\mathcal{C}$ of **objects**,
- for each pair of objects $X, Y \in \mathrm{Ob}\,\mathcal{C}$ a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of **morphisms** ,
- for any triple of objects X, Y, Z a **composition map**

$$\circ : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z),$$

such that the following axioms are satisfied:

(C1) The sets $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of morphisms are pairwise disjoint,

(C2) The composition is associative: $f \circ (g \circ h) = (f \circ g) \circ h$ for all objects W, X, Y, Z and morphisms $h \in \mathrm{Hom}_{\mathcal{C}}(W, X)$, $g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, $f \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$,

(C3) For any object X there is a morphism $1_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ with $1_X \circ f = f$ and $g \circ 1_X = g$ for all $f \in \mathrm{Hom}_{\mathcal{C}}(W, X)$, $g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. The morphisms 1_X are called **identity morphisms**.

Instead of $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, we also write $f : X \to Y$. The object X is called **source** of f, and the object Y is called **target** of f. A morphism $f : X \to X$ is called an **endomorphism**.

A morphism $f : X \to Y$ is called an **isomorphism**, if there is a morphism $g : Y \to X$ with $g \circ f = 1_X$ and $f \circ g = 1_Y$. In this case, we write $f : X \xrightarrow{\sim} Y$ and call the objects X and Y **isomorphic**.

Example 2.1.2:

1. The category Set: the objects of Set are sets, and the morphisms are maps $f : X \to Y$. The composition is the composition of maps and the identity morphisms are the identity maps. Isomorphisms are bijective maps.

   Note that the definition of a category requires that morphisms between two objects in a category form a set, but not that the objects of the category do so. Requiring that the objects of a category form a set would force one to consider sets of sets when defining the category Set, which is known to lead to problems and contradictions. A category in which not only the morphisms between given objects but also the objects form a set is called a **small category**.
2. The category Set^* of **pointed sets**: the objects are pairs (X, x) of a set X and a point $x \in X$, and morphisms $f : (X, x) \to (Y, y)$ are maps $f : X \to Y$ with $f(x) = y$. The composition of morphisms is the composition of maps, the identity morphisms are the identity maps and isomorphisms $f : (X, x) \to (Y, y)$ are bijections with $f(x) = y$.
3. The category Top of topological spaces. Objects are topological spaces, morphisms $f : X \to Y$ are continuous maps. The isomorphisms in this category are homeomorphisms.
4. The category Top^* of **pointed topological spaces**: Objects are pairs (X, x) of a topological space X and a point $x \in X$, morphisms $f : (X, x) \to (Y, y)$ are continuous maps $f : X \to Y$ with $f(x) = y$.
5. The category $Top(2)$ of **pairs of topological spaces**: Objects are pairs (X, A) of a topological space X and a subspace $A \subset X$, morphisms $f : (X, A) \to (Y, B)$ are continuous maps $f : X \to Y$ with $f(A) \subset B$. Isomorphisms are homeomorphisms $f : X \to Y$ with $f(A) = B$.
6. Many examples of categories we will encounter in the following are categories of algebraic structures. This includes the following:
   - the category $\mathrm{Vect}_{\mathbb{F}}$ of vector spaces over a field $\mathbb{F}$:
     objects: vector spaces over $\mathbb{F}$, morphisms: $\mathbb{F}$-linear maps,
   - the category $\mathrm{Vect}_{\mathbb{F}}^{fin}$ of finite dimensional vector spaces over a field $\mathbb{F}$:
     objects: vector spaces over $\mathbb{F}$, morphisms: $\mathbb{F}$-linear maps,
   - the category Grp of groups:
     objects: groups, morphisms: group homomorphisms,
   - the category Ab of abelian groups:
     objects: abelian groups, morphisms: group homomorphisms,
   - the category Ring of rings:
     objects: rings, morphisms: ring homomorphisms,
   - the category URing of unital rings:
     objects: unital rings, morphisms: unital ring homomorphisms,
   - the category Field of fields:
     objects: fields, morphisms: field monomorphisms,
   - the category $\mathbb{F}$-Alg of algebras over a field $\mathbb{F}$:
     objects: algebras over $\mathbb{F}$, morphisms: algebra homomorphisms,

- the categories R-Mod (Mod-R) of left (right) modules over a ring R:
  objects: R-left (right) modules, morphisms: R-left (right) module homomorphisms.

- the **representation category** $\mathrm{Rep}_{\mathbb{F}}(G)$ of a group G over a field $\mathbb{F}$:
  objects: representations $\rho_V : G \to \mathrm{Aut}_{\mathbb{F}}(V)$ of G on vector spaces V over $\mathbb{F}$, morphisms: linear maps $f : V \to W$ with $\rho_W(g) \circ f = f \circ \rho_V(g)$ for all $g \in G$.

7. The category $TopGrp$ of topological groups:
   objects: topological groups, morphisms: continuous group homomorphisms,
8. The category $G - Space$ of G-spaces for a topological group G:
   objects: G-spaces $(X, \rhd_X)$, morphisms $f : (X, \rhd_X) \to (Y, \rhd_Y)$: continuous maps $f : X \to Y$ with $g \rhd_Y f(x) = f(g \rhd_X x)$ for all $x \in X$, $g \in G$.
9. The category CW of CW-complexes:
   objects: CW-complexes $X = \cup_{n \geq 0} X_n$,
   morphisms: **cellular maps** $f : X = \cup_{n \geq 0} X_n \to Y = \cup_{n \geq 0} Y_n$, i. e. continuous maps $f : X \to Y$ with $f(X^n) \subset Y^n$ for all $n \in \mathbb{N}$.

Some important examples and basic constructions for categories that are used extensively are given in the next examples. The second example is particulary useful for developing an intuition on categorical notions and relating them to known concepts and constructions.

Example 2.1.3:

1. A small category $\mathcal{C}$ in which all morphisms are isomorphisms is called a **groupoid**.
2. A category with a single object X is a **monoid**, and a groupoid $\mathcal{C}$ with a single object X is a group. Group elements are identified with endomorphisms $f : X \to X$ and the composition of morphisms is the group multiplication. More generally, one can show that for any object X in a groupoid $\mathcal{C}$, the set $\mathrm{End}_{\mathcal{C}}(X) = \mathrm{Hom}_{\mathcal{C}}(X, X)$ with the composition $\circ : \mathrm{End}_{\mathcal{C}}(X) \times \mathrm{End}_{\mathcal{C}}(X) \to \mathrm{End}_{\mathcal{C}}(X)$ is a group.
3. For every category $\mathcal{C}$, one has an **opposite category** $\mathcal{C}^{op}$, which has the same objects as $\mathcal{C}$, whose morphisms are given by $\mathrm{Hom}_{\mathcal{C}^{op}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$ and in which the order of the composition is reversed.
4. The **Cartesian product** of two categories $\mathcal{C}, \mathcal{D}$ is the category $\mathcal{C} \times \mathcal{D}$ whose objects are pairs (C, D) of objects C in $\mathcal{C}$ and D in $\mathcal{D}$, with $\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((C, D), (C', D')) = \mathrm{Hom}_{\mathcal{C}}(C, C') \times \mathrm{Hom}_{\mathcal{D}}(D, D')$ and the composition of morphisms $(h, k) \circ (f, g) = (h \circ f, k \circ g)$ for all $f : C \to C'$, $h : C' \to C''$, $g : D \to D'$, $k : D' \to D''$.
5. **Quotient categories:** Let $\mathcal{C}$ be a category with an equivalence relation $\sim_{X,Y}$ on $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ for all objects X, Y. Suppose that the equivalence relations are compatible with the composition of morphisms, i. e. $f \sim_{X,Y} g$ in $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $h \sim_{Y,Z} k$ in $\mathrm{Hom}_{\mathcal{C}}(Y, Z)$ implies $h \circ f \sim_{X,Z} k \circ g$ in $\mathrm{Hom}_{\mathcal{C}}(X, Z)$. Then one obtains a new category $\mathcal{C}'$, called a **quotient category** of $\mathcal{C}$, with the same objects as $\mathcal{C}$ and equivalence classes of morphisms in $\mathcal{C}$ as morphisms.

   The composition of morphisms in $\mathcal{C}'$ is given by $[h] \circ [f] = [h \circ f]$, and the identity morphisms by the equivalence classes $[1_X]$ of the identity morphisms in $\mathcal{C}$. Isomorphisms in $\mathcal{C}'$ are equivalence classes of morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ for which there exists a morphism $g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$ with $f \circ g \sim_{Y,Y} 1_Y$ and $g \circ f \sim_{X,X} 1_X$.

The construction in the last example plays a fundamental role in classification problems, in particular in the context of topological spaces. Classifying the objects of a category $\mathcal{C}$ usually means classifying them up to isomorphism, obtaining a list of objects in $\mathcal{C}$ such that every object in $\mathcal{C}$ is isomorphic to exactly one object in this list. While this is possible in some contexts - for the category $\mathrm{Vect}_{\mathbb{F}}^{fin}$ of finite dimensional vector spaces over $\mathbb{F}$, the list contains the vector spaces $\mathbb{F}^n$ with $n \in \mathbb{N}_0$ - it is often too difficult to solve this problem in full generality. In this case, it is sometimes simpler to consider instead the category $\mathcal{C}'$ and to attempt a partial classification. If two objects are isomorphic in $\mathcal{C}$, they are by definition isomorphic in $\mathcal{C}'$ since for any objects X, Y in $\mathcal{C}$ and any isomorphism $f : X \to Y$ with inverse $g : Y \to X$, one has $[g] \circ [f] = [g \circ f] = [1_X]$ and $[f] \circ [g] = [f \circ g] = [1_Y]$. However, the converse does not hold in general - the category $\mathcal{C}'$ yields a weaker classification result than $\mathcal{C}$.

We will now consider mathematical concepts that allow one to relate different categories. From the discussion above, it is clear that they must not only relate their objects but also their morphisms, in a way that is compatible with their source and target objects, the identity morphisms and the composition of morphisms. This leads to the notion of a functor.

Definition 2.1.4: Let $\mathcal{C}, \mathcal{D}$ be categories. A **functor** $F : \mathcal{C} \to \mathcal{D}$ consists of:

- an assignment of an object $F(X)$ in $\mathcal{D}$ to every object X in $\mathcal{C}$,
- for any pair of objects X, Y in $\mathcal{C}$, a map

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y)), \quad f \mapsto F(f),$$

which is compatible with the composition of morphisms and with the identity morphisms

$$\begin{aligned} &F(f \circ g) = F(f) \circ F(g) && \forall f \in \mathrm{Hom}_{\mathcal{C}}(X, Y), g \in \mathrm{Hom}_{\mathcal{C}}(W, X) \\ &F(1_X) = 1_{F(X)} && \forall X \in \mathrm{Ob}\, \mathcal{C}. \end{aligned}$$

A functor $F : \mathcal{C} \to \mathcal{C}$ is sometimes called an **endofunctor**. If $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{B} \to \mathcal{C}$ are functors, then their **composite** $FG : \mathcal{B} \to \mathcal{D}$ is the functor given by the assignment $X \mapsto FG(X)$ for all $X \in \mathrm{Ob}\, \mathcal{B}$ and the maps $\mathrm{Hom}_{\mathcal{B}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(FG(X), FG(Y))$, $f \mapsto F(G(f))$.

Example 2.1.5:

1. For any category $\mathcal{C}$, identity functor $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$, that assigns each object and morphism in $\mathcal{C}$ to itself is an endofunctor of $\mathcal{C}$.
2. The functor $\mathrm{Vect}_{\mathbb{F}} \to \mathrm{Ab}$, which assign to each vector space the underlying abelian group and to each linear map the associated group homomorphism, and the functors $\mathrm{Vect}_{\mathbb{F}} \to \mathrm{Set}$, $\mathrm{Ring} \to \mathrm{Set}$, $\mathrm{Grp} \to \mathrm{Set}$, $\mathrm{Top} \to \mathrm{Set}$ etc which assign to each vector space, ring, group, topological space the underlying set and to each morphism the underlying map between sets are functors. A functor of this type is called **forgetful functor**.
3. The functor $* : \mathrm{Vect}_{\mathbb{F}} \to \mathrm{Vect}_{\mathbb{F}}^{op}$, which assigns to a vector space V its dual V^* and to a linear map $f : V \to W$ its adjoint $f^* : W^* \to V^*$, $\alpha \mapsto \alpha \circ f$.
4. Let G, H be topological groups and $f : G \to H$ a continuous group homomorphism. Then one obtains a functor $F : H{-}space \to G{-}space$, which assigns to every H-space $(X, \rhd_{H,X})$ a G-space $(X, \rhd_{G,X})$ with G-action $g \rhd_{G,X} x = f(g) \rhd_{H,X} x$.

   Morphisms $\phi : (X, \rhd_H^X) \to (Y, \rhd_H^Y)$ in H- space are continuous maps $\phi : X \to Y$ with $h \rhd_{H,Y} \phi(x) = \phi(h \rhd_{H,X} x)$ for all $h \in H$, $x \in X$. They are mapped to morphisms

$\phi : (X, \rhd_{G,X}) \to (Y, \rhd_{G,Y})$, since one has $g \rhd_{G,Y} \phi(x) = f(g) \rhd_{H,Y} \phi(x) = \phi(f(g) \rhd_{H,X} x) = \phi(g \rhd_{G,X} x)$ for all $g \in G, x \in X$.

5. If $\mathcal{C}$ is a groupoid with a single object X, i. e. a group $G = (\mathrm{End}_{\mathcal{C}}(X), \circ)$, then a functor $F : \mathcal{C} \to$ Set is a group action of G on the set $F(X)$ and a functor $F : \mathcal{C} \to \mathrm{Vect}_{\mathbb{F}}$ a representation of the group G on the vector space $F(X)$.
6. The **Hom-functors**:
   For any category $\mathcal{C}$ and object X in $\mathcal{C}$, one obtains a functor $\mathrm{Hom}(X, -) : \mathcal{C} \to$ Set, which assigns to an object Y in $\mathcal{C}$ the set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ and to a morphism $f : Y \to Z$ in $\mathcal{C}$ the map $\mathrm{Hom}(X, f) : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$, $g \mapsto f \circ g$.

   Similarly, one obtains a functor $\mathrm{Hom}(-, X) : \mathcal{C}^{op} \to$ Set, which assigns to an object W in $\mathcal{C}^{op}$ the set $\mathrm{Hom}_{\mathcal{C}}(W, X)$ and to a morphism $f : V \to W$ in $\mathcal{C}$ the map $\mathrm{Hom}(f, X) : \mathrm{Hom}_{\mathcal{C}}(W, X) \to \mathrm{Hom}_{\mathcal{C}}(V, X)$, $g \mapsto g \circ f$.

It will become apparent in the following that it is not sufficient to consider functors between different categories but one also needs to to introduce a structure that relates different functors. As a functor $F : \mathcal{C} \to \mathcal{D}$ involves maps between different sets, namely the Hom-spaces $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $\mathrm{Hom}_{\mathcal{D}}(F(x), F(Y))$, this concept must relate the Hom spaces $\mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ and $\mathrm{Hom}_{\mathcal{D}}(G(X), G(Y))$. The simplest way to do this is to assign to each object X in $\mathcal{C}$ a morphism $\eta_X : F(X) \to G(X)$ in $\mathcal{D}$. By requiring that this assignment of morphisms is compatible with the images of morphisms $f : X \to Y$ in $\mathcal{C}$ under F and G, one obtains the notion of a natural transformation.

Definition 2.1.6: A **natural transformation** $\eta : F \to G$ between functors $F, G : \mathcal{C} \to \mathcal{D}$ is an assignment of a morphism $\eta_X : F(X) \to G(X)$ in $\mathcal{D}$ to every object X in $\mathcal{C}$ such that the following diagram commutes:

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"] & G(Y). \end{tikzcd}$$

If for all objects X in $\mathcal{C}$ the morphisms $\eta_X : F(X) \to G(X)$ are isomorphisms, then $\eta : F \to G$ is called a **natural isomorphism** and denoted $\eta : F \xrightarrow{\sim} G$. Two functors that are related by a natural isomorphism are called **naturally isomorphic**.

Example 2.1.7:

1. Consider the category $\mathcal{C} = \mathrm{Vect}(\mathbb{F})$ with functors $\mathrm{id} : \mathrm{Vect}(\mathbb{K}) \to \mathrm{Vect}(\mathbb{K})$ and $** : \mathrm{Vect}(\mathbb{K}) \to \mathrm{Vect}(\mathbb{K})$. Then there is a canonical natural transformation $\mathrm{can} : \mathrm{id} \to **$, which assigns to a vector space $\eta_V : V \to (V^*)^*$ the vector space $(V^*)^*$.
2. If G is a group and $g \in G$ fixed, then $G \to G$, $h \mapsto g \cdot h \cdot g^{-1}$ is a group isomorphism, which gives rise to an endofunctor $F_g : \mathrm{Rep}_{\mathbb{F}}(G) \to \mathrm{Rep}_{\mathbb{F}}(G)$.

   This functor assigns to a representation $\rho_V : G \to \mathrm{Aut}_{\mathbb{F}}(V)$ the representation $F_g(\rho_V) = \rho_V(g) \circ \rho_V \circ \rho_V(g)^{-1} : h \mapsto \rho_V(g \cdot h \cdot g^{-1})$ and to a morphism of representations $f : V \to W$ with $\rho_W \circ f = f \circ \rho_V$ the linear map $f : V \to W$, which is also a morphism of representations between $F_g(\rho_V)$ and $F_g(\rho_W)$, since $F_g(\rho_W) \circ f = \rho_V(g) \circ \rho_V \circ \rho_V(g)^{-1} \circ f = f \circ \rho_W(g) \circ \rho_W \circ \rho_W(g)^{-1} = f \circ F_g(\rho_V)$.

If one assigns to each representation $\rho_V : G \to \mathrm{Aut}_{\mathbb{F}}(V)$ the morphism of representations $\eta_{\rho_V} = \rho_V(g) : V \to V$, then one obtains a natural transformation $\eta : F_g \to \mathrm{id}$, since for all morphisms of representations $f : V \to W$ the following diagram commutes by definition:

$$\begin{tikzcd} \rho_V \arrow[r, "\eta_{\rho_V}"] \arrow[d, "f"'] & F_g(\rho_V) \arrow[d, "f=F_g(f)"] \\ \rho_W \arrow[r, "\eta_{\rho_W}"] & F_g(\rho_W). \end{tikzcd}$$

3. Consider the category CRing of commutative unital rings and ring homomorphisms and the category Grp of group homomorphisms.

   Let F : CRing $\to$ Grp the functor that assigns to a ring R the group $\mathrm{GL}_n(R)$ of invertible matrices with entries in R and to a ring homomorphism $f : R \to S$ the group homomorphism $\mathrm{GL}_n(f) : \mathrm{GL}_n(R) \to \mathrm{GL}_n(S)$, $(r_{ij})_{i,j=1,\ldots,n} \mapsto (f(r_{ij}))_{i,j=1,..,n}$.

   Let G : CRing $\to$ Grp be the functor which assigns to every ring R the group $G(R) = R^\times = \{r \in R : \exists s \in R\ r \cdot r' = r' \cdot r = 1\}$ of its units and to every ring homomorphism $f : R \to S$ the induced group homomorphism $G(f) = f|_{R^\times} : R^\times \to S^\times$. (Note that the image of a unit under a ring homomorphism is a unit, since $r \cdot r' = r' \cdot r = 1$ implies $f(r) \cdot f(r') = f(r') \cdot f(r) = f(r \cdot r') = f(r' \cdot r) = f(1) = 1$.)

   Then $\eta : F \to G$ with $\eta_R = \det : \mathrm{GL}_n(R) \to R^\times$ is natural transformation, since for every ring homomorphism $f : R \to S$ the following diagram commutes

$$\begin{tikzcd} \mathrm{GL}_n(R) \arrow[r, "\det"] \arrow[d, "\mathrm{GL}_n(f)"'] & R^\times \arrow[d, "f|_{R^\times}"] \\ \mathrm{GL}_n(S) \arrow[r, "\det"] & S^\times. \end{tikzcd}$$

Remark 2.1.8: If $\mathcal{C}$ is a small category and $\mathcal{D}$ a category, then the functors $\mathcal{C} \to \mathcal{D}$ and natural transformations between them form a category, which is called the **functor category** and denoted $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

The notions of natural transformations and natural isomorphisms are particularly important as they allow one to generalise the notion of an inverse map and of a bijection to functors. While it is possible to define an inverse of a functor $F : \mathcal{C} \to \mathcal{D}$ as a functor $G : \mathcal{D} \to \mathcal{C}$ with $GF = \mathrm{id}_{\mathcal{C}}$ and $FG = \mathrm{id}_{\mathcal{D}}$ and an isomorphism as a functor with an inverse, it turns out that this is not useful in practice because it is too strict. There are very few non-trivial examples of functors with an inverse. A more useful generalisation is obtained by weakening this requirement. Instead of requiring $FG = \mathrm{id}_{\mathcal{D}}$ and $GF = \mathrm{id}_{\mathcal{C}}$, one requires *naturally isomorphic* to the identity functors. This leads to the concept of an equivalence of categories.

Definition 2.1.9: A functor $F : \mathcal{C} \to \mathcal{D}$ is called an **equivalence of categories** if there is a functor $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\kappa : GF \xrightarrow{\sim} \mathrm{id}_{\mathcal{C}}$ $\eta : FG \xrightarrow{\sim} \mathrm{id}_D$. In this case, the categories $\mathcal{C}$ and $\mathcal{D}$ are called **equivalent**.

Sometimes it is easier to use a more direct characterisation of an equivalences of categories in terms of its behaviour on objects and morphisms. The proof of the following lemma makes use of the axiom of choice and an be found for instance in [K], Chapter XI, Prop XI.1.5.

Lemma 2.1.10: A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if it is:

1. **essentially surjective**:
   for every object A in $\mathcal{D}$ there is an object X of $\mathcal{C}$ such that A is isomorphic to $F(X)$.
2. **fully faithful**:
   for all objects X, Y in $\mathcal{C}$ the map $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$, $f \mapsto F(f)$ is a bijection.

We will now see that many constructions and concepts from topological or algebraic settings can be generalised straightforwardly to categories. This is true whenever it is possible to characterise them in terms of universal properties involving morphisms. In particular, the notion of an object similar to the empty topological space or the one-point space can be formulated in any category. We will also see that there are generalisations of the notions of topological sums and products, namely categorical products and coproducts.

Definition 2.1.11: An object X in a category $\mathcal{C}$ is called:

1. **final** or **terminal** if for every object W in $\mathcal{C}$ there is exactly one morphism $f_W : W \to X$,
2. **cofinal** or **initial** if for eery object Y in $\mathcal{C}$ there is exactly one morphism $k_Y : X \to Y$,
3. **null object**, if it is final and initial; null objects are denoted 0.

Final, initial and null objects are unique up to isomorphisms. This follows directly from their characterisation by a universal property. If A and B are both initial objects in a category $\mathcal{C}$, then there is a unique morphism $f : A \to B$ and a unique morphism $g : B \to A$. The uniqueness property then implies that the composites $g \circ f : A \to A$ and $f \circ g : B \to B$ must agree with the identity morphisms $1_A : A \to A$ and $1_B : B \to B$ and hence $f : A \to B$ is an isomorphism with inverse $g : B \to A$.

Example 2.1.12:

1. The empty set is an initial object in Set and the empty topological space an initial object in Top. Any one point set is a final object in Set and any one point topological space an initial object in Top. The categories Set and Top do not have null objects.
2. The null vector space $\{0\}$ is a null object in the category $\mathrm{Vect}_{\mathbb{F}}$. More generally, for any ring R, the trivial R-module $\{0\}$ is a null object in R-Mod (Mod-R).
3. The trivial group $G = \{e\}$ is a null object in Grp and in Ab.
4. The ring $\mathbb{Z}$ is a initial object in the category URing of unital rings since for every unital ring R, there is exactly one ring homomorphism $f : \mathbb{Z} \to R$, namely

$$f(n) = \begin{cases} \underbrace{1 + \ldots + 1}_{n\times} & n \in \mathbb{N} \\ 0 & n = 0 \\ -\underbrace{(1 + \ldots + 1)}_{(-n)\times} & -n \in \mathbb{N}, \end{cases}$$

where 0 and 1 denote the additive and multiplicative unit of R and $-$ the additive inverse. The zero ring $R = \{0\}$ with $0 = 1$ is a final object in the category of unital rings, but not an initial one.

5. The category Field does not have initial or final objects.

In analogy to the definition of initial, final and zero objects, we will now generalise topological sums and products to general categories $\mathcal{C}$. This can be done straightfowardly by using the corresponding universal properties.

Definition 2.1.13: Let $\mathcal{C}$ be a category and $(X_j)_{j\in J}$ a family of objects in $\mathcal{C}$.

1. A **coproduct** is an object $\amalg_{j\in J} X_j$ in $\mathcal{C}$ together with a family of morphisms $i_j : X_j \to \amalg_{j\in J} X_j$, such that for every family of morphisms $f_j : X_j \to Y$ there is a unique morphism $f : \amalg_{j\in J} X_j \to Y$ such that the diagram
$$\begin{tikzcd} Y & \amalg_{j\in J} X_j \arrow[l, dashed, "\exists! f"'] \\ & X_j \arrow[ul, "f_j"] \arrow[u, "i_j"'] \end{tikzcd} \tag{12}$$
commutes for all $j \in J$. This is called the **universal property** of the coproduct.

2. A **product** is an object $\Pi_{j\in J} X_j$ in $\mathcal{C}$ together with a family of morphisms $\pi_j : \Pi_{j\in J} X_j \to X_j$, such that for all families of morphisms $f_j : W \to X_j$ there is a unique morphism $f : W \to \Pi_{j\in J} X_j$ such that the diagram
$$\begin{tikzcd} W \arrow[r, dashed, "\exists! f"] \arrow[dr, "f_j"'] & \Pi_{j\in J} X_j \arrow[d, "\pi_j"] \\ & X_j \end{tikzcd} \tag{13}$$
commutes for all $j \in J$. This is called the **universal property** of the product.

Remark 2.1.14: Products or coproducts do not necessarily exist for a given family of objects $(X_j)_{j\in J}$ in a category $\mathcal{C}$, but if they exist, they are unique up to isomorphisms. This follows directly from the universal property:

If $(\Pi_{j\in J} X_j, (\pi_j)_{j\in J})$ and $(\Pi'_{j\in J} X_j, (\pi'_j)_{j\in J})$ are two products for a family of objects $(X_j)_{j\in J}$ in $\mathcal{C}$, then for the family of morphisms $f_k = \pi'_k : \Pi'_{j\in J} X_j \to X_k$ there is a unique morphism $f : \Pi'_{j\in J} X_j \to \Pi_{k\in K} X_j$ such that $\pi_k \circ f = \pi'_k$ for all $k \in J$. Similarly, for the family of morphisms $f'_k = \pi_k : \Pi_{j\in J} X_j \to X_k$ there is a unique morphism $f' : \Pi_{j\in J} X_j \to \Pi'_{j\in J} X_j$ with $\pi'_k \circ f' = \pi_k$ for all $k \in J$. It then follows that $f \circ f' : \Pi_{j\in J} X_j \to \Pi_{j\in J} X_j$ is a morphism with $\pi_k \circ f \circ f' = f_k \circ f' = \pi'_k \circ f' = \pi_k$ for all $k \in J$. Since the identity morphism is another morphism with this property, the uniqueness implies that $f \circ f' = 1_{\Pi_{j\in J} X_j}$. By an analogous argument, one obtains $f' \circ f = 1_{\Pi'_{j\in J} X_j}$ and hence the objects $\Pi'_{j\in J} X_j$ and $\Pi_{j\in J} X_j$ are isomorphic. The reasoning for the coproduct is similar.

Example 2.1.15:

1. The disjoint union of sets is a coproduct in Set and the cartesian product of sets a product in Set. The topological sum is a coproduct in Top and the product of topological spaces a product in Top. In Set and Top, products and coproducts exist for all families of objects.
2. The direct sum of vector spaces is a coproduct and the direct product of vector spaces a product in $\mathrm{Vect}_{\mathbb{F}}$. More generally, direct sums and products of R-left (right) modules over a unital ring R are coproducts and products in R-Mod (Mod-R). Again, products and coproducts exist for all families of objects in R-Mod (Mod-R).
3. The wedge sum is a coproduct in the category Top^* of pointed topological spaces.
4. The direct product of groups is a product in Grp. Given a family $(G_j)_{j\in J}$ of groups G_j, one defines their direct product as the set

$$\times_{j\in J}G_j = \{(g_j)_{j\in J}\, g_j \in G_j\}$$

with group multiplication law $(g_j)_{j\in J}\cdot(h_j)_{j\in J} = (g_j\cdot h_j)_{j\in J}$. It is easy to check that $\times_{j\in J}G_j$ is a group and that the projection maps $\pi_k : (g_j)_{j\in J} \to g_k$ are group homomorphisms. Moreover, for any family $(f_j)_{j\in J}$ of group homomorphisms $f_j : H \to G_j$, there is a unique group homomorphism $f : H \to \times_{j\in J}G_j$ with $\pi_k\circ f(h) = f_k(h)$, namely $f : h \mapsto (f_j(h))_{j\in J}$.

We will now show that the category Grp is also equipped with coproducts. The coproduct in the category Grp is called the *free product* of groups and plays an important role in the characterisation of groups in terms of generators and relations. A concrete formulation of the direct product of two groups is obtained by considering the set of alternating tuples of non-trivial group elements and to define a product in terms of their concatenation and the group product in the individual groups.

Definition 2.1.16: Let $(G_i)_{i\in I}$ be a collection of groups with $G_i \cap G_j = \emptyset$ for $i, j \in I$, $i \neq j$[1]. Then the **free product** of the groups G_i, $i \in I$, is the set of reduced **free words** in

$$\star_{i\in I}G_i = \{(h_1, ..., h_n) :\ n \in \mathbb{N}_0, h_i \in G_{k(i)} \setminus \{e\}, k(i) \neq k(i+1)\ \forall i \in \{1, ..., n-1\}\}$$

with the multiplication map $\cdot : \star_{i\in I}G_i \times \star_{i\in I}G_i \to \star_{i\in I}G_i$ defined inductively by

$$(h_1, ..., h_n)\cdot(h'_1, ..., h'_m) = \begin{cases} (h_1, ..., h_n, h'_1, ...h'_m) & h_n \in G_i, h'_1 \notin G_i \\ (h_1, ..., h_n\cdot h'_1, h'_2, ..., h'_m) & h_n, h'_1 \in G_i, h'_1 \neq h_n^{-1} \\ (h_1, ..., h_{n-1})\cdot(h'_2, ..., h'_m) & h'_1 = h_n^{-1} \in G_i. \end{cases} \tag{14}$$

The properties of the free product of groups resemble the properties of topological sums in the category Top. It is is a group which is defined in precisely such a way that the inclusion maps $i_j : G_j \to \star_{i\in I}G_i$ become group homomorphisms and exhibits a universal property, which involves group homomorphisms from the groups G_i to another group H. In other words: the following lemma shows that the free product of groups is a coproduct in the category Grp.

[1]This is assumed for formal reasons and does not restrict generality. If this condition is not satisfied, one can replace the groups G_i by isomorphic groups which satisfy the condition.

Lemma 2.1.17: Let $(G_i)_{i\in I}$ be groups with $G_i \cap G_j = \emptyset$ for $i \neq j$ and denote by $i_j : G_j \to \star_{i\in I} G_i$, $g \mapsto (g)$ the canonical inclusions. Then:

1. $\star_{i\in I} G_i$ is a group with neutral element $()$ and inverse $(h_1, ..., h_n) = (h_n^{-1}, ..., h_1^{-1})$.
2. The inclusion maps $i_j : G_j \to \star_{i\in I} G_i$ are injective group homomorphisms.
3. **Universal property** For any family $(f_i)_{i\in I}$ of group homomorphisms $f_i : G_i \to H$ there is a unique group homomorphism $\star_{i\in I} f_i : \star_{i\in I} G_i \to H$ such that the following diagram commutes for all $j \in I$

$$\begin{tikzcd} H & \star_{i\in I} G_i \arrow[l, dashed, "\exists! \star_{i\in I} f_i"'] \\ & G_j . \arrow[ul, "f_j"] \arrow[u, "i_j"'] \end{tikzcd} \tag{15}$$

Proof:
That the group multiplication $\cdot : \star_{i\in I} G_i \times \star_{i\in I} G_i \to \star_{i\in I} G_i$ is associative with neutral element $()$ and inverse $(h_1, ..., h_n)^{-1} = (h_n^{-1}, ..., h_1^{-1})$ is verified by a direct computation, and the inclusion maps $i_j : G_j \to \star_{i\in I} G_i$ are group homomorphisms by definition. The group homomorphism $\star_{i\in I} f_i : \star_{i\in I} G_i \to H$ is given by

$$\begin{aligned} &\star_{i\in I}\, f_i(()) = e, \quad \star_{i\in I} f_i((h)) = \star_{i\in I} f_i(i_j(h)) = f_j(h) \quad \text{for}\, h \in G_j \\ &\star_{i\in I}\, f_i(h_1, ..., h_n) = (\star_{i\in I} f_i(h_1)) \cdot (\star_{i\in I} f_i(h_2)) \cdots (\star_{i\in I} f_i((h_n))). \end{aligned}$$

If $\phi : \star_{i\in I} G_i \to H$ is another group homomorphism with the universal property, we have $\phi \circ i_j(h) = \phi((h)) = f_j(h)$ for $h \in G_j$ and since ϕ is a group homomorphism $\phi(()) = e$ and $\phi((h_1, ..., h_n)) = \phi(h_1) \cdot \phi(h_2) \cdots \phi(h_n)$, which implies $\phi = \star_{i\in I} f_i : \star_{i\in I} G_i \to H$. □

It is also simple to verify the following properties, which are simple generalisations of properties of topological sums and follow directly from the definitions.

Lemma 2.1.18: The free product of groups has the following properties:

1. **Associativity:** for all groups G_1, G_2, G_3 the groups $G_1 \star (G_2 \star G_3)$ and $(G_1 \star G_2) \star G_3$ are isomorphic.
2. **Generators:** the subgroups $\iota_i(G_i) \cong G_i \subset G_1 \star G_2$ generate $G_1 \star G_2$.
3. **Trivial products:** If $G_1 = \{e\}$ $(G_2 = \{e\})$, then $G_1 \star G_2 \cong G_2$ $(G_1 \star G_2 \cong G_1)$.

Similarly to the notions of topological sums and products, the notions of topological pullbacks and pushouts can also be generalised straightforwardly to categories. For this, one uses their universal property to define them and replaces topological spaces and continuous maps, respectively, by objects and morphisms in the category under consideration.

Definition 2.1.19: Let $\mathcal{C}$ be a category and X_1, X_2, W, Y objects in $\mathcal{C}$.

1. A **pullback** or **fibre product** of two morphisms $f_i : X_i \to Y$ is an object P in $\mathcal{C}$ together with morphisms $\pi_i : P \to X_i$ such that the inner rectangle in the following diagram commutes and for any pairs of morphisms $g_i : W \to X_i$ that make the outer

quadrilateral commute there is a unique morphism $g : W \to P$ that makes the two triangles commute

$$\begin{tikzcd}
W \arrow[dr, dashed, "\exists! g"] \arrow[drr, bend left, "g_2"] \arrow[ddr, bend right, "g_1"'] & & \\
& P \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] & X_2 \arrow[d, "f_2"] \\
& X_1 \arrow[r, "f_1"'] & Y.
\end{tikzcd} \tag{16}$$

This is called the **universal property** of the pullback.

2. A **pushout** of two morphisms $g_i : W \to X_i$ is an object P together with morphisms $\iota_i : X_i \to P$ such that the inner rectangle in the following diagram commutes and for all pairs of morphisms $f_i : X_i \to Y$ for which outer quadrilateral commutes there is a unique morphism $f : P \to Y$ such that two triangles commute

$$\begin{tikzcd}
Y & & \\
& P \arrow[ul, dashed, "\exists! f"'] & X_2 \arrow[l, "\iota_2"'] \arrow[ull, bend right, "f_2"'] \\
& X_1 \arrow[u, "\iota_1"] \arrow[uul, bend left, "f_1"] & W. \arrow[l, "g_1"] \arrow[u, "g_2"']
\end{tikzcd} \tag{17}$$

This is called the **universal property** of the pushout.

Just as in the case of categorical (co)products, pullbacks and pushouts do not necessarily exist for all pairs of morphisms in a given category. However, if they do exist, their universal property characterises them uniquely up to isomorphism. The proof is analogous to the one for the uniqueness of products and coproducts.

Example 2.1.20:

1. In the category Set, the pullback of two maps $f_i : X_i \to Y$ is the set $P = \{(x_1, x_2) \in X_1 \times X_2 : f_1(x_1) = f_2(x_2)\}$ with the projection maps $\pi_i : P \to X_i$, $(x_1, x_2) \mapsto x_i$. The pushout of two maps $g_i : W \to X_i$ is the set $P = X_1 \amalg X_2 / \sim$, where $\sim$ is given by $i_1 \circ g_1(x_1) \sim i_2 \circ g_2(x_2)$ for all $x_1 \in X_1$, $x_2 \in X_2$, together with the maps $\pi \circ i_j : X_j \to P$, $x_j \mapsto [i_j(x_i)]$.
2. Pullbacks and pushouts in topological space are pullbacks and pushouts in Top. Note that the corresponding pullbacks and pushouts in Set are obtained by replacing topological spaces with the underlying sets and omitting the word continuous in all definitions.
3. Attaching is a pushout in Top.

As the direct product of groups $G_1 \times G_2$ is a product in the category Grp, one expects to obtain pullbacks in Grp from pairs of group homomorphisms $f_i : G_i \to H$. Similarly, the fact that the free product of groups is a coproduct in Grp leads one to expect that pushouts in

Grp can be obtained by taking quotients with respect to suitable subgroups of $G_1 \star G_2$. Note, however, that the set of cosets gN for a subgroup $N \subset G$ does in general not have the structure of a group unless we require the subgroup N to be *normal*: $g \cdot N \cdot g^{-1} \subset N$ for all $g \in G$. In this case, G/N becomes a group with multiplication $(gN) \cdot (hN) = (gh)N$ for all $g, h \in G$, and the canonical surjection $\pi : G \to G/N$, $g \mapsto gN$ is a group homomorphism. Moreover, for any subset $A \subset G$, we can consider the **normal subgroup generated by** A

$$\langle A \rangle_N = \bigcap_{\substack{N \subset G \text{ normal} \\ A \subset N}} N,$$

with the group multiplication induced by the multiplication of G. This yields a suitable notion of pushout in the category Grp.

Proposition 2.1.21: Let F, G_1, G_2, H be groups and $f_i : F \to G_i$, $h_i : G_i \to H$ group homomorphisms.

1. Then the set $G_1 \times_H G_2 = \{(g_1, g_2) \in G_2 \times G_2 : \ h_1(g_1) = h_2(g_2)\}$ is a subgroup of $G_1 \times G_2$. Together with the projection maps $\pi_i : G_1 \times_H G_2 \to G_i$, $(g_1, g_2) \mapsto g_i$ this becomes a pullback in Grp.
2. Consider the free product $G_1 \star G_2$ with inclusion maps $i_j : G_j \to G_1 \star G_2$ and the normal subgroup $N = \langle (i_1 \circ f_1(x)) \cdot (i_2 \circ f_2(x^{-1})) : x \in F \rangle \subset G_1 \star G_2$. Then $G_1 \star G_2/N$ is a group, and together with the inclusion maps $\iota_j = \pi \circ i_j : G_j \to (G_1 \star G_2)/N$, $g \mapsto i_j(g)N$, a pushout in Grp.

Proof:
1. It is clear that $G_1 \times_H G_2$ is a subgroup of $G_1 \times G_2$ since $h_1(e_1) = e_H = h_2(e_2)$ and for all $g_i, g_i' \in G_i$, with $h_1(g_1) = h_2(g_2)$ and $h_1(g_1') = h_2(g_2')$, one has $h_1(g_1 \cdot g_1') = h_1(g_1) \cdot h_1(g_1') = h_2(g_2) \cdot h_2(g_2') = h_2(g_2 \cdot g_2')$, which shows that $G_1 \times_H G_2$ is closed under the multiplication. The diagram

$$\begin{tikzcd} G_1 \times_H G_2 \arrow[r, "\pi_2"] \arrow[d, "h_1"'] & G_2 \arrow[d, "h_2"] \\ G_1 \arrow[r, "h_1"'] & H \end{tikzcd}$$

commutes by definition. It remains to verify the universal property. For this, consider a pair of group homomorphisms $f_i : F \to G_i$ with $h_1 \circ f_1 = h_2 \circ f_2$ and note that a group homomorphism $f : F \to G_1 \times G_2$ with $\pi_j \circ f = f_j$ is determined uniquely by the requirement $\pi_j \circ f = f_j$, since this implies $f(k) = (f_1(k), f_2(k))$ for all $k \in F$. To show existence, it is sufficient to show that f takes values in $G_1 \times_H G_2$, which follows from the condition $h_1 \circ f_1 = h_2 \circ f_2$.

2. The set $G_1 \star G_2/N$ is a group since $N \subset G_1 \star G_2$ is normal. The diagram

$$\begin{tikzcd} G_1 \star G_2/N & G_2 \arrow[l, "\iota_2"'] \\ G_1 \arrow[u, "\iota_1"] & F \arrow[l, "f_1"] \arrow[u, "f_2"'] \end{tikzcd}$$

commutes by definition since $i_1 \circ f_1(x) \cdot i_2 \circ f_2(x)^{-1} \in N$ for all $x \in F$. This implies

$$\iota_1 \circ f_1(x) = \pi \circ i_1 \circ f_1(x) = i_1 \circ f_1(x)N = i_2 \circ f_2(x)N = \pi \circ i_1 \circ f_2(x) = \iota_2 \circ f_2(x).$$

To verify the universal property, consider two group homomorphisms $\phi_j : G_j \to H$ with $\phi_1 \circ f_1 = \phi_2 \circ f_2$. Then the group homomorphism $\phi_1 \star \phi_2 : G_1 \star G_2 \to H$ satisfies

$$(\phi_1 \star \phi_2)(i_1 \circ f_1(x) \cdot i_2 \circ f_2(x^{-1})) = (\phi_1 \circ f_1(x)) \cdot (\phi_2 \circ f_2(x^{-1})) = (\phi_1 \circ f_1(x)) \cdot (\phi_1 \circ f_1(x))^{-1} = e$$

for all $x \in F$. This implies $N \subset \ker(\phi_1 \star \phi_2)$ and hence $\phi_1 \star \phi_2 : G_1 \star G_2 \to H$ induces a group homomorphism $\phi : G_1 \star G_2/N \to H$, $gN \mapsto (\phi_1 \star \phi_2)(g)$ with $\phi \circ \iota_k(g) = \phi \circ \pi \circ i_k(g) = (\phi_1 \star \phi_2)(g) = \phi_k(g)$ for all $g \in G$. On the other hand, the requirement $\phi \circ \iota_j = \phi \circ \pi \circ i_j = \phi_j$ determines the group homomorphism $\phi : G_1 \star G_2/N \to H$ uniquely since $\pi : G_1 \star G_2 \to G_1 \star G_2/N$ is surjective and the subset $i_1(G_1) \cup i_2(G_2) \subset G_1 \star G_2$ generates $G_1 \star G_2$. $\square$

The pushouts in Grp are important since they allow one to characterise groups in terms of generators and relations, which is often convenient. Note however, that such a characterisation is highly non-unique and it is in general difficult to decide if two groups that are given in terms of generators and relations are isomorphic.

Definition 2.1.22:

1. For any set A, the free product $\langle A \rangle = \star_{a \in A} \mathbb{Z}$ is called the **free group generated by** A.
2. The n-fold free product $F_n = \langle \{1, .., n\} \rangle = \mathbb{Z} \star \ldots \star \mathbb{Z}$ is called the **free group** with n generators.
3. If a group G is given as a quotient $\langle A \rangle / N$ with a normal subgroup $N \subset \langle A \rangle$ and $B \subset \langle A \rangle$ generates N, one writes $G = \langle A | b = 1 \; \forall b \in B \rangle$ and speaks of a **presentation** of G. If $A = \{a_1, ..., a_n\} \cong \{1, ..., n\}$ and $B = \{r_1, ..., r_k\}$ are finite, the group G is called **finitely presented**, and one writes $G = \langle a_1, ..., a_n | \; r_1 = 1, ..., r_k = 1 \rangle$.

Example 2.1.23:

1. The group $\mathbb{Z}/n\mathbb{Z}$ has a presentation $\mathbb{Z}/n\mathbb{Z} \cong \langle a | \; a^n \rangle$.
2. The group $\mathbb{Z} \times \mathbb{Z}$ has a presentation $\mathbb{Z} \times \mathbb{Z} = \langle a, b : [a, b] \rangle$, where $[a, b] = a \cdot b \cdot a^{-1} \cdot b^{-1}$ is the **group commutator**.

2.2 Modules

In this section we summarise some basic facts and definitions about modules over (unital) rings. Roughly speaking, a module is a useful concept in algebraic topology because it unifies various algebraic structures such as abelian groups, commutative rings and vector spaces over fields, which are all used to define different versions of homology. By working with modules, we can relate these different notions of homologies and treat them in a common framework. At the same time, it becomes apparent which aspects on the resulting homology theories are universal and which depend on the choice of the underlying ring.

Definition 2.2.1: Let R be a unital ring.

1. A **(left) module** over R or R-(left) module is an abelian group $(M, +)$ together with a map $\rhd : R \times M \to M$, $(r, m) \mapsto r \rhd m$, the **structure map**, such that for all $m, m' \in M$ and $r, r' \in R$

$$\begin{aligned} r \rhd (m + m') &= r \rhd m + r \rhd m' & \qquad (r + r') \rhd m &= r \rhd m + r' \rhd m \\ (r \cdot r') \rhd m &= r \rhd (r' \rhd m) & 1 \rhd m &= m. \end{aligned}$$

2. A **morphism of R-modules** or **R-linear map** between R-modules $(M, +, \rhd_M)$ and $(N, +, \rhd_N)$ is a group homomorphism $\phi : M \to N$ with

$$\phi(r \rhd_M m) = r \rhd_N \phi(m) \qquad \forall m \in M, r \in R.$$

A bijective R-module morphism $f : M \to N$ is called a R-module **isomorphism**, and one writes $M \cong N$. The set of R-module morphisms $\phi : M \to N$ is denoted $\mathrm{Hom}_R(M, N)$.

Remark 2.2.2:

1. Analogously, one defines a **right module** over R as an abelian group $(M, +)$ together with a map $\lhd : M \times R \to M$, $(m, r) \mapsto m \lhd r$, such that for all $m, m' \in M$ and $r, r' \in R$:

$$\begin{aligned} (m + m') \lhd r &= m \lhd r + m' \lhd r & \qquad m \lhd (r + r') &= m \lhd r + m \lhd r' \\ m \lhd (r \cdot r') &= (m \lhd r) \lhd r' & m \lhd 1 &= m. \end{aligned}$$

An R-left (right) module is the same as an R^{op}- right (left) module, where R^{op} is the ring with the opposite multiplication. This implies in particular that left and right modules over a commutative ring coincide.

2. The set $\mathrm{Hom}_R(M, N)$ of R-module morphisms $f : M \to N$ has a canonical R-module structure given by

$$(f+g)(m) = f(m)+g(m), \quad (r \rhd f)(m) = r \rhd f(m) \qquad \forall m \in M, f, g \in \mathrm{Hom}_R(M, N), r \in R.$$

3. For any unital ring R, the left (right) modules over R and left (right) module homomorphisms form a category R-Mod (Mod-R). Isomorphisms in R-Mod (Mod-R) are left (right) module isomorphisms. The trivial module $\{0\}$ is a zero object in R-Mod (Mod-R).
4. If R, S are unital rings and $\phi : R \to S$ is a unital ring homomorphism, then every S-module M becomes an R-module with structure map $\rhd_R : R \times M \to M$, $r \rhd m = \phi(r) \rhd_S m$.

In algebraic topology, we mainly consider modules over commutative rings and module morphisms between them. Some examples that are particularly relevant are the following:

Example 2.2.3:

1. A module over $\mathbb{Z}$ is the same as an abelian group. This follows because any abelian group M has a unique $\mathbb{Z}$-module structure determined by $0 \rhd m = 0$ and $1 \rhd m = m$ for all $m \in M$. A morphism of $\mathbb{Z}$-modules is a group homomorphism between abelian groups. As $\mathbb{Z}$ is an initial object in URing, for every unital ring R, there is a unique ring homomorphism $\mathbb{Z} \to R$ given by $1 \mapsto 1_R$. The induced $\mathbb{Z}$-module structure on an R-module $(M, +, \rhd)$ is its abelian group structure.

2. A module over a field $\mathbb{F}$ is a vector space over $\mathbb{F}$, and a morphism of $\mathbb{F}$-modules an $\mathbb{F}$-linear map.
3. Every ring is a left (right) module over itself with the left (right) multiplication as a structure map.

We will now study in more depth the category of modules over unital ring R. An essential fact that makes modules useful for algebraic topology is that the four basic constructions in the category Top of topological spaces - subspaces, quotients, sums and products - all have counterparts in the category of modules over a ring R, namely submodules, quotients, direct sums and products of modules. We recall their main definitions and properties.

Definition 2.2.4: Let R be a unital ring and M a module over R. A **submodule** of M is a subgroup $N \subset M$ that is closed under the operation of R: $r \rhd n \in N$ for all $r \in R$ and $n \in N$.

Example 2.2.5:
1. For any R-module M, the trivial module $\{0\} \subset M$ and $M \subset M$ are submodules. All other submodules are called proper submodules.
2. For any module morphism $\phi : M \to N$, $\mathrm{Ker}\,(\phi) \subset M$ and $\mathrm{Im}\,(\phi) \subset N$ are submodules.
3. If M is an abelian group, i. e. a $\mathbb{Z}$-module, then a submodule of M is a subgroup.
4. Submodules of modules over a field $\mathbb{F}$ are linear subspaces.
5. Submodules of a ring R as a left (right) module over itself are its left (right) ideals.

Definition 2.2.6: Let M be a module over a unital ring R, $N \subset M$ a submodule and $\pi : M \to M/N, m \mapsto m+N$ the canonical surjection. Then $\rhd : R \times M/N \to M/N, r \rhd (mN) \mapsto (r \rhd m)N$ defines an R-module structure on the factor group M/N, the **quotient module** structure.

Remark 2.2.7:
1. The quotient module structure on M/N is the unique R-module structure on the abelian group M/N that makes the canonical surjection $\pi : M \to M/N$ an R-module morphism.
2. If $\phi : M \to M'$ is a module morphism with $N \subset \mathrm{Ker}\,(\phi)$, then there is a unique module morphism $\tilde{\phi} : M/N \to M'$ such that the diagram

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[d, "\pi"'] & M' \\ M/N \arrow[ur, dashed, "\exists!\tilde{\phi}"'] & \end{tikzcd}$$

commutes. This is called the **universal property** of the quotient module.
3. If $\phi : M \to N$ is a morphism of R-modules, then $M/\,\mathrm{Ker}\,(\phi) \xrightarrow{\sim} \mathrm{Im}\,(\phi)$, $m + \ker(\phi) \mapsto \phi(m)$. is a canonical isomorphism of R-modules.
4. If M is a module over R with submodules $U \subset V \subset M$ then V/U is a submodule of M/U, and there is a canonical isomorphism $(M/U)/(V/U) \xrightarrow{\sim} M/V$.

Definition 2.2.8: Let R be a unital ring and $(M_i)_{i\in I}$ a family of modules over R. Then the **direct sum** $\oplus_{i\in I} M_i$ and the **direct product** $\Pi_{i\in I} M_i$ are the sets

$$\begin{aligned}\oplus_{i\in I} M_i &= \{(m_i)_{i\in I} : \ m_i \in M_i, m_i = 0 \text{ for almost all } i \in I\}\\ \Pi_{i\in I} M_i &= \{(m_i)_{i\in I} : \ m_i \in M_i\}\end{aligned}$$

with the R-module structures given by

$$(m_i)_{i\in I} + (m_i')_{i\in I} := (m_i + m_i')_{i\in I} \qquad r \rhd (m_i)_{i\in I} := (r \rhd m_i)_{i\in I}.$$

Lemma 2.2.9: The direct product and the direct sum of modules are products and coproducts in the category R-Mod. More precisely:

1. **Universal property of direct sums:** the direct sum module structure is the unique R-module structure on $\oplus_{i\in I} M_i$ for which all inclusion maps $\iota_i : M_i \to M$, $m \mapsto (0, ..., 0, m, 0, ...)$ are module morphisms. For a family $(\phi)_{i\in I}$ of module morphisms $\phi_i : M_i \to N$ there is a unique module morphism $\phi : \oplus_{i\in I} M_i \to N$ such that for all $i \in I$ the following diagram commutes.

$$\begin{tikzcd} M_i \arrow[r, "\phi_i"] \arrow[d, "\iota_i"'] & N \\ \oplus_{j\in I} M_j. \arrow[ur, dashed, "\exists!\phi"'] & \end{tikzcd}$$

2. **Universal property of products:** the product module structure is the unique R-module structure on $\Pi_{i\in I} M_i$ for which all projection maps $\pi_i : \Pi_{i\in I} M_i \to M_i$, $(m_1, m_2, ...) \mapsto m_i$ are module morphisms. For a family $(\psi)_{i\in I}$ of module morphisms $\psi_i : L \to M_i$ there is a unique module morphism $\psi : L \to \Pi_{i\in I} M_i$ such that for all $i \in I$ the following diagram commutes

$$\begin{tikzcd} M_i & L \arrow[l, "\psi_i"'] \arrow[dl, dashed, "\exists!\psi"] \\ \Pi_{j\in I} M_j. \arrow[u, "\pi_i"] & \end{tikzcd}$$

We have thus shown that modules over a unital ring R form an additive category. The Hom-sets $\mathrm{Hom}_R(M, N)$ in R-Mod are abelian groups with respect to the pointwise addition of module morphisms and with the trivial module morphism as the unit. The composition of module morphisms is bilinear and finite products and coproducts exist for all objects in R-Mod. Moreover, every module morphism $f : M \to N$ has a kernel, namely the inclusion morphism $\iota : \ker(f) \to M$ and a cokernel, namely the canonical surjection $\pi : N \to N/\mathrm{Im}\,(f)$. It is easy to see that every monomorphism is the kernel of its cokernel and every epimorphism the cokernel of its kernel. Together, this implies

Remark 2.2.10: Modules over a unital ring R and module morphisms between them form an abelian category.

Given the four basic constructions for modules - submodules, quotients, direct sums and products - it is easy to construct pullbacks for a pair of R-module morphisms $f_i : X_i \to Y$ and pushouts for any pair of R-module morphisms $g_i : W \to X_i$. This yields the following lemma whose proof is left as an exercise for the reader.

Lemma 2.2.11: Let R be a unital ring.

1. For two R-module morphisms $f_i : X_i \to Y$, $i = 1, 2$, the submodule

$$P = \ker(f_2 \circ \pi_2 - f_1 \circ \pi_1) \subset X_1 \times X_2,$$

together with the projection maps $\pi_i|_P : P \to X_i$ is a pullback in R-Mod, where $\pi_i : X_1 \times X_2 \to X_i$ denotes the canonical projections.

2. For two R-module morphisms $g_i : W \to X_i$ is the quotient module

$$P' = (X_1 \oplus X_2)/\mathrm{Im}\,(\iota_2 \circ g_2 - \iota_1 \circ g_1)$$

together with the maps $\pi \circ \iota_i : X_i \to P'$ is a pushout in R-Mod, where where $\iota_i : X_i \to X_1 \oplus X_2$ denotes the inclusion maps and $\pi : X_1 \oplus X_2 \to P'$ the canonical surjection.

While the four basic constructions for modules are straightforward generalisations of the corresponding constructions for vector spaces, there is one fundamental way in which modules over general rings differ from vector spaces, namely the existence of a basis and the existence of complements. While every vector space has a basis and every linear subspace has a complement, this does not hold for general modules. Although there are always generating sets - subsets of a module such that every element can be written as a (finite) R-linear combination of these elements - there is general no linearly independent generating set. In contrast to vector spaces, which are determined uniquely up to isomorphisms by the choice of a basis, modules therefore need to be characterised by more elaborate data, namely a presentation.

Definition 2.2.12: Let R be a unital ring.

1. For a subset $A \subset M$, the **submodule generated by** A is the smallest submodule of M containing A, the subset $\langle A \rangle_M = \{\Sigma_{a \in A} r_a \rhd a : \ r_a \in R, r_a = 0 \text{ for almost all } a \in A\}$ with the induced R-module structure.
2. A subset $A \subset M$ is called **generating set** of M if $\langle A \rangle_M = M$ and a **basis** of M if it is a generating set and linearly independent: $\Sigma_{a \in A} r_a \rhd a = 0$ with $r_a \in R$ and $r_a = 0$ for almost all $a \in A$ implies $r_a = 0$ for all $a \in A$. A module with a finite generating set is called **finitely generated**, and a module with a generating set that contains only one element is called **cyclic**. A module is called **free** if it has a basis.
3. The **free R-module** generated by a set A is the direct sum $\bigoplus_{a \in R} R$. Equivalently, it can be characterised as the set $\langle A \rangle_R = \{f : A \to R : \ f(a) = 0 \text{ for almost all } a \in A\}$ with the canonical R-module structure

$$(f+g)(a) = f(a) + g(a) \qquad (r \rhd f)(a) = r \cdot f(a) \qquad \forall f, g \in \langle A \rangle_R, r \in R, a \in A.$$

The maps $\delta_a : A \to R$ with $\delta_a(a) = 1_R$ and $\delta_a(a') = 0$ for $a' \neq a$ are a basis of $\langle A \rangle_R$, since every map $f : A \to R$ with $f(a) = 0$ for almost all $a \in A$ can be expressed as a finite R-linear combination $f = \sum_{a \in A, f(a) \neq 0} f(a) \rhd \delta_a$. Instead of $\sum_{a \in A} r_a \rhd \delta_a$ with $r_a = 0$ for almost all $a \in A$, we write $\sum_{a \in A} r_a\, a$.

4. For a subset $B \subset \langle A \rangle_R$, we denote by $\langle A|B \rangle_R$ the quotient module $\langle A|B \rangle_R = \langle A \rangle_R / \langle B \rangle_{\langle A \rangle_R}$. If $M = \langle A|B \rangle_R$, then $\langle A|B \rangle_R$ is called a **presentation** of M, the elements of A are called **generators** and the elements of B **relations**.

Remark 2.2.13:

1. Every module has a presentation $M = \langle A|B\rangle_R$.
2. Presentations of modules are characterised by a **universal property**:
   For any R-module M and any map $\phi : A \to M$, there is a unique map $\tilde{\phi} : \langle A\rangle_R \to M$ with $\tilde{\phi}|_A = \phi$. If $B \subset \ker(\tilde{\phi})$, then the universal property of the quotient implies that there is a unique map $\bar{\phi} : \langle A|B\rangle_R \to M$ with $\bar{\phi} \circ \pi = \tilde{\phi}$, where $\pi : \langle A\rangle_R \to \langle A|B\rangle_R$ is the canonical surjection.
3. If R is a commutative unital ring and M a free module over R, then any two bases of M have the same number of elements. This number is called **rank** of M and denoted $\mathrm{rk}(M)$. This notion makes no sense for non-commutative rings since one can have $R^n \cong R^m$ as R modules for $n \neq m$.

Example 2.2.14:

1. Every ring R is a cyclic free module as a left or right module over itself: $R = \langle 1_R\rangle_R$.
2. If $\mathbb{F}$ is a field, then every module over $\mathbb{F}$ is free, since a module over $\mathbb{F}$ is a vector space, and every vector space has a basis.
3. If M is a free module over a principal ideal ring R, then every submodule $U \subset M$ is free with $\mathrm{rk}(U) \leq \mathrm{rk}(M)$. (For a proof, see [JS]).
4. The $\mathbb{Z}$-module $M = \mathbb{Z}/2\mathbb{Z}$ is not a free module. Any generating set of $\mathbb{Z}/2\mathbb{Z}$ must contain the element $\bar{1}$, but $2 \rhd \bar{1} = \bar{1} + \bar{1} = \bar{0}$ and hence a generating set cannot be free. A presentation of $\mathbb{Z}/2\mathbb{Z}$ is given by $\langle A \,|\, B\rangle_{\mathbb{Z}} = \langle 1|2\rangle$,

In the context of vector spaces, an important consequence of the existence of bases is the existence of a complement for any linear subspace $U \subset V$, a linear subspace $W \subset V$ with $V = U \oplus W$. This does not hold for modules over more general rings. A simple example is the submodule $n\mathbb{Z} \subset \mathbb{Z}$ for $n \in \mathbb{N}$, $n \geq 2$. As $1 \notin n\mathbb{Z}$, any complement of this module would need to contain the element $1 \in \mathbb{Z}$ and hence be equal to $\mathbb{Z}$. A similar argument shows that a proper submodule of a cyclic module can never have a complement. In many applications, one needs practical criteria to determine if a given submodule has a complement. A sufficient one is given in the following lemma, which involves structures that can be viewed as the module counterparts of retractions.

Lemma 2.2.15: Let R be a unital ring and M a module over R.

1. If $\phi : M \to F$ is a surjective R-module morphism into a free module F, then there is a R-module morphism $\psi : F \to M$ with $\phi \circ \psi = \mathrm{id}_F$ and $M \cong \mathrm{Im}\,(\psi) \oplus \ker(\phi)$. One says that ψ **splits** the module morphism $\phi : M \to F$.
2. If $N \subset M$ is a submodule such that M/N is free, then there is a submodule $P \subset M$ with $P \cong M/N$ and $M \cong N \oplus P$.

Proof:
1. Choose a basis B of F and for every $b \in B$ an element $m_b \in \phi^{-1}(b) \subset M$. Define the R-module morphism $\psi : F \to M$ by $\psi(b) = m_b$ and R-linear extension to F. As $\phi \circ \psi = \mathrm{id}_F$, we have $m = \psi \circ \phi(m) + (m - \phi \circ \psi(m))$ with $\psi \circ \phi(m) \in \mathrm{Im}\,(\psi)$ and $m - \phi \circ \psi(m) \in \ker(\phi)$ for all $m \in M$. As $\phi \circ \psi = \mathrm{id}_F$, we have $\ker(\phi) \cap \mathrm{Im}\,(\psi) = \{0\}$ and hence $M = \ker(\phi) \oplus \mathrm{Im}\,(\psi)$.

2. By 1., there is a R-module morphism $\psi : M/N \to M$ which splits the surjective module morphism $\pi : M \to M/N$ and hence $M \cong \ker(\pi) \oplus \mathrm{Im}\,(\psi) \cong N \oplus \mathrm{Im}\,(\psi)$. The R-module morphism $\pi|_{\mathrm{Im}\,(\psi)} : \mathrm{Im}\,(\psi) \to M/N$ is surjective by definition and injective since $\pi \circ \psi = \mathrm{id}_{M/N}$, hence an isomorphism. □

The fact that a module M over a ring R does not need to have a basis is closely related to the presence of elements $m \in M$ for which there is an $r \in R \setminus \{0\}$ with $r \rhd m = 0$, the so-called *torsion elements*. It is clear that a module with non-trivial torsion elements cannot be free, since an R-linear combination of basis elements cannot be a torsion element unless it is trivial.

Definition 2.2.16: Let R be a unital ring and M an R-module. An element $m \in M$ is called a **torsion element** if there is an $r \in R \setminus \{0\}$ with $r \rhd m = 0$. The set of torsion elements in M is denoted $\mathrm{Tor}_R(M)$, and M is called **torsion free** if $\mathrm{Tor}_R(M) = 0$.

Example 2.2.17:

1. If $M = R$ is a commutative unital ring considered as a module over itself, then torsion elements are precisely the zero divisors of R. In particular, every integral domain R considered as a module over itself is torsion free. This applies in particular to $\mathbb{Z}$, to any field $\mathbb{F}$ and to the ring $\mathbb{F}[X]$ of polynomials over a field $\mathbb{F}$.
2. In the $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$ for $n \in \mathbb{N}$, every element is a torsion element since $n \rhd \overline{k} = \overline{n \cdot k} = \overline{0}$ for all $k \in \mathbb{Z}$. The ring $\mathbb{Z}/n\mathbb{Z}$ as a module over itself is torsion free if and only if n is a prime.

It is natural to expect that the set of torsion elements in an R-module M should be a submodule of M. However, this does not hold in general unless R is commutative and without zero divisors, i. e. an integral domain. IN this case the torsion elements form a submodule and by taking the quotient with respect to this submodule, one obtains a module that is torsion free.

Lemma 2.2.18: If M is a module over an integral domain R then $\mathrm{Tor}_R(M) \subset M$ is a submodule and the module $M/\mathrm{Tor}_R(M)$ is torsion free.

Proof:
Let $m, m' \in \mathrm{Tor}_R(M)$ torsion elements and $r, r' \in R \setminus \{0\}$ with $r \rhd m = r' \rhd m' = 0$. Then $(r \cdot r') \rhd (m + m') = r' \rhd (r \rhd m) + r \rhd (r' \rhd m') = 0$. As R is an integral domain, $r \cdot r' \neq 0$ and hence $m + m' \in \mathrm{Tor}_R(M)$. Similarly, for all $s \in R$, one has $r \rhd (s \rhd m) = (r \cdot s) \rhd m = s \rhd (r \rhd m) = 0$, which implies $s \rhd m \in \mathrm{Tor}_R(M)$, and hence $\mathrm{Tor}_R(M) \subset M$ is a submodule. If $[m] \in M/\mathrm{Tor}_R(M)$ is a torsion element, then there is an $r \in R \setminus \{0\}$ with $r \rhd [m] = [r \rhd m] = 0$. This implies $r \rhd m \in \mathrm{Tor}_R(M)$, and there is an $r' \in R$ with $r' \rhd (r \rhd m) = (r \cdot r') \rhd m = 0$. As R is an integral domain, one has $r \cdot r' \neq 0$, which implies $m \in \mathrm{Tor}_R(M)$ and $[m] = 0$. □

If R is an integral domain, it is natural to ask if the torsion submodule $\mathrm{Tor}_R(M)$ of an R-module M has a complement, i. e. if there is a an R-module N with $M \cong \mathrm{Tor}_R(M) \oplus N$. A sufficient condition that ensures the existence of such a complement is that R is a principal ideal ring and

M is finitely generated. In this case, the classification theorem for finitely generated modules over principal ideal rings allows one to identify the torsion elements and their complement. In particular, this implies to finitely generated abelian groups, i. e. finitely generated modules over the principal ideal ring $\mathbb{Z}$.

Lemma 2.2.19: Let R be a principal ideal ring. Then every finitely generated R-module M is of the form $M \cong \mathrm{Tor}_R(M) \oplus R^n$ with a unique $n \in \mathbb{N}_0$. In particular, every finitely generated torsion free R-module is free.

Proof:
This follows from the classification theorem for finitely generated modules over principal ideal rings, which states that every such module is of the form $M \cong R^n \times R/q_1R \times \ldots \times R/q_mR$ with prime powers $q_1, ..., q_m \in R$ and $n \in \mathbb{N}_0$. Every element $m \in R/q_1R \times \ldots \times R/q_lR$ is a torsion element since $(q_1 \cdots q_m) \rhd m = 0$, and $r \rhd (m_1 + m_2) = 0$ with $m_1 \in R^n$ and $m_2 \in R/q_1R \times \ldots \times R/q_mR$ implies $r = 0$ or $m_1 = 0$, i. e. $\mathrm{Tor}_R(M) = R/q_1R \times \ldots \times R/q_mR$. $\square$

In the following, we will require another fundamental construction involving modules over unital rings, namely the *tensor product*. This generalises the tensor product of vector spaces, i. e. modules over fields. It is obtained as a quotient of a free module generated by the Cartesian products of the underlying sets.

Definition 2.2.20: Let R be a unital ring, M an R-right module and N an R-left module. The **tensor product** $M \otimes_R N$ is the abelian group generated by the set $M \times N$ with relations

$$\begin{aligned} &(m, n) + (m', n) = (m + m', n), \quad &&(m, n) + (m, n') = (m, n + n'), \\ &(m \lhd r, n) = (m, r \rhd n) &&\forall m, m' \in M, n, n' \in N, r \in R. \end{aligned}$$

We denote by $m \otimes n = \otimes(m, n)$ the images of the elements $(m, n) \in \langle M \times N\rangle_{\mathbb{Z}}$ under the map $\otimes = \pi \circ \iota : M \times N \to M \otimes_R N$, where $\iota : M \times N \to \langle M \times N\rangle_{\mathbb{Z}}$, $(m, n) \to (m, n)$ is the canonical inclusion and $\pi : \langle M \times N\rangle_{\mathbb{Z}} \to M \otimes_R N$ the canonical surjection.

Remark 2.2.21:

1. The set $\{m \otimes n : \ m \in M, n \in N\}$ generates $M \otimes_R N$, since the elements (m, n) generate the free group $\langle M \times N\rangle_{\mathbb{Z}}$ and the group homomorphism $\pi : \langle M \times N\rangle_{\mathbb{Z}} \to M \otimes_R N$ is surjective. The relations in Definition 2.2.20 induce the following identities in $M \otimes_R N$:

$$\begin{aligned} &(m + m') \otimes n = m \otimes n + m' \otimes n, \quad &&m \otimes (n + n') = m \otimes n + m \otimes n', \\ &(m.r) \otimes n = m \otimes (r.n) &&\forall m, m' \in M, n, n' \in N, r \in R. \end{aligned}$$

2. If M is an R-right-module and N an (R, S)-bimodule, then $M \otimes_R N$ has a canonical S-right-module structure given by $(m \otimes n) \lhd s := m \otimes (n \lhd s)$. Similarly, if M is a (Q, R)-bimodule and N an R-left module then $M \otimes_R N$ has a canonical Q-left module structure given by $q \rhd (m \otimes n) := (q \rhd m) \otimes n$.
3. As every left module over a commutative ring R is an (R, R)-bimodule, it follows from 2. that the tensor product $M \otimes_R N$ of modules over a commutative unital ring R has a canonical (R, R)-bimodule structure, given by

$$r \rhd (m \otimes n) = (r \rhd m) \otimes n = (m \lhd r) \otimes n = m \otimes (r \rhd n) = m \otimes (n \lhd r) = (m \otimes n) \lhd r.$$

4. As every module is an abelian group, it is always possible to tensor two modules over the ring $\mathbb{Z}$. In this case, the last relation in Definition 2.2.20 is a consequence of the first two.

Example 2.2.22:

1. If $R = \mathbb{F}$ is a field, the tensor product of R-modules is the tensor product of vector spaces.
2. For any unital ring R and $R^k := R \oplus R \oplus \ldots \oplus R$, one has $R^m \otimes R^n \cong R^{nm}$.
3. If R is commutative and $R[X, Y]$ the polynomial ring over R in the variables X, Y, then $R[X] \otimes_R R[Y] \cong R[X, Y]$.
4. The tensor product of the abelian groups $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}/m\mathbb{Z}$ for $n, m \in \mathbb{N}$ is
$$\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/\mathrm{gcm}(m, n)\mathbb{Z},$$
where $\mathrm{gcm}(m, n)$ is the greatest common divisor of m and n (see Exercise 13).
5. One has $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} \cong 0$. More generally, if R is an integral domain with associated quotient field $Q(R)$ and M an R-module, then $\mathrm{Tor}_R(M)\otimes_R Q(R) \cong 0$ since for every torsion element $m \in M$ there is an $r \in R \setminus \{0\}$ with $r \rhd m = 0$. This implies $m \otimes q = m \otimes (r \cdot q/r) = (m \lhd r) \otimes q/r = 0$ for all $q \in Q$.

Just as submodules, quotients, direct sums and products of modules, tensor products of R-modules can be characterised by a universal property. As tensor products are defined in terms of a presentation, this universal property is obtained by applying the one in Remark 2.2.13 to the relations in Definition 2.2.20. The special form of these relations allows one to characterise the universal property in terms of bilinear maps $M \times N \to A$ into abelian groups A.

Definition 2.2.23: Let R be a unital ring, M an R-right module and N an R-left module. A map $f : M \times N \to A$ into an abelian group A is called R**-bilinear** if

$$\begin{aligned} &f(m + m', n) = f(m, n) + f(m', n), && f(m, n + n') = f(m, n) + f(m, n'), \\ &f(m \lhd r, n) = f(m, r \rhd n) && \forall m, m' \in M, n, n' \in N, r \in R. \end{aligned}$$

Lemma 2.2.24: Let R be a unital ring, M an R-right module and N an R-left module. Then the map $\otimes : M \times N \to M \otimes_R N$, $(m, n) \mapsto m \otimes n$ is R-bilinear, and for any R-bilinear map $f : M \times N \to A$ into an abelian group A, there is a unique group homomorphism $\tilde{f} : M \otimes_R N \to A$, so such that the following diagram commutes

$$\begin{tikzcd} M \times N \arrow[r, "f"] \arrow[d, "\otimes"'] & A \\ M \otimes_R N. \arrow[ur, dashed, "\exists! \tilde{f}"'] & \end{tikzcd}$$

This is called the **universal property of the tensor product** . If R is commutative, then $\tilde{f} : M \otimes_R N \to A$ is an R-module homomorphism.

Proof:
The first statement holds per Definition, since these are the defining relations of the tensor product. For the second claim, we define $\tilde{f}: M \otimes_R N \to A$ by $\tilde{f}(m \otimes n) = f(m,n)$ and additive extension to the abelian group $M \otimes_R N$. As the elements $m \otimes n$ generate $M \otimes_R N$, this defines a unique group homomorphism $\tilde{f}$. The bilinearity of f guarantees that $\tilde{f}$ is well-defined since

$$\begin{aligned}
&\tilde{f}((m+m')\otimes n) = f(m+m',n) = f(m,n) + f(m',n) = \tilde{f}(m\otimes n) + \tilde{f}(m'\otimes n)\\
&\tilde{f}(m\otimes(n+n')) = f(m,n+n') = f(m,n) + f(m,n') = \tilde{f}(m\otimes n) + \tilde{f}(m\otimes n')\\
&\tilde{f}((m\lhd r)\otimes n) = f(m\lhd r,n) = f(m,r\rhd n) = \tilde{f}(m\otimes(r\rhd n)),
\end{aligned}$$

and one obtains a group homomorphism $\tilde{f}: M \otimes_R N \to A$ with $\tilde{f}\circ\otimes = f$. If $\tilde{f}': M\otimes_R N \to A$ is another group homomorphism with this property, then for all $m \in M$, $n \in N$ one has $(\tilde{f}'-\tilde{f})(m\otimes n) = \tilde{f}'\circ\otimes(m,n) - \tilde{f}\circ\otimes(m,n) = f(m,n) - f(m,n) = 0$ and hence $\tilde{f}' = \tilde{f}$. □

To conclude our discussion of tensor products, we assemble some important properties of tensor products that are a direct consequence of the definitions and the universal property.

Lemma 2.2.25: Let R, S be unital rings, I an index set, M, M_i R-right modules, N, N_i R-left modules for all $i \in I$, P a (R,S)-bimodule and Q an S-left module. Then:

1. **tensor products with the trivial module:** $0\otimes_R N \cong M \otimes_R 0 \cong 0$,
2. **tensor product with the underlying ring:** $M \otimes_R R \cong M$, $R \otimes_R N \cong N$,
3. **direct sums:** $(\oplus_{i\in I} M_i)\otimes_R N \cong \oplus_{i\in I} M_i \otimes_R N$, $M \otimes_R \left(\bigoplus_{i\in I} N_i\right) \cong \bigoplus_{i\in I} M\otimes_R N_i$,
4. **associativity:** $(M\otimes_R P)\otimes_S Q \cong M \otimes_R (P \otimes_S Q)$.

Proof:
1. follows directly from the universal property of the tensor product. For 2., note that group homomorphism $M \to M\otimes_R R$, $m \mapsto m\otimes 1$ has an inverse, namely $M\otimes_R R \to M$, $m\otimes r \mapsto m\lhd r$ The proof for $R\otimes_R N \cong N$ is analogous.

3. Consider the group homomorphisms $\phi_i : M_i\otimes_R N \to (\oplus_{i\in I} M_i)\otimes_R N$, $\phi_i(m_i\otimes n) = \iota_i(m)\otimes n$, where $\iota_i : M_i \to \oplus_{i\in I} M_i$ is the canonical inclusion. By the universal property of the direct sum this defines a unique group homomorphism $\phi: \oplus_{i\in I} M_i\otimes_R N \to (\oplus_{i\in I} M_i)\otimes_R N$ with $\phi\circ j_i = \phi_i$ for the inclusion maps $j_i : M_i \otimes_R N \to \oplus_{i\in I} M_i \otimes_R N$. This group homomorphism has an inverse $\psi : (\oplus_{i\in I} M_i)\otimes_R N \to \oplus_{i\in I} M_i \otimes_R N$ given by $\psi(\iota_i(m_i)\otimes n) = j_i(m_i\otimes n)$ and hence is an isomorphism. The proof for the other identity is analogous.

4. A group isomorphism $\phi: (M\otimes_R P)\otimes_S Q \to M\otimes_R (P\otimes_S Q)$ is given by

$$\phi((m\otimes p)\otimes q) = m\otimes(p\otimes q) \qquad \forall m \in M, p \in P, q \in Q.$$

□

It remains to investigate the behaviour of tensor products under module morphisms, i. e. to determine how the tensor products $M\otimes_R N$ and $M'\otimes_R N'$ are related if there are module morphisms $\phi: M \to M'$ and $\psi: N \to N'$. Clearly, the module morphisms $\phi: M \to M'$ and

$\psi : N \to N'$ induce a map $\phi \times \psi : M \times N \to M' \times N'$. By composing it with the map $\otimes : M' \times N' \to M'\otimes_R N'$, we obtain an R-bilinear map $\otimes \circ (\phi \times \psi) : M \times N \to M'\otimes_R N'$. The universal property of the tensor products yields a map $\phi\otimes\psi : M\otimes_R N \to M'\otimes_R N'$ with $(\phi\otimes\psi) \circ \otimes = \otimes \circ (\phi \times \psi)$, and this allows one to extend the tensor product to a functor $\otimes : \text{Mod-}R \times R\text{-Mod} \to \text{Ab}$.

Theorem 2.2.26: Let R be a unital ring. Then the tensor product of R-modules defines a functor $\otimes : \text{Mod-}R \times R\text{-Mod} \to \text{Ab}$ that assigns to a pair of R modules (M, N) the abelian group $M\otimes_R N$ and to a pair of module morphisms $(\phi, \psi) : (M, N) \to (M', N')$ the unique group homomorphism $\phi\otimes\psi : M\otimes_R N \to M'\otimes_R N'$ for which the following diagram commutes

$$\begin{tikzcd} M \times N \arrow[r, "\phi\times\psi"] \arrow[d, "\otimes"] & M' \times N' \arrow[d, "\otimes"] \\ M \otimes_R N \arrow[r, dashed, "\exists!\phi\otimes\psi"'] & M' \otimes_R N'. \end{tikzcd}$$

For each R-right module M, this defines an functor $M\otimes- : R\text{-Mod} \to \text{Ab}$. If R is commutative, this yields functors $\otimes : R\text{-Mod} \times R\text{-Mod} \to R\text{-Mod}$ and $M\otimes- : R\text{-Mod} \to R\text{-Mod}$.

Proof:
The map $\otimes \circ (\phi \times \psi) : M \times N \to M' \otimes_R N'$ is R-bilinear, and by the universal property of the tensor product there is a unique group homomorphism $\phi \otimes \psi : M \otimes_R N \to M' \otimes_R N'$ with $(\phi \otimes \psi)(m \otimes n) = \phi(m) \otimes \psi(n)$ for all $m \in M, n \in N$. That this defines a functor $\otimes : \text{Mod-}R \times R\text{-Mod} \to \text{Ab}$ follows from the fact the the following two diagrams commute

$$\begin{tikzcd} M \times N \arrow[r, "\mathrm{id}_M\times\mathrm{id}_N"] \arrow[d, "\otimes"] & M \times N \arrow[d, "\otimes"] \\ M \otimes_R N \arrow[r, "\mathrm{id}_{M\otimes_R N}"'] & M \otimes_R N \end{tikzcd} \qquad \begin{tikzcd} M \times N \arrow[r, "\phi\times\psi"] \arrow[d, "\otimes"] & M' \times N' \arrow[r, "\phi'\times\psi'"] \arrow[d, "\otimes"] & M'' \times N'' \arrow[d, "\otimes"] \\ M \otimes_R N \arrow[r, "\phi\otimes\psi"'] & M' \otimes_R N' \arrow[r, "\phi'\otimes\psi'"'] & M''\otimes_R N''. \end{tikzcd}$$

The functor $M\otimes- : R\text{-Mod} \to \text{Ab}$ assigns to an R-left module N the abelian group $M\otimes_R N$ and to a module morphism $\psi : N \to N'$ the group homomorphism $\mathrm{id}_M\otimes\psi : M\otimes_R N \to M\otimes_R N'$. $\square$

2.3 Homological algebra

In this section we assemble the algebraic background for homology theory. The fundamental notion is that of a chain complex - a sequence of modules over a unital ring R and R-module morphisms such that the composition of two subsequent module morphisms is the trivial map. We will see later that each topological space gives rise to a chain complex and continuous maps between topological spaces induce chain maps.

Definition 2.3.1: Let R be a unital ring.

1. A **chain complex** $(X_\bullet, d_\bullet)$ in R-Mod is a sequence $\ldots \xrightarrow{d_{n+2}} X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} \ldots$ of R-modules X_n and R-module morphisms $d_n : X_n \to X_{n-1}$ with $d_{n-1} \circ d_n = 0\ \forall n \in \mathbb{Z}$.

2. Elements of X_n are called n-**chains**, elements of $Z_n(X_\bullet) := \ker(d_n) \subset X_n$ n-**cycles** and elements of $B_n(X_\bullet) := \mathrm{Im}\,(d_{n+1}) \subset Z_n(X_\bullet)$ n-**boundaries** in $(X_\bullet, d_\bullet)$.
3. A **chain map** $f_\bullet : (X_\bullet, d_\bullet) \to (Y_\bullet, d'_\bullet)$ between chain complexes $(X_\bullet, d_\bullet)$ and $(Y_\bullet, d'_\bullet)$ is a family $(f_n)_{n\in\mathbb{Z}}$ of morphisms $f_n : X_n \to Y_n$ for which the following diagram commutes

$$\begin{tikzcd}
\ldots \arrow[r, "d_{n+2}"] & X_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] & X_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & X_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "f_{n-1}"'] & \ldots \\
\ldots \arrow[r, "d'_{n+2}"] & Y_{n+1} \arrow[r, "d'_{n+1}"] & Y_n \arrow[r, "d'_n"] & Y_{n-1} \arrow[r, "d'_{n-1}"] & \ldots
\end{tikzcd}$$

To keep notation simple, one omits subsequences of trivial modules and trivial module morphisms between them in a chain complex. The sequence $0 \to X_m \xrightarrow{d_m} X_{m-1} \xrightarrow{d_{m-1}} ...$ stands for a chain complex with $X_k = 0$ for all $k > m$ and $... \xrightarrow{d_{m+2}} X_{m+1} \xrightarrow{d_{m+1}} X_m \to 0$ for a chain complex with $X_k = 0$ for all $k > m$. If there are $l, m \in \mathbb{Z}$ with $X_k = 0$ for all $k < m$, $k > l$, one writes $0 \to X_l \xrightarrow{d_l} X_{l-1} \xrightarrow{d_{l-1}} ... \xrightarrow{d_{m+2}} X_{m+1} \xrightarrow{d_{m+1}} X_m \to 0$ and calls the chain complex **finite**.

Remark 2.3.2:

1. For any unital ring R, chain complexes and chain maps in R-Mod form a category $\mathrm{Ch}_{R\text{-Mod}}$. The identity morphism $1_{X_\bullet} : (X_\bullet, d_\bullet) \to (X_\bullet, d_\bullet)$ is given by the family of identity morphisms $(\mathrm{id}_{X_n})_{n\in\mathbb{Z}}$. The composite $g_\bullet \circ f_\bullet$ of two chain maps $f_\bullet : (X_\bullet, d_\bullet) \to (Y_\bullet, d'_\bullet)$ and $g_\bullet : (Y_\bullet, d'_\bullet) \to (Z_\bullet, d''_\bullet)$ is given by the family of module morphisms $(f_n \circ g_n)_{n\in\mathbb{Z}}$. By composing the associated diagrams, it is easy to see that this defines a chain map:

$$\begin{tikzcd}
\ldots \arrow[r, "d_{n+2}"] & X_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] & X_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & X_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "f_{n-1}"'] & \ldots \\
\ldots \arrow[r, "d'_{n+2}"] & Y_{n+1} \arrow[r, "d'_{n+1}"] \arrow[d, "g_{n+1}"'] & Y_n \arrow[r, "d'_n"] \arrow[d, "g_n"'] & Y_{n-1} \arrow[r, "d'_{n-1}"] \arrow[d, "g_{n-1}"'] & \ldots \\
\ldots \arrow[r, "d''_{n+2}"] & Z_{n+1} \arrow[r, "d''_{n+1}"] & Z_n \arrow[r, "d''_n"] & Z_{n-1} \arrow[r, "d''_{n-1}"] & \ldots
\end{tikzcd}$$

2. One can show that the category $\mathrm{Ch}_{R\text{-Mod}}$ is abelian. The relevant structures such as the zero objects and morphisms, products, coproducts, the abelian structures on the Hom-sets, kernels and cokernels are all induced in the obvious way by the corresponding structures in R-Mod.

The condition $d_n \circ d_{n+1} = 0$, which characterises a chain complex $... \xrightarrow{d_{n+2}} X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} ...$ implies for all $n \in \mathbb{Z}$ that $B_n(X_\bullet) = \mathrm{Im}\,(d_{n+1}) \subset \ker(d_n) = Z_n(X_\bullet)$ is a submodule. The associated quotient module $H_n(X_\bullet) = Z_n(X_\bullet)/B_n(X_\bullet)$ consists of equivalence classes of n-cycles module n-boundaries and is called the nth homology of $X_\bullet$.

Definition 2.3.3: Let R be a unital ring and $(X_\bullet, d_\bullet) = ...X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} ...$ a chain complex in R-Mod.

1. The nth **homology** of $(X_\bullet, d_\bullet)$ is the quotient module $H_n(X_\bullet) = Z_n(X_\bullet)/B_n(X_\bullet)$.
2. A chain complex $(X_\bullet, d_\bullet)$ is called **exact** in X_n if $H_n(X_\bullet) = 0$ and **exact** if $H_n(X_\bullet) = 0$ for all $n \in \mathbb{Z}$. An exact chain complex in R-Mod is also called a **long exact sequence** of R-modules, and a finite exact chain complex in R-Mod of the form $0 \to W \xrightarrow{\iota} X \xrightarrow{\pi} Y \to 0$ is called a **short exact sequence** of R-modules.

It is clear that a chain complex is exact if and only if for all $n \in \mathbb{Z}$ one has $\ker(d_n) = \mathrm{Im}\,(d_{n+1})$. The name *short exact sequence* for an exact chain complex of the form $0 \to W \xrightarrow{\iota} X \xrightarrow{\pi} Y \to 0$ is motivated by the fact that it is the shortest non-trivial exact chain complex. In an exact sequence of the form $0 \to X \to 0$ one has $X = 0$, and in a short exact sequence of the form $0 \to X \xrightarrow{f} Y \to 0$ one has $\ker(f) = 0$ and $\mathrm{Im}\,(f) = Y$, which implies that $f : X \to Y$ is a module isomorphism.

In contrast, a short exact sequence $0 \to W \xrightarrow{\iota} X \xrightarrow{\pi} Y \to 0$ corresponds to the choice of a submodule $W \subset X$. This follows because $0 \to W \xrightarrow{\iota} X \xrightarrow{\pi} Y \to 0$ is exact if and only if $\ker(\iota) = 0$, $\mathrm{Im}\,(\pi) = Y$ and $\mathrm{Im}\,(\iota) = \ker(\pi)$, i. e. $\iota : W \to X$ is injective, $\pi : X \to Y$ is surjective and $\mathrm{Im}\,(\iota) = \ker(\pi)$. This gives rise to an isomorphism of R-modules $Y \cong X/\ker(\pi) \cong X/\mathrm{Im}\,(\iota) \cong X/W$. Conversely, for any submodule $W \subset X$ the inclusion map $\iota : W \to X$ and the canonical surjection $\pi : X \to X/W$ define an exact sequence of R-modules.

It remains to investigate the algebraic properties of the homologies, in particular their transformation behaviour under chain maps. A chain map $f_\bullet : (X_\bullet, d_\bullet) \to (Y_\bullet, d'_\bullet)$ consists of a set of module morphisms $f_n : X_n \to Y_n$ for $n \in \mathbb{Z}$ with $d'_n \circ f_n = f_{n-1} \circ d_n$ for all $n \in \mathbb{Z}$. This implies $d'_n \circ f_n(x) = f_{n-1} \circ d_n(x) = 0$ for all $x \in Z_n(X_\bullet) = \ker(d_n)$, i. e. $f_n(Z_n(X_\bullet)) \subset Z_n(Y_\bullet)$. Moreover, if there is a $x' \in X_{n+1}$ with $x = d_{n+1}(x')$ then $d'_{n+1}(f_{n+1}(x')) = f_n(d_{n+1}(x')) = f_n(x)$, i. e. $f_n(B_n(X_\bullet)) \subset B_n(Y_\bullet)$. We obtain a module morphism

$$H_n(f_\bullet) : H_n(X_\bullet) \to H_n(Y_\bullet), \quad [x] \mapsto [f_n(x)]$$

where $[x]$ is the equivalence class of $x \in Z_n(X_\bullet)$ in $H_n(X_\bullet)$ and $[f_n(x)]$ the equivalence class of $f_n(x) \in Z_n(Y_\bullet)$ in $H_n(Y_\bullet)$. As

$$\begin{aligned} &H_n(\mathrm{id}_{X_\bullet})([x]) = [\mathrm{id}_{X_n}(x)] = [x] = \mathrm{id}_{H_n(X_\bullet)}([x]) \\ &H_n(g_\bullet \circ f_\bullet)([x]) = [g_n \circ f_n(x)] = H_n(g_\bullet)([f_n(x)]) = H_n(g_\bullet) \circ H_n(f_\bullet)([x]), \end{aligned}$$

for all $x \in Z_n(X_\bullet)$ and chain maps $f_\bullet : (X_\bullet, d_\bullet) \to (Y_\bullet, d'_\bullet)$, $g_\bullet : (Y_\bullet, d'_\bullet) \to (Z_\bullet, d''_\bullet)$, we obtain the following proposition.

Proposition 2.3.4: The nth homology defines a functor $H_n : \mathrm{Ch}_{R\text{-Mod}} \to R$-Mod, which assigns to a chain complex $(X_\bullet, d_\bullet)$ the R-module $H_n(X_\bullet)$ and to a chain map $f_\bullet : (X_\bullet, d_\bullet) \to (Y_\bullet, d'_\bullet)$ the module morphism $H_n(f_\bullet) : H_n(X_\bullet) \to H_n(Y_\bullet)$, $[x] \mapsto [f_n(x)]$.

We will see later that continuous maps between topological spaces induce chain maps between the associated chain complexes. However, there is another layer of structure that relates continuous maps between topological spaces, namely homotopies, which also have a counterpart in the category $\mathrm{Ch}_{R\text{-Mod}}$ - the so-called chain homotopies. We will show later that homotopies between continuous maps induce chain homotopies between the associated chain maps.

Definition 2.3.5: Let R be a unital ring and $(X_\bullet, d_\bullet)$ and $(Y_\bullet, d'_\bullet)$ chain complexes in R-Mod.

1. A **chain homotopy** $h_\bullet : f_\bullet \Rightarrow f'_\bullet$ from a chain map $f_\bullet : (X_\bullet, d_\bullet) \to (Y_\bullet, d'_\bullet)$ to a chain map $g_\bullet : (X_\bullet, d_\bullet) \to (Y_\bullet, d'_\bullet)$ is a family $(h_n)_{n\in\mathbb{Z}}$ of R-module morphisms $h_n : X_n \to Y_{n+1}$ with
$$g_n - f_n = h_{n-1} \circ d_n + d'_{n+1} \circ h_n \qquad \text{for all } n \in \mathbb{Z}.$$
If there is a chain homotopy from $f_\bullet : (X_\bullet, d_\bullet) \to (Y_\bullet, d'_\bullet)$ to $g_\bullet : (X_\bullet, d_\bullet) \to (Y_\bullet, d'_\bullet)$, then $f_\bullet$ and $g_\bullet$ are called **chain homotopic** and one writes $f_\bullet \sim g_\bullet$.
2. Two chain complexes $(X_\bullet, d_\bullet)$ and $(Y_\bullet, d'_\bullet)$ are called **chain homotopy equivalent**, if there are chain maps $f_\bullet : (X_\bullet, d_\bullet) \to (Y_\bullet, d'_\bullet)$ and $g_\bullet : (Y_\bullet, d'_\bullet) \to (X_\bullet, d_\bullet)$ with $g_\bullet \circ f_\bullet \sim \mathrm{id}_{X_\bullet}$ and $f_\bullet \circ g_\bullet \sim \mathrm{id}_{Y_\bullet}$. Such chain maps are called **chain homotopy equivalences**.

Remark 2.3.6: A chain homotopy $h_\bullet : f_\bullet \Rightarrow g_\bullet$ between chain maps $f_\bullet, g_\bullet : X_\bullet \to Y_\bullet$ can be viewed as a chain map in a natural way. Denote by $Z_\bullet$ the chain complex consisting of R-modules and R-module morphisms
$$Z_n = X_n \oplus X_n \oplus X_{n-1}, \qquad d^Z_n : Z_n \to Z_{n-1}, (x, x', x'') \mapsto (d^X_n(x) - x'', d^X_n(x') + x'', -d^X_{n-1}(x'')).$$
Then by the universal property of the direct sum, a chain map $k_\bullet : Z_\bullet \to Y_\bullet$ is given by triples of module morphisms $k_n = (f_n, g_n, h_{n-1}) : X_n \oplus X_n \oplus X_{n-1} \to Y_n$ for all $n \in \mathbb{Z}$. The condition that $k_\bullet : Z_\bullet \to Y_\bullet$ is a chain map - $d^Y_n \circ k_n = k_{n-1} \circ d^Z_n$ for all $n \in \mathbb{Z}$ - is equivalent to
$$\begin{aligned}&d^Y_n \circ f_n(x) + d^Y_n \circ g_n(x') + d^Y_n \circ h_{n-1}(x'')\\ &= f_{n-1} \circ d^X_n(x) - f_{n-1}(x'') + g_{n-1} \circ d^X_n(x') + g_{n-1}(x'') - h_{n-2} \circ d^X_{n-1}(x'') \quad \forall x, x' \in X_n, x'' \in X_{n-1}.\end{aligned}$$
This implies that $f_\bullet, g_\bullet : X_\bullet \to Y_\bullet$ are chain maps and that $h_\bullet : f_\bullet \Rightarrow g_\bullet$ is a chain homotopy.
$$\begin{aligned}&d^Y_n \circ f_n(x) = f_{n-1} \circ d^X_n(x), \qquad d^Y_n \circ g_n(x) = g_{n-1} \circ d^X_n(x)\\ &g_n(x) - f_n(x) = d^Y_{n+1} \circ h_n(x) + h_{n-1} \circ d^X_n(x) \qquad \forall x \in X_n.\end{aligned}$$

Remark 2.3.7:

1. For given chain complexes $X_\bullet$, $Y_\bullet$, chain maps $f_\bullet, f'_\bullet : X_\bullet \to Y_\bullet$ and chain homotopies $h_\bullet : f_\bullet \Rightarrow f'_\bullet$ form a groupoid.

   The composite of two chain homotopies $h : f_\bullet \Rightarrow f'_\bullet$ and $h'_\bullet : f'_\bullet \Rightarrow f''_\bullet$ is the chain homotopy $(h_\bullet + h'_\bullet) = (h_n + h'_n)_{n\in\mathbb{Z}}$ from $f_\bullet$ to $f''_\bullet$. The composition of chain homotopies is associative, and the identity morphism on a chain map $f_\bullet : X_\bullet \to Y_\bullet$ is the trivial homotopy $1_{f_\bullet} = (0)_{n\in\mathbb{Z}}$. The inverse of a chain homotopy $h : f_\bullet \Rightarrow f'_\bullet$ is the chain homotopy $-h_\bullet = (-h_n)_{n\in\mathbb{Z}}$ from $f'_\bullet$ to $f''_\bullet$.
2. It follows that *chain homotopic* is an equivalence relation on each Hom-set $\mathrm{Hom}_{\mathrm{Ch}_R\mathrm{Mod}}((X_\bullet, d_\bullet), (Y_\bullet, d'_\bullet))$, and this equivalence relation is compatible with the composition of morphisms:

   If $h_\bullet$ is a chain homotopy from $f_\bullet : X_\bullet \to Y_\bullet$ to $f'_\bullet : X_\bullet \to Y_\bullet$ and $h'_\bullet$ a chain homotopy from $g_\bullet : Y_\bullet \to Z_\bullet$ to $g'_\bullet : Y_\bullet \to Z_\bullet$, then $h''_\bullet = (g'_{n+1} \circ h_n + h'_n \circ f_n)_{n\in\mathbb{Z}}$ is a chain homotopy from $g_\bullet \circ f_\bullet$ to $g'_\bullet \circ f'_\bullet$ since
$$\begin{aligned}g'_n \circ f'_n - g_n \circ f_n =& g'_n \circ (f'_n - f_n) + (g'_n - g_n) \circ f_n\\ =& g'_n \circ (h_{n-1} \circ d_n + d'_{n+1} \circ h_n) + (h'_{n-1} \circ d'_n + d''_{n+1} \circ h'_n) \circ f_n\\ =& g'_n \circ h_{n-1} \circ d_n + d''_{n+1} \circ g'_{n+1} \circ h_n + h'_{n-1} \circ f_{n-1} \circ d_n + d''_{n+1} \circ h'_n \circ f_n\\ =& (g'_n \circ h_{n-1} + h'_{n-1} \circ f_{n-1}) \circ d_n + d''_{n+1} \circ (g'_{n+1} \circ h_n + h'_n \circ f_n).\end{aligned}$$

3. We obtain a category $\mathrm{hCh}_{R\text{-Mod}}$, the **homotopy category of chain complexes** in R-Mod, whose objects are chain complexes in R-Mod and whose morphisms are chain homotopy classes of chain maps. The isomorphisms in $\mathrm{hCh}_{R\text{-Mod}}$ are chain homotopy equivalences.

The fact that chain homotopic is an equivalence relation on the set of chain maps $f_\bullet : X_\bullet \to Y_\bullet$ between given chain complexes $X_\bullet$, $Y_\bullet$ allows one to identify all chain maps that are chain homotopic. This amounts to a classification of chain complexes up to chain homotopy equivalences rather than isomorphisms of chain complexes. A key motivation to do so is the fact that chain homotopic chain maps induce the same module morphisms on the homologies.

Proposition 2.3.8: Let R be a unital ring and $(X_\bullet, d_\bullet)$, $(Y_\bullet, d'_\bullet)$ chain complexes in R-Mod.

1. If $f_\bullet, f'_\bullet : X_\bullet \to Y_\bullet$ are chain homotopic, then $H_n(f_\bullet) = H_n(f'_\bullet) : H_n(X_\bullet) \to H_n(Y_\bullet)$.
2. If $f_\bullet : X_\bullet \to Y_\bullet$ is a chain homotopy equivalence, then $H_n(f_\bullet) : H_n(X_\bullet) \to H_n(Y_\bullet)$ is an R-module isomorphism.

The nth homology functor $H_n : \mathrm{Ch}_{R\text{-Mod}} \to R\text{-Mod}$ induces a functor $H_n : \mathrm{hCh}_{R\text{-Mod}} \to R\text{-Mod}$ that assigns to a chain complex $X_\bullet$ the R-module $H_n(X_\bullet)$ and to a chain homotopy class of chain maps $f : X_\bullet \to Y_\bullet$ the R-module morphism $H_n(f_\bullet) : H_n(X_\bullet) \to H_n(Y_\bullet)$.

Proof:
1. Let $f_\bullet, f'_\bullet : (X_\bullet, d_\bullet) \to (Y_\bullet, d'_\bullet)$ chain maps and $h_\bullet : f_\bullet \Rightarrow f'_\bullet$ a chain homotopy. Then we have for all $x \in Z_n(X_\bullet)$

$$H_n(g_\bullet)([x]) = [g_n(x)] = [f_n(x)] + [h_{n-1} \circ \underbrace{d_n(x)}_{=0}] + [\underbrace{d'_{n+1} \circ h_n(x)}_{\in B_n(Y_\bullet)}] = [f_n(x)] = H_n(f)([x]).$$

2. If $f_\bullet : X_\bullet \to Y_\bullet$ is a chain homotopy equivalence, then there is a chain map $g_\bullet : Y_\bullet \to X_\bullet$ such that $g_\bullet \circ f_\bullet \sim \mathrm{id}_{X_\bullet}$ and $f_\bullet \circ g_\bullet \sim \mathrm{id}_{Y_\bullet}$. With 1. this implies $H_n(g_\bullet) \circ H_n(f_\bullet) = \mathrm{id}_{H_n(X_\bullet)}$, $H_n(f_\bullet) \circ H_n(g_\bullet) = \mathrm{id}_{H_n(Y_\bullet)}$, and hence $H_n(g_\bullet)$ and $H_n(f_\bullet)$ are isomorphisms. □

As chain complexes and chain maps form an abelian category, it is possible to perform all constructions of homological algebra with chain maps and chain complexes instead of R-modules and module morphisms. In the following, we will not need this in full generality, but we require the notion of a (short exact) sequence of chain complexes.

Definition 2.3.9:

1. A **sequence** of chain complexes is a family $(X^{(n)}_\bullet d^{(n)}_\bullet)_{n\in\mathbb{Z}}$ of chain complexes $(X^{(n)}_\bullet, d^{(n)}_\bullet)$ together with a family $(f^{(n)}_\bullet)_{n\in\mathbb{Z}}$ of chain maps $f^{(n)}_\bullet : (X^{(n)}_\bullet, d^{(n)}_\bullet) \to (X^{(n-1)}_\bullet, d^{(n-1)}_\bullet)$
$$\ldots \xrightarrow{f^{(n+2)}_\bullet} X^{(n+1)}_\bullet \xrightarrow{f^{(n+1)}_\bullet} X^{(n)}_\bullet \xrightarrow{f^{(n)}_\bullet} X^{(n-1)}_\bullet \xrightarrow{f^{(n-2)}_\bullet} \ldots.$$
2. A sequence of chain complexes is called **exact** if for all $m \in \mathbb{Z}$ the sequence
$$\ldots \xrightarrow{f^{(n+2)}_m} X^{(n+1)}_m \xrightarrow{f^{(n+1)}_m} X^{(n)}_m \xrightarrow{f^{(n)}_m} X^{(n-1)}_m \xrightarrow{f^{(n-1)}_m} \ldots$$
is an exact sequence in R-Mod, i. e. $\ker(f^{(n)}_m) = \mathrm{Im}\,(f^{(n+1)}_m)$ for all $n, m \in \mathbb{Z}$.

3. A **short exact sequence of chain complexes** is an exact sequence of chain complexes

$$0 \to L_\bullet \xrightarrow{\iota_\bullet} M_\bullet \xrightarrow{\pi_\bullet} N_\bullet \to 0.$$

We will see later that pairs of topological spaces (X, A) give rise to short exact sequences of chain complexes. The key point about short exact sequence of chain complexes is that they allows one to relate the associated homologies, which is highly useful in computations. Clearly, each sequence $0 \to L_\bullet \xrightarrow{\iota_\bullet} M_\bullet \xrightarrow{\pi_\bullet} N_\bullet \to 0$. gives rise to R-module morphisms $H_n(\iota_\bullet) : H_n(L_\bullet) \to H_n(M_\bullet)$ and $H_n(\pi_\bullet) : H_n(M_\bullet) \to H_n(N_\bullet)$ for all $n \in \mathbb{Z}$. However, we will see in the following that the exactness of the sequence yields another R-module morphism $\overline{\partial}_n : H_n(N_\bullet) \to H_{n-1}(L_\bullet)$, the *connecting homomorphism*. This allows one to organise the homologies $H_n(L_\bullet)$, $H_n(M_\bullet)$, $H_n(N_\bullet)$ into an exact sequence that involves the homologies of all three chain complexes and all indices $n \in \mathbb{Z}$. The proof of this statement requires a rather technical lemma known under the name *snake lemma*.

Lemma 2.3.10: (Snake lemma) Let R be a commutative unital ring, and suppose the following diagram in R-Mod commutes and has exact rows

$$\begin{tikzcd} & L \arrow[r, "\iota"] \arrow[d, "f"] & M \arrow[r, "\pi"] \arrow[d, "g"] & N \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & L' \arrow[r, "\iota'"'] & M' \arrow[r, "\pi'"'] & N'. & \end{tikzcd}$$

Then there is a unique morphism $\partial : \ker(h) \to L/\mathrm{Im}\,(f)$, the **connecting homomorphism** such that the following sequence of R-modules and R-module morphisms is exact

$$\ker(f) \xrightarrow{\iota|_{\ker(f)}} \ker(g) \xrightarrow{\pi|_{\ker(g)}} \ker(h) \xrightarrow{\partial} L'/\mathrm{Im}\,(f) \xrightarrow{\tilde{\iota}'} M'/\mathrm{Im}\,(g) \xrightarrow{\tilde{\pi}'} N'/\mathrm{Im}\,(h).$$

If $\iota : L \to M$ is injective, then $\iota|_{\ker(f)} : \ker(f) \to \ker(g)$ is injective, and if $\pi' : M' \to N'$ is surjective, then $\tilde{\pi}' : M'/\mathrm{Im}\,(g) \to N'/\mathrm{Im}\,(h)$ is surjective.

Proof:
1. As the commutativity of the diagram implies $g \circ \iota = \iota' \circ f$ and $h \circ \pi = \pi' \circ g$, we have $\iota(\ker(f)) \subset \ker(g)$ and $\pi(\ker(g)) \subset \ker(h)$. Denoting by $p_f : L' \to L'/\mathrm{Im}\,(f)$, $p_g : M' \to M'/\mathrm{Im}\,(g)$, $p_h : N' \to N'/\mathrm{Im}\,(h)$ the canonical surjections, we obtain from the commutativity of the diagram the identity $p_g \circ \iota' \circ f = p_g \circ g \circ \iota = 0$, i. e. $\mathrm{Im}\,(f) \subset \ker(p_g \circ \iota')$. The universal property of the quotient then implies that there is unique R-module morphism $\tilde{\iota}' : L'/\mathrm{Im}\,(f) \to M'/\mathrm{Im}\,(g)$ with $\tilde{\iota}' \circ p_f = p_g \circ \iota'$. Analogously, we obtain a unique R-module morphism $\tilde{\pi}' : M'/\mathrm{Im}\,(g) \to N'/\mathrm{Im}\,(h)$ with $\tilde{\pi}' \circ p_g = p_h \circ \pi'$. This yields the commuting diagram

$$\begin{tikzcd} & \ker(f) \arrow[r, "\iota|_{\ker(f)}"] \arrow[d, hook, "i_f"] & \ker(g) \arrow[r, "\pi|_{\ker g}"] \arrow[d, hook, "i_g"] & \ker(h) \arrow[d, hook, "i_h"] & \\ & L \arrow[r, "\iota"] \arrow[d, "f"] & M \arrow[r, "\pi"] \arrow[d, "g"] & N \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & L' \arrow[r, "\iota'"'] \arrow[d, "p_f"] & M' \arrow[r, "\pi'"'] \arrow[d, "p_g"] & N' \arrow[d, "p_h"] & \\ & L'/\mathrm{Im}\,(f) \arrow[r, dashed, "\exists!\tilde{\iota}'"'] & M'/\mathrm{Im}\,(g) \arrow[r, dashed, "\exists!\tilde{\pi}'"'] & N'/\mathrm{Im}\,(h) & \end{tikzcd}$$

with exact columns and in which the second and third row are exact. It is clear that for any injective module morphism $\iota : L \to M$, the module morphism $\iota : \ker(f) \to \ker(g)$ is injective, and for any surjective module morphism $\pi : M \to N$ the module morphism $\tilde{\pi}' : M'/\mathrm{Im}\,(g) \to N'/\mathrm{Im}\,(h)$ is surjective.

2. We show that the first and fourth row of the diagram are exact:

The commutativity of the diagram implies $i_h \circ (\pi|_{\ker(g)}) \circ (\iota|_{\ker(f)}) = \pi \circ \iota \circ i_f = 0$. As the map $i_h : \ker(h) \to N$ is injective it follows that $(\pi|_{\ker(g)}) \circ (\iota|_{\ker(f)}) = 0$, i. e. $\mathrm{Im}\,(\iota|_{\ker(f)}) \subset \ker(\pi|_{\ker g})$. If $m \in \ker(g)$ with $\pi(m) = 0$, the exactness of the second row implies that there is an $l \in L$ with $m = \iota(l)$. The commutativity of the diagram implies $\iota' \circ f(l) = g \circ \iota(l) = g(m) = 0$, and since $\iota' : L' \to M'$ is injective it follows that $f(l) = 0$, i. e. $l \in \ker(f)$. This shows that $m \in \mathrm{Im}\,(\iota|_{\ker(f)})$ and hence $\mathrm{Im}\,(\iota|_{\ker(f)}) = \ker(\pi|_{\ker g})$. The proof of the exactness of the fourth row is analogous and is left as an exercise.

3. We construct the connecting homomorphism:

Consider an element $n \in \ker(h)$. Then the surjectivity of π implies that there is an $m \in M$ with $\pi(m) = n$. If $m' \in M$ is another element with this property, then the exactness of the second row implies that there is an $l \in L$ with $m' = m + \iota(l)$. As we have $\pi' \circ g(m) = h \circ \pi(m) = h(n) = 0$ it follows that $g(m) \in \ker(\pi')$, and analogously $g(m') \in \ker(\pi')$. The exactness of the third row implies that there is an $l' \in L'$ with $\iota'(l') = g(m)$, and since $\iota' : L' \to M'$ is injective, this element $l' \in L'$ is unique. Similarly, we obtain a unique element $l'' \in L'$ with $\iota'(l'') = g(m') = g(m) + g \circ \iota(l) = \iota'(l') + \iota' \circ f(l) = \iota'(l' + f(l))$. As ι' is injective, this implies $l'' = l' + f(l)$, and $p_f(l) = p_f(l')$. We obtain a well-defined map

$$\partial : \ker(h) \to L'/\mathrm{Im}\,(f), \quad n \mapsto p_f(l') \quad \text{where} \quad n = \pi(m), \iota'(l') = g(m), \tag{18}$$

which is an R-module morphisms since all ingredients involved in the construction are R-linear.

4. The connecting homomorphism in (18) yields a sequence

$$\ker(f) \xrightarrow{\iota|_{\ker(f)}} \ker(g) \xrightarrow{\pi|_{\ker(g)}} \ker(h) \xrightarrow{\partial} L'/\mathrm{Im}\,(f) \xrightarrow{\tilde{\iota}'} M'/\mathrm{Im}\,(g) \xrightarrow{\tilde{\pi}'} N'/\mathrm{Im}\,(h).$$

which is already exact in all entries except $\ker(h)$ and $L'/\mathrm{Im}\,(f)$. It remains to show the exactness in $\ker(h)$ and $L'/\mathrm{Im}\,(f)$. For the former, consider an element $n \in \mathrm{Im}\,(\pi|_{\ker(g)})$. Then there is an element $m \in M$ with $g(m) = 0$ and $n = \pi(m)$ and 3. yields an element $l' \in L$ with $\iota'(l') = g(m) = 0$. The injectivity of ι' implies $l' = 0$ and by definition of the connecting homomorphism in (18), we have $\partial(n) = p_f(l') = 0$. This shows that $\mathrm{Im}\,(\pi|_{\ker(g)}) \subset \ker(\partial)$.

Conversely, if $n \in \ker(\partial)$, then by (18) there are elements $m \in M$, $l' \in L'$ with $n = \pi(m)$, $g(m) = \iota'(l')$ and $p_f(l') = 0$. This implies $l' \in \ker(p_f) = \mathrm{Im}\,(f)$, and hence there is an $l \in L$ with $l' = f(l)$. This implies $g(m) = \iota' \circ f(l) = g \circ \iota(l)$, and hence $m - \iota(l) \in \ker(g)$. It follows that $\pi(m - \iota(l)) = \pi(m) = n$ and consequently $n \in \mathrm{Im}\,(\pi|_{\ker(g)})$. This shows that $\ker(\partial) \subset \mathrm{Im}\,(\pi|_{\ker(g)})$ and proves the exactness of the sequence in $\ker(h)$. The proof of the exactness in $L'/\mathrm{Im}\,(f)$ is analogous. □

The name *snake lemma* stems from the fact that the connecting homomorphism $\partial : \ker(h) \to L'/\mathrm{Im}\,(f)$ has to be represented by an snakelike arrow in the commutative diagram with the short exact sequences of chain complexes and the associated chain maps, as shown below

$$
\begin{tikzcd}
 & \ker(f) \arrow[r, "\iota|_{\ker(f)}"] \arrow[d, hook, "i_f"] & \ker(g) \arrow[r, "\pi|_{\ker(g)}"] \arrow[d, hook, "i_g"] & \ker(h) \arrow[d, hook, "i_h"] \arrow[dddll, rounded corners, "\partial"] & \\
 & L \arrow[r, "\iota"] \arrow[d, "f"] & M \arrow[r, "\pi"] \arrow[d, "g"] & N \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & L' \arrow[r, "\iota'"] \arrow[d, "p_f"] & M' \arrow[r, "\pi'"] \arrow[d, "p_g"] & N' \arrow[d, "p_h"] & \\
 & L'/\mathrm{Im}\,(f) \arrow[r, "\tilde{\iota}'"'] & M'/\mathrm{Im}\,(g) \arrow[r, "\tilde{\pi}'"'] & N'/\mathrm{Im}\,(h) &
\end{tikzcd}
$$

By means of the snake lemma we can relate the homologies $H_n(L_\bullet)$, $H_n(M_\bullet)$, $H_n(N_\bullet)$ for any short exact sequence of chain complexes $0 \to (L_\bullet, d^L_\bullet) \xrightarrow{\iota_\bullet} (M_\bullet, d^M_\bullet) \xrightarrow{\pi_\bullet} (N_\bullet, d^N_\bullet) \to 0$. The result is the so-called *long exact sequence of homologies* which will serve as an important tool to determine the homologies of topological spaces.

Theorem 2.3.11: (Long exact sequence of homologies)

A short exact sequence $0 \to L_\bullet \xrightarrow{\iota_\bullet} M_\bullet \xrightarrow{\pi_\bullet} N_\bullet \to 0$ of chain complexes in R-Mod induces an exact sequence

$$\ldots \xrightarrow{H_{n+1}(\pi_\bullet)} H_{n+1}(N_\bullet) \xrightarrow{\overline{\partial}_{n+1}} H_n(L_\bullet) \xrightarrow{H_n(\iota_\bullet)} H_n(M_\bullet) \xrightarrow{H_n(\pi_\bullet)} H_n(N_\bullet) \xrightarrow{\overline{\partial}_n} H_{n-1}(L_\bullet) \xrightarrow{H_{n-1}(\iota_\bullet)} \ldots,$$

of homologies, the **long exact sequence of homologies**. The morphisms $\overline{\partial}_n : H_n(N_\bullet) \to H_{n-1}(L_\bullet)$ are called **connecting homomorphisms**.

Proof:
For all $n \in \mathbb{Z}$, the short exact sequence of chain complexes determines a commuting diagram with exact rows

$$
\begin{tikzcd}
0 \arrow[r] & L_n \arrow[r, "\iota_n"] \arrow[d, "d^L_n"] & M_n \arrow[r, "\pi_n"] \arrow[d, "d^M_n"] & N_n \arrow[r] \arrow[d, "d^N_n"] & 0 \\
0 \arrow[r] & L_{n-1} \arrow[r, "\iota_{n-1}"'] & M_{n-1} \arrow[r, "\pi_{n-1}"'] & N_{n-1} \arrow[r] & 0,
\end{tikzcd}
$$

and the snake lemma yields a unique R-module morphism $\partial_n : \ker(d^N_n) \to L_{n-1}/\mathrm{Im}\,(d^L_n)$ and an exact sequence

$$\ker(d^L_n) \xrightarrow{\iota_n|_{\ker(d^L_n)}} \ker(d^M_n) \xrightarrow{\pi_n|_{\ker(d^M_n)}} \ker(d^N_n) \xrightarrow{\partial_n} L_{n-1}/\mathrm{Im}\,(d^L_n) \xrightarrow{\tilde{\iota}_{n-1}} M_{n-1}/\mathrm{Im}\,(d^M_n) \xrightarrow{\tilde{\pi}_{n-1}} N_{n-1}/\mathrm{Im}\,(d^N_n)$$

As all module morphisms $\iota_n : L_n \to M_n$ are injective and all module morphisms $\pi_n : M_i \to N_n$ surjective the module morphisms $\iota_n|_{\ker(d^L_n)} : \ker(d^L_n) \to \ker(d^M_n)$ are injective and the module morphisms $\tilde{p}_{n-1} : M_{n-1}/\mathrm{Im}\,(d^M_n) \to N_{n-1}/\mathrm{Im}\,(d^N_n)$ are surjective. This yields the following commuting diagram with exact rows

$$\begin{tikzcd}
& {\begin{array}{c} L_n/B_n(L_\bullet) \\ = L_n/\mathrm{Im}\,(d^L_{n+1}) \end{array}} \arrow[r, "\tilde\iota_n"] \arrow[d, "\tilde d^L_n"] & {\begin{array}{c} M_n/B_n(M_\bullet) \\ = M_n/\mathrm{Im}\,(d^M_{n+1}) \end{array}} \arrow[r, "\tilde\pi_n"] \arrow[d, "\tilde d^M_n"] & {\begin{array}{c} N_n/B_n(N_\bullet) \\ = N_n/\mathrm{Im}\,(d^N_{n+1}) \end{array}} \arrow[r] \arrow[d, "\tilde d^N_n"] & 0\,, \\
0 \arrow[r] & {\begin{array}{c} \ker(d^L_{n-1}) \\ = Z_{n-1}(L_\bullet) \end{array}} \arrow[r, "\iota_{n-1}|_{\ker(d^L_{n-1})}"] & {\begin{array}{c} \ker(d^M_{n-1}) \\ = Z_{n-1}(M_\bullet) \end{array}} \arrow[r, "\pi_{n-1}|_{\ker(d^M_{n-1})}"] & {\begin{array}{c} \ker(d^N_{n-1}) \\ = Z_{n-1}(N_\bullet) \end{array}} &
\end{tikzcd}$$

in which the morphisms $\tilde{d}^X_n : X_n/\mathrm{Im}\,(d^X_{n+1}) \to \ker(d^X_{n-1})$ for $X = L, M, N$ are the unique morphisms with $\tilde{d}^X_n([x]) = d^X_n(x)$ for all $x \in X_n$ determined by the universal property of the quotient and the identities $d^X_{n-1} \circ d^X_n = 0$. This implies for $X = L, M, N$ and all $n \in \mathbb{Z}$

$$H_n(X_\bullet) = Z_n(X_\bullet)/B_n(X_\bullet) \cong \ker(\tilde{d}^X_n), \qquad H_{n-1}(X_\bullet) = Z_{n-1}(X_\bullet)/B_{n-1}(X_\bullet) \cong Z_{n-1}(X_\bullet)/\mathrm{Im}\,(\tilde{d}^X_n).$$

Applying again the snake lemma, we obtain an exact sequence

$$H_n(L_\bullet) \xrightarrow{H_n(\iota_\bullet)} H_n(M_\bullet) \xrightarrow{H_n(\pi_\bullet)} H_n(N_\bullet) \xrightarrow{\overline{\partial}_n} H_{n-1}(L_\bullet) \xrightarrow{H_{n-1}(\iota_\bullet)} H_{n-1}(M_\bullet) \xrightarrow{H_{n-1}(\pi_\bullet)} H_{n-1}(N_\bullet),$$

and by combining these sequences for different $n \in \mathbb{Z}$ the long exact sequence of homologies. The connecting homomorphism $\overline{\partial}_k : H_k(N_\bullet) \to H_{k-1}(L_\bullet)$ is given by

$$\overline{\partial}_k([n]_k) = [l]_{k-1} \text{ where } l \in Z_{k-1}(L_\bullet) \text{ with } \iota_{k-1}(l) = d^M_k(m) \text{ for an } m \in M_k \text{with } \pi_k(m) = n. \tag{19}$$

□

The connecting homomorphism is called connecting homomorphism because it connects the homologies $H_n(L_\bullet)$, $H_n(M_\bullet)$, $H_n(N_\bullet)$ for different $n \in \mathbb{N}$, as shown below. Although its construction appears to be technical and not very intuitive, it has a direct interpretation as a natural transformation of between certain functors. As it assigns to a short exact sequence of chain complexes $0 \to L_\bullet \xrightarrow{\iota_\bullet} M_\bullet \xrightarrow{\pi_\bullet} N \to 0$ a collection of morphisms $\overline{\partial}_k : H_k(N_\bullet) \to H_{k-1}(N_\bullet)$, the relevant functors should associate to the short exact sequence of chain complexes the homologies $H_k(N_\bullet)$ and $H_{k-1}(L_\bullet)$ and to a triple of chain maps $f_\bullet : L_\bullet \to L'_\bullet$, $g_\bullet : M_\bullet \to M'_\bullet$, $h : N_\bullet \to N'_\bullet$ that satisfy a compatibility condition with the structure of the short exact sequence the associated R-module morphisms $H_k(h_\bullet) : H_k(N_\bullet) \to H_k(N'_\bullet)$ and $H_{k-1}(f_\bullet) : H_{k-1}(L_\bullet) \to H_{k-1}(L'_\bullet)$. The following lemma shows that the defining condition of a natural transformation is indeed satisfied by the connecting homomorphisms.

$$
\begin{tikzcd}
& \dots \arrow[r, "H_{n+2}(\pi_\bullet)"] & H_{n+2}(N_\bullet) \arrow[dll, rounded corners, "\overline{\partial}_{n+2}"] \\
H_{n+1}(L_\bullet) \arrow[r, "H_{n+1}(\iota_\bullet)"] & H_{n+1}(M_\bullet) \arrow[r, "H_{n+1}(\pi_\bullet)"] & H_{n+1}(N_\bullet) \arrow[dll, rounded corners, "\overline{\partial}_{n+1}"] \\
H_n(L_\bullet) \arrow[r, "H_n(\iota_\bullet)"] & H_n(M_\bullet) \arrow[r, "H_n(\pi_\bullet)"] & H_n(N_\bullet) \arrow[dll, rounded corners, "\overline{\partial}_n"] \\
H_{n-1}(L_\bullet) \arrow[r, "H_{n-1}(\iota_\bullet)"] & H_{n-1}(M_\bullet) \arrow[r, "H_{n-1}(\pi_\bullet)"] & H_{n-1}(N_\bullet) \arrow[dll, rounded corners, "\overline{\partial}_{n-1}"] \\
H_{n-2}(L_\bullet) \arrow[r, "H_{n-2}(\iota_\bullet)"] & \dots &
\end{tikzcd}
$$

The connecting homomorphism.

Lemma 2.3.12: Let $0 \to L_\bullet \xrightarrow{\iota_\bullet} M_\bullet \xrightarrow{\pi_\bullet} N \to 0$ and $0 \to L'_\bullet \xrightarrow{\iota'_\bullet} M'_\bullet \xrightarrow{\pi'_\bullet} N' \to 0$ short exact sequences of chain complexes in R-Mod and $f_\bullet : L_\bullet \to L'_\bullet$, $g_\bullet : M_\bullet \to M'_\bullet$ and $h : N_\bullet \to N'_\bullet$ chain maps such that the diagram

$$
\begin{tikzcd}
0 \arrow[r] & L_\bullet \arrow[r, "\iota_\bullet"] \arrow[d, "f_\bullet"] & M_\bullet \arrow[r, "\pi_\bullet"] \arrow[d, "g_\bullet"] & N_\bullet \arrow[r] \arrow[d, "h_\bullet"] & 0 \\
0 \arrow[r] & L'_\bullet \arrow[r, "\iota'_\bullet"] & M'_\bullet \arrow[r, "\pi'_\bullet"] & N'_\bullet \arrow[r] & 0.
\end{tikzcd}
\tag{20}
$$

commutes. Denote by $\overline{\partial}_k : H_k(N_\bullet) \to H_{k-1}(L_\bullet)$ and $\overline{\partial}'_k : H_k(N'_\bullet) \to H_{k-1}(L'_\bullet)$ the associated connecting homomorphisms. Then for all $k \in \mathbb{Z}$ the following diagram commutes

$$
\begin{tikzcd}
H_k(N_\bullet) \arrow[r, "\overline{\partial}_k"] \arrow[d, "H_k(h_\bullet)"'] & H_{k-1}(L_\bullet) \arrow[d, "H_{k-1}(f_\bullet)"] \\
H_k(N'_\bullet) \arrow[r, "\overline{\partial}'_k"'] & H_{k-1}(L'_\bullet).
\end{tikzcd}
$$

Proof:
The commutativity of diagram (20) implies $h_k \circ \pi_k = \pi'_k \circ g_k$, $g_k \circ \iota_k = \iota'_k \circ f_k$ for all $k \in \mathbb{Z}$. For any $n \in Z_k(N_\bullet)$, we have by definition (19) of the connecting homomorphism $\partial_k([n]_k) = [l]_{k-1}$, where $l \in Z_{k-1}(L_\bullet)$ satisfies $\iota_{k-1}(l) = d^M_k(m)$ for an $m \in M_k$ with $\pi_k(m) = n$. This gives $H_{k-1}(f_\bullet) \circ \overline{\partial}_k([n_k]) = [f_{k-1}(l)]_{k-1}$. On the other hand, the commutativity of (20) implies $h_k(n) = h_k \circ \pi_k(m) = \pi'_k \circ g_k(m)$ and $d^{M'}_k \circ g_k(m) = g_{k-1} \circ d^M_k(m) = g_{k-1} \circ \iota_{k-1}(l) = \iota'_{k-1} \circ f_{k-1}(l)$. With definition (19) of the connecting morphism, this gives

$$\overline{\partial}'_k \circ H_k(h_\bullet)([n]_k) = \overline{\partial}'_k([h_k(n)]) = [f_{k-1}(l)]_{k-1} = H_{k-1}(f_\bullet) \circ \overline{\partial}_k([n]_k).$$

□

Corollary 2.3.13: Let $\mathcal{C}$ be the category with short exact sequences of chain complexes in R-Mod $0 \to L_\bullet \xrightarrow{\iota_\bullet} M_\bullet \xrightarrow{\pi_\bullet} N_\bullet \to 0$ as objects and triples of chain maps $f_\bullet : L_\bullet \to L'_\bullet$, $g_\bullet : M_\bullet \to M'_\bullet$, $h_\bullet : N_\bullet \to N'_\bullet$ for which the diagram (20) commutes as morphisms.

Then for all $k \in \mathbb{Z}$, the homologies define functors $H_k^{(s)}, H_k^{(t)} : \mathcal{D} \to R$-Mod, that assign to a short exact sequence $0 \to L_\bullet \xrightarrow{\iota_\bullet} M_\bullet \xrightarrow{\pi_\bullet} N_\bullet \to 0$, the kth homologies $H_k(L_\bullet)$ and $H_k(N_\bullet)$, respectively, and to a triple of chain maps $(f_\bullet, g_\bullet, h_\bullet)$ the morphisms $H_k(f_\bullet) : H_k(L_\bullet) \to H_k(L'_\bullet)$ and $H_k(h_\bullet) : H_k(N_\bullet) \to H_k(N'_\bullet)$. The connecting homomorphism $\overline{\partial}_k : H_k(N_\bullet) \to H_{k-1}(L_\bullet)$ is a natural transformation $\overline{\partial}_k : H_k^{(t)} \to H_{k-1}^{(s)}$.

Lemma 2.3.12 also allows us to relate the long exact sequences of homologies associated to two short exact sequences with compatible triples of chain maps between them. One finds that the induced R-module morphisms between the homologies are chain maps between the associated long exact sequences of homologies.

Corollary 2.3.14: Let $0 \to L_\bullet \xrightarrow{\iota_\bullet} M_\bullet \xrightarrow{\pi_\bullet} N \to 0$ and $0 \to L'_\bullet \xrightarrow{\iota'_\bullet} M'_\bullet \xrightarrow{\pi'_\bullet} N' \to 0$ be short exact sequences of chain complexes in R-Mod and $f_\bullet : L_\bullet \to L'_\bullet$, $g_\bullet : M_\bullet \to M'_\bullet$, $h : N_\bullet \to N'_\bullet$ chain maps such that diagram (20) commutes. Then for all $k \in \mathbb{Z}$, the following diagram commutes

$$\begin{tikzcd}
\ldots \arrow[r, "H_{k+1}(\iota_\bullet)"] & H_{k+1}(M_\bullet) \arrow[r, "H_{k+1}(\pi_\bullet)"] \arrow[d, "H_{k+1}(g_\bullet)"'] & H_{k+1}(N_\bullet) \arrow[r, "\overline{\partial}_{k+1}"] \arrow[d, "H_{k+1}(h_\bullet)"'] & H_k(L_\bullet) \arrow[r, "H_k(\iota_\bullet)"] \arrow[d, "H_k(f_\bullet)"'] & H_k(M_\bullet) \arrow[r, "H_k(\pi_\bullet)"] \arrow[d, "H_k(g_\bullet)"'] & H_k(N_\bullet) \arrow[r, "\overline{\partial}_k"] \arrow[d, "H_k(h_\bullet)"'] & \ldots \\
\ldots \arrow[r, "H_{k+1}(\iota'_\bullet)"'] & H_{k+1}(M'_\bullet) \arrow[r, "H_{k+1}(\pi'_\bullet)"'] & H_{k+1}(N'_\bullet) \arrow[r, "\overline{\partial}'_{k+1}"'] & H_k(L'_\bullet) \arrow[r, "H_k(\iota'_\bullet)"'] & H_k(M'_\bullet) \arrow[r, "H_k(\pi'_\bullet)"'] & H_k(N'_\bullet) \arrow[r, "\overline{\partial}'_k"'] & \ldots .
\end{tikzcd}$$

Proof:
The commutativity of diagram (20) implies $H_k(h_\bullet) \circ H_k(\pi_\bullet) = H_k(\pi'_\bullet) \circ H_k(g_\bullet)$ and $H_k(g_\bullet) \circ H_k(\iota_\bullet) = H_k(\iota'_\bullet) \circ H_k(f_\bullet)$ for all $k \in \mathbb{Z}$, which means that the two outer squares in the diagram commute. The two inner squares commute by Lemma 2.3.12. □

2.4 Exercises for Chapter 2

Exercise 1: Let $\mathcal{C}$ be a groupoid with a single object X and set $G = (\mathrm{End}_\mathcal{C}(X), \circ)$.

(a) Show that a functor $F : \mathcal{C} \to \mathrm{Set}$ is the same as a group action of G on the set $F(X)$.
(b) Show that a functor $F : \mathcal{C} \to \mathrm{Vect}_\mathbb{F}$ is the same as a representation of G on $F(X)$.
(c) Show that a functor $F : \mathcal{C} \to \mathrm{Grp}$ is the same as a group homomorphism from G to the group $\mathrm{Aut}(H)$ of group automorphisms of the group $H = F(X)$.
(d) For (a), (b), (c), characterise the natural transformations between functors $F, G : \mathcal{C} \to$ Set, $\mathrm{Vect}_\mathbb{F}$, Grp.

Exercise 2: Let G be a group and $\rhd : G \times X \to X$ a group action of G on a set X, i. e.

$$(g \cdot h) \rhd x = g \rhd (h \rhd x), \quad e \rhd x = x \qquad \forall g, h \in G, x \in X.$$

Show that the category $\mathcal{C}$ with elements of X as objects, morphisms $\mathrm{Hom}_\mathcal{C}(x, y) = \{g \in G : g \rhd x = y\}$ and the composition of morphisms given by the group multiplication is a groupoid, the so-called **action groupoid** for the group action of G on X.

Exercise 3: Let $\mathcal{C}$ be a groupoid. Show that for any two objects X, Y of $\mathcal{C}$ with $\mathrm{Hom}_{\mathcal{C}}(X,Y) \neq \emptyset$, the groups $\mathrm{Hom}_{\mathcal{C}}(X,X)$ and $\mathrm{Hom}_{\mathcal{C}}(Y,Y)$ with the composition of morphisms as group multiplication are isomorphic.

Exercise 4: Let $\mathcal{C}, \mathcal{D}$ be groupoids and $F, G : \mathcal{C} \to \mathcal{D}$ a functors. Show that for all objects X of $\mathcal{C}$ the associated maps $F : \mathrm{Hom}_{\mathcal{C}}(X,X) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(X))$, $G : \mathrm{Hom}_{\mathcal{C}}(X,X) \to \mathrm{Hom}_{\mathcal{D}}(G(X), G(X))$ are group homomorphisms and determine the relation between these group homomorphisms that arises from a natural transformation $\eta : F \to G$.

Exercise 5: Let $\mathcal{C}$ be a category and $g_i : W \to X_i$ ($h_i : X_i \to Y$) for $i = 1, 2$ two morphisms in $\mathcal{C}$. Show that the pushout (pullback) of these morphisms is determined uniquely up to isomorphism by its universal property.

Exercise 6: Let G be a group. The **commutator subgroup** of G is the subgroup $[G,G]$ generated by the elements $[g,h] = g \cdot h \cdot g^{-1} \cdot h^{-1}$ for $g, h \in G$

$$[G,G] = \{[g_n, h_n] \cdot [g_{n-1}, h_{n-1}] \cdots [g_1, h_1] :\ n \in \mathbb{N}, g_i, h_i \in G \text{ for all } i \in \{1, ..., n\}\}.$$

(a) Show that the commutator subgroup is a normal subgroup of G and that the factor group $G/[G,G]$ is abelian.
(b) Denote by $\pi : G \to G/[G,G]$, $g \to g[G,G]$ the canonical surjection. Show that for every group homomorphism $f : G \to H$ there is a unique group homomorphism $\tilde{f} : G/[G,G] \to H/[H,H]$ with $\tilde{f} \circ \pi_G = \pi_H \circ f$.
(c) Show that the assignments $G \to G/[G,G]$, $f \to \tilde{f}$ define a functor $F :$ Grp $\to$ Ab and a functor $F :$ Grp $\to$ Grp, the **abelianisation functor**.
(d) Show that the canonical surjections $\pi_G : G \to G/[G,G]$ define a natural transformation between the identity functor $\mathrm{id}_{\mathrm{Grp}} : \mathrm{Grp} \to \mathrm{Grp}$ and the functor $F : \mathrm{Grp} \to \mathrm{Grp}$.

Exercise 7: Give a presentation of the following groups in terms of generators and relations:
(a) $\mathbb{Z}/n\mathbb{Z}$ with $n \in \mathbb{N}$,
(b) $\mathbb{Z} \times \mathbb{Z}$,
(c) $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ with $n, m \in \mathbb{N}$,
(d) the symmetric group S_3,
(e) the commutator subgroup $[F_2, F_2]$, where F_2 is the free group with two generators.

Exercise 8: Prove the **5-Lemma**: For any unital ring R and a commutative diagram of R-Modules and R-Modulhomomorphisms of the form

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r, "h"] \arrow[d, "\gamma"] & D \arrow[r, "k"] \arrow[d, "\delta"] & E \arrow[d, "\epsilon"] \\
A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & D' \arrow[r, "k'"] & E'
\end{tikzcd}$$

with exact rows, the following implications hold:

(i) If β, δ are monomorphisms and α an epimorphism, then γ is a monomorphism.

(ii) If β, δ are epimorphisms and ϵ a monomorphism, then γ is an epimorphism.

(iii) If α an epimorphism, ϵ a monomorphism and β, δ are isomorphisms, then γ is an isomorphism.

Exercise 9: Prove the **9-Lemma**: Let R be a unital ring and

$$\begin{tikzcd} A' \arrow[r, hook, "\iota_A"] \arrow[d, "\phi_1"] & A \arrow[r, two heads, "\pi_A"] \arrow[d, "\phi_2"] & A'' \arrow[d, "\phi_3"] \\ B' \arrow[r, hook, "\iota_B"] \arrow[d, "\psi_1"] & B \arrow[r, two heads, "\pi_B"] \arrow[d, "\psi_2"] & B'' \arrow[d, "\psi_3"] \\ C' \arrow[r, hook, "\iota_C"] & C \arrow[r, two heads, "\pi_C"] & C'', \end{tikzcd}$$

a commutative diagram with exact rows in R-Mod such that $\psi_i \circ \phi_i = 0$ für $i = 1, 2, 3$. If two columns are short exact sequences, then the third column is also a short exact sequence.

Exercise 10: Let R be a unital ring, $\mathrm{Ch}_{R\text{-Mod}}$ the category of chain complexes and chain maps in R-Mod and $\mathcal{C}$ a category.

(a) Prove that a functor $F : \mathcal{C} \to \mathrm{Ch}_{R\text{-Mod}}$ is the same as a family of functors $F_n : \mathcal{C} \to R$-Mod, $n \in \mathbb{Z}$, together with natural transformations $\eta_n : F_n \to F_{n-1}$ satisfying $\eta_{n-1} \circ \eta_n = 0$.
(b) Prove that a natural transformation between functors $F, F' : \mathcal{C} \to \mathrm{Ch}_{r\text{-Mod}}$ is the same as a family of natural transformations $\kappa_n : F_n \to F'_n$ such that $\kappa_{n-1} \circ \eta_n = \eta_n \circ \kappa_n$ for all $n \in \mathbb{Z}$.

Exercise 11: Let R be a unital ring. One says a short exact sequence $0 \to A \xrightarrow{\iota} M \xrightarrow{p} B \to 0$ of R-modules and R-module morphisms **splits** if there is a R-module isomorphism $\phi : M \to A \oplus B$ such that the following diagram commutes

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\iota"] \arrow[dr, "\iota'"'] & M \arrow[r, "p"] \arrow[d, "\phi"] & B \arrow[r] & 0, \\ & & A \oplus B \arrow[ur, "p'"'] & & \end{tikzcd}$$

where $\iota' : A \to A \oplus B$, $a \mapsto (a, 0)$ and $p' : A \oplus B \to B$, $(a, b) \to b$. Show that the following statements are equivalent:

(i) The short exact sequence $0 \to A \xrightarrow{\iota} M \xrightarrow{p} B \to 0$ splits.
(ii) The projection $p : M \to B$ has a right inverse.
(iii) the injection $\iota : A \to M$ has a left inverse.

Exercise 12: Let R be a unital ring, $(X_\bullet, d_\bullet), (Y_\bullet, d'_\bullet)$ chain complexes in R-Mod and $p \in \mathbb{Z}$. We define a chain complex $(T_p(X_\bullet), T_p(d_\bullet))$ as follows

$$T_p(X_\bullet)_n = X_{n+p}, \qquad T_p(d_\bullet)_n = (-1)^p d_{n+p} \qquad \forall n \in \mathbb{Z}.$$

and for every chain map $f_\bullet : (X_\bullet, d_\bullet) \to (Y_\bullet, d'_\bullet)$ a chain map

$$T_p(f_\bullet) : (T_p(X_\bullet), T_p(d_\bullet)) \to (T_p(Y_\bullet), T_p(d'_\bullet)), \qquad T_p(f_\bullet)_n = f_{n+p}.$$

Show that this defines a functor $T_p : \mathrm{Ch}_{R\text{-Mod}} \to \mathrm{Ch}_{R\text{-Mod}}$ that satisfies

$$H_n(T_p(X_\bullet)) = H_{n+p}(X_\bullet).$$

This functor is called the **translation functor**.

Exercise 13: Let $n, m \in \mathbb{N}$. Prove that the tensor product of the abelian groups $\mathbb{Z}/m\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$ is given by

$$\mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/\mathrm{gcm}(m, n)\mathbb{Z},$$

where $\mathrm{gcm}(m, n)$ is the greatest comon divisor of n and m.

3 The fundamental groupoid and the fundamental group

3.1 Homotopies and homotopy equivalences

When considering topological spaces, one generally does not distinguish homeomorphic topological spaces - as their topologies are in bijection, there is no possibility to distinguish them by their topological properties. This corresponds to isomorphism classes of objects in the category Top. In this section, we will find an equivalence relation on the set of continuous maps $f : X \to Y$ that allows us to weaken this classification as described in Example 2.1.3, 5. by replacing the category of topological spaces with a certain quotient category.

Concretely, two continuous maps $f : X \to Y$ will be regarded as equivalent, if there is a one-parameter family of continuous maps from X to Y that relates them, possibly subject to the additional requirement that the value on certain subspaces remains constant. This leads to the notion of homotopies and homotopy equivalences.

Definition 3.1.1: Let X, Y be topological spaces, $A \subset X$ a subspace with inclusion $i_A : A \to X$ and $f, g : X \to Y$ continuous maps with $f \circ i_A = g \circ i_A$.

1. A **homotopy from f to g relative to A** is a continuous map $h : [0,1] \times X \to Y$, $(t, x) \mapsto h(t, x)$ with
$$h(0,x) = f(x), \quad h(1,x) = g(x), \quad h(t,a) = f(a) = g(a) \qquad \forall x \in X, a \in A, t \in [0,1].$$
2. If there is a homotopy relative to A from f to g, then f, g are called **homotopic relative to A**, and one writes $f \sim_A g$.
3. For $A = \emptyset$, one speaks of **homotopy** and **homotopic** instead of homotopy relative to $\emptyset$ and homotopic relative to $\emptyset$, and one writes $f \sim g$ instead of $f \sim_\emptyset g$.
4. Continuous maps $f : X \to Y$ that are homotopic to a constant map $g : X \to Y$ are called **null homotopic**.

Example 3.1.2:

1. Let $\exp : [0,1] \to S^1$, $x \mapsto e^{2\pi i x} = \cos(2\pi x) + i \sin(2\pi x)$ be the exponential map. Then $\exp : [0,1] \to S^1$ is homotopic to the constant map $c_1 : [0,1] \to S^1$, $t \mapsto 1$ with the homotopy
$$h : [0,1] \times [0,1] \to S^1, \quad h(t,x) = e^{2\pi i(1-t)x}.$$
We will see later that exp is not homotopic to c_1 relative to the subspace $\{0,1\} \subset [0,1]$.
2. Let X be a topological space and $A \subset X$ a subspace. If $Y \subset \mathbb{R}^n$ is convex, then any two maps $f, g : X \to Y$ with $f|_A = g|_A$ are homotopic relative to A with the homotopy
$$h : [0,1] \times X \to Y, \quad h(t,x) = t\, f(x) + (1-t)\, g(x).$$
3. If $X \subset \mathbb{R}^n$ is **star shaped** with respect to $p \in X$, i. e. for all $x \in X$ the line segment $[x,p] = \{t\,x + (1-t)p : t \in [0,1]\}$ connecting x and p is contained in X, then any continuous map $f : X \to Y$ is null homotopic with the homotopy
$$h : [0,1] \times X \to Y, \quad h(t,x) = f(t\,x + (1-t)\,p).$$

4. If $f, g : X \to Y$ are homotopic relative to $A \subset X$, then for any continuous map $k : Y \to Z$, the maps $k \circ f, k \circ g : X \to Z$ are homotopic relative to A. If $h : [0,1] \times X \to Y$ is a homotopy from f to g relative to A, then $k \circ h : [0,1] \times X \to Z$ is a homotopy from $k \circ f$ to $k \circ g$ relative to A.
5. If $f, g : Y \to Z$ are homotopic relative to $B \subset Y$ and $d : X \to Y$ is a continuous map with $d(A) \subset B$, then the maps $f \circ d, g \circ d : X \to Z$ are homotopic relative to A. If $h : [0,1] \times Y \to Z$ is a homotopy from f to g relative to B, then $h \circ d : [0,1] \times X \to Z$ is a homotopy from $f \circ d$ to $g \circ d$ relative to A.
6. If $h_1 : [0,1] \times X_1 \to Y_1$ is a homotopy from f_1 to g_1 relative to $A_1 \subset X_1$ and $h_2 : [0,1] \times X_2 \to Y_2$ is a homotopy from f_2 to g_2 relative to $A_2 \subset X_2$, then $h : [0,1] \times X_1 \times X_2 \to Y_1 \times Y_2$, $(t, x_1, x_2) \mapsto (h_1(t, x_1), h_2(t, x_2))$ is a homotopy from $f_1 \times f_2$ to $g_1 \times g_2$ relative to $A_1 \times A_2$.

Before considering specific examples of homotopies, we investigate their general properties. The first observation is that homotopies define an equivalence relation on the set of continuous maps $f : X \to Y$ and the second is that this equivalence relation is compatible with the compositions of maps in the sense of Example 2.1.3, 5.

Proposition 3.1.3: Let X, Y be topological spaces and $A \subset X$ a subspace. Then $\sim_A$ is an equivalence relation on the set $C(X, Y)$ of continuous maps $f : X \to Y$.

Proof:
• **Symmetry:** Every map $f : X \to Y$ is homotopic to itself relative to A with the trivial homotopy $h : [0,1] \times X \to Y$, $h(t, x) = f(x)$ for all $t \in [0,1]$, $x \in X$.

• **Reflexivity:** If $h : [0,1] \times X \to Y$ is a homotopy from $f : X \to Y$ to $g : X \to Y$ relative to A, then $\overline{h} : [0,1] \times X \to Y$, $\overline{h}(t, x) = h(1-t, x)$ is a homotopy from g to f relative to A. It is continuous as a composite of continuous functions and satisfies for all $t \in [0,1]$, $x \in X$, $a \in A$

$$\overline{h}(0, x) = h(1, x) = g(x), \quad \overline{h}(1, x) = h(0, 1) = f(x), \quad \overline{h}(t, a) = h(1-t, a) = f(a) = g(a)$$

• **Transitivity:** If $h, h' : [0,1] \times X \to Y$ are homotopies relative to A from $f : X \to Y$ to $g : X \to Y$ and from $g : X \to Y$ to $k : X \to Y$, respectively, then the map

$$h'' : [0,1] \times X \to Y, \quad h''(t, x) = \begin{cases} h(2t, x) & t \in [0, \frac{1}{2}] \\ h'(2t-1, x) & t \in (\frac{1}{2}, 1]. \end{cases}$$

is a homotopy relative to A from f to k. It is continuous since h, h' are continuous and $h(1, x) = g(x) = h'(0, x)$. Moreover, it satisfies for all $t \in [0,1]$, $a \in A$, $x \in X$

$$h''(0, x) = h(0, x) = f(x), \quad h''(1, x) = h'(1, x) = k(x), \quad h''(t, a) = f(a) = k(a).$$

This shows that h' is a homotopy from f to k relative to A. □

Proposition 3.1.4: Let X, Y, Z be topological spaces and $A \subset X$, $B \subset Y$ subspaces. If $f, f' : X \to Y$ are continuous maps with $f(A), f'(A) \subset B$ that are homotopic relative to A and $g, g' : Y \to Z$ continuous maps that are homotopic relative to B, then $g \circ f, g' \circ f' : X \to Z$ are homotopic relative to A.

Proof:
Let $h_f : [0,1] \times X \to Y$ be a homotopy from f to f' relative to A and $h_g : [0,1] \times Y \to Z$ a homotopy from g to g' relative to B. Then the map

$$h : [0,1] \times X \to Z, \quad h(t,x) = h_g(t, h_f(t,x))$$

is continuous as a composite of continuous functions and satisfies for all $t \in [0,1]$, $x \in X$, $a \in A$:

$$\begin{aligned}
&h(0,x) = h_g(0, h_f(0,x)) = g(h_f(0,x)) = g \circ f(x) \\
&h(1,x) = h_g(1, h_f(1,x)) = g'(h_f(1,x)) = g' \circ f'(x) \\
&h(t,a) = h_g(t, h_f(t,a)) = h_g(f(a)) = h_g(f'(a)) = g \circ f(a) = g' \circ f'(a).
\end{aligned}$$

This shows that $h : [0,1] \times X \to Z$ is a homotopy from $g \circ f$ to $g' \circ f'$ relative to A. □

By combining Propositions 3.1.3 and 3.1.4, we find that *homotopic* defines an equivalence relation on each set $\mathrm{Hom}(X,Y)$ in the category Top that is compatible with the composition of morphisms in the sense of Example 2.1.3, 5. This allows us to construct a quotient category hTop whose objects are topological spaces and whose morphisms between objects X and Y are homotopy equivalence classes of morphisms $f : X \to Y$. This category is called the **homotopy category**. The composition in hTop is given by the composition of continuous maps, $[g] \circ [f] = [g \circ f]$ for all continuous maps $f : X \to Y$ and $g : Y \to Z$. The identity morphisms are the homotopy equivalence classes of the identity maps, $1_X = [\mathrm{id}_X]$ for all topological spaces X. Isomorphisms in hTop are homotopy equivalence classes of continuous maps $f : X \to Y$ for which there exists a continuous map $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are homotopic to, respectively id_X and id_Y. Such maps are called homotopy equivalences.

Similarly, we can consider the category $Top(2)$ of pairs of topological spaces, where objects are pairs (X,A) of a topological space X and a subspace $A \subset X$ and morphisms from (X,A) to (Y,B) continuous maps $f : X \to Y$ with $f(A) \subset B$. Then Propositions 3.1.3 and 3.1.4 imply that *homotopic relative to* A defines an equivalence on each set $\mathrm{Hom}((X,A),(Y,B))$ that is compatible with the composition of morphisms. We obtain an associated quotient category, the category hTop(2) with the same objects as Top(2) and homotopy classes relative to A of morphisms $f : (X,A) \to (Y,B)$ as morphisms. Isomorphisms in hTop(2) are equivalence classes of homotopy equivalences relative to A, i. e. of continuous maps $f : X \to Y$ with $f(A) \subset B$ such that there is a $g : Y \to X$ with $g(B) \subset A$ and $g \circ f \sim_A \mathrm{id}_X$, $f \circ g \sim_B \mathrm{id}_Y$.

Definition 3.1.5: Let X, Y be topological spaces.

1. A continuous map $f : X \to Y$ is called a **homotopy equivalence** if there is a continuous map $g : X \to Y$ with $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$.
2. If there is a homotopy equivalence $f : X \to Y$, then X and Y are called **homotopy equivalent** or **of the same homotopy type**, and we write $X \simeq Y$.
3. A topological space X is called **contractible** if it is homotopy equivalent to a point.

Remark 3.1.6:

1. Homotopy equivalence is an equivalence relation.

2. Any homeomorphism $f : X \to Y$ is a homotopy equivalence between X and Y but the converse is not true. This implies that classification of topological spaces up to homotopy type is a weaker classification than their classification up to homeomorphism.

Example 3.1.7: Any convex subspace $X \subset \mathbb{R}^n$ is contractible, since for any $p \in X$ map $f : X \to \{p\}$, $x \mapsto p$ is a homotopy equivalence. With $g : \{p\} \to X$, $p \mapsto p$, one has $f \circ g = \mathrm{id}_{\{p\}}$ and $h : [0,1] \times X \to X$, $h(t,x) = tx + (1-t)p$ is a homotopy from $g \circ f : X \to X$, $x \mapsto p$ to id_X by Example 3.1.2, 2. This implies in particular $\mathbb{R}^n \simeq \mathbb{R}^m$ for all $n, m \in \mathbb{N}$. However, we will see later that $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$ for $n \neq m$.

In order to consider more interesting topological spaces and to show that they are homotopy equivalent, we need simple examples of homotopy equivalences that are easily visualised. Such examples are given by deformation retracts, which can be imagined as subspaces of a topological spaces that are obtained by melting or compressing it to the subspace.

Definition 3.1.8: Let X be a topological space and $A \subset X$ a subspace with inclusion map $i_A : A \to X$. Then A is called

1. a **retract** of X if there exists a **retraction** $r : X \to A$, a continuous map $r : X \to A$ with $r \circ i_A = \mathrm{id}_A$,
2. a **deformation retract** of X, if there is a retraction $r : X \to A$ for which $i_A \circ r \simeq_A \mathrm{id}_X$.

Remark 3.1.9:

1. As a retraction $r : X \to A$ satisfies $r \circ i_A \circ r = r$, retractions can be viewed as the topological counterparts of projection operators.
2. A retraction $r : X \to A$ with $i_A \circ r \simeq \mathrm{id}_X$ is a homotopy equivalence, which implies in particular that all deformation retractions are homotopy equivalences.

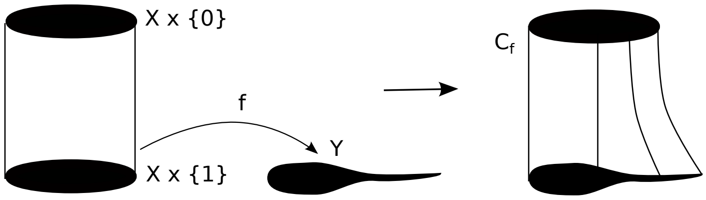
Example 3.1.10:

1. For any topological space X and any point $p \in X$, the set $\{p\} \subset X$ is a retract of X with retraction $r : X \to \{p\}$, $x \mapsto p$. The subspace $\{p\} \subset X$ is homotopy equivalent to X if and only if X is contractible.

2. The n-sphere S^n is a deformation retract of $(\mathbb{R}^{n+1})^\times$. The map $r : (\mathbb{R}^{n+1})^\times \to S^n$, $x \mapsto x/||x||$ is a retraction since $r \circ i_{S^n} = \mathrm{id}_{S^n}$, and
$$h : [0,1] \times (\mathbb{R}^{n+1})^\times \to (\mathbb{R}^{n+1})^\times, \quad h(t,x) = tx + (1-t)x/||x||$$
is a homotopy relative to S^n from $i_{S^n} \circ r : (\mathbb{R}^{n+1})^\times \to (\mathbb{R}^{n+1})^\times$ to $\mathrm{id}_{(\mathbb{R}^{n+1})^\times}$. This shows that S^n and $(\mathbb{R}^{n+1})^\times$ are of the same homotopy type but they cannot be homeomorphic since S^1 is compact and $(\mathbb{R}^{n+1})^\times$ is not.

3. $S^1 \times \{0\}$ is a deformation retract of the cylinder $S^1 \times \mathbb{R}$. The map $r : S^1 \times \mathbb{R} \to S^1 \times \{0\}$, $(x,y,z) \mapsto (x,y,0)$ is a retraction since $r \circ i_{S^1} = \mathrm{id}_{S^1 \times \{0\}} : S^1 \times \{0\} \to S^1 \times \{0\}$, and
$$h : [0,1] \times S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}, \quad h(t,x,y,z) = (x,y,tz)$$
is a homotopy from $i_{S^1 \times \mathbb{R}} \circ r : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ to $\mathrm{id}_{S^1 \times \mathbb{R}}$ relative to $S^1 \times \{0\}$.

The last example is a special case of a more general construction, namely mapping cylinders of continuous maps $f : X \to Y$. They are obtained by attaching the "cylinder" $[0,1] \times X$ to the topological space Y with the associated map $\tilde{f} : X \times \{1\} \to Y$, $(x,1) \mapsto f(x)$ and can be easily visualised. As the topological space Y is a deformation retract of the mapping cylinder, the mapping cylinder is homotopy equivalent to Y.



Example 3.1.11: The **mapping cylinder** of a continuous map $f : X \to Y$ is the topological space $C_f = (X \times [0,1]) \cup_{\tilde{f}} Y$ obtained by attaching $X \times [0,1]$ to Y with the attaching map $\tilde{f} : X \times \{1\} \to Y$, $(x,1) \mapsto f(x)$. Y is a deformation retract of C_f.

To see this, consider the continuous maps $r_1 : X \times [0,1] \to Y$, $(x,t) \mapsto f(x)$ and $r_2 = \mathrm{id}_Y : Y \to Y$. As $r_1(x,1) = r_2 \circ \tilde{f}(x,1) = r_2 \circ f(x)$ for all $x \in X$, the universal property of attaching implies that there is a unique map $r : C_f \to Y$ with $r \circ \pi \circ i_{X \times [0,1]} = r_1$ and $r \circ \pi \circ i_Y = r_2 = \mathrm{id}_Y$. Hence $r : C_f \to Y$ is a retraction. Consider now the maps

$$h_1 : [0,1] \times X \times [0,1] \to X \times [0,1], \; (s,x,t) \mapsto (x, (1-s)t + s),$$
$$h_2 : [0,1] \times Y \to Y, \; (s,y) \mapsto y.$$

By composing the sum map $h_1 + h_2 : [0,1] \times ((X \times [0,1]) + Y) \to (X \times [0,1]) + Y$ with the canonical surjection $\pi : (X \times [0,1]) + Y \to C_f$, one obtains a continuous map $\pi \circ (h_1 + h_2) : [0,1] \times ((X \times [0,1]) + Y) \to C_f$, which satisfies

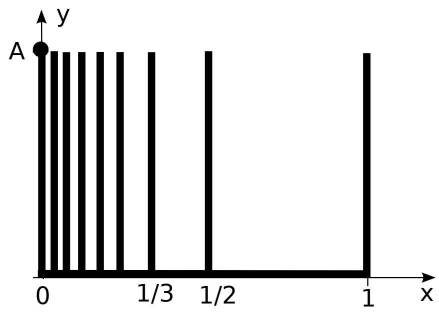
$$\pi \circ (h_1 + h_2)(s,x,1) = [(x,1)] = [f(x)] = \pi \circ (h_1 + h_2)(s, f(x)) \quad \forall x \in X, s \in [0,1]$$

and hence induces a unique continuous map $h : [0,1] \times C_f \to C_f$ with $h \circ (\mathrm{id}_{[0,1]} \times \pi) = \pi \circ (h_1 + h_2)$. We have for all $x \in X$, $s \in [0,1]$

$$h(0, [(x,t)]) = \pi \circ h_1(0,x,t) = [(x,t)],$$
$$h(1, [(x,t)]) = \pi \circ h_1(1,x,t) = [(x,1)] = [f(x)],$$
$$h(s, [(x,1)]) = \pi \circ h_1(s,x,1) = [(x,1)] = [f(x)].$$

This shows that $h : [0,1] \times C_f \to C_f$ is a homotopy from $\mathrm{id}_{C_f} : C_f \to C_f$ to $\pi \circ i_Y \circ r : C_f \to C_f$ relative to $\pi \circ i_Y(Y) \subset C_f$. As $\pi \circ i_Y : Y \to C_f$ is an embedding, this shows that Y and C_f are homotopy equivalent.

The following example shows that a subspace $A \subset X$ that is a retract of X with a retraction $r : X \to A$ that is a homotopy equivalence does not have to be a deformation retract of X.



Example 3.1.12: The **comb space** is the topological space

$$C = (\{0\} \times [0,1]) \cup (\cup_{n\in\mathbb{N}}\{1/n\} \times [0,1]) \cup ([0,1] \times \{0\}) \subset \mathbb{R}^2$$

The subspace $A = \{(0,1)\} \subset C$ is a retract of C with a retraction $r : C \to A$ that is a homotopy equivalence, but A is not a deformation retract of C.

To show that A is a retract of C, it is sufficient to construct a retraction $r : C \to \{0\} \times [0,1]$, since $\{0\} \times [0,1] \approx [0,1]$ is convex and therefore contractible to any point $p \in [0,1]$ by Example 3.1.7. Such a retraction is given by $r : C \to \{0\} \times [0,1]$, $(x,y) \mapsto (x,0)$. The continuous map $h : [0,1] \times C \to C$, $(t,x,y) \mapsto (x,(1-t)y)$ is a homotopy from $\mathrm{id}_C : C \to C$ to $i_{[0,1]\times\{(1,0)\}} \circ r : C \to C$, and therefore C is homotopy equivalent to $\{0\} \times [0,1]$ and to A.

Suppose now that $h : [0,1] \times C \to C$ is a homotopy from $\mathrm{id}_C : C \to C$ to $i_A \circ r : C \to C$ relative to $A = \{(0,1)\}$. Then h satisfies $h(0,x,y) = (x,y)$ and $h(1,x,y) = h(t,0,1) = (0,1)$ for all $t \in [0,1]$, $(x,y) \in C$. As $[0,1]$ is compact and $C \subset \mathbb{R}^2$ is compact as a closed and bounded subset of $\mathbb{R}^2$, it follows that $h : [0,1] \times C \to C$ is uniformly continuous and hence there is a $\delta > 0$ with $||h(t,x,y) - h(t',x',y')|| < 1/2$ for all $(t,x,y), (t',x',y') \in [0,1] \times C$ with $|t-t'|, |x-x'|, |y-y'| < \delta$. This implies $||h(t,1/n,1) - h(t,0,1)|| < 1/2$ for all $n > 1/\delta$ and all $t \in [0,1]$, which is a contradiction to $h(0,1/n,1) = (1/n,1)$ and the continuity of h. Hence $A = \{(0,1)\}$ is not a deformation retract of C.

3.2 Fundamental group(oid)s

In the following, we will use continuous maps and homotopies between them to characterise and distinguish topological spaces. The simplest possible continuous map into a topological space X is a map $\{p\} \to X$ from the point space into X. However, such maps do not contain sufficient information - they only allow one to determine the number of components of X. This leads us to the consideration of continuous maps $\gamma : [0,1] \to X$ from the unit interval into X, i. e. paths in X. As we are not interested in the exact shape of the paths nor in their parametrisation, we will consider them up to continuous deformations which keep their endpoints fixed, i. e. homotopies $h : [0,1] \times [0,1] \to X$ relative to $\{0,1\}$.

The basic idea is to use such paths to to probe the topological space. We will see later that "holes" in a topological space X present an obstruction to homotopies between certain paths in X. Although there are higher-dimensional analogues, the advantage of working with low-dimensional objects such as paths is that they are much simpler to compute with than their higher-dimensional analogues. We start by considering the basic operations on paths.

Definition 3.2.1: Let X be a topological space.

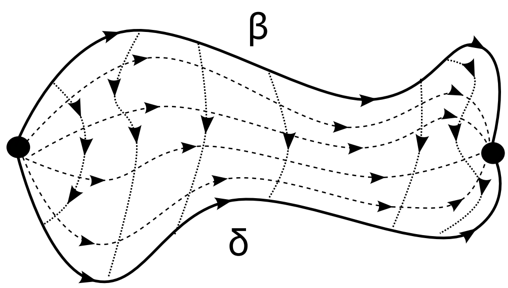
1. A **path** in X is a continuous map $\gamma : [0,1] \to X$. It is called **closed path** or **loop** if $\gamma(0) = \gamma(1)$ and **open path** otherwise.
2. If $\gamma, \delta : [0,1] \to X$ are paths with $\delta(0) = \gamma(1)$, then their **composite** is the path

$$\delta \star \gamma : [0,1] \to X, \quad \delta \star \gamma(t) = \begin{cases} \gamma(2t) & t \in [0, \frac{1}{2}) \\ \delta(2t-1) & t \in [\frac{1}{2}, 1]. \end{cases}$$

   The **reversed path** for $\gamma : [0,1] \to X$ is the path $\overline{\gamma} : [0,1] \to X$, $\overline{\gamma}(t) = \gamma(1-t)$, and the **trivial path** based at $p \in X$ is the path $\gamma_p : [0,1] \to X$, $\gamma_p(t) = p$ for all $t \in [0,1]$.
3. A topological space X is called **path connected** if for any $p, q \in X$, there is a path $\gamma : [0,1] \to X$ with $\gamma(0) = p$ and $\gamma(1) = q$.

The existence of a composition law, of reversed paths and of trivial paths suggests a group or groupoid structure associated with paths in a topological space X. However, we will see in the following that the composition of paths is not associative, composing a path with a trivial path changes the path and composing a path and a reversed path does not yield a trivial path - all of these hold only up to parametrisation. The other disadvantage of the paths is that they contain too much irrelevant information. We are interested in the paths only up to parametrisation and continuous deformations that keep the endpoints fixed. For this reason, we consider the corresponding homotopy equivalence classes.

Definition 3.2.2: Let $\gamma, \gamma' : [0,1] \to X$ be paths with $\gamma(0) = \gamma'(0)$ and $\gamma(1) = \gamma'(1)$. Then a **homotopy** (with fixed endpoints) from γ to γ' is a homotopy from γ to γ' relative to $\{0,1\} \subset [0,1]$. If there exists a homotopy between paths γ, γ' then γ, γ' are called **homotopic**, $\gamma \sim \gamma'$. Paths that are homotopic to a trivial path are called **null homotopic**.



Homotopy with fixed endpoints from $\beta : [0,1] \to X$ to $\delta : [0,1] \to X$. The dashed lines correspond to the paths $h_s : [0,1] \to X$, $h_s(t) = h(t,s)$ and the dotted lines to the paths $h_t : [0,1] \to X$, $h_t(s) = h(t,s)$.

Remark 3.2.3:

1. For all $p, q \in X$ "homotopic" is an equivalence relation on the set $P(p,q,X)$ of paths $\gamma : [0,1] \to X$ with starting point $\gamma(0) = p$ and end point $\gamma(1) = q$.

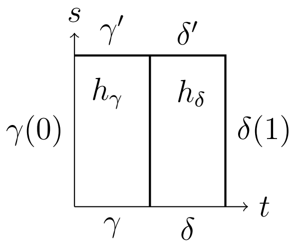
2. Reparametrisations are homotopies. For any path $\gamma : [0,1] \to X$ and any continuous map $f : [0,1] \to [0,1]$ with $f(0) = 0$ and $f(1) = 1$ the map $h : [0,1] \times [0,1] \to X$, $(s,t) \mapsto \gamma(s\, f(t) + (1-s)\, t)$ is a homotopy from γ to $\gamma \circ f$ relative to $\{0,1\}$.

If we replace paths in the topological space X by the associated homotopy classes (with fixed endpoints) we obtain the structure of a groupoid with points of X as objects and homotopy classes of paths as morphisms. In particular, for each point $x \in X$, the homotopy classes of loops based at X form a group.

Proposition 3.2.4: Let X be a topological space. By taking points in X as objects, homotopy classes of paths in X as morphisms and the composition of morphisms induced by the composition of paths, one obtains a groupoid $\Pi_1(X)$, the **fundamental groupoid** of X. In particular, for any $x \in X$, the homotopy classes of loops based at x form a group, the **fundamental group** $\pi_1(x, X) = \mathrm{Hom}_{\Pi_1(X)}(x, x)$.

Proof:
1. **Composition of morphisms**: Let $\gamma, \gamma', \delta, \delta' : [0,1] \to X$ be paths with $\delta(0) = \delta'(0) = \gamma(1) = \gamma'(1)$ and $h_\gamma, h_\delta : [0,1] \times [0,1] \to X$ homotopies from γ to γ' and δ to δ'. Define $h : [0,1] \times [0,1] \to X$ by

$$h(s,t) = \begin{cases} h_\gamma(s, 2t) & t \in [0, \frac{1}{2}) \\ h_\delta(s, 2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$ or, graphically,

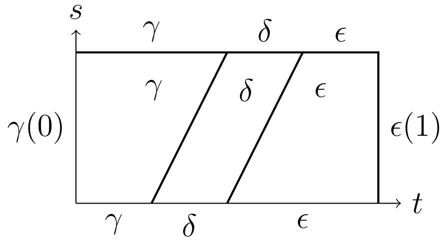


Then h is a homotopy from $\delta \star \gamma$ to $\delta' \star \gamma'$ relative to $\{0,1\}$, and hence $\delta \star \gamma \sim_{\{0,1\}} \delta' \star \gamma'$. This shows that the composition of morphisms, $[\delta] \circ [\gamma] = [\delta \star \gamma]$, is well-defined.

2. **Associativity**: To show that the composition of morphisms is associative, consider paths $\epsilon, \delta, \gamma : [0,1] \to X$ with $\epsilon(0) = \delta(1)$ and $\delta(0) = \gamma(1)$. Then one has

$$\epsilon \star (\delta \star \gamma)(t) = \begin{cases} \gamma(4t) & t \in [0, \frac{1}{4}) \\ \delta(4t-1) & t \in [\frac{1}{4}, \frac{1}{2}) \\ \epsilon(2t-1) & t \in [\frac{1}{2}, 1] \end{cases} \qquad (\epsilon \star \delta) \star \gamma(t) = \begin{cases} \gamma(2t) & t \in [0, \frac{1}{2}) \\ \delta(4t-2) & t \in [\frac{1}{2}, \frac{3}{4}) \\ \epsilon(4t-3) & t \in [\frac{3}{4}, 1], \end{cases} \tag{21}$$

and a homotopy $h : [0,1] \times [0,1] \to X$ from $\epsilon \star (\delta \star \gamma)$ to $(\epsilon \star \delta) \star \gamma$ relative to $\{0,1\}$ is given by

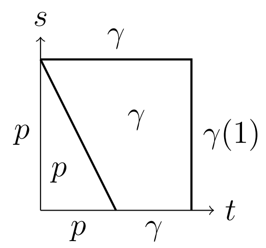


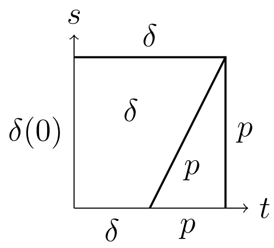
This shows that $\epsilon \star (\delta \star \gamma) \sim_{\{0,1\}} (\epsilon \star \delta) \star \gamma$ and hence the composition of morphisms is associative.

3. **Identity morphisms**: The identity morphism 1_p for $p \in X$ is given by the equivalence class $[\gamma_p]$ of the trivial path based at p. For paths $\gamma, \delta : [0,1] \to X$ with $\gamma(0) = \delta(1) = p$, we have

$$\gamma \star \gamma_p(t) = \begin{cases} \gamma(0) = p & t \in [0, \frac{1}{2}) \\ \gamma(2t-1) & t \in [\frac{1}{2}, 1] \end{cases} \qquad \gamma_p \star \delta(t) = \begin{cases} \delta(2t) & t \in [0, \frac{1}{2}) \\ \delta(1) = p & t \in [\frac{1}{2}, 1] \end{cases}, \tag{22}$$

and homotopies from $\gamma \star \gamma_p$ to γ and $\gamma_p \star \delta$ to δ relative to $\{0, 1\}$ are given by



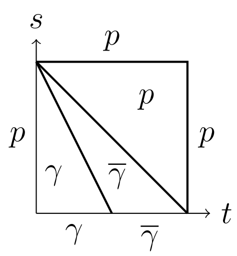


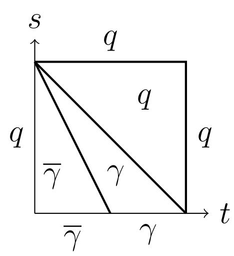
This proves $\gamma \star \gamma_p \sim_{\{0,1\}} \gamma$ and $\gamma_p \star \delta \sim_{\{0,1\}} \delta$ and shows that $1_p = [\gamma_p]$ has the properties of an identity morphism. Hence, $\Pi_1(X)$ is a category.

4. **Inverses:** To show that every morphism in $\Pi_1(X)$ has an inverse, consider a path $\gamma : [0,1] \to X$ with $\gamma(0) = p$, $\gamma(1) = q$ and the reversed path $\overline{\gamma} : [0,1] \to X$. Then one has

$$\overline{\gamma} \star \gamma(t) = \begin{cases} \gamma(2t) & t \in [0, \frac{1}{2}) \\ \overline{\gamma}(2t-1) & t \in [\frac{1}{2}, 1] \end{cases} \qquad \gamma \star \overline{\gamma}(t) = \begin{cases} \overline{\gamma}(2t) & t \in [0, \frac{1}{2}) \\ \gamma(2t-1) & t \in [\frac{1}{2}, 1], \end{cases} \tag{23}$$

and the following are homotopies from $\gamma \star \overline{\gamma}$ to γ_q and from $\overline{\gamma} \star \gamma$ to γ_q





This proves $\overline{\gamma} \star \gamma \sim_{\{0,1\}} \gamma_p$ and $\gamma \star \overline{\gamma} \sim_{\{0,1\}} \gamma_q$ and, consequently, $[\overline{\gamma}] \circ [\gamma] = 1_p$, $[\gamma] \circ [\overline{\gamma}] = 1_q$. This shows that $\Pi_1(X)$ is a groupoid and therefore for any object x, the set $\pi_1(x, X) = \mathrm{Hom}_{\Pi_1(X)}(x, x)$ with the composition of morphisms is a group. $\square$

The advantage of the fundamental groupoid compared to the fundamental group is that the former is more canonical. It does not involve the choice of a basepoint and hence cannot depend on such such a choice, while the fundamental groups associated two different basepoints may differ. However, it turns out that the dependence of the fundamental group on the basepoint is mild, when the underlying space is path connected. In this case, the fundamental groups at different basepoints can be related by paths connecting the basepoints and one finds that they are isomorphic.

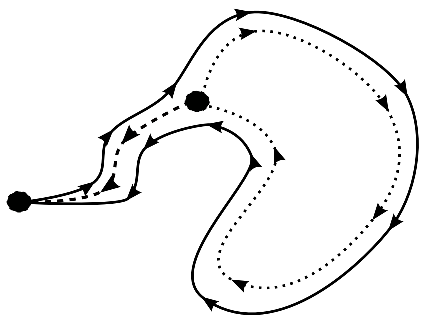
Proposition 3.2.5: Let X be a topological space. Then for any path $\gamma : [0,1] \to X$, the map

$$\Phi_{[\gamma]} : \pi_1(\gamma(0), X) \to \pi_1(\gamma(1), X), \quad [\beta] \mapsto [\gamma] \circ [\beta] \circ [\overline{\gamma}]$$

is a group isomorphism with the following properties:

1. It depends only on the homotopy class of γ relative to $\{0,1\}$.
2. $\Phi_{1_x} = \mathrm{id}_{\pi_1(x,X)}$
3. $\Phi_{[\delta]\circ[\gamma]} = \Phi_{[\delta]} \circ \Phi_{[\gamma]}$ for all paths γ, δ with $\delta(0) = \gamma(1)$.

If X is path connected, this implies $\pi_1(x, X) \cong \pi_1(x', X)$ for all $x, x' \in X$. One speaks of the fundamental group of X and writes $\pi_1(X)$ instead of $\pi_1(x, X)$.



Group isomorphism $\Phi_{[\gamma]} : \pi_1(\gamma(0), X) \to \pi_1(\gamma(1), X)$, $[\beta] \mapsto [\gamma] \circ [\beta] \circ [\overline{\gamma}]$. The dashed path denotes γ, the dotted path β and the solid path $\gamma \star \beta \star \overline{\gamma}$.

Proof:
That $\Phi_{[\gamma]}$ depends only on the homotopy class of γ relative to $\{0,1\}$ follows directly from its definition. The other statements follow from the the fact that $\Pi_1(X)$ is a groupoid and $\pi_1(x, X) = \mathrm{Hom}_{\Pi_1(X)}(x, x)$. For any groupoid $\mathcal{C}$ and any morphism $g : X \to Y$ in $\mathcal{C}$, the map $\Phi_g : \mathrm{Hom}_{\mathcal{C}}(X, X) \to \mathrm{Hom}_{\mathcal{C}}(Y, Y)$, $f \mapsto g \circ f \circ g^{-1}$ is a group homomorphism since

$$\Phi_g(f \circ f') = g \circ f \circ f' \circ g^{-1} = g \circ f \circ g^{-1} \circ g \circ f' \circ g^{-1} = \Phi_g(f) \circ \Phi_g(f')$$
$$\Phi_g(1_X) = g \circ 1_X \circ g^{-1} = 1_Y.$$

Moreover, for all morphisms $f : X \to X$, $h : Y \to Z$, one has

$$\Phi_{1_X}(f) = 1_X \circ f \circ 1_X^{-1} = f, \quad \Phi_{h\circ g}(f) = h \circ g \circ f \circ g^{-1} \circ h^{-1} = \Phi_h(g \circ f \circ g^{-1}) = \Phi_h \circ \Phi_g(f).$$

Hence, the group homomorphism $\Phi_{g^{-1}} : \mathrm{Hom}_{\mathcal{C}}(Y, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, X)$ is the inverse of the group homomorphism $\Phi_g : \mathrm{Hom}_{\mathcal{C}}(X, X) \to \mathrm{Hom}_{\mathcal{C}}(Y, Y)$ and both group homomorphisms are isomorphisms. □

A particularly simple groupoid is a groupoid which contains exactly one morphism $f : x \to x'$ for any objects x, x'. A topological space X whose fundamental groupoid is of this form is called 1-connected. It is clear that this condition is equivalent to the requirement that X is path connected and that the fundamental group $\pi_1(x, X)$ is trivial for all $x \in X$. The former holds if and only if for any two objects x, x' in $\Pi_1(X)$ there is at least one morphism $f : x \to x'$. As any pair of morphisms $f : x \to x'$ and $g : x' \to x$ yields a morphisms $g \circ f \in \pi_1(x, X)$, the fundamental group $\pi_1(x, X)$ is trivial for all $x \in X$ if and only if $\mathrm{Hom}_{\Pi_1(X)}(x, x')$ contains at most one morphism for all $x, x' \in X$.

Definition 3.2.6: A topological space X is called **simply connected** if all of its fundamental groups $\pi_1(x, X)$ are trivial and **1-connected** if for any two objects x, x' in the fundamental groupoid $\Pi_1(x)$, $\mathrm{Hom}_{\Pi_1(X)}(x, x')$ contains exactly one morphism. This is equivalent to the condition that X is path-connected and $\pi_1(x, X)$ is trivial for all $x \in X$.

Example 3.2.7: If $X \subset \mathbb{R}^n$ is star shaped, then X is 1-connected.

As there is a point $p \in X$ with $[p, x] \subset X$ for all $x \in X$, one has a path $P_x : [0,1] \to X$, $P_x(t) = (1-t)\,p + t\,x$ from p to x and, consequently, a path $P_y \star \overline{P_x} : [0,1] \to X$ from x to y for all $x, y \in X$. If $\gamma : [0,1] \to X$ is a loop based at p, then $h : [0,1] \times [0,1] \to X$, $h(s,t) = (1-s)\,p + s\gamma(t)$ is a homotopy from the trivial path $\gamma_p : [0,1] \to X$ to γ relative to $\{0,1\}$, which shows that $\pi_1(x, X) \cong \pi_1(p, X) = \{e\}$ for all $x \in X$.

To determine the fundamental group(oid)s of more complicated topological spaces, we first need to consider continuous maps between such spaces and to clarify how they interact with the associated fundamental group(oid)s. The basic observation is that for any path $\gamma : [0,1] \to X$ and any continuous map $f : X \to Y$, we obtain a path $f \circ \gamma : [0,1] \to Y$ in Y.

This assignment is compatible with homotopies between paths and allows one to define a map $[\gamma] \to [f \circ \gamma]$ that associates homotopy equivalence classes of paths in Y to homotopy equivalence classes of paths in X. Moreover, one finds that this assignment is compatible with the composition of continuous maps and with homotopies between them. We obtain the following theorem.

Theorem 3.2.8: Let X,Y,Z be topological spaces. Then:

1. The assignments $x \mapsto f(x)$, $[\gamma] \mapsto [f \circ \gamma]$ associate a functor $\Pi_1(f) : \Pi_1(X) \to \Pi_1(Y)$ to each continuous map $f : X \to Y$.
2. These functors satisfy: $\Pi_1(k \circ f) = \Pi_1(k)\Pi_1(f)$ for all continuous maps $f : X \to Y$, $k : Y \to Z$, and $\Pi_1(\mathrm{id}_X) = \mathrm{id}_{\Pi_1(X)}$.
3. A homotopy $h : [0,1] \times X \to Y$ from $f : X \to Y$ to $g : X \to Y$ induces a natural isomorphism $\Pi_1(h) : \Pi_1(f) \xrightarrow{\sim} \Pi_1(g)$.

Proof:
1. $\Pi_1(f)$ assigns an object $f(x)$ in the groupoid $\Pi_1(Y)$ to every object x of $\Pi_1(X)$ and a morphism $[f \circ \gamma]$ from $f(x)$ to $f(y)$ to every morphism $[\gamma]$ from x to y in $\Pi_1(X)$. To show that this assignment is compatible with the composition of morphisms and identity morphisms, consider paths $\gamma, \delta : [0,1] \to X$ with $\gamma(1) = \delta(0)$ and a trivial path $\gamma_x : [0,1] \to X$. Then the identities $f \circ (\delta \star \gamma) = (f \circ \delta) \star (f \circ \gamma)$ and $f \circ \gamma_x = \gamma_{f(x)}$ imply

$$\Pi_1(f)([\delta] \circ [\gamma]) = [f \circ (\delta \star \gamma)] = [(f \star \delta) \star (f \circ \gamma)] = [f \circ \delta] \circ [f \circ \gamma] = \Pi_1(f)([\delta]) \circ \Pi_1(f)([\gamma]),$$
$$\Pi_1(f)(1_x) = \Pi_1(f)([\gamma_x]) = [f \circ \gamma_x] = [\gamma_{f(x)}] = 1_{f(x)} = 1_{\Pi_1(x)}.$$

This proves that $\Pi_1(f) : \Pi_1(X) \to \Pi_1(Y)$ is a functor.

2. To prove the second claim, note that $\Pi_1(k \circ f)(x) = k \circ f(x) = \Pi_1(k)(\Pi_1(f)(x))$ and $\Pi_1(\mathrm{id}_X)(x) = x$ for all $x \in X$. Moreover, for all paths $\gamma : [0,1] \to X$, one has

$$\Pi_1(k \circ f)([\gamma]) = [k \circ f \circ \gamma] = \Pi_1(k)([f \circ \gamma]) = \Pi_1(k)(\Pi_1(f)([\gamma]))$$
$$\Pi_1(\mathrm{id}_X)([\gamma]) = [\mathrm{id}_X \circ \gamma] = [\gamma].$$

3. Define for all $x \in X$ a path $h_x : [0,1] \to Y$, $h_x(t) = h(t,x)$ in Y from $f(x) = h(0,x)$ to $g(x) = h(1,x)$ and assign to the object x in $\Pi_1(X)$ the morphism $[h_x] : f(x) \to g(x)$ in $\Pi_1(Y)$. To prove that this defines a natural transformation $\Pi_1(h) : \Pi_1(f) \to \Pi_1(g)$, we have to show that the diagram

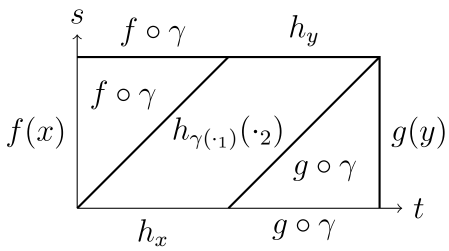
$$\begin{tikzcd} f(x) \arrow[r, "{[h_x]}"] \arrow[d, "\Pi_1(f)([\gamma])"'] & g(x) \arrow[d, "\Pi_1(g)([\gamma])"] \\ f(y) \arrow[r, "{[h_y]}"'] & g(y) \end{tikzcd} \tag{24}$$

commutes for all paths $\gamma : [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. As this is equivalent to

$$[(g \circ \gamma) \star h_x] = \Pi_1(g)([\gamma]) \circ [h_x] = [h_y] \circ \Pi_1(f)([\gamma]) = [h_y \star (f \circ \gamma)],$$

it is sufficient to construct a homotopy relative to $\{0,1\}$ from $(g \circ \gamma) \star h_x$ to $h_y \star (f \circ \gamma)$. Such a homotopy is given by

$$h(s,t) = \begin{cases} f \circ \gamma(2t) & t \in [0, \frac{s}{2}) \\ h(2t - s, \gamma(s)) & t \in [\frac{s}{2}, \frac{s}{2} + \frac{1}{2}) \\ g \circ \gamma(2t-1) & t \in [\frac{s}{2} + \frac{1}{2}, 1] \end{cases}$$



This shows that $\Pi_1(h) : \Pi_1(f) \to \Pi_1(g)$ is a natural transformation. That it is a natural isomorphism follows directly from the fact that $\Pi_1(Y)$ is a groupoid, i. e. $[h_x] : f(x) \to g(x)$ has an inverse $[\overline{h_x}] : g(x) \to f(x)$ for all $x \in X$. □

In particular, we can apply Theorem 3.2.8 to homotopy equivalences $f : X \to Y$. In this case, one has a continuous map $g : Y \to X$ with $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$. Theorem 3.2.8 then yields a pair of functors $\Pi_1(f) : \Pi_1(X) \to \Pi_1(Y)$, $\Pi_1(g) : \Pi_1(Y) \to \Pi_1(X)$ together with natural isomorphisms $\kappa : \Pi_1(g)\Pi_1(f) \xrightarrow{\sim} \mathrm{id}_X$ and $\eta : \Pi_1(f)\Pi_1(g) \xrightarrow{\sim} \Pi_1(\mathrm{id}_Y)$. In other words:

Corollary 3.2.9: If $f : X \to Y$ is a homotopy equivalence, then the associated functor $\Pi_1(f) : \Pi_1(X) \to \Pi_1(Y)$ is an equivalence of categories. If two topological spaces X and Y are of the same homotopy type, then their fundamental groupoids $\Pi_1(X)$ and $\Pi_1(Y)$ are equivalent.

We will now consider the implications of Theorem 3.2.8 and Corollary 3.2.9 for the fundamental group $\pi_1(x, X) = \mathrm{Hom}_{\Pi_1(X)}(x, x)$ of a topological space X based at $x \in X$. In this case, the statement that $\Pi_1(f) : \Pi_1(X) \to \Pi_1(Y)$ is a functor implies that the map $\pi_1(x, f) = \Pi_1(f) : \pi_1(x, X) \to \pi_1(f(x), Y)$ is a group homomorphism.

The second claim in Theorem 3.2.8 states that the composition of continuous maps corresponds to the composition of group homomorphisms and that the identity map to the identity homomorphism. In other words: we obtain a functor $\pi_1 : \mathrm{Top}^* \to \mathrm{Grp}$ that assigns to each pointed topological space (x, X) the fundamental group $\pi_1(x, X)$ and to each continuous map $f : X \to Y$ with $f(x) = y$ the group homomorphism $\pi_1(x, f) : \pi_1(x, X) \to \pi_1(y, Y)$, $[\gamma] \mapsto [f \circ \gamma]$.

The third claim in Theorem 3.2.8 states that for homotopic maps $f, g : X \to Y$, the associated group homomorphisms $\pi_1(x, f)$ and $\pi_1(x, g)$ are related by conjugation. More specifically, if $h : [0,1] \times X \to Y$ is a homotopy from $f : X \to Y$ to $g : X \to Y$, diagram (24) implies

$$\pi_1(x, g)([\gamma]) = [h_x] \circ \pi_1(x, f)([\gamma]) \circ [\overline{h}_x] \text{ with } h_x : [0,1] \to X, h_x(t) = h(t, x)$$

for all closed paths $\gamma : [0,1] \to X$ based at $x \in X$. By combining these statements with Corollary 3.2.9, we obtain the following theorem.

Theorem 3.2.10:

1. The assignments $(x, X) \to \pi_1(x, X)$ and $f \to \pi_1(x, f)$ define a functor $\pi_1 :$ Top$^* \to$ Grp.
2. If $f, g : X \to Y$ are homotopic, then the group homomorphisms $\pi_1(x, f) : \pi_1(x, X) \to \pi_1(f(x), Y)$ and $\pi_1(x, g) : \pi_1(x, X) \to \pi_1(g(x), Y)$ are related by conjugation.
3. If $f : X \to Y$ is a homotopy equivalence, then the associated group homomorphism $\pi_1(x, f) : \pi_1(x, X) \to \pi_1(f(x), Y)$ is a group isomorphism. Topological spaces of the same homotopy type have isomorphic fundamental groups.

The results in Theorem 3.2.8 and 3.2.10 allow one to relate the fundamental groups of topological spaces X, Y via group homomorphisms induced by continuous maps $f : X \to Y$. In particular, they allow one to relate the fundamental group of a topological space X to the one of a subspace $A \subset X$ that is a retract or deformation retract of X. In this case, the associated retraction identifies the fundamental group $\pi_1(a, A)$ with a subgroup of $\pi_1(a, X)$. If $A \subset X$ is a deformation retract, then the retraction induces a group isomorphism. This often allows one to reduce the problem of determining the fundamental group of a topological space X to a few simple examples.

Corollary 3.2.11: Let X be a topological space, $A \subset X$ a subspace and $a \in A$.

1. If A is a retract of X, then the inclusion map $i_A : A \to X$ induces an injective group homomorphism $\pi_1(a, A) \to \pi_1(a, X)$.
2. If A is a deformation retract of X, then the inclusion map $i_A : A \to X$ induces a group isomorphism $\pi_1(a, A) \to \pi_1(a, X)$.

Proof:
If $A \subset X$ is a retract of X with retraction $r : X \to A$, then $r \circ i_A = \mathrm{id}_A$ and consequently $\pi_1(a, r) \circ \pi_1(a, i_A) = \pi_1(a, r \circ i_A) = \pi_1(a, \mathrm{id}_A) = \mathrm{id}_{\pi_1(a,A)}$ This implies that the group homomorphism $\pi_1(a, i_A) : \pi_1(a, A) \to \pi_1(a, X)$ is injective. If additionally, $i_A \circ r \simeq_A \mathrm{id}_X$, then the group homomorphism $\pi_1(a, i_A) \circ \pi_1(a, r) = \pi_1(a, i_A \circ r)$ is related to $\pi_1(a, \mathrm{id}_X) = \mathrm{id}_{\pi_1(a,X)}$ by conjugation and hence an isomorphism. This implies that $\pi_1(a, i_A) : \pi_1(a, A) \to \pi_1(a, X)$ is surjective and hence a group isomorphism. □

Example 3.2.12:

1. The fundamental groups of the ring $\{x \in \mathbb{R}^2 : 1 \leq ||x|| \leq 2\}$, the cylinder $\mathbb{R} \times S^1$, the circle S^1, the Möbius strip and $S^1 \times D^2$ are all isomorphic, since these spaces are homotopy equivalent (see Example 3.1.10 and Exercise 15).

2. For all $n \in \mathbb{N}$, the fundamental groups of $(\mathbb{R}^{n+1})^\times$ and of S^n are isomorphic since by Example 3.1.10 S^n is a deformation retract of $(\mathbb{R}^n)^\times$.
3. For connected topological spaces X, Y and any continuous map $f : X \to Y$, the fundamental group $\pi_1(C_f)$ of the mapping cylinder $C_f - (X \times [0,1]) \cup_{\tilde{f}} Y$ is isomorphic to the fundamental group $\pi_1(Y)$ since Y is a deformation retract of the mapping cylinder by Example 3.1.11.
4. Contractible topological spaces are simply connected since they are homotopy equivalent to a point. This applies to Bing's house (Exercise 7) and to the comb space (Example 3.1.12).

Similarly to the fundamental group of a topological space with base point x, one can also define higher homotopy groups. In this case, one considers continuous maps $f : [0,1]^n \to X$ from the unit n-cube $[0,1]^n = [0,1] \times \ldots \times [0,1]$ to X with $f(\partial[0,1]^n) = \{x\}$. As $S^n \approx [0,1]^n/\sim$, where $\sim$ is the equivalence relation that identifies all points on $\partial[0,1]^n$, such maps are in bijection with continuous maps $f : S^n \to X$. Just as in the case of the fundamental group, one finds that the homotopy classes of such maps relative to $\partial[0,1]^n$ form a group, which for $n \geq 2$ turns out to be abelian.

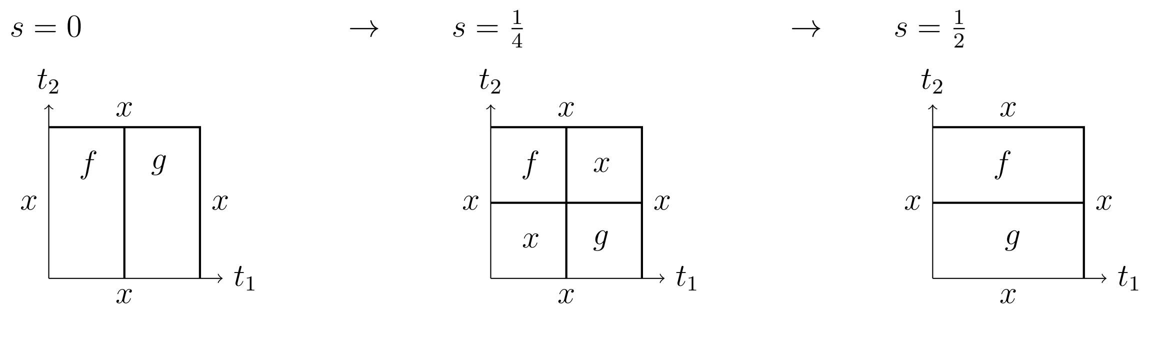
Proposition 3.2.13: Let X be a topological space, $x \in X$ and $n \geq 2$. Then the homotopy classes relative to $\partial[0,1]^n$ of continuous maps $f : [0,1]^n \to X$ with $f(\partial[0,1]^n) = \{x\}$ with the composition law

$$[g] \circ [f] = [g \star f] \qquad (g \star f)(t_1, ..., t_n) = \begin{cases} f(2t_1, t_2, ..., t_n) & t_k \in [0, \frac{1}{2}) \\ g(2t_1 - 1, t_2, ..., t_n) & t_k \in [\frac{1}{2}, 1]. \end{cases}$$

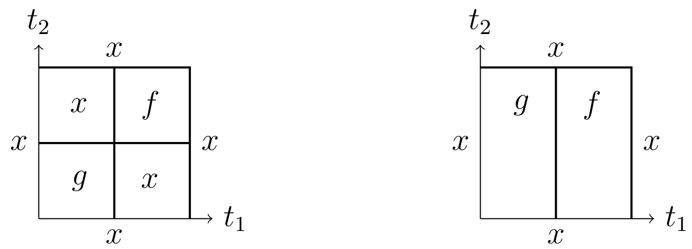
form an abelian group, called the n**th homotopy group** $\pi_n(x, X)$.

Proof:
It is clear that the composition is associative with the homotopy class of the constant map $f_x : [0,1]^n \to X$, $(t_1, ..., t_n) \mapsto x$ as unit and inverses $[f]^{-1} = [\overline{f}]$ with $\overline{f} : [0,1]^n \to X$, $(t_1, ..., t_n) \mapsto f(1 - t_1, t_2, ..., t_n)$. This follows analogously to the proof of Proposition 3.2.4.

It remains to show that the group $\pi_n(x, X)$ is abelian, e. .g $f \star g \sim_{\partial[0,1]^n} g \star f$ for all continuous maps $f, g : [0,1]^n \to X$ with $f(\partial[0,1]^n) = g(\partial[0,1]^n) = \{x\}$. For this, consider the map $h : [0,1]^{n+1} \to X$ pictured below



$\rightarrow \quad s = \frac{3}{4} \qquad\qquad\qquad \rightarrow \qquad s = 1$



which is given explicitly by

$$h(s, t_1, ..., t_n) = \begin{cases} x & t_1 \in [0, \frac{1}{2}), t_2 \in [0, 2s) \\ f(2t_1, \frac{t_2-2s}{1-2s}, t_3, ..., t_n) & t_1 \in [0, \frac{1}{2}), t_2 \in [2s, 1] \\ g(2t_1 - 1, \frac{t_2}{1-2s}, t_3, ..., t_n) & t_1 \in [\frac{1}{2}, 1], t_2 \in [0, 1-2s) \\ x & t_1 \in [\frac{1}{2}, 1], t_2 \in [1-2s, 1] \end{cases} \qquad s \in [0, \tfrac{1}{4})$$

$$h(s, t_1, ..., t_n) = \begin{cases} x & t_1 \in [0, 1-2s), t_2 \in [0, \frac{1}{2}) \\ g(\frac{t_1-1+2s}{2s}, 2t_2, t_3, ..., t_n) & t_1 \in [1-2s, 1], t_2 \in [0, \frac{1}{2}) \\ f(\frac{t_1}{2s}, 2t_2 - 1, t_3, ..., t_n) & t_1 \in [0, 2s), t_2 \in [\frac{1}{2}, 1] \\ x & t_1 \in [1-2s, 1], t_2 \in [\frac{1}{2}, 1] \end{cases} \qquad s \in [\tfrac{1}{4}, \tfrac{1}{2})$$

$$h(s, t_1, ..., t_n) = \begin{cases} g(\frac{t_1}{2-2s}, 2t_2, t_3, ..., t_n) & t_1 \in [0, 2-2s), t_2 \in [0, \frac{1}{2}) \\ x & t_1 \in [2-2s, 1], t_2 \in [0, \frac{1}{2}) \\ x & t_1 \in [0, 2s-1), t_2 \in [\frac{1}{2}, 1] \\ f(\frac{t_1+1-2s}{2-2s}, 2t_2 - 1, t_3, ..., t_n) & t_1 \in [2s-1, 1], t_2 \in [\frac{1}{2}, 1] \end{cases} \qquad s \in [\tfrac{1}{2}, \tfrac{3}{4})$$

$$h(s, t_1, ..., t_n) = \begin{cases} g(2t_1, \frac{t_2}{2s-1}, t_3, ..., t_n) & t_1 \in [0, \frac{1}{2}), t_2 \in [0, 2s-1) \\ x & t_1 \in [\frac{1}{2}, 1], t_2 \in [0, 2-2s) \\ x & t_1 \in [0, \frac{1}{2}), t_2 \in [2s-1, 1] \\ f(2t_1 - 1, \frac{t_2-2s+2}{2s-1} t_3, ..., t_n) & t_1 \in [\frac{1}{2}, 1], t_2 \in [2-2s, 1] \end{cases} \qquad s \in [\tfrac{3}{4}, 1].$$

Clearly, h is continuous with $h(0, t_1, ..., t_n) = g \star f$, $h(1, t_1, ..., t_n) = f \star g$ und $h(s, t_1, ..., t_n) = x$ for all $(x_1, ..., x_n \in \partial W^n)$, and therefore h is a homotopy from $f \star g$ to $g \star f$ relative to $\partial[0, 1]^n$. $\square$

Remark 3.2.14: The proof of Proposition 3.2.13 also shows that the choice of the first coordinate for the composition in $\pi_n(x, X)$ is arbitrary and that the composition law

$$[g] \circ_k [f] = [g \star_k f] \qquad (g \star_k f)(t_1, ..., t_n) = \begin{cases} f(t_1, ..., t_{k-1}, 2t_k, t_{k+1}, ..., t_n) & t_k \in [0, \frac{1}{2}) \\ g(t_1, ..., t_{k-1}, 2t_k - 1, t_{k+1}, ..., t_n) & t_k \in [\frac{1}{2}, 1]. \end{cases}$$

would yield the same result.

Remark 3.2.15: One has analogues of the statements in Theorem 3.2.10:

1. The nth homotopy group defines a functor π_n : Top* $\to$ Ab from the category Top* of pointed topological spaces into the category Ab of abelian groups that assigns to a continuous map $f : X \to Y$ with $f(x) = y$ a group homomorphism $\pi_n(x, f) : \pi_n(x, X) \to \pi_n(y, Y)$.
2. If $f, g : X \to Y$ with $f(x) = g(x) = y$ are homotopic relative to $\{x\}$, then $\pi_n(x, f) = \pi_n(x, g) : \pi_n(x, X) \to \pi_n(y, Y)$. This means that the functor $\pi_n :$ Top$^* \to$ Ab is constant on the homotopy equivalence classes of morphisms in Top* and induces a functor $\pi_n :$ hTop$^* \to$ Ab.
3. If $f : X \to Y$ is a homotopy equivalence, then $\pi_n(x, f) : \pi_n(x, X) \to \pi_n(f(x), Y)$ is a group isomorphism. Topological spaces of the same homotopy type have isomorphic homotopy groups.

Unlike fundamental group(oid)s, higher homotopy groups are rather difficult to compute. Even in the case of the n-spheres S^n, there is no general formula or result that characterises all homotopy groups $\pi_k(S^n)$ for general $k, n \in \mathbb{N}$. We will therefore not pursue the computation of higher homotopy groups further in this lecture but instead characterise higher-dimensional topological manifolds in terms of homology.

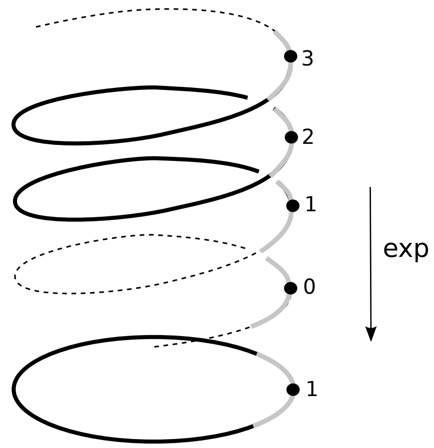
3.3 The fundamental group of the circle

The aim is now to compute fundamental group(oid)s of topological spaces also for those cases, where it does not follow immediately that they are trivial. We will see later that this can be achieved by building up topological spaces gradually from simple building blocks and relating their fundamental group(oid)s to the ones of the building blocks.

The first essential building block in this construction is the circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, and the central ingredient in the computation of its fundamental group is the exponential map

$$\exp : \mathbb{R} \to S^1, \quad x \mapsto e^{2\pi \mathrm{i} x} = \cos(2\pi x) + \mathrm{i}\, \sin(2\pi x).$$

The basic idea is to relate homotopy classes of paths $\gamma : [0, 1] \to S^1$ with $\gamma(0) = \gamma(1) = 1$ to homotopy classes of paths $\tilde{\gamma} : [0, 1] \to \mathbb{R}$ with $\tilde{\gamma}(0), \tilde{\gamma}(1) \in \mathbb{Z}$. Clearly, every continuous path $\tilde{\gamma} : [0, 1] \to \mathbb{R}$ with $\tilde{\gamma}(0), \tilde{\gamma}(1) \in \mathbb{Z}$ defines a continuous path $\gamma = \exp \circ \tilde{\gamma} : [0, 1] \to S^1$ with $\gamma(0) = \gamma(1) = 1$. However, it is not clear a priori which paths $\tilde{\gamma}$ yield a given path γ (or a path homotopic to it) or if every path $\gamma : [0, 1] \to S^1$ with $\gamma(0) = \gamma(1) = 1$ arises this way. To clarify this, we first consider continuous maps of the circle $S^1 \to S^1$ and relate them to continuous maps $\mathbb{R} \to \mathbb{R}$. As S^1 is homeomorphic to the quotient $[0, 1]/\sim$ where $\sim$ is the equivalence relation on $[0, 1]$ that identifies the endpoints, we can easily extend these results to closed paths $\gamma : [0, 1] \to S^1$ later.



Lemma 3.3.1: Let $f : S^1 \to S^1$ be continuous with $f(1) = 1$. For every $t_0 \in \mathbb{Z}$ there is a unique continuous map $\tilde{f} : \mathbb{R} \to \mathbb{R}$ with $\tilde{f}(0) = t_0$ and $\exp \circ \tilde{f} = f \circ \exp$. The map $\tilde{f}$ is called a **lift** of f.

Proof:
Uniqueness: If $\tilde{f}_1, \tilde{f}_2 : \mathbb{R} \to \mathbb{R}$ are two two lifts of f, then

$$\exp \circ (\tilde{f}_1(t) - \tilde{f}_2(t)) = \frac{\exp \circ \tilde{f}_1(t)}{\exp \circ \tilde{f}_2(t)} = \frac{f \circ \exp(t)}{f \circ \exp(t)} = 1 \qquad \forall t \in \mathbb{R},$$

which implies $\tilde{f}_1(t) - \tilde{f}_2(t) \in \mathbb{Z}$ for all $t \in \mathbb{R}$. As $\tilde{f}_1 - \tilde{f}_2 : \mathbb{R} \to \mathbb{R}$ is continuous, it follows that $\tilde{f}_2 - \tilde{f}_1$ is constant, and as $\tilde{f}_1(0) - \tilde{f}_2(0) = t_0 - t_0 = 0$, we have $\tilde{f}_1 = \tilde{f}_2$.

Existence: As S^1 is compact and $f : S^1 \to S^1$ continuous, f is uniformly continuous. The same holds for $\exp : \mathbb{R} \to S^1$ since S^1 and $[0, 1]$ are compact and exp is periodic with period 1. Hence $f \circ \exp : \mathbb{R} \to S^1$ is uniformly continuous, and there is an $\epsilon > 0$ such that $f \circ \exp(I)$ is contained in an open half-circle for all intervals I of length $\leq \epsilon$. For any open half-circle $H \subset S^1$, one has $\exp^{-1}(H) = \dot{\bigcup}_{k \in \mathbb{Z}} I_k$ with $I_k = (s + k, s + k + \frac{1}{2})$ for some $s \in \mathbb{R}$, and $\exp|_{I_k} : I_k \to H$ is a homeomorphism.

As $f \circ \exp([0, \epsilon])$ is contained in an open half-circle H_0, there is a unique interval I_0 of length $1/2$ with $\exp(I_0) = H_0$ and $t_0 \in I_0$. Define

$$\tilde{f}(t) = (\exp|_{I_0})^{-1} \circ f \circ \exp(t) \qquad \forall t \in [0, \epsilon].$$

Then $\tilde{f} : [0, \epsilon] \to \mathbb{R}$ is continuous with $\exp \circ \tilde{f}|_{[0,\epsilon]} = \exp \circ f|_{[0,\epsilon]}$ and $\tilde{f}(0) = t_0$. Now apply the same procedure to the interval $[\epsilon, 2\epsilon]$ and $t_1 = \tilde{f}(1)$. As $f \circ \exp([\epsilon, 2\epsilon])$ is contained in a half-circle H_1, there is a unique interval I_1 of length $1/2$ with $\exp(I_1) = H_1$ and $t_1 = \tilde{f}(\epsilon) \in I_1$. We obtain a continuous function $\tilde{f} : [0, 2\epsilon] \to \mathbb{R}$ with $\exp \circ \tilde{f}|_{[0,2\epsilon]} = \exp \circ f|_{[0,2\epsilon]}$ by setting

$$\tilde{f}(t) = (\exp|_{I_1})^{-1} \circ f \circ \exp(t) \qquad \forall t \in [\epsilon, 2\epsilon].$$

By iterating this step and extending it to negative numbers, we obtain a continuous function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ with $\exp \circ \tilde{f} = \exp \circ f$ and $\tilde{f}(0) = t_0$. $\square$

Given a lift $\tilde{f}: \mathbb{R} \to \mathbb{R}$ of a continuous map $f: S^1 \to S^1$, we can consider the quantity $\tilde{f}(1) - \tilde{f}(0)$. As $\exp \circ \tilde{f} = f \circ \exp$ and $\exp(0) = \exp(1) = 1$, we have $\exp(\tilde{f}(1) - \tilde{f}(0)) = 1$ and hence $\tilde{f}(1) - \tilde{f}(0) \in \mathbb{Z}$. Moreover, this quantity does not depend on the choice of the lift, since any two lifts differ by a constant. This allows one to assign to each map $f: S^1 \to S^1$ the quantity $\deg(f) = \tilde{f}(1) - \tilde{f}(0)$.

To obtain an analogous quantity for general paths $\gamma : [0,1] \to S^1$ with $\gamma(0) = \gamma(1) = 1$, recall that S^1 is homeomorphic to the quotient $[0,1]/\sim$ where $\sim$ is the equivalence relation on $[0,1]$ that identifies the endpoints. As $\gamma(0) = \gamma(1)$ and $\exp(0) = \exp(1)$, the universal property of the quotient implies that there is a unique path $\gamma_\sim : [0,1]/\sim \to S^1$ with $\gamma_\sim \circ \pi = \gamma$, and the unique map $\exp_\sim : [0,1]/\sim \to S^1$ with $\exp_\sim \circ \pi = \exp|_{[0,1]}$ is a homeomorphism. Hence, we can define the degree of γ as the degree of $\gamma_\sim \circ \exp_\sim^{-1} : S^1 \to S^1$.

$$\begin{tikzcd}
S^1 & {[0,1]} \arrow[l, "\exp|_{[0,1]}"'] \arrow[r, "\gamma"] \arrow[d, "\pi"] & S^1 \\
& {[0,1]/\sim} \arrow[ul, "\exp_\sim", "\approx"'] \arrow[ur, "\gamma_\sim"'] &
\end{tikzcd} \tag{25}$$

Definition 3.3.2:

1. Let $f: S^1 \to S^1$ be continuous and $\tilde{f}: \mathbb{R} \to \mathbb{R}$ of f a lift of f. Then the **degree** of f is $\deg(f) = \tilde{f}(1) - \tilde{f}(0) \in \mathbb{Z}$.
2. Let $\gamma : [0,1] \to S^1$ be a path with $\gamma(1) = \gamma(0) = 1$. Then the **degree** of γ is $\deg(\gamma) = \deg(\gamma_\sim \circ \exp_\sim^{-1})$, where $\gamma' \circ \exp_\sim^{-1} : S^1 \to S^1$ is the map defined by (25).

Example 3.3.3: For $n \in \mathbb{Z}$, the map $f: S^1 \to S^1$, $z \mapsto z^n$ has degree $\deg(f) = n$.
The map $\tilde{f}: \mathbb{R} \to \mathbb{R}$, $t \mapsto nt$ is a lift of f since $\exp \circ \tilde{f}(t) = \exp(nt) = \exp(t)^n = f \circ \exp(t)$ for all $t \in \mathbb{R}$. This yields $\deg(f) = \tilde{f}(1) - \tilde{f}(0) = n$.

We will make use of the notion of degree to determine the fundamental group of the circle. This requires that we first determine its basic properties, which also have many useful applications in geometry. They follow from basic computations together with the properties of the exponential map and the lifts.

Lemma 3.3.4:

1. For any lift $\tilde{f}: \mathbb{R} \to \mathbb{R}$ of $f: S^1 \to S^1$, $t \in \mathbb{R}$ and $k \in \mathbb{Z}$, one has $\tilde{f}(t+k) - \tilde{f}(t) = k \deg(f)$.
2. $\deg(f \cdot g) = \deg(f) + \deg(g)$ for all continuous maps $f, g : S^1 \to S^1$.
3. $\deg(f \circ g) = \deg(f) \cdot \deg(g)$ for all continuous maps $f, g : S^1 \to S^1$.
4. If $f: S^1 \to S^1$ is continuous with $f(-z) = -f(z)$ for all $z \in S^1$, then $\deg(f)$ is odd.
5. Any continuous map $f: S^1 \to S^1$ with $\deg(f) \neq 0$ is surjective.

Proof:
1. We have $\tilde{f}(t+k) - \tilde{f}(t) = \sum_{j=0}^{k-1} \tilde{f}(t+j+1) - \tilde{f}(t+j) = k \deg(f)$.

2. If $\tilde{f}$ is a lift of f and $\tilde{g}$ a lift of g, then $\tilde{f} + \tilde{g} : \mathbb{R} \to \mathbb{R}$ is a lift of $f \cdot g : S^1 \to S^1$ since $\exp \circ (\tilde{f} + \tilde{g}) = (\exp \circ \tilde{f}) \cdot (\exp \circ \tilde{g}) = (f \circ \exp) \cdot (g \circ \exp) = (f \cdot g) \circ \exp$.

3. If $\tilde{f}$ is a lift of f and $\tilde{g}$ a lift of g, then $\tilde{f} \circ \tilde{g} : \mathbb{R} \to \mathbb{R}$ is a lift of $f \circ g : S^1 \to S^1$, since $\exp \circ \tilde{f} \circ \tilde{g} = f \circ \exp \circ \tilde{g} = f \circ g \circ \exp$.

4. For any lift $\tilde{f} : \mathbb{R} \to \mathbb{R}$ of f and $t \in \mathbb{R}$, one has

$$\exp(\tilde{f}(t) + \tfrac{1}{2}) = -\exp \circ \tilde{f}(t) = -f \circ \exp(t) = f \circ (-\exp(t)) = f \circ \exp(t + \tfrac{1}{2}) = \exp(\tilde{f}(t + \tfrac{1}{2})).$$

This implies that the continuous map $g : \mathbb{R} \to \mathbb{R}$, $g(t) = \tilde{f}(t + \frac{1}{2}) - \tilde{f}(t) - \frac{1}{2}$ takes values in $\mathbb{Z}$ and hence is constant. It follows that $\deg(f) = \tilde{f}(1) - \tilde{f}(0) = g(\frac{1}{2}) + g(0) + 1 = 2g(0) + 1$.

5. From $\deg(f) \neq 0$ it follows that for any lift $\tilde{f}$ of f we have $|\tilde{f}(1) - \tilde{f}(0)| \geq 1$ and hence the image $\tilde{f}(\mathbb{R})$ contains an interval I of length one. This implies $f(S^1) = f(\exp(\mathbb{R})) = \exp(\tilde{f}(\mathbb{R})) \supset \exp(I) = S^1$. □

By considering the degree of paths $\gamma : [0,1] \to S^1$ with $\gamma(0) = \gamma(1) = 1$, we are able to compute the fundamental group of the circle and to show that it is isomorphic to $\mathbb{Z}$. The essential steps are to show that the degree of such a path depends only on its homotopy class relative to $\{0,1\}$ and that the composition of paths adds their degrees.

Theorem 3.3.5: The degree induces a group isomorphism $\deg : \pi_1(1, S^1) \to \mathbb{Z}$, $[\gamma] \mapsto \deg(\gamma)$.

Proof:
1. We show that the degree of γ depends only on the homotopy class of γ:
Let $f, g : S^1 \to S^1$ be continuous and $h : [0,1] \times S^1 \to S^1$ be a homotopy from f to g. Consider for $t, t' \in [0,1]$ the maps

$$h_t : S^1 \to S^1,\ z \mapsto h(t,z), \qquad H_{t,t'} : S^1 \to S^1,\ z \mapsto \frac{h(t,z)}{h(t',z)}.$$

As h is continuous and $[0,1] \times S^1$ is compact, h is uniformly continuous, and there is a $\delta > 0$ with $|h(t,z) - h(t',z)| < 1$ for $|t - t'| < \delta$. In this case, $H_{t,t'} : S^1 \to S^1$ is not surjective and hence $\deg(H_{t,t'}) = 0$ by Lemma 3.3.4, 5. It then follows from Lemma 3.3.4, 2. that

$$\deg(h_t) = \deg H_{t,t'} + \deg(h_{t'}) = \deg(h_{t'}) \qquad \forall t, t' \in [0,1] \text{ with } |t - t'| < \delta.$$

By choosing a subdivision $0 = t_0 < \ldots < t_n = 1$ with $|t_i - t_{i-1}| < \delta$ for $i \in \{1, ..., n\}$, we obtain

$$\deg(f) = \deg(h_{t_0}) = \deg(h_{t_1}) = \ldots = \deg(h_{t_n}) = \deg(g).$$

If $\gamma, \delta : [0,1] \to S^1$ are paths that are homotopic relative to $\{0,1\}$ via $h : [0,1] \times [0,1] \to S^1$, the commuting diagram (25) yields a homotopy $h_\sim : [0,1] \times S^1 \to S^1$, $h_\sim(t,z) = h(t, \exp_\sim^{-1}(z))$ from $\gamma_\sim \circ \exp_\sim^{-1} : S^1 \to S^1$ to $\delta_\sim \circ \exp_\sim^{-1} : S^1 \to S^1$, which implies

$$\deg(\gamma) = \deg(\gamma_\sim \circ \exp_\sim^{-1}) = \deg(\delta_\sim \circ \exp_\sim^{-1}) = \deg(\delta).$$

Hence the degree induces a map $\pi_1(1, S^1) \to \mathbb{Z}$.

2. To show that $\deg : \pi_1(1, S^1) \to \mathbb{Z}$, $[\gamma] \mapsto \deg(\gamma)$ is a group homomorphism, it is sufficient to show that $\deg(\delta \star \gamma) = \deg(\delta) + \deg(\gamma)$ for all paths $\gamma, \delta : [0,1] \to S^1$ with $\gamma(0) = \gamma(1) = \delta(0) = \delta(1) = 1$. This follows from the fact that S^1 is a topological group with the group multiplication induced by the multiplication in $\mathbb{C}$. By Exercise 12, the group structures on the

set $\pi_1(1, S^1)$ that are induced by the pointwise multiplication of paths and their composition coincide: $[\delta] \circ [\gamma] = [\delta \star \gamma] = [\delta \cdot \gamma]$. Together with Lemma 3.3.4, 2. this implies

$$\begin{aligned}&\deg(\delta \star \gamma) = \deg(\delta \cdot \gamma) = \deg((\delta \cdot \gamma)_\sim \circ \exp_\sim^{-1}) = \deg((\delta_\sim \circ \exp_\sim^{-1}) \cdot (\gamma_\sim \circ \exp_\sim^{-1})) \\ &= \deg(\delta_\sim \circ \exp_\sim^{-1}) + \deg(\gamma_\sim \circ \exp_\sim^{-1}) = \deg(\delta) + \deg(\gamma).\end{aligned}$$

3. To show that $\deg : \pi_1(1, S^1) \to \mathbb{Z}$ is surjective, note that for $n \in \mathbb{Z}$, the path $\gamma_n : [0,1] \to S^1$, $t \mapsto \exp(nt)$ has $\deg(\gamma_n) = n$ by Example 3.3.3. To prove that $\deg : \pi_1(1, S^1) \to \mathbb{Z}$ is injective, suppose that $\gamma : [0,1] \to S^1$ is a path with $\gamma(0) = \gamma(1) = 1$ and $\deg(\gamma) = 0$. Let $\tilde{\gamma} : \mathbb{R} \to \mathbb{R}$ be a lift of $\gamma_\sim \circ \exp_\sim^{-1} : S^1 \to S^1$ with $\tilde{\gamma}(0) = 0$. Then

$$h : [0,1] \times [0,1] \to S^1, \quad h(s,t) = \exp(s\tilde{\gamma}(t))$$

is a homotopy from the constant path based at 1 to the path $\gamma = \exp \circ \tilde{\gamma}|_{0,1} : [0,1] \to S^1$ relative to $\{0,1\}$. This shows that $\deg : \pi_1(1, S^1) \to \mathbb{Z}$ is injective. $\square$

Theorem 3.3.5 determines the fundamental group of the circle and all other topological spaces of the same homotopy type, such as the punctured plane $\mathbb{C}^\times$ and the cylinder $S^1 \times \mathbb{R}$. Moreover, it has applications to many geometrical, topological and algebraic questions. This becomes evident in the following corollaries.

Corollary 3.3.6:

1. If $f : S^1 \to S^1$ is the restriction of a continuous map $g : D^2 \to S^1$, then $\deg(f) = 0$.

   This follows since $h : [0,1] \times S^1 \to S^1$, $h(t,z) = g(tz)$ is a homotopy from the constant map $S^1 \to g(0)$ to f.
2. S^1 is not a retract of D^2.

   This can be seen as follows. If $r : D^2 \to S^1$ is a retraction, then $r \circ i_{S^1} = \mathrm{id}_{S^1} : S^1 \to S^1$. In this case, 1. implies $\deg(r \circ i_{S^1}) = \deg(\mathrm{id}_{S^1}) = 0$. On the other hand, we have $\deg(\mathrm{id}_{S^1}) = 1$ by Example 3.3.3.
3. A continuous map $f : D^2 \to D^2$ has a fix point (Brouwer's fix point theorem in $d = 2$).

   Suppose there is a continuous map $f : D^2 \to D^2$ without a fix point. Then for any $x \in D^2$, there is a unique straight line through x and $f(x)$. Let $r : D^2 \to S^1$ be the map that assigns to $x \in D^2$ the intersection point of this line with $S^1 = \partial D^2$ that is closer to x than to $f(x)$. Then r is continuous due to the continuity of f and hence a retraction from D^2 to S^1. Such a retraction cannot exist by 2.

Theorem 3.3.5 and Corollary 3.3.6 also give rise to a very simple, purely topological proof of the fundamental theorem of algebra and allows us to prove that S^2 is not homeomorphic to a subset of $\mathbb{R}^2$, which implies in turn that $\mathbb{R}^2$ and $\mathbb{R}^3$ cannot be homeomorphic.

Corollary 3.3.7 (fundamental theorem of algebra):
Every non-constant polynomial with coefficients in $\mathbb{C}$ has a zero.

Proof:
Suppose there is a non-constant polynomial with coefficients in $\mathbb{C}$ and without a zero. Without restriction of generality suppose that p is normalised, $p = z^n + a_{n-1}z^{n-1} + \ldots + a_0$ with $n \geq 1$. Then the map

$$f : \mathbb{C} \to S^1, \quad z \mapsto \frac{p(z)}{|p(z)|}$$

is continuous and by Corollary 3.3.6, 1. we have $\deg(f \circ i_{S^1}) = 0$. However, the map

$$h : [0,1] \times S^1 \to S^1, \quad h(t,z) = \frac{z^n + ta_{n-1}z^{n-1} + t^2 a_{n-2}z^{n-1} \ldots + t^n a_0}{||z^n + ta_{n-1}z^{n-1} + t^2 a_{n-2}z^{n-2} + \ldots + t^n a_0||}$$

is a homotopy from $g : S^1 \to S^1$, $z \mapsto z^n$ to $f \circ i_{S^1} : S^1 \to S^1$. This implies $\deg(f \circ i_{S^1}) = \deg(g) = n$. □

Corollary 3.3.8 (Borsuk-Ulam):
If $f : S^2 \to \mathbb{R}^2$ is continuous with $f(-x) = -f(x)$ for all $x \in S^2$, then f has a zero.

Proof:
Suppose f does not have a zero. Then the maps

$$g : S^2 \to S^1,\ x \mapsto \frac{f(x)}{||f(x)||}, \qquad G : D^2 \to S^1,\ z \mapsto g(z, \sqrt{1 - |z|^2})$$

are continuous with $g(-x) = -g(x)$ for all $x \in S^2$ and $G \circ i_{S^1} = g \circ i_{S^1}$. By Corollary 3.3.6, 1. this implies $\deg(g \circ i_{S^1}) = 0$, while Lemma 3.3.4, 4. implies that $\deg(g \circ i_{S^1})$ is odd. Contradiction. □

Corollary 3.3.9: S^2 is not homeomorphic to a subset of $\mathbb{R}^2$. $\mathbb{R}^3$ is not homeomorphic to $\mathbb{R}^2$.

Proof:
As any continuous map $\mathbb{R}^3 \to \mathbb{R}^2$ restricts to a continuous map $S^2 \to \mathbb{R}^2$, it is sufficient to show that a continuous map $g : S^2 \to \mathbb{R}^2$ cannot be injective. This follows from Corollary 3.3.8 since the associated map $f : S^2 \to \mathbb{R}^2$, $x \mapsto g(x) - g(-x)$ satisfies $f(-x) = -f(x)$. By Corollary 3.3.8, there is a point $x \in S^2$ with $f(x) = g(x) - g(-x) = 0$ and hence g is not injective. □

Another nice but slightly less serious application of the degree and Corollary 3.3.8 is that it gives an answer to the following question:

> Is it possible to cut a sandwich, consisting of bread and two toppings, in two pieces such that both pieces contain an equal amount of bread and of each topping in only one cut?

To answer this question, we interpret the bread and each topping as a subset of $\mathbb{R}^3$ and model the knife by an affine plane in $\mathbb{R}^3$. To give a precise interpretation to "equal amount" , we suppose that the three sets are Lebesgue measurable, for instance open, bounded, and require that the resulting pieces have equal volume. The answer to the question is then given by the following Lemma.

Lemma 3.3.10 (The sandwich lemma): Let $A, B, C \subset \mathbb{R}^3$ open and bounded subsets. Then there is an affine plane that cuts each of the subsets A, B, C into two subsets of equal volume.

Proof:
For $x \in S^2$ and $t \in \mathbb{R}$, we denote by $H_{x,t}$ the affine hyperplane normal to x through tx and by $H^+_{x,t}$ the associated half plane that does not contain the origin

$$H_{x,t} = \{y \in \mathbb{R}^3 : \langle y, x\rangle = t\}, \qquad H^+_{x,t} = \{y \in \mathbb{R}^3 : \langle y, x\rangle \geq t\}.$$

For fixed $x \in S^2$, we consider the map

$$a_x : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \mathrm{vol}(A \cap H^+_{x,t}).$$

Clearly, a_x decreases monotonically. As A is bounded, there is an $r > 0$ with $||a|| < r$ for all $a \in A$, which implies $a_x(t) = 0$ for $t > r$ and $a_x(t) = \mathrm{vol}(A)$ for $t < -r$. Moreover, a_x is (Lipschitz) continuous since

$$|a_x(t) - a_x(t')| \leq \mathrm{vol}(A \cap (H^+_{x,\min(t,t')} \setminus H^+_{x,\max(t,t')})) \leq \pi r^2 |t - t'| \qquad \forall t, t' \in \mathbb{R}$$

By continuity and monotonicity of a_x, we have $a_x^{-1}(\frac{1}{2}\mathrm{vol}(A)) = [t_1, t_2]$ for some $t_1, t_2 \in \mathbb{R}$, and we set $t_A(x) = \frac{1}{2}(t_1 + t_2)$. This yields a continuous map $t_A : S^2 \to \mathbb{R}$ with $t_A(-x) = -t_A(x)$ since $a_{-x}(t) = \mathrm{vol}(A) - a_x(-t)$. Analogously, we obtain contain (Lipschitz) continuous maps $b_x, c_x : \mathbb{R} \to \mathbb{R}$ and continuous maps $t_B, t_C : S^2 \to \mathbb{R}$ for the sets B, C. The continuous map

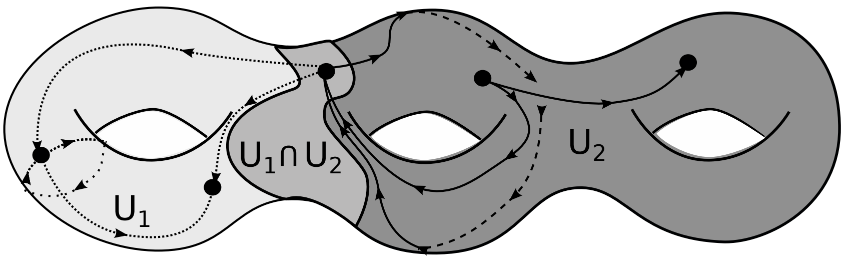
$$f : S^2 \to \mathbb{R}^2, \quad x \mapsto (t_B(x) - t_A(x), t_C(x) - t_A(x)).$$

satisfies $f(-x) = -f(x)$ and Corollary 3.3.8 implies that there is an $x \in S^2$ with $f(x) = 0$. It follows that $t_A(x) = t_B(x) = t_C(x) =: t$ and $a_x(t) = \frac{1}{2}\mathrm{vol}(A)$, $b_x(t) = \frac{1}{2}\mathrm{vol}(B)$, $c_x(t) = \frac{1}{2}\mathrm{vol}(C)$.
□

The sandwich lemma does not hold for more than three sets. This can be seen easily by choosing sets A, B, C for which the affine plane $H_{x,t}$ that cuts them into two pieces of equal volume is determined uniquely, for instance three balls of radius 1 centred on different points of the x-axis. It is then easy to choose a fourth set D, which the plane $H_{x,t}$ does not cut into two pieces of equal volume.

3.4 The theorem of Seifert and van Kampen

In this section, we develop an important tool that will allows us to compute the fundamental group(oid) of many topological spaces - the theorem of Seifert and van Kampen. The basic idea is to cut a topological space X into two overlapping pieces, whose fundamental group(oid)s are simpler to compute than the one of X and then to build up the fundamental group(oid) of X from the fundamental group(oid)s of these pieces.



To implement this idea, one considers two open subsets $U_1, U_2 \subset X$ with $X = U_1 \cup U_2$, the associated inclusion maps $i_k : U_k \to X$, $j_k : U_1 \cap U_2 \to U_k$ for $k = 1, 2$, and their fundamental groupoids $\Pi_1(U_1 \cap U_2)$, $\Pi_1(U_1)$, $\Pi_1(U_2)$ and $\Pi_1(X)$.

By Theorem 3.2.8, the inclusion maps $i_k : U_k \to X$, $j_k : U_1 \cap U_2 \to U_k$ induce functors

$$\begin{aligned}
\Pi_1(i_k) : \Pi_1(U_k) &\to \Pi_1(X) & \qquad \Pi_1(j_k) : \Pi_1(U_1 \cap U_2) &\to \Pi_1(U_k)\\
x &\mapsto x & x &\mapsto x\\
[\gamma]_k &\mapsto [i_k \circ \gamma]_X & [\delta]_{12} &\mapsto [j_k \circ \delta]_k
\end{aligned}$$

where $[\quad]_{12}$, $[\quad]_k$, $[\quad]_X$ denote the homotopy classes relative to $\{0, 1\}$ in $U_1 \cap U_2$, U_k, X, respectively. Although the action of these functors on the objects of the fundamental groupoids (points) is rather trivial, this is not the case for their action on morphisms. The homotopy classes of a given path in different topological spaces can look quite different, and it is essential to distinguish them.

The aim is now to relate the fundamental groupoids $\Pi_1(U_1 \cap U_2)$, $\Pi_1(U_1)$, $\Pi_1(U_2)$ and $\Pi_1(X)$ via these functors. To find the appropriate algebraic structure, it is helpful to note that the subsets U_1 and U_2 give rise to a description of X as a pushout of topological spaces. By considering the topological spaces U_1, U_2 and the topological space $U_1 \cap U_2$, together with continuous maps $j_k : U_1 \cap U_2 \to U_k$, one obtains a pushout $U_1 +_{U_1 \cap U_2} U_2 = U_1 + U_2/\sim$ where $\sim$ is the equivalence relation on $U_1 + U_2$ given by $j_1(w) \sim j_2(w)$ for all $w \in U_1 \cap U_2$. From the universal property of this pushout and the continuous maps $i_k : U_k \to X$, which satisfy $i_1 \circ j_1 = i_2 \circ j_2$, one obtains a unique continuous map $\phi : U_1 +_{U_1 \cap U_2} U_2 \to X$ with $\phi \circ j_k = i_k$. One can show that this is a homeomorphism and hence $X \approx U_1 +_{U_1 \cap U_2} U_2$.

This suggests that the fundamental groupoids $\Pi_1(X)$, $\Pi_1(U_1)$, $\Pi_1(U_2)$ and $\Pi_1(U_1 \cap U_2)$ together with the functors $\Pi_1(j_k) : \Pi_1(U_1 \cap U_2) \to \Pi_1(U_k)$ and $\Pi_1(i_k) : U_k \to X$ should form a pushout in an appropriate category. This category is the category of groupoids, whose objects are groupoids and whose morphisms are functors between them, with the identity functors as identity morphisms.

Theorem 3.4.1 (Seifert-van Kampen): Let X be a topological space, $U_1, U_2 \subset X$ open with $U_1 \cup U_2 = X$ and $i_k : U_k \to X$ and $j_k : U_1 \cap U_2 \to U_k$ the associated inclusion maps. Then the following diagram is a pushout in the category of groupoids

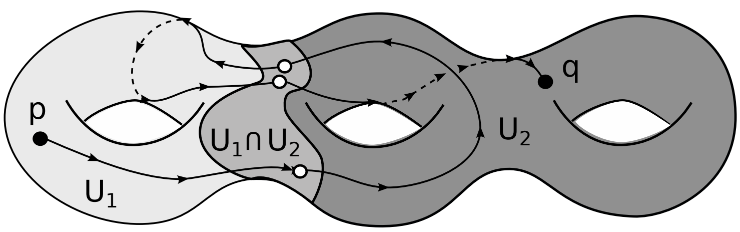
$$\begin{tikzcd}
\Pi_1(X) & \Pi_1(U_1) \arrow[l, "\Pi_1(i_1)"'] \\
\Pi_1(U_2) \arrow[u, "\Pi_1(i_2)"] & \Pi_1(U_1 \cap U_2). \arrow[l, "\Pi_1(j_2)"] \arrow[u, "\Pi_1(j_1)"']
\end{tikzcd}$$

Proof:
1. By Theorem 3.2.8 the continuous maps $i_k : U_k \to X$, $j_k : U_1 \cap U_2 \to U_k$ induce functors $\Pi_1(i_k) : \Pi_1(U_k) \to \Pi_1(X)$, $\Pi_1(j_k) : \Pi_1(U_1 \cap U_2) \to \Pi_1(U_k)$. As $i_1 \circ j_1 = i_2 \circ j_2 : U_1 \cap U_2 \to X$, they satisfy $\Pi_1(i_1)\Pi_1(j_1) = \Pi_1(i_1 \circ j_1) = \Pi_1(i_2 \circ j_2) = \Pi_1(i_2)\Pi_1(j_2)$, and the diagram commutes.

2. It remains to verify the universal property. For this, consider a groupoid $\mathcal{C}$ and two functors $F_k : \Pi_1(U_k) \to \mathcal{C}$ with $F_1\Pi_1(j_1) = F_2\Pi_1(j_2)$. We have to show that there is a unique functor $F : \Pi_1(X) \to \mathcal{C}$ with $F\Pi_1(i_k) = F_k$ for $k = 1, 2$.

$$\begin{tikzcd}
\mathcal{C} & & \\
& \Pi_1(X) \arrow[ul, dashed, "\exists! F"'] & \Pi_1(U_1) \arrow[l, "\Pi_1(i_1)"] \arrow[ull, bend right, "F_1"'] \\
& \Pi_1(U_2) \arrow[u, "\Pi_1(i_2)"'] \arrow[uul, bend left, "F_2"] & \Pi_1(U_1 \cap U_2). \arrow[l, "\Pi_1(j_2)"] \arrow[u, "\Pi_1(j_1)"']
\end{tikzcd}$$

To define F on objects, we use $X = U_1 \cup U_2$ and set $F(x) = F_k(x)$ for $x \in U_k$. If $x \in U_1 \cap U_2$, the identity $F_1\Pi_1(j_1) = F_2\Pi_1(j_2)$ implies $F_1(x) = F_2(x)$, which shows that this definition is consistent.



Decomposition of a path $\gamma : [0, 1] \to X$ into paths in U_1 and U_2.

To define $F : \Pi_1(X) \to \mathcal{C}$ on morphisms, we first show that the universal property of the pushout determines the functor completely. For this, we consider a path $\gamma : [0, 1] \to X$ and subdivide it into pieces that are contained entirely in U_1 or U_2. As $\gamma^{-1}(U_1 \cup U_2)$ is an open cover of the compact interval $[0, 1]$, by Lebesgue's lemma there is a finite decomposition $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = 1$ of $[0, 1]$ such that $[t_{i-1}, t_i] \subset \gamma^{-1}(U_{k(i)})$, where $k(i) \in \{1, 2\}$. By choosing homeomorphisms $\tau_i : [0, 1] \to [t_{i-1}, t_i]$ with $\tau_i(0) = t_{i-1}$ and $\tau_i(1) = t_i$, we obtain paths $\gamma_i = \gamma \circ \tau_i : [0, 1] \to U_{k(i)}$. By composing the associated paths $\delta_i = i_{k(i)} \circ \gamma_i : [0, 1] \to X$, we obtain $[\gamma]_X = [\delta_n]_X \circ \ldots \circ [\delta_1]_X$. The universal property of the pushout then implies

$$\begin{aligned}
&F([\delta_i]_X) = F([j_{k(i)} \circ \gamma_i]_X) = F\Pi_1(j_{k(i)})([\gamma_i]_{k(i)}) = F_{k(i)}([\gamma_i]_{k(i)}), \\
&F([\gamma]_X) = F([\delta_n]_X) \circ \ldots \circ F([\delta_1]_X) = F_{k(n)}([\gamma_n]_{k(n)}) \circ \cdots \circ F_{k(1)}([\gamma_1]_{k(1)}),
\end{aligned} \tag{26}$$

and hence determines $F([\gamma]_X)$ uniquely.

3. To prove the existence of F, we can define F by (26) if we can show that right hand-side of the second equation does not depend on the choice of the decomposition $0 = t_0 < t_1 < \ldots < t_n = 1$ nor on the homeomorphisms $\tau_i : [0, 1] \to [t_{i-1}, t_i]$ and depends only on the homotopy class $[\gamma]_X$. Independence of the choice of homeomorphisms is clear, since different choices are related by

reparametrisations, which are homotopies. Independence of the decomposition follows because for any two finite decompositions of $[0,1]$ there is a finite sub decomposition, and the identities

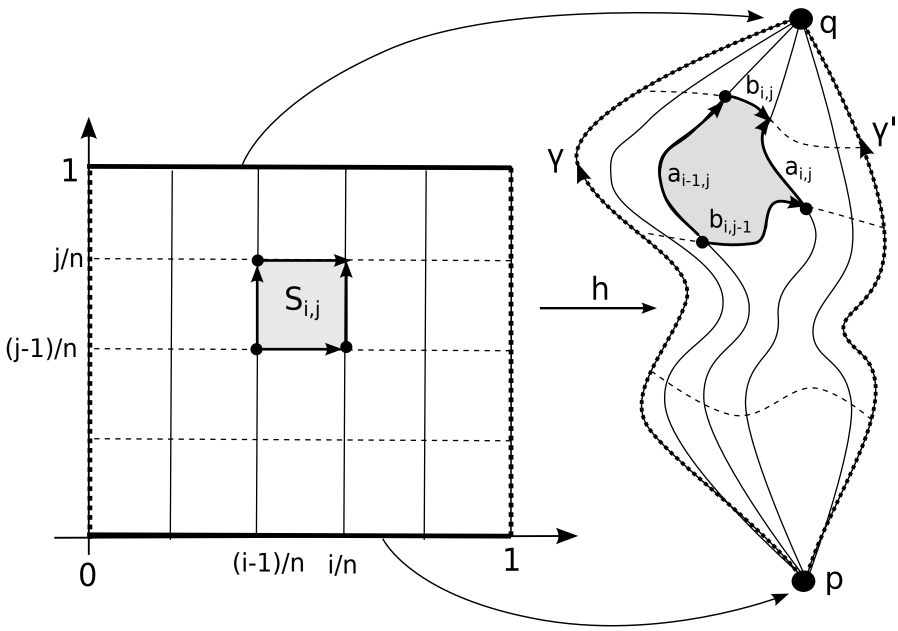
$$F_2([\gamma_i]_2) = F_1([\gamma_i]_1) \quad \text{if } \gamma_i([0,1]) \subset U_1 \cap U_2$$
$$F_{k(i)}([\gamma_i]_{k(i)}) \circ F_{k(i-1)}([\gamma_{i-1}]_{k(i-1)}) = F_{k(i)}([\gamma_i]_{k(i)} \circ [\gamma_{i-1}]_{k(i)}) \text{ for } k(i) = k(i-1)$$

ensure that the associated expressions in (26) agree.

To show that (26) depends only on the homotopy class of γ, consider two paths $\gamma, \gamma' : [0,1] \to X$ with $\gamma(0) = \gamma'(0) = p$, and $\gamma(1) = \gamma'(1) = q$ and a homotopy $h : [0,1] \times [0,1] \to X$ from γ to γ' relative to $\{0,1\}$. Because h is continuous and $h^{-1}(U_1), h^{-1}(U_2)$ form an open cover of the compact set $[0,1] \times [0,1]$, Lebesgue's lemma ensures that there is an $n \in \mathbb{N}$ such that all squares $S_{i,j} = [\frac{i-1}{n}, \frac{i}{n}] \times [\frac{j-1}{n}, \frac{j}{n}]$ with $i, j \in \{1, ..., n\}$ are contained in a set $h^{-1}(U_{k(i,j)})$ with $k(i,j) \in \{1,2\}$. Choose homeomorphisms $\tau_i : [0,1] \to [\frac{i-1}{n}, \frac{i}{n}]$ with $\tau_i(0) = \frac{i-1}{n}$ and $\tau_i(1) = \frac{i}{n}$ and consider for $i, j \in \{0, ..., n\}$ the paths

$$a_{i,j} : [0,1] \to U_{k(i,j)} \cap U_{k(i+1,j)},\ t \mapsto h\left(\tfrac{i}{n}, \tau_j(t)\right), \quad b_{i,j} : [0,1] \to U_{k(i,j)} \cap U_{k(i,j+1)},\ t \mapsto h\left(\tau_i(t), \tfrac{j}{n}\right),$$

with images in, respectively, $h(S_{i,j} \cap S_{i+1,j})$ and $h(S_{i,j} \cap S_{i,j+1})$.



Consider now the two paths from $(0,0)$ to $(1,1)$ in the boundary of $[0,1] \times [0,1]$ pictured below and their images under h, which are obtained by composing the paths $a_{i,j}$ and $b_{i,j}$. By definition, the paths on the left and right boundary of the square correspond to

$$[\gamma]_X = [a_{0,n}]_X \circ \ldots \circ [a_{0,1}]_X, \qquad [\gamma']_X = [a_{n,n}]_X \circ \ldots \circ [a_{n,1}]_X.$$

As h is a homotopy relative to $\{0,1\}$, we have $b_{(j,0)}(t) = p$, $b_{(j,n)}(t) = q$ for $j \in \{0, ..., n\}$, $t \in [0,1]$ - the homotopy classes of the paths along the top and bottom of the square are trivial

$$[\gamma_q]_X = [b_{n,n}]_X \circ \ldots \circ [b_{1,n}]_X, \qquad [\gamma_p]_X = [b_{n,0}]_X \circ \ldots \circ [b_{1,0}]_X.$$

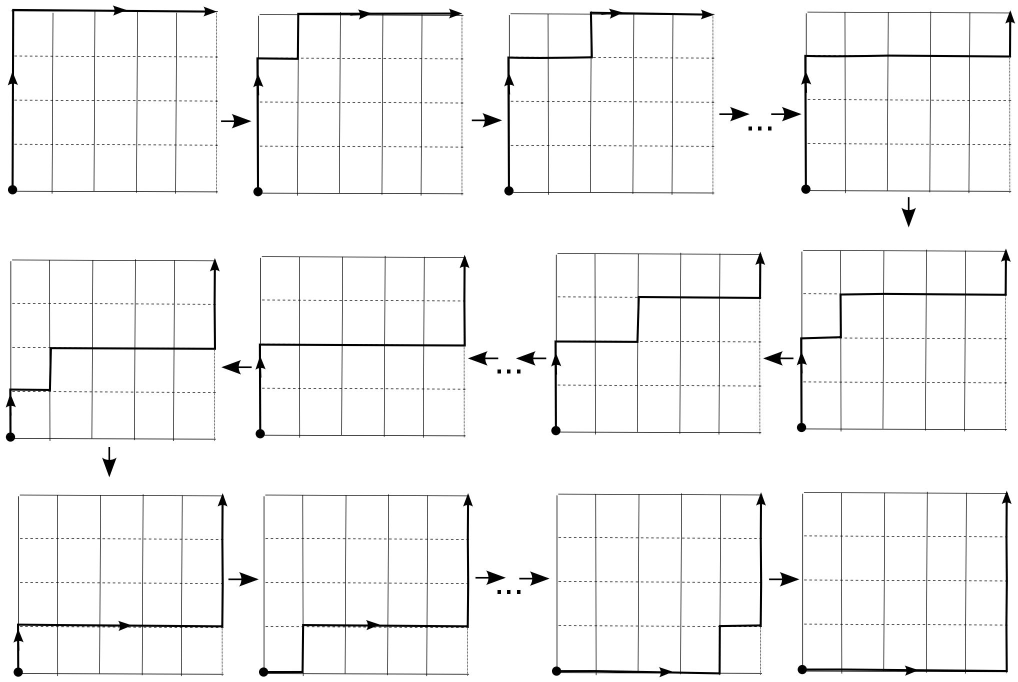
Using (26), we compute

$$
\begin{aligned}
&F([a_{0,n}]_X \circ \ldots \circ [a_{0,1}]_X) = F_{k(0,n)}([a_{0,n}]_{k(0,n)}) \circ \ldots \circ F_{k(0,1)}([a_{0,1}]_{k(0,1)}) \\
&= F([\gamma_q]) \circ F([a_{0,n}]_X \circ \ldots \circ [a_{0,1}]_X) = 1_{F(q)} \circ F([a_{0,n}]_X \circ \ldots \circ [a_{0,1}]_X) \\
&= F_{k(n,n)}([b_{n,n}]_{k(n,n)}) \circ \ldots \circ F_{k(1,n)}([b_{1,n}]_{k(1,n)}) \circ F_{k(0,n)}([a_{0,n}]_{k(0,n)}) \circ \ldots F_{k(0,1)}([a_{0,1}]_{k(0,1)})
\end{aligned}
$$

and

$$
\begin{aligned}
&F([a_{n,n}]_X \circ \ldots \circ [a_{n,1}]_X) = F_{k(n,n)}([a_{n,n}]_{k(n,n)}) \circ \ldots F_{k(n,1)}([a_{n,1}]_{k(n,1)}) \\
&= F([a_{n,n}]_X \circ \ldots \circ [a_{n,1}]_X) \circ F([\gamma_p]) = F([a_{n,n}]_X \circ \ldots \circ [a_{n,1}]_X) \circ 1_{F(p)} \\
&= F_{k(n,n)}([a_{n,n}]_{k(n,n)}) \circ \ldots F_{k(n,1)}([a_{n,1}]_{k(n,1)}) \circ F_{k(n,0)}([b_{n,0}]_{k(n,0)}) \circ \ldots \circ F_{k(1,0)}([b_{1,0}]_{k(1,0)}).
\end{aligned}
$$

By considering the images $h(S_{i,j})$ and the associated paths $a_{i,j}$, $b_{i,j}$, we can transform the two paths in X representing the two terms in the last two lines into each other as indicated in the picture below.



We will show that this does not change the right hand side of equation (26). For this, note that because the square $S_{ij} \subset [0,1] \times [0,1]$ is convex and $h : [0,1] \times [0,1] \to X$ continuous, Example 3.1.2, 3. implies

$$[a_{i,j}]_{k(i,j)} \circ [b_{i,j-1}]_{k(i,j)} = [b_{i,j}]_{k(i,j)} \circ [a_{i-1,j}]_{k(i,j)} \quad \forall i,j \in \{1,...,n\}.$$

As $F_1\Pi_1(j_1) = F_2\Pi_2(j_2)$ one has for all $i,j \in \{0,...,n-1\}$

$$F_{k(i,j)}([a_{i,j}]_{k(i,j)}) = F_{k(i+1,j)}([a_{i,j}]_{k(i+1,j)}), \qquad F_{k(i,j)}([b_{i,j}]_{k(i,j)}) = F_{k(i,j+1)}([b_{i,j}]_{k(i,j+1)}).$$

Using the shorthand notation $A_{i,j} = F_{k(i,j)}([a_{i,j}]_{k(i,j)})$ and $B_{i,j} = F_{k(i,j)}([b_{i,j}]_{k(i,j)})$, we find that these identities imply

$$A_{i,j} \circ B_{i,j-1} = B_{i,j} \circ A_{i-1,j},$$

and we obtain a chain of equations

$$
\begin{aligned}
&F([\gamma]_X) = B_{n,n} \circ \ldots \circ B_{1,n} \circ A_{0,n} \circ \ldots \circ A_{0,1} = B_{n,n} \circ \ldots \circ B_{2,n} \circ A_{1,n} \circ B_{1,n-1} \circ A_{0,n-1} \circ \ldots \circ A_{0,1} \\
&= B_{n,n} \circ \ldots \circ B_{2,n} \circ A_{1,n} \circ A_{1,n-1} \circ B_{1,n-2} \circ A_{0,n-2} \circ \ldots \circ A_{0,1} = \ldots = \\
&= B_{n,n} \circ \ldots \circ B_{2,n} \circ A_{1,n} \circ \ldots A_{1,1} \circ B_{1,0} = B_{n,n} \circ \ldots \circ B_{3,n} \circ A_{2,n} \circ B_{2,n-1} \circ A_{1,n-1} \circ \ldots A_{1,1} \circ B_{1,0} \\
&= \ldots = B_{n,n} \circ \ldots \circ B_{3,n} \circ A_{2,n} \circ \ldots \circ A_{2,1} \circ B_{2,0} \circ B_{1,0} \\
&= \ldots = A_{n,n} \circ \ldots \circ A_{n,1} \circ B_{n,0} \circ \ldots \circ B_{1,0} = F([\gamma']_X)
\end{aligned}
$$

which shows that the right hand side of (26) depends only on the homotopy class of γ. $\square$

Theorem 3.4.1 gives an explicit description of the fundamental groupoid $\Pi_1(X)$ in terms of the fundamental groupoids $\Pi_1(U_1)$, $\Pi_1(U_2)$ and $\Pi_1(U_1 \cap U_2)$. However, this description is not very practical for computations, since it contains as objects all points $x \in X$. If one is interested only in the associated Hom sets, this is not necessary, since for any two objects X, X' in a groupoid $\mathcal{C}$ for which there is a morphism $f : X \to X'$, one has $\mathrm{Hom}_{\mathcal{C}}(X,Y) \cong \mathrm{Hom}_{\mathcal{C}}(X',Y)$ and $\mathrm{Hom}_{\mathcal{C}}(Y,X) \cong \mathrm{Hom}(Y,X')$ for all other objects Y. in other words, one is interested in isomorphism classes of objects in $\Pi_1(X)$ rather than in the objects themselves.

This suggests one could work with a reduced fundamental groupoid, in which only a few objects from each isomorphism class of objects, i. e. from each path-component of the topological space X, are included. By selecting certain objects and considering only the Hom-sets between those selected objects, one obtains a new groupoid, which still contains all relevant information about the topological space X. By applying this procedure to the fundamental groupoids $\Pi_1(X)$, $\Pi_1(U_1)$, $\Pi_1(U_2)$ and $\Pi_1(U_1 \cap U_2)$ in Theorem 3.4.1, one obtains the following reduced version of the Theorem of and van Kampen.

Theorem 3.4.2: Let X be a topological space and $A, U_1, U_2 \subset X$ subspaces with U_1, U_2 open, $U_1 \cup U_2 = X$ and such that every path-component of $U_1, U_2, U_1 \cap U_2$ contains a point in A. Denote for $V \subset X$ by $\Pi_1^A(V)$ the full subcategory of $\Pi_1(V)$ with points in $A \cap V$ as objects and $\mathrm{Hom}_{\Pi_1^A(V)}(a,a') = \mathrm{Hom}_{\Pi_1(V)}(a,a')$ for all $a, a' \in A \cap V$. Then the following diagram is a pushout in the category of groupoids

$$
\begin{tikzcd}
\Pi_1^A(X) & \Pi_1^A(U_1) \arrow[l, "\Pi_1^A(i_1)"'] \\
\Pi_1^A(U_2) \arrow[u, "\Pi_1^A(i_2)"] & \Pi_1^A(U_1 \cap U_2). \arrow[l, "\Pi_1^A(j_2)"] \arrow[u, "\Pi_1^A(j_1)"']
\end{tikzcd}
$$

Proof:
For $V = U_1 \cap U_2, U_1, U_2$, denote by $I_V : \Pi_1^A(V) \to \Pi_1(V)$ the inclusion functors, that assign all points $a, a' \in A \cap V$ and morphisms $[\gamma] : a \to a'$ to themselves. We construct retraction functors $R_V : \Pi_1(V) \to \Pi_1^A(V)$ with $R_V I_V = \mathrm{id}_{\Pi_1^A(V)}$ as follows.

Choose for every point $x \in U_1 \cap U_2$ a point $p_x \in A \cap U_1 \cap U_2$ and path $\gamma^x : [0,1] \to U_1 \cap U_2$ from x to p_x such that for every $a \in U_1 \cap U_2 \cap A$, $p_a = a$ and γ^a is the trivial path based at a. Assign to each point $x \in U_1 \cap U_2$ the point p_x and to each morphism $[\gamma] : x \to y$ in $\Pi_1(U_1 \cap U_2)$ the morphism $R_{U_1 \cap U_2}([\gamma]) = [\gamma^y] \circ [\gamma] \circ [\gamma^x]^{-1} : p_x \to p_y$. Then apply the same procedure to every point of $U_1 \setminus U_2$ and $U_2 \setminus U_1$. This defines functors $R_V : \Pi_1(V) \to \Pi_1^A(V)$ with $R_V I_V = \mathrm{id}_{\Pi_1^A(V)}$ for $V = U_1 \cap U_2, U_1, U_2, X$ such that all quadrilaterals in the following diagram commute

$$\begin{tikzcd}
\Pi_1(X) \arrow[dr, bend left, "R_X"] & & & \Pi_1(U_1) \arrow[lll, bend right, "\Pi_1(i_1)"'] \arrow[dl, bend right, "R_{U_1}"'] \arrow[ddd, phantom] \\
& \Pi_1^A(X) \arrow[ul, bend left, "I_X"] & \Pi_1^A(U_1) \arrow[l, "\Pi_1^A(i_1)"'] \arrow[ur, bend right, "I_{U_1}"'] & \\
& \Pi_1^A(U_2) \arrow[u, "\Pi_1^A(i_2)"] \arrow[dl, bend right, "I_{U_1}"'] & \Pi_1^A(U_1 \cap U_2) \arrow[l, "\Pi_1^A(j_2)"] \arrow[u, "\Pi_1^A(j_1)"'] \arrow[dr, bend left, "I_{U_1 \cap U_2}"] & \\
\Pi_1(U_2) \arrow[uuu, bend left, "\Pi_1(i_2)"] \arrow[ur, bend right, "R_{U_2}"'] & & & \Pi_1(U_1 \cap U_2). \arrow[lll, bend left, "\Pi_1(j_2)"] \arrow[ul, bend left, "R_{U_1 \cap U_2}"] \arrow[uuu, bend right, "\Pi_1(j_1)"']
\end{tikzcd}$$

Then any pair of functors $F_k : \Pi_1^A(U_k) \to Y$ with $F_1 \Pi_1^A(j_1) = F_2 \Pi_1^A(j_2)$ yields a pair of functors $F'_k = F_k R_{U_k} : \Pi_1(U_k) \to Y$ with

$$F'_1 \Pi_1(j_1) = F_1 R_{U_1} \Pi_1(j_1) = F_1 \Pi_1^A(j_1) R_{U_1 \cap U_2} = F_2 \Pi_2^A(j_2) R_{U_1 \cap U_2} = F_2 R_{U_2} \Pi_1(j_2) = F'_2 \Pi_1(j_2).$$

By Theorem 3.4.1, there is a unique morphism $F' : \Pi_1(X) \to Y$ with $F'\Pi_1(i_k) = F'_k$. Then $F = F' I_X : \Pi_1^A(X) \to Y$ satisfies

$$F \Pi_1^A(i_k) = F' I_X \Pi_1^A(i_k) = F' \Pi_1(i_k) I_{U_k} = F'_k I_{U_k} = F_k R_{U_k} I_{U_k} = F_k.$$

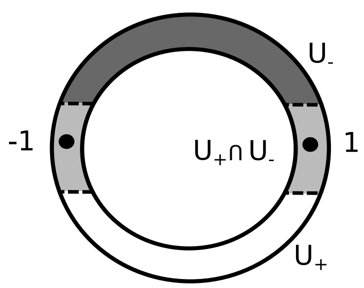
Conversely, for every functor $G : \Pi_1^A(X) \to Y$ with $G\Pi_1^A(i_k) = F_k$, the functor $G' = GR_X : \Pi_1(X) \to Y$ satisfies $G'\Pi_1(i_k) = GR_X \Pi_1(i_k) = G\Pi_1^A(i_k) R_{U_k} = F_k R_{U_k} = F'_k$. The universal property of the pushout in Theorem 3.4.1 implies $G' = F'$ and

$$G = GR_X I_X = G' I_X = F' I_X = FR_X I_X = F.$$

□

Theorem 3.4.2 allows us to compute the fundamental group of the circle in a much simpler way than the method used in Section 3.3. For this, we only need to cover the circle by overlapping open subsets U_1 and U_2 and to choose a collection of points $z \in S^1$ such that every path-component of $U_1 \cap U_2$, U_1 and U_2 contain at least one of these points. This illustrates the power if the abstract approach to fundamental groups based on groupoids.

Example 3.4.3 (Fundamental group of the circle): Consider $X = S^1$ and for fixed $\epsilon \in (0, 1)$ the open sets $U_\pm = \{z \in S^1 : \pm \mathrm{Im}\,(z) < \epsilon\}$.



$$S^1 = U_+ \cup U_- \text{ with } U_\pm = \{z \in S^1 : \pm\mathrm{Re}\,(z) < \epsilon\}.$$

Then the subspaces $U_\pm$ are 1-connected, and $U_+ \cap U_-$ has two contractible path components, one containing 1 and one containing -1. We can therefore choose $A = \{1, -1\}$ and obtain the following groupoids

- $\Pi_1^A(U_1 \cap U_2)$ has two objects, ± 1, and two identity morphisms $1_{\pm 1} : \pm 1 \to \pm 1$.
- $\Pi_1^A(U_\pm)$ have two objects, ± 1, two identity morphisms $1_{\pm 1} : \pm 1 \to \pm 1$, an isomorphism $\alpha_\pm : 1 \to -1$ and its inverse $\alpha_\pm^{-1} : -1 \to 1$.
- $\Pi_1^A(X)$ has two objects, ± 1, two identity morphisms $1_{\pm 1} : \pm 1 \to \pm 1$ and some other morphisms, which are to be determined.

The functors in the pushout in Theorem 3.4.2 are given as follows:

- The functors $\Pi_1^A(j_\pm) : \Pi_1^A(U_+ \cap U_-) \to \Pi_1^A(U_\pm)$ map the objects 1, -1 and the identity morphisms 1_1, 1_{-1} in $\Pi_1^A(U_+ \cap U_-)$ to the objects 1, -1 and the identity morphisms 1_1, 1_{-1} in $\Pi_1^A(U_\pm)$.
- The functors $\Pi_1^A(i_\pm) : \Pi_1^A(U_\pm) \to \Pi_1^A(X)$ map the objects 1, -1 and identity morphisms 1_1, 1_{-1} in $\Pi_1^A(U_\pm)$ to the corresponding objects and identity morphisms in $\Pi_1^A(X)$ and the morphisms $\alpha_\pm : 1 \to -1$ to some morphisms $\beta_\pm = \Pi_1^A(i_\pm)(\alpha_\pm) : 1 \to -1$ in $\Pi_1^A(X)$.

We claim that the morphisms $\beta_\pm : 1 \to -1$ in $\Pi_1^A(X)$ are different and the Hom sets in $\Pi_1^A(X)$ are given by

$$\begin{aligned} &\mathrm{Hom}(1,1) = \{(\beta_-^{-1}\beta_+)^n : n \in \mathbb{Z}\} && \mathrm{Hom}(-1,-1) = \{(\beta_+\beta_-^{-1})^n : n \in \mathbb{Z}\} \\ &\mathrm{Hom}(1,-1) = \{(\beta_+\beta_-)^n\beta_+ : n \in \mathbb{Z}\} && \mathrm{Hom}(-1,1) = \{\beta_+(\beta_-^{-1}\beta_+)^n : n \in \mathbb{Z}\}. \end{aligned} \tag{27}$$

It is clear that the objects ± 1 together with these Hom sets define a groupoid $\mathcal{B}$. A functor $F : \mathcal{B} \to \mathcal{C}$ into another groupoid $\mathcal{C}$ is determined uniquely by the objects $F(\pm 1)$ in $\mathcal{C}$ and the morphisms $F(\beta_\pm) : F(1) \to F(-1)$, since all morphisms in $\mathcal{B}$ are obtained by composing β_+ and β_-. Conversely, for any two objects $C_\pm$ in $\mathcal{C}$ and two morphisms $f_\pm : C_+ \to C_-$ in $\mathcal{C}$ there is a functor $F : \mathcal{B} \to \mathcal{C}$ with $F(\pm 1) = C_\pm$ and $F(\beta_\pm) = f_\pm$.

Similarly, a pair of functors $F_\pm : \Pi_1(U_\pm) \to \mathcal{C}$ into another groupoid $\mathcal{C}$ that satisfies the condition $F_+\Pi_1^A(j_+) = F_-\Pi_1^A(j_-)$ is determined uniquely by the objects $F_+(1) = F_-(1)$, $F_+(-1) = F_-(-1)$ and the morphisms $F(\alpha_\pm) : F_\pm(1) \to F_\pm(-1)$ in $\mathcal{C}$. Conversely, for any two objects $C_\pm$ and morphisms $f_\pm : C_+ \to C_-$ in $\mathcal{C}$ there is such a pair of functors with $F_+(\pm 1) = F_-(\pm 1) = C_\pm$ and $F_\pm(\alpha_\pm) = f_\pm$.

This shows that for every pair of functors $F_\pm : \Pi_1(U_\pm) \to \mathcal{C}$ with $F_+\Pi_1^A(j_+) = F_-\Pi_1^A(j_-)$, there is exactly one functor $\mathcal{B} \to \mathcal{C}$ with with $F(1) = F_\pm(1)$, $F(-1) = F_\pm(-1)$ and $F(\beta_\pm) = F_\pm(\alpha_\pm)$. This is equivalent to the conditions $F\Pi_1^A(i_\pm) = F_\pm$, and by the uniqeness of the pushout it follows that $\Pi_1^A(X)$ is isomorphic to $\mathcal{B}$. In particular, we have $\pi_1(1, S^1) = \mathrm{Hom}_{\Pi_1^A(X)}(1,1) \cong \mathbb{Z}$.

Another important consequence of Theorem 3.4.2 arises when the subspaces U_1, U_2 and their overlap $U_1 \cap U_2$ of the covering subsets are path connected. In this case, it is possible to choose a subset $A \subset X$ which consists of a single point $x \in U_1 \cap U_2$. We can then replace the groupoids $\Pi_1^A(V)$ in Theorem 3.4.2 by fundamental groups $\pi_1(x, V)$, all functors $\Pi_1^A(f) : \Pi_1^A(V) \to \Pi_1^A(V')$ by group homomorphisms $\pi_1(x, f) : \pi_1(x, V) \to \pi_1(x, V')$, and the pushout becomes a pushout in the category of groups. As pushouts are unique up to isomorphisms, this implies that the fundamental group $\pi_1(x, X)$ is given as a quotient of the free product of $\pi_1(x, U_1)$ and $\pi_1(x, U_2)$ by the normal subgroup generated by the inclusions of $\pi_1(x, U_1 \cap U_2)$ (see Lemma 2.1.21).

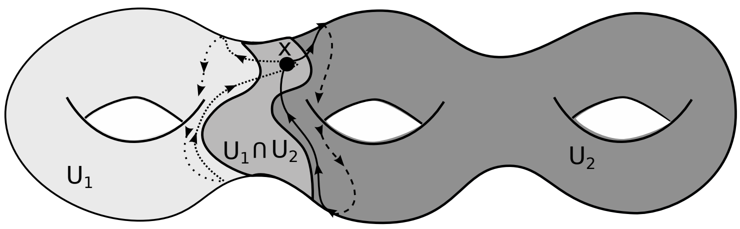
Corollary 3.4.4 (**Seifert-van Kampen, fundamental groups**):
Let X be a path-connected topological space, $U_1, U_2 \subset X$ open such that $X = U_1 \cup U_2$ and U_1, U_2, $U_1 \cap U_2$ are path connected. Denote by $i_k : U_k \to X$ and $j_k : U_1 \cap U_2 \to U_k$ the inclusion maps. Then for all $x \in U_1 \cap U_2$, the following diagram of groups and group homomorphisms is a pushout in *Grp*

$$\begin{tikzcd}
\pi_1(x,X) & \pi_1(x,U_1) \arrow[l, "\pi_1(x,i_1)"'] \\
\pi_1(x,U_2) \arrow[u, "\pi_1(x,i_2)"] & \pi_1(x,U_1\cap U_2). \arrow[l, "\pi_1(x,j_2)"] \arrow[u, "\pi_1(x,j_1)"']
\end{tikzcd}$$

This implies $\pi_1(x, X) \cong (\pi_1(x, U_1) \star \pi_1(x, U_2))/N$ where

$$N = \langle i_{1,12}([j_1 \circ \gamma]) \cdot i_{2,12}([j_2 \circ \overline{\gamma}]) : \ [\gamma] \in \pi_1(x, U_1 \cap U_2)\rangle$$

is the normal subgroup of $\pi_1(x, U_1) \star \pi_1(x, U_2)$ generated by the the elements $i_{1,12}([j_1 \circ \gamma]) \cdot i_{2,12}([j_2 \circ \overline{\gamma}])$ for paths γ in $U_1 \cap U_2$ and $i_{j,12} : \pi_1(x, U_j) \to \pi_1(x, U_1) \star \pi_1(x, U_2)$ denote the inclusions for the free product of groups.

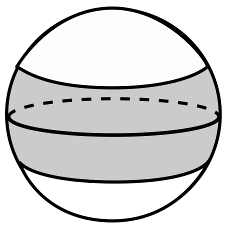


The theorem of Seifert and van Kampen for fundamental groups: The dotted and the solid path in, respectively, U_1 and U_2 represent the same element of $\pi_1(x, U_1 \cap U_2)$ and of $\pi_1(x, X)$. They are identified by taking the quotient $\pi_1(x, U_1) \star \pi_1(x, U_2)/N$.

We will see in the following that this Corollary of the Theorem of and van Kampen allows one to directly compute the fundamental groups for many topological spaces. The key point is to choose the sets U_1 and U_2 in such a way that their fundamental groups and the one of their overlap $U_1 \cap U_2$ become as simple as possible. The simplest situation arises, when the fundamental groups of U_1 and U_2 are trivial. In this case, X has a trivial fundamental group, since the free product of two trivial groups is trivial.

Example 3.4.5: For $n \geq 2$, the fundamental group of the n-sphere S^n is trivial.

To show this, choose $\epsilon \in (0, 1)$ and consider the open sets $U_\pm = \{x \in S^n : \pm x_1 < \epsilon\}$. Then $S^n = U_+ \cup U_-$, and the sets $U_\pm$, $U_+ \cap U_-$ are path-connected. As $U_\pm \simeq D^n \simeq \{p\}$, one has $\pi_1(U_\pm) = \{e\}$ and Corollary 3.4.4 implies $\pi_1(S^n) \cong \pi_1(D^n) \star \pi_1(D^n)/N = \{e\}$.



Covering of th n-sphere by open sets $U_\pm = \{x \in S^n : \pm x_1 < \epsilon\}$.

This example also shows that the condition that $U_1 \cap U_2$ is path-connected in Theorem 3.4.4 is necessary. In the case of the circle, we also have $U_\pm \simeq D^1 \simeq \{p\}$, but $U_+ \cap U_- \simeq \{1\}\dot{\cup}\{-1\}$ is not path-connected and $\pi_1(S^1) \cong \mathbb{Z}$.

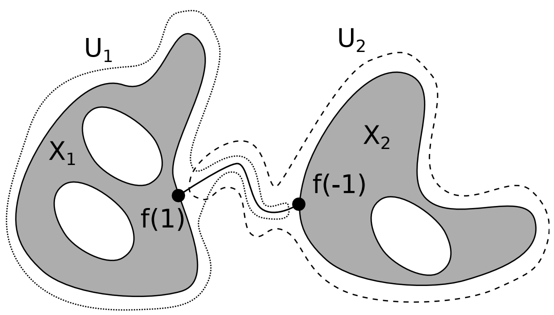
Another situation, in which it is very simple to compute the fundamental group $\pi_1(x, X)$ from the pushout in Corollary 3.4.4 is the case where the intersection $U_1 \cap U_2$ is 1-connected. In this case, the normal subgroup generated by the inclusions of $\pi_1(x, U_1 \cap U_2) \cong \{e\}$ is trivial, and the associated quotient reduces to the free product of the groups $\pi_1(x, U_1) \star \pi_1(x, U_2)$.

Corollary 3.4.6: Let X be a topological space, $U_1, U_2 \subset X$ open and path-connected such that $X = U_1 \cup U_2$ and $U_1 \cap U_2$ is 1-connected. Then for all $x \in U_1 \cap U_2$, one has

$$\pi_1(x, X) = \pi_1(x, U_1) \star \pi_1(x, U_2).$$

Corollary 3.4.6 can be applied to a wide class of examples of topological spaces and combined with familiar constructions such as the attaching of n-cells, deformation retracts and wedge sums.

Example 3.4.7: Let X_1, X_2 be path-connected topological spaces and $Y = D^1 \cup_f (X_1 + X_2)$, where $f : \{1, -1\} \to X_1 + X_2$ satisfies $f(1) \in i_1(X_1)$, $f(-1) \in i_2(X_2)$. Then $\pi_1(Y) \cong \pi_1(X_1) \star \pi_1(X_2)$.



This can be shown as follows. Denote by $i_j : X_j \to X_1 + X_2$, $\iota_{12} : X_1 + X_2 \to Y$, $\iota_{D^1} : D^1 \to Y$ the inclusion maps and by $\pi : D^1 + X_1 + X_2 \to Y$ the canonical surjection. Choose $U_j = \iota_{D^1}((-1, 1)) \cup \iota_{12} \circ i_j(X_j)$. Then $U_1 \cup U_2 = Y$ and $U_j \subset Y$ are open since $\pi^{-1}(U_1) = (-1, 1] \cup X_1$ and $\pi^{-1}(U_2) = [-1, 1) \cup X_2$ are open in $D^1 + X_1 + X_2$. It also follows directly that Y, U_1 and U_2 are path-connected. As $\iota_{D^1}|_{(-1,1)} : (-1, 1) \to Y$ is an embedding, $U_1 \cap U_2 = \iota_{D^1}((-1, 1)) \simeq (-1, 1)$ is 1-connected and Corollary 3.4.6 proves the claim.

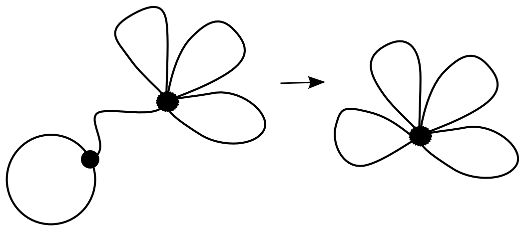
Example 3.4.8 (Bouquets):

For $n \in \mathbb{N}$, a **bouquet with** n **circles** is the wedge sum $\vee^n S^1 = S^1 \vee S^1 \vee \ldots \vee S^1$ of n circles with respect to a single point, as pictured below. Its fundamental group is

$$\pi_1(\vee^n S^1) \cong F_n = \mathbb{Z} \star \ldots \star \mathbb{Z}.$$

This follows since the wedge sum $\vee^n S^1$ is homotopy equivalent to the topological space obtained by attaching a 1-cell to the disjoint sum $S^1 + \vee^{n-1} S^1$ as in Example 3.4.7. This implies

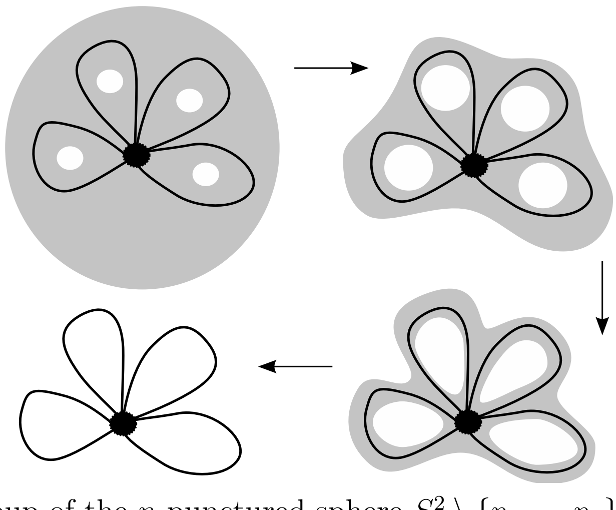
$$\pi_1(\vee^n S^1) \cong \mathbb{Z} \star \pi_1(\vee^{n-1} S^1) = \mathbb{Z} \star \mathbb{Z} \star \pi_1(\vee^{n-2} S^1) = \ldots = \mathbb{Z} \star \ldots \star \mathbb{Z} = F_n.$$



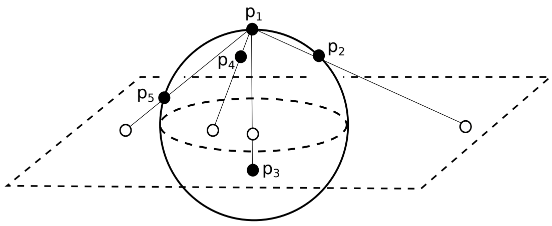
It is clear that Example 3.4.8 not only allows us to compute the fundamental groups of bouquets but also of all topological spaces that are homotopy equivalent to bouquets. In particular, this includes the following examples, which motivates the consideration given to them and, more generally, wedge sums.

Example 3.4.9:

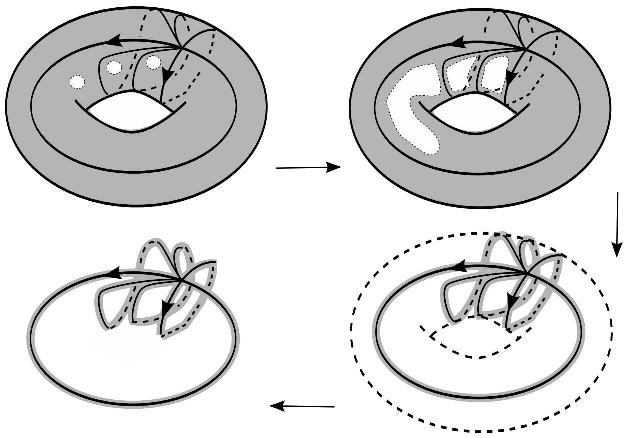
1. The fundamental group of the n-punctured plane $\mathbb{R}^2 \setminus \{p_1, ..., p_n\}$ or of the n-punctured disc $D^2 \setminus \{p_1, ..., p_n\}$ is the free group F_n with n generators. This follows because the bouquet with n circles is a deformation retract of $\mathbb{R}^2 \setminus \{p_1, ..., p_n\}$ and of $D^2 \setminus \{p_1, ..., p_n\}$, as pictured below.



2. The fundamental group of the n-punctured sphere $S^2 \setminus \{p_1, ..., p_n\}$ is the free group F_{n-1} with $n-1$ generators. This follows because the stereographic projection with respect to p_1 defines a homeomorphism $S^2 \setminus \{p_1, ..., p_n\} \to \mathbb{R}^2 \setminus \{p_2, .., p_n\}$.



3. The fundamental group of the n-punctured torus $T \setminus \{p_1, ..., p_n\}$ is the free group F_{n+1} with $n+1$ generators. Again, this follows because the bouquet with $n+1$ circles is a deformation retract of $T \setminus \{p_1, ..., p_n\}$.



The fact that the fundamental group $\pi_1(S^n)$ is trivial for all $n \geq 2$ also allows one to apply Corollary 3.4.6 to the attaching of n-cells with $n \geq 3$. Open coverings of spaces obtained by attaching an n-cell to a topological space X typically give rise to open sets U_1, U_2 whose intersection is homotopy equivalent to an $n-1$-sphere. If $n \geq 3$, this $n-1$-sphere is 1-connected and Corollary 3.4.6 yields the following result.

Lemma 3.4.10: Let X be a path-connected topological space, $n \geq 3$ and $f : \partial D^n = S^{n-1} \to X$ continuous. Then $\pi_1(D^n \cup_f X) \cong \pi_1(X)$.

Proof:
The pushout diagram for attaching D^n to X with $f : S^{n-1} \to X$ is given by

$$\begin{tikzcd} D^n \cup_f X & X \arrow[l, "\iota_X"'] \\ D^n \arrow[u, "\iota_{D^n}"] & S^{n-1}. \arrow[l, "i"] \arrow[u, "f"'] \end{tikzcd}$$

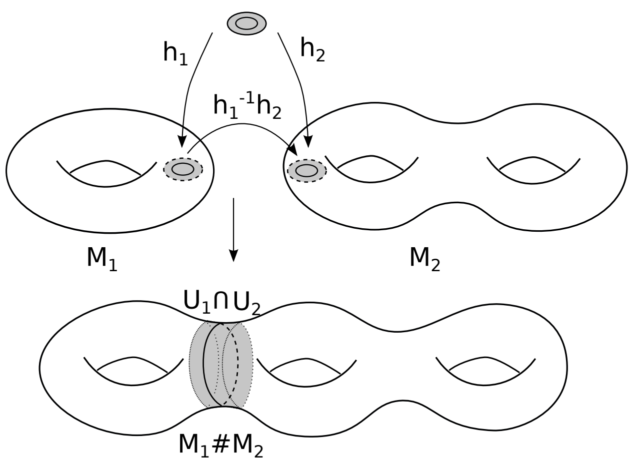
Consider the subsets $U_1 = \iota_{D^n}(\mathring{D}^n)$ and $U_2 = (D^n \cup_f X) \setminus \{\iota_{D^n}(0)\}$. Then U_1 is open and contractible since $\pi^{-1}(U_1) = i_{D^n}(\mathring{D}^n) \approx \mathring{D}^n \simeq \{p\}$, which implies $\pi_1(U_1) \cong \{e\}$. The set U_2 is open since $\pi^{-1}(U_2) = i_{D^n}(D^n \setminus \{0\}) \cup i_X(X)$ and $i_{D^n}(D^N \setminus \{0\}) \subset D^n$ and X are open. Moreover, U_2 is homeomorphic to the mapping cylinder $C_g = (S^{n-1} \times (0,1]) \cup_f X$, which implies $U_2 \simeq X$ and $\pi_1(U_2) \cong \pi_1(X)$ (see Example 3.1.11). The intersection $U_1 \cap U_2$ is homotopy equivalent to S^{n-1}, since $\mathring{D}^n \setminus \{0\} \simeq S^{n-1}$ and $\iota_{D^n}|_{\mathring{D}^n} : \mathring{D}^n \to D^n \cup_f X$ is an embedding. Hence $U_1 \cap U_2$ is 1-connected by Corollary 3.4.5, and Corollary 3.4.6 implies

$\pi_1(D^n \cup_f X) \cong \pi_1(D^n) \star \pi_1(X) \cong \pi_1(X)$. □

In particular, we can use this Lemma to compute the fundamental group of a finite path-connected CW-complex $X = \bigcup_{k=0}^n X^k$. As X is path-connected, X^k is path-connected for all $k \geq 1$. As X^k is obtained from X^{k-1} by attaching k-cells, it follows with Lemma 3.4.10 that $\pi_1(X) = \pi_1(X^k) = \pi_1(X^{k-1}) = \pi_1(X^2)$ for all $k \geq 3$. This shows that all information about the fundamental group of X is encoded in its 2-skeleton.

Corollary 3.4.11: Let $X = \bigcup_{k=0}^n X^k$ be a finite path-connected CW-complex. Then $\pi_1(X) = \pi_1(X^2)$.

A similar phenomenon occurs when one considers the connected sum of manifolds. In this case, two n-dimensional manifolds M_1, M_2 are glued along an $(n-1)$-sphere, which is obtained by cutting two n-discs from M_1 and M_2 and identifying their boundaries. By covering the connected sum with open subspaces $U_1, U_2 \subset M_1\#M_2$ that overlap along this $(n-1)$-sphere, one obtains again that $U_1 \cap U_2$ is one-connected if $n \geq 3$. Corollary 3.4.6 then shows that the fundamental group of the connected sum $M_1\#M_2$ of two n-dimensional manifolds is the free product of the fundamental groups $\pi_1(M_1)$ and $\pi_1(M_2)$.



Lemma 3.4.12: Let M_1, M_2 be path-connected topological manifolds of dimension $\dim(M_1) = \dim(M_2) \geq 3$. Then $\pi_1(M_1\#M_2) \cong \pi_1(M_1) \star \pi_1(M_2)$.

Proof:
Choose homeomorphisms $h_i : U_{2\epsilon} = \{x \in \mathbb{R}^n : ||x|| < 2\epsilon\} \to h_i(U_{2\epsilon}) \subset M_i$ and attach $N_1 := M_1 \setminus h_1(U_\epsilon)$ to $N_2 := M_2 \setminus h_2(U_\epsilon)$ with the homeomorphism $h_2 \circ (h_1|_{\partial U_\epsilon})^{-1} : h_1(\partial U_\epsilon) \to h_2(\partial U_\epsilon)$. This gives the pushout diagram

$$\begin{tikzcd} M_1\#M_2 & N_2 = M_2 \setminus h_2(U_\epsilon) \arrow[l, "\iota_2"'] \\ N_1 = M_1 \setminus h_1(U_\epsilon) \arrow[u, "\iota_1"] & h_1(\partial U_\epsilon) \simeq S^{n-1}, \arrow[l, "i_{\partial U_\epsilon}"] \arrow[u, "h_2\circ(h_1|_{\partial U_\epsilon})^{-1}"'] \end{tikzcd}$$

in which $\iota_i : N_i \to M_1\#M_2$ are embeddings since $h_2 \circ (h_1|_{\partial U_\epsilon})^{-1} : h_1(\partial U_\epsilon) \to h_2(\partial U_\epsilon)$ is a homeomorphism. Denote by $\pi : N_1 + N_2 \to M_1\#M_2$ the canonical surjection. Then the sets

$$U_1 = \iota_1(N_1) \cup \iota_2 \circ h_2(U_{2\epsilon} \setminus U_\epsilon) \qquad U_2 = \iota_2(N_2) \cup \iota_1 \circ h_1(U_{2\epsilon} \setminus U_\epsilon).$$

are open, since $\pi^{-1}(U_i) \approx N_i + h_j(U_{2\epsilon} \setminus U_\epsilon)$ for $i \neq j$ and N_i, $h_j(U_{2\epsilon} \setminus U_\epsilon)$ are open in, respectively N_i and N_j. Analogously, it follows that the set $U_1 \cap U_2 = \iota_1 \circ h_1(U_{2\epsilon} \setminus U_\epsilon) \cup \iota_2 \circ h_2(U_{2\epsilon} \setminus U_\epsilon)$ is open. It is also apparent that U_1, U_2 and $U_1 \cap U_2$ are path-connected and $U_1 \cap U_2 \simeq h_i(U_{2\epsilon} \setminus U_\epsilon) \simeq S^{n-1}$, which is 1-connected for $n \geq 3$ by Corollary 3.4.5. Corollary 3.4.6 then implies $\pi_1(M_1\#M_2) = \pi_1(N_1) \star \pi_1(N_2)$. As M_i is obtained by gluing an n-cell to N_i with the continuous map $f_i : S^{n-1} \to h_i(\partial U_\epsilon)$, $x \mapsto h_i(\epsilon x)$ and $n \geq 3$, we have $\pi_1(M_i) \cong \pi_1(D^n \cup_{f_i} N_i) \cong \pi_1(N_i)$, which proves the claim. $\square$

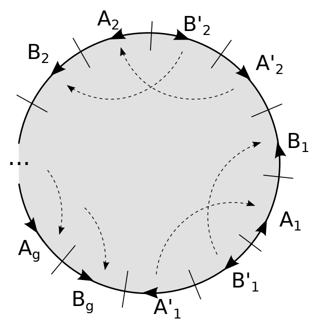
It remains to consider the connected sum $M_1\#M_2$ two path-connected two-dimensional topological manifolds M_1 and M_2. In this case, we can choose open subsets $U_1, U_2 \subset M_1\#M_2$ with $U_1 \cup U_2 = M_1\#M_2$ as in Lemma 3.4.12 and again apply Corollary 3.4.4 to compute the fundamental group $\pi_1(x, X)$ for $x \in U_1 \cap U_2$. However, the the conclusion that $\pi_1(U_1 \cap U_2)$ is trivial no longer holds. This means that $\pi_1(M_1\#M_2)$ is no longer a free product. Instead, Corollary 3.4.4 yields a presentation of $\pi_1(M_1\#M_2)$ in terms of generators and relations. We compute this presentation explicitly for surfaces of genus g, i. e. connected sums $M = T^{\#g} = T\#T\#\ldots\#T$.

Theorem 3.4.13: For $g \in \mathbb{N}$, the fundamental group of an oriented genus g surface $T^{\#g} = (S^1 \times S^1)\#\ldots\#(S^1 \times S^1)$ has a presentation

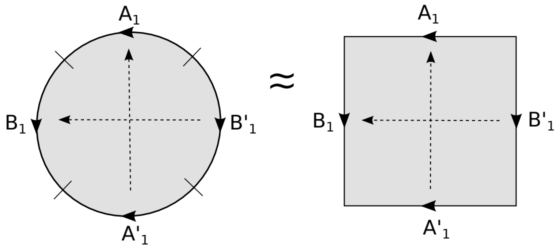
$$\pi_1(T^{\#g}) = \langle a_1, b_1, ..., a_g, b_g \mid [b_g, a_g] \cdot \cdots \cdot [b_2, a_2] \cdot [b_1, a_1] = 1 \rangle,$$

where $[b, a] = b \cdot a \cdot b^{-1} \cdot a^{-1}$ is the group commutator.

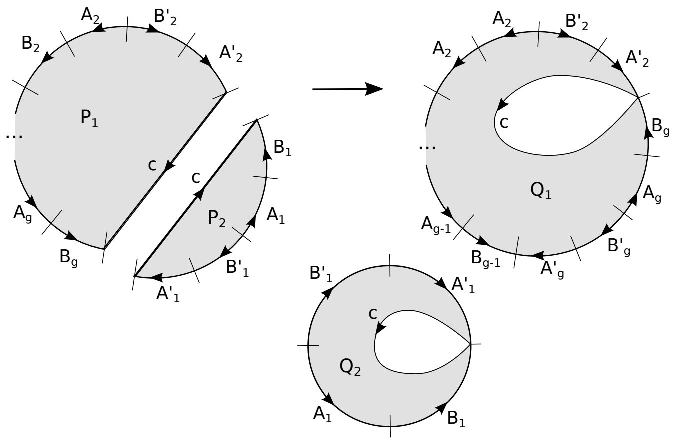
Proof:
1. We first show that $T^{\#g}$ is homeomorphic to a $D^2/\sim$, where the equivalence relation $\sim$ identifies points at the boundary of $S^1 = \partial D^2$ as shown below.



This follows by induction over g. For $g = 1$, there is a homeomorphism $\tilde{h} : D^2 \to [0,1] \times [0,1]$, which induces a homeomorphism $D^2/\sim \to S^1 \times S^1$, as shown below.



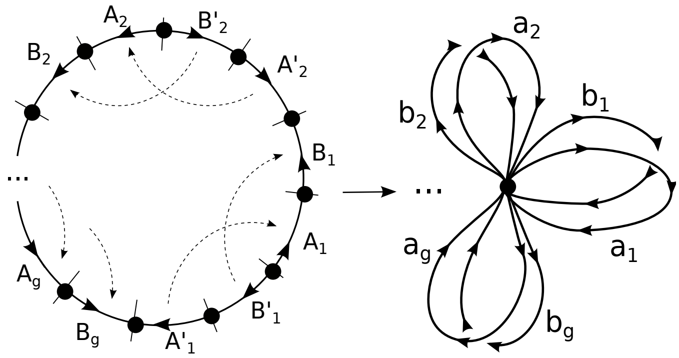
Suppose the statement is shown for $T^{\#k}$ with $k \leq g-1$. Then we can cut the disc D^2 along a straight line labeled with c into two pieces P_1 and P_2 as shown below. As the two endpoints of c on the boundary $S^1 = \partial D^2$ are identified, the quotients $P_1/\sim$ and $P_2/\sim$ are homeomorphic to the quotients of $Q_1/\sim$ and $Q_2/\sim$ with the same boundary identification and a disc bordered by c removed from them. By induction hypothesis, $Q_1/\sim$ is homeomorphic to $M_1 = T^{\#(g-1)} \setminus h_1(U_\epsilon)$ and $Q_2/\sim$ to $M_2 = T \setminus h_2(U_\epsilon)$, where $h_1 : U_{2\epsilon} \to h_1(U_{2\epsilon}) \subset T^{\#(g-1)}$ and $h_2 : U_{2\epsilon} \to h_2(U_{2\epsilon}) \subset T$ are homeomorphisms. By attaching M_1 to M_2 with the homeomorphism $h_2 \circ (h_1|_{\partial U_\epsilon})^{-1} : h_1(\partial U_\epsilon) \to h_2(\partial U_\epsilon)$, we obtain $T^{\#g} = M_1 \# M_2 \approx D^2/\sim$.



2. We compute the fundamental group of $T^{\#g} = D^2/\sim$ with the Theorem of van Kampen. For this we choose the open and subspaces

$$U_1 = \mathring{D}^2/\sim \approx \mathring{D}^2 \simeq \{p\}, \qquad U_2 = D^2 \setminus \{0\}/\sim$$

with $T^{\#g} = U_1 \cup U_2$ and a point $x \in U_1 \cap U_2$. Then we have $U_1 \cap U_2 \approx \mathring{D}^2 \setminus \{0\} \simeq S^1$ and, consequently, $\pi_1(x, U_1 \cap U_2) \cong \pi_1(S^1) \cong \mathbb{Z}$. As $S^1 = \partial D^2$ is a deformation retract of $D^2 \setminus \{0\}$, there is a retraction $r : D^2 \setminus \{0\} \to S^1$ and a homotopy $h : [0,1] \times D^2 \setminus \{0\} \to D^2 \setminus \{0\}$ from $i_{S^1} \circ r$ to $\mathrm{id}_{D^2 \setminus \{0\}}$ relative to S^1. As the equivalence relation $\sim$ only identifies different points $x \neq x' \in D^2 \setminus \{0\}$ if both $x, x' \in S^1$, the universal property of quotients yields a retraction $r_\sim : U_2 \to S^1/\sim$ and a homotopy $h_\sim : [0,1] \times U_2 \to U_2$ from $i_{S^1/\sim} \circ r_\sim$ to id_{U_2}. This implies $U_2 \simeq S^1/\sim$. As $S^1/\sim$ is a bouquet with $2g$ circles, it follows from Example 3.4.8 that $\pi_1(x, U_2) \cong F_{2g}$.



Corollary 3.4.4 then yields the following pushout in Grp

$$\begin{tikzcd}
\pi_1(x,X) & \pi_1(x,U_1)=\{e\} \arrow[l, "\pi_1(x,i_1)"', "e\mapsto e"] \\
F_{2g}\cong\pi_1(x,U_2) \arrow[u, "\pi_1(x,i_2)"] & \pi_1(x,U_1\cap U_2)\cong\mathbb{Z}. \arrow[l, "\pi_1(x,j_2)"] \arrow[u, "\mathbb{Z}\mapsto e", "\pi_1(x,j_1)"']
\end{tikzcd}$$

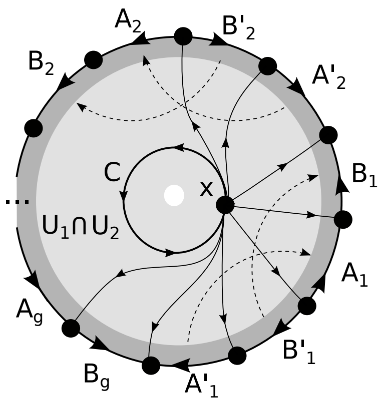
As the diagram commutes, the group homomorphism $\pi_1(x,i_2)\circ\pi_1(x,j_2)=\pi_1(x,i_1)\circ\pi_1(x,j_1):\pi_1(x,U_1\cap U_2)\to\pi_1(x,X)$ is trivial. Moreover, as any group homomorphism $\pi_1(x,U_1)\cong\{e\}\to H$ into a group H is trivial, the universal property of this pushout states that for any group homomorphism $\phi_2:\pi_1(x,U_2)\cong F_{2g}\to H$ for which $\phi_2\circ\pi_1(x,j_2):\mathbb{Z}\to H$ is trivial, there is a unique group homomorphism $\phi:\pi_1(x,X)\to H$ with $\phi\circ\pi_1(x,i_2)=\phi_2$. This implies

$$\pi_1(x,X)\cong\frac{\pi_1(x,U_2)}{\pi_1(x,j_2)(\pi_1(x,U_1\cap U_2))}\cong F_{2g}/\mathbb{Z}.$$

To obtain a presentation of $\pi_1(x,X)$ we need to determine the group homomorphism

$$\pi_1(x,j_2):\pi_1(x,U_1\cap U_2)\cong\mathbb{Z}\to\pi_1(x,U_2)\cong F_{2g}.$$

For this, we consider the paths $A_i,A_i',B_i,B_i':[0,1]\to D^2\setminus\{0\}$ pictured below and their images $a_i=p\circ A_i, b_i=p\circ B_i:[0,1]\to U_2$ under the canonical surjection $p:D^2\setminus\{0\}\to U_2$. Then the homotopy classes of $a_i,b_i:[0,1]\to U_2$ freely generate $\pi_1(x,U_2)=\langle a_1,b_1,\ldots,a_g,b_g\rangle=F_{2g}$.



As $U_1\cap U_2\simeq S^1$, the fundamental group $\pi_1(x,U_1\cap U_2)$ is isomorphic to $\pi_1(S^1)\cong\mathbb{Z}$ and generated by the homotopy class of the path $c=p\circ(\frac{1}{2}\exp|_{[0,1]}):[0,1]\to U_1\cap U_2$. As the homotopy class of the associated path $C = \frac{1}{2}\exp|_{[0,1]} : [0,1] \to D^2 \setminus \{0\}$ is given by

$$\begin{aligned}[C] &= [B_g \star A_g \star \overline{B'_g} \star \overline{A'_g} \star \ldots B_1 \star A_1 \star \overline{B'_1} \star \overline{A'_1}] \\ &= [B_g] \circ [A_g] \circ [B'_g]^{-1} \circ [A'_g]^{-1} \circ \ldots \circ [B_1] \circ [A_1] \circ [B'_1]^{-1} \circ [A'_1]^{-1},\end{aligned}$$

it follows that the group homomorphism $\pi_1(x, j_2) : \pi_1(x, U_1 \cap U_2) \to \pi_1(x, U_2)$ is given by

$$\pi_1(x, j_2) : [c] \mapsto [b_g] \circ [a_g] \circ [b_g]^{-1} \circ [a_g]^{-1} \circ \ldots \circ [b_1] \circ [a_1] \circ [b_1]^{-1} \circ [a_1]^{-1},$$

and we obtain the presentation

$$\pi_1(x, X) \cong \pi_1(x, U_2)/\pi_1(x, j_2)(\pi_1(x, U_1 \cap U_2) = \langle a_1, b_1, ..., a_g, b_g \,|\, [b_g, a_g] \cdots [b_1, a_1] = 1\rangle.$$

□

Example 3.4.14: For the torus, Theorem 3.4.13 yields $\pi_1(T) = \langle a, b | \, [b, a] = 1\rangle \cong \mathbb{Z}\times\mathbb{Z}$, while the torus $T \setminus D$ with a disc removed has the fundamental group $\pi_1(T \setminus D) = \langle a, b\rangle = F_2$. On the punctured torus, the group element $[b, a]$ corresponds to the curve that winds counterclockwise along the disc, as pictured in Figure 3.4. If the hole is patched by attaching a 2-cell along the dashed circle, then one obtains a torus in which the correponding curve is contractible.

Corollary 3.4.15: For $g_1 \neq g_2$, the surfaces $T^{\#g_1}$ and $T^{\#g_2}$ are not of the same homotopy type and, consequently, not homeomorphic.

Proof:
If two path-connected topological spaces X_1, X_2 are of the same homotopy type, then their fundamental groups are isomorphic $\pi_1(X_1) \cong \pi_1(X_2)$. This implies that their abelianisations $\mathrm{Ab}(\pi_1(X_i)) = \pi_1(X_i)/[\pi_1(X_i), \pi_1(X_i)]$ are isomorphic, too. However, by Theorem 3.4.13 we have $\pi_1(T^{\#g}) = \langle a_1, b_1, ..., a_g, b_g | \, [b_g, a_g] \cdots [b_1, a_1] = 1\rangle$, which implies

$$\mathrm{Ab}(T^{\#g}) = \underbrace{\mathbb{Z} \times \mathbb{Z} \times \ldots \times \mathbb{Z}}_{2g\,\times}$$

and hence $\pi_1(T^{\#g_1}) \not\cong \pi_1(T^{\#g_2})$ for $g_1 \neq g_2$. □

By attaching m 2-cells to a bouquet with n circles, one can construct a topological space X with fundamental group $\pi_1(X) \cong G$ for every finitely presented group G. The statement holds more generally for groups with a presentation in terms of infinitely many generators and relations, but the proof becomes more complicated due to topological subtleties of CW-complexes.

The principal idea is to start with a topological space Y whose fundamental group is a free group, for instance a bouquet, and to attach 2-cells with the relations as attaching maps. Each 2-cell attached with a continuous map $f : S^1 \to Y$ "destroys" the corresponding element $[f] \in \pi_1(Y)$, that is, identifies it with the unit element, and one obtains a topological space with with fundamental group $\pi_1(X) \cong G$.

Proposition 3.4.16: Let $G = \langle a_1, ..., a_n \,|\, r_1 = 1, ..., r_m = 1\rangle$ be a presentation of a group G with generators $a_1, ..., a_n$ and relations $r_1, ..., r_m$. There is a path-connected topological space X with $\pi_1(X) \cong G$.

Proof:
We prove the claim by induction over the number m of relations. For $m = 0$ we can choose X as a bouquet with n circles, since $\pi_1(X) = F_n = \langle a_1, ..., a_n\rangle$ by Example 3.4.8.

Suppose the statement is shown for $m \leq k$ and $G = \langle a_1, ..., a_n \,|\, r_1, ..., r_{k+1}\rangle$. Consider the group $H = \langle a_1, ..., a_n \,|\, r_1, .., r_k\rangle$ with the canonical surjection $\pi_H : F_n \to H = F_n/\langle\{r_1, ..., k_k\}\rangle$. Then by induction hypothesis, there is a path-connected topologuical space Y with $\pi_1(Y) \cong H$. Choose a point $y \in Y$ and a closed path γ based at y with $[\gamma] = \pi_H(r_{k+1})$. As $\gamma(0) = \gamma(1) = y$, the path γ induces a continuous map $f : S^1 \to Y$ that can be used to attach a 2-cell to Y.

We consider the topological space $X := D^2 +_f Y$ and compute its fundamental group with the thorem of Seiffert and van Kampen. Consider the open and path-connected subspaces

$$U_1 = \iota_1(\mathring{D}^2) \qquad U_2 = \iota_1'(D^2 \setminus \{0\}) \cup \iota_2'(Y),$$

where $\iota_i = \pi_X \circ \iota_i$, $\iota_1 : D^2 \to D^2 + Y$ and $\iota_2 : Y \to D^2 + Y$ denote the inclusions for the topological sum and $\pi_X : D^2 + Y \to X$ the canonical surjection. The subspace U_1 is homotopy equivalent to $\mathring{D}^2$, which implies $\pi_1(U_2) = \{e\}$. The subspace $U_1 \cap U_2 = \iota_1(\mathring{D}^2 \setminus \{0\})$ is homotopy equivalent to S^1, and this implies $\pi_1(U_1 \cap U_2) = \mathbb{Z}$. The subspace U_2 is homotopy equivalent to Y, since an argument analogous to the proof of Lemma 3.4.10 shows that $\iota_2'(Y)$ is a deformation retract of U_2. This implies $\pi_1(x, U_2) \cong \pi_1(y, Y)$ for all $x \in U_1 \cap U_2$. With Corollary 3.4.4 we obtain the pushout

$$\begin{tikzcd} \pi_1(x, X) & \{e\} \arrow[l, "e\mapsto e"'] \\ \pi_1(Y) \arrow[u, "\pi_1(i_2)"] & \mathbb{Z}. \arrow[l, "\pi_1(j_2)"] \arrow[u, "z\mapsto e"'] \end{tikzcd} \quad ,$$

with $\pi_1(j_2) : \mathbb{Z} \to \pi_1(Y)$, $1 \mapsto \pi_H(r_{k+1})$. This implies

$$\pi_1(x, X) \cong \frac{\pi_1(Y)}{\pi_1(j_2)(\mathbb{Z})} \cong \frac{H}{\langle\{\pi_H(r_{k+1})\}\rangle}$$

It is now sufficoent to show that the groups $G = F_n/\langle\{r_1, ..., r_{k+1}\}\rangle$ and $K := H/\langle\{\pi_H(r_{k+1})\}$ are isomorphic. For this, denote by $\pi_G : F_n \to G$ and $\pi_K : H \to K$ the canonical surjections. As $\{r_1, ..., r_{k+1}\} \subseteq \ker(\pi_K \circ \pi_H)$, by the universal property of the factor group there is a unique group homomorphism $\phi : G \to K$ with $\phi \circ \pi_G = \pi_K \circ \pi_H$. The group homomorphism $\pi_G : F_n \to G$ satisfies $\{r_1, ..., r_k\} \subseteq \ker(\pi_G)$, and hence there is a unique group homomorphism $\psi' : H \to G$ with $\psi' \circ \pi_H = \pi_G$. As $\pi_H(r_{k+1}) \in \ker(\psi')$, we also obtain a unique group homomorphism $\psi : K \to G$ mit $\psi \circ \pi_K = \psi'$. This implies $\psi \circ \phi \circ \pi_G = \psi \circ \pi_K \circ \pi_H = \psi' \circ \pi_H = \pi_G$ and $\phi \circ \psi \circ \pi_K \circ \pi_H = \phi \circ \psi' \circ \pi_H = \phi \circ \pi_G = \pi_K \circ \pi_H$. With the universal properties of the factor groups we obtain $\psi \circ \phi = \mathrm{id}_G$ and $\psi \circ \phi = \mathrm{id}_K$. This shows that $\psi = \phi^{-1}$ and G and K are isomorphic. □

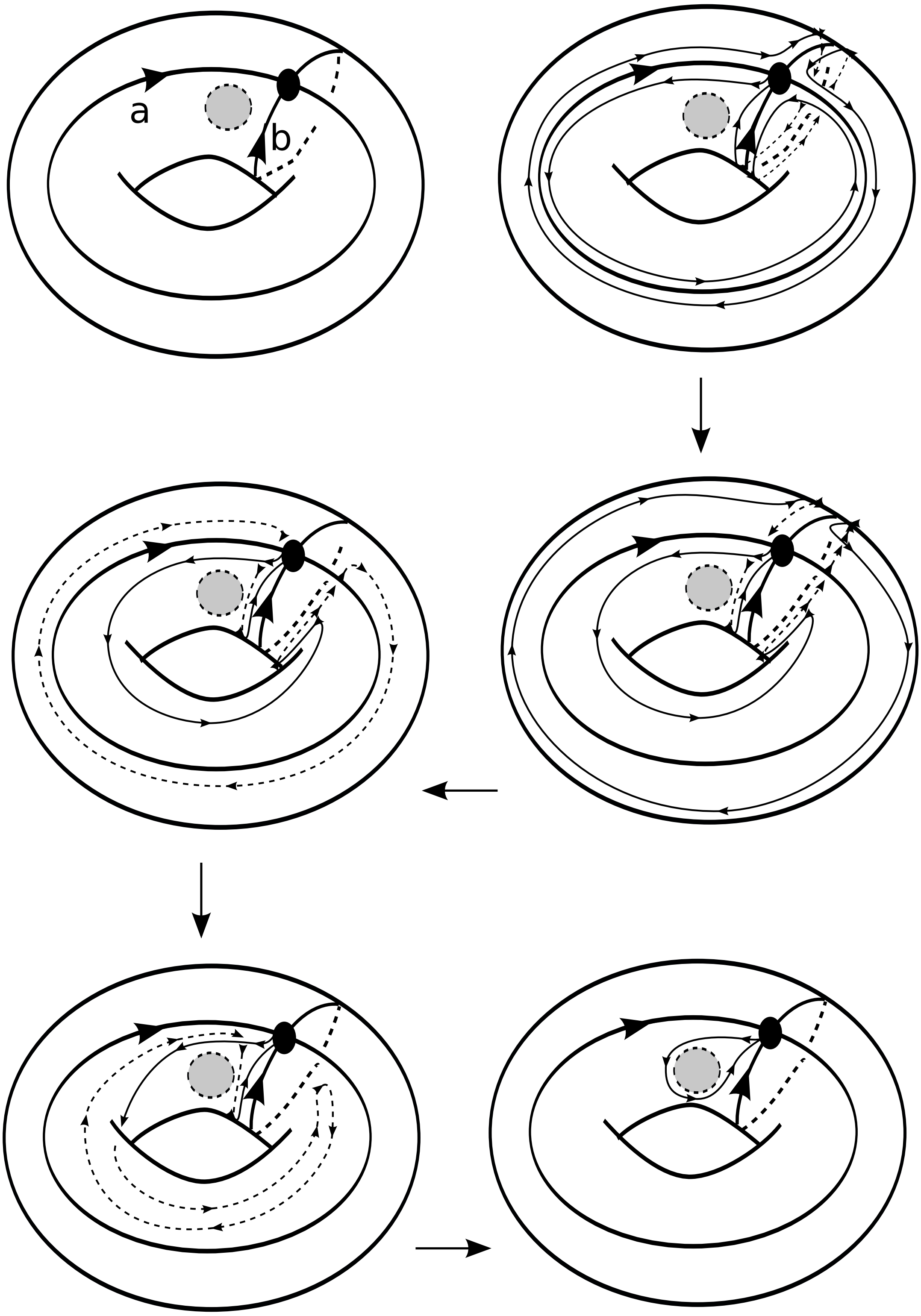


Figure 1: The defining relation for the fundamental group of the torus.

3.5 Supplement*: paths, homotopies and fundamental groupoids from the higher category perspective

The relation between topological spaces, continuous maps, homotopies and fundamental groupoids, functors and natural transformations between them in Theorem 3.2.8 can be formulated in a more concise way in the language of higher categories. For this, we first introduce the notion of a 2-category. While a category involves two layers of structure, objects and morphisms between them, a 2-category involves three layers, objects, morphisms between them and morphisms between morphisms. It also exhibits two compositions and certain coherence axioms between them.

Definition 3.5.1: A **2-category** $\mathcal{C}$ consists of

1. A collection of objects $X, Y, ...$
2. For each pair of objects X, Y in $\mathcal{C}$, a category $\mathcal{C}(X, Y)$.

   The objects in $\mathcal{C}(X, Y)$ are called **1-morphisms** and denoted $f : X \to Y$. The morphisms in $\mathcal{C}(X, Y)$ are called **2-morphisms** and denoted $\alpha : f \Rightarrow g$, and the identity morphisms for an object f in $\mathcal{C}(X, Y)$ by $1_f : f \Rightarrow f$. The composition of morphisms in $\mathcal{C}(X, Y)$ is denoted $\cdot$ and called **vertical composition**.
3. For any triple of objects X, Y, Z a functor $\bullet_{X,Y,Z} : \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \to \mathcal{C}(X, Z)$, called the **horizontal composition**. It is required to be strictly associative, i. e. to satisfy for all objects W, X, Y, Z

$$\bullet_{W,X,Z}(\bullet_{X,Y,Z} \times \mathrm{id}_{\mathcal{C}(W,X)}) = \bullet_{W,Y,Z}(\mathrm{id}_{\mathcal{C}(Y,Z)} \times \bullet_{W,X,Y})$$

4. For each object X, a **unit functor** $I_X : \mathcal{I} \to \mathcal{C}(X, X)$, where $\mathcal{I}$ is the trivial category with one object X and one identity morphism 1_X. This data is equivalent to the choice of a 1-morphism $e_X : X \to X$ in each category $\mathcal{C}(X, X)$ together with the associated identity morphism $1_{e_X} : e_X \Rightarrow e_X$. The unit functors are required to satisfy the following compatibility condition with the horizontal composition

$$\bullet_{X,Y,Y}(I_Y \times \mathrm{id}_{\mathcal{C}(X,Y)}) = \mathrm{id}_{\mathcal{C}(x,Y)} = \bullet_{X,X,Y}(\mathrm{id}_{\mathcal{C}(X,Y)} \times I_X).$$

Remark 3.5.2: In a 2-category $\mathcal{C}$, the objects and 1-morphisms also form a category, with the composition of 1-morphisms given by the horizontal composition and the identity morphisms by the unit functors. A 2-category in which all 1- and 2-morphisms are isomorphisms is called a **2-groupoid**.

Remark 3.5.3: To illustrate structures in a 2-category with diagrams, one often uses the following notation. A 1-morphism $f : X \to Y$ between from an object X to an object Y is denoted

$$X \xrightarrow{f} Y,$$

and the unit 1-morphism $e_X : X \to X$ for an object X is omitted, i. e.

$$X \xrightarrow{e_X} X \qquad = \qquad X.$$

A 2-morphism $h : f_1 \Rightarrow f_2$ from a 1-morphism $f_1 : X \to Y$ to a 1-morphism $f_2 : X \to Y$ corresponds to a 2-cell

$$\begin{tikzcd} X \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] \arrow[r, phantom, "\Downarrow h"] & Y, \end{tikzcd}$$

and the identity 2-morphism $1_f : f \Rightarrow f$ from a 1-morphism $f : X \to Y$ to itself is omitted

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow 1_f"] & Y \end{tikzcd} \qquad = \qquad X \xrightarrow{f} Y.$$

The vertical composite $h_2 \cdot h_1 : f_1 \Rightarrow f_3$ of two 2-morphisms $h_1 : f_1 \Rightarrow f_2$, $h_2 : f_2 \Rightarrow f_3$ between 1-morphisms $f_1, f_2, f_3 : X \to Y$ is denoted

$$\begin{tikzcd} X \arrow[r, bend left=60, "f_1"] \arrow[r, "f_2" description] \arrow[r, bend right=60, "f_3"'] \arrow[r, phantom, shift left=3, "\Downarrow h_1"] \arrow[r, phantom, shift right=3, "\Downarrow h_2"] & Y \end{tikzcd} .$$

The horizontal composite $g \bullet_{X,Y,Z} f : X \to Y$ of 1-morphisms $f : X \to Y$, $g : Y \to Z$ is denoted

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

and the horizontal composite $k \bullet_{X,Y,Z} h$ of 2-morphisms $h : f_1 \Rightarrow f_2$, $k : g_1 \Rightarrow g_2$ between 1-morphisms $f_1, f_2 : X \to Y$ and $g_1, g_2 : Y \to Z$ by

$$\begin{tikzcd} X \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] \arrow[r, phantom, "\Downarrow h"] & Y \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] \arrow[r, phantom, "\Downarrow k"] & Z \end{tikzcd}$$

The associativity of the vertical composition and the properties of the identity morphisms ensure that diagrams involving multiple vertical composites and identity morphisms have a unique interpretation. The requirement that the horizontal composites are functors ensures that diagrams involving both vertical and horizontal composition have a unique interpretation. The associativity and unit condition for the horizontal composition ensure that diagrams involving multiple horizontal composites have a unique interpretation.

To develop an intuition for the structures in a higher category, it is always useful to first consider the situation where there is only one object and to determine to what the categorical data reduces in this case. For a 2-category, this yields a monoidal category or tensor category.

Example 3.5.4: A 2-category $\mathcal{C}$ with a single object X is a strict monoidal category $(\mathcal{D}, \otimes, e)$.

In this case, one has a single category $\mathcal{D} = \mathcal{C}(X, X)$ together with a functor $\otimes = \bullet_{X,X,X} : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$, a distinguished object $e \in \mathcal{D}$ and a 1-morphism $\epsilon_X = 1_e : e \to e$ such that $\otimes(\otimes \times \mathrm{id}_{\mathcal{D}}) = \otimes(\mathrm{id}_{\mathcal{D}} \times \otimes)$, $e \otimes D = D \otimes e = D$ for all objects D of $\mathcal{D}$ and $1_e \otimes f = f \otimes 1_e = f$ for all morphisms $f : D \to D'$ in $\mathcal{D}$. This is precisely the data for a strict monoidal category.

Example 3.5.5: Let R be a unital ring. Then chain complexes, chain maps and chain homotopies in R-Mod form a 2-category Ch(R-Mod):

1. **objects:** chain complexes $X_\bullet, Y_\bullet, ...$ in R-Mod,
2. **1- and 2-morphisms:** The objects of the categories $\mathrm{Ch}(R\text{-Mod})(X_\bullet, Y_\bullet)$ (1-morphisms) are chain maps $f_\bullet, g_\bullet, ... : X_\bullet \to Y_\bullet$. A morphism from $f_\bullet : X_\bullet \to Y_\bullet$ to $g_\bullet : X_\bullet \to Y_\bullet$ (2-morphism) is a chain homotopy $h_\bullet : f_\bullet \Rightarrow g_\bullet$. The composition of morphisms in $\mathrm{Ch}(R\text{-Mod})(X_\bullet, Y_\bullet)$ (vertical composition) is given by the pointwise addition of chain homotopies $h_\bullet + h'_\bullet = (h_n + h'_n)_{n\in\mathbb{Z}}$. The identity morphisms in $\mathrm{Ch}(R\text{-Mod})(X_\bullet, Y_\bullet)$ are trivial chain homotopies $1_{f_\bullet} = (0)_{n\in\mathbb{Z}}$.
3. **horizontal composition**: The horizontal composition functor
$$\circ_{X_\bullet, Y_\bullet, Z_\bullet} : \mathrm{Ch}(R\text{-Mod})(Y_\bullet, Z_\bullet) \times \mathrm{Ch}(R\text{-Mod})(X_\bullet, Y_\bullet) \to \mathrm{Ch}(R\text{-Mod})(X_\bullet, Z_\bullet)$$
is given by the composition of chain maps $g_\bullet \circ f_\bullet = (g_n \circ f_n)_{n\in\mathbb{Z}}$ and the horizontal composition of chain homotopies $h'_\bullet \circ h_\bullet = (g'_{n+1} \circ h_n + h'_n \circ f_n)_{n\in\mathbb{Z}} = (g_{n+1} \circ h_n + h'_n f'_n)_{n\in\mathbb{Z}}$ for chain homotopies $h_\bullet : f_\bullet \Rightarrow f'_\bullet$ and $h'_\bullet : g_\bullet \Rightarrow g'_\bullet$ and chain maps $f_\bullet, f'_\bullet : X_\bullet \to Y_\bullet$, $g_\bullet, g'_\bullet : Y_\bullet \to Z_\bullet$.
4. **unit functors**: The unit functor are given by the identity chain maps $e_{X_\bullet} = \mathrm{id}_{X_\bullet} = (\mathrm{id}_{X_n})_{n\in\mathbb{Z}}$ and the trivial chain homotopy $1_{e_{X_\bullet}}$ from $e_{X_\bullet}$ to itself.

It is clear that the vertical composition is associative and unital, and hence the sets $\mathrm{Ch}(R\text{-Mod})(X_\bullet, Y_\bullet)$ form categories. It also follows directly that the horizontal composition is a functor, since

$$\begin{aligned}
&k'_\bullet \circ h'_\bullet + k_\bullet \circ h_\bullet = (g''_{n+1} \circ h'_n + k'_n \circ f'_n + g'_{n+1} \circ h_n + k_n \circ f_n)_{n\in\mathbb{Z}} \\
&= (g''_{n+1} \circ h'_n + k'_n \circ f_n + g''_{n+1} \circ h_n + k_n \circ f_n)_{n\in\mathbb{Z}} = (g''_{n+1} \circ (h'_n + h_n) + (k'_n + k_n) \circ f_n)_{n\in\mathbb{Z}} \\
&= (k'_\bullet + h'_\bullet) \circ (k_\bullet + h_\bullet) \\
&1_{g_\bullet} \circ 1_{f_\bullet} = (g_\bullet \circ 0 + 0 \circ f_\bullet)_{n\in\mathbb{Z}} = (0)_{n\in\mathbb{Z}} = 1_{g_\bullet \circ f_\bullet}.
\end{aligned}$$

for all chain maps $f_\bullet, f'_\bullet, f''_\bullet : X_\bullet \to Y_\bullet$, $g_\bullet, g'_\bullet, g''_\bullet : Y_\bullet \to Z_\bullet$ and chain homotopies $h_\bullet : f_\bullet \Rightarrow f'_\bullet$, $h'_\bullet : f'_\bullet \Rightarrow f''_\bullet$, $k_\bullet : g_\bullet \Rightarrow g'_\bullet$ $k'_\bullet : g'_\bullet \Rightarrow g''_\bullet$. The horizontal compoisition functor is strictly associative since for all chain maps $f^1_\bullet, g^1_\bullet : W_\bullet \to X_\bullet$, $f^2_\bullet, g^2_\bullet : X_\bullet \to Y_\bullet$, $f^3_\bullet, g^3_\bullet : Y_\bullet \to Z_\bullet$ and chain homotopies $h^k_\bullet : f^k_\bullet \Rightarrow g^k_\bullet$, we have $f^3_\bullet \circ (f^2_\bullet \circ f^1_\bullet) = (f^3_n \circ f^2_n \circ f^1_n)_{n\in\mathbb{Z}} = (f^3_\bullet \circ f^2_\bullet) \circ f^1_\bullet$ and

$$\begin{aligned}
&h^3_\bullet \circ (h^2_\bullet \circ h^1_\bullet) = (g^3_{n+1} \circ (h^2_\bullet \circ h^1_\bullet)_n + h^3_n \circ f^2_n \circ f^1_n)_{n\in\mathbb{Z}} \\
&= (g^3_{n+1} \circ (g^2_{n+1} \circ h^1_n + h^2_n \circ f^1_n) + h^3_n \circ f^2_n \circ f^1_n)_{n\in\mathbb{Z}} \\
&= (g^3_{n+1} \circ g^2_{n+1} \circ h^1_n + (g^3_{n+1} \circ h^2_n + h^3_n \circ f^2_n) \circ f^1_n)_{n\in\mathbb{Z}} \\
&= (g^3_{n+1} \circ g^2_{n+1} \circ h^1_n + (h^3_\bullet \circ h^2_\bullet)_n \circ f^1_n)_{n\in\mathbb{Z}} = (h^3_\bullet \circ h^2_\bullet) \circ h^1_\bullet.
\end{aligned}$$

The compatibility conditions between the horizontal composition and units are satisfied since for all chain maps $f_\bullet, g_\bullet : X_\bullet \to Y_\bullet$ and chain homotopies $h_\bullet : f_\bullet \Rightarrow g_\bullet$, we have $e_{Y_\bullet} \circ f_\bullet = (\mathrm{id}_{Y_n} \circ f_n)_{n\in\mathbb{Z}} = (f_n \circ \mathrm{id}_{X_n})_{n\in\mathbb{Z}} = f_\bullet \circ e_{X_\bullet}$ and

$$h_\bullet \circ 1_{e_{X_\bullet}} = (g_{n+1} \circ 0 + h_n \circ \mathrm{id}_{X_n})_{n\in\mathbb{Z}} = (h_n)_{n\in\mathbb{Z}} = (\mathrm{id}_{Y_{n+1}} \circ h_n + 0 \circ f_n)_{n\in\mathbb{Z}} = 1_{e_{Y_\bullet}} \circ h_\bullet$$

We will now show that the notion of a 2-category is also appropriate for topological spaces, continuous maps and homotopies between them. This is rather intuitive up to one detail. To ensure that the vertical composition of 2-morphisms is associative, one cannot simply work with homotopies but has to consider instead homotopy classes of homotopies. This is familiar from the construction of the fundamental groupoid, where one has to take homotopy classes of paths instead of paths to obtain a category.

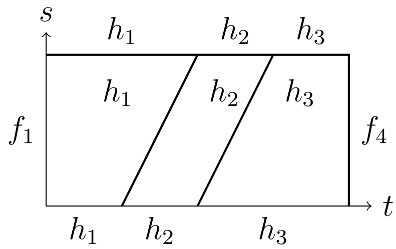
Example 3.5.6: The 2-category 2Top involves the following structures:

1. **objects**: topological spaces $X, Y, ...,$
2. **1- and 2-morphisms**: for given topological spaces X, Y, the category $\mathrm{Top}(X, Y)$ has as objects continuous maps $f : X \to Y$. Morphisms from $f : X \to Y$ to $g : X \to Y$ are homotopy classes relative to $\{0, 1\} \times X$ of homotopies from f to g. The vertical composition is given by the composition of homotopies. For homotopies $h_i : [0, 1] \times X \to Y$ from $f_i : X \to Y$ to $f_{i+1} : X \to Y$, $i = 1, 2$, we set
$$[h_2] \cdot [h_1] := [h_2 \cdot h_1] \quad \text{with} \quad h_2 \cdot h_1(t, x) = \begin{cases} h_1(2t, x) & t \in [0, \frac{1}{2}) \\ h_2(2t - 1, x) & t \in [\frac{1}{2}, 1] \end{cases}$$
The unit 2-morphisms are the homotopy classes relative to $\{0, 1\} \times X$ of trivial homotopies $h_f : [0, 1] \times X \to Y$, $h_f(t, x) = f(x)$ for all $t \in [0, 1]$, $x \in X$.
3. **horizontal composition**: The horizontal composition is given by the composition of maps and homotopies
$$f_2 \bullet_{X,Y,Z} f_1 = f_2 \circ f, \qquad [h_2] \bullet_{X,Y,Z} [h_1] := [h_2 \bullet h_1] \quad \text{with } (h_2 \bullet h_1)(t, x) = h_2(t, h_1(t, x))$$
for all continuous maps $f_1, f_1' : X \to Y$, $f_2, f_2' : Y \to Z$ and homotopies $h_1 : [0, 1] \times X \to Y$ from f_1 to f_1' and $h_2 : [0, 1] \times Y \to Z$ from f_2 to f_2'.
4. **unit functors:** The unit functor for a topological space X is given by the identity map $e_X = \mathrm{id}_X : X \to X$ and the homotopy class of the identity homotopy $\epsilon_X = [h_{\mathrm{id}_X}]$ with $h_{\mathrm{id}_X} : [0, 1] \times X \to X$, $h(t, x) = x$ for all $t \in [0, 1]$, $x \in X$.

We will now show that these structures satisfy the axioms of a 2-category.

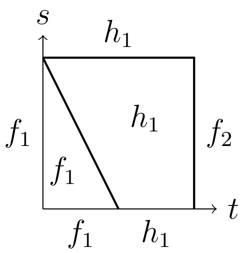
1. The vertical composition is well-defined. If $h_i, h_i' : [0, 1] \times X \to Y$ are homotopies from $f_i : X \to Y$ to $f_{i+1} : X \to Y$ and $H_i : [0, 1] \times [0, 1] \times X \to Y$ are homotopies from h_i to h_i' relative to $\{0, 1\} \times X$ for $i = 1, 2$, then $H : [0, 1] \times [0, 1] \times X \to Y$
$$H(s, t, x) = \begin{cases} H_1(s, 2t, x) & s \in [0, \frac{1}{2}) \\ H_2(s, 2t - 1, x) & t \in [\frac{1}{2}, 1] \end{cases}$$
is a homotopy relative to $\{0, 1\} \times X$ from $h_2 \cdot h_1$ to $h_2' \cdot h_1'$ and hence $h_2' \sim_{\{0,1\}\times X} h_2$, $h_1' \sim_{\{0,1\}\times X} h_1$ implies $[h_2' \cdot h_1'] = [h_2 \cdot h_1]$.

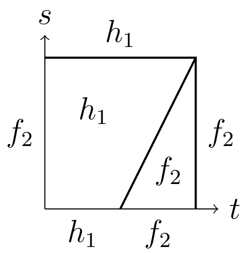
2. The vertical composition is associative and unital. For all continuous maps $f_1, f_2, f_3, f_4 : X \to Y$ and homotopies h_i from f_i to f_{i+1}, one has $h_3 \cdot (h_2 \cdot h_1) \sim_{\{0,1\}\times X} (h_3 \cdot h_2) \cdot h_1$ and $h_{f_2} \cdot h_1 \sim_{\{0,1\}\times X} h_1 \sim_{\{0,1\}\times X} h_1 \cdot h_{f_1}$. Explicitly, a homotopy $H : [0, 1] \times [0, 1] \times X \to Y$ from $h_3 \cdot (h_2 \cdot h_1)$ to $(h_3 \cdot h_2) \cdot h_1$ relative to $\{0, 1\} \times X$ is given by
$$H(s, t, x) = \begin{cases} h_1\left(\frac{4t}{s+1}, x\right) & t \in [0, \frac{s+1}{4}) \\ h_2(4t - s - 1, x) & t \in [\frac{s+1}{4}, \frac{s+2}{4}) \\ h_3\left(\frac{4t-s-2}{2-s}, x\right) & t \in [\frac{s+2}{4}, 1]. \end{cases}$$



Homotopies $H_R, H_L : [0,1] \times [0,1] \times X \to Y$ relative to $\{0,1\} \times X$ from $h_1 \cdot h_{f_1}$ and $h_{f_2} \cdot h_1$, respectively, to h_1 are given by

$$H_L(s,t,x) = \begin{cases} f_1(x) & t \in [0, \frac{1-s}{2}) \\ h_1\left(\frac{2t-1+s}{1+s}, x\right) & t \in [\frac{1-s}{2}, 1] \end{cases}$$
$$H_R(s,t,x) = \begin{cases} h_1\left(\frac{2t}{1+s}, x\right) & t \in [0, \frac{1+s}{2}) \\ f_2(x) & t \in [\frac{1+s}{2}, 1] \end{cases}$$





3. The horizontal composition is a functor since for all continuous maps $f, f'f'' : X \to Y$, $g, g', g'' : Y \to Z$ and all homotopies h from f to f', h' from f' to f'', k from g to g', k' from g' to g'', one has for all $t \in [0,1]$, $x \in X$

$$(h_g \bullet_{X,Y,Z} h_f)(t,x) = h_g(t, h(t,x)) = g \circ f(x) = h_{g \circ f}(t,x)$$
$$(k' \cdot k) \bullet_{X,Y,Z} (h' \cdot h)(t,x) = (k' \bullet_{X,Y,Z} h') \cdot (k \bullet_{X,Y,Z} h)(t,x) = \begin{cases} k(2t, h(2t,x)) & t \in [0, \frac{1}{2}) \\ k'(2t-1, h'(2t-1,x)) & t \in [\frac{1}{2}, 1] \end{cases}.$$

The horizontal composition is associative and unital, since for all continuous maps $f : W \to X$, $g : X \to Y$, $h : Y \to Z$

$$(h \bullet_{X,Y,Z} g) \bullet_{W,X,Y} f = (h \circ g) \circ f = h \circ g \circ f = h \circ (g \circ f) = h \bullet_{X,Y,Z} (g \bullet_{W,X,Y} f)$$
$$e_Y \bullet_{X,Y,Y} f = \mathrm{id}_Y \circ f = f = f \circ \mathrm{id}_X = f \bullet_{X,X,Y} e_Y,$$

and for all continuous maps $f_1, f_1' : W \to X$, $f_2, f_2' : X \to Y$, $f_3, f_3' : Y \to Z$ and homotopies h_i from f_i to f_i', one has

$$(h_3 \cdot (h_2 \cdot h_1))(t,x) = h_3(t, (h_2 \cdot h_1)(t,x)) = h_3(t, h_2(t, h_1(t,x))) = (h_3 \cdot h_2)(t, h_1(t,x))$$
$$= (h_3 \cdot h_2) \cdot h_1)(t,x)$$
$$(h_Y \cdot h)(t,x) = h_Y(t, h(t,x)) = h(t,x) = h(t, h_X(t,x)) = (h \cdot h_X)(t,x).$$

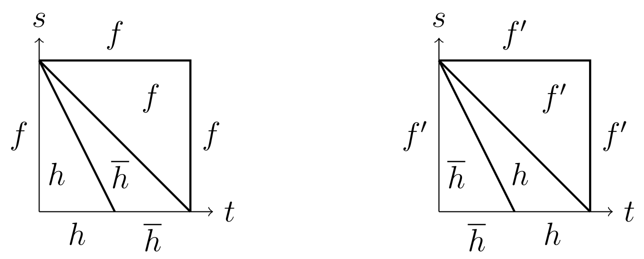
for all $t \in [0,1]$ and $x \in X$. This shows that 2Top is a 2-category.

Note that all 2-morphisms in 2Top are invertible since for any homotopy h from $f : X \to Y$ to $f' : X \to Y$ one has a homotopy $\overline{h} : [0,1] \times X \to Y$, $\overline{h}(t,x) = h(1-t,x)$ from f' to f and

$\overline{h}\cdot h \sim_{\{0,1\}\times X} h_f$, $h\cdot\overline{h} \sim_{\{0,1\}\times X} h_{f'}$. Homotopies $H, H' : [0,1]\times[0,1]\times X \to Y$ from $h\cdot\overline{h}$ to h_f and $\overline{h}\cdot h$ to $h_{f'}$ relative to $\{0,1\}\times X$ are given by

$$H(s,t,x) = \begin{cases} h(\frac{2t}{1-s}, x) & t\in[0,\frac{1-s}{2}) \\ \overline{h}(\frac{2t-1+s}{1-s}, x) & t\in[\frac{1-s}{2}, 1-s) \\ f(x) & t\in[1-s,1] \end{cases}$$

$$H'(s,t,x) = \begin{cases} \overline{h}(\frac{2t}{1-s}, x) & t\in[0,\frac{1-s}{2}) \\ h(\frac{2t-1+s}{1-s}, x) & t\in[\frac{1-s}{2}, 1-s) \\ f'(x) & t\in[1-s,1] \end{cases}$$



The aim is now to relate this 2-category to fundamental groupoids of topological spaces, functors between them and natural transformations. The fact that there are again three layers of structure strongly suggests that groupoids, functors between them and natural transformations between such functors form a 2-category. In fact, this not only holds for groupoids but, more generally, for small categories, functors and natural transformations.

Example 3.5.7: The 2-category **Cat** consists of the following:

1. **objects**: small categories $\mathcal{C}, \mathcal{D}, ...,$
2. **1- and 2-morphisms**: the category $\mathrm{Cat}(\mathcal{C},\mathcal{D})$ is the functor category $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ with functors $F, G : \mathcal{C}\to\mathcal{D}$ as objects and natural transformations $\kappa : F\to G$ as morphisms. The vertical composition is the composition of 2-morphisms in $\mathcal{D}$, i. e. $(\kappa\cdot\eta)_C = \kappa_C\circ\eta_C$ for all natural transformations $\eta : F\to G$, $\kappa : G\to H$, functors $F, G, H : \mathcal{C}\to\mathcal{D}$ and objects C in $\mathcal{C}$. The identity 2-morphisms are the identity natural transformations $1_F : F\Rightarrow F$ with $(1_F)_C = 1_{F(C)}$ for all objects C in $\mathcal{C}$.
3. **horizontal composition**: The horizontal composition is given by the composition of functors and their action on natural transformations

$$G \bullet_{\mathcal{C},\mathcal{D},\mathcal{E}} F = GF : \mathcal{C}\to\mathcal{E}, \qquad (\beta\bullet_{\mathcal{C},\mathcal{D},\mathcal{E}}\alpha)_C = \beta_{F'(C)}\circ G(\alpha_C) : FG\Rightarrow F'G'$$

   for functors $F, F' : \mathcal{C}\to\mathcal{D}$, $G, G' : \mathcal{D}\to\mathcal{E}$ and natural transformations $\alpha : F\to F'$, $\beta : G\to G'$.
4. **unit functors:** The unit functor for a small category $\mathcal{C}$ is given by the identity functor $e_{\mathcal{C}} = \mathrm{id}_{\mathcal{C}} : \mathcal{C}\to\mathcal{C}$ and the identity natural transformation $\epsilon_{\mathcal{C}} = 1_{\mathrm{id}_{\mathcal{C}}} : \mathrm{id}_{\mathcal{C}}\to\mathrm{id}_{\mathcal{C}}$.

We verify that these structures satisfy the axioms of a 2-category.

1. It is easy to see that the vertical composition is strictly associative and unital, since for functors $F, G, H, K : \mathcal{C}\to\mathcal{D}$ and natural transformations $\alpha : F\to G$, $\beta : G\to H$, $\gamma : H\to K$,

one has $(\gamma\cdot(\beta\cdot\alpha))_C = \gamma_C\circ\beta_C\circ\alpha_C = ((\gamma\cdot\beta)\cdot\alpha)_C$ and $(1_G\cdot\alpha)_C = 1_{G(C)}\circ\alpha_C = \alpha_C = \alpha_C\circ 1_{F(C)} = (\alpha_C\cdot 1_F)_C$.

2. The requirement that the horizontal composition is a functor reads

$$(\delta\cdot\gamma)\bullet_{\mathcal{C},\mathcal{D},\mathcal{E}}(\beta\cdot\alpha) = (\delta\bullet_{\mathcal{C},\mathcal{D},\mathcal{E}}\beta)\cdot(\gamma\bullet_{\mathcal{C},\mathcal{D},\mathcal{E}}\alpha) \qquad 1_G\bullet_{\mathcal{C},\mathcal{D},\mathcal{E}} 1_F = 1_{GF}$$

for all functors $F, F', F'' : \mathcal{C}\to\mathcal{D}$, $G, G', G'' : \mathcal{D}\to\mathcal{E}$ and natural transformations $\alpha : F\to F'$, $\beta : F'\to F''$, $\gamma : G\to G'$, $\delta : G'\to G''$. This is equivalent to the conditions

$$\begin{aligned}
&\delta_{F''(C)}\circ\gamma_{F''(C)}\circ G(\beta_C)\circ G(\alpha_C) = \delta_{F''(C)}\circ G'(\beta_C)\circ\gamma_{F'(C)}\circ G(\alpha_C)\\
&1_{GF(C)}\circ G(1_{F(C)}) = 1_{GF(C)}
\end{aligned}$$

for all objects C in $\mathcal{C}$. The first condition follows from the naturality of the natural transformation $\gamma : G\to G'$, which implies $G'(\beta_C)\circ\gamma_{F'(C)} = \gamma_{F''(C)}\circ G(\beta_C)$ for all objects C in $\mathcal{C}$. The second from the fact that for any functor $G : \mathcal{C}\to\mathcal{D}$ and any object D in $\mathcal{D}$, one has $G(1_D) = 1_{G(D)}$.

3. The horizontal composition is strictly associative and unital, since for functors $F : \mathcal{C}\to\mathcal{D}$, $G : \mathcal{D}\to\mathcal{E}$ and $H : \mathcal{E}\to\mathcal{F}$ one has

$$\begin{aligned}
&(H\bullet_{\mathcal{D},\mathcal{E},\mathcal{F}} G)\bullet_{\mathcal{C},\mathcal{D},\mathcal{E}} = HGF = H\bullet_{\mathcal{D},\mathcal{E},\mathcal{F}}(G\bullet_{\mathcal{C},\mathcal{D},\mathcal{E}} F)\\
&e_{\mathcal{D}}\bullet_{\mathcal{C},\mathcal{D},\mathcal{D}} F = \mathrm{id}_{\mathcal{D}}F = F = F\mathrm{id}_{\mathcal{C}} = F\bullet_{\mathcal{C},\mathcal{C},\mathcal{D}}\mathrm{id}_{\mathcal{C}}
\end{aligned}$$

and for all natural transformations $\alpha : F\to F'$, $\beta : G\to G'$, $\gamma : H\to H'$ and objects C in $\mathcal{C}$

$$\begin{aligned}
&(\gamma\bullet_{\mathcal{D},\mathcal{E},\mathcal{F}}(\beta\bullet_{\mathcal{C},\mathcal{D},\mathcal{E}}\alpha))_C = \gamma_{G'F'(C)}\circ H(\beta_{F'(C)}\circ G(\alpha_C)) = \gamma_{G'F'(C)}\circ H(\beta_{F'(C)})\circ HG(\alpha_C)\\
&= (\gamma_{G'(F'(C))}\circ H(\beta_{F'(C)}))\circ HG(\alpha_C) = (\gamma\bullet_{\mathcal{D},\mathcal{E},\mathcal{F}}\beta)_{F'(C)}\circ HG(\alpha_C)\\
&= ((\gamma\bullet_{\mathcal{D},\mathcal{E},\mathcal{F}}\beta)\bullet_{\mathcal{C},\mathcal{D},\mathcal{E}}\alpha)_C\\
&(\epsilon_{\mathcal{D}}\bullet_{\mathcal{C},\mathcal{D},\mathcal{D}}\alpha)_C = 1_{F'(C)}\circ\alpha_C = \alpha_C = \alpha_C\circ 1_{F(C)} = \alpha_C\circ F(1_C) = (\alpha\bullet_{\mathcal{C},\mathcal{C},\mathcal{D}}\epsilon_{\mathcal{C}})_C
\end{aligned}$$

This shows that Cat is a 2-category.

Just as in the case of a category, one can construct full sub 2-categories of a 2-category $\mathcal{C}$ by discarding certain objects X in $\mathcal{C}$ and all categories $\mathcal{C}(X,Y)$ and $\mathcal{C}(W,X)$ involving the discarded objects. This allows one to restrict attention to a subcategory of Cat, whose objects are groupoids instead of general categories.

Example 3.5.8: The 2-category Grpd consists of:

1. **objects**: groupoids.
2. **1- and 2-morphisms**: functors between groupoids and natural transformations between them. Vertical composition and identity 2-morphisms as in Cat.
3. **horizontal composition** and **unit functors**: as in Cat.

The structures introduced so far suggest that there should be a generalisation of the notion of a functor to 2-categories. This generalisation should involve all layers of structure, i. e. the objects as well as the categories $\mathcal{C}(X,Y)$ and should be compatible with both, vertical and horizontal composition and the associated identity 2-morphisms and unit 1-morphisms.

Definition 3.5.9: A **2-functor** $F : \mathcal{C} \to \mathcal{D}$ between 2-categories $\mathcal{C}, \mathcal{D}$ consists of the following data:

1. An assignment $X \to F_0(X)$ of an object $F_0(X)$ in $\mathcal{D}$ to each object X in $\mathcal{C}$.
2. For each pair of objects X, Y in $\mathcal{C}$, a functor $F_{X,Y} : \mathcal{C}(X,Y) \to \mathcal{D}(F_0(X), F_0(Y))$.
3. For each triple of objects X, Y, Z in $\mathcal{C}$ a natural isomorphism
$$\phi_{X,Y,Z} : \bullet_{F_0(X),F_0(Y),F_0(Z)}(F_{Y,Z} \times F_{X,Y}) \xrightarrow{\sim} F_{X,Z} \bullet_{X,Y,Z} .$$
This determines for all 1-morphisms $f : X \to Y$, $g : Y \to Z$ a 2-morphism $\phi_{g,f} : F_{Y,Z}(g) \bullet_{F_0(X),F_0(Y),F_0(Z)} F_{X,Y}(f) \Rightarrow F_{X,Z}(g \bullet_{X,Y,Z} f)$.
4. For each object X in $\mathcal{C}$, an invertible 2-morphism $\phi_X : e_{F_0(X)} \Rightarrow F_{X,X}(e_X)$,

such that the following consistency conditions are met:

1. For all 1-morphisms $f : X \to Y$, the following diagram commutes
$$\begin{tikzcd}
F_{X,Y}(f) = F_{X,Y}(f) \bullet e_{F_0(X)} = e_{F_0(Y)} \bullet F_{X,Y}(f) \arrow[r, "1\bullet\phi_X"] \arrow[d, "\phi_Y\bullet 1"] \arrow[dr, "1"] & F_{X,Y}(f) \bullet F_{X,X}(e_X) \arrow[d, "\phi_{f,e_X}"] \\
F_{X,Y}(e_Y \bullet f) \arrow[r, "\phi_{e_Y,g}"'] & F_{X,Y}(e_Y \bullet f) = F_{X,Y}(f) = F_{X,Y}(f \bullet e_X)
\end{tikzcd}$$
2. For all 1-morphisms $f : W \to X$, $g : X \to Y$, $h : Y \to Z$, the following diagram commutes
$$\begin{tikzcd}
F_{Y,Z}(h) \bullet F_{X,Y}(g) \bullet F_{W,X}(f) \arrow[r, "1\bullet\phi_{g,f}"] \arrow[d, "\phi_{h,g}\bullet 1"] & F_{Y,Z}(h) \bullet F_{W,Y}(g \bullet f) \arrow[d, "\phi_{h,g\bullet f}"] \\
F_{X,Y}(h \bullet g) \bullet F_{W,X}(f) \arrow[r, "\Phi_{h\bullet g,f}"'] & F_{W,Z}(h \bullet g \bullet f).
\end{tikzcd}$$

The 2-functor is called a **strict 2-functor** if all 2-morphisms $\phi_{g,f} : F_{Y,Z}(g) \bullet F_{X,Y}(f) \Rightarrow F_{X,Z}(g \bullet f)$ and $\phi_X : e_{F_0(X)} \to F_{X,X}(e_X)$ are identity 2-morphisms.

Using the notion of a 2-functor, we can now formulate the results from Theorem 3.2.8 (and some additional ones) more concisely, namely as the existence of a 2-functor from the 2-category of topological spaces, continuous maps and homotopy classes of homotopies to the 2-category of groupoids, functors and natural transformations between them.

Theorem 3.5.10: The assignments

$$\begin{array}{rlcl} F_0 : X \mapsto \Pi_1(X) & & & \\ F_{X,Y} : & f : X \to Y & \mapsto & \Pi_1(f) : \Pi_1(X) \to \Pi_1(Y) \\ & [h] : f \Rightarrow g & \mapsto & \Pi_1(h) : \Pi_1(f) \to \Pi_1(g) \end{array}$$

from Theorem 3.2.8 define a strict 2-functor $F :$ 2Top$\to$ Grpd.

Proof:
1. The assignments $F_{X,Y} :$ 2Top$(X,Y) \to$ Fun$(\Pi_1(X), \Pi_1(Y))$ are well-defined:

If $h : [0,1] \times X \to Y$ is a homotopy from f to g, then the natural transformation $\Pi_1(h) : \Pi_1(f) \to \Pi_1(g)$ is given by a collection of morphisms $\Pi_1(h)_x = [h_x]_{0,1} : f(x) \to g(x)$, where $h_x : [0,1] \to Y$ is the path given by $h_x(t) = h(t,x)$.

If $h, h' : [0,1]\times X \to Y$ are homotopies from f to g and $H : [0,1]\times[0,1]\times X \to Y$ is a homotopy from h to h' relative to $\{0,1\} \times X$, then $H_x : [0,1] \times [0,1] \to Y$, $H_x(s,t) = H(s,t,x)$ is a homotopy from $h_x : [0,1] \to Y$ to $h'_x : [0,1] \to Y$ relative to $\{0,1\}$. This implies $[h_x]_{0,1} = [h'_x]_{0,1}$ for all $x \in X$ and hence $\Pi_1(h) = \Pi_1(h')$.

2. The assignments $F_{X,Y} :$ 2Top$(X,Y) \to$ Fun$(\Pi_1(X), \Pi_1(Y))$ are functors:

For fixed topological spaces X, Y, continuous maps $f_1, f_2, f_3 : X \to Y$ and homotopies $h_1 : [0,1] \times X \to Y$ from f_1 to f_2 and $h_2 : [0,1] \times X \to Y$ from f_2 to f_3, we have

$$(h_2 \cdot h_1)(t,x) = \begin{cases} h_1(2t,x) & t \in [0, \frac{1}{2}) \\ h_2(2t-1,x) & t \in [\frac{1}{2}, 1] \end{cases}$$

and, consequently, $(h_2 \cdot h_1)_x = (h_2)_x \star (h_1)_x$. This proves that for all $x \in X$

$$\Pi_1(h_2 \cdot h_1)_x = [(h_2 \cdot h_1)_x]_{0,1} = [(h_2)_x \star (h_1)_x]_{0,1} = [(h_2)_x]_{0,1} \circ [(h_1)_x]_{0,1} = \Pi_1(h_2)_x \circ \Pi_1(h_1)_x$$

and hence $\Pi_1(h_2 \cdot h_1) = \Pi_1(h_2)\Pi_1(h_1)$. Similarly, we find for the trivial homotopies $h_f : [0,1] \times X \to Y$, $h_f(t,x) = f(x)$ for all $x \in X$ that $\Pi_1(h_f)_x = [\gamma_{f(x)}] = 1_{f(x)}$ for all $x \in X$ and hence $\Pi_1([h_f]) = \mathrm{id}_{\Pi_1(F)} = 1_{\Pi_1(F)}$. This shows that for all topological spaces X, Y, the assignment $F_{X,Y} :$ 2Top$(X,Y) \to$ Fun$(\Pi_1(X), \Pi_1(Y))$ is a functor.

3. The functors $F_{X,Y} :$ 2Top$(X,Y) \to$ Fun$(\Pi_1(X), \Pi_1(Y))$ are compatible with the horizontal composition and the unit functors:

For the 1-morphisms $e_X = \mathrm{id}_X$ in 2Top and the 2-morphisms $1_{e_X} = [h_{\mathrm{id}_X}]$ given by the trivial homotopy h_{id_X} from id_X to id_X, we obtain

$$\begin{aligned} &F_{X,X}(e_X) = F_{X,X}(\mathrm{id}_X) = \Pi_1(\mathrm{id}_X) = \mathrm{id}_{\Pi_1(X)} = e_{\Pi_1(X)}, \\ &F_{X,X}([h_{\mathrm{id}_X}]) = [(h_{\mathrm{id}_X})]_x = [\gamma_x]_x = 1_{e_{\Pi(X,X)}} \end{aligned}$$

This shows that the functors $F_{X,X}$ are compatible with units. To check compatibility with the horizontal composition, consider continuous maps $f_1, f'_1 : X \to Y$, $f_2, f'_2 : Y \to Z$ and homotopies $h_1 : [0,1] \times X \to Y$ from f_1 to f'_1 relative to $\{0,1\} \times X$ and $h_2 : [0,1] \times Y \to Z$ from f_2 to f'_2 relative to $\{0,1\} \times Y$, respectively. Then it follows directly that the functors $F_{X,Y} :$ 2Top$(X,Y) \to$ Fun$(\Pi_1(X), \Pi_1(Y))$ satisfy

$$F_{X,Z}(f_2 \bullet_{X,Y,Z} f_1) = F_{X,Z}(f_2 \circ f_1) = \Pi_1(f_2 \circ f_1) = \Pi_1(f_2)\Pi_1(f_1) = F_{Y,Z}(f_2) \bullet_{\Pi_1(Y),\Pi_1(Z)} F_{X,Y}(f_1).$$

Moreover, from $h_1 \sim_{\{0,1\}\times X} h_{f'_1} \cdot h_1$ and $h_2 \sim_{\{0,1\}\times Y} h_2 \cdot h_{f_2}$ and the identities

$$(h_2 \cdot h)f_2) \bullet (h_{f'_1} \cdot h_1)(t,x) = (h_2 \cdot h_{f_2})(t, (h_{f'_1} \cdot h_1)(t,x)) = \begin{cases} f_2 \circ h_1(2t,x) & t \in [0, \frac{1}{2}) \\ h_2(2t-1, f'_1(x)) & t \in [\frac{1}{2}, 1], \end{cases}$$

$$\Rightarrow \; ((h_2 \cdot h_{f_2}) \bullet (h_{f'_1} \cdot))_x = (h_2)_{f'_1(x)} \star (f_2 \circ h_1)_x$$

we obtain for all $x \in X$

$$\begin{aligned} &(F_{Y,Z}([h_2] \bullet_{X,Y,Z} [h_1]))_x = [(h_2)_{f'_1(x)} \star (f_2 \circ h_1)_x]_{0,1} = [(h_2)_{f'_1(x)}]_{0,1} \circ [(f_2 \circ h_1)_x]_{0,1} \\ &= \Pi_1([h_2])_{f'_1(x)} \circ \Pi_1(f_2)(\Pi_1([h_1])_x) = F_{Y,Z}([h_2])_{f'_1(x)} \circ \Pi_1(f_2)(F_{X,Y}([h_1]))_x \\ &= (F_{Y,Z}([h_2]) \bullet_{\Pi_1(X),\Pi_1(Y),\Pi_1(Z)} F_{X,Y}([h_1]))_x. \end{aligned}$$

This shows that $F_{Y,Z}([h_2] \bullet_{X,Y,Z} [h_1]) = F_{Y,Z}([h_2]) \bullet_{\Pi_1(X),\Pi_1(Y),\Pi_1(Z)} F_{X,Y}([h_1])$, and hence all 2-morphisms $\phi_{g,f} = 1_{\Pi_1(g\circ f)} : \Pi_1(g \circ f) \to \Pi_1(g \circ f)$ and $\phi_X = 1_{\mathrm{id}_{\Pi_1(x)}} : \mathrm{id}_{\Pi_1(X)} \to \mathrm{id}_{\Pi_1(X)}$ are trivial. This shows that $F :$ 2Top $\to$ Grpd is a strict 2-functor. $\square$

This theorem shows that the notion of 2-categories and 2-functors between them give rise to a more conceptual formulation of Theorem 3.2.8. However, this is not the end of the story. A closer inspection of Definition 3.5.1 reveals that a 2-category is not the most general structure that involves three layers of structure and two composition laws. More precisely, the associativity condition on the horizontal composition and the compatibility condition for horizontal composition and units in Definition 3.5.1 are unnecessarily strict. The corresponding equations impose that certain functors are equal, which is a rather unnatural requirement. By weakening this requirement to the condition that these functors are naturally isomorphic, one obtains the notion of a bicategory.

Definition 3.5.11: A **bicategory** $\mathcal{C}$ consists of

1. A collection of objects $X, Y, ...$
2. For any pair of objects X, Y, a category $\mathcal{C}(X, Y)$. The objects in $\mathcal{C}(X, Y)$ are called **1-morphisms**, the morphisms in $\mathcal{C}(X, Y)$ **2-morphisms**. 1-morphisms in $\mathcal{C}(X, Y)$ are denoted $f : X \to Y$ and we write $\alpha : f \Rightarrow g$ for a 2-morphism between 1-morphisms $f, g : X \to Y$. The composition of morphisms in $\mathcal{C}(X, Y)$ is denoted $\cdot$ and called **vertical composition**.
3. For any object X a **unit functor** $I_X : \mathcal{I} \to \mathcal{C}(X, X)$ where $\mathcal{I}$ is the trivial category with one object X and one 1-morphism 1_X. This is equivalent to the choice of a 1-morphism e_X and a 2-morphism $\epsilon_X : e_X \Rightarrow e_X$ for all objects X.
4. For any triple of objects X, Y, Z a functor $\bullet_{X,Y,Z} : \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \to \mathcal{C}(X, Z)$, called the **horizontal composition functor**.
5. For any pair of objects X, Y, natural isomorphisms
$$l_{X,Y} : \bullet_{X,Y,Y}(I_Y \times \mathrm{id}_{\mathcal{C}(X,Y)}) \xrightarrow{\sim} \mathrm{id}_{\mathcal{C}(X,Y)}, \qquad r_{X,Y} : \bullet_{X,X,Y}(\mathrm{id}_{\mathcal{C}(X,Y)} \times I_Y) \xrightarrow{\sim} \mathrm{id}_{\mathcal{C}(X,Y)},$$
the **left and right unit constraints**
6. For any quadruple of objects W, X, Y, Z, natural isomorphisms, called **associators**,
$$a_{W,X,Y,Z} : \bullet_{W,X,Z}(\bullet_{X,Y,Z} \times \mathrm{id}_{\mathcal{C}(W,X)}) \xrightarrow{\sim} \bullet_{W,Y,Z}(\mathrm{id}_{\mathcal{C}(Y,Z)} \times \bullet_{W,X,Y}),$$

such that the following axioms are satisfied

(B1) **pentagon axiom**: for all objects f, g, h, k in, respectively, $\mathcal{C}(V, W)$, $\mathcal{C}(W, X)$, $\mathcal{C}(X, Y)$, $\mathcal{C}(Y, Z)$ the following diagram commutes

$$\begin{tikzcd}
(k \bullet_{X,Y,Z} h) \bullet_{V,X,Z} (g \bullet_{V,W,X} f) \arrow[dr, "a_{V,X,Y,Z}"] & \\
((k \bullet_{X,Y,Z} h) \bullet_{W,X,Z} g) \bullet_{V,W,Z} f \arrow[u, "a_{V,W,X,Z}"] \arrow[d, "a_{W,X,Y,Z} \bullet 1_f"'] & k \bullet_{V,Y,Z} (h \bullet_{V,X,Y} (g \bullet_{V,W,X} f)) \\
(k \bullet_{W,Y,Z} (h \bullet_{W,X,Y} g)) \bullet_{V,W,Z} f \arrow[r, "a_{V,W,Y,Z}"'] & k \bullet_{V,Y,Z} ((h \bullet_{W,X,Y} g) \bullet_{V,W,Y} f)) \arrow[u, "1_k \bullet a_{V,W,X,Y}"']
\end{tikzcd},$$

where we write $a_{V,W,X,Y}$ for the morphisms $(a_{V,W,X,Y})_{(h,g,f)} : (h \bullet_{W,X,Y} g) \bullet_{V,W,Y} f \to h \bullet_{V,X,Y} (g \bullet_{V,W,X} f)$ in $\mathcal{C}(V,Y)$.

(B2) **triangle axiom**: for all objects f, g in, respectively, $\mathcal{C}(V,W)$, $\mathcal{C}(W,X)$, the following diagram commutes

$$\begin{tikzcd}
(g \bullet_{W,W,X} I_W) \bullet_{V,W,X} f \arrow[r, "a_{V,W,X,Y}"] \arrow[d, "r_{W,X}\bullet 1_f"'] & g \bullet_{V,X,Y} (I_W \bullet_{V,W,W} f) \arrow[dl, "(1_g \bullet l_{V,X})"] \\
g \bullet_{V,W,X} f, &
\end{tikzcd}$$

where we write $r_{W,X}$ for the morphism $(r_{W,X})_g : g \bullet_{W,W,X} I_W \to g$.

Clearly, a bicategory in which all natural isomorphisms $l_{X,Y}$, $r_{X,Y}$ and $a_{W,X,Y,Z}$ are identity natural transformations is a 2-category. However, the notion of a bicategory is more general which makes it much easier to find examples. This is obvious already in the simplest example, namely a bicategory with a single object.

Example 3.5.12: A bicategory with one object is a monoidal category. Examples of monoidal categories are the category $\mathrm{Vect}_{\mathbb{F}}$ with the tensor product of vector spaces and the unit object $e = \mathbb{F}$. More generally, the category R-Mod-R of (R,R)-bimodules over a unital ring R is a monoidal category with the tensor product of bimodules as horizontal composition and the unit bimodule R with the bimodule structure given by left and right multiplication as the tensor unit.

Beyond the fact that a bicategory is a more natural notion than a 2-category and that there are many algebraic structures that form bicategories but not 2-categories, there is another motivation for bicategories that arises directly from algebraic topology. This is the fact that bicategories provide a different perspective on the fundamental groupoid and relate its definition to the familiar construction of the homotopy category of topological spaces. To see this, we first consider a bicategory $\Pi_2(X)$, which can be viewed as a higher analogue of the fundamental groupoid.

Example 3.5.13: Let X be topological space. The bicategory $\Pi_2(X)$ consists of:

1. **objects**: points $x \in X$,
2. **1- and 2-morphisms**: the category $\Pi_2(x,y)$ for $x, y \in X$ has as objects paths $\gamma : [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. A morphism $[h] : \gamma \Rightarrow \gamma'$ from $\gamma : [0,1] \to X$ to $\gamma' : [0,1] \to X$ is a homotopy class relative to $\{0,1\} \times [0,1]$ of a homotopy h relative to $\{0,1\}$ from γ to γ'. The vertical composition is given by the composition of homotopies

$$[h'] \cdot [h] = [h \cdot h'] \qquad h' \cdot h(s,t) = \begin{cases} h(2s,t) & s \in [0, \tfrac{1}{2}) \\ h'(2s-1,t) & t \in [\tfrac{1}{2}, 1] \end{cases}$$

and the identity morphisms are the homotopy classes $[h_\gamma]$ relative to $\{0,1\} \times [0,1]$ of trivial homotopies $h_\gamma : [0,1] \times [0,1] \to X$, $h_\gamma(s,t) = \gamma(t)$ for all $s, t \in [0,1]$.

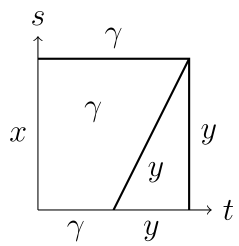
3. **horizontal composition**: The horizontal composition functor is given by the composition of paths and homotopies

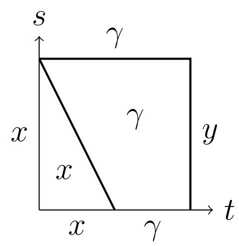
$$(\gamma_2 \bullet_{x,y,z} \gamma_1)(t) = \gamma_2 \star \gamma_1(t) \begin{cases} \gamma_2(2t) & t \in [0, \frac{1}{2}) \\ \gamma_2(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$

$$[h_2] \bullet_{x,y,z} [h_1] = [h_2 \star h_1] \quad \text{with} \quad (h_2 \star h_1)(s,t) = \begin{cases} h_1(s, 2t) & t \in [0, \frac{1}{2}) \\ h_2(s, 2t-1) & t \in [\frac{1}{2}, 1]. \end{cases}$$

4. **unit functors**: The unit functors are given by the trivial paths γ_x with $\gamma_x(t) = x$ for all $t \in [0,1]$ and the homotopy classes of trivial homotopies $[h_x]$ with $h_x(s,t) = x$ for all $s, t \in [0,1]$.
5. **left and right unit constraints**: The left and right unit constraints associate to each path $\gamma : x \to y$ the 2-morphisms $[h_L] : \gamma_y \star \gamma \Rightarrow \gamma$ and $[h_R] : \gamma \star \gamma_x \Rightarrow \gamma$ given by

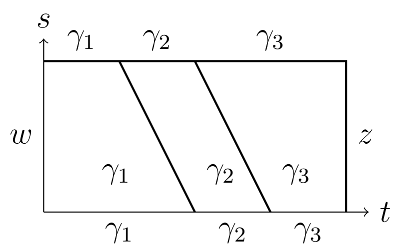
$$h_L(s,t) = \begin{cases} \gamma(\frac{2t}{1+s}) & t \in [0, \frac{1+s}{2}) \\ y & t \in [\frac{1+s}{2}, 1] \end{cases} \quad h_R(s,t) = \begin{cases} x & t \in [0, \frac{1-s}{2}) \\ \gamma(\frac{2t-1+s}{1+s}) & t \in [\frac{1-s}{2}, 1] \end{cases}$$





6. **associators**: the associators associate to each triple of paths $\gamma_1 : w \to x$, $\gamma_2 : x \to y$, $\gamma_3 : y \to z$ the 2-morphism $[h] : (\gamma_3 \bullet_{x,y,z} \gamma_2) \bullet_{w,x,z} \gamma_1 \Rightarrow \gamma_3 \bullet_{w,y,z} (\gamma_2 \bullet_{w,x,y} \gamma_1)$ given by

$$h(s,t,x) = \begin{cases} \gamma_1(\frac{4t}{2-s}) & t \in [0, \frac{2-s}{4}] \\ \gamma_2(\frac{4t-2+s}{4}) & t \in [\frac{2-s}{4}, \frac{3-s}{4}) \\ \gamma_3(\frac{4t-3+s}{1+s}) & t \in [\frac{3-s}{4}, 1]. \end{cases}$$



That the vertical composition of 2-morphisms is associative and unital follows in the same way as in Example 3.5.6. The horizontal composition functor is not longer associative and unital since it is paths rather than homotopy classes of paths that are composed. The associators and left and right unit constraints are then given by the associated homotopies. That the horizontal composition and unit functors are functors follows directly from the fact that the vertical composition of 2-morphisms is based on the first argument of the associated homotopies while the horizontal composition makes use of the second argument. That the associators and unit constraints satisfy the pentagon and triangle identities follows by a lengthy but straightforward computation.

We will now clarify how the bicategory $\Pi_2(X)$ is related to the fundamental groupoid $\Pi_1(X)$ for a topological space X. Clearly, the two have the same objects, namely the points in X. A morphism $[\gamma] : x \to Y$ in $\Pi_1(X)$ is a homotopy class relative to $\{0,1\}$ of a path $\gamma : [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$, i. e. a 1-morphism in $\Pi_2(X)$. Moreover, two such paths $\gamma, \gamma' : [0,1] \to X$ are homotopic relative to $\{0,1\}$ if and only if they are isomorphic in the category $\Pi_2(X)(x,y)$. The composition of morphisms in $\Pi_1(X)$ is related to the horizontal composition in $\Pi_2(X)$ and the identity morphisms to the associated units. This shows that the fundamental groupoid $\Pi_1(X)$ is obtained from $\Pi_2(X)$ via the general construction in the following lemma.

Lemma 3.5.14: Let $\mathcal{C}$ be a bicategory. Then there is a category $\mathrm{h}\mathcal{C}$ whose objects are the objects of $\mathcal{C}$ and whose morphisms are 2-isomorphism classes of 1-morphisms $f : X \to Y$ in $\mathcal{C}$, with the composition of morphisms given by $[g] \circ [f] = [g \bullet_{X,Y,Z} f]$ and with identity morphisms $1_X = [e_X] : X \to X$.

Proof:
The composition of morphisms is well-defined, since for any 1-morphisms $f, f' : X \to Y$, $g, g' : Y \to Z$ in $\mathcal{C}$ for which there are 2-isomorphisms $\alpha : f \to f'$, $\beta : g \to g'$, the 2-morphism $\beta \bullet_{X,Y,Z} \alpha : g \bullet_{X,Y,Z} f \to g' \bullet_{X,Y,Z} f'$ is a 2-isomorphism. The composition of morphisms is associative since for any triple of 1-morphisms $f : W \to X$, $g : X \to Y$, $h : Y \to Z$ the associator gives a 2-morphism $(h \bullet_{X,Y,Z} g) \bullet_{W,Y,Z} f \Rightarrow h \bullet_{W,Y,Z} (g \bullet_{W,X,Y} f)$, which is an isomorphism in $\mathcal{C}(W,Z)$. The identity morphisms satisfy $1_X \circ f = f = f \circ 1_Y$ since the left and right unit constraints define 2-morphisms $e_X \bullet_{W,X,X} f \Rightarrow f$ and $f \bullet_{W,W,X} e_Y \Rightarrow f$, which are isomorphisms in $\mathcal{C}(X,Y)$. $\square$

The usefulness of this construction is illustrated by the fact that it places two known and well-motivated concepts from algebraic topology in a common framework, namely the homotopy category of topological spaces and the fundamental groupoid of a topological space. Both are applications of Lemma 3.5.14 to different bicategories.

Example 3.5.15:

1. For the bicategory $\Pi_2(X)$ in Example 3.5.13, the associated category $\mathrm{h}\Pi_2(X)$ is the fundamental groupoid $\Pi_1(X)$.
2. For the 2-category 2Top in Example 3.5.6, the associated category h2Top is the homotopy category of topological spaces hTop with topological spaces as objects and homotopy classes of continuous maps as morphisms.

3.6 Exercises for Chapter 3

Exercise 1: Consider the category $Top(2)$ with pairs (X, A) of a topological space X and a subspace $A \subset X$ as objects and continuous maps $f : X \to Y$ with $f(A) \subset B$ as morphisms from (X, A) to (Y, B).

(a) Show that $\sim_A$ defines an equivalence relation on the set of morphisms $\mathrm{Hom}_{\mathrm{Top(2)}}((X,A),(Y,B))$ that is compatible with the composition of morphisms and the identity morphisms. Construct the associated quotient category $hTop(2)$.

(b) Characterise the isomorphisms in $hTop(2)$.

Exercise 2: Let X be a topological space and $\mathrm{Homeo}(X)$ the group of homeomorphisms $f: X \to X$

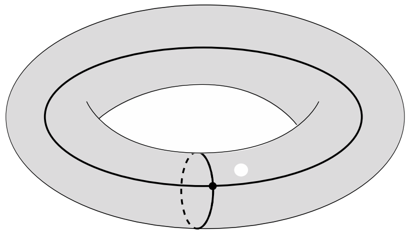
(a) Show that the homeomorphisms $f: X \to X$ that are homotopic to the identity map $\mathrm{id}_X: X \to X$ form a normal subgroup $\mathrm{Homeo}_0(X) \subset \mathrm{Homeo}(X)$.
(b) Show that the homotopy classes of homeomorphisms $f: X \to X$ form a group $\mathrm{Map}(X)$.
(c) Show that the group $\mathrm{Map}(X)$ is isomorphic to the quotient $\mathrm{Homeo}(X)/\mathrm{Homeo}_0(X)$.

Exercise 3: Let G be a topological group and $\rhd: G \times X \to X$ an action of G on a topological space X. Consider for $g \in G$ the continuous map $l_g: X \to X$, $x \mapsto g \rhd x$. Prove that if G is path-connected, then l_g is homotopic to the identity map id_X for all $g \in G$.

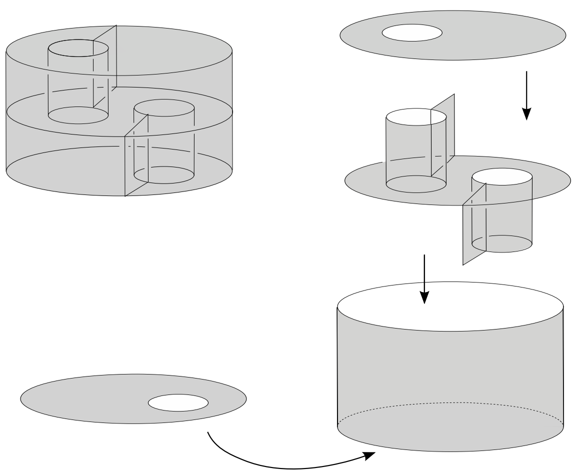
Exercise 4: Prove that a retract of a contractible topological space is contractible.

Exercise 5: Let X be a topological space. Prove the following statements:
(a) X is contractible if and only if every continuous map $f: X \to Y$ in a topological space Y is null homotopic.
(b) X is contractible if and only if every continuous map $f: W \to X$ from a topological space W is null homotopic.

Exercise 6: Prove that the graph pictured below (bold lines and bold vertex) is a deformation retract of the punctured torus.



Exercise 7: Show that the subset of $\mathbb{R}^3$ pictured below (**Bing's house**) is contractible.



Exercise 8: Consider the Möbius strip $M = [0,1] \times [0,1]/\sim$ with the equivalence relation $(x,0) \sim (1-x,1)$ for $x \in [0,1]$. Show that the Möbius strip is homeomorphic to a mapping cylinder.

Exercise 9: Let X be a topological space, $[0,1]^n = [0,1] \times \ldots \times [0,1]$ for $n \in \mathbb{N}$, $[0,1]^0 := \{0\}$ and $f^{(n)} : [0,1]^n \to X$ constant maps. Show that homotopies from $f^{(n)}$ to $f^{(n)}$ relative to $\partial[0,1]^n$ are in bijection with continuous maps $h^{(n)} : S^n \to X$ with $h^{(n)}(p) = f^{(n)}([0,1]^n)$ for a point $p \in S^n$.

Exercise 10: Let X be a path connected topological space. Show that the fundamental group $\pi_1(x, X)$ is abelian if and only if for all paths $\gamma : [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$ the group isomorphism $\Phi_{[\gamma]} : \pi_1(x, X) \to \pi_1(y, X)$, $[\beta] \mapsto [\gamma] \circ [\beta] \circ [\overline{\gamma}]$ does not depend on the choice of $\gamma : [0,1] \to X$.

Exercise 11: The **Cartesian product** of two categories $\mathcal{C}$ and $\mathcal{D}$ is the category $\mathcal{C} \times \mathcal{D}$ whose objects are pairs (C, D) of objects C in $\mathcal{C}$ and D in $\mathcal{D}$, morphisms $\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((C,D),(C',D')) = \mathrm{Hom}_{\mathcal{C}}(C, C') \times \mathrm{Hom}_{\mathcal{D}}(D, D')$ and the composition of morphisms given by $(f,g) \circ_{\mathcal{C}\times\mathcal{D}} (h,k) = (f \circ_{\mathcal{C}} h, g \circ_{\mathcal{D}} k)$.

(a) Show that for any two topological spaces X, Y the fundamental groupoid of the product space $X \times Y$ is given by $\Pi_1(X \times Y) = \Pi_1(X) \times \Pi_1(Y)$.
(b) Relate the fundamental group $\pi_1((x,y), X \times Y)$ to $\pi_1(x, X)$ and $\pi_1(y, Y)$.

Exercise 12: Let G be a topological group. For two paths $\gamma, \delta : [0,1] \to G$ with $\gamma(0) = \delta(0) = \gamma(1) = \delta(1) = e$, define a pointwise product by

$$(\gamma \bullet \delta)(t) := \gamma(t) \cdot \delta(t) \qquad \forall t \in [0,1].$$

(a) Show that $\gamma \sim_{\{0,1\}} \gamma'$ and $\delta \sim_{\{0,1\}} \delta'$ implies $\delta \bullet \gamma \sim_{\{0,1\}} \delta' \bullet \gamma'$. Conclude that $[\delta] \bullet [\gamma] := [\delta \bullet \gamma]$ defines a group structure on the set of homotopy classes of paths $\gamma : [0,1] \to G$ with $\gamma(0) = \gamma(1) = e$.
(b) Show that $\gamma \sim_{\{0,1\}} \delta$ if and only if $\gamma \bullet \delta^{-1} \sim_{\{0,1\}} \gamma_e$.
(c) Show that $[\delta] \bullet [\gamma] = [\gamma] \bullet [\delta]$ for all $\gamma, \delta : [0,1] \to G$ with $\gamma(0) = \delta(0) = \gamma(1) = \delta(1) = e$.
(d) Show that $[\delta] \circ [\gamma] = [\delta] \bullet [\gamma]$ for all $\gamma, \delta : [0,1] \to G$ with $\gamma(0) = \delta(0) = \gamma(1) = \delta(1) = e$. Conclude that for any topological group G, the fundamental group $\pi_1(e, G)$ is abelian.

Hint: For 3. and 4., use that $\gamma \star \gamma_e \sim_{\{0,1\}} \gamma \sim_{\{0,1\}} \gamma_e \star \gamma$ and $\gamma(0) = \gamma(1) = e$.

Exercise 13: Let G be a path connected topological group and X a G-space.

(a) Show that for every $x \in X$, the action $\rhd : G \times X \to X$ defines a group homomorphism $\pi_1(e, G) \to \pi_1(x, X)$.
(b) Show that the image of this group homomorphism is in the center of $\pi_1(x, X)$.

Exercise 14: Let X be a topological space. Show that the following statements are equivalent:

(a) Every continuous map $f : S^1 \to X$ is homotopic to a constant map.
(b) Every continuous map $f : S^1 \to X$ can be extended to a map $\overline{f} : D^2 \to X$ with $\overline{f}|_{\partial D^2} = f$.
(c) Any two paths paths $\gamma, \gamma' : [0,1] \to X$ with $\gamma(0) = \gamma'(0)$ and $\gamma(1) = \gamma'(1)$ are homotopic.

Exercise 15: Show that all of the following topological spaces are homotopy equivalent but none of them is homeomorphic to one of the others:

(a) the circle S^1,
(b) the ring $R = \{x \in \mathbb{R}^2 : \ 1 \leq x_1^2 + x_2^2 \leq 2\}$,
(c) the Möbius strip,
(d) $S^1 \times D^2$

Exercise 16: Prove the following statements:

(a) If $f, g : S^1 \to S^1$ are continuous with $f(z) \neq g(z)$ for all $z \in S^1$ then $\deg(f) = \deg(g)$.

(b) If $f : S^1 \to S^1$ is continuous with $\deg(f) = 0 \mod n$ then there is a continuous map $g : S^1 \to S^1$ with $f = g^n$.

(c) If $w = \exp(1/n)$ with $n \in \mathbb{N}$ and $f : S^1 \to S^1$ continuous with $f(wz) = f(z)$ for all $z \in S^1$ then $\deg(f) = 0 \mod n$.

Exercise 17: Show that for the following topological spaces X and subspaces $A \subset X$ there is no retraction $r : X \to A$:

(a) $X = S^1 \times D^2$, $A = S^1 \times \partial D^2 \approx S^1 \times S^1$.

(b) $X = D^2 \vee D^2$, $A = \partial D^2 \vee \partial D^2 \approx S^1 \vee S^1$.

(c) $X = D^2/\sim$ with $1 \sim -1$, $A = \partial X \approx S^1 \vee S^1$.

Exercise 18: For a continuous map $f : \mathbb{C} \to \mathbb{C}$ without zeros on the unit circle we set

$$W(f) = \deg(\bar{f}) \quad \text{where } \bar{f} : S^1 \to S^1, \bar{f}(z) = \frac{f(z)}{|f(z)|}.$$

Show that for any polynomial p without zeros on the unit circle $W(p)$ is the sum of the multiplicities of all zeros of p in the unit disc.

Exercise 19: Real projective space $\mathbb{R}\mathrm{P}^2$ is obtained as a quotient $D^2/\sim$, where $x \sim -x$ for all $x \in S^1 = \partial D^2$. Use the theorem of Seifert and van Kampen to obtain a presentation of $\pi_1(\mathbb{R}\mathrm{P}^2)$.

Exercise 20: The Klein bottle is the quotient $K = [0,1] \times [0,1]/\sim$, where $(x,0) \sim (x,1)$ and $(0,y) \sim (1, 1-y)$ for all $x, y \in [0,1]$. Use the theorem of Seifert and van Kampen to give a presentation of the fundamental group $\pi_1(K)$.

Exercise 21: Let $f : S^1 \to S^1$ be a continuous map with $\deg(f) = n \in \mathbb{N}$. Use the theorem of Seifert and van Kampen to compute the fundamental group $\pi_1(D^2 \cup_f S^1)$ of the topological space $D^2 \cup_f S^1$ obtained by attaching a 2-cell to S^1 with f.

Exercise 22: Show that the fundamental group of an n-punctured surface of genus g has a presentation

$$\pi_1(T^{\#g} \setminus \{p_1, ..., p_n\}) = \langle m_1, ..., m_n, a_1, b_1, ..., a_g, b_g | \, [b_g, a_g] \cdots [b_1, a_1] m_n \cdots m_1 = 1 \rangle$$

Exercise 23: Consider the union $X \subset \mathbb{R}^3$ of $n \in \mathbb{N}$ lines through the origin in $\mathbb{R}^3$. Determine the fundamental group $\pi_1(\mathbb{R}^3 \setminus X)$.

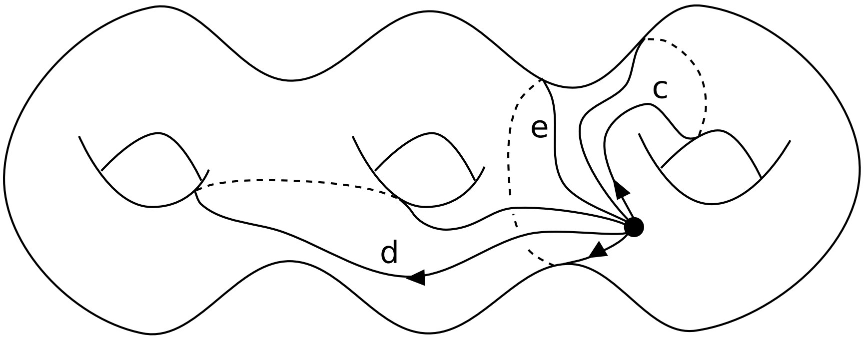
Exercise 24: Let X be the union of $n \in \mathbb{N}$ parallel lines in $\mathbb{R}^3$. Determine the fundamental group $\pi_1(\mathbb{R}^3 \setminus X)$.

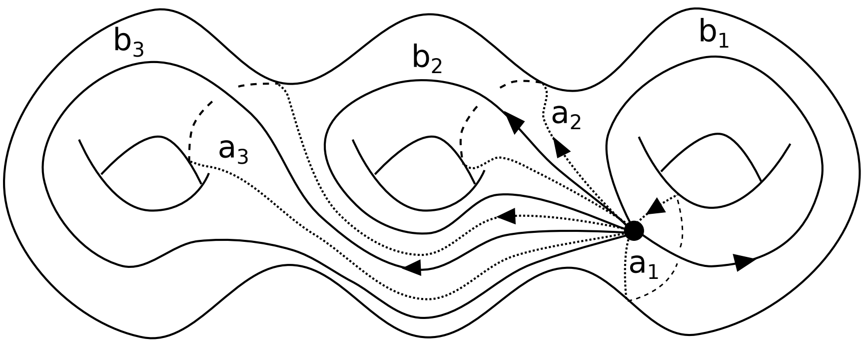
Exercise 25: Let $X_1, ..., X_n$ pairwise disjoint path connected topological spaces, $x_i \in X_i$ for $i \in \{1, ..., n\}$ and suppose that $x_i \in X_i$ has a contractible open neighbourhood $U_i \subset X_i$. Show that the fundamental group of the wedge sum $X_1 \vee \ldots \vee X_n$ with respect to $x = [x_1] = \ldots = [x_n]$ is given by

$$\pi_1(x, X_1 \vee \ldots \vee X_n) = \pi_1(x_1, X_1) \star \ldots \star \pi_1(x_n, X_n).$$

Hint: Consider the sets $V_j := X_j \vee \left(\bigvee_{i \in \{1,\ldots,n\}, i \neq j} U_i \right)$.

Exercise 26: Express the homotopy classes of the paths c, d, e as products of the homotopy classes of the paths a_i, b_i and their inverses.

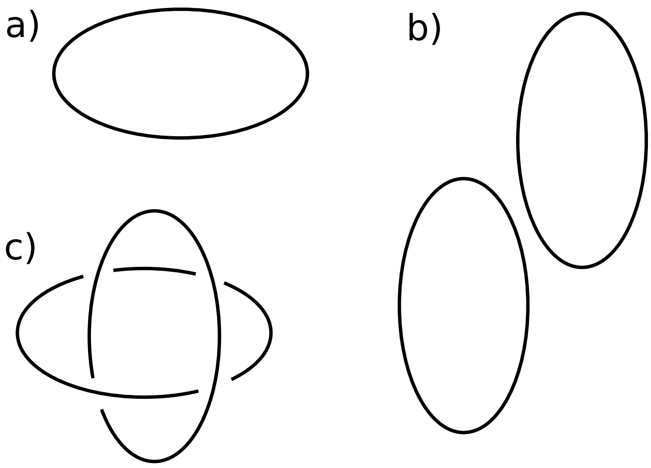




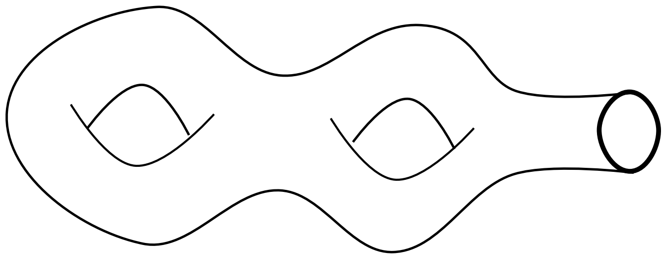
Exercise 27: Consider the subsets $A \subset \mathbb{R}^3$ pictured below:

(a) a ring S^1
(b) two separate rings
(c) two linked rings

Determine the fundamental group of $\mathbb{R}^3 \setminus A$.



Exercise 28: Show that the boundary of the surface (in boldface) cannot be a retract of the surface.



Exercise 29: Let $X = \bigcup_{i=0}^{2} X^n$ a path-connected CW-complex of cellular dimension 2.

(a) Consider the 1-skeleton X^1. A **maximal tree** $T \subset X^1$ is a maximal 1-connected subcomplex of X^1. Show that the topological space $X/\sim$ obtained by collapsing a maximal tree $T \subset X^1$ is homotopy equivalent to X^1.

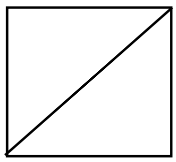
(b) Determine the fundamental group of $X^1/\sim$.

(c) Compute the fundamental group $\pi_1(X)$. For this, determine first how the topological space obtained by collapsing $T \subset X$ can be obtained by attaching 2-cells to $X^1/\sim$.

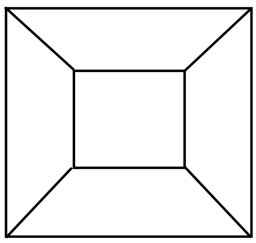
Exercise 30: Determine the fundamental groups of the graphs $X = X^0 \cup X^1$ pictured below.

Hint: Use Exercise 29.

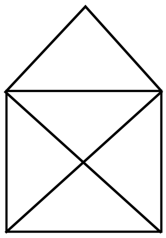
a)



b)



c)



4 Simplicial and singular homology

4.1 Singular simplexes and Δ-complexes

In homology theories topological spaces are described in terms of continuous maps from simplexes (convex hulls of certain sets of points in $\mathbb{R}^n$) into a topological space. This looks similar at first sight to the fundamental group(oid), in which one characterises a topological space X in terms of open or closed paths, i. e. continuous maps from the interval or a circle to X, and to the higher homotopy groups, in which one considers continuous maps from the n-spheres or the unit n-cube with $n \geq 2$. The use of simplexes instead of spheres or discs is motivated mainly by combinatorial simplifications, and the fundamental difference between homotopy and homology lies elsewhere, namely in the algebraic structures that are used to organise these continuous maps from cubes, spheres or simplexes.

In homotopy theory, the relevant structures are group(oid)s and arise from the concatenation of paths or, more generally, unit cubes. In homology theories, one instead works with linear combinations by organising these continuous maps into a module over a commutative ring. Roughly speaking, this amounts to neglecting the order in which the relevant structures are composed. As a consequence, (higher) homologies are much easier to handle than (higher) homotopy groups. Given a well-behaved topological space and a nice decomposition into the images of certain simplexes, all homologies can be computed straightforwardly, while to computation of homotopy groups is a more difficult problem.

To introduce homology theories, we first have to assemble some basic background on simplexes in $\mathbb{R}^n$. It should be kept in mind, however, that it is the algebraic structures used to organise them rather than the combinatorial properties of the simplexes which are essential.

Definition 4.1.1:

1. An **affine n-simplex** $[v_0, ..., v_n]$ is the **convex hull** of $(n+1)$ points $v_0, v_1, ..., v_n \in \mathbb{R}^m$

$$[v_0, ..., v_n] = \text{conv}(\{v_0, ..., v_n\}) = \bigcap_{\substack{C \subset \mathbb{R}^m \text{ convex} \\ v_0, ..., v_n \in C}} C = \{\Sigma_{i=0}^n t_i v_i \,:\, t_i \geq 0,\, \Sigma_{i=0}^n t_i = 1\}\,.$$

   The points $v_0, ..., v_n$ are called **vertices** and the point $b = \frac{1}{n+1}\sum_{i=0}^n v_i$ the **barycentre** of the simplex $[v_0, ..., v_n]$. The convex hull of a subset $V \subset \{v_0, ..., v_n\}$ with $|V| = k+1$ is called a k-**face** of $[v_0, ..., v_n]$.

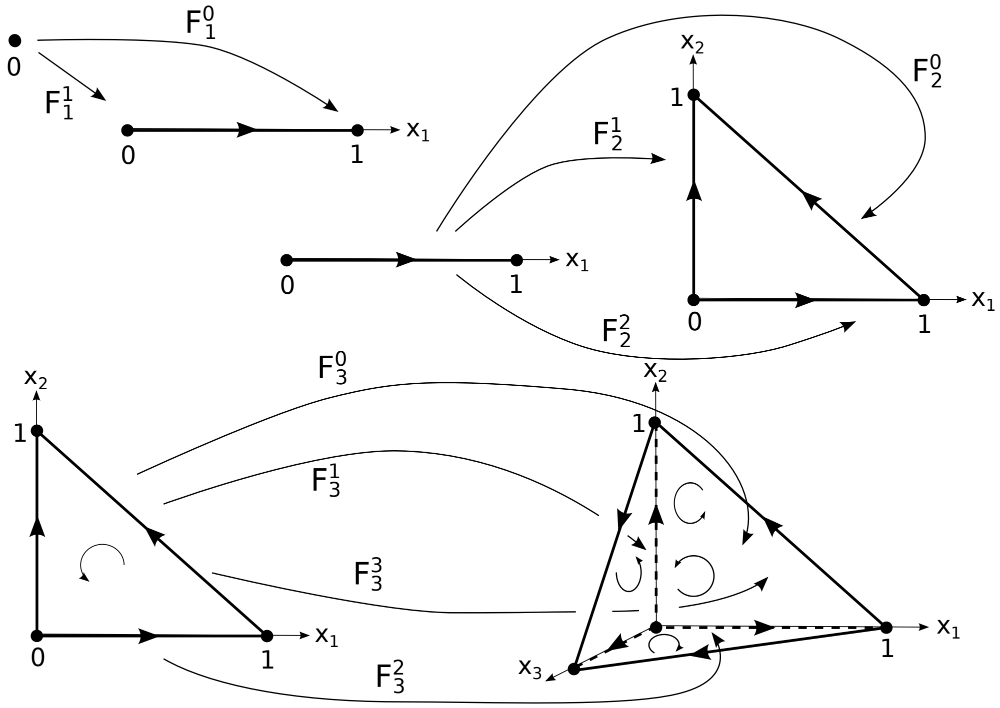
2. An affine simplex $[v_0, ..., v_n]$ is called **regular** if it is not contained in an affine plane of dimension $< n$, i. e. if $v_1 - v_0, v_2 - v_0, ..., v_n - v_0 \in \mathbb{R}^m$ are linearly independent, and **singular** otherwise.

3. The **standard** n**-simplex** $\Delta^n \subset \mathbb{R}^n$ is the set

$$\Delta^n = [e_0, e_1, ..., e_n] = \{\Sigma_{i=0}^n t_i e_i \in \mathbb{R}^n \,:\, t_i \geq 0,\, \Sigma_{i=0}^n t_i = 1\} \subset \mathbb{R}^n,$$

   where $e_1, ..., e_n$ is the standard basis of $\mathbb{R}^n$ and $e_0 = 0$ the null vector.

4. For $i \in \{0, ..., n\}$ the **face map** is the affine linear map $F_n^i : \Delta^{n-1} \to \Delta^n$ that maps the standard $(n-1)$-simplex Δ^{n-1} to the face $[e_0, ..., e_{i-1}, e_{i+1}, ..., e_n]$ opposite to e_i in the standard simplex Δ^n. It is given by $F_n^i(e_j) = e_j$ for $j < i$ and $F_n^i(e_j) = e_{j+1}$ for $j \geq i$.



Standard n-simplexes and face maps.

Remark 4.1.2:

1. Affine simplexes in $\mathbb{R}^m$ are equipped with the subspace topology. The **boundary** of an affine simplex $[v_0, ..., v_n]$ is the the union of its $(n-1)$-faces

$$\partial[v_0, ..., v_n] = \cup_{i=0}^n [v_0, ..., v_{i-1}, v_{i+1}, ..., v_n].$$

   Its **interior** the set

$$[v_0, ..., v_n] \setminus \partial[v_0, ..., v_n] = \{\Sigma_{i=0}^n t_i v_i : \, t_i \in (0,1), \Sigma_{i=0}^n t_i = 1\}.$$

2. An affine linear map $\phi : [v_0, ..., v_n] \to [w_0, ..., w_m]$ is determined uniquely by the images of the vertices $v_0, ..., v_n$. In particular, for an n-simplex $[v_0, ..., v_n]$, there is a unique affine linear map $\phi : \Delta^n \to [v_0, ..., v_n]$ with $\phi(e_i) = v_i$, namely $\sum_{i=0}^n t_i e_i \mapsto \sum_{i=0}^n t_i v_i$. The coefficients $t_0, ..., t_i$ are called **barycentric coordinates**.

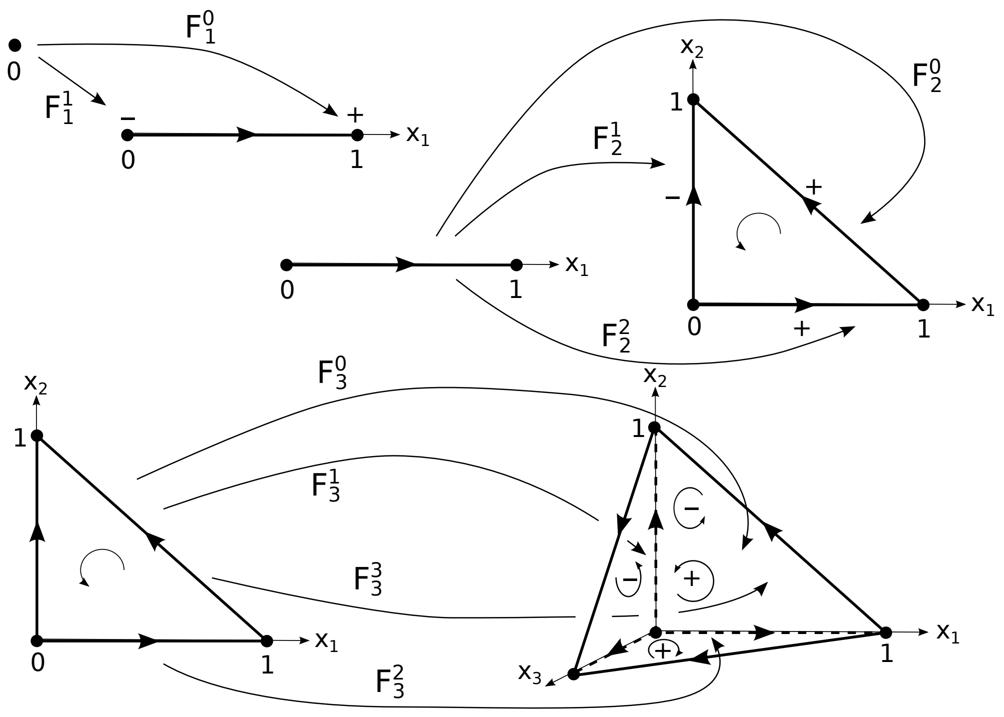
In the following, we will not only consider affine simplexes as subsets of $\mathbb{R}^n$ but also equip these simplexes and their faces with an orientation. For this, recall that an oriented vector space is a (finite-dimensional) vector space V together with the choice of an ordered basis $v_1, ..., v_n$. A bijective linear map $V \to V$ is called order preserving if its representing matrix with respect to the ordered basis $v_1, ..., v_n$ has determinant 1 and order reversing if it has determinant -1.

If $[v_0, ..., v_n] \subset \mathbb{R}^n$ is a regular affine n-simplex with ordered vertices, we can thus assign an orientation to $[v_0, ..., v_n]$ by requiring that the linear map $\Phi : \mathbb{R}^n \to \mathbb{R}^n$, $e_i \mapsto v_i - v_0$ is orientation preserving, which is the case if and only if $\det(v_1 - v_0, ..., v_n - v_0) > 0$. We then equip each $(n-1)$-face $[v_0, ..., v_{i-1}, v_{i+1}, ..., v_n]$ with the orientation induced by the normal vector that points inwards into the n-simplex. The orientation induced by requiring that the face map $\Phi \circ F_n^i : \Delta^{n-1} \to [v_0, .., v_{i-1}, v_{i+1}, ..., v_n]$ is orientation preserving agrees with this orientation if i is even and is the opposite of this orientation if i is odd. It is also clear that the orientation depends on the ordering of a simplex only up to even permutations.

Definition 4.1.3:

1. An **ordered n-simplex** is an affine n-simplex together with a linear ordering of its vertices. We write $[v_0, ..., v_n]$ for the affine n-simplex with ordering $v_0 < v_1 < \ldots < v_n$.
2. The **orientation** of a regular ordered affine n-simplex $[v_0, ..., v_n] \subset \mathbb{R}^n$ is defined as $\epsilon([v_0, ..., v_n]) = \mathrm{sgn}\det(v_1 - v_0, ..., v_n - v_0)$. The orientation of $[v_0, ..., v_n]$ induces an orientation on each face of $[v_0, ..., v_n]$, which is defined inductively by

$$\epsilon([v_0, ..., v_{i-1}, v_{i+1}, ..., v_n]) = \begin{cases} \epsilon([v_0, ..., v_n]) & i \,\text{even} \\ -\epsilon([v_0, ..., v_n]) & i \,\text{odd.} \end{cases}$$



Face maps and orientations for the standard n-simplexes.

To describe a topological space X in terms of continuous maps $\sigma : \Delta^n \to X$, one must decide which continuous maps are considered - all continuous maps, embeddings or only continuous maps that satisfy certain compatibility conditions. Different choices lead to different versions of homology. In the following, we focus on two main examples, namely *singular* and *simplicial homology*. The former admits all continuous maps $\sigma : \Delta^n \to X$ - even very singular ones that map the entire simplex to a single point. The latter is based on collections of maps that are homeomorphisms onto their image when restricted to the interior of the standard n-simplex and satisfy certain matching conditions.

Definition 4.1.4: Let X be a topological space and R a commutative unital ring.

1. For $n \in \mathbb{N}_0$, a **singular n-simplex** in X is a continuous map $\sigma : \Delta^n \to X$. We denote by $S_n(X)$ the free R-module generated by the set of singular n-simplexes $\sigma : \Delta^n \to X$. Elements of $S_n(X)$ are called **singular n-chains**.
2. For $n \in \mathbb{N}$, the **boundary operator** $\partial_n : S_n(X) \to S_{n-1}(X)$ is the R-module morphism defined by
$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \, \sigma \circ F_n^i$$
for singular n-simplexes $\sigma : \Delta^n \to X$, where $F_n^i : \Delta^{n-1} \to \Delta^n$ are the face maps from Definition 4.1.1. For $n = 0$, we set $\partial_0 = 0$.

Notational convention: Here and in the following, we we write rm instead of $r \rhd m$ for the structure map $\rhd : R \times M \to M$ of an R-module M. We write 1 for the multiplicative unit of R, -1 for its additive inverse and, more generally $\pm n = -(1 + \ldots + 1)$ for the n-fold sum of 1 and its additive inverse.

The idea of the boundary operator is that it describes the boundary of a simplex as an R-linear combination of its faces. The coefficients of this linear combination tell one how often a given face arises in the boundary and their signs characterise the orientation.

Example 4.1.5: Consider $R = \mathbb{Z}$, $X = \mathbb{R}^3$ and choose vertices $v_0, v_1, v_2, v_3 \in \mathbb{R}^3$. Denote by $\sigma_n : \Delta^n \to \mathbb{R}^3$ the affine linear map with $\sigma_n(e_i) = v_i$ for $i \in \{0, ..., n\}$ and $n = 1, 2, 3$.

1. Then $\partial_1(\sigma_1) = \sigma_1 \circ F_1^0 - \sigma_1 \circ F_1^1$, which implies $\partial_1(\sigma_1)([e_0]) = [v_1] - [v_0]$. This is a linear combination of the endpoints of the oriented line segment from v_0 to v_1 with a positive sign for the endpoint and a negative sign for the starting point.
2. For $\sigma_2 : \Delta^2 \to \mathbb{R}^3$, one obtains $\partial_2(\sigma_2) = \sigma_2 \circ F_2^0 - \sigma_2 \circ F_2^1 + \sigma_2 \circ F_2^2$, which implies $\partial_2(\sigma_2)([e_0, e_1]) = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$. This is a linear combination of the faces in the triangle with coefficients $+1$ if their orientation agrees with the one of triangle and -1 if it is opposite. By composing two boundary operators, one obtains
$$\partial_1 \circ \partial_2(\sigma_2)([e_0]) = [v_2] - [v_1] - ([v_2] - [v_0]) + [v_1] - [v_0] = 0.$$
This is because each vertex occurs in exactly two edges, once as the starting point and once as the end point, if these edges are oriented in the sense of the triangle.
3. Similarly, one obtains for $n = 3$:
$$\begin{aligned}
&\partial_3(\sigma_3) = \sigma_3 \circ F_3^0 - \sigma_3 \circ F_3^1 + \sigma_3 \circ F_3^2 - \sigma_3 \circ F_3^3 \\
&\partial_3(\sigma_3)([e_0, e_1, e_2]) = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2]
\end{aligned}$$
This is a linear combination of the 2-faces of tetrahedron Δ^3 with a coefficient 1 if the orientation agrees with the one of the tetrahedron and -1 if it is the opposite. By composing this with the boundary operator ∂_2, we obtain
$$\begin{aligned}
\partial_2 \circ \partial_3(\sigma_3)([e_0, e_1]) =& [v_2, v_3] - [v_1, v_3] + [v_1, v_2] - ([v_2, v_3] - [v_0, v_3] + [v_0, v_2]) \\
&+ [v_1, v_3] - [v_0, v_3] + [v_0, v_1] - ([v_1, v_2] - [v_0, v_2] + [v_0, v_1]) = 0
\end{aligned}$$

This is because every edge occurs in exactly two triangles, once with positive and once with negative orientation, if the triangles are equipped with the orientation induced by the one of the tetrahedron.

4. Let v_4 be the image of v_0 under reflection on the affine hyperplane through v_1, v_2, v_3 and $\tau_3 : \Delta^3 \to \mathbb{R}^3$ the affine linear map with $\tau_3(e_0) = v_1$, $\tau_3(e_1) = v_2$, $\tau_3(e_2) = v_3$, $\tau_3(e_3) = v_4$. This corresponds to two tetrahedra which intersect in the triangle $[v_1, v_2, v_3]$. We obtain
$$\begin{aligned}\partial_3(\sigma_3 + \tau_3)([e_0, e_1, e_2]) = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2]\\ + [v_2, v_3, v_4] - [v_1, v_3, v_4] + [v_1, v_2, v_4] - [v_1, v_2, v_3]\\ = -[v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2] + [v_2, v_3, v_4] - [v_1, v_3, v_4] + [v_1, v_2, v_4].\end{aligned}$$
This is a linear combination of the faces of the polyhedron $P = \mathrm{conv}\{v_0, v_1, v_2, v_3, v_4\}$, where again the coefficients are given by the orientation of the faces - the contribution of the inner triangle $[v_1, v_2, v_3]$ which is common to both tetrahedra cancels. By composing with ∂_2, we obtain again $\partial_2 \circ \partial_3(\sigma_3 + \tau_3)([e_0, e_1]) = 0$.

Example 4.1.5 provides a geometrical interpretation of the term *boundary operator*. By applying the boundary operator to a linear combination of n-simplexes one obtains their boundaries, realised as an R-linear combination of their faces. Moreover, the example suggests that the boundaries of $(n+1)$-dimensional polyhedra define n-dimensional polyhedra without boundaryi. e. that that $\partial_{n-1} \circ \partial_n = 0$ for all $n \in \mathbb{N}$. The following proposition shows that this is indeed the case and that simplexes and their boundaries define a chain complex in R-Mod.

Proposition 4.1.6: Let X be a topological space and R a commutative unital ring. Then the boundary operator satisfies $\partial_{n-1} \circ \partial_n = 0$ for all $n \in \mathbb{N}$. The singular n-chains in $S_n(X)$ with the boundary operators $\partial_n : S_n(X) \to S_{n-1}(X)$ form a chain complex $S_\bullet(X)$ in R-Mod:
$$\ldots \xrightarrow{\partial_{n+2}} S_{n+1}(X) \xrightarrow{\partial_{n+1}} S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_3} S_2(X) \xrightarrow{\partial_2} S_1(X) \xrightarrow{\partial_1} S_0(X) \to 0.$$
The n-cycles in $Z_n(X) := \ker(\partial_n) \subset S_n(X)$ are called **singular n-cycles** and the n-boundaries in $B_n(X)_R = \mathrm{Im}\,(\partial_{n+1}) \subset Z_n(X)$ **singular n-boundaries** in X. The nth homology $H_n(X) := Z_n(X)/B_n(X)$ is called the nth **singular homology** of X.

Proof:
A direct computation shows that the face maps from Definition 4.1.1 satisfy the relations
$$F^i_n \circ F^j_{n-1} = F^j_n \circ F^{i-1}_{n-1} : \Delta^{n-2} \to \Delta^n \qquad \forall 0 \le j < i \le n.$$
Using these relations, we obtain for all continuous maps $\sigma : \Delta^n \to X$:
$$\begin{aligned}\partial_{n-1} \circ \partial_n(\sigma) =&\partial_{n-1}\left(\Sigma_{i=0}^n (-1)^i\, \sigma \circ F^i_n\right) = \Sigma_{i=0}^n (-1)^i \Sigma_{j=0}^{n-1} (-1)^j \sigma \circ F^i_n \circ F^j_{n-1}\\ =&\Sigma_{0\le j<i\le n} (-1)^{i+j} \sigma \circ F^i_n \circ F^j_{n-1} + \Sigma_{0\le i\le j\le n-1} (-1)^{i+j} \sigma \circ F^i_n \circ F^j_{n-1}\\ =&\Sigma_{0\le j<i\le n} (-1)^{i+j} \sigma \circ F^i_n \circ F^j_{n-1} + \Sigma_{0\le j\le i\le n-1} (-1)^{i+j} \sigma \circ F^j_n \circ F^i_{n-1}\\ =&\Sigma_{0\le j<i\le n} (-1)^{i+j} \sigma \circ F^j_n \circ F^{i-1}_{n-1} + \Sigma_{0\le j<i\le n} (-1)^{i+j-1} \sigma \circ F^j_n \circ F^{i-1}_{n-1} = 0.\end{aligned}$$
As $\partial_n : S_n(X) \to S_{n-1}(X)$ is R-linear, this proves the claim. □

Singular homologies are in general rather difficult to compute. However, in some cases it is possible to show that they are trivial straightforwardly. Examples of this are the following.

Example 4.1.7: Let R be a commutative unital ring and $X = \{p\}$ a single point. Then for each $n \in \mathbb{N}_0$, there is a single n-simplex $\sigma_n : \Delta^n \to \{p\}$. This implies $S_n(X) = \langle \sigma_n \rangle_R$ and $\partial\sigma_n = \Sigma_{i=0}^n (-1)^i \sigma_{n-1}$. For n odd or $n = 0$, this yields $\partial_n \sigma_n = 0$, in the other cases $\partial_n \sigma_n = \sigma_{n-1}$. This implies

$$B_n(X) \cong \begin{cases} 0 & n \,\text{even} \\ \langle \sigma_n \rangle_R & n \,\text{odd} \end{cases} \qquad Z_n(X) \cong \begin{cases} 0 & n \neq 0 \,\text{even} \\ \langle \sigma_n \rangle_R & n = 0 \,\text{or}\, n \,\text{odd}, \end{cases}.$$

and $H_n(X) = 0$ for $n > 0$ and $H_0(X) \cong R$.

Example 4.1.8: If $X = \amalg_{i \in I} X_i$ is a decomposition of X into path components X_i then

$$H_n(X) \cong \bigoplus_{i \in I} H_n(X_i).$$

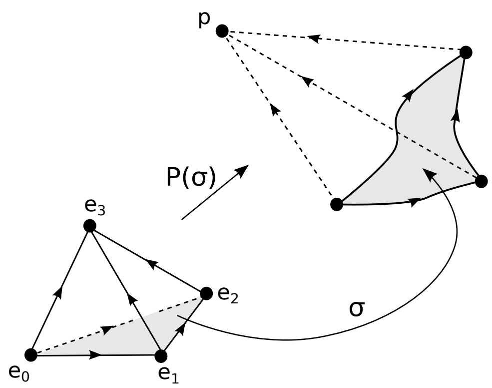
As Δ^n is path-connected, $\sigma(\Delta^n)$ is path-connected for any singular n-simplex $\sigma : \Delta^n \to X$ and hence contained in a path-component X_i. It follows that $S_n(X) = \bigoplus_{i \in I} S_n(X_i)$. As $\sigma(\Delta^n) \subset X_i$ implies $\partial_n \sigma(\Delta^n) \subset X_i$, we also have $Z_n(X) = \bigoplus_{i \in I} Z_n(X_i)$, $B_n(X) = \bigoplus_{i \in I} B_n(X_i)$ with $B_n(X_i) \subset Z_n(X_i) \subset S_n(X_i)$ and $H_n(X) = \bigoplus_{i \in I} H_n(X_i)$.

Example 4.1.9: If $\emptyset \neq X \subset \mathbb{R}^n$ is star-shaped with respect to $p \in X$, then $H_n(X) = 0$ for all $n > 0$ and $H_0(X) \cong R$.

Define for $n \geq 0$ the module morphism $P_n : S_n(X) \to S_{n+1}(X)$ by

$$P_n(\sigma)\left(\Sigma_{i=0}^{n+1} t_i e_i\right) = t_{n+1}\, p + (1 - t_{n+1})\, \sigma\left(\Sigma_{i=0}^n \tfrac{t_i}{1-t_{n+1}} e_i\right).$$

for all singular simplexes $\sigma : \Delta^n \to X$.



The 3-simplex $P(\sigma) : \Delta^3 \to X$ for a 2-simplex $\sigma : \Delta^2 \to X$ (gray area).

Then $P_n(\sigma)$ is continuous and satisfies

$$P_n(\sigma) \circ F_{n+1}^{n+1} = \sigma \quad P_n(\sigma) \circ F_{n+1}^i = P_{n-1}(\sigma \circ F_n^i) \qquad \forall \sigma \in S_n(X), 0 \leq i \leq n.$$

From this, we obtain

$$\begin{aligned} \partial_{n+1}(P_n(\sigma)) =& \Sigma_{i=0}^{n+1} (-1)^i P_n(\sigma) \circ F_{n+1}^i = (-1)^{n+1}\sigma + \Sigma_{i=0}^n (-1)^i P_n(\sigma) \circ F_{n+1}^i \\ =& (-1)^{n+1}\sigma + \Sigma_{i=0}^n (-1)^i P_{n-1}(\sigma \circ F_n^i) = (-1)^{n+1}\sigma + P_{n-1}(\partial_n \sigma). \end{aligned}$$

This implies $\sigma = (-1)^i \partial_{n+1}(P_n(\sigma))$ for all $\sigma \in Z_n(X)$ and $B_n(X) = Z_n(X)$ for all $n \in \mathbb{N}$.

As it is still difficult to compute singular homologies, we consider another type of homology, namely simplicial homology. Its advantage is that it is very easy to use in computations, since one does not need to take into account all singular n-simplexes $\sigma : \Delta^n \to X$ but only a finite or infinite subset. This must be chosen in such a way that it is closed under the action of the boundary operators. If it consists of infinitely many singular simplexes, one also needs to impose a condition that ensures that the topologies on the images of the simplexes are compatible with the topology of X. This leads to the notion of a Δ-complex.

Definition 4.1.10: Let X be a topological space. A (finite) **simplicial complex** or **Δ-complex**[2] structure on X is a (finite) family $\{\sigma_i\}_{i\in I}$ of singular simplexes $\sigma_i : \Delta^{n(i)} \to X$ that satisfies the following axioms:

(S1) Every point $x \in X$ is contained in exactly one image $\sigma_i(\mathring{\Delta}^{n(i)})$, and $\sigma_i|_{\mathring{\Delta}^{n(i)}} : \mathring{\Delta}^{n(i)} \to X$ is injective for all $i \in I$.

(S2) For every simplex $\sigma_i : \Delta^{n(i)} \to X$ and every $j \in \{0, ..., n(i)\}$ there is a $k \in I$ with $\sigma_i \circ F^j_{n(i)} = \sigma_k : \Delta^{n(i)-1} \to X$.

(S3) X is equipped with the final topology induced by the family $\{\sigma_i\}_{i\in I}$: a subset $A \subset X$ is open if and only if $\sigma_i^{-1}(A)$ is open in $\Delta^{n(i)}$ for all $i \in I$.

The **k-skeleton** X^k of a Δ-complex $(X, \{\sigma_i\}_{i\in I})$ is the subspace

$$X^k = \bigcup_{i\in I_0\cup...\cup I_k} \sigma_i(\Delta^k) \quad \text{with} \quad I_k = \{i \in I : n(i) = k\} \subset I.$$

A **sub complex** of a Δ-complex $(X, \{\sigma_i\}_{i\in I})$ is a subspace $A \subset X$ together with a subset $\{\sigma_j\}_{j\in J}$, $J \subset I$, of simplexes such that $(A, \{\sigma_j\}_{j\in J})$ is a Δ-complex. A **simplicial map** $f : (X, \{\sigma_i\}_{i\in I}) \to (Y, \{\tau_j\}_{j\in J})$ between Δ-complexes $(X, \{\sigma_i\}_{i\in I})$, $(Y, \{\tau_j\}_{j\in J})$ is a continuous map $f : X \to Y$ such that for each $i \in I$ there is a $j \in J$ with $f \circ \sigma_i = \tau_j$.

Remark 4.1.11:

1. If X is a Hausdorff space, then the restrictions $\sigma_i|_{\mathring{\Delta}^{n(i)}} : \Delta^{n(i)} \to X$ are homeomorphisms onto their image. This follows because closed subsets $A \subset \Delta^{n(i)}$ are compact and hence mapped to compact subsets $\sigma_i(A) \subset X$, which are closed since X is hausdorff. Hence $\sigma_i|_{\mathring{\Delta}^{n(i)}} : \mathring{\Delta}^{n(i)} \to \sigma_i(\mathring{\Delta}^{n(i)})$ is continuous, bijective and closed, i. e. a homeomorphism.
2. One can show that if X is hausdorff, then a simplicial complex structure $\{\sigma_i\}_{i\in I}$ on X defines a CW-complex structure on X in such a way that the k-skeleta of $(X, \{\sigma_i\}_{i\in I})$ and the k-skeleta of the CW-complex agree. The k-skeleton X^k is obtained from X^{k-1} by attaching k-cells $D^k \approx \Delta^k$ with the attaching maps $\sigma_i|_{\partial\Delta^k} : \partial\Delta^k \to X^{k-1}$ for $i \in I_k$.
3. Δ-complex structures on a topological space are not unique. A given topological space X can have several different Δ-complex structures.
4. Every two- or three-dimensional topological manifold M can be triangulated and hence has a structure of a Δ-complex. If additionally M is compact, it has the structure of a finite Δ-complex.

[2]There are several different notions of simplicial complexes. The Δ-complexes considered here were introduced in 1950 by Eilenberg and Zilber as **semisimplicial complexes**.

5. The traditional notion of simplicial complex imposes the additional condition that all vertices of an n-simplex $\sigma_i : \Delta^n \to X$, i. e. 0-simplexes obtained from $\sigma_i : \Delta^n \to X$ by applying (S2), are different and that the sets of vertices of any two n-simplexes $\sigma_i, \sigma_j : \Delta^n \to X$ with $i \neq j$ are different.

   It can be shown that every Δ-complex structure on X can be transformed into a simplicial complex structure in this sense via subdivisions. However, the additional requirements make the stricter notion of simplicial complex impractical for computations, since it requires a much larger number of vertices than a Δ-complex structure.

The boundary operator for a simplicial complex is defined in the same way as in singular homology and has the same properties. The second axiom ensures that the set of simplexes $\{\sigma_i\}_{i\in I}$ is closed under the action of the boundary operator and one obtains a chain complex.

Lemma 4.1.12: Let R be a commutative unital ring and $(X, \{\sigma_i\}_{i\in I})$ a simplicial complex. Denote for $n \in \mathbb{N}_0$ by $S^\Delta_n(X)$ the free R-module generated by the set of n-simplexes $\{\sigma_i\}_{i\in I_n}$. Then the restriction of the boundary operator to $S^\Delta_n(X) \subset S_n(X)$ defines a module morphism $\partial^\Delta_n : S^\Delta_n(X) \to S^\Delta_{n-1}(X)$ with $\partial^\Delta_{n-1} \circ \partial^\Delta_n = 0$. This yields a chain complex $S^\Delta_\bullet(X)$ in R-Mod

$$\ldots \xrightarrow{\partial^\Delta_{n+2}} S^\Delta_{n+1}(X) \xrightarrow{\partial^\Delta_{n+1}} S^\Delta_n(X) \xrightarrow{\partial^\Delta_n} S^\Delta_{n-1}(X) \xrightarrow{\partial^\Delta_{n-1}} \ldots \xrightarrow{\partial^\Delta_3} S^\Delta_2(X) \xrightarrow{\partial^\Delta_2} S^\Delta_1(X) \xrightarrow{\partial^\Delta_1} S^\Delta_0(X) \to 0.$$

Elements of $S^\Delta_n(X)$ are called **simplicial n-chains**, elements of $Z^\Delta_n(X) := \ker(\partial^\Delta_n)$ **simplicial n-cycles**, elements of $B^\Delta_n(X) := \mathrm{Im}\,(\partial^\Delta_{n+1})$ **simplicial n-boundaries**, and the associated homologies $H^\Delta_n(X) = Z^\Delta_n(X)/B^\Delta_n(X)$ **simplicial homologies**.

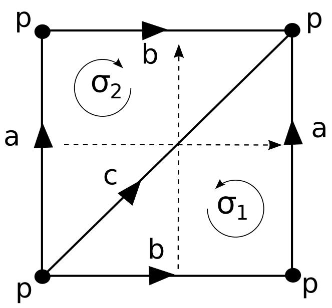
Example 4.1.13:

1. The circle S^1 has the structure of a simplicial complex with a 1-simplex $\sigma_1 : [0,1] \to S^1$ with $\sigma_1(0) = \sigma_1(1) = 1$ and a 0-simplex $\sigma_0 : \{0\} \to S_1$, $\sigma(0) = 1$. They satisfy $\partial^\Delta_0(\sigma_0) = 0$ and $\partial^\Delta_1\sigma_1(0) = \sigma_1 \circ F^0_1(0) - \sigma_1 \circ F^1_1(0) = \sigma_1(1) - \sigma_1(0) = 0$. This implies $Z^\Delta_0(S^1) \cong Z^\Delta_1(S^1) \cong R$. As $S^\Delta_m(X) = 0$ for $m \neq 0, 1$, we have $B^\Delta_1(S^1) = \{0\}$ and $H^\Delta_1(S^1) \cong Z^\Delta_1(S^1) \cong R$. As $B^\Delta_0(S^1) = \mathrm{Im}\,(\partial^\Delta_1) = 0$, it follows that $H^\Delta_0(S^1) \cong Z^\Delta_0(S^1) \cong R$. The simplicial homologies are given by

$$H^\Delta_m(S^1) \cong \begin{cases} R & m \in \{0,1\} \\ 0 & m \notin \{0,1\}. \end{cases}$$

2. The torus $T = S^1 \times S^1$ has the structure of a simplicial complex with two 2-simplexes, three 1-simplexes and one 0-simplex. They are obtained by composing the canonical surjection $\pi : [0,1] \times [0,1] \to [0,1] \times [0,1]/\sim$ with the affine simplexes

$$\begin{aligned}
&\sigma_1 : [e_0, e_1, e_2] \to [e_0, e_1, e_1 + e_2], \qquad \sigma_2 : [e_0, e_1, e_2] \to [e_0, e_2, e_1 + e_2],\\
&a : [e_0, e_1] \to [e_0, e_2], \quad b : [e_0, e_1] \to [e_0, e_1], \quad c : [e_0, e_1] \to [e_0, e_1 + e_2]\\
&p : [e_0] \to [e_0].
\end{aligned}$$



We have $S_2^\Delta(T) = \langle \pi\circ\sigma_1, \pi\circ\sigma_2\rangle_R$, $S_1^\Delta(T) = \langle \pi\circ a, \pi\circ b, \pi\circ c\rangle_R$, $S_0^\Delta(T) = \langle \pi\circ p\rangle_R$. As there are no 3-simplexes, we have $B_2^\Delta(T) = 0$. The 2-simplexes satisfy

$$\partial_2^\Delta(\pi\circ\sigma_2) = \partial_2^\Delta(\pi\circ\sigma_1) = \pi\circ a + \pi\circ b - \pi\circ c,$$

which implies $Z_2^\Delta(T) = \langle \pi\circ\sigma_2 - \pi\circ\sigma_1\rangle_R \cong R$ and $B_1^\Delta(T) = \langle \pi\circ a + \pi\circ b - \pi\circ c\rangle_R \cong R$. As $\pi\circ a, \pi\circ b$ and $\pi\circ c$ are closed paths, we obtain

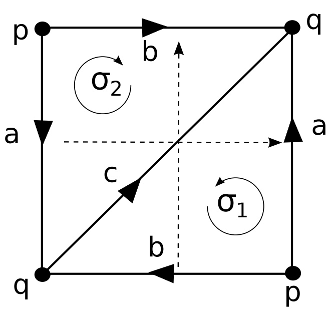
$$\partial_1^\Delta(\pi\circ a)(0) = \partial_1^\Delta(\pi\circ b)(0) = \partial_1^\Delta(\pi\circ c)(0) = \pi\circ p - \pi\circ p = 0$$

and therefore $Z_1^\Delta(T) = \langle \pi\circ a, \pi\circ b, \pi\circ c\rangle_R$. As $Z_0^\Delta(T) = \langle \pi\circ p\rangle_R \cong R$, this implies

$$H_m^\Delta(T) \cong \begin{cases} R\oplus R & m=1 \\ R & m\in\{0,2\} \\ 0 & m\notin\{0,1,2\}. \end{cases}$$

3. Real projectice space $\mathbb{RP}^2$ can be realised as a quotient $[0,1]\times[0,1]/\sim$ where $(x,1)\sim(1-x,0)$ and $(0,y)\sim(1,1-y)$ for all $x,y\in[0,1]$. It has a CW-structure with two 2-simplexes, three 1-simplexes and two 0 -simplexes which are obtained by composing the canonical surjection $\pi : [0,1]\times[0,1]\to\mathbb{RP}^2$ with the affine simplexes

$$\begin{aligned} &\sigma_1 : [e_0,e_1,e_2]\to[e_0,e_1,e_1+e_2], \qquad \sigma_2 : [e_0,e_1,e_2]\to[e_0,e_2,e_1+e_2],\\ &a : [e_0,e_1]\to[e_2,e_0], \quad b : [e_0,e_1]\to[e_1,e_0], \quad c : [e_0,e_1]\to[e_0,e_1+e_2]\\ &p : [e_0]\to[e_1], \quad q : [e_0]\to[e_0] \end{aligned}$$



We have $S_2^\Delta(\mathbb{RP}^2) = \langle \pi\circ\sigma_1, \pi\circ\sigma_2\rangle_R$, $S_1^\Delta(\mathbb{RP}^2) = \langle \pi\circ a, \pi\circ b, \pi\circ c\rangle_R$, $S_0^\Delta(\mathbb{RP}^2) = \langle \pi\circ p, \pi\circ q\rangle_R$. The action of the boundary operators is gven by

$$\begin{aligned} &\partial_2^\Delta(\pi\circ\sigma_1) = -\pi\circ c - \pi\circ b + \pi\circ a, \qquad &&\partial_2^\Delta(\pi\circ\sigma_2) = -\pi\circ c + \pi\circ b - \pi\circ a\\ &\partial_1^\Delta(\pi\circ a) = \partial_1^\Delta(\pi\circ b) = \pi\circ q - \pi\circ p &&\partial_1^\Delta(\pi\circ c) = \pi\circ q - \pi\circ q = 0. \end{aligned}$$

As $B_2^\Delta(\mathbb{RP}^2) = 0$ and $\partial_2(r_1\pi\circ\sigma_1 + r_2\pi\circ\sigma_2) = -(r_1+r_2)\pi\circ c + (r_1-r_2)(\pi\circ a - \pi\circ b) = 0$ if and only if $r_1 = r_2$, $2r_1 = 0$, we have $H_2^\Delta(\mathbb{RP}^2) = Z_2^\Delta(\mathbb{RP}^2) \cong \{r\in R : 2r = 0\}$. Similarly,

$$Z_1^\Delta(\mathbb{RP}^2) = \langle \pi\circ c, \pi\circ b - \pi\circ a\rangle_R, \quad B_1^\Delta(\mathbb{RP}^2) = \langle \pi\circ c + \pi\circ b - \pi\circ a, \pi\circ c - \pi\circ b + \pi\circ a\rangle_R$$

and therefore

$$\begin{aligned}
H_1^\Delta(\mathbb{R}\mathrm{P}^2) &= \frac{\langle \pi\circ c, \pi\circ b - \pi\circ a\rangle_R}{\langle \pi\circ c + \pi\circ b - \pi\circ a, \pi\circ c - \pi\circ b + \pi\circ a\rangle_R}\\
&\cong \frac{\langle \pi\circ c, \pi\circ b - \pi\circ a\rangle_R / \langle \pi\circ c - \pi\circ b + \pi\circ a\rangle_R}{\langle \pi\circ c + \pi\circ b - \pi\circ a, \pi\circ c - \pi\circ b + \pi\circ a\rangle_R / \langle \pi\circ c - \pi\circ b + \pi\circ a\rangle_R}\\
&\cong \frac{\pi\circ b - \pi\circ a\rangle_R}{\langle 2(\pi\circ b - \pi\circ a)\rangle_R} \cong R/2R.
\end{aligned}$$

As $B_0^\Delta(\mathbb{R}\mathrm{P}^2) = \langle \pi\circ q - \pi\circ p\rangle_R$, it follows that

$$H_0^\Delta(\mathbb{R}\mathrm{P}^2) = \frac{\langle \pi\circ p, \pi\circ q\rangle_R}{\langle \pi\circ q - \pi\circ p\rangle_R} \cong R$$

Remark 4.1.14: We will show later that for any topological space X that has the structure of a simplicial complex, the simplicial and the singular homology agree: $H_n(X) \cong H_n^\Delta(X)$ for all $n \in \mathbb{N}_0$. This implies in particular that the simplicial homologies cannot depend on the choice of the simplicial complex structure.

It is obvious from the preceding definitions that the singular and simplicial homologies $H_n(X)$, $H_n^\Delta(X)$ of a topological space X are obtained by applying the functors $H_n : \mathrm{Ch}_{R\text{-Mod}} \to R$-Mod from Proposition 2.3.4 to the associated chain complexes $S_\bullet(X)$ and $S_\bullet^\Delta(X)$ in R-Mod. In view of Remark 2.3.2, Definition 2.3.5, Remarks 2.3.6 and 2.3.7 this suggests that the correspondence between topological spaces and chain complexes should also include continuous maps and homotopies between them. More precisely, one expects an assignment of a chain map $S_\bullet(f) : S_\bullet(X) \to S_\bullet(Y)$ to each continuous map $f : X \to Y$ and of a chain homotopy $S_\bullet(h) : S_\bullet(f) \Rightarrow S_\bullet(f')$ to a homotopy $h : [0,1]\times X \to Y$ from $f : X \to Y$ to $f' : X \to Y$.

Theorem 4.1.15: Let R be a commutative unital ring and X, Y, Z topological spaces.

1. A continuous map $f : X \to Y$ induces a chain map $S_\bullet(f) : S_\bullet(X) \to S_\bullet(Y)$ given by $S_n(f)(\sigma) = f\circ\sigma$ for singular n-simplexes $\sigma : \Delta^n \to X$. These chain maps satisfy $S_\bullet(\mathrm{id}_X) = \mathrm{id}_{S_\bullet(X)}$ and $S_\bullet(g\circ f) = S_\bullet(g)\circ S_\bullet(f)$ for all continuous maps $f : X \to Y$, $g : Y \to Z$. This defines a functor $S_\bullet : \mathrm{Top} \to \mathrm{Ch}_{R\text{-Mod}}$ that assigns to a topological space X the chain complex $S_\bullet(X)$ and to a continuous map $f : X \to Y$ the chain map $S_\bullet(f) : X_\bullet \to Y_\bullet$.
2. A homotopy $h : [0,1]\times X \to Y$ from $f : X \to Y$ to $f' : X \to Y$ induces a chain homotopy $S_\bullet(h) : S_\bullet(f) \Rightarrow S_\bullet(f')$. The functor $S_\bullet : \mathrm{Top} \to \mathrm{Ch}_{R\text{-Mod}}$ induces a functor $\mathrm{hTop} \to \mathrm{hCh}_{R\mathrm{Mod}}$ from the homotopy category of topological spaces into the homotopy category of chain complexes.

Proof:
1. We define for $n \in \mathbb{N}_0$ an R-module morphisms $S_n(f) : S_n(X) \to S_n(Y)$ by $S_n(f)(\sigma) = f\circ\sigma$ for all n-simplexes $\sigma : \Delta^n \to X$ and R-linear continuation. Then for all singular n-simplexes σ

$$\partial'_n \circ S_n(f)(\sigma) = \partial_n(f\circ\sigma) = \sum_{i=0}^n (-1)^i (f\circ\sigma)\circ F_n^i = \sum_{i=0}^n (-1)^i f\circ(\sigma\circ F_n^i) = S_{n-1}(f)\circ\partial_n(\sigma),$$

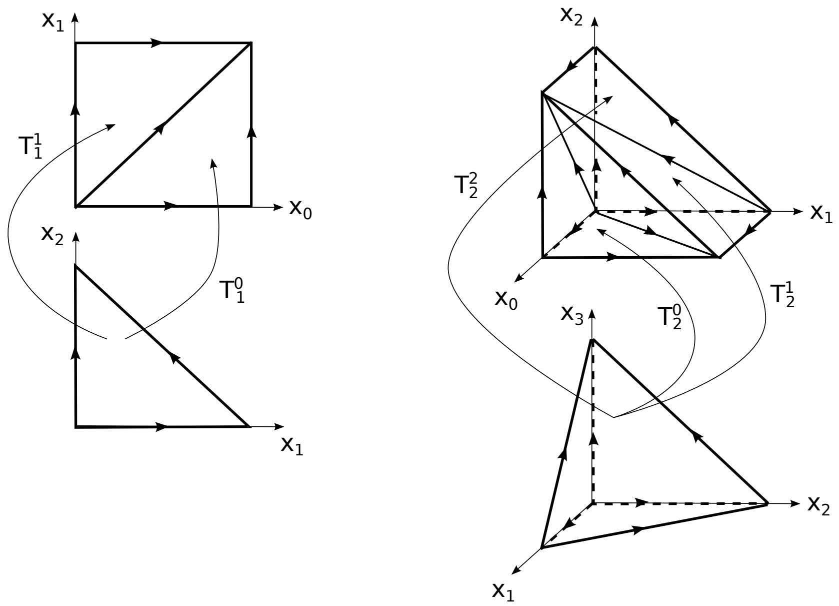
which shows that $S_\bullet(f) = (S_n(f))_{n\in\mathbb{N}_0}$ is a a chain map. By definition, we have for all $n \in \mathbb{N}_0$

$$S_n(g \circ f)(\sigma) = g \circ f \circ \sigma = S_n(g)(S_n(f)(\sigma)) = S_n(g) \circ S_n(f)(\sigma) \qquad S_n(\mathrm{id}_X)(\sigma) = \mathrm{id}_X \circ \sigma = \sigma.$$

As the composite of the chain maps $S_\bullet(g) : S_\bullet(Y) \to S_\bullet(Z)$, $S_\bullet(f) : S_\bullet(X) \to S_\bullet(Y)$ is given by the module morphisms $S_n(g) \circ S_n(f) : S_n(X) \to S_n(Z)$, $S_\bullet : \mathrm{Top} \to \mathrm{Ch}_{R-\mathrm{Mod}}$ is a functor.

2. As a chain homotopy $S_\bullet(h) : S_\bullet(f) \Rightarrow S_\bullet(f')$ is a collection of R-module morphisms $S_n(h) : S_n(X) \to S_{n+1}(Y)$, we need to associate to each n-simplex $\sigma : \Delta^n \to X$ an $(n+1)$-simplex $\sigma' : \Delta^{n+1} \to Y$. The homotopy $h : [0,1] \times X \to Y$ yields a continuous map $h \circ (\mathrm{id} \times \sigma) : [0,1] \times \Delta^n \to Y$. By composing this map with suitable affine linear maps $\Delta^{n+1} \to [0,1] \times \Delta^n$ we obtain the desired $(n+1)$-simplex $\sigma' : \Delta^{n+1} \to Y$.

These affine linear maps are obtained from geometrical considerations, namely a decomposition of the prism $[0,1] \times \Delta^n$ into affine $(n+1)$-simplexes Δ^{n+1}. This decomposition is given by the **prism maps** $T_n^j : \Delta^{n+1} \to [0,1] \times \Delta^n$, $0 \leq j \leq n$, which are the unique affine linear maps with $T_n^j(e_k) = (0, e_k)$ for $k \leq j$ and $T_n^j(e_k) = (1, e_{k-1})$ for $k > j$. It is is easy to see that every point in $[0,1] \times \Delta^n$ is contained in an image $T_n^j(\Delta^{n+1})$ and that $T_n^j(\Delta^{n+1}) \cap T_n^k(\Delta^{n+1})$ for $k \neq j$ is a lower-dimensional face in both simplexes.



The prism maps $T_n^j : \Delta^{n+1} \to [0,1] \times \Delta^n$ for $n = 1, 2$.

A direct computation shows that the the prism maps satisfy the following relations

$$\begin{aligned}
&T_n^j \circ F_{n+1}^i = (\mathrm{id}_{[0,1]} \times F_n^i) \circ T_{n-1}^{j-1} \quad \forall j > i \quad T_n^j \circ F_{n+1}^i = (\mathrm{id}_{[0,1]} \times F_n^{i-1}) \circ T_{n-1}^j \quad \forall j < i-1\\
&T_n^i \circ F_{n+1}^i = T_n^{i-1} \circ F_{n+1}^i \quad \forall i \in \{1, ..., n\} \quad T_n^0 \circ F_{n+1}^0 = i_1,\ T_n^n \circ F_{n+1}^{n+1} = i_0,
\end{aligned} \tag{28}$$

where $i_t : \Delta^n \to [0,1] \times \Delta^{n+1}$, $x \mapsto (t, x)$ is the inclusion map and $F_{n+1}^j : \Delta^n \to \Delta^{n+1}$ the face maps from Definition 4.1.1. This yields for $n \in \mathbb{N}_0$ R-module morphisms

$$h_n : S_n(X) \to S_{n+1}(Y), \qquad \sigma \mapsto \Sigma_{j=0}^n (-1)^j h \circ (\mathrm{id}_{[0,1]} \times \sigma) \circ T_n^j. \tag{29}$$

As $h(0,x)=f(x)$ and $h(1,x)=f'(x)$ we have for all singular n-simplexes $\sigma:\Delta^n\to X$

$$\begin{aligned} h\circ(\mathrm{id}_{[0,1]}\times\sigma)\circ i_0 &= h\circ(\{0\}\times\sigma)=f\circ\sigma=S_n(f)(\sigma) \\ h\circ(\mathrm{id}_{[0,1]}\times\sigma)\circ i_1 &= h\circ(\{1\}\times\sigma)=f'\circ\sigma=S_n(f')(\sigma), \end{aligned} \tag{30}$$

and by applying the boundary operator $\partial^Y_{n+1}:S_{n+1}(Y)\to S_n(Y)$, we obtain

$$\partial^Y_{n+1}h_n(\sigma)=\Sigma_{i=0}^{n+1}\Sigma_{j=0}^{n}(-1)^{i+j}h\circ(\mathrm{id}_{[0,1]}\times\sigma)\circ T_n^j\circ F^i_{n+1}.$$

Splitting this sum into terms with $j<i-1$, $j=i-1$, $j=i$, $j>i+1$ and the identities (28) and (30) yield

$$\begin{aligned} \partial^Y_{n+1}h_n(\sigma)=&\Sigma_{i=2}^{n+1}\Sigma_{j=0}^{i-2}(-1)^{i+j}h\circ(\mathrm{id}_{[0,1]}\times\sigma)\circ T_n^j\circ F^i_{n+1}-\Sigma_{i=1}^{n+1}h\circ(\mathrm{id}_{[0,1]}\times\sigma)\circ T_n^{i-1}\circ F^i_{n+1}\\ &+\Sigma_{i=0}^{n}h\circ(\mathrm{id}_{[0,1]}\times\sigma)\circ T_n^i\circ F^i_{n+1}+\Sigma_{i=0}^{n-1}\Sigma_{j=i+1}^{n}(-1)^{i+j}h\circ(\mathrm{id}_{[0,1]}\times\sigma)\circ T_n^j\circ F^i_{n+1}\\ \overset{(28)}{=}&\Sigma_{i=2}^{n+1}\Sigma_{j=0}^{i-2}(-1)^{i+j}h\circ(\mathrm{id}_{[0,1]}\times\sigma\circ F_n^{i-1})\circ T^j_{n-1}-h\circ(\mathrm{id}_{[0,1]}\times\sigma)\circ i_0\\ &+h\circ(\mathrm{id}_{[0,1]}\times\sigma)\circ i_1+\Sigma_{i=0}^{n-1}\Sigma_{j=i+1}^{n}(-1)^{i+j}h\circ(\mathrm{id}_{[0,1]}\times\sigma\circ F^i_n)\circ T^{j-1}_{n-1}\\ \overset{(30)}{=}&\Sigma_{i=1}^{n}\Sigma_{j=0}^{i-1}(-1)^{i+j+1}(\mathrm{id}_{[0,1]}\times\sigma\circ F^i_n)\circ T^j_{n-1}\\ &+\Sigma_{i=0}^{n-1}\Sigma_{j=i}^{n-1}(-1)^{i+j+1}(\mathrm{id}_{[0,1]}\times\sigma\circ F^i_n)\circ T^j_{n-1}-S_n(f)(\sigma)+S_n(f')(\sigma)\\ =&\Sigma_{i=0}^{n}\Sigma_{j=0}^{n-1}(-1)^{i+j+1}(\mathrm{id}_{[0,1]}\times\sigma\circ F^i_n)\circ T^j_{n-1}-S_n(f)(\sigma)+S_n(f')(\sigma)\\ =&-h_{n-1}(\partial^X_n\sigma)-S_n(f)(\sigma)+S_n(f')(\sigma). \end{aligned}$$

This proves that $S_\bullet(h):S_\bullet(f)\Rightarrow S_\bullet(f')$ is a chain homotopy. □

By composing the functor $S_\bullet:\mathrm{Top}\to\mathrm{Ch}_{R\text{-Mod}}$ with the homology functors $H_n:\mathrm{Ch}_{R\text{-Mod}}\to R\text{-Mod}$, we obtain a family of functors $H_n\circ S_\bullet:\ \mathrm{Top}\to R\text{-Mod}$ that assign to a topological space X the R-module $H_n(X)$ and to a continuous map $f:X\to Y$ an R-module morphism $H_n(f):H_n(X)\to H_n(Y)$. As homotopies $h:[0,1]\times X\to Y$ from $f:X\to Y$ to $f':X\to Y$ induce chain homotopies $S_\bullet(h):S_\bullet(f)\to S_\bullet(f')$, we obtain from Proposition 2.3.8 that the R-module morphisms $H_n(f),H_n(f'):X\to Y$ agree if $f,f':X\to Y$ are homotopic. The functors $H_n:\ \mathrm{Top}\to R\text{-Mod}$ are constant on the homotopy classes of morphisms in Top and induce functors $H_n:\mathrm{hTop}\to R\text{-Mod}$. In particular, a homotopy equivalence $f:X\to Y$ induces a chain homotopy equivalence $S_\bullet(f):S_\bullet(X)\to S_\bullet(Y)$ and hence an R-module isomorphism between the homologies $H_n(X)$ and $H_n(Y)$.

Corollary 4.1.16: Let R be a commutative unital ring and X,Y topological spaces. Then:

1. Homotopic maps $f,f':X\to Y$ induce the same R-module morphisms on the homologies: $H_n(f)=H_n(f'):H_n(X)\to H_n(Y)$ for all $n\in\mathbb{N}_0$.
2. If $f:X\to Y$ is a homotopy equivalence, then $H_n(f):H_n(X)\to H_n(Y)$ is an R-module isomorphism for all $n\in\mathbb{N}_0$.
3. If X and Y are of the same homotopy type, then they have isomorphic homologies: $H_n(X)\cong H_n(Y)$ for all $n\in\mathbb{N}_0$.

The functors $H_n\circ S_\bullet:\mathrm{Top}\to R\text{-Mod}$ are constant on the homotopy classes of continuous maps $f:X\to Y$ and induce a family of functors $\mathrm{hTop}\to R\text{-Mod}$ that assign to a topological space X the nth homology $H_n(X)$ and to a homotopy class of continuous maps $f:X\to Y$ the R-module morphism $H_n(f):H_n(X)\to H_n(Y)$.

To develop an intuition for homologies, it is helpful to investigate the relation between the fundamental group and the first homology group of a topological space X. Clearly, closed paths $\gamma : [0,1] \to X$ in a topological space coincide with singular 1-simplexes $\sigma : [0,1] \to X$. This makes it plausible that there is a relation between the fundamental group $\pi_1(x,X)$ and the first homology group $H_1(X)$. As modules are abelian groups, this relation should be given by a group homomorphism and involve the abelianisation of the fundamental group $\pi_1(x,X)$.

Theorem 4.1.17: Let (x,X) be a pointed topological space and R a commutative unital ring. Then the map

$$\mathrm{Hu} : \pi_1(x,X) \to H_1(X), \quad [\gamma]_{\pi_1} \mapsto [\gamma]_{H_1},$$

is a group homomorphism. If X is path-connected and $R = \mathbb{Z}$, it induces a group isomorphism $\mathrm{Ab}(\mathrm{Hu}) : \mathrm{Ab}(\pi_1(x,X)) \to H_1(X)$ between the abelianisation $\mathrm{Ab}(\pi_1(x,X))$ of the fundamental group and $H_1(X)$, the **Huréwicz isomorphism**.

Proof:
1. $Hu : \pi_1(x,X) \to H_1(X)$, $[\gamma]_{\pi_1(X)} \to [\gamma]_{H_1(X)}$ is well-defined: The canonical surjection $\pi : [0,1] \to S^1 \approx [0,1]/\sim$ can be viewed as a 1-cycle in $Z_1(S^1)$. For every closed path $\gamma : [0,1] \to X$ there is a unique map $\gamma_\sim : S^1 \to X$ with $\gamma_\sim \circ \pi = \gamma$. This implies $[\gamma]_{H_1(X)} = [\gamma_\sim \circ \pi]_{H_1(X)} = H_1(\gamma_\sim)([\pi]_{H_1(S^1)})$. If $\gamma' : [0,1] \to X$ is homotopic to γ relative to $\{0,1\}$, then $\gamma_\sim, \gamma'_\sim : S^1 \to X$ are homotopic and induce the same R-module morphisms on the homologies. This implies $[\gamma]_{H_1(X)} = H_1(\gamma_\sim)([\pi]_{H^1(S^1)}) = H_1(\gamma'_\sim)([\pi]_{H^1(S^1)}) = [\gamma']_{H_1(X)}$ and shows that the map $\mathrm{Hu} : \pi_1(x,X) \to H_1(X)$, $[\gamma]_{\pi_1(X)} \to [\gamma]_{H_1(X)}$ is well-defined.

2. To show that $\mathrm{Hu} : \pi_1(x,X) \to H_1(X)$, $[\gamma]_{\pi_1(X)} \to [\gamma]_{H_1(X)}$ is a group homomorphism, consider composable paths $\beta, \gamma : [0,1] \to X$ with $\beta(0) = \gamma(1)$ and the map $g : \Delta^2 \to [0,1]$, $\Sigma_{i=0}^2 t_i e_i \mapsto \frac{1}{2}t_1 + t_2$. Then for all $t \in [0,1]$, we have

$$\begin{aligned}
&\partial_2((\beta \star \gamma) \circ g)(t) = \\
&= (\beta \star \gamma) \circ g((1-t)e_1 + te_2) - (\beta \star \gamma) \circ g((1-t)e_0 + te_2) + (\beta \star \gamma) \circ g((1-t)e_0 + te_1) \\
&= (\beta \star \gamma)(\tfrac{1}{2} + \tfrac{t}{2}) - (\beta \star \gamma)(t) + (\beta \star \gamma)(\tfrac{t}{2}) = \beta(t) - \beta \star \gamma(t) + \gamma(t).
\end{aligned}$$

This implies $[\beta]_{H_1(X)} + [\gamma]_{H_1(X)} - [\beta \star \gamma]_{H_1(X)} = 0$, and consequently $\mathrm{Hu} : \pi_1(x,X) \to H_1(X)$ is a group homomorphism. By applying the abelianisation functor, we obtain a group homomorphism $\mathrm{Ab}(\mathrm{Hu}) : \mathrm{Ab}(\pi_1(x,X)) \to H_1(X)$.

3. Let now X be path-connected and $R = \mathbb{Z}$. We show that $\mathrm{Ab}(\mathrm{Hu}) : \mathrm{Ab}(\pi_1(x,X)) \to H_1(X)$ is a group isomorphism by constructing its inverse. For this, choose for every point $y \in X$ a path $\gamma^y : [0,1] \to X$ with $\gamma^y(0) = y$, $\gamma^y(1) = x$ and define a morphism of abelian groups[3]

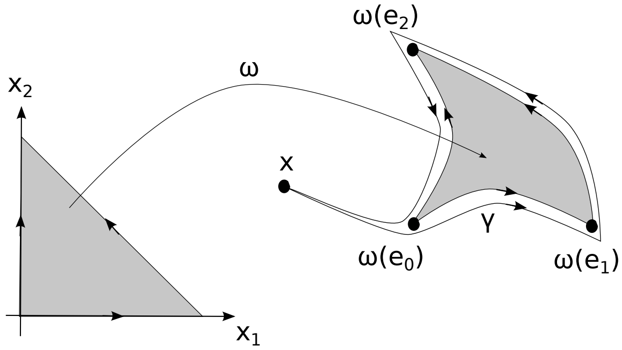
$$K : S_1(X) \to \mathrm{Ab}(\pi_1(x,X)), \qquad \sigma \mapsto [\gamma^{(\sigma(e_1))} \star \sigma \star \overline{\gamma}^{(\sigma(e_0))}]_{\mathrm{Ab}(\pi_1)}.$$

For any 2-simplex $\omega : \Delta^2 \to X$ we obtain

$$\begin{aligned}
K(\partial_2\omega) =& [\gamma^{\omega(e_2)} \star (\omega \circ F_0^2) \star \overline{\gamma}^{\omega(e_1)}]_{\mathrm{Ab}(\pi_1)} - [\gamma^{\omega(e_2)} \star (\omega \circ F_2^1) \star \overline{\gamma}^{\omega(e_0)}]_{\mathrm{Ab}(\pi_1)} \\
&+ [\gamma^{\omega(e_1)} \star (\omega \circ F_2^2) \star \overline{\gamma}^{\omega(e_0)}]_{\mathrm{Ab}(\pi_1)} = [\gamma]_{\mathrm{Ab}(\pi_1)},
\end{aligned}$$

where $\gamma : [0,1] \to X$ is a loop with base point x that circles the boundary $\omega(\partial\Delta^2) \subset X$ counterclockwise. As γ is null homotopic, we have $K(\partial_2\omega) = 0$. This implies $B_1(X) \subset \ker(K)$, and K induces a group homomorphism $K : H_1(X) \to \mathrm{Ab}(\pi_1(x,X))$.

[3] In this step we need the assumption $R = \mathbb{Z}$. Like any abelian group, the abelianisation of $\pi_1(x,X)$ has a canonical $\mathbb{Z}$-module structure, and the $\mathbb{Z}$-module homomorphism K can be defined by $\mathbb{Z}$-linear continuation. For other commutative unital rings R, there is no R-linear continuation of K to an R-module morphism.



For any loop $\delta : [0,1] \to X$ based at x, we have

$$K \circ \mathrm{Hu}([\delta]_{\mathrm{Ab}(\pi_1)}) = [\gamma^x \star \delta \star \overline{\gamma}^x]_{\mathrm{Ab}(\pi_1)} = [\gamma^x]_{\mathrm{Ab}(\pi_1)} - [\gamma^x]_{\mathrm{Ab}(\pi_1)} + [\delta]_{\mathrm{Ab}(\pi_1)} = [\delta]_{\mathrm{Ab}(\pi_1)},$$

and hence $K \circ \mathrm{Hu} = \mathrm{id}_{\mathrm{Ab}(\pi_1(x,X))}$. Conversely, for any 1-cycle $\sigma : \Delta^1 \to X \in Z_1(X)$ one obtains

$$\mathrm{Hu} \circ K([\sigma]_{H_1(X)}) = [\gamma^{\sigma(e_1)} \star \sigma \star \overline{\gamma}^{\sigma(e_0)}]_{H_1(X)} = [\gamma^{\sigma(e_1)}]_{H_1(X)} + [\overline{\gamma}^{\sigma(e_0)}]_{H_1(X)} + [\sigma]_{H_1(X)} = [\sigma]_{H_1(X)},$$

because $\partial_1\sigma(e_0) = \sigma(e_1) - \sigma(e_0) = 0$ implies $[\gamma^{\sigma(1)}]_{H_1(X)} = -[\gamma^{\sigma(1)}]_{H_1(X)} = -[\gamma^{\sigma(0)}]_{H_1(X)}$. This shows that $K = \mathrm{Hu}^{-1}$ and $\mathrm{Hu} : \mathrm{Ab}(\pi_1(x,X)) \to H_1(X)$ is a group isomorphism. □

Remark 4.1.18: There are generalisations of the Huréwicz isomorphism for higher homotopy and homology groups. Under the same assumptions as in Theorem 4.1.17, one obtains a group homomorphism $\mathrm{Hu}^{(n)} : \pi_n(x,X) \to H_n(X)$ for all $n \in \mathbb{N}$. If $R = \mathbb{Z}$ and X is $(n-1)$-**connected**, i. e. path-connected and $\pi_k(X) = 0$ for all $1 \leq k \leq n-1$, then $\mathrm{Hu}^{(n)} : \pi_n(x,X) \to H_n(X)$ is an isomorphism.

4.2 Subspaces and relative homology

In this section, we relate the homologies of a subspace $A \subset X$ to the homologies of X. This will not only give rise to R-module morphisms between $H_n(X)$, $H_n(A)$ and the quotient module $H_n(X)/H_n(A)$ for fixed $n \in \mathbb{N}$ but also relate these homologies for different values of $n \in \mathbb{N}$.

The appropriate categorical setting is the category Top(2) whose objects are pairs (X,A) of a topological space X and a subspace $A \subset X$ and whose morphisms $f : (X,A) \to (Y,B)$ are continuous maps $f : X \to Y$ with $f(A) \subset B$. Just as the category Top, this category can be equipped with notions of homotopy, which is a homotopy in the usual sense that satisfies a compatibility condition with respect to subspaces.

Definition 4.2.1: Consider the category Top(2) of pairs of topological spaces.

1. Two morphisms $f, f' : (X,A) \to (Y,B)$ in Top(2) are called **homotopic**, $f \sim_{(X,A)} f'$, if there is a homotopy $h : [0,1] \times X \to Y$ from f to f' with $h([0,1] \times A) \subset B$.

2. This is an equivalence relation on each set $\mathrm{Hom}_{\mathrm{Top}(2)}((X,A),(Y,B))$ which is compatible with the composition of morphisms. The associated category hTop(2) with pairs of topological spaces as objects and homotopy classes of morphisms in Top(2) as morphisms is called the **homotopy category of pairs of topological spaces**.
3. A morphism $f:(X,A)\to(Y,B)$ is called a **homotopy equivalence** if there is a morphism $g:(Y,B)\to(X,A)$ with $g\circ f\sim_{(X,A)}\mathrm{id}_X$ and $f\circ g\sim_{(Y,B)}\mathrm{id}_Y$. In this case, the pairs (X,A) and (Y,B) are called **homotopy equivalent**.

With these definitions, we can generalise the notions of n-chains, n-cycles and n-boundaries to the category Top(2). The basic idea is to identify n-chains in X that differ by an n-chain in A, i. e. to to consider the quotients $S_n(X)/S_n(A)$ with respect to the submodule $S_n(A)\subset S_n(X)$. Similarly, we weaken the conditions on n-cycles and n-boundaries. Instead of requiring that an n-cycle (n-boundary) lies in the kernel (image) of the boundary operator $\partial_n: S_n(X)\to S_{n-1}(X)$, one only imposes these conditions up to n-chains in A. That this is consistent is guaranteed by the following lemma.

Lemma 4.2.2: Let R be a commutative unital ring and (X,A) a pair of topological spaces. Denote for $n\in\mathbb{N}_0$ by $S_n(X,A)=S_n(X)/S_n(A)$ the quotient module of **relative n-chains** and by $\pi_n: S_n(X)\to S_n(X,A)$ the canonical surjection. Then the boundary operator $\partial_n: S_n(X)\to S_{n-1}(X)$ induces an R-module morphism $\partial_n^{(X,A)}: S_n(X,A)\to S_{n-1}(X,A)$ with $\partial_n^{(X,A)}\circ\pi_n=\pi_{n-1}\circ\partial_n$, the **relative boundary operator**, and one obtains a chain complex

$$S_\bullet(X,A)=\quad\ldots\xrightarrow{\partial_{n+2}^{(X,A)}} S_{n+1}(X,A)\xrightarrow{\partial_{n+1}^{(X,A)}} S_n(X,A)\xrightarrow{\partial_{n}^{(X,A)}} S_{n-1}(X,A)\xrightarrow{\partial_{n-1}^{(X,A)}}\ldots$$

Elements of $Z_n(X,A):=\ker(\partial_n^{(X,A)})\subset S_n(X,A)$ and $B_n(X,A)=\mathrm{Im}\,(\partial_{n+1}^{(X,A)})\subset Z_n(X,A)$ are called **relative n-cycles** and **relative n-boundaries**. The **nth relative homology** of $S_\bullet(X,A)$ is the quotient module $H_n(X,A)=Z_n(X,A)/B_n(X,A)$.

Proof:
The definition of the boundary operator implies $\partial_n(S_n(A))\subset S_{n-1}(A)$ and hence $S_n(A)\subset\ker(\pi_{n-1}\circ\partial_n)$. By the universal property of the quotient, there is a unique R-module morphism $\partial_n^{(X,A)}: S_n(X,A)\to S_{n-1}(X,A)$ with $\partial_n^{(X,A)}\circ\pi_n=\pi_{n-1}\circ\partial_n$. Using again the universal property of the quotient, we obtain

$$\partial_{n-1}^{(X,A)}\circ\partial_n^{(X,A)}\circ\pi_n=\partial_{n-1}^{(X,A)}\circ\pi_{n-1}\circ\partial_n=\pi_{n-2}\circ\partial_{n-1}\circ\partial_n=0\quad\Rightarrow\quad\partial_{n-1}^{(X,A)}\circ\partial_n^{(X,A)}=0\;\forall n\in\mathbb{N}.$$

□

Remark 4.2.3:

1. For $A=\emptyset$, relative n-chains, n-cycles, n-boundaries and homologies reduce to singular n-chains, n-cycles, n-boundaries and homologies:
$$S_n(X,\emptyset)\cong S_n(X)\quad Z_n(X,\emptyset)\cong Z_n(X)\quad B_n(Z,\emptyset)\cong B_n(X)\quad H_n(X,\emptyset)\cong H_n(X).$$
2. For $A=X$, one has $S_n(X,X)\cong Z_n(X,X)\cong B_n(X,X)\cong H_n(X,X)\cong 0$ for all $n\in\mathbb{N}_0$.

3. For a Δ-complex $(X, \{\sigma_i\}_{i\in I})$ and a sub complex $(A, \{\sigma_i\}_{i\in J})$ with $J \subset I$ we define analogously the R-modules of relative simplicial n-chains and the relative simplicial boundary operator as well as relative simplicial n-cycles, n-boundaries and homologies:

$$S_n^\Delta(X,A) = S_n^\Delta(X)/S_n^\Delta(A), \qquad \partial_n^{(X,A)} : S_n^\Delta(X,A) \to S_{n-1}^\Delta(X,A), \quad [\beta] \mapsto [\partial_n^\Delta \beta]$$
$$Z_n^\Delta(X,A) = \ker(\partial_n^{(X,A)}), \quad B_n^\Delta(X,A) = \mathrm{Im}\,(\partial_{n+1}^{(X,A)}), \quad H_n^\Delta(X,A) = Z_n^\Delta(X,A)/B_n^\Delta(X,A).$$

Just as in singular homology, one finds that morphisms in Top(2) give rise to chain maps between the relative chain complexes and relative homotopies induce chain homotopies between them. Both of these statements follow directly from the corresponding statements for singular homology, because all relevant structures in Top(2) are compatible with subspaces.

Lemma 4.2.4: Let R be a commutative unital ring.

1. Relative n-chains define a functor $S_\bullet^{(2)} : \mathrm{Top}(2) \to \mathrm{Ch}_{R\text{-Mod}}$ that assigns to a pair of topological spaces (X,A) the chain complex $S_\bullet^{(2)}(X,A) = S_\bullet(X,A)$ and to a morphism $f : (X,A) \to (Y,B)$ the chain map $S_\bullet^{(2)}(f) : S_\bullet(X,A) \to S_\bullet(Y,B)$. The nth relative homology is obtained by composing this functor with the functor $H_n : \mathrm{Ch}_{R-\mathrm{Mod}} \to R\text{-Mod}$.
2. If $f, f' : (X,A) \to (Y,B)$ are homotopic, then they induce chain homotopic chain maps $S_\bullet^{(2)}(f), S_\bullet^{(2)}(f') : S_\bullet(X,A) \to S_\bullet(Y,B)$ and the same module morphisms on the relative homologies $H_n(f_\bullet) = H_n(f'_\bullet) : H_n(X,A) \to H_n(Y,B)$. In particular, if (X,A) and (Y,B) are homotopy equivalent, then $H_n(X,A) \cong H_n(Y,B)$ for all $n \in \mathbb{N}_0$.
3. The relative homologies induce a functor $H_n : \mathrm{hTop}(2) \to R\text{-Mod}$.

Proof:
1. Denote by $\pi_n : S_n(X) \to S_n(X,A)$ and $\pi'_n : S_n(Y) \to S_n(Y,B)$ the cnonical surjections. As $f(A) \subset B$ for any morphism $f : (X,A) \to (Y,B)$, we have $S_n(f)(S_n(A)) \subset S_n(B)$ and hence $S_n(A) \in \ker(\pi'_n \circ S_n(f))$. By the universal property of the quotient, there is a unique R-module morphism $S_n^{(2)}(f) : S_n(X,A) \to S_n(Y,B)$ with $S_n^{(2)}(f) \circ \pi_n = \pi'_n \circ S_n(f)$. Compatibility with identities and composition follows from the corresponding properties of singular chain maps. We have $S_n^{(2)}(\mathrm{id}_X) \circ \pi_n = \pi_n \circ S_n(\mathrm{id}_X) = \pi_n$ and for any morphism $g : (Y,B) \to (Z,C)$

$$S_n^{(2)}(g\circ f) \circ \pi_n = \pi''_n \circ S_n(g\circ f) = \pi''_n \circ S_n(g) \circ S_n(f) = S_n^{(2)}(g) \circ \pi'_n \circ S_n(f) = S_n^{(2)}(g) \circ S_n^{(2)}(f) \circ \pi_n,$$

where $\pi''_n : S_n(Z) \to S_n(Z,C)$ is the canonical surjection. By the universal property of the quotient, this implies $S_n^{(2)}(\mathrm{id}_X) = \mathrm{id}_{S_n^{(2)}(X)}$, $S_n^{(2)}(g\circ f) = S_n^{(2)}(g) \circ S_n^{(2)}(f)$.

2. The proof is analogous to the one of Theorem 4.1.15. We construct a relative counterpart of the chain homotopy $h_n : S_n(X) \to S_{n+1}(Y)$ from (29) for any relative homotopy $h : [0,1] \times X \to Y$ between morphisms $f, f' : (X,A) \to (Y,B)$. As any relative homotopy satisfies $h([0,1] \times A) \subset B$, it follows from (29) that $h_n(S_n(A)) \subset S_{n+1}(B)$ and hence $S_n(A) \subset \ker(\pi'_{n+1} \circ h_n)$. By the universal property of the quotient, there is a unique R-module morphism $h_n^{(2)} : S_n(X,A) \to S_{n+1}(Y,B)$ with $h_n^{(2)} \circ \pi_n = \pi'_{n+1} \circ h_n$. That the morphisms $h_n^{(2)}$ for $n \in \mathbb{N}_0$ define a chain homotopy follows from the universal property of the quotient, since

$$(\partial_{n+1}^{(Y,B)} \circ h_n^{(2)} + h_{n-1}^{(2)} \circ \partial_n^{(X,A)}) \circ \pi_n = \partial_{n+1}^{(Y,B)} \circ \pi_{n+1} \circ h_n + h_{n-1}^{(2)} \circ \pi_n \circ \partial_n$$
$$= \pi_n \circ (\partial'_{n+1} \circ h_n + h_{n-1} \circ \partial_n) = \pi'_n \circ (S_n(f') - S_n(f)) = (S_n^{(2)}(f') - S_n^{(2)}(f)) \circ \pi_n.$$

$\square$

We will now clarify the relation between the homologies of a topological space X, of a subspace $A \subset X$ and the associated relative homologies. For this, we need chain maps that relate the associated chain complexes

$$
\begin{aligned}
S_\bullet(X): &\quad \ldots \xrightarrow{\partial_{n+2}} S_{n+1}(X) \xrightarrow{\partial_{n+1}} S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \xrightarrow{\partial_{n-1}} \ldots \\
S_\bullet(A): &\quad \ldots \xrightarrow{\partial_{n+2}} S_{n+1}(A) \xrightarrow{\partial_{n+1}} S_n(A) \xrightarrow{\partial_n} S_{n-1}(A) \xrightarrow{\partial_{n-1}} \ldots \\
S_\bullet(X,A): &\ldots \xrightarrow{\partial^{(X,A)}_{n+2}} S_{n+1}(X,A) \xrightarrow{\partial^{(X,A)}_{n+1}} S_n(X,A) \xrightarrow{\partial^{(X,A)}_n} S_{n-1}(X,A) \xrightarrow{\partial_{n-1}} \ldots
\end{aligned}
$$

and that arise from morphisms in Top(2). The relevant morphisms in Top(2) are the morphism $\iota : (A, \emptyset) \to (X, \emptyset)$ induced by the inclusion $i_A : A \to X$ and the morphism $\pi : (X, \emptyset) \to (X, A)$. The associated chain maps are the inclusion map and the canonical surjection

$$S^{(2)}_n(\iota) : S_n(A) \to S_n(X), \sigma \mapsto \sigma \qquad S^{(2)}_n(\pi) = \pi_n : S_n(X) \to S_n(X,A), \sigma \mapsto [\sigma],$$

Clearly, $S^{(2)}_n(\iota)$ is injective, $S^{(2)}_n(\pi)$ is surjective and by definition of the relative n-chains, we have $\ker(S^{(2)}_n(\pi)) = S_n(A) = \mathrm{Im}\,(S^{(2)}_n(\iota))$. In other words,

$$0 \to S_\bullet(A) \xrightarrow{S^{(2)}_\bullet(\iota)} S_\bullet(X) \xrightarrow{S^{(2)}_\bullet(\pi)} S_\bullet(X,A) \to 0$$

is a short exact sequence of chain complexes in R-Mod. By applying Theorem 2.3.11 and Lemma 2.3.12, we obtain the following theorem.

Theorem 4.2.5: Let R be a commutative unital ring.

1. For every pair (X, A) of topological spaces, there is a long exact sequence of homologies

$$\ldots \xrightarrow{H_{n+1}(\pi)} H_{n+1}(X,A) \xrightarrow{\overline{\partial}_{n+1}} H_n(A) \xrightarrow{H_n(\iota)} H_n(X) \xrightarrow{H_n(\pi_\bullet)} H_n(X,A) \xrightarrow{\overline{\partial}_{n+1}} \ldots$$

2. For any morphism $f : (X, A) \to (Y, B)$ in Top(2), the associated long exact sequences of homologies form a commutative diagram

$$
\begin{array}{ccccccccccccc}
\ldots & \xrightarrow{H_{n+1}(\iota)} & H_{n+1}(X) & \xrightarrow{H_{n+1}(\pi)} & H_{n+1}(X,A) & \xrightarrow{\overline{\partial}_{n+1}} & H_n(A) & \xrightarrow{H_n(\iota)} & H_n(X) & \xrightarrow{H_n(\pi)} & H_n(X,A) & \xrightarrow{\overline{\partial}_n} & \ldots \\
 & & \downarrow{\scriptstyle H_{n+1}(f)} & & \downarrow{\scriptstyle H_{n+1}(f)} & & \downarrow{\scriptstyle H_n(f\circ i_A)} & & \downarrow{\scriptstyle H_n(f)} & & \downarrow{\scriptstyle H_n(f)} & & \\
\ldots & \xrightarrow[H_{n+1}(\iota)]{} & H_{n+1}(Y) & \xrightarrow[H_{n+1}(\pi)]{} & H_{n+1}(Y,B) & \xrightarrow[\overline{\partial}_{n+1}]{} & H_n(B) & \xrightarrow[H_n(\iota)]{} & H_n(Y) & \xrightarrow[H_n(\pi)]{} & H_n(Y,B) & \xrightarrow[\overline{\partial}_n]{} & \ldots.
\end{array}
$$

In the following, we will use this theorem systematically to compute the homologies of topological spaces. In particular, we can apply it to subspaces $A \subset X$ that are retracts or deformation retracts of X.

Example 4.2.6: Let R be a commutative unital ring, X a topological space and $A \subset X$ a retract of X. Then for all $n \in \mathbb{N}_0$ the R-module morphism $H_n(\iota) : H_n(A) \to H_n(X)$ is injective. If A is a deformation retract of X, then $H_n(r) : H_n(X) \to H_n(A)$ and $H_n(\iota) : H_n(A) \to H_n(X)$ are isomorphisms.

This follows because the retraction induces a morphism $r : (X, A) \to (A, A)$ in Top(2) with $r \circ i_A = \mathrm{id}_A$. As $H_n(A, A) = 0$ for all $n \in \mathbb{N}_0$, Theorem 4.2.5 yields the following commutative diagram with exact rows

$$\begin{tikzcd}
\ldots \arrow[r, "H_{n+1}(\iota)"] & H_{n+1}(X) \arrow[r, "H_{n+1}(\pi)"] \arrow[d, "H_{n+1}(r)"'] & H_{n+1}(X,A) \arrow[r, "\overline{\partial}_{n+1}"] \arrow[d, "H_{n+1}(r)"'] & H_n(A) \arrow[r, "H_n(\iota)"] \arrow[d, "H_n(r|_A)"', "=\mathrm{id}"] & H_n(X) \arrow[r, "H_n(\pi)"] \arrow[d, "H_n(r)"] & H_n(X,A) \arrow[r, "\overline{\partial}_n"] \arrow[d, "H_n(r)"] & \ldots \\
\ldots \arrow[r, "\mathrm{id}"'] & H_{n+1}(A) \arrow[r] & 0 \arrow[r, "\overline{\partial}_{n+1}"'] & H_n(A) \arrow[r, "\mathrm{id}"'] & H_n(A) \arrow[r] & 0 \arrow[r, "\overline{\partial}_n"'] & \ldots
\end{tikzcd}$$

This implies $\ker(H_n(\iota)) = \mathrm{Im}\,(\overline{\partial}_{n+1}) = \mathrm{Im}\,(\overline{\partial}_{n+1} \circ H_{n+1}(r)) = 0$. If $A \subset X$ is a deformation retract, then $r : X \to A$ is a homotopy equivalence, and by Corollary 4.1.16 all R-module morphisms $H_n(r) : H_n(X) \to H_n(A)$ and $H_n(\iota) : H_n(A) \to H_n(X)$ are isomorphisms.

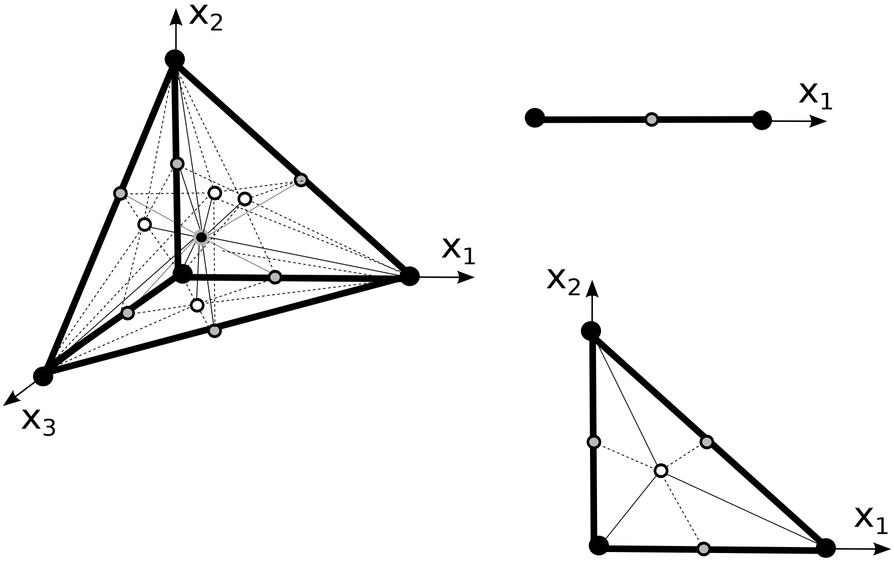
4.3 Excision

In this section, we develop a homology counterpart for the theorem of Seifert and van Kampen. The idea is to consider a covering of a topological space X by open subsets $U_i \subset X$, $i \in I$, and to relate the (relative) homologies of X (with respect to a subspace $A \subset X$) to the homologies of U_i (and $U_i \cap A$). In particular, we will show that the homologies $H_n(X, A)$ do not change if one imposes the condition that all singular simplexes take values in one of the subsets $U_i \subset X$.

As in the proof of the theorem of Seifert and van Kampen, this will be achieved by subdividing the relevant structures - the singular n-simples $\Delta^n \to X$ - into singular simplexes $\Delta^n \to U_i$. This requires a systematic subdivision procedure for singular n-simplexes that yield simplexes of arbitrarily small diameter. As we can view a singular n-simplex $\sigma : \Delta^n \to X$ as the image of the affine n-simplex id_{Δ^n} under the chain map $S_n(\sigma) : S_n(\Delta^n) \to S_n(X)$, this problem can be reduced to a subdivision procedure for affine linear simplexes $\Delta^n \to \Delta^j$.

A minimum requirement for such a subdivision procedure is that it is compatible with affine maps - it should not matter if an affine simplex is first subdivided and then mapped to another affine simplex or first mapped to an affine simplex and subdivided afterwards. This means that all vertices in the subdivided simplex must be affine linear functions of the original vertices. Natural candidates for the vertices in a subdivision of an affine n-simplex are its barycentre and the barycentres of its $(n-1)$-, $(n-2)-$,..., 1-faces. This leads to the concept of the *barycentric subdivision*, in which an affine n-simplex is divided into $(n+1)!$ sub simplexes that each contain its barycentre, the barycentre of one $(n-1)$-face, the barycentre of an adjacent $(n-2)$-face, a barycentre of an adjacent $(n-3)$-face,..., and one vertex of the simplex, as shown below.

As we want to apply this construction to singular n-simplexes, i. e. continuous maps $\sigma : \Delta^n \to X$, we need a description of the barycentric subdivision in terms of affine maps $\Delta^n \to \Delta^j$ between the standard simplexes. This is simlar to the description of the faces and the boundary of a singular simplex in terms of the face maps $\Delta^{n-1} \to \Delta^n$. We thus require an R-module morphism $U_n : S_n^{\mathrm{aff}}(\Delta^n) \to S_n^{\mathrm{aff}}(\Delta^n)$ that maps the submodule $S_n^{\mathrm{aff}}(\Delta^n) \subset S_n(\Delta^n)$ of affine n-simplexes $\sigma : \Delta^n \to \Delta^n$ to itself and assigns to each affine n-simplex an R-linear combination of affine n-simplexes in its barycentric subdivision.



Barycentric subdivision for the standard simplexes $\Delta^3, \Delta^2, \Delta^1$.

Definition 4.3.1: Let R be a unital ring. For $j, n \in \mathbb{N}_0$ the **barycentric map** is the R-module morphism $C_n^j : S_n^{\text{aff}}(\Delta^j) \to S_{n+1}^{\text{aff}}(\Delta^j)$ given by

$$C_n^j(\sigma) : \Delta^{n+1} \to \Delta^j, \qquad C_n^j(\sigma)(e_k) = \begin{cases} b_j = \frac{1}{j+1}\Sigma_{m=0}^j e_m & k = 0 \\ \sigma(e_{k-1}) & 1 \leq k \leq n+1. \end{cases}$$

for all affine n-simplexes $\sigma : \Delta^n \to \Delta^j$. The **barycentric subdivision operator** is the R-module morphism $U_n : S_n(X) \to S_n(X)$ defined inductively by $U_0 = \text{id}_{S_0(X)}$ and

$$U_n(\sigma) = S_n(\sigma) \circ U_n(\text{id}_{\Delta^n}) = S_n(\sigma) \circ C_{n-1}^n \circ U_{n-1} \circ \partial_n(\text{id}_{\Delta^n})$$

for all singular simplexes $\sigma : \Delta^n \to X$.

Remark 4.3.2:

1. The barycentric subdivision operator is well-defined because $U_0(S_0^{\text{aff}}(\Delta^0)) \subset S_0^{\text{aff}}(\Delta^0)$ and $U_{n-1}(S_{n-1}^{\text{aff}}(\Delta^{n-1})) \subset S_{n-1}^{\text{aff}}(\Delta^{n-1})$ imply $U_n(S_n^{\text{aff}}(\Delta^n)) \subset S_n^{\text{aff}}(\Delta^n)$.
2. The image of the face map $F_n^j : \Delta^{n-1} \to \Delta^n$ under the barycentric map $C_{n-1}^n : S_{n-1}^{\text{aff}}(\Delta^n) \to S_n^{\text{aff}}(\Delta^n)$ is given by

$$C_{n-1}^n(F_n^j)(e_k) = \begin{cases} b_n = \frac{1}{n+1}\Sigma_{j=0}^n e_j & k = 0 \\ e_{k-1} & 1 \leq k \leq j \\ e_k & j < k \leq n. \end{cases} \tag{31}$$

   It maps the standard n-simplex $[e_0, ..., e_n]$ to the simplex $[b_n, e_0, ..., e_{j-1}, e_{j+1}, ..., e_n]$, in which the vertex e_j is replaced by the barycentre b_n of Δ^n.
3. The barycentric maps satisfy the identities

$$\partial_1(C_0^1(\sigma)) = \sigma - b_1, \qquad \partial_{n+1}(C_n^j(\sigma)) = \sigma - C_{n-1}^{j-1}(\partial_n(\sigma)). \tag{32}$$

   for all singular n-simplexes $\sigma : \Delta^n \to \Delta^j$.

It remains to show that the barycentric subdivision operator indeed implements the geometrical notion of barycentric subdivision. For this, we need to show that it maps an affine n-simplex $\sigma : \Delta^n \to \mathbb{R}^m$ to an R-linear combination of affine simplexes, which involves precisely the simplexes in its barycentric subdivision and with coefficients ± 1, depending on the orientation of these simplexes.

Lemma 4.3.3: The barycentric subdivision operator is given by

$$U_n(\sigma)([e_0, ..., e_n]) = \Sigma_{\pi \in S_{n+1}} \text{sgn}(\pi)[v_0^\pi, ..., v_n^\pi] \quad \text{with } v_r^\pi = \tfrac{1}{n-r+1} \Sigma_{j=r}^n v_{\pi(j)}$$

for all affine n-simplexes $\sigma : \Delta^n \to \mathbb{R}^m$, $e_i \mapsto v_i$.

Proof:
For $n = 0$ the claim is trivial. Suppose it holds for all $n \leq k-1$ and let $\sigma : \Delta^k \to \mathbb{R}^m$, $e_i \mapsto v_i$ be an affine k-simplex. Then we obtain

$$\begin{aligned} U_k(\sigma)([e_0, ..., e_k]) &= S_k(\sigma) \circ C_{k-1}^k \circ U_{k-1}(\partial_k \text{id}_{\Delta^k})([e_0, ..., e_k]) \\ &= \Sigma_{j=0}^k (-1)^j S_k(\sigma) \circ C_{k-1}^k \circ S_{k-1}(F_k^j) \circ U_{k-1}(\text{id}_{\Delta^{k-1}})([e_0, ..., e_k]) \\ &\overset{k-1}{=} \Sigma_{\pi \in S_k} \Sigma_{j=0}^k (-1)^j \text{sgn}(\pi) S_k(\sigma) \circ C_{k-1}^k (F_k^j)([e_0^\pi, ..., e_{k-1}^\pi]) \overset{\star}{=} \Sigma_{\tau \in S_{k+1}} \text{sgn}(\tau)[v_0^\tau, ..., v_k^\tau], \end{aligned}$$

where $\star$ follows from (31) and the fact that for any $\pi \in S_k$ the permutation $\tau \in S_{k+1}$

$$\tau(i) = \begin{cases} j & i = 0 \\ \pi(i-1) & i \in \{1, ..., k\} \text{ and } \pi(i-1) < j \\ \pi(i-1) + 1 & i \in \{1, ..., k\} \text{ and } \pi(i-1) \geq j \end{cases}$$

satisfies $\text{sgn}(\tau) = (-1)^j \text{sgn}(\sigma)$. $\square$

Lemma 4.3.3 shows that the barycentric subdivision operator $U_n : S_n(X) \to S_n(X)$ from Definition 4.3.1 indeed has the desired geometrical interpretation. It maps an affine n-simplex $\sigma : \Delta^n \to \mathbb{R}^m$, $e_i \mapsto v_i$ to an R-linear combination of $(n+1)!$ affine n-simplexes whose images $[v_0^\pi, ..., v_n^\pi]$ have as vertices the barycentre v_0^π of $[v_0, ..., v_n]$, the barycentre v_1^π of the $(n-1)$-face $[v_{\pi(1)}, ..., v_{\pi(n)}]$, the barycentre v_2^π of the adjacent $(n-2)$-face $[v_{\pi(2)}, ..., v_{\pi(n)}]$,..., the barycentre v_{n-1}^π of the 1-face $[v_{\pi(n-1)}, v_{\pi(n)}]$ and the vertex $v_{\pi(n)}$.

To determine how the barycentric subdivision operator affects the homologies $H_n(X)$, we have to clarify its algebraic properties and give an interpretation in terms of chain complexes and chain maps.

Proposition 4.3.4: Let R be a commutative unital ring and X a topological space.

1. The barycentric subdivision operators $U_n(X) : S_n(X) \to S_n(X)$ define a natural transformation from the functor $S_\bullet : \text{Top} \to \text{Ch}_{R\text{-Mod}}$ to itself.
2. The chain map $U_\bullet(X) : S_\bullet(X) \to S_\bullet(X)$ is naturally chain homotopic to the identity map $\text{id}_{S_\bullet(X)} : S_\bullet(X) \to S_\bullet(X)$, i. e. there are natural transformations $T_n : S_n \to S_{n+1}$ between the functors $S_n, S_{n+1} : \text{Top} \to R\text{-Mod}$ such that $T_{n,X} : S_n(X) \to S_{n+1}(X)$ form a chain homotopy $T_{\bullet,X} : U_\bullet(X) \Rightarrow \text{id}_{S_\bullet(X)}$.

Proof:
1. A natural transformation from $S_\bullet : \text{Top} \to \text{Ch}_{R\text{-Mod}}$ to itself is an assignment of a chain map $U_\bullet(X) : S_\bullet(X) \to S_\bullet(X)$ to each topological space X such that $S_\bullet(f) \circ U_\bullet(X) = U_\bullet(Y) \circ S_\bullet(f)$ for all continuous maps $f : X \to Y$. We thus have to show that for all $n \in \mathbb{N}_0$, the barycentric subdivision operator satisfies

$$\partial_n \circ U_n(\sigma) = U_{n-1} \circ \partial_n(\sigma) \qquad S_n(f) \circ U_n(\sigma) = U_n \circ S_n(f)(\sigma)$$

for all singular n-simplexes $\sigma : \Delta^n \to X$ and all continuous maps $f : X \to Y$. The second identity follows directly from the definition

$$S_n(f) \circ U_n(\sigma) = S_n(f) \circ S_n(\sigma) \circ U_n(\text{id}_{\Delta^n}) = S_n(S_n(f)(\sigma)) \circ U_n(\text{id}_{\Delta^n}) = U_n(S_n(f)(\sigma)).$$

As $S_\bullet(\sigma) : S_\bullet(\Delta^n) \to S_\bullet(X)$ is a chain map and the second identity holds, it is sufficient to prove the first identity for $\sigma = \text{id}_{\Delta^n} : \Delta^n \to \Delta^n$. For $n = 0$, it is obvious, and if it holds for U_k with $k \leq n-1$, we obtain

$$\begin{aligned}\partial_n \circ U_n(\text{id}_{\Delta^n}) =& \partial_n(C^n_{n-1} \circ U_{n-1} \circ \partial_n(\text{id}_{\Delta^n})) \overset{(32)}{=} U_{n-1} \circ \partial_n(\text{id}_{\Delta^n}) - C^{n-1}_{n-2} \circ \partial_{n-1} \circ U_{n-1} \circ \partial_n(\text{id}_{\Delta^n})\\ \overset{n-1}{=}& U_{n-1} \circ \partial_n(\text{id}_{\Delta^n}) - C^{n-1}_{n-2} \circ U_{n-2} \circ \partial_{n-1} \circ \partial_n(\text{id}_{\Delta^n}) = U_{n-1} \circ \partial_n(\text{id}_{\Delta^n}).\end{aligned}$$

2. We define the R-module morphisms $T_{n,X} : S_n(X) \to S_{n+1}(X)$ inductively by

$$\begin{aligned}&T_{0,X} = 0 \\ &T_{n,X}(\sigma) = S_{n+1}(\sigma) \circ T_{n,\Delta^n}(\text{id}_{\Delta^n}) = S_{n+1}(\sigma) \circ C^n_n(\text{id}_{\Delta^n} - U_n(\text{id}_{\Delta_n}) - T_{n-1,\Delta^n}(\partial_n \text{id}_{\Delta^n}))\end{aligned} \tag{33}$$

for all singular n-simplexes $\sigma : \Delta^n \to X$. For any continuous map $f : X \to Y$, we obtain

$$\begin{aligned}T_{n,Y} \circ S_n(f)(\sigma) =& S_{n+1}(S_n(f)(\sigma)) \circ T_{n,\Delta^n}(\text{id}_{\Delta^n}) = S_{n+1}(f \circ \sigma) \circ T_{n,\Delta^n}(\text{id}_{\Delta^n})\\ =& S_{n+1}(f) \circ S_{n+1}(\sigma) \circ T_{n,\Delta^n}(\text{id}_{\Delta^n}) = S_{n+1}(f) \circ T_{n,X}(\sigma).\end{aligned}$$

This proves that the R-module morphisms $T_{n,X} : S_n(X) \to S_{n+1}(X)$ define a natural transformation $T_n : S_n \to S_{n+1}$. To show that $T_{\bullet,X} = (T_{n,X})_{n \in \mathbb{N}_0}$ is a chain homotopy between $U_\bullet(X) : S_\bullet(X) \to S_\bullet(X)$ and $\text{id}_{S_\bullet(X)} : S_\bullet(X) \to S_\bullet(X)$, we show that

$$\partial_{n+1} \circ T_{n,X} + T_{n-1,X} \circ \partial_n = \text{id}_{S_n(X)} - U_n(X) \qquad \forall n \in \mathbb{N}.$$

Due to the naturality of T_n and because $S_n(\sigma) : S_n(\Delta^n) \to S_n(X)$ is a chain map, it is sufficient to prove this for T_{n,Δ^n}. For $n = 0$, it is obvious. If it holds for T_{k,Δ^n} with $k \leq n-1$, we obtain

$$\begin{aligned}&\partial_{n+1} T_{n,\Delta^n}(\text{id}_{\Delta^n}) = \partial_{n+1} \circ C^n_n(\text{id}_{\Delta^n} - U_n(\text{id}_{\Delta_n}) - T_{n-1,\Delta^n}(\partial_n \text{id}_{\Delta^n}))\\ &\overset{(32)}{=} \text{id}_{\Delta^n} - U_n(\text{id}_{\Delta_n}) - T_{n-1,\Delta^n}(\partial_n \text{id}_{\Delta^n}) - C^{n-1}_{n-1} \circ \partial_n(\text{id}_{\Delta^n} - U_n(\text{id}_{\Delta_n}) - T_{n-1,\Delta^n}(\partial_n \text{id}_{\Delta^n}))\\ &\overset{n-1}{=} \text{id}_{\Delta^n} - U_n(\text{id}_{\Delta_n}) - T_{n-1,\Delta^n}(\partial_n \text{id}_{\Delta^n})\\ &\quad - C^{n-1}_{n-1}(\partial_n \text{id}_{\Delta^n} - U_{n-1}(\partial_n \text{id}_{\Delta_n}) - \partial_n \text{id}_{\Delta^n} + U_{n-1}(\partial_n \text{id}_{\Delta^n}) + T_{n-2,\Delta^n} \circ \partial_{n-1} \circ \partial_n(\text{id}_{\Delta^n}))\\ &= \text{id}_{\Delta^n} - U_n(\text{id}_{\Delta_n}) - T_{n-1,\Delta^n}(\partial_n \text{id}_{\Delta^n}).\end{aligned}$$

□

As the chain maps $U_\bullet(X) : S_\bullet(X) \to S_\bullet(X)$ defined by the barycentric subdivision operator are chain homotopic to the identity map $\text{id}_{S_\bullet(X)} : S_\bullet(X) \to S_\bullet(X)$, applying the barycentric subdivision operator to an n-cycle yields an n-cycle in the same homology class. Moreover, Proposition 4.3.4 states that the chain maps $U_\bullet : S_\bullet \to S_\bullet$ commute with the chain maps $S_\bullet(f) : S_\bullet(X) \to S_\bullet(Y)$ in the sense of a natural transformation. It is clear that analogous statements hold if one applies the barycentric subdivision operators repeatedly, since the composite of natural transformations is a natural transformation, the composite of chain maps is a chain map and chain homotopies are compatible with the composition of chain maps. It is also clear that such composites again map affine simplexes to affine simplexes.

Remark 4.3.5:

1. The r-fold composites $U_\bullet^r = U_\bullet \circ ... \circ U_\bullet : S_\bullet \to S_\bullet$ are natural transformations from $S_\bullet : \mathrm{Top} \to \mathrm{Ch}_{R\text{-Mod}}$ to itself and the associated chain maps $U_\bullet^r(X) : S_\bullet(X) \to S_\bullet(X)$ are naturally chain homotopic to $\mathrm{id}_{S_\bullet(X)}$. The chain homotopy $T_{n,X}^{(r)} : U_\bullet^r(X) \Rightarrow \mathrm{id}_{S_\bullet(X)}$ is given by the R-module morphisms $T_{n,X}^{(r)} = (U_{n+1}^{r-1} + U_{n+1}^{r-2} + \ldots + U_{n+1} + \mathrm{id}_{S_{n+1}(X)}) \circ T_{n,X}$. This follows because one has for all singular n-simplexes $\sigma : \Delta^n \to X$

$$\begin{aligned}\sigma - U_n^r(\sigma) &= (\mathrm{id}_{S_n(X)} + U_n + \ldots + U_n^{r-1})(\sigma - U_n(\sigma))\\ &= (\mathrm{id}_{S_n(X)} + U_n + \ldots + U_n^{r-1})(\partial_{n+1} T_{n,X}(\sigma) + T_{n-1,X}(\partial_n \sigma)) = \partial_n(T_{n,X}^{(r)}(\sigma)) + T_{n-1,x}^{(r)}(\partial_n \sigma).\end{aligned}$$

2. For all $r \in \mathbb{N}$ and any affine n-simplex $\sigma : \Delta^n \to \Delta^j$, all n-simplexes in $U_n^r(\sigma)$ and all $(n+1)$-simplexes in $T_{n,\Delta^j}^{(r)}(\sigma)$ are again affine.

We will now consider a covering of a topological space by open sets U_i, $i \in I$ and subdivide each singular n-simplex $\sigma : \Delta^n \to X$ by applying the barycentric subdivision operator repeatedly until the image $\sigma(\Delta^n)$ is contained in one of the sets U_i. To determine the effect of this procedure on the homologies, we have to consider the submodule of $S_n(X)$ that is generated by the affine n-simplexes whose image is contained in one of the sets U_i and to show that these submodules form a chain complex.

Definition 4.3.6: Let R be a commutative unital ring, X a topological space, $A \subset X$ a subspace and $\mathcal{U} = \{U_i\}_{i \in I}$ an open cover of X.

1. An n-chain $\sigma = \sum_{j=1}^m r_j \sigma_j \in S_n(X)$ is called **$\mathcal{U}$-small** if for every $j \in \{1, ..., m\}$ there is an $i \in I$ with $\sigma_j(\Delta^n) \subset U_i$. We denote by $S_n^{\mathcal{U}}(X) \subset S_n(X)$ the submodule of $\mathcal{U}$-small n-chains.
2. As $\partial_n(S_n^{\mathcal{U}}(X)) \subset S_{n-1}^{\mathcal{U}}(X)$, $\mathcal{U}$-small n-chains define a chain complex $(S_\bullet^{\mathcal{U}}(X), \partial_n^{\mathcal{U}})$. Elements of $Z_n^{\mathcal{U}}(X) = \ker(\partial_n^{\mathcal{U}}) \subset Z_n(X)$ and $B_n^{\mathcal{U}}(X) = \mathrm{Im}\,(\partial_{n+1}^{\mathcal{U}}) \subset B_n(X)$ are called **$\mathcal{U}$-small n-cycles** and **$\mathcal{U}$-small n-boundaries**. The nth **$\mathcal{U}$-small homology** is the quotient $H_n^{\mathcal{U}}(X) = Z_n^{\mathcal{U}}(X)/B_n^{\mathcal{U}}(X)$.
3. Analogously, we define the R-module $S_n^{\mathcal{U}}(X, A) = S_n^{\mathcal{U}}(X)/S_n^{\mathcal{U}}(A)$ of **$\mathcal{U}$-small relative n-chains**, the submodules $Z_n^{\mathcal{U}}(X, A)$ and $B_n^{\mathcal{U}}(X, A)$ of **$\mathcal{U}$-small relative n-cycles** and **$\mathcal{U}$-small relative n-boundaries** and the **$\mathcal{U}$-small relative homologies** $H_n^{\mathcal{U}}(X, A) = Z_n^{\mathcal{U}}(X, A)/B_n^{\mathcal{U}}(X, A)$.

We can now prove that any singular n-simplex $\sigma : \Delta^n \to X$ can be mapped to a $\mathcal{U}$-small n-chain by applying the barycentric subdivision operator repeatedly. In this, we use that Δ^n

is compact and apply Lebesgue's lemma to obtain an $\epsilon > 0$ such that the image of any subset $A \subset \Delta^n$ of diameter $< \epsilon$ is contained in one if the sets U_i. It is then sufficient to prove that by applying the barycentric subdivision operator repeatedly to a standard n-simplex, one obtains a set of diameter $< \epsilon$.

Lemma 4.3.7: Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open covering of a topological space X and consider a singular n-simplex $\sigma : \Delta^n \to X$. Then there is an $r \in \mathbb{N}$, such that for $m \geq r$ all simplexes in $U_n^m(\sigma) \in S_n(X)$ are contained in one of the sets U_i. In particular, $U_n^m(\sigma) \in S_n^{\mathcal{U}}(X)$ for all $m \geq r$.

Proof:
1. We show that for any affine n-simplex $\sigma : \Delta^n \to \Delta^n$, all affine simplexes τ in $U_n^r(\sigma)$ satisfy

$$\mathrm{diam}(\tau(\Delta^n)) \leq \left(\tfrac{n}{n+1}\right)^r \mathrm{diam}(\sigma(\Delta^n)).$$

This follows inductively from the statement for $r = 1$. To prove the latter, we use induction over n. For $n = 0$, the claim holds trivially. Suppose now the claim is proven for $r = 1$ and $k \leq n-1$, and let $\tau : \Delta^n \to \Delta^n$ be an affine n-simplex in $U_n(\sigma)$. Then there are vertices $p, q \in \tau(\Delta^n)$ with $||p - q|| = \mathrm{diam}(\tau(\Delta^n))$. If p, q are contained in $\sigma(\partial\Delta^n)$, then the induction hypothesis implies

$$d(p,q) \leq \tfrac{n-1}{n}\mathrm{diam}(\sigma(\Delta^n)) < \tfrac{n}{n+1}\mathrm{diam}(\sigma(\Delta^n)).$$

Otherwise, one of the two vertices p, q must be the barycenter of $\sigma(\Delta^n) = [v_0, ..., v_n]$. If q is the barycenter of $\sigma(\Delta^n)$, then

$$d(p,q) = \left|\left|p - \tfrac{1}{n+1}\Sigma_{i=0}^n v_i\right|\right| = \tfrac{1}{n+1}\,||\Sigma_{i=0}^n (p - v_i)|| \leq \tfrac{1}{n+1}\Sigma_{i=0}^n ||p - v_i|| \leq \tfrac{n}{n+1}\mathrm{diam}(\sigma(\Delta^n)).$$

2. Consider now a singular n-simplex $\sigma : \Delta^n \to X$. Then the sets $\sigma^{-1}(U_i)$, $i \in I$ form an open cover of the compact metric space Δ^n. By Lebesgue's lemma there exists an $\epsilon > 0$ such that for all $p \in \Delta^n$, the set $U_\epsilon(p) = \{q \in \Delta^n : ||p - q|| < \epsilon\}$ is contained in one of the sets $\sigma^{-1}(U_i)$. If r is sufficiently large, then by 1. $\mathrm{diam}(\tau(\Delta^n)) < \epsilon$ for all affine simplexes τ in $U_n^r(\mathrm{id}_{\Delta^n})$ and hence $\tau(\Delta^n)$ is contained in one of the sets $\sigma^{-1}(U_i)$. It follows that for all simplexes in τ in $U_n^r(\sigma)$ the image $\tau(\Delta^n)$ is contained in one of the sets U_i, $i \in I$. □

Lemma 4.3.7 allows us to construct $\mathcal{U}$-small n-simplexes for any covering $\mathcal{U} = (U_i)_{i\in I}$ of X by applying the barycentric subdivision operator repeatedly. Proposition 4.3.4 suggests that applying the barycentric subdivision operator should not change the equivalence class of a given element $x \in Z_n(X)$ in $H_n(X)$. This leads one to expect that for all $n \in \mathbb{N}$, the R-modules $H_n(X)$ should should be isomorphic to the $\mathcal{U}$-small R-modules $H_n^{\mathcal{U}}(X)$.

Theorem 4.3.8: Let X be a topological space with a subspace $A \subset X$ and $\mathcal{U} = \{U_i\}_{i\in I}$ an open cover of X. Then the inclusion maps $j_\bullet : S_\bullet^{\mathcal{U}}(X, A) \to S_\bullet(X, A)$, $\sigma \mapsto \sigma$ induce R-module isomorphisms $H_n(j_\bullet) : H_n^{\mathcal{U}}(X, A) \xrightarrow{\sim} H_n(X, A)$.

Proof:
1. We prove the claim for $A = \emptyset$: Let $x \in Z_n^{\mathcal{U}}(X)$ a $\mathcal{U}$-small n-cycle with $H_n(j_\bullet)([x]_{\mathcal{U}}) = [x] = 0$. Then there is an n-chain $y \in S_{n+1}(X)$ with $x = \partial_{n+1}(y)$, and for all $r \in \mathbb{N}$, one has

$$\partial_{n+1} \circ U_{n+1}^r(y) = U_n^r \circ \partial_{n+1}(y) = U_n^r(x) = x - \partial_{n+1} \circ T_{n,X}^{(r)}(x) - T_{n-1,X}^{(r)} \circ \partial_n(x) = x - \partial_{n+1} \circ T_{n,X}^{(r)}(x).$$

This implies $x = \partial_{n+1}(U^r_{n+1}(y) + T^{(r)}_{n+1,X}(x))$. By Lemma 4.3.7, we have $U^r_{n+1}(y) \in S^{\mathcal{U}}_{n+1}(X)$ for r sufficiently large. Together with Lemma 4.3.7, the definitions of $T^{(r)}_{n,X}$ and $T_{n,X}$ in Remark 4.3.5 and in (33) also imply that $T^{(r)}_{n,X}(x) \in S^{\mathcal{U}}_{n+1}(X)$ for all $x \in S^{\mathcal{U}}_n(X)$. Hence, we have $x \in B^{\mathcal{U}}_n(X)$ and $[x]_{\mathcal{U}} = 0$. This shows that $H_n(j_\bullet)$ is injective.

Similarly, for any $x \in Z_n(X)$, we have $U^r_n(x) \in S^{\mathcal{U}}_n(X)$ for r sufficiently large and

$$H_n(j_\bullet)([U^r_n(x)]_{\mathcal{U}}) = [U^r_n(x)] = [x - \partial_{n+1} \circ T^{(r)}_{n,X}(x) - T^{(r)}_{n-1,X} \circ \partial_n(x)] = [x].$$

This shows that $H_n(j_\bullet) : H^{\mathcal{U}}_n(X) \to H_n(X)$ is surjective.

2. For $n \in \mathbb{N}$ and a subspace $A \subset X$, we obtain a commutative diagram with exact rows

$$\begin{tikzcd}
H^{\mathcal{U}}_n(A) \arrow[r, "H_n(\iota)"] \arrow[d, "\cong"'] & H^{\mathcal{U}}_n(X) \arrow[r, "H_n(\pi)"] \arrow[d, "\cong"'] & H^{\mathcal{U}}_n(X,A) \arrow[r, "\partial^{\mathcal{U}}_n"] \arrow[d, "H_n(j_\bullet)"] & H^{\mathcal{U}}_{n-1}(A) \arrow[r, "H_{n-1}(\iota)"] \arrow[d, "\cong"'] & H^{\mathcal{U}}_{n-1}(X) \arrow[d, "\cong"'] \\
H_n(A) \arrow[r, "H_n(\iota)"] & H_n(X) \arrow[r, "H_n(\pi)"] & H_n(X,A) \arrow[r, "\partial_n"] & H_{n-1}(A) \arrow[r, "H_{n-1}(\iota)"] & H_{n-1}(X),
\end{tikzcd}$$

and the 5-Lemma implies that $H_n(j_\bullet) : H^{\mathcal{U}}_n(X,A) \to H_n(X,A)$ is an isomorphism. □

In particular, we can apply Theorem 4.3.8 to compute the (relative) homologies of a topological space obtained by removing an open set $U \subset X$ with $\overline{U} \subset \mathring{A}$ from a pair of topological spaces (X,A). It turns out that removing such an open set $U \subset X$ does not affect the homologies, and we obtain the excision theorem.

Theorem 4.3.9: (**Excision theorem**)

Let (X,A) be a pair of topological spaces and $U \subset \mathring{A}$ a subspace with $\overline{U} \subset \mathring{A}$. Then the inclusion map $i_U : (X \setminus U, A \setminus U) \to (X,A)$ induces isomorphisms $H_n(i_U) : H_n(X \setminus U, A \setminus U) \xrightarrow{\sim} H_n(X,A)$.

Proof:
The sets $U_1 := \mathring{A}$ and $U_2 := X \setminus \bar{U}$ form an open covering of X, and by definition we have

$$S^{\mathcal{U}}_n(X,A) = \frac{S^{\mathcal{U}}_n(X)}{S^{\mathcal{U}}_n(A)} = \frac{S_n(\mathring{A}) \oplus S_n(X \setminus \bar{U})}{S_n(\mathring{A}) \oplus S_n(A \setminus \bar{U})} = \frac{S_n(X \setminus \bar{U})}{S_n(A \setminus \bar{U})},$$

$$S^{\mathcal{U}}_n(X \setminus U, A \setminus U) = \frac{S^{\mathcal{U}}_n(X \setminus U)}{S^{\mathcal{U}}_n(A \setminus U)} = \frac{S_n(\mathring{A}) \oplus S_n(X \setminus \bar{U})}{S_n(\mathring{A}) \oplus S_n(A \setminus \bar{U})} = \frac{S_n(X \setminus \bar{U})}{S_n(A \setminus \bar{U})}.$$

Theorem 4.3.8 then implies $H_n(X,A) \cong H^{\mathcal{U}}_n(X,A) \cong H^{\mathcal{U}}_n(X \setminus U, A \setminus U) \cong H_n(X \setminus U, A \setminus U)$. □

Theorem 4.3.8 also provides a very effective method to compute the (relative) homologies for a pair of topological spaces (X,A) - the *Mayer-Vietoris sequence* which can be viewed as a homological counterpart of the theorem of Seifert and van Kampen. The basic idea is to cover the topological space X by a pair of open subsets U_1, U_2 such that the homologies of U_1, U_2 and $U_1 \cap U_2$ are as simple as possible. The Mayer-Vietoris sequence then combines these homologies in a long exact sequence with the homologies of (X,A) and allows one to compute the latter.

Theorem 4.3.10: Let (X, A) be a pair of topological spaces, $U_1, U_2 \subset X$ open subsets with $U_1 \cup U_2 = X$ and $A_i := U_i \cap A$. Denote by $\iota_i : (U_1 \cap U_2, A_1 \cap A_2) \to (U_i, A_i)$ and $j_i : (U_i, A_i) \to (X, A)$ the associated inclusion morphisms in Top(2). Then there is a natural R-module homomorphism $\partial_n^{MV} : H_n(X, A) \to H_{n-1}(U_1 \cap U_2, A_1 \cap A_2)$ such that the diagram

$$\begin{tikzcd} H_n(U_1, A_1) \oplus H_n(U_2, A_2) \arrow[r, "H_n(j_1)+H_n(j_2)"] \arrow[dr, "H_n(j_1)+H_n(j_2)"'] & H_n^{\mathcal{U}}(X, A) \arrow[r, "\overline{\partial}_n"] \arrow[d, "H_n(j_\bullet)"] & H_{n-1}(U_1 \cap U_2, A_1 \cap A_2) \\ & H_n(X, A) \arrow[ur, "\partial_n^{MV}"'] & \end{tikzcd} \tag{34}$$

commutes and an exact and natural sequence, the **Mayer-Vietoris sequence**

$$\begin{tikzcd} \ldots \arrow[r, "\partial_{n+1}^{MV}"] & H_n(U_1 \cap U_2, A_1 \cap A_2) \arrow[r, "(H_n(\iota_1), -H_n(\iota_2))"] & H_n(U_1, A_1) \oplus H_n(U_2, A_2) \arrow[d, "H_n(j_1)+H_n(j_2)"] \\ \ldots & H_{n-1}(U_1 \cap U_2, A_1 \cap A_2) \arrow[l, "(H_{n-1}(\iota_1), -H_{n-1}(\iota_2))"'] & H_n(X, A) \arrow[l, "\partial_n^{MV}"'] \end{tikzcd}$$

Proof:
The sets U_1, U_2 form an open covering $\mathcal{U} = \{U_1, U_2\}$ of X and the sets A_1, A_2 an open covering $\mathcal{A} = \{A_1, A_2\}$ of A. This yields a sequence of chain complexes

$$0 \to S_\bullet(U_1 \cap U_2, A_1 \cap A_2) \xrightarrow{(S_\bullet(\iota_1), -S_\bullet(\iota_2))} S_\bullet(U_1, A_1) \oplus S_\bullet(U_2, A_2) \xrightarrow{S_\bullet(j_1)+S_\bullet(j_2)} S_\bullet^{\mathcal{U}}(X, A) \to 0.$$

As every $\mathcal{U}$-small relative n-chain in (X, A) is a linear combination of relative n-chains in (U_1, A_1), (U_2, A_2), the module morphisms $S_n(j_1) + S_n(j_2) : S_n(U_1, A_1) \oplus S_n(U_2, A_2) \to S_n^{\mathcal{U}}(X, A)$ are surjective. The morphisms $(S_n(\iota_1), -S_n(\iota_2)) : S_n(U_1 \cap U_2, A_1 \cap A_2) \to S_n(U_1, A_1) \oplus S_n(U_2, A_2)$ are injective and $\ker(S_n(j_1) + S_n(j_2)) = \mathrm{Im}\,((S_n(\iota_1), -S_n(\iota_2)))$. This shows that the sequence is exact. From Theorem 2.3.11 we obtain a long exact sequence of homologies

$$\begin{tikzcd} \ldots \arrow[r, "\overline{\partial}_{n+1}"] & H_n(U_1 \cap U_2, A_1 \cap A_2) \arrow[r, "(H_n(\iota_1), -H_n(\iota_2))"] & H_n(U_1, A_1) \oplus H_n(U_2, A_2) \arrow[d, "H_n(j_1)+H_n(j_2)"] \\ \ldots & H_{n-1}(U_1 \cap U_2, A_1 \cap A_2) \arrow[l, "(H_{n-1}(\iota_1), -H_{n-1}(\iota_2))"'] & H_n^{\mathcal{U}}(X, A) \arrow[l, "\overline{\partial}_n"'] \end{tikzcd}$$

Theorem 4.3.8 implies that $H_n(j_\bullet) : H_n^{\mathcal{U}}(X, A) \to H_n(X, A)$ is an isomorphism. The R-module morphism $\partial_n^{MV} = \overline{\partial}_n \circ H_n(j_\bullet)^{-1} : H_n^{\mathcal{U}}(X, A) \to H_{n-1}(U_1 \cap U_2, A_1 \cap A_2)$ makes the diagram (34) commute and has the same naturality properties as the connection homomorphism (see Lemma 2.3.12, Corollary 2.3.13 and Corollary 2.3.14). The claim follows by inserting diagram (34) into the long exact sequence of homologies. □

Corollary 4.3.11: Let X be a topological space $U_1, U_2 \subset X$ open and path-connected with $X = U_1 \cup U_2$ and $U_1 \cap U_2$ path-connected. Then $H_0(U_1) \cong H_0(U_2) \cong H_0(U_1 \cap U_2) \cong H_0(X) \cong R$, and the Mayer-Vietoris sequence for $(X, \emptyset)$ and $\mathcal{U} = \{U_1, U_2\}$ is the long exact sequence

$$\begin{tikzcd} \ldots \arrow[r, "\partial_2^{MV}"] & H_1(U_1 \cap U_2) \arrow[r, "(H_1(\iota_1), -H_1(\iota_2))"] & H_1(U_1) \oplus H_1(U_2) \arrow[r, "H_1(j_1)+H_1(j_2)"] & H_1(X) \arrow[d, "\partial_1^{MV}"] \\ 0 & \underbrace{H_0(X)}_{\cong R} \arrow[l] & \underbrace{H_0(U_1) \oplus H_0(U_2)}_{\cong R \oplus R} \arrow[l, "H_0(j_1)+H_0(j_2)"'] & \underbrace{H_0(U_1 \cap U_2)}_{\cong R} \arrow[l, "(H_0(\iota_1), -H_0(\iota_2))"'] \end{tikzcd}$$

As $(H_0(\iota_1), -H_0(\iota_2))$ is injective, this implies $\mathrm{Im}\,(\partial_1^{MV}) = \ker(H_0(\iota_1), -H_0(\iota_2)) = 0$ and hence $\partial_1^{MV} = 0$ and $H_1(j_1) + H_1(j_2)$ surjective. We obtain

$$H_1(X) = \frac{H_1(U_1) \oplus H_1(U_2)}{\ker(H_1(j_1) \oplus H_1(j_2))} = \frac{H_1(U_1) \oplus H_1(U_2)}{\mathrm{Im}\,(H_1(\iota_1), -H_1(\iota_2))}$$

which can be viewed as the abelian version of the Theorem of Seifert and van Kampen.

Example 4.3.12: We compute the singular homologies of the n-Spheres S^n with the Mayer-Vietoris sequence. As $S^0 = \{1, -1\}$ is the topological sum of two one-point spaces, we have

$$H_k(S^0) = H_k(\{1\}) \oplus H_k(\{-1\}) \cong \begin{cases} 0 & k > 0 \\ R \oplus R & k = 0 \end{cases}$$

To compute the homologies of the spheres S^n with $n \geq 1$, we consider for an $\epsilon \in (0,1)$ the open subspaces $U_\pm = \{x \in S^n \mid \mp x_{n+1} < \epsilon\}$. Then we have $S^n = U_+ \cup U_-$, $U_\pm \simeq D^n \simeq \{p\}$ and $U_+ \cap U_- = \{x \in S^n \mid -\epsilon < x_{n+1} < \epsilon\} \simeq S^{n-1}$. This implies $H_k(U_\pm) = 0$ for $k \geq 1$, $H_0(U_\pm) = R$ and $H_k(U_+ \cap U_-) = H_k(S^{n-1})$. As S^n is path-connected for $n \geq 1$, we have $H_0(S^n) = R$ for $n \geq 1$. From Corollary 4.3.11 we obtain the Mayer-Vietoris sequence

$$\begin{tikzcd}
\ldots \to H_k(S^{n-1}) \arrow[r] & \underbrace{H_k(D^n) \oplus H_k(D^n)}_{=0} \arrow[r] & H_k(S^n) \arrow[r, "\partial_k^{MV}"] & H_{k-1}(S^{n-1}) \arrow[d] \\
H_1(S^{n-1}) \leftarrow \ldots \arrow[d] & H_{k-2}(S^{n-1}) \arrow[l] & H_{k-1}(S^n) \arrow[l, "\partial_{k-1}^{MV}"'] & \underbrace{H_{k-1}(D^n) \oplus H_{k-1}(D^n)}_{=0} \arrow[l] \\
\underbrace{H_1(D^n) \oplus H_1(D^n)}_{=0} \arrow[r] & H_1(S^n) \arrow[r, "\partial_1^{MV}"] & H_0(S^{n-1}) \arrow[r, "\phi"] & \underbrace{H_0(D^n) \oplus H_0(D^n)}_{=R\oplus R} \arrow[d, "\psi:(r_1,r_2)\mapsto r_1+r_2"'] \\
 & & 0 & \underbrace{H_0(S^n)}_{=R} \arrow[l]
\end{tikzcd}$$

The exactness of the sequence implies that $\partial_k^{MV} : H_k(S^n) \to H_{k-1}(S^{n-1})$ is an isomorphism for all $n \geq 1$, $k \geq 2$. Similarly, we find that $\partial_1^{MV} : H_1(S^n) \to H_0(S^{n-1})$ is injective, and this implies $H_1(S^n) \cong \mathrm{Im}\,(\partial_1^{MV}) \cong \ker(\phi)$.

If $n > 1$, we have $H_0(S^{n-1}) = R$, and the map $\phi : H_0(S^{n-1}) \to H_0(D^n) \oplus H_0(D^n)$ is given by $\phi : R \to R \oplus R$, $r \mapsto (r, -r)$. As this map is injective, we obtain $H_1(S^n) = 0$. If $n = 1$, we have $H_0(S^{n-1}) \cong R \oplus R$ and $H_1(S^1) \cong \ker(\phi) \cong R \oplus R/\mathrm{Im}\,(\phi) = R \oplus R/\ker(\psi) \cong R$. Hence, we have for the homologies of te n-spheres S^n with $n \geq 1$

$$H_k(S^n) \cong H_{k-1}(S^{n-1}) \cong ... \cong \begin{cases} H_{k-n}(S^0) = 0 & k > n \\ H_1(S^1) \cong R & k = n \geq 1 \\ H_1(S^{n-k+1}) = 0 & 0 < k < n \\ R & 0 = k < n. \end{cases}$$

Example 4.3.13: Let (x,X), (y,Y) be pointed topological spaces with open neighbourhoods $U \subset X$ and $V \subset Y$ such that $\{x\}$ is a deformation retract of U and $\{y\}$ a deformation retract of V. Consider the wedge sum $X \vee Y$ obtained by identifying $x \in X$ and $y \in Y$. Then $H_n(X \vee Y) \cong H_n(X) \oplus H_n(Y)$ for all $n \in \mathbb{N}$.

Proof: Choose $U_1 = \iota_X(X) \cup \iota_Y(V)$, $U_2 = \iota_Y(Y) \cup \iota_X(U)$, where $\iota_X : X \to X \vee Y$ and $\iota_Y : Y \to X \vee Y$ are the canonical inclusions. Then U_1, U_2 are open with $X \vee Y = U_1 \cup U_2$, and $U_1 \cap U_2 = \iota_X(U) \cup \iota_Y(V)$ is homotopy equivalent to a point. This implies $H_n(U_1 \cap U_2) = 0$ for all $n \in \mathbb{N}$. As $\{x\}$ and $\{y\}$ are deformation retracts of $U \subset X$ and $V \subset Y$, respectively, we have $U_1 \simeq X$ and $U_2 \simeq Y$ and therefore $H_n(U_1) \cong H_n(X)$, $H_n(U_2) \cong H_n(Y)$ for all $n \in \mathbb{N}_0$. The Mayer-Vietoris sequence takes the form

$$\begin{tikzcd}
\ldots \arrow[r, "\partial^{MV}_{n+1}"] & \underbrace{H_n(U_1 \cap U_2)}_{=0} \arrow[r, "(H_n(\iota_1), -H_n(\iota_2))"] & \underbrace{H_n(U_1) \oplus H_n(U_2)}_{\cong H_n(X) \oplus H_n(Y)} \arrow[r, "H_n(j_1)+H_n(j_2)"] & H_n(X \vee Y) \arrow[d, "\partial^{MV}_n"] \\
\ldots & H_{n-1}(X \vee Y) \arrow[l, "\partial^{MV}_{n-1}"'] & \underbrace{H_{n-1}(U_1) \oplus H_{n-1}(U_2)}_{\cong H_{n-1}(X) \oplus H_{n-1}(Y)} \arrow[l, "H_{n-1}(j_1)+H_{n-1}(j_2)"'] & \underbrace{H_{n-1}(U_1 \cap U_2)}_{=0} \arrow[l, "(H_{n-1}(\iota_1), -H_{n-1}(\iota_2))"']
\end{tikzcd}$$

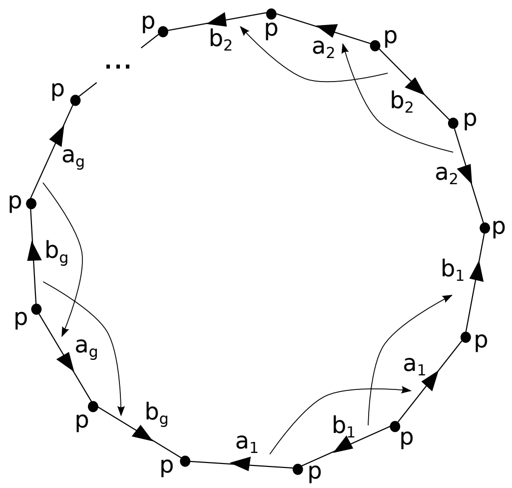
For $n \in \mathbb{N}$, this implies that $H_n(j_1) + H_n(j_2) : H_n(U_1) \oplus H_n(U_2) \cong H_n(X) \oplus H_n(Y) \to H_n(X \vee Y)$ is an isomorphism.

4.4 Exercises for Chapter 4

Exercise 1: Let R be a commutative unital ring.

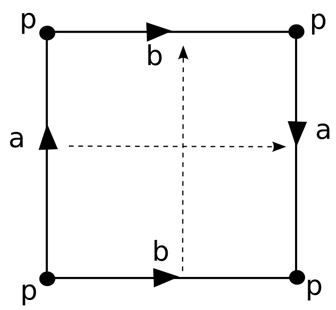
(a) Compute the simplicial homologies $H_n^\Delta(S^2)$ over R by making use of the fact that S^2 is homeomorphic to the surface of a tetrahedron.

(b) Compute the simplicial homologies $H_n^\Delta(T^{\#g})$ over R for an oriented surface of genus $g \geq 2$ by realising it as a quotient of a $4g$-gon and choosing a suitable triangulation.



Exercise 2: Compute the simplicial homologies $H_n^\Delta(S_{g,n})_R$ for an oriented surface $S_{g,n} := T^{\#g} \setminus \{B_1, ..., B_n\}$ of genus g with n points removed.

Exercise 3: The **Klein bottle** K is the quotient of the unit square with respect to the equivalence relation that identifies its boundary as shown below.



Compute the simplicial homologies $H_n^\Delta(K)$ over R by choosing a suitable triangulation. Consider the cases:

(a) R is a commutative unital ring of characteristic $\mathrm{char}(R) = 2$

(b) $R = \mathbb{Z}$

(c) $R = \mathbb{F}$ is a field of characteristic $\mathrm{char}(\mathbb{F}) = 0$.

Exercise 4: Let R be a unital commutative ring and X a path-connected graph with V vertices and E edges. Show that X has a Δ-complex structure and that its first homology group over R is

$$H_1^\Delta(X) \cong R^{E-V+1} = \underbrace{R \oplus R \ldots \oplus R}_{(E-V+1)\times}.$$

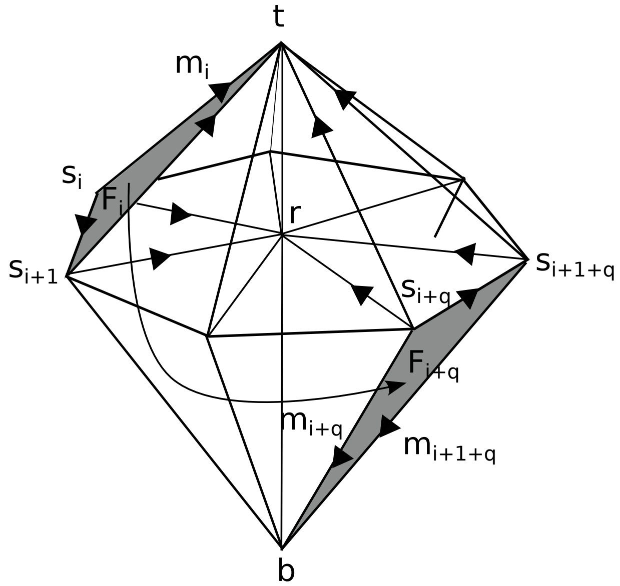
Hint: Prove that X is a tree if and only if $E - V + 1 = 0$. Then construct the graph X by adding edges to a maximal tree in X.

Exercise 5: Consider real projective space $\mathbb{RP}^2$. Show that $\mathbb{RP}^2$ has the structure of a Δ-complex and compute its simplicial homologies $H_n^\Delta(\mathbb{RP}^2)$ over:

(a) a commutative unital ring R of characteristic $\mathrm{char}(R) = 2$.
(b) $R = \mathbb{Z}$.
(c) a field $R = \mathbb{F}$ of characteristic $\mathrm{char}(\mathbb{F}) = 0$.

Hint: Realise $\mathbb{RP}^2$as a quotient of the disc $D^2/\sim$.

Exercise 6: **(lens spaces)** Let $p, q \in \mathbb{N}$ with $p > q \geq 1$ relatively prime. Consider the polyhedron obtained as the convex hull $P_p = [t, b, s_1, ..., s_p] \subset \mathbb{R}^3$ of a convex p-gon and two points t and b above and below its center r, as shown below.



The **lens space** $L(p, q)$ is the quotient of P_p with respect to the equivalence relation that identifies for $i \in \{1, ..., m\}$ the face F_i of the upper pyramid with the $(i + q) \bmod(p)$th face $F_{(i+q) \bmod(p)}$ of the lower pyramid, as indicated below. Choose a Δ-complex structure and compute the simplicial homologies $H_n^\Delta(L(p, q))$ over $\mathbb{Z}$.

Exercise 7: Let X be a topological space with n path components and R a commutative unital ring. Show that the 0th singular homology over R is given by $H_0(X) \cong \underbrace{R \oplus R \ldots \oplus R}_{n\times}$.

Exercise 8: Consider the canonical Δ-complex structure on the standard n-Simplex $\Delta^n = [e_0, ..., e_n]$. Determine the associated chain complex $S_\bullet^\Delta(\Delta^n)$ explicitly and determine its homologies $H_k^\Delta(\Delta^n)$ for $0 \leq k \leq n$.

Exercise 9: Let $X = \amalg_{i \in I} X_i$ be a topological space with path-components X_i and $A \subset X$ a subspace. Prove the following statements

(a) $H_0(X, A) = 0$ if and only if $A \cap X_i \neq \emptyset$ for all $i \in I$.

(b) $H_1(X,A)=0$ if and only if the map $H_1(\iota): H_1(A)\to H_1(X)$ is surjective and every path component of X contains at most one path-component of A.

Exercise 10: Equip the cylinder $X = S^1 \times [0,1]$ with the structure of a Δ-complex such that the circle $A = S^1 \times \{0\}$ is realised as a subcomplex and compute the relative homologies $H_n^\Delta(X,A)$.

Exercise 11: Compute the relative homologies $H_n^\Delta(X,A)$ of the Δ-complex $X = \Delta^3$ and the subcomplex $A = \Delta^2 \subset \Delta^3$.

Exercise 12: Let X be a topological space $A, B \subset X$ subspaces such that B is a deformation retract of A. Prove that $H_n(X,A) \cong H_n(X,B)$ for all $n \in \mathbb{N}_0$.

Exercise 13: We denote by $[v_0, ..., v_k]$ the affine k-simplex $\sigma : \Delta^k \to \mathbb{R}^n$ with $\sigma(e_j) = v_j$ for $j \in \{0, ..., k\}$. For $(s+1)$ points $v_0, ..., v_s \in \mathbb{R}^n$ we denote by $b(v_0, ..., v_s) = \frac{1}{s+1}\sum_{j=0}^s v_j$ the barycentre. Show that for all $k \in \mathbb{N}$ the barycentric subdivision operator is given by

$$U_k([v_0, ..., v_k]) = \sum_{\pi \in S_{k+1}} \mathrm{sgn}(\pi)[v_0, ..., v_k]^\pi,$$

where $[v_0, ..., v_k]^\pi = [v_0^\pi, ..., v_k^\pi]$ and $v_r^\pi = b(v_{\pi(r)}, ..., v_{\pi(k)})$.

Exercise 14: Let R be a commutative unital ring. Use the Mayer-Vietoris sequence to prove the identity $H_k(S^n) \cong H_{k-1}(S^{n-1})$ for all $n, k \in \mathbb{N}$.

Hint: Consider for $0 < \epsilon < 1$ the sets $S^n_\pm = \{x \in S^n : \mp x_{n+1} < \epsilon\}$.

Exercise 15: Compute for $R = \mathbb{Z}$ the homologies of the Klein bottle by covering it with two overlapping Möbius strips and using the Mayer-Vietoris sequence.

Exercise 16: Let X be a topological space and $A \subset X$ a retract of X. Show that the short exact sequence $0 \to S_\bullet(A) \xrightarrow{S_\bullet(\iota)} S_\bullet(X) \xrightarrow{S_\bullet(\pi)} S_\bullet(X,A) \to 0$ splits and therefore one has $H_n(X) \cong H_n(A) \oplus H_n(X,A)$ for all $n \in \mathbb{N}_0$.

Exercise 17: Compute the homologies of the n-punctured sphere $S^2 \setminus \{p_1, ..., p_n\}$ and of the n-punctured plane $\mathbb{R}^2 \setminus \{q_1, .., q_n\}$.

5 Homology theories and their applications

5.1 The axiomatic formulation

Historically, there were many different homology theories such as simplicial homologies, homologies of Δ-complexes and singular homologies. In 1945 Samuel Eilenberg and Norman Steenrod gave an abstract formulation in terms of axioms that unified and related these approaches. By using the language of categories, functors and natural transformations, they developed a formulation that was independent of details such as the choice of simplexes under consideration and showed that the homologies of a topological space can be largely computed from these axioms.

Definition 5.1.1: Let R be a commutative unital ring. A **homology theory** with coefficients in R consists of a family $(H_n)_{n\in\mathbb{Z}}$ of functors $H_n : \mathrm{Top}(2) \to R\text{-Mod}$ together with a family $(\partial_n)_{n\in\mathbb{Z}}$ of natural transformations $\partial_n : H_n^{\mathrm{rel}} \to H_{n-1}^{\mathrm{sub}}$, $H_n(X,A) \to H_{n-1}(A)$, that satisfies the **Eilenberg-Steenrod axioms**:

(H1) **homotopy axiom**: if $f, g : (X,A) \to (Y,B)$ are homotopic, then $H_n(f) = H_n(g) : H_n(X,A) \to H_n(Y,B)$ for all $n \in \mathbb{Z}$.

(H2) **exact sequence axiom**: for a pair of topological spaces (X,A) with inclusion morphisms $\iota : (A,\emptyset) \to (X,A)$ and $\pi : (X,\emptyset) \to (X,A)$, the following sequence is exact

$$\ldots \xrightarrow{\partial_{n+1}} H_n(A) \xrightarrow{H_n(\iota)} H_n(X) \xrightarrow{H_n(\pi)} H_n(X,A) \xrightarrow{\partial_n} H_{n-1}(A) \xrightarrow{H_{n-1}(\iota)} H_{n-1}(X) \xrightarrow{H_{n-1}(\pi)} \ldots$$

(H3) **excision axiom**: for any pair (X,A) and any subset $U \subset A$ with $\overline{U} \subset \mathring{A}$ the inclusion morphism $j : (X \setminus U, A \setminus U) \to (X,A)$ induces isomorphisms

$$H_n(j) : H_n(X \setminus U, A \setminus U) \xrightarrow{\sim} H_n(X,A).$$

(H4) **additivity axiom**: If $X = \amalg_{i\in I} X_i$ is a decomposition of X into its path-components X_i, then $H_n(X) \cong \bigoplus_{i\in I} H_n(X)$.

(H5) **dimension axiom**: for the one point space $H_n(\{p\}) = 0$ for all $n \neq 0$, $H_0(\{p\}) \cong R$.

The functors $H_n : \mathrm{Top}(2) \to R\text{-Mod}$ are called **homologies** and the natural transformations $\partial_n : H_n(X,A) \to H_{n-1}(A)$ **boundary operators**.

Note that the homotopy axiom implies that homotopy equivalent topological spaces have the same homologies. The dimension axiom is sometimes omitted. In that case, the zeroth homologies of the one point space play the role of coefficients in the homology theory. To ensure that the homology theory contains non-trivial information about the topological spaces, one must require that the homologies of the one-point space should not all be zero.

Clearly, singular homology is a homology theory,. The homotopy axiom is contained in Corollary 4.1.16 and Lemma 4.2.4, the dimension axiom in Example 4.1.7. The additivity axiom corresponds to Example 4.1.8, the exact sequence axiom to Theorem 4.2.5 and the excision axiom to Theorem 4.3.9. However, we will now show in an example that the homologies of a topological space can be computed directly from the axioms, without making use of the concrete definition of singular homology. For this, we consider the homologies of the n-spheres S^n, the

n-discs D^n and the relative homologies of the pairs (S^n, D^n) and (D^n, S^{n-1}). These are not only important in their own right but will later result in a simple procedure for the computation of homologies of CW-complexes.

Theorem 5.1.2: Let R be a commutative unital ring. Then for any homology theory and any $n \in \mathbb{N}_0$, the homologies of the n-sphere S^n and the relative homologies of the n-disc D^n are

$$H_m(S^n) \cong \begin{cases} R & m = 0, n \neq 0 \,\text{or}\, m = n \neq 0 \\ 0 & m \notin \{0, n\} \\ R \oplus R & m = n = 0 \end{cases} \qquad H_m(D^n, S^{n-1}) = \begin{cases} R & m = n \\ 0 & m \neq n. \end{cases} \tag{35}$$

Proof:
We consider for $n \in \mathbb{N}_0$ the pairs (S^n, S^n_+) where $S^n_+ = \{x \in S^n \,|\, x_{n+1} > 0\}$ and the subspace $U = \{x \in S^n \,|\, x_{n+1} > 1/2\} \subset S^n_+$. For $n \geq 1$, the spaces S^n_+ and $S^n \setminus U$ are homotopy equivalent to D^n and hence to a one-point space $\{p\}$, the space $S^n_+ \setminus U$ is homotopy equivalent to S^{n-1}.

For all $n \in \mathbb{N}$ we have $\overline{U} \subset \mathring{S}^n_+$, and by the excision axiom (H3) $j : (S^n \setminus U, S^n_+ \setminus U) \to (S^n, S^n_+)$ induces an isomorphism $H_k(j) : H_k(S^n \setminus U, S^n_+ \setminus U) \to H_k(S^n, S^n_+)$ for all $k \in \mathbb{N}_0$.

We consider the exact sequences for the pairs $(S^n, S^n_+) \simeq (S^n, D^n)$ with the inclusion and projection morphisms $\iota : (D^n, \emptyset) \to (S^n, \emptyset)$ and $\pi : (S^n, \emptyset) \to (S^n, D^n)$ and for the pairs $(S^n \setminus U, S^n_+ \setminus U) \simeq (D^n, S^{n-1})$ with inclusion and projection morphisms $\iota' : (S^{n-1}, \emptyset) \to (D^n, \emptyset)$ and $\pi' : (D^n, \emptyset) \to (D^n, S^{n-1})$. This yields the following commuting diagram with exact rows

$$\begin{tikzcd}
\ldots \arrow[r] & H_{k+1}(S^n) \arrow[r, "H_{k+1}(\pi)"] & H_{k+1}(S^n, D^n) \arrow[r, "\partial_{k+1}"] & \overbrace{H_k(D^n)}^{=0} \arrow[r, "H_k(\iota)"] & H_k(S^n) \arrow[r, "H_k(\pi)"] & H_k(S^n, D^n) \arrow[r, "\partial_k"] & \overbrace{H_{k-1}(D^n)}^{=0} \arrow[r] & \ldots \\
\ldots \arrow[r] & \underbrace{H_{k+1}(D^n)}_{=0} \arrow[u, "H_{k+1}(\iota)"] \arrow[r, "H_{k+1}(\pi')"] & H_{k+1}(D^n, S^{n-1}) \arrow[u, "H_{k+1}(j)", "\cong"'] \arrow[r, "\partial'_{k+1}"] & H_k(S^{n-1}) \arrow[u, "H_k(\iota')"] \arrow[r, "H_k(\iota')"] & \underbrace{H_k(D^n)}_{=0} \arrow[u, "H_k(\iota)"] \arrow[r, "H_k(\pi')"] & H_k(D^n, S^{n-1}) \arrow[u, "H_k(j)", "\cong"'] \arrow[r, "\partial'_k"] & H_{k-1}(S^{n-1}) \arrow[u, "H_{k-1}(\iota')"] \arrow[r] & \ldots
\end{tikzcd}$$

where $k \geq 2$ and we used the fact that D^n is homotopy equivalent to the one point space, which implies $H_k(D^n) = 0$ for all $k \geq 1$ and $H_0(D^n) = R$ by the homotopy and the dimension axiom. The exactness of the first row then implies that (a) $H_k(\pi) : H_k(S^n) \to H_k(S^n, D^n)$ is an isomorphism for all $k \geq 2$. The exactness of the second row implies that (b) $\partial'_k : H_k(D^n, S^{n-1}) \to H_{k-1}(S^{n-1})$ is an isomorphism for all $k \geq 2$. Hence, we obtain inductively for $k \geq 2$

$$H_k(S^n) \cong H_k(S^n, D^n) \cong H_k(D^n, S^{n-1}) \cong H_{k-1}(S^{n-1}) = \begin{cases} H_1(S^{n-k+1}) & n \geq k \\ H_{k-n}(S^0) = 0 & n < k. \end{cases} \tag{36}$$

It remains to compute the homologies $H_k(S^n)$, $H_k(D^n, S^{n-1})$, $H_k(S^n, D^n)$ for $k \leq 1$. For this, we consider the last terms on the right of the diagram, which are given by the following comuting diagram with exact rows

$$\begin{tikzcd}
\ldots \arrow[r] & \overbrace{H_1(D^n)}^{=0} \arrow[r, "H_1(\iota)"] & H_1(S^n) \arrow[r, "H_1(\pi)"] & H_1(S^n, D^n) \arrow[r, "\partial_1"] & \overbrace{H_0(D^n)}^{=R} \arrow[r, "H_0(\iota)"] & H_0(S^n) \arrow[r, "H_0(\pi)"] & H_0(S^n, D^n) \arrow[r] & 0 \\
\ldots \arrow[r] & H_1(S^{n-1}) \arrow[u, "H_1(\iota')"] \arrow[r, "H_1(\iota')"] & \underbrace{H_1(D^n)}_{=0} \arrow[u, "H_1(\iota)"] \arrow[r, "H_1(\pi')"] & H_1(D^n, S^{n-1}) \arrow[u, "H_1(j)", "\cong"'] \arrow[r, "\partial'_1"] & H_0(S^{n-1}) \arrow[u, "H_0(\iota')"] \arrow[r, "H_0(\iota')"] & \underbrace{H_0(D^n)}_{=R} \arrow[u, "H_0(\iota)"] \arrow[r, "H_0(\pi')"] & H_0(D^n, S^{n-1}) \arrow[u, "H_0(j)", "\cong"'] \arrow[r] & 0
\end{tikzcd}$$

By exactness of the first row, $H_1(\pi)$ is injective and $H_0(\pi)$ surjective. By exactness of the second row, ∂_1' is injective and $H_0(\pi')$ surjective. By the exactness of the second row we have $\ker(H_0(\iota')) = \mathrm{Im}\,(\partial_1')$, and because of the commutativity, this implies $\partial_1 \circ H_0(j) = H_0(\iota')\circ\partial_1' = 0$. As $H_0(j)$ is an isomorphism, it follows that $\partial_1 = 0$. Hence $H_0(\iota)$ is injective, $H_1(\pi)$ is surjective and hence an isomorphism. This shows that (c) $H_1(S^n) \cong H_1(S^n, D^n) \cong H_1(D^n, S^{n-1})$.

The exactness of the first row also implies $\ker(H_0(\pi)) = \mathrm{Im}\,(H_0(\iota))$, and with the commutativity of the diagram $0 = H_0(\pi) \circ H_0(\iota) = H_0(j) \circ H_0(\pi')$. As $H_0(j)$ is an isomorphism, it follows that $H_0(\pi') = 0$. As $H_0(\pi')$ is surjective, this implies (d) $H_0(D^n, S^{n-1}) \cong H_0(S^n, D^n) = 0$ for all $n \in \mathbb{N}$. From this, it follows that that $H_0(\iota')$ and $H_0(\iota)$ are surjective, and hence $H_0(\iota)$ is an isomorphism, which implies (e) $H_0(S^n) \cong R$ for all $n \in \mathbb{N}$.

For $n = 0$ the additivity axiom and the dimension axiom imply $H_1(S^0) = 0$, $H_0(S^0) = R \oplus R$ and $H_0(\iota') : R \oplus R \to R$, $(r_1, r_2) \mapsto r_1 + r_2$. With the injectivity of ∂_1', this implies (f)

$$H_1(S^1) \cong H_1(S^1, D^1) \cong H_1(D^1, S^0) \cong \mathrm{Im}\,(\partial_1') = \ker(H_0(\iota')) = \{(r, -r) \,|\, r \in R\} \cong R.$$

For $n \geq 2$, we have $H_0(S^{n-1}) \cong R$ by (e). As $H_0(\iota') : H_0(S^{n-1}) \cong R \to R$ is surjective, it follows that it is an isomorphism and hence (g) $\ker(H_0(\iota')) = \mathrm{Im}\,(\partial_1') = H_1(S^n) \cong H_1(S^n, D^n) \cong H_1(D^n, S^{n-1}) = 0$ for $n \geq 2$. Combining the identities (d)-(g) with (36) then proves the claim. □

As in the case of the fundamental group $\pi_1(S^1)$ in Section 3.3, this result on the homologies of the spheres S^n has many geometrical applications. Some of them are higher-dimensional analogues of the results in Section 3.3 and their proofs are quite similar to the ones in in Section 3.3. The only difference is that they make use of the homology groups $H_k(S^n)$ instead of the fundamental group $\pi_1(S^1)$.

Corollary 5.1.3: For $n, m \in \mathbb{N}$, $n \neq m$, $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic.

Proof:
A homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^m$ induces a homeomorphism $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^m \setminus \{f(0)\}$. As $\mathbb{R}^n \setminus \{0\} \simeq S^{n-1}$ and $\mathbb{R}^m \setminus \{f(0)\} \simeq S^{m-1}$ this yields an isomorphism $H_k(S^{n-1}) \xrightarrow{\sim} H_k(S^{m-1})$ for all $k \in \mathbb{N}$, but for $n \neq m$ this is a contradiction to Theorem 5.1.2. □

Corollary 5.1.4: (Brouwer's fix point theorem)
For $n \geq 1$, every continuous map $f : D^n \to D^n$ has a fix point.

Proof:
For $n = 1$, the claim follows from the midpoint theorem, since the continuous map $g : D^1 = [1, -1] \to \mathbb{R}$, $x \mapsto f(x) - x$ satisfies $g(-1) \geq 0$ and $g(1) \leq 0$, which implies that there is an $x \in [-1, 1]$ with $g(x) = 0$, i. e. a fix point of f. Suppose $n \geq 2$ and $f : D^n \to D^n$ is continuous without a fix point. Then for all $x \in D^n$ there is a unique straight line g_x through x and $f(x)$. By assigning to $x \in D^n$ the intersection point of g_x with $\partial D^n = S^{n-1}$ that is closer to x than to $f(x)$, we obtain a continuous map $g : D^n \to S^{n-1}$ with $g|_{S^{n-1}} = \mathrm{id}_{S^{n-1}}$. This yields a commutative diagram with a contradiction

$$\begin{tikzcd}
 & H_{n-1}(D^n) = 0 \arrow[dr, "H_{n-1}(g)"] & \\
H_{n-1}(S^{n-1}) \cong R \arrow[ur, "H_{n-1}(\iota)"] \arrow[rr, "H_{n-1}(\mathrm{id})=\mathrm{id}"] & & H_{n-1}(S^{n-1}) \cong R.
\end{tikzcd}$$

$\square$

The fact that the homologies of many topological spaces can be computed directly from the Eilenberg-Steenrod axioms without making use of concrete simplexes $\Delta^n \to X$ suggests that different homology theories should give rise to isomorphic homologies, at least for well-behaved topological spaces. This was one of the central motivations for introducing these axioms, and we can use them to show that simplicial and singular homology agree.

Theorem 5.1.5: Let $X_\Delta = (X, \{\sigma_i\}_{i\in I})$ be a Δ-complex. Then for all $n \in \mathbb{N}_0$ the inclusion maps $i_n^\Delta : S_n^\Delta(X) \to S_n(X)$ induce isomorphisms $H_n(i_\bullet^\Delta) : H_n^\Delta(X) \xrightarrow{\sim} H_n(X)$.

Proof:
1. We prove that $H_n^\Delta(X^k, X^{k-1}) \cong H_n(X^k, X^{k-1})$ for all $k, n \in \mathbb{N}_0$:

(a) Denote by $I_k \subset I$ the index set for the k-simplexes in X_Δ. Then the k-skeleta of X are given by $X^k = \cup_{j=0}^k \left(\cup_{i\in I_j} \sigma_i(\Delta^j)\right)$ have an induced Δ-complex structure $(X^k, \{\sigma_i\}_{i\in I_0\cup\ldots\cup I_k})$. For $n > k$, the set $\{\sigma_i\}_{i\in I_0\cup\ldots\cup I_k}$ does not contain n-simplexes and $S_n^\Delta(X^k) = S_n^\Delta(X^k, X^{k-1}) = 0$. For $n < k$ all n-simplexes in $\{\sigma_i\}_{i\in I_0\cup\ldots\cup I_k}$ are also contained in $\{\sigma_i\}_{i\in I_0\cup\ldots\cup I_{k-1}}$, which implies $S_n^\Delta(X^k) = S_n^\Delta(X^{k-1})$ and $S^\Delta(X^k, X^{k-1}) = 0$. This yields $H_n^\Delta(X_k, X_{k-1}) = 0$ for $n \neq k$. For $n = k$, we have

$$H_k^\Delta(X^k, X^{k-1}) \cong \ker(\partial_k^\Delta)/\mathrm{Im}\,(\partial_{k-1}^\Delta) \cong S_k^\Delta(X_k, X_{k-1})/0 \cong S_k^\Delta(X^k)/S_k^\Delta(X^{k-1}) = \oplus_{i\in I_k} R$$

since $\partial_k(\sigma_i) \in S_{k-1}(X^{k-1})$ for all $i \in I_k$. This implies

$$H_n^\Delta(X^k, X^{k-1}) \cong \begin{cases} \oplus_{i\in I_k} R & n = k \\ 0 & n \neq k. \end{cases}$$

(b) To compute $H_n(X^k, X^{k-1})$, we consider the *thickened* $(k-1)$*-skeleton* of U^k, that is obtained by removing all barycentres of k-simplexes

$$U^k = X^k \setminus \bigcup_{i\in I_k} \sigma_i(b_k).$$

The $(k-1)$-skeleton X^{k-1} is a deformation retract of U^k. A retraction $r_k : U^k \to X^{k-1}$ and the associated homotopy $h_k : [0,1] \times U^k \to U^k$ from $i_{X^{k-1}} \circ r_k$ to id_{U^k} are obtained as follows. Let $r : \Delta^k \setminus \{b_k\} \to \partial\Delta^k$ be a retraction and $h : [0,1] \times \Delta^k \setminus \{b_k\} \to \Delta^k \setminus \{b_k\}$ a homotopy from $i_{\partial\Delta^k} \circ r$ to $\mathrm{id}_{\Delta^k\setminus\{b_k\}}$ relative to $\partial\Delta^k$. Define

$$r_k(x) = \begin{cases} x & x \in X^{k-1} \\ \sigma_i \circ r \circ (\sigma_i|_{\overset{\circ}{\Delta}{}^k})^{-1}(x) & x \in \sigma_i(\overset{\circ}{\Delta}{}^k \setminus \{b_k\}) \end{cases}$$

$$h_k(t,x) = \begin{cases} x & x \in X^{k-1} \\ \sigma_i \circ h(t, \circ(\sigma_i|_{\overset{\circ}{\Delta}{}^k})^{-1}(x)) & x \in \sigma_i(\overset{\circ}{\Delta}{}^k \setminus \{b_k\}). \end{cases}$$

Then $r_k : U^k \to X^{k-1}$ is a retraction and $h_k : [0,1] \times U^k \to U^k$ a homotopy from $i_{X^{k-1}} \circ r_k$ to id_{U_k} relative to X^{k-1}. This implies $H_n(X^k, X^{k-1}) \cong H_n(X^k, U^k)$ (see Exercise 12, Section 4.4).

(c) By the excision axiom the inclusion $(X^k \setminus X^{k-1}, U^k \setminus X^{k-1}) \to (X^k, U^k)$ induces an isomorphism $H_n(X^k \setminus X^{k-1}, U^k \setminus X^{k-1}) \xrightarrow{\sim} H_n(X^k, U^k)$. As excising the $(k-1)$-skeleton yields a topological sum $(X^k \setminus X^{k-1}, U^k \setminus X^{k-1}) = \amalg_{i \in I_k}(\sigma_i(\mathring{\Delta}^k), \sigma_i(\mathring{\Delta}^k \setminus b_k))$, the additivity axiom implies $H_n(X^k \setminus X^{k-1}, U^k \setminus X^{k-1}) \cong \oplus_{i \in I_k} H_n(\sigma_i(\mathring{\Delta}^k), \sigma_i(\mathring{\Delta}^k \setminus b_k))$. As $\sigma_i|_{\mathring{\Delta}^k} : \mathring{\Delta}^k \to \sigma_i(\mathring{\Delta}^k)$ is a homeomorphism for all $i \in I_k$, we have

$$(\sigma_i(\mathring{\Delta}^k), \sigma_i(\mathring{\Delta}^k \setminus b_k)) \simeq (\mathring{\Delta}^k, \mathring{\Delta}^k \setminus \{b_k\}) \simeq (\mathring{D}_k, \mathring{D}_k \setminus \{0\}) \simeq (D^k, S^{k-1}) \quad \forall i \in I_k,$$

and by combining these results, we obtain

$$\begin{aligned} H_n(X^k, X^{k-1}) &\overset{\text{(b)}}{\cong} H_n(X^k, U^k) \overset{\text{(c)}}{\cong} H_n(X^k \setminus X^{k-1}, U^k \setminus X^{k-1}) \overset{\text{(c)}}{\cong} \oplus_{i \in I_k} H_n(\sigma_i(\mathring{\Delta}^k), \sigma_i(\mathring{\Delta}^k \setminus b_k)) \\ &\cong \oplus_{i \in I_k} H_n(D^k, S^{k-1}) \cong \begin{cases} \oplus_{i \in I_k} R & n = k \\ 0 & n \neq k. \end{cases} \end{aligned}$$

2. We prove that $H_n(X^k) \cong H_n^\Delta(X^k)$ by induction over $n, k \in \mathbb{N}$. Clearly, $H_0^\Delta(X^k) \cong H_0(X^k)$ for all $k \in \mathbb{N}$. Suppose we have $H_m(X^k) \cong H_m^\Delta(X^k)$ for all $k \in \mathbb{N}$ and $m \leq n-1$. As X^0 is discrete, we have $H_n(X^0) = H_n^\Delta(X^0)$. Suppose now that $H_n(X^l) \cong H_n^\Delta(X^l)$ for all $0 \leq l \leq k-1$. Then we obtain the following commuting diagram with exact rows

$$\begin{tikzcd} H^\Delta_{n+1}(X^k, X^{k-1}) \arrow[r, "\partial_{n+1}"] \arrow[d, "\cong"] & H^\Delta_n(X^{k-1}) \arrow[r, "H_n(\iota)"] \arrow[d, "\cong"] & H^\Delta_n(X^k) \arrow[r, "H_n(j)"] \arrow[d, "H_n(i^\Delta_\bullet)"] & H^\Delta_n(X^k, X^{k-1}) \arrow[r, "\partial_n"] \arrow[d, "\cong"] & H^\Delta_{n-1}(X^{k-1}) \arrow[d, "\cong"] \\ H_{n+1}(X^k, X^{k-1}) \arrow[r, "\partial_{n+1}"] & H_n(X^{k-1}) \arrow[r, "H_n(\iota)"] & H_n(X^k) \arrow[r, "H_n(j)"] & H_n(X^k, X^{k-1}) \arrow[r, "\partial_n"] & H_{n-1}(X^{k-1}) \end{tikzcd}$$

The R-module morphisms on the first and fourth vertical arrow are isomorphisms by 1., the ones on the second and fifth vertical arrow are isomorphisms by induction hypothesis. The 5-lemma then implies that $H_n(i^\Delta_\bullet) : H_n^\Delta(X) \to H_n(X)$ is an isomorphism. □

5.2 The mapping degree

Another important application of the (relative) homologies of spheres is the notion of *mapping degree*, which generalises the mapping degree of a function $f : S^1 \to S^1$ from Section 3.3 to higher dimensions. For this, one considers the homologies S^n and of the pairs (D^n, S^{n-1}) with coefficients in $R = \mathbb{Z}$. By Theorem 5.1.2, one has $H_n(S^n) \cong H_n(D^n, S^{n-1}) \cong \mathbb{Z}$, and any morphism $f : (D^n, S^{n-1}) \to (D^n, S^{n-1})$ or $f : (S^n, \emptyset) \to (S^n, \emptyset)$ in Top(2) induces a group homomorphism $H_n(f) : \mathbb{Z} \to \mathbb{Z}$. As every group homomorphism $\phi : \mathbb{Z} \to \mathbb{Z}$ is of the form $\phi : z \mapsto mz$ with $m = \phi(1) \in \mathbb{Z}$, we obtain a generalisation of the mapping degree to the n-spheres.

Definition 5.2.1: Consider the ring $R = \mathbb{Z}$. Then a morphism $f : (D^n, S^{n-1}) \to (D^n, S^{n-1})$ or $f : (S^n, \emptyset) \to (S^n, \emptyset)$ in Top(2) induces a group homomorphism $H_n(f) : \mathbb{Z} \to \mathbb{Z}$, $z \mapsto mz$ for an $m \in \mathbb{Z}$. The number $m = H_n(f)(1) = \deg(f) \in \mathbb{Z}$ is called the **mapping degree** of f.

Remark 5.2.2:

1. As the homologies define a functor $H_n : \mathrm{Top}(2) \to \mathbb{Z}$, one has $\deg(\mathrm{id}_{S^n}) = \deg(\mathrm{id}_{D^n}) = 1$ and $\deg(g \circ f) = \deg(g) \cdot \deg(f)$.
2. The degree $\deg(f)$ depends only on the homotopy class[4] of f. In particular, if f is a homotopy equivalence, then $\deg(f) \in \{\pm 1\}$.
3. The naturality of the connecting homomorphism for the pair (D^n, S^{n-1}) implies that for all morphisms $f : (D^n, S^{n-1}) \to (D^n, S^{n-1})$ one has $\deg(f) = \deg(f|_{S^{n-1}})$ since the following diagram commutes

$$\begin{tikzcd} H_n(D^n, S^{n-1}) \arrow[r, "\partial_n"] \arrow[d, "H_n(f)"] & H_{n-1}(S^{n-1}) \arrow[d, "H_{n-1}(f|_{S^{n-1}})"] \\ H_n(D^n, S^{n-1}) \arrow[r, "\partial_n"] & H_{n-1}(S^{n-1}). \end{tikzcd}$$

4. If $f : S^n \to S^n$ is constant, then $\deg(f) = 0$ since $f = f_2 \circ f_1$ with $f_1 : S^n \to \{p\}$, $f_2 : \{p\} \to S^n$ and $H_k(\{p\}) = 0$ for all $k \in \mathbb{N}$. This implies $H_k(f) = H_k(f_2) \circ H_k(f_1) = 0$ for all $k \in \mathbb{N}$.
5. It follows from the Huréwicz isomorphism (Theorem 4.1.17) that for $n = 1$ the mapping degree from Definition 5.2.1 agrees with the one from Definition 3.3.2.

Note that the first, second and fourth claim in Remark 5.2.2 are direct generalisations of Lemma 3.3.4 for the mapping degree of a map $f : S^1 \to S^1$ and reduce to them for $n = 1$. However the mapping degree of maps $f, g : S^1 \to S^1$ also satisfies the identity $\deg(f \cdot g) = \deg(f) + \deg(g)$ (see Lemma 3.3.4. 2.), and it is natural to ask if this also has a higher-dimensional counterpart.

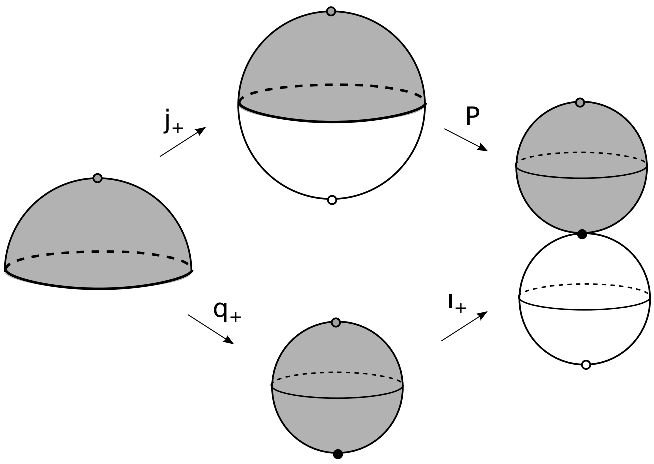
While the pointwise product $f \cdot g$ of continuous maps $f, g : S^1 \to S^1$ has no analogue in higher dimensions, we can nevertheless generalise this identity to continuous maps $f : S^n \to S^n$. For this, recall that the pointwise product $f \cdot g$ of continuous maps $f, g : S^1 \to S^1$ and the concatenation of the associated paths closed paths in S^1 induce the same group homomorphism $\pi_1(S^1) \to \pi_1(S^1)$. The latter has an analogue for S^n which is obtained from a decomposition of S^n into half-spheres $S^n_\pm := \{x \in S^n : \pm x_{n+1} \geq 0\}$.

As $S^n_\pm / S^{n-1} \approx S^n$ and $S^n/S^{n-1} \approx S^n \vee S^n$, the associated canonical surjections induce continuous maps $q_\pm : S^n_\pm \to S^n$, a continuous map $P : S^n \to S^n \vee S^n$, the **pinch map** such that the following diagram commutes

$$\begin{tikzcd} S^n_\pm \arrow[r, "j_\pm"] \arrow[d, "q_\pm"'] & S^n \arrow[d, "P"] \\ S^n \arrow[r, "\iota_\pm"] & S^n \vee S^n, \end{tikzcd} \tag{37}$$

where $j_\pm : S^n_\pm \to S^n$ are the canonical inclusions and $\iota_\pm : S^n \to S^n \vee S^n$ the inclusion into the first and second component in $S^n \vee S^n$.

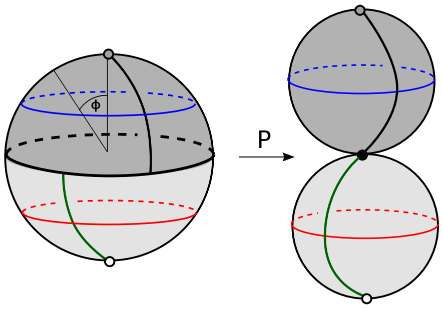
[4]**Hopf's theorem** states that the converse of this statement is also true. If $f, f' : S^n \to S^n$ are two continuous maps of the same degree, then f and f' are homotopic.



The pinch map $P : S^n \to S^n \vee S^n$

If we identify $(0, -1) = -e_{n+1}$ on the first copy of S^n with $(0, -1) = -e_{n+1}$ on the second copy, then the pinch map $P : S^n \to S^n \vee S^n$ is given by

$$P(x \sin \phi, \cos \phi) = \begin{cases} \iota_+(x \sin(2\phi), \cos(2\phi)) & \phi \in [0, \frac{\pi}{2}) \\ \iota_-(x \sin(2\phi), \cos(2\phi)) & \phi \in [\frac{\pi}{2}, \pi] \end{cases} \qquad \forall x \in S^{n-1}.$$



The pinch map $P : S^n \to S^n \vee S^n$ **in coordinates**

For a pair of morphisms $f_\pm : (S^n, \{-e_{n+1}\}) \to (X, \{x\})$ in Top(2), the universal property of the wedge sum yields a continuous map $f_+ \vee f_- : S^n \vee S^n \to X$ with $(f_+ \vee f_-) \circ \iota_\pm = f_\pm : S^n \to X$. By composing it with the pinch map, we obtain a continuous map $f_+ + f_- := (f_+ \vee f_-) \circ P : S^n \to X$

$$(f_+ + f_-)(x \sin \phi, \cos \phi) = \begin{cases} f_+(x \sin(2\phi), \cos(2\phi)) & \phi \in [0, \frac{\pi}{2}) \\ f_-(x \sin(2\phi), \cos(2\phi)) & \phi \in [\frac{\pi}{2}, \pi]. \end{cases}$$

The map $f_+ + f_- : S^n \to S^n$ can be viewed as a higher-dimensional analogue of the composition of paths in S^1. Moreover, it has the same properties with respect to homology, as it adds the mapping degrees of f_+ and f_-.

Lemma 5.2.3: For morphisms $f_+, f_- : (S^n, \{-e_{n+1}\}) \to (X, x)$ in Top(2) and all $n, k \geq 1$, the continuous map $f_+ + f_- : S^n \to X$ satisfies $H_k(f_+ + f_-) = H_k(f_+) + H_k(f_-) : H_k(S^n) \to H_k(X)$.

Proof:
By Example 4.3.13, the canonical inclusions $\iota_\pm : S^n \to S^n \vee S^n$ that map S^n to the first and

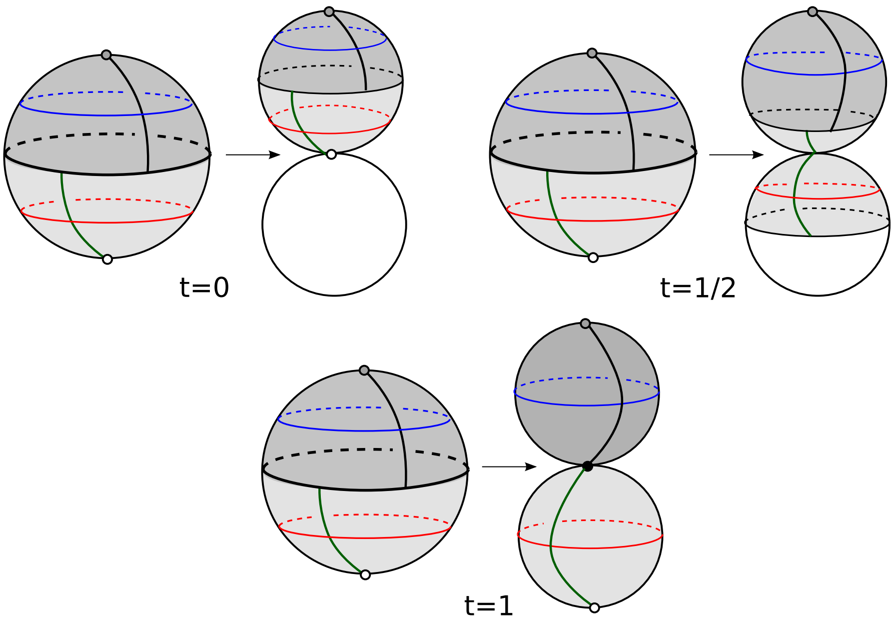
second component induce an isomorphism $H_k(\iota_+)+H_k(\iota_-) : H_k(S^n)\oplus H_k(S^n) \xrightarrow{\sim} H_k(S^n \vee S^n)$. By combining it with diagram (37), we obtain the diagram

$$\begin{tikzcd}
H_k(S^n) \arrow[d, "H_k(P)"'] \arrow[dr, "H_k(f_+{+}f_-)"] \arrow[dd, bend right=70, "(H_k(\mathrm{id}){,}H_k(\mathrm{id}))"'] & \\
H_k(S^n \vee S^n) \arrow[r, "H_k(f_+\vee f_-)"] & H_k(X) \\
H_k(S^n)\oplus H_k(S^n) \arrow[u, "H_k(\iota_+)+H_k(\iota_-)", "\cong"'] \arrow[ur, "H_k(f_+)+H_k(f_-)"'] &
\end{tikzcd}$$

in which the two triangles on the right commute. It is therefore sufficient to show that the triangle on the left commutes. This follows from the fact that the pinch map $P : S^n \to S^n \vee S^n$ is homotopic to the canonical inclusions $\iota_\pm : S^n \to S^n \vee S^n$. In the parametrisation above, a homotopy from $\iota_+ : S^n \to S^n \vee S^n$ to $P : S^n \to S^n \vee S^n$ is given by

$$h : [0,1]\times S^n \to S^n \vee S^n, \quad h(t, x\sin\phi, \cos\phi) = \begin{cases} \iota_+(x\sin(\phi+t\phi), \cos(\phi+t\phi)) & \phi\in[0,\frac{\pi}{1+t}) \\ \iota_-(x\sin(\phi+t\phi), \cos(\phi+t\phi)) & \phi\in[\frac{\pi}{1+t},\pi]. \end{cases}$$

□



homotopy from the the inclusion map $\iota_+ : S^n \to S^n \vee S^n$ to the pinch map $P : S^n \to S^n \vee S^n$.

We can now use the mapping degree of continuous maps $f : S^n \to S^n$ to derive the higher-dimensional counterparts of the geometrical statements in Section 3.3, which made use of the reflection and the antipodal map of the circle. For this, we consider the reflections $s : S^n \to S^n$, $x \mapsto x - 2\langle x, v\rangle v$ on a hyperplane through the origin with normal vector $v \in S^n$ and the antipodal map $a : S^n \to S^n$, $x \mapsto -x$. The first step is to compute their degrees.

Lemma 5.2.4: For $n \geq 1$, the reflection $s : S^n \to S^n$ on a hyperplane through the origin in $\mathbb{R}^{n+1}$ has $\deg(s) = -1$, and the antipodal map $a : S^n \to S^n$ has $\deg(a) = (-1)^{n+1}$.

Proof:
The second claim follows from the first, since the antipodal map is obtained by composing $(n+1)$ reflections. To prove the first, we can restrict attention to the reflection $s : S^n \to S^n$ on the hyperplane orthogonal to e_1, since every other hyperplane can be obtained from this by applying a rotation in $\mathrm{SO}(n+1)$, which is a homeomorphism. In the parametrisation above, the map $\mathrm{id}_{S^n} + s : S^n \to S^n$ is given by

$$(\mathrm{id}_{S^n}+s)(x_1\sin\phi, x_\perp \sin(\phi), \cos\phi) = \begin{cases} (x_1\sin(2\phi), x_\perp\sin(2\phi), \cos(2\phi)) & \phi\in[0,\frac{\pi}{2}) \\ (-x_1\sin(2\phi), x_\perp\sin(2\phi)\cos(2\phi)) & \phi\in[\frac{\pi}{2},\pi], \end{cases}$$

where $(x_1, x_\perp) \in S^{n-1}$ and $\phi\in[0,\pi]$. It is homotopic to the constant map $f : S^n \to S^n$, $x \mapsto (0,0,1)$ via the homotopy $h : [0,1]\times S^n \to S^n$

$$h(t, x_1\sin\phi, x_\perp\sin\phi, \cos\phi) = \begin{cases} (x_1\sin(2\phi), x_\perp\sin(2\phi), \cos(2\phi)) & \phi\in[0,\frac{\pi(1-t)}{2}) \\ (\alpha(t,\phi,x_1)x_1\sin(\pi t), \beta(t,\phi)x_\perp\sin(\pi t), -\cos(\pi t)) & \phi\in[\frac{\pi(1-t)}{2},\frac{\pi(1+t)}{2}) \\ (-x_1\sin(2\phi), x_\perp\sin(2\phi), \cos(2\phi)) & \phi\in[\frac{\pi(1+t)}{2},\pi] \end{cases}$$

with $\quad \alpha(t,\phi,x_1) = \dfrac{\sqrt{t^2-(\pi-2\phi)^2(1-x_1^2)}}{t|x_1|} \qquad \beta(t,\phi) = \dfrac{\pi-2\phi}{t}.$

This implies $0 = \deg(f) = \deg(s+\mathrm{id}_{S^n}) = \deg(s) + \deg(\mathrm{id}_{S^n}) = 1 + \deg(s)$. □

By making use of this lemma, we can prove a well-known statement that is known under the name of **combing a hedgehog**. It states that combing a hedgehog always leads to the appearance of a bald spot. The hedgehog is modelled by a sphere S^2 and its quills correspond to a (continuous) vector field on S^2. The bald spot represents a zero of the vector field, and the claim is that every continuous vector field on S^2 has a zero. More generally, one considers **vector fields** on $S^n \subset \mathbb{R}^{n+1}$ for $n \geq 1$, i. e. continuous maps $v : S^n \to \mathbb{R}^{n+1}$ with $\langle v(x), x\rangle = 0$ for all $x \in S^n$, where $\langle\ ,\ \rangle$ denotes the euclidean scalar product on $\mathbb{R}^{n+1}$.

Corollary 5.2.5: For $n \geq 1$, S^n admits a vector field without zeros if and only if n is odd.

Proof:
For n odd, a vector field without zeros on S^n is given by

$$v : S^n \to \mathbb{R}^{n+1}, \quad v(x_1, ..., x_{n+1}) = (-x_2, x_1, -x_4, x_3, ..., -x_{n+1}, x_n).$$

Conversely, if $v : S^n \to \mathbb{R}^{n+1}$ is a vector field with $v(x) \neq 0$ for all $x \in S^n$, then

$$h : [0,1]\times S^n \to S^n, \quad h(x,t) = x\cos(\pi t) + \frac{v(x)}{||v(x)||}\sin(\pi t).$$

is a homotopy from id_{S^n} to the antipodal map, which implies $H_n(\mathrm{id}) = H_n(a) = (-1)^{n+1} = -1$ and hence n odd. □

By a similar argument, namely the construction of homotopies from a given map $f : S^n \to S^n$ to the identity map or the antipodal map via convex combinations, we can draw conclusions about the degree of maps without fix points or antipodal points.

Corollary 5.2.6:

1. If $f : S^n \to S^n$ is a continuous map without a fix point, then $\deg(f) = (-1)^{n+1}$.
2. If $f : S^n \to S^n$ is a continuous map without an antipodal point, then $\deg(f) = 1$.
3. If n is even, every continuous map $f : S^n \to S^n$ has a fix point or an antipodal point.

Proof:
If $f : S^n \to S^n$ has no fix point (antipodal point), then $h : [0,1] \times S^n \to S^n$

$$h(t,x) = \frac{(1-t)f(x)\overset{-}{_{(+)}}tx}{||(1-t)f(x)\overset{-}{_{(+)}}tx||}$$

is a homotopy from f to the antipodal map $a : S^n \to S^n$ (the identity map $\mathrm{id}_{S^n} : S^n \to S^n$), and $\deg(f) = \deg(a) = (-1)^{n+1}$ ($\deg(f) = \deg(\mathrm{id}_{S^n}) = 1$). If f has neither fix nor antipodal points, then $\deg(f) = \deg(a) = (-1)^{n+1} = \deg(\mathrm{id}_{S^n}) = 1$ and hence n odd. □

5.3 Homologies of CW-complexes

In this section, we develop a procedure to compute the homologies of CW-complexes. This is important because it is often much simpler than a computation of homologies via Δ-complexes or the homology axioms. As every topological manifold is homotopy equivalent to a CW-complex and every topological space is weakly homotopy equivalent to a CW-complex, this procedure is quite general and can be applied to many examples. On the other hand, it is conceptually important because it shows that homology theories do not need to be based on simplexes and can be just as well be formulated in terms of spheres and discs.

The central observation is that for a CW-complex $(X, \cup_{n\geq 0} X^n)$ the characteristic maps of the n-cells relate the relative homologies $H_k(X^n, X^{n-1})$ to the relative homologies $H_k(D^n, S^{n-1})$. If $f_j : S^{n-1} \to X^{n-1}$ for $j \in J_n$ are the attaching maps for the n-cells, then X^n is obtained from X^{n-1} by attaching the topological sum $\amalg_{j\in J_n} D^n$ to X^{n-1} with $\amalg_{j\in J_n} f_j : \amalg_{j\in J_n} D^n \to X^{n-1}$. The characteristic maps $\iota : \amalg_{j\in J_n} D^n \to X^n$ and $\iota_j = \iota \circ i_j : D^n \to X^n$ then induce an isomorphism $H_n(\iota) : \oplus_{j\in J_n} H_i(D^n, S^{n-1}) \to H_i(X^n, X^{n-1})$. As the homologies $H_i(D^n, S^{n-1})$ were determined in Theorem 5.1.2 this reduces the computation of the relative homologies $H_k(X^n, X^{n-1})$ to the question how many n-cells are attached to X^{n-1} to obtain X^n.

Lemma 5.3.1: Let R be a unital ring and $(X, \cup_{n\geq 0} X^n)$ a CW-complex. Then

$$H_i(X^n, X^{n-1}) \cong \oplus_{j\in J_n} H_i(D^n, S^{n-1}) = \begin{cases} \oplus_{j\in J_n} R & i = n \\ 0 & i \neq n. \end{cases} \qquad \forall i \in \mathbb{N}_0, n \in \mathbb{N}.$$

Proof:
The idea is to consider the punctured n-skeleta $\dot{X}^n = X^n \setminus \{\iota_j(0) : j \in J_n\}$ and to prove that $H_i(X^n, \dot{X}^n) \cong H_i(X^n, X^{n-1})$ by constructing a homotopy equivalence. From the excision axiom, we obtain $H_i(X^n, X^{n-1}) \cong H_i(X^n \setminus X^{n-1}, \dot{X}^n \setminus X^{n-1})$. We then use the fact that $X^n \setminus X^{n-1} \simeq \amalg_{j\in J_n} \iota_j(\mathring{D}^n)$ and $\dot{X}^n \setminus X^{n-1} \simeq \amalg_{j\in J_n} \iota_j(\mathring{D}^n \setminus \{0\})$ to show that the latter are given by $H_i(X^n \setminus X^{n-1}, \dot{X}^n \setminus X^{n-1}) \cong \oplus_{j\in J_n} H_i(D^n, S^{n-1})$.

1. Recall that S^{n-1} is a deformation retract of the punctured disc $\dot{D}^n = D^n \setminus \{0\}$. The inclusion $\dot{D}^n \to S^{n-1}$ induces a homotopy equivalence $\iota_j(S^{n-1}) \to \iota_j(\dot{D}^n)$ for all $j \in J_n$ and a homotopy

equivalence $X^{n-1} \to \dot{X}^n$. This shows that $H_i(\dot{X}^n) \cong H_i(X^{n-1})$ for all $i \in \mathbb{N}_0$. By applying the 5-Lemma to the long exact homology sequences for the pairs (X^n, X^{n-1}) and $(X^n, \dot{X}^n)$

$$\begin{tikzcd}
H_i(X^{n-1}) \arrow[r, "H_i(\iota'_n)"] \arrow[d, "\cong"'] & H_i(X^n) \arrow[r, "H_i(\pi'_n)"] \arrow[d, "\mathrm{id}"'] & H_i(X^n, X^{n-1}) \arrow[r, "\partial_i"] \arrow[d, "H_i(j')"'] & H_{i-1}(X^{n-1}) \arrow[r, "H_{i-1}(\iota'_n)"] \arrow[d, "\cong"'] & H_{i-1}(X^n) \arrow[d, "\mathrm{id}"'] \\
H_i(\dot{X}^n) \arrow[r, "H_i(\tilde{\iota}'_n)"'] & H_i(X^n) \arrow[r, "H_i(\tilde{\pi}'_n)"'] & H_i(X^n, \dot{X}^n) \arrow[r, "\tilde{\partial}_i"'] & H_{i-1}(\dot{X}^n) \arrow[r, "H_{i-1}(\tilde{\iota}'_n)"'] & H_{i-1}(X^n)
\end{tikzcd}$$

we obtain $H_i(X^n, X^{n-1}) \cong H_i(X^n, \dot{X}^n)$.

2. The excision axiom implies $H_i(X^n, \dot{X}^n) \cong H_i(X^n \setminus X^{n-1}, \dot{X}^n \setminus X^{n-1})$ for all $i \in \mathbb{N}_0$. As $X^n \setminus X^{n-1} = \amalg_{j\in J_n} \iota_j(\mathring{D}^n)$, $\dot{X}^n \setminus X^{n-1} = \amalg_{j\in J_n} \iota_j(\mathring{D}^n \setminus \{0\})$ and $\iota_j|_{\mathring{D}^n} : \mathring{D}^n \to X^n$ are embeddings, we obtain a homotopy equivalence $(X^n \setminus X^{n-1}, \dot{X}^n \setminus X^{n-1}) \to (\amalg_{j\in J_n} D^n, \amalg_{j\in J_n} \dot{D}^n)$ and hence $H_i(X^n, X^{n-1}) \cong H_i(\amalg_{j\in J_n} D^n, \amalg_{j\in J_n} \dot{D}^n)$.

3. We abbreviate $Y^n := \amalg_{j\in J_n} D^n$, $\dot{Y}^n := \amalg_{j\in J_n} \dot{D}^n$ and $Y^{n-1} := \amalg_{j\in J_n} S^{n-1}$. To show that $H_i(Y^n, \dot{Y}^n) \cong H_i(Y^n, Y^{n-1})$, note that by the universal property of topological sums the inclusion $S^{n-1} \to \dot{D}^n$ induces a homotopy equivalence $Y^{n-1} \to \dot{Y}^n$. This implies $H_i(Y^{n-1}) \cong H_i(\dot{Y}^n)$. To pass to the relative homologies, we consider the long exact homology sequences for the pairs (Y^n, Y^{n-1}) and $(Y^n, \dot{Y}^n)$. This yields a commutative diagram with exact rows

$$\begin{tikzcd}
H_i(Y^{n-1}) \arrow[r, "H_i(\iota_n)"] \arrow[d, "\cong"'] & H_i(Y^n) \arrow[r, "H_i(\pi_n)"] \arrow[d, "\mathrm{id}"'] & H_i(Y^n, Y^{n-1}) \arrow[r, "\partial_i"] \arrow[d, "H_i(j_n)"'] & H_{i-1}(Y^{n-1}) \arrow[r, "H_{i-1}(\iota_n)"] \arrow[d, "\cong"'] & H_{i-1}(Y^n) \arrow[d, "\mathrm{id}"'] \\
H_i(\dot{Y}^n) \arrow[r, "H_i(\tilde{\iota}_n)"'] & H_i(Y^n) \arrow[r, "H_i(\tilde{\pi}_n)"'] & H_i(Y^n, \dot{Y}^n) \arrow[r, "\tilde{\partial}_i"'] & H_{i-1}(\dot{Y}^n) \arrow[r, "H_{i-1}(\tilde{\iota}_n)"'] & H_{i-1}(Y^n)
\end{tikzcd}$$

where $\iota_n : (Y^{n-1}, \emptyset) \to (Y^n, \emptyset)$, $\pi_n : (Y^n, \emptyset) \to (Y^n, Y^{n-1})$, $\tilde{\iota}_n : (\dot{Y}^n, \emptyset) \to (Y^n, \emptyset)$, $\tilde{\pi}_n : (Y^n, \emptyset) \to (Y^n, \dot{Y}^n)$ the canonical morphisms for the pairs (Y^n, Y^{n-1}), $(Y, \dot{Y}^n)$ and $j_n : (Y^n, Y^{n-1}) \to (Y^n, \dot{Y}^n)$ the inclusion morphisms in Top(2). The 5-Lemma then implies that $H_i(j_n) : H_i(Y^n, Y^{n-1}) \to H_i(Y^n, \dot{Y}^n)$ is an isomorphism, and with the additivity axiom we obtain $H_i(X^n, X^{n-1}) \cong H_i(Y^n, Y^{n-1}) \cong H_i(Y^n, \dot{Y}^n) \cong \oplus_{j\in J_n} H_i(D^n, S^{n-1})$. With Theorem 5.1.2 it follows that $H_n(X^n, X^{n-1}) \cong \oplus_{j\in J_n} R$ and $H_i(X^n, X^{n-1}) = 0$ for $i \neq n$. □

To compute the homologies of CW-complexes, we will now organise the relative homologies $H_n(X^n, X^{n-1})$ into a chain complex, the *cellular complex*. This is motivated by the fact that the relative homologies $H_k(X^n, X^{n-1})$ are trivial for $k \neq n$ and hence all information is contained in the relative homologies $H_n(X^n, X^{n-1})$. To define a boundary operator between these homologies, consider the connection homomorphisms $\partial_n^{(n)} : H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1})$ and the module morphisms on the homologies induced by the morphisms $\pi_n : (X^n, \emptyset) \to (X^n, X^{n-1})$, $\iota_n : (X^{n-1}, \emptyset) \to (X^n, \emptyset)$ in Top(2). By composing them appropriately, we obtain natural module morphisms $H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$ which organise the homologies $H_n(X^n, X^{n-1})$ into a chain complex.

Lemma 5.3.2: Let R be a unital ring and $(X, \cup_{n\geq 0} X^n)$ a CW-complex. Then the R-modules and R-module morphisms

$$C_n(X) = H_n(X^n, X^{n-1}) \qquad d_n = H_{n-1}(\pi_{n-1}) \circ \partial_n^{(n)} : C_n(X) \to C_{n-1}(X),$$

form a chain complex, the **cellular complex** $C_\bullet(X)$.

Proof:
By combining the long exact sequences of homologies for all pairs (X^n, X^{n-1}), we obtain the commutative diagram with exact rows

$$\begin{tikzcd}
\ldots \arrow[r, "\partial^{(n+2)}_{n+2}"] \arrow[dr, dashed, "d_{n+2}"] & H_{n+1}(X^{n+1}) \arrow[r, "H_{n+1}(\iota_{n+2})"] \arrow[d, "H_{n+1}(\pi_{n+1})"] & \ldots & & \\
\ldots \arrow[r, "H_{n+1}(\pi_{n+1})"] & H_{n+1}(X^{n+1}, X^n) \arrow[r, "\partial^{(n+1)}_{n+1}"] \arrow[d, "\partial^{(n+1)}_{n+1}"] \arrow[dr, dashed, "d_{n+1}"] & H_n(X^n) \arrow[r, "H_n(\iota_{n+1})"] \arrow[d, "H_n(\pi_n)"] & H_n(X^{n+1}) \arrow[r, "H_n(\pi_{n+1})"] & \ldots \\
\ldots \arrow[r, "H_n(\iota_n)"] & H_n(X^n) \arrow[r, "H_n(\pi_n)"] & H_n(X^n, X^{n-1}) \arrow[r, "\partial^{(n)}_n"] \arrow[d, "\partial^{(n)}_n"] \arrow[dr, dashed, "d_n"] & H_{n-1}(X^{n-1}) \arrow[r, "H_{n-1}(\iota_n)"] \arrow[d, "H_{n-1}(\pi_{n-1})"] & \ldots \\
\ldots \arrow[r, "\partial^{(n-1)}_n"] & H_{n-1}(X^{n-2}) \arrow[r, "H_{n-1}(\iota_{n-1})"] & H_{n-1}(X^{n-1}) \arrow[r, "H_{n-1}(\pi_{n-1})"] & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r, "\partial^{(n-1)}_{n-1}"] \arrow[d, "\partial^{(n-1)}_{n-1}"] \arrow[dr, dashed, "d_{n-1}"] & \ldots \\
 & & \ldots \arrow[r, "H_{n-2}(\iota_{n-2})"'] & H_{n-2}(X^{n-2}) \arrow[r, "H_{n-2}(\pi_{n-2})"'] & \ldots
\end{tikzcd}$$

As all rows in the diagram are exact, we have $\partial^{(n-1)}_{n-1} \circ H_{n-1}(\pi_{n-1}) = 0$ and

$$d_{n-1} \circ d_n = H_{n-2}(\pi_{n-1}) \circ \partial^{(n-1)}_{n-1} \circ H_{n-1}(\pi_{n-1}) \circ \partial^{(n)}_n = 0 \qquad \forall n \in \mathbb{N}.$$

□

The aim is now to relate the homologies of the cellular complex $C_\bullet(X)$ to the homologies of the underlying topological space X. This requires a computation of the relative homologies $H_n(X, X^k)$ and the homologies $H_n(X^k)$ for the k-skeleta of the CW-complex. As both, n-simplexes and n-cells are n-dimensional structures, it is natural to expect that the information about n-simplexes in X is mostly contained in the n-skeleton X^n. This leads one to expect that the homologies $H_n(X^k)$ should be trivial for $k < n$, since X^k does not contain any n-cells. Similarly, the relative homologies $H_n(X, X^k)$ should vanish for $k \geq n$, since $S_n(X, X^k) = S_n(X)/S_n(X^k)$ and all relevant information about n-simplexes is already contained in X^k. By the same line of reasoning, one expects $H_n(X) = H_n(X^k)$ and $H_n(X, X^q) = H_n(X^k, X^q)$ for $k > n$, since the k-cells with $k > n$ should not contain any additional information about n-simplexes in X. This is made precise in the following lemma.

Lemma 5.3.3: Let R be a unital ring and $(X, \cup_{n\geq 0} X^n)$ a CW-complex. Then:

1. $H_n(X^p, X^q) = 0$ for all $n, p, q \in N_0$ with $p \geq q \geq n$ or $q \leq p < n$,
2. $H_n(X^p) = 0$ for all $n > p$,
3. $H_n(X, X^q) = 0$ for all $q \geq n$,
4. For $r > n$ the inclusion $i : X^r \to X$ induces an isomorphism $H_n(i) : H_n(X^r) \xrightarrow{\sim} H_n(X)$,
5. For $r > n$ and $r \geq q$ the morphism $\iota : (X^r, X^q) \to (X, X^q)$ in Top(2) induces an isomorphism $H_n(\iota) : H_n(X^r, X^q) \xrightarrow{\sim} H_n(X, X^q)$.

Proof:
1. Induction over $p-q$: For $p-q=0$ it is obvious. Now assume it holds for $p-q\leq k$. Then for $p-q=k+1$, the morphisms $\iota:(X^{q+1},X^q)\to(X^p,X^q)$, $\pi:(X^p,X^q)\to(X^p,X^{q+1})$ in Top(2) induce a short exact sequence of chain complexes

$$0\to S_\bullet(X^{q+1},X^q)\xrightarrow{S^{(2)}_\bullet(\iota)} S_\bullet(X^p,X^q)\xrightarrow{S^{(2)}_\bullet(\pi)} S_\bullet(X^p,X^{q+1})\to 0$$

and an associated long exact sequence of homologies

$$\ldots\to 0\overset{5.3.1}{=} H_n(X^{q+1},X^q)\xrightarrow{H_n(\iota)} H_n(X^p,X^q)\xrightarrow{H_n(\pi)} H_n(X^p,X^{q+1})\overset{\text{p-q=k}}{=} 0\to\ldots.$$

From the latter, we conclude that $H_n(X^p,X^{q+1})=\ker(H_n(\pi))=\mathrm{Im}\,(H_n(\iota))=0$.

2. This follows from the exact sequence $0=H_n(X^0)\xrightarrow{H_n(\iota)} H_n(X^p)\xrightarrow{H_n(\pi)} H_n(X^p,X^0)=0$.

3. As the characteristic maps $\iota_j:D^n\to X$, $j\in J_n$, of the n-cells are embeddings when restricted to $\mathring{D}^n$ and every point in X is contained in an image $\iota_j(\mathring{D}^n)$ the sets $\iota_j\mathring{D}^n$ form an open covering of X and hence of the image $\sigma(\Delta^n)$ for every n-simplex $\sigma:\Delta^n\to X$. As $\Sigma(\Delta^n)$ is compact, there is a finite sub-covering and hence $\sigma(\Delta^n)$ intersects only the interiors of finitely many cells in X. It follows that there is a $k=k(\sigma)\in\mathbb{N}_0$ with $\sigma(\Delta^n)\subset X^k$. As elements of $S_n(X)$ are finite linear combinations of n-simplexes, it follows that every element of $S_n(X,X^q)$ is contained in $S_n(X^p,X^q)$ for some $p\geq q\geq n$. It follows that every element of $H_n(X,X^q)$ is contained in $H_n(X^p,X^q)$ for or some $p\geq q\geq n$, and since by 1. $H_n(X^p,X^q)=0$ for all $p\geq q\geq n$, we have $H_n(X,X^q)=0$.

4. This follows from 3. and the long exact sequence for the pair (X,X^r)

$$\ldots\to 0\overset{3.}{=} H_{n+1}(X,X^r)\xrightarrow{\partial_{n+1}} H_n(X^r)\xrightarrow{H_n(\iota)} H_n(X)\xrightarrow{H_n(\pi)} H_n(X,X^r)\overset{3.}{=} 0\to\ldots$$

5. The morphisms $\iota:(X^r,X^q)\to(X,X^q)$, $\pi:(X,X^q)\to(X,X^r)$ in Top(2) induce a short exact sequence of chain complexes

$$0\to S_\bullet(X^r,X^q)\xrightarrow{S^{(2)}_\bullet(\iota)} S_\bullet(X,X^q)\xrightarrow{S^{(2)}_\bullet(\pi)} S_n(X,X^r)\to 0,$$

and an associated long exact sequence of homologies

$$\ldots\to 0\overset{3.}{=} H_{n+1}(X,X^r)\xrightarrow{\partial_{n+1}} H_n(X^r,X^q)\xrightarrow{H_n(\iota)} H_n(X,X^q)\xrightarrow{H_n(\pi)} H_n(X,X^r)\overset{3.}{=} 0\to\ldots,$$

which implies that $H_n(\iota):H_n(X^r,X^q)\to H_n(X,X^q)$ is an isomorphism. □

With this lemma, it is simple to show that the homologies of the cellular complex agree with the homologies of the underlying topological space X. This implies in particular that the homologies of the cellular complex cannot depend on the choice of the CW-structure on X.

Theorem 5.3.4: Let R be a unital ring and $(X,\cup_{n\geq 0}X^n)$ a CW-complex. Then the homologies of X and the homologies of $C_\bullet(X)$ agree: $H_n(C_\bullet(X))\cong H_n(X)$ for all $\forall n\in\mathbb{N}_0$.

Proof:
Consider the following commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & H_{n+1}(X^{n+1},X^n) \arrow[d, "\partial^{(n+1)}_{n+1}"'] \arrow[dr, "d_{n+1}"] & & H_{n-1}(X^{n-2}) \overset{5.3.3,2.}{=} 0 \arrow[d, "H_{n-1}(\iota_{n-1})"] \\
0 \overset{5.3.3,2.}{=} H_n(X^{n-1}) \arrow[r, "H_n(\iota_n)"] & H_n(X^n) \arrow[r, "H_n(\pi_n)"] \arrow[d, "H_n(\iota_{n+1})"'] & H_n(X^n, X^{n-1}) \arrow[r, "\partial^{(n)}_n"] \arrow[dr, "d_n"] & H_{n-1}(X^{n-1}) \arrow[d, "H_{n-1}(\pi_{n-1})"] \\
 & H_n(X^{n+1}) \arrow[d, "H_n(\pi_{n+1})"'] & & H_{n-1}(X^{n-1}, X^{n-2}) \\
 & H_n(X^{n+1}, X^n) \overset{5.3.3,1.}{=} 0. & &
\end{tikzcd}$$

Then $H_n(\pi_{n-1})$ and $H_n(\pi_n)$ are injective and $H_n(\iota_{n+1})$ is surjective. Together with the exactness of the first column this implies $H_n(X^{n+1}) \cong H_n(X^n)/\ker(\iota_{n+1}) \cong H_n(X^n)/\text{Im }(\partial^{(n+1)}_{n+1})$, and the exactness of the row yields $\ker(\partial^{(n)}_n) \cong \text{Im }(H_n(\pi_n))$. By combining these results, we obtain

$$H_n(C_\bullet(X)) = \frac{\ker(d_n)}{\text{Im }(d_{n+1})} \cong \frac{\ker(\partial^{(n)}_n)}{\text{Im }(d_{n+1})} \cong \frac{\text{Im }(H_n(\pi_n))}{\text{Im }(d_{n+1})} \cong \frac{H_n(X^n)}{\text{Im }(\partial^{(n+1)}_{n+1})} \cong H_n(X^{n+1}) \overset{5.3.3,4.}{\cong} H_n(X).$$

□

Corollary 5.3.5:

1. If $(X, \cup_{n\geq 0} X^n)$ is a CW-complex without n-cells, then $H_n(X) = 0$.
2. In particular, if $(X, \cup_{n\geq 0} X^n)$ is a CW-complex of cellular dimension k, then $H_n(X) = 0$ for all $k > n$.

This theorem gives a simple way to compute the homologies of a topological space X that has the structure of a CW-complex. The first step is to determine the associated cellular complex $C_\bullet(X)$. From Theorem 5.3.1 we know that each R-module $C_n(X) = H_n(X^n, X^{n-1})$ is of the form $C_n(X) = R^{\oplus|J_n|}$, where $|J_n|$ is the number of n-cells attached to X^{n-1} to form X^n. In many cases, in particular for CW-complexes that have k-cells only in few dimensions, this information is sufficient to determine the homologies. In other cases, one has to explicitly compute the boundary operators $d_n : C_n(X) \to C_{n-1}(X)$, but this task can often be reduced to computations involving spheres and discs. This is illustrated by the following examples.

Example 5.3.6: Consider for $n \geq 1$ the n-sphere with the CW-complex structure from Example 1.3.11, 3.

$$X^0 = X^1 = \ldots = X^{n-1} = \{1\}, \quad X^n = S^n,$$

where X^n is obtained from X^{n-1} by attaching an n-cell with the attaching map $f : S^{n-1} \to \{1\}$. By definition of the cellular complex and Lemma 5.3.1 we have $C_k(S^n) = H_k(X^k, X^{k-1}) = \{0\}$ for $k \neq n, 0$, $C_0(S^n) = H_0(X^0) \cong R$ and $C_n(S^n) = H_n(S^n, \{1\}) \cong H_n(D^n, S^{n-1}) \cong R$. The cellular complex and the homologies are given by

$$0 \xrightarrow{d_{n+1}} H_n(S^n, \{1\}) \cong R \xrightarrow{d_n} 0 \to 0 \ldots \to 0 \xrightarrow{d_1} H_0(\{1\}) \cong R \to 0,$$

$$H_j(S^n) \cong H_j(C_\bullet(S^n)) \cong \begin{cases} R & j \in \{0, n\} \\ 0 & j \neq \{0, n\} \end{cases}.$$

Example 5.3.7: Consider complex projective space $\mathbb{C}\mathrm{P}^n$ with the CW-complex structure from Example 1.3.11, 6.

$$X^0 = X^1 = \{1\}, \quad X^2 = X^3 = \mathbb{C}\mathrm{P}^1, \ldots, \quad X^{2n-2} = X^{2n-1} = \mathbb{C}\mathrm{P}^{n-1}, \quad X^{2n} = \mathbb{C}\mathrm{P}^n,$$

where X^{2k} is obtained from X^{2k-1} by attaching a $2k$-cell with the map $f_k : S^{2k-1} \to \mathbb{C}\mathrm{P}^{k-1}$, $[(x_1, ..., x_{2k}) \mapsto (x_1 + \mathrm{i}x_2, ..., x_{2k-1} + \mathrm{i}x_{2k})]$. This implies $C_{2k}(\mathbb{C}\mathrm{P}^n) = H_{2k}(X^{2k}, X^{2k-1}) \cong H_{2k}(D^{2k}, S^{2k-1}) \cong R$ and $C_k(\mathbb{C}\mathrm{P}^n) \cong H_k(X^k, X^{k-1}) = 0$ for $k > 2n$ or $0 < k < 2n$, k odd. The cellular complex and the homologies are given by

$$0 \xrightarrow{d_{2n+1}} C_{2n}(\mathbb{C}\mathrm{P}^n) \cong R \xrightarrow{d_{2n}} 0 \xrightarrow{d_{2n-1}} C_{2n-2}(\mathbb{C}\mathrm{P}^n) \cong R \xrightarrow{d_{2n-2}} 0 \to \ldots \to 0 \xrightarrow{d_1} C_0(\mathbb{C}\mathrm{P}^n) \cong R \to 0,$$

$$H_j(\mathbb{C}P^n) \cong H_j(C_\bullet(\mathbb{C}\mathrm{P}^n)) = \frac{\ker(d_j)}{\mathrm{Im}\,(d_{j+1})} = \begin{cases} R & j \in \{0, 2, 4, \ldots, 2n\} \\ 0 & \text{else.} \end{cases}$$

Example 5.3.8: We compute the homologies of $\mathbb{R}\mathrm{P}^n \cong S^n/\sim$ for $R = \mathbb{Z}$. For this, equip $\mathbb{R}\mathrm{P}^n$ with the the CW-complex structure from Example 1.3.11, 5.

$$X^0 = \{1\}, \quad X^1 = \mathbb{R}\mathrm{P}^1, \quad X^2 = \mathbb{R}\mathrm{P}^2, \ldots, \quad X^n = \mathbb{R}\mathrm{P}^n,$$

where X^k is obtained by attaching a k-cell to X^{k-1} with the canonical surjection $f_{k-1} : S^{k-1} \to \mathbb{R}\mathrm{P}^{k-1}$. Then we have $C_k(\mathbb{R}\mathrm{P}^n) = H_k(X^k, X^{k-1}) \cong H_k(D^k, S^{k-1}) \cong \mathbb{Z}$ for all $0 \leq k \leq n$ and $C_k(\mathbb{R}\mathrm{P}^n) = 0$ for $k > n$. The cellular complex takes the form

$$0 \to C_n(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z} \xrightarrow{d_n} C_{n-1}(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z} \xrightarrow{d_{n-1}} \ldots \xrightarrow{d_2} C_1(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z} \xrightarrow{d_1} C_0(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z} \to 0.$$

To compute the homologies, we determine the boundary operators $d_n : \mathbb{Z} \to \mathbb{Z}$, which are given by multiplication with an integer. For this, note that the quotient $\mathbb{R}\mathrm{P}^k/\mathbb{R}\mathrm{P}^{k-1}$ is homeomorphic to S^k and the quotient S^k/S^{k-1} is homeomorphic to $S^k \vee S^k$. We obtain a commutative diagram

$$\begin{tikzcd} S^k \arrow[r, "P_k"] \arrow[d, "f_k"'] \arrow[dr, "f_k + f_k \circ a"] & S^k \vee S^k \arrow[d, "\mathrm{id} \vee a"] \\ \mathbb{R}P^k \arrow[r, "\pi'_k"'] & S^k \cong \mathbb{R}\mathrm{P}^k/\mathbb{R}\mathrm{P}^{k-1}, \end{tikzcd}$$

where $S^k_\pm = \{x \in S^{k-1} : \pm x_k > 0\}$ are the upper- and lower half-sphere, $P_k : S^k \to S^k \vee S^k$ is the pinch map and $a : S^k \to S^k$ the antipodal map. By Lemma 5.2.3, $\pi'_k \circ f_k = \mathrm{id} + a$ satisfies $H_j(\pi'_k \circ f_k) = H_j(\mathrm{id}) + H_j(a)$ for all $j \in \mathbb{N}$. Inserting these identities into the definition of the cellular complex yields the commuting diagram

$$\begin{tikzcd}
\begin{array}{l} H_k(\mathbb{R}\mathrm{P}^k, \mathbb{R}\mathrm{P}^{k-1}) \\ \cong H_k(S^k) \cong \mathbb{Z} \end{array} \arrow[d, "\partial_k^{(k)}"'] & \begin{array}{l} H_k(S^k, S^{k-1}) \\ \cong H_k(S^k \vee S^k) \cong \mathbb{Z} \oplus \mathbb{Z} \end{array} \arrow[l, "(\mathrm{id},a)"'] \arrow[d, "\partial'^{(k)}_k"', "(\mathrm{id},a)"] \\
H_{k-1}(\mathbb{R}\mathrm{P}^{k-1}) \arrow[d, "H_{k-1}(\pi'_{k-1})"'] & H_{k-1}(S^{k-1}) \cong \mathbb{Z} \arrow[l, "H_{k-1}(f_{k-1})"'] \arrow[ul, "\mathrm{id}"'] \arrow[dl, "\deg(\mathrm{id}+a)"'] \arrow[d, "H_{k-1}(P_{k-1})"] \\
\begin{array}{l} H_{k-1}(\mathbb{R}\mathrm{P}^{k-1}, \mathbb{R}\mathrm{P}^{k-2}) \\ \cong H_{k-1}(S^{k-1}) \cong \mathbb{Z} \end{array} & \begin{array}{l} H_{k-1}(S^{k-1}, S^{k-2}) \\ \cong H_{k-1}(S^{k-1} \vee S^{k-1}) \cong \mathbb{Z} \oplus \mathbb{Z} \end{array} \arrow[l, "(\mathrm{id},a)"]
\end{tikzcd}.$$

This shows that the boundary operator $d_k : \mathbb{Z} \to \mathbb{Z}$ acts by multiplication with the degree $\deg(\pi'_{k-1} \circ f_{k-1}) = \deg(\mathrm{id} + a) = 1 + \deg(a) = 1 + (-1)^k$. The cellular complex is given by

$$... \to 0 \xrightarrow{d_{n+1}} \mathbb{Z} \xrightarrow{\cdot 2} \cong \mathbb{Z} \xrightarrow{\cdot 0} \mathbb{Z} \xrightarrow{\cdot 2} \ldots \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 0} \mathbb{Z} \to 0 \qquad n \text{ even}$$
$$... \to 0 \xrightarrow{d_{n+1}} \mathbb{Z} \xrightarrow{\cdot 0} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 0} \ldots \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 0} \mathbb{Z} \to 0 \qquad n \text{ odd}$$

and the homologies by

$$H_k(\mathbb{R}P^n) \cong H_k(C_\bullet(\mathbb{R}\mathrm{P}^n)) = \frac{\ker d_k}{\mathrm{Im}\ (d_{k+1})} = \begin{cases} \mathbb{Z}/2\mathbb{Z} & k \text{ odd},\ 0 < k < n \\ \mathbb{Z} & k = n \text{ odd or } k = 0 \\ 0 & \text{else.} \end{cases}$$

These examples show that cellular homology is a very efficient way of computing the homologies of CW-complexes. Moreover, it allows one to compute two important topological invariants, namely the *Betti numbers* and the *Euler-Poincaré characteristic*. For this, one restricts attention to finite CW-complexes and to fields $R = \mathbb{F}$. In this case, the R-modules $C_n(X)$ are finite-dimensional vector spaces over $\mathbb{F}$ and the boundary operators $d_n : C_n(X) \to C_{n-1}(X)$ $\mathbb{F}$-linear maps. This implies that the homologies $H_n(C_\bullet(X)) = H_n(X) = \ker(d_n)/\mathrm{Im}\ (d_{n+1})$ can be be characterised by their dimensions[5].

Definition 5.3.9: Let $R = \mathbb{F}$ be a field and $(X, \cup_{k=0}^n X^k)$ a finite CW-complex. Then

- $b_k(X) = \dim_{\mathbb{F}} H_k(X) \in \mathbb{N}_0$ is called the kth **Betti-number** of X.
- $\chi(X) = \Sigma_{i=0}^n (-1)^i \dim_{\mathbb{F}} C_i(X) \in \mathbb{Z}$ is called the **Euler number** or **Euler-Poincaré characteristic** of X.

It is clear that the Betti numbers are topological invariants, i. e. constant on the homeomorphism classes of topological spaces, and do not depend choice of the CW-structure, since this holds already for the homologies. For the Euler-Poincaré characteristic the latter is not immediately apparent, but it follows from the fact that it can be computed from the Betti numbers.

[5]Note that this does not make sense for modules over a general (commutative unital) ring R since modules over R do not need to be free and in that case there is no sensible notion of dimension.

Lemma 5.3.10: Let $R = \mathbb{F}$ be a field and $(X, \cup_{k=0}^n X^k)$ a finite CW-complex. Then the Euler-Poincaré characteristic is given in terms of the Betti numbers by

$$\chi(X) = \Sigma_{i=0}^n (-1)^i b_i(X).$$

Proof:
As $X = \cup_{k=0}^n X^k$ is finite, the cellular complex takes the form

$$C_\bullet(X) = 0 \to C_n(X) \xrightarrow{d_n} C_{n-1}(X) \xrightarrow{d_{n-1}} \ldots \xrightarrow{d_2} C_1(X) \xrightarrow{d_1} C_0(X) \to 0$$

where $C_k(X) = H_k(X^k, X^{k-1})$ is a finite-dimensional vector space over $\mathbb{F}$. This implies

$$\begin{aligned}\chi(X) =& \Sigma_{i=0}^n (-1)^i \dim_{\mathbb{F}} C_i(X) = \Sigma_{i=0}^n (-1)^i (\dim_{\mathbb{F}} \ker(d_i) + \dim_{\mathbb{F}} \mathrm{Im}\,(d_i)) \\ =& \Sigma_{i=0}^n (-1)^i \dim_{\mathbb{F}} \ker(d_i) - \Sigma_{i=0}^n (-1)^i \mathrm{Im}\,(d_{i+1}) = \Sigma_{i=0}^n (-1)^i \dim_{\mathbb{F}} (\ker(d_i)/\mathrm{Im}\,(d_{i+1})) \\ =& \Sigma_{i=0}^n (-1)^i \dim_{\mathbb{F}} H_i(C_\bullet(X)) = \Sigma_{i=0}^n (-1)^i \dim_{\mathbb{F}} H_i(X) = \Sigma_{i=0}^n (-1)^i b_i(X)\end{aligned}$$

□

Example 5.3.11:

1. For $X = S^n$, $n \geq 1$, with the CW-complex structure from Example 1.3.11, 3., the Betti numbers over $\mathbb{F}$ are given by $b_0(S^n) = 1$, $b_n(S^n) = 1$ and $b_j(S^n) = 0$ for all $j \neq 0, n$. The Euler-Poincaré characteristic is given by
$$\chi(S^n) = \begin{cases} 2 & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$$
2. For $X = \mathbb{C}\mathrm{P}^n$ with the CW-complex structure from Example 1.3.11, 6., we obtain from Example 5.3.11:
$$b_j(\mathbb{C}\mathrm{P}^n) = \begin{cases} 1 & j \in \{0, 2, 4, ..., 2n\} \\ 0 & \text{else,} \end{cases}$$
and the Euler-Poincaré characteristic is given by $\chi(\mathbb{C}\mathrm{P}^n) = n + 1$.
3. For $X = \mathbb{R}\mathrm{P}^n$ with the CW-complex structure from Example 1.3.11, 5., we have $C_k(X) = H_k(X^k, X^{k-1}) \cong \mathbb{F}$ for all $k \in \{0, ..., n\}$. This implies
$$\chi(\mathbb{R}P^n) = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$$
The Betti numbers $b_k(\mathbb{R}\mathrm{P}^n)$ depend on the choice of the field $\mathbb{F}$.
4. If $(X, \{\sigma_i\}_{i \in I})$ is a finite Δ-complex and X hausdorff, then X has the structure of a CW-complex, where the k-simplexes in X are the characteristic maps of the k-cells and the skeleta agree. Denoting by $I_k \subset I$ the index-set for the k-simplexes $\sigma_i : \Delta^k \to X$, we obtain $\dim_{\mathbb{F}} C_k(X) = |I_k|$ and $\chi(X) = \Sigma_{j=0}^n (-1)^j |I_j|$. If $(X, \{\sigma_i\}_{i \in I})$ consists only of 2-, 1- and 0-simplexes, this reduces to **Euler's polyhedron formula**
$$\chi(X) = V(X) - E(X) + F(X),$$
where $V(X)$, $E(X)$ and $F(X)$ are, respectively, the number of vertices (0-simplexes), edges (1-simplexes) and faces (2-simplexes) in X.

5.4 Homology with coefficients

In this section, we will clarify how the homologies of a topological space X depend on the choice of the underlying ring R. On the one hand, most aspects of homology theories can be formulated quite generally, for modules over unital rings. On the other hand, the concrete results obtained when computing the homologies of a topological space X depend on the choice of the ring - see for instance the homologies of $\mathbb{R}\mathrm{P}^2$ in Example 4.1.13. This raises the question if the homologies with respect to some rings contain more information about the topological spaces than the homologies with respect to others, and if there is a unital ring for which the homologies contain a maximum amount of information.

It seems plausible that if this is the case, this distinguished unital ring should be the ring $\mathbb{Z}$, since $\mathbb{Z}$ is an initial object in URing. As every unital ring R is an abelian group, i. e. a $\mathbb{Z}$-module, it should be possible to relate the homologies of a topological space with respect to the ring R to its homologies with coefficients in $\mathbb{Z}$. More precisely, we will show that the homologies of a topological space with respect to a ring R can be expressed as a function of its homologies in $\mathbb{Z}$ and certain purely algebraic data that depends only on R but not on the underlying topological space.

The central idea is to interpret the free R-modules $S_n(X)_R \cong \oplus_{i\in I} R$ of singular n-chains with values in a ring $R \cong R\otimes_{\mathbb{Z}}\mathbb{Z}$ as a tensor product $S_n(X)_R \cong \oplus_{i\in I}(R\otimes_Z\mathbb{Z}) \cong R\otimes_{\mathbb{Z}}(\oplus_{i\in I}\mathbb{Z})$ and to investigate how tensoring with R affects the n-cycles, n-boundaries and homologies. To address this question, we do not restrict attention to unital rings but consider more general structures, namely singular n-chains, n-cycles and n-boundaries with values in an R-module M.

Definition 5.4.1: Let R be a commutative unital ring, M a module over R and X a topological space. Denote by $C(\Delta^n, X)$ the set of singular n-simplexes $\sigma : \Delta^n \to X$. The R-module of **singular n-chains with coefficients in** M is the R-module $S_n(X;M) = \oplus_{C(\Delta^n,X)} M$. Boundary operators $\partial_n : S_n(X;M) \to S_{n-1}(X;M)$, the submodules $Z_n(X;M) = \ker(\partial_n)$, $B_n(X;M) = \mathrm{Im}\,(\partial_{n+1})$ and the homologies $H_n(X;M) = Z_n(X;M)/B_n(X,M)$ are defined as for singular homology with coefficients in R. Analogously, one defines relative and simplicial n-chains, n-cycles, n-boundaries and homologies with coefficients in M.

Remark 5.4.2: As tensor products of modules are compatible with direct sums, we can interpret the module of singular n-chains with coefficients in M as a tensor product

$$S_n(X;M) = \oplus_{C(\Delta^n,X)} M \cong \oplus_{C(\Delta^n,X)}(M\otimes_R R) \cong M\otimes_R(\oplus_{C(\Delta^n,X)} R) = M\otimes_R S_n(X).$$

The singular chain complex with coefficients in M is therefore given by

$$S_n(X;M) = \ldots \xrightarrow{\mathrm{id}_M\otimes\partial_{n+1}} M\otimes_R S_n(X) \xrightarrow{\mathrm{id}_M\otimes\partial_n} \ldots \xrightarrow{\mathrm{id}_M\otimes\partial_2} M\otimes_R S_1(X) \xrightarrow{\mathrm{id}_M\otimes\partial_1} M\otimes_R S_0(X) \to 0$$

with $Z_n(X;M) = \ker(\mathrm{id}_M\otimes\partial_n) \cong M\otimes_R Z_n(X)$, $B_n(X;M) \cong \mathrm{Im}\,(\mathrm{id}_M\otimes\partial_{n+1}) \cong M\otimes_R B_n(X)$. It follows that the homologies with coefficients in M take the form

$$H_n(X;M) \cong \frac{M\otimes_R Z_n(X)}{M\otimes_R B_n(X)}.$$

Remark 5.4.2 expresses the homologies $H_n(X;M)$ with coefficients in M in terms of n-cycles and n-boundaries with values in R. To express them in terms of the homologies

$H_n(X) = Z_n(X)/B_n(X)$, we need to relate quotients of modules of the form $(M\otimes_R U)/(M\otimes_R V)$ to quotients U/V, which is a purely algebraic problem. To address it, it is advantageous to work with exact sequences instead of quotients. We consider the short exact sequence

$$0 \to B_n(X) \xrightarrow{i_n} Z_n(X) \xrightarrow{\pi_n} H_n(X) \to 0,$$

where $i_n : B_n(X) \to Z_n(X)$ is the canonical inclusion and $\pi_n : Z_n(X) \to H_n(X)$ the canonical surjection. By tensoring this sequence with an R-module M, we obtain a sequence

$$0 \to M\otimes_R B_n(X) \xrightarrow{\mathrm{id}_M\otimes i_n} M\otimes_R Z_n(X) \xrightarrow{\mathrm{id}_M\otimes \pi_n} M\otimes_R H_n(X) \to 0.$$

If this sequence was exact, we would obtain

$$M\otimes_R H_n(X) \cong \frac{M\otimes_R Z_n(X)}{M\otimes_R B_n(X)} \cong H_n(X; M)$$

However, it is in general not true that for an exact sequence $0 \to X \xrightarrow{\iota} Y \xrightarrow{\pi} Z \to 0$ the sequence $0 \to M\otimes_R X \xrightarrow{\mathrm{id}_M\otimes\iota} M\otimes_R Y \xrightarrow{\mathrm{id}_M\otimes\pi} M\otimes_R Z \to 0$ is exact, since $\mathrm{id}_M\otimes\iota : M\otimes_R X \to M\otimes_R Y$ need not be injective. A counterexample is the short exact sequence $0 \to \mathbb{Z} \xrightarrow{\iota} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \to 0$ with the injective map $\iota : \mathbb{Z} \to \mathbb{Z}$, $z \mapsto 2z$, for which the map $\mathrm{id}_{\mathbb{Z}/2\mathbb{Z}}\otimes\iota : \mathbb{Z}/2\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}$ is trivial since $\overline{z}\otimes 2z' = \overline{2z}\otimes z' = \overline{0}\otimes z' = 0$ for all $z, z' \in \mathbb{Z}$. We therefore need to investigate how the functor $M\otimes_R - : R\text{-Mod} \to R\text{-Mod}$ interacts with short exact sequences. This can be done more generally for right modules over a unital ring R and functors $M\otimes_R - : R\text{-Mod} \to \mathrm{Ab}$.

Lemma 5.4.3: Let R be a unital ring and A an R-right module. Then the functor $A \otimes_R - : R\text{-Mod} \to \mathrm{Ab}$ from Lemma 2.2.26 is

- **additive**: $\mathrm{id}_A\otimes(f + f') = \mathrm{id}_A\otimes f + \mathrm{id}_A\otimes f'$ for all module morphisms $f, f' : B \to C$,
- **right exact**: for all exact sequences $B \xrightarrow{f} C \xrightarrow{\pi} D \to 0$ in R-Mod the sequence $A\otimes_R B \xrightarrow{\mathrm{id}_A\otimes f} A\otimes_R C \xrightarrow{\mathrm{id}_A\otimes\pi} A\otimes_R D \to 0$ is exact.

If R is commutative, this yields an additive right exact functor $A\otimes_R - : R\text{-Mod} \to R\text{-Mod}$.

Proof:
That $A\otimes_R - : R\text{-Mod} \to \mathrm{Ab}$ is additive follows directly from its the definition in Lemma 2.2.26. To see that it is right exact, consider R-linear maps $\pi : C \to D$ and $f : B \to C$ with $\ker(\pi) = \mathrm{Im}\,(f)$ and π surjective. The latter implies that the group homomorphism $\mathrm{id}_A\otimes\pi : A\otimes_R C \to A\otimes_R D$, $a\otimes c \mapsto a\otimes\pi(c)$ is surjective as well and the former implies $\mathrm{Im}\,(\mathrm{id}_A\otimes f) \subset \ker(\mathrm{id}_A\otimes\pi)$.

We show that $\mathrm{Im}\,(\mathrm{id}_A\otimes f) \supset \ker(\mathrm{id}_A\otimes\pi)$. For this, we consider the canonical surjection $p : A\otimes_R C \to A\otimes_R C/\mathrm{Im}\,(\mathrm{id}_A\otimes f)$ and construct an R-linear map $q' : A\otimes_R D \to A\otimes_R C/\mathrm{Im}\,(\mathrm{id}_A\otimes f)$ with $q' \circ (\mathrm{id}_A\otimes\pi) = p$. The last equation then implies $\mathrm{Im}\,(\mathrm{id}_A\otimes f) = \ker(p) \supset \ker(\mathrm{id}_A\otimes\pi)$.

As π is surjective, we can choose for every element $d \in D$ an element $i(d) \in \pi^{-1}(d)$ and obtain a map $i : C \to D$ with $\pi \circ i = \mathrm{id}_D$. The map $q : A \times D \to A\otimes_R C/\mathrm{Im}\,(\mathrm{id}\otimes f)$, $(a, d) \mapsto p(a\otimes i(d))$ satisfies

$$\begin{aligned}
q(a + a', d) &= p((a + a')\otimes i(d)) = p(a\otimes i(d) + a'\otimes i(d)) = p(a\otimes i(d)) + p(a'\otimes i(d)) \\
&= q(a, d) + q(a', d) \\
q(a, d + d') &= p(a\otimes i(d + d')) = p(a\otimes(i(d) + i(d'))) + p(a\otimes(i(d + d') - i(d) - i(d'))) \\
&= q(a, d) + q(a, d') + p(a\otimes(i(d + d') - i(d) - i(d'))) \\
q(a \lhd r, d) &= p((a \lhd r)\otimes i(d)) = p(a\otimes(r \rhd i(d))) = p(a\otimes i(r \rhd d)) + p(a\otimes(r \rhd i(d) - i(r \rhd d))) \\
&= q(a, r \rhd d) + p(a\otimes(r \rhd i(d) - i(r \rhd d))).
\end{aligned}$$

The identity $\pi \circ i = \mathrm{id}_D$ implies

$$\begin{aligned}
&\pi(i(d+d') - i(d) - i(d')) = \pi \circ i(d+d') - \pi \circ i(d) - \pi \circ i(d') = d + d' - d - d' = 0 \\
&\pi(r \rhd i(d) - i(r \rhd d)) = r \rhd \pi \circ i(d) = \pi \circ i(r \rhd d) = r \rhd d - r \rhd d = 0, \\
\Rightarrow\quad & i(d+d') - i(d) - i(d'),\ r \rhd i(d) - i(r \rhd d) \in \ker(\pi) = \mathrm{Im}\,(f) \\
\Rightarrow\quad & a \otimes (i(d+d') - i(d) - i(d')),\ a \otimes (r \rhd i(d) - i(r \rhd d)) \in \mathrm{Im}\,(\mathrm{id}_A \otimes f) = \ker(p)
\end{aligned}$$

for all $a \in A$, $d, d' \in D$ and $r \in R$. This shows that the map q is R-bilinear, and by the universal property of the tensor product, it induces a unique group homomorphism

$$q' : A \otimes_R D \to A \otimes_R C / \mathrm{Im}\,(\mathrm{id} \otimes f),\ a \otimes d \mapsto q(a, d) = p(a \otimes i(d)).$$

This group homomorphism satisfies

$$q' \circ (\mathrm{id}_A \otimes \pi)(a \otimes c) = p(a \otimes i(\pi(c))) = p(a \otimes c) + p(a \otimes (i(\pi(c)) - c)) = p(a \otimes c) \quad \forall a \in A, c \in C,$$

since we have $\pi(i(\pi(c)) - c) = \pi \circ i \circ \pi(c) - \pi(c) = \pi(c) - \pi(c) = 0$ for all $c \in C$, which implies $i \circ \pi(c) - c \in \ker(\pi) = \mathrm{Im}\,(f)$ and $a \otimes (\pi \circ i(c) - c) \in \mathrm{Im}\,(\mathrm{id}_A \otimes f) = \ker(p)$ for all $a \in A$, $c \in C$.

□

Remark 5.4.4:

1. A right exact functor can be viewed as a functor that preserves cokernels. The condition in Lemma 5.4.3 is equivalent to the claim that $Z \cong Y/\mathrm{Im}\,(\iota)$ implies $M \otimes_R Z \cong M \otimes_R Y / \mathrm{Im}\,(\mathrm{id}_M \otimes \iota)$.
2. As shown above, the functor $M \otimes_R - : R\text{-Mod} \to \mathrm{Ab}$ is in general not **left exact**, i. e. for an exact sequence $0 \to X \xrightarrow{\iota} Y \xrightarrow{\pi} Z$, the sequence $0 \to M \otimes_R X \xrightarrow{\mathrm{id}_M \otimes \iota} M \otimes_R Y \xrightarrow{\mathrm{id}_M \otimes \pi} M \otimes_R Z$ does not need to be exact. In other words, $M \otimes_R -$ does not preserve kernels.

To characterise the non-left exactness of the functor $M \otimes_R - : R\text{-Mod} \to \mathrm{Ab}$ more precisely, we consider exact sequences of R-modules and R-module morphisms which are of a particularly simple form and contain sufficient information about the exactness properties of F. As tensor products are compatible with direct sums and tensor products of free modules are free, it is advantageous to work with free modules. For a (not necessarily free) module M, we choose an exact sequencewith M as the last entry from the right in which all modules except M are free.

Definition 5.4.5: Let M be a module over a unital ring R. A **free resolution** of M is an exact sequence $M_\bullet = \ldots \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} \ldots \xrightarrow{d_1} M_0 \xrightarrow{d_0} M \to 0$, in which all modules M_i for $i \geq 0$ are free.

Lemma 5.4.6: Let R be a unital ring. Then every R-module M has a free resolution. If R is a principal ideal ring, then M has a free resolution of the form $0 \to M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} M \to 0$.

Proof:
Choose $M_0 = \langle M \rangle_R$ and for $d_0 : M_0 \to M$ the canonical surjection $d_0 = \pi : M_0 \to M$, $m \mapsto m$.

Then define inductively for $i \geq 0$ $M_{i+1} = \langle \ker(d_i) \rangle_R$ and $d_{i+1} = \iota_i \circ \pi_i : M_{i+1} \to M_i$, $m \mapsto m$, where $\iota_i : \ker(d_i) \to M_i$ is the canonical inclusion and $\pi_i : M_{i+1} \to \ker(d_i)$ the canonical surjection. Then M_{i+1} is free by definition and since π_i is surjective, $\ker(d_i) = \mathrm{Im}\,(d_{i+1})$ for all $i \geq -1$. If R is a principal ideal ring, then $\ker(d_0) \subset \langle M \rangle_R$ is already free as a submodule of a free module, and one can choose $M_1 = \ker(f_0)$, $d_1 = \iota_1 : M_1 \to M_0$, $m \mapsto m$. □

Generally, the minimal length of free resolution of an R-module M tells one how far the module M is from a free module or, more precisely, and how many free modules one needs to express M as an iterated quotient of free modules. A module M is free if and only if it has a free resolution of length two, namely $0 \to M \xrightarrow{\mathrm{id}} M \to 0$. Similarly, M has a free resolution of length three $0 \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$ if and only if it is isomorphic to a quotient $M \cong F_0/F_1$ of a free module F_0 by a free submodule F_1. Similarly, M has a free resolution $0 \to F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$ of length four if and only if $M \cong \mathrm{Im}\,(d_0) \cong F_0/\ker(d_0) \cong F_0/\mathrm{Im}\,(d_1) \cong F_0/(F_1/\ker(d_1)) \cong F_0/(F_1/\mathrm{Im}\,(d_2)) \cong F_0/(F_1/F_2)$ with free modules F_0, F_1, F_2. Generally, the existence of a free resolution of given length depends on the ring R as well as on the chosen module M. This is illustrated by the following examples.

Example 5.4.7:

1. If $R = \mathbb{F}$ is a field, any R-module M is free and hence has a free resolution of the form $0 \to M \xrightarrow{\mathrm{id}_M} M \to 0$.
2. The abelian group $\mathbb{Z}/n\mathbb{Z}$ with $n \in \mathbb{N}$ has a free resolution $0 \to \mathbb{Z} \xrightarrow{n\cdot} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \to 0$.
3. For $R = \mathbb{Z}/p^2\mathbb{Z}$ with $p \in \mathbb{N}$ prime, the R-module $\mathbb{Z}/p\mathbb{Z}$ has an infinite free resolution of the form $\ldots \xrightarrow{p\cdot} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{p\cdot} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/p\mathbb{Z} \to 0$. There is no finite free resolution.

It remains to clarify the uniqueness properties of free resolutions. As free resolutions are chain complexes, one expects that different free resolutions of a module should be related by certain chain maps. As each chain map $f_\bullet : M_\bullet \to N_\bullet$ involves a module morphism $f_{-1} : M \to N$ it is natural to ask if any module morphism $f : M \to N$ can be extended to a chain map $f_\bullet : M_\bullet \to N_\bullet$ with $f_{-1} = f$ and in how many different ways. The answer to this question is given by the following lemma.

Lemma 5.4.8: Let R be a unital ring and M, M' modules over R with free resolutions $M_\bullet = \ldots \xrightarrow{d_1} M_0 \xrightarrow{d_0} M \to 0$ and $M'_\bullet = \ldots M'_1 \xrightarrow{d'_1} M'_0 \xrightarrow{d'_0} M' \to 0$. Then every R-module homomorphism $f : M \to M'$ extends to a chain map $f_\bullet : M_\bullet \to M'_\bullet$, and any two such extensions are chain homotopic.

Proof:
1. **Existence:** We construct the R-module morphisms $f_n : M_n \to M'_n$ in $f_\bullet$ inductively. Set $f_{-1} = f$. Suppose we have constructed R-module morphisms $f_i : M_i \to M'_i$ for $-1 \leq i \leq n-1$ such that all squares in the following diagram commute

$$\begin{tikzcd}
M_n \arrow[r, "d_n"] & M_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "f_{n-1}"] & M_{n-2} \arrow[r, "d_{n-2}"] \arrow[d, "f_{n-2}"] & \ldots \arrow[r, "d_2"] & M_1 \arrow[r, "d_1"] \arrow[d, "f_1"] & M_0 \arrow[r, "d_{-1}"] \arrow[d, "f_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\
M'_n \arrow[r, "d'_n"] & M'_{n-1} \arrow[r, "d'_{n-1}"] & M'_{n-2} \arrow[r, "d'_{n-2}"] & \ldots \arrow[r, "d'_2"] & M'_1 \arrow[r, "d'_1"] & M'_0 \arrow[r, "d'_{-1}"'] & M' \arrow[r] & 0.
\end{tikzcd}$$

As M_n is free, there is a subset $B_n \subset M_n$ with $M_n = \langle B_n \rangle_R$. For all $b \in B_n$, one has $d'_{n-1} \circ f_{n-1} \circ d_n = f_{n-2} \circ d_{n-1} \circ d_n = 0$ since the square involving f_{n-1} and f_{n-2} commutes. This implies $f_{n-1} \circ d_n(b) \in \ker(d'_{n-1}) = \mathrm{Im}\,(d'_n)$ and hence there is a $b' \in M'_n$ with $d'_n(b') = f_{n-1} \circ d_n(b)$. By assigning to each $b \in B_n$ such a $b' \in M'_n$ and R-linear continuation to M_n, we obtain an R-module morphism $f_n : M_n \to M'_n$ such that the following diagram commutes

$$\begin{tikzcd}
M_n \arrow[r, "d_n"] \arrow[d, dashed, "f_n"] & M_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "f_{n-1}"] & M_{n-2} \arrow[r, "d_{n-2}"] \arrow[d, "f_{n-2}"] & \ldots \arrow[r, "d_2"] & M_1 \arrow[r, "d_1"] \arrow[d, "f_1"] & M_0 \arrow[r, "d_{-1}"] \arrow[d, "f_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\
M'_n \arrow[r, "d'_n"] & M'_{n-1} \arrow[r, "d'_{n-1}"] & M'_{n-2} \arrow[r, "d'_{n-2}"] & \ldots \arrow[r, "d'_2"] & M'_1 \arrow[r, "d'_1"] & M'_0 \arrow[r, "d'_{-1}"'] & M' \arrow[r] & 0.
\end{tikzcd}$$

2. **Uniqueness:** Suppose there are chain maps $f_\bullet, f'_\bullet : M_\bullet \to M'_\bullet$ with $f_{-1} = f'_{-1} = f : M \to M'$. We construct a chain homotopy from $f_\bullet$ to $f'_\bullet$ by induction. Set $h_{-1} = 0$ and suppose we constructed R-module morphisms $h_i : M_i \to M'_{i+1}$ with $f'_i - f_i = d'_{i+1} \circ h_i + h_{i-1} \circ d_i$ for all $-1 \leq i \leq n-1$. As by induction hypothesis

$$d'_n \circ (f'_n - f_n - h_{n-1} \circ d_n) = (f'_{n-1} - f_{n-1} - d'_n \circ h_{n-1}) \circ d_n = h_{n-2} \circ d_{n-1} \circ d_n = 0,$$

we have $f'_n - f_n - h_{n-1} \circ d_n \in \ker(d'_n) = \mathrm{Im}\,(d'_{n+1})$. Hence for each $b \in B_n$ there is a $b' \in M'_{n+1}$ with $d'_{n+1}(b') = f'_n(b) - f_n(b) - h_{n-1} \circ d_n(b)$. By assigning such an element $b' \in M'_{n+1}$ to each $b \in B_n$ and R-linear continuation, we obtain an R-module morphism $h_n : M_n \to M'_{n+1}$ with $f'_n - f_n = d'_{n+1} \circ h_n + h_{n-1} \circ d_n$. $\square$

In particular, Lemma 5.4.8 clarifies the question about the uniqueness of a free resolution. Given a module M over R and two different free resolutions $M_\bullet$, $M'_\bullet$ of M, we can extend the identity map id_M to a chain map $f_\bullet : M_\bullet \to M'_\bullet$ and to a chain map $f'_\bullet : M'_\bullet \to M_\bullet$. Then $f'_\bullet \circ f_\bullet : M_\bullet \to M_\bullet$ and $f_\bullet \circ f'_\bullet : M'_\bullet \to M'_\bullet$ both extend the identity map and hence are chain homotopic to, respectively, $\mathrm{id}_{M_\bullet}$ and $\mathrm{id}_{M'_\bullet}$. This means that $M_\bullet$ and $M'_\bullet$ are chain homotopy equivalent.

Corollary 5.4.9: Let R be a unital ring and M a module over R. Then any two free resolutions of M are chain homotopy equivalent.

We now use free resolutions to describe additive, right exact functors $F : R\text{-Mod} \to S\text{-Mod}$. We choose for each R-module X a free resolution $X_\bullet = \ldots \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} X \to 0$. Applying the functor F to $X_\bullet$ yields a chain complex $F(X_\bullet) = \ldots \xrightarrow{F(d_2)} F(X_1) \xrightarrow{F(d_1)} F(X_0) \xrightarrow{F(d_0)} F(X) \to 0$.

As F is right exact, the sequence $F(X_1) \xrightarrow{F(d_1)} F(X_0) \xrightarrow{F(d_0)} F(X) \to 0$ is exact. If F is also left exact, it follows that the entire chain complex $F(X_\bullet)$ is exact. Hence the homologies $H_n(F(X_\bullet))$ measure the failure of F to be left exact. As $H_{-1}(F(X_\bullet)) = 0$ due to the exactness of $F(X_1) \xrightarrow{F(d_1)} F(X_0) \xrightarrow{F(d_0)} F(X) \to 0$, we can omit the entry $F(X)$ from the chain complex $F(X_\bullet)$ without losing information and consider instead the chain complex

$$F(X_\bullet)_{\geq 0} = \ldots \xrightarrow{F(d_3)} F(X_2) \xrightarrow{F(d_2)} F(X_1) \xrightarrow{F(d_1)} F(X_0) \to 0$$

By assigning to each R-module X the homology $H_n(F(X_\bullet)_{\geq 0})$ and to a module morphism $f : X \to X'$ with an extension $f_\bullet : X_\bullet \to X'_\bullet$ the module morphism $H_n(F(f_\bullet)_{\geq 0}) : H_n(F(X_\bullet)_{\geq 0}) \to H_n(F(X'_\bullet)_{\geq 0})$, we obtain a functors from R-Mod to S-Mod.

Lemma 5.4.10: Let R, S be unital rings and $F : R\text{-Mod} \to S\text{-Mod}$ an additive, right exact functor. Assign to an R-module X with a free resolution $X_\bullet$ the S-module $L_nF(X) := H_n(F(X_\bullet)_{\geq 0})$ and to an R-module morphism $f : X \to X'$ between R-modules X, X' with free resolutions $X_\bullet, X'_\bullet$ the S-module morphism $L_nF(f) := H_n(F(f_\bullet)_{\geq 0}) : H_n(F(X_\bullet)_{\geq 0}) \to H_n(F(X'_\bullet)_{\geq 0})$, where $f_\bullet : X_\bullet \to X'_\bullet$ is a chain map that extends f as in Lemma 5.4.8. Then for all $n \in \mathbb{N}_0$, this defines functors $L_nF : R\text{-Mod} \to S\text{-Mod}$, the **left derived functors** of F.

Proof:
1. We first show that the assignments depend only on the functor F and are independent of the choices made in the definition. First, note that each functor $F : R\text{-Mod} \to S\text{-Mod}$ induces a functor $F : \mathrm{Ch}_{R\text{-Mod}} \to \mathrm{Ch}_{S\text{-Mod}}$ obtained by applying F to each R-module and R-module morphism in a chain complex and a chain map. As F is additive, applying F to the morphisms $h_n : X_n \to X'_{n+1}$ in a chain homotopy $h_\bullet : f_\bullet \Rightarrow g_\bullet$ yields a chain homotopy $F(h_\bullet) : F(f_\bullet) \Rightarrow F(g_\bullet)$. Hence F maps chain homotopic chain maps to chain homotopic chain maps and homotopy equivalences to homotopy equivalences.

As different free resolutions $X_\bullet, X'_\bullet$ of a module X are chain homotopy equivalent by Corollary 5.4.9, this also holds for the associated chain complexes $F(X_\bullet)_{\geq 0}, F(X'_\bullet)_{\geq 0}$, and their homologies are isomorphic. Hence the homologies $H_n(F(X_\bullet))$ depend only on the module X and not on the choice of the free resolution $X_\bullet$.

If X, X' are R-modules with associated free resolutions $X_\bullet$ and $X'_\bullet$, then any module morphism $f : X \to X'$ extends to a chain map $f_\bullet : X_\bullet \to X'_\bullet$ by Lemma 5.4.8. By applying the functor F, we obtain a chain map $F(f_\bullet) : F(X_\bullet) \to F(X'_\bullet)$ and an associated S-module morphism $H_n(F(f_\bullet)) : H_n(F(X_\bullet)) \to H_n(F(X'_\bullet))$. As different extensions of $f : X \to X'$ are chain homotopic by Lemma 5.4.8 and the image of a chain homotopy under an additive functor is a chain homotopy, the S-module morphism $H_n(F(f_\bullet)) : H_n(F(X_\bullet)) \to H_n(F(X'_\bullet))$ does not depend on the choice of the extension of f.

2. It remains to show that the assignments are compatible with the identity morphisms and the composition of morphisms in R-Mod. This follows because since the identity morphism $\mathrm{id}_X : X \to X$ extends to the trivial chain map and F is a functor, which implies $\mathrm{id}_{X_\bullet} : X_\bullet \to X_\bullet$ and $F(\mathrm{id}_{X_\bullet}) = \mathrm{id}_{F(X_\bullet)}$. Moreover, for module morphisms $f : X \to X'$ and $g : X' \to X''$ with extensions $f_\bullet : X_\bullet \to X'_\bullet$, $g_\bullet : X'\bullet \to X''_\bullet$ the composite $g_\bullet \circ f_\bullet : X_\bullet \to X''_\bullet$ extends $g \circ f : X \to X''$ and $F(g_\bullet \circ f_\bullet) = F(g_\bullet) \circ F(f_\bullet)$ since F is a functor. $\square$

We can now investigate the left derived functors of the additive right exact functor $M \otimes_R - : R\text{-Mod} \to \mathrm{Ab}$ for each R-right module M. These functors are known under the name of *torsion functors* since for finitely generated modules over a principal ideal ring, they are closely related to the torsion submodule.

Definition 5.4.11: Let R be a unital ring and M an R-right module. Then the left derived functors $\mathrm{Tor}^R_n(M, -) := L_n(M \otimes_R -) : R\text{-Mod} \to \mathrm{Ab}$ are called the **torsion functors** and $L_n(M \otimes_R -)(N) = \mathrm{Tor}^R_n(M, N)$ is called the **torsion product** of M and N. If R is commutative one obtains functors $\mathrm{Tor}_n^R(M,-) : R\text{-Mod} \to R\text{-Mod}$ and for each R-module N an R-module $\mathrm{Tor}_n^R(M,N)$.

Remark 5.4.12:

1. The torsion functors are compatible with direct sums. If $(X^i_\bullet, d^i_\bullet)$ is a free resolution of X^i for $i \in I$, then $(\oplus_{i\in I}X^i_\bullet, \oplus_{i\in I}d^i_\bullet)$ is a free resolution of $\oplus_{i\in I}X^i$ and $M\otimes_R(\oplus_{i\in I}X^i_\bullet) \cong \oplus_{i\in I}M\otimes_R X^i_\bullet$. This implies $\mathrm{Tor}_n^R(M, \oplus_{i\in I}X^i) \cong \oplus_{i\in I}\mathrm{Tor}_n^R(M, X^i)$.
2. One can show that for any commutative unital ring R the torsion product is commutative: $\mathrm{Tor}_n^R(M,N) \cong \mathrm{Tor}_n^R(N,M)$ for all R-modules M, N.

Example 5.4.13: Let R be a principal ideal ring and M, N modules over R. Then N has a finite free resolution $N_\bullet = 0 \to N_1 \xrightarrow{d_1} N_0 \xrightarrow{d_0} N \to 0$, and the sequence $0 \to M\otimes_R N_1 \xrightarrow{\mathrm{id}_M\otimes d_1} M\otimes_R N_0 \xrightarrow{\mathrm{id}_M\otimes d_0} M\otimes_R N \to 0$ is exact in $M\otimes_R N_0$ and $M\otimes_R N$. By removing the entry $M\otimes_R N$ we obtain the sequence $(M\otimes_R F_\bullet)_{\geq 0} = 0 \to M\otimes_R F_1 \xrightarrow{\mathrm{id}_M\otimes d_1} M\otimes_R F_0 \to 0$, and the torsion is

$$\begin{aligned}
\mathrm{Tor}_n^R(M,N) =& H_n((M\otimes_R F_\bullet)_{\geq 0}) = 0 \quad \forall n \geq 2,\\
\mathrm{Tor}_1^R(M,N) =& H_1((M\otimes_R F_\bullet)_{\geq 0}) = \ker(\mathrm{id}_M\otimes d_1),\\
\mathrm{Tor}_0^R(M,N) =& H_0((M\otimes_R F_\bullet)_{\geq 0}) = M\otimes_R F_0/\mathrm{Im}\,(\mathrm{id}_M\otimes d_1) = M\otimes_R F_0/\ker(\mathrm{id}_M\otimes d_0)\\
\cong& \mathrm{Im}\,(\mathrm{id}_M\otimes d_0) \cong M\otimes_R N.
\end{aligned}$$

We compute $\mathrm{Tor}_1^R(M,N)$ for different modules M, N:

1. If the module N is free, we can set $N_0 = N$, $N_1 = 0$, $d_0 = \mathrm{id}_M$, $d_1 = 0$. This implies $\mathrm{Tor}_n^R(M,N) = 0$ for all $n \neq 0$.
2. If the module M is free then $M \cong \oplus_{i\in I}R$ for an index set I. This implies $M\otimes_R N_\bullet \cong \oplus_{i\in I}R\otimes_R N_\bullet \cong \oplus_{i\in I}N_\bullet$ and we obtain $\mathrm{Tor}_n^R(M,N) \cong \oplus_{i\in I}\mathrm{Tor}_n^R(R,N) \cong 0$ for all $n \neq 0$.
3. If $N = R/qR$ and $M = R/pR$ with $p, q \in R$, then $0 \to R \xrightarrow{q\cdot} R \xrightarrow{\pi} R/qR \to 0$ is a free resolution of N. We have $M\otimes_R N \cong R/\gcd(p,q)R$, and the sequence $(M\otimes_R N_\bullet)_{\geq}0$ is given by $0 \to R/pR\otimes_R R \cong R/pR \xrightarrow{q\cdot} R/pR \cong R/pR\otimes_R R \to 0$. This yields

$$\mathrm{Tor}_1^R(M,N) = \ker(R/pR \xrightarrow{\overline{z}\mapsto q\cdot\overline{z}} R/pR) \cong R/\gcd(p,q)R \cong \mathrm{Tor}(R/pR) \otimes \mathrm{Tor}(R/qR).$$

4. As every finitely generated R-module N is of the form $R/qR \oplus R^n$ with $q \in R$, $n \in \mathbb{N}$ and $\mathrm{Tor}(N) \cong R/qR$, one can show with Remark 5.4.12 that $\mathrm{Tor}_1^R(M,N) \cong \mathrm{Tor}(M)\otimes\mathrm{Tor}(N)$ for all finitely generated R-modules M, N. This explains the name torsion.

In the following we will focus on principal ideal rings R. As in that case $\mathrm{Tor}_0^R(M,N) \cong M\otimes_R N$ and $\mathrm{Tor}_n^R(M,N) \cong 0$ for $n \geq 2$, all non-trivial information about the chain complex $M\otimes_R N_\bullet$ is contained in the torsion products $\mathrm{Tor}_1^R(M,N)$. We will now use the torsion products to relate the homologies $H_n(X;M)$ of a topological space X with respect to an R-module M to the homologies $H_n(X)$. For this, we consider the singular chain complex $S_\bullet(X)$ and for each $n \in \mathbb{N}$ the short exact sequence $0 \to Z_n(X) \xrightarrow{\iota_n} S_n(X) \xrightarrow{\partial_n} B_{n-1}(X) \to 0$. Tensoring this short exact sequence with an R-module M yields a sequence $0 \to M\otimes_R Z_n(X) \xrightarrow{\mathrm{id}_m\otimes\iota_n} M\otimes_R S_n(X) \xrightarrow{\mathrm{id}_M\otimes d_n}$

$M\otimes B_n(X) \to 0$, which is exact in the last two entries. By interpreting it as a long exact sequence of chain complexes (each equipped with the boundary operator $\mathrm{id}_M\otimes\partial_n$), we obtain a long exact sequence of homologies that relates the homologies $H_n(X;M)$ to $H_n(X)$.

Theorem 5.4.14: (Universal coefficients)

Let R be a principal ideal ring, M an R-module and X a topological space. Then there is a natural short exact sequence

$$0 \to M\otimes_R H_n(X) \to H_n(X;M) \to \mathrm{Tor}_1^R(M, H_{n-1}(X)) \to 0.$$

This sequence splits, but not naturally, and $H_n(X;M) \cong (M\otimes_{\mathbb{R}} H_n(X)) \oplus \mathrm{Tor}_1^R(M, H_{n-1}(X))$.

Proof:
We show that for every chain complex $F_\bullet = \ldots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to 0$ in R-Mod in which every R-module F_i is free and for every $n \in \mathbb{N}$, there is a natural short exact sequence

$$0 \to M\otimes_R H_n(F_\bullet) \to H_n(M\otimes_R F_\bullet) \to \mathrm{Tor}_1^R(M, H_{n-1}(F_\bullet)) \to 0,$$

which splits. The claim then follows by choosing $F_n = S_n(X)$.

Consider for $n \in \mathbb{N}$ the short exact sequence $0 \to Z_n(F_\bullet) \xrightarrow{\iota_n} F_n \xrightarrow{d_n} B_{n-1}(F_\bullet) \to 0$. As R is a principal ideal ring, the modules $Z_n(F_\bullet) \subset F_n$, $B_{n-1}(F_\bullet) \subset F_{n-1}$ are free as submodules of free modules. As $d_n : F_n \to B_{n-1}(F_\bullet)$ is surjective, by Lemma 2.2.15 there is an R-module morphism $\psi : B_{n-1}(F_\bullet) \to F_n$ which splits the sequence, and $F_n \cong \ker(d_n) \oplus \mathrm{Im}\,(\psi) \cong Z_n(F_\bullet) \oplus B_{n-1}(F_\bullet)$. Tensoring the short exact sequence with M yields a short exact sequence

$$0 \to M\otimes_R Z_n(F_\bullet) \xrightarrow{\mathrm{id}_M\otimes\iota_n} (M\otimes_R Z_n(F_\bullet)) \oplus (M\otimes_R B_{n-1}(F_\bullet)) \xrightarrow{\mathrm{id}_M\otimes d_n} M\otimes_R B_{n-1}(F_\bullet) \to 0.$$

By considering the chain complexes

$$\begin{aligned}
M\otimes_R Z_\bullet &= \ldots \xrightarrow[=0]{\mathrm{id}_M\otimes d_{n+2}} M\otimes_R Z_{n+1}(F_\bullet) \xrightarrow[=0]{\mathrm{id}_M\otimes d_{n+1}} M\otimes_R Z_n(F_\bullet) \xrightarrow[=0]{\mathrm{id}_M\otimes d_n} M\otimes_R Z_{n-1}(F_\bullet) \xrightarrow[=0]{\mathrm{id}_M\otimes d_{n-1}} \ldots \\
M\otimes_R F_\bullet &= \ldots \xrightarrow{\mathrm{id}_M\otimes d_{n+2}} M\otimes_R F_{n+1} \xrightarrow{\mathrm{id}_M\otimes d_{n+1}} M\otimes_R F_n \xrightarrow{\mathrm{id}_M\otimes d_n} M\otimes_R F_{n-1} \xrightarrow{\mathrm{id}_M\otimes d_{n-1}} \ldots \\
M\otimes_R B_\bullet &= \ldots \xrightarrow[=0]{\mathrm{id}_M\otimes d_{n+1}} M\otimes_R B_n(F_\bullet) \xrightarrow[=0]{\mathrm{id}_M\otimes d_n} M\otimes_R B_{n-1}(F_\bullet) \xrightarrow[=0]{\mathrm{id}_M\otimes d_{n-1}} M\otimes_R B_{n-2}(F_\bullet) \xrightarrow[=0]{\mathrm{id}_M\otimes d_{n-2}} \ldots
\end{aligned}$$

we can interpret this short exact sequence as a short exact sequence of chain complexes $0 \to M\otimes_R Z_\bullet \xrightarrow{\mathrm{id}_M\otimes\iota_\bullet} M\otimes_R F_\bullet \xrightarrow{\mathrm{id}_M\otimes d_\bullet} M\otimes_R B_\bullet \to 0$ and obtain the long exact sequence of homologies

$$\ldots \xrightarrow{\partial_{n+1}} M\otimes_R Z_n(F_\bullet) \xrightarrow{H_n(\mathrm{id}_M\otimes\iota_\bullet)} H_n(M\otimes_R F_\bullet) \xrightarrow{H_n(\mathrm{id}_M\otimes d_n)} M\otimes_R B_{n-1}(F_\bullet) \xrightarrow{\partial_n} M\otimes_R Z_{n-1}(F_\bullet) \to \ldots$$

By Theorem 2.3.11 the connecting homomorphism $\partial_n : M\otimes_R B_{n-1}(F_\bullet) \to M\otimes_R Z_{n-1}(F_\bullet)$ is given by $\partial_n(m\otimes f) = m\otimes f'$, where $f' \in Z_{n-1}(F_\bullet) \subset F_{n-1}$ satisfies $\iota_{n-1}(f') = d_n(f'')$ for an $f'' \in F_n$ with $d_n(f'') = f$. This implies $\partial_n = \mathrm{id}_M\otimes i_{n-1} : M\otimes_R B_{n-1}(F_\bullet) \to M\otimes_R Z_{n-1}(F_\bullet)$ is given by the canonical inclusion $i_{n-1} : B_{n-1}(F_\bullet) \to Z_{n-1}(F_\bullet)$. We can thus rewrite the long exact sequence of homologies as

$$\ldots \xrightarrow{\mathrm{id}_M\otimes i_n} M\otimes_R Z_n(F_\bullet) \xrightarrow{H_n(\mathrm{id}_M\otimes\iota_\bullet)} H_n(M\otimes_R F_\bullet) \xrightarrow{H_n(\mathrm{id}_M\otimes d_\bullet)} M\otimes_R B_{n-1}(F_\bullet) \xrightarrow{\mathrm{id}_M\otimes i_{n-1}} M\otimes_R Z_{n-1}(F_\bullet) \to \ldots$$

Its exactness implies

$$\begin{aligned}&\ker(\mathrm{id}_M\otimes i_{n-1}) \cong \mathrm{Im}\,(H_n(\mathrm{id}_M\otimes d_n)) \cong H_n(M\otimes_R F_\bullet)/\ker(H_n(\mathrm{id}_M\otimes d_n))\\ &\cong H_n(M\otimes_R F_\bullet)/\mathrm{Im}\,(H_n(\mathrm{id}_M\otimes \iota_\bullet)) \cong H_n(M\otimes_R F_\bullet)/(M\otimes_R Z_n(F_\bullet)/\ker(H_n(\mathrm{id}_M\otimes\iota_\bullet)))\\ &\cong H_n(M\otimes_R F_\bullet)/(M\otimes_R Z_n(F_\bullet)/\mathrm{Im}\,(\mathrm{id}_M\otimes i_n))),\end{aligned}$$

and hence we have for all $n\in\mathbb{N}$ a short exact sequence

$$0\to (M\otimes_R Z_n(F_\bullet))/\mathrm{Im}\,(\mathrm{id}_M\otimes i_n)\to H_n(M\otimes_R F_\bullet)\to \ker(\mathrm{id}_M\otimes i_{n-1})\to 0.$$

By applying Example 5.4.13 to $0\to B_n(F_\bullet)\xrightarrow{i_n} Z_n(F_\bullet)\xrightarrow{\pi_n} H_n(F_\bullet)\to 0$, which is a free resolution of $H_n(F_\bullet)$, we obtain

$$\begin{aligned}&\ker(\mathrm{id}_M\otimes i_{n-1}) = \mathrm{Tor}_1^R(M, H_{n-1}(F_\bullet))\\ &M\otimes_R Z_n(F_\bullet)/\mathrm{Im}\,(\mathrm{id}_M\otimes i_n) = \mathrm{Tor}_0^R(M, H_n(F_\bullet)) = M\otimes_R H_n(F_\bullet),\end{aligned}$$

and this produces the short exact sequence in the claim. Its naturality follows directly from the naturality of the sequence

$$...\to M\otimes_R B_n(F_\bullet)\xrightarrow{\mathrm{id}_M\otimes i_n} M\otimes_R Z_n(F_\bullet)\xrightarrow{H_n(\mathrm{id}_M\otimes\iota_\bullet)} H_n(M\otimes_R F_\bullet)\xrightarrow{H_n(\mathrm{id}_M\otimes d_n)} M\otimes_R B_{n-1}(F_\bullet)\to ...$$

To show that this sequence splits, we construct a map $\phi:\ker(\mathrm{id}_M\otimes i_{n-1})\to H_n(M\otimes_R F_\bullet)$ with $H_n(\mathrm{id}_M\otimes d_\bullet)\circ\phi = \mathrm{id}_{\ker(\mathrm{id}_M\otimes i_{n-1})}$. As $\ker(\mathrm{id}_M\otimes i_{n-1})\subset M\otimes_R B_{n-1}(F_\bullet)$ we use the non-natural splitting $\psi: B_{n-1}(F_\bullet)\to F_n$ constructed in the first step of the proof. As $(\mathrm{id}_M\otimes d_n)\circ(\mathrm{id}_M\otimes\psi)(m\otimes f) = m\otimes f = (\mathrm{id}_M\otimes\iota_{n-1})\circ(\mathrm{id}_M\otimes i_{n-1})(m\otimes f)$, we have $\psi(\ker(\mathrm{id}_M\otimes i_{n-1}))\subset\ker(\mathrm{id}_M\otimes d_n)$ and hence ψ induces a map $\ker(\mathrm{id}_M\otimes i_{n-1})\to H_n(M\otimes_R F_\bullet)$ with $H_n(\mathrm{id}_M\otimes d_\bullet)\circ\psi = \mathrm{id}_{\ker(\mathrm{id}_M\otimes i_{n-1})}$. The sequence splits, but the splitting is not natural since $\psi: B_{n-1}(F_\bullet)\to F_n$ is not natural. □

Remark 5.4.15: The proof of Theorem 5.4.14 only requires that there is a free chain complex $F_\bullet$ with homologies $H_n(F_\bullet)\cong H_n(X)$ and such that $M\otimes_R F_\bullet$ is a chain complex with homologies $H_n(M\otimes_R F_\bullet) = H_n(X;M)$. Using the same arguments as in the proof of Theorem 5.4.14 one can show that there is a short exact sequence

$$0\to M\otimes_R H_k(X,A)\to H_k(X,A;M)\to \mathrm{Tor}_1^R(M, H_{k-1}(X,A))\to 0$$

for the relative homologies and a short exact sequence

$$0\to M\otimes_R H_k(C_\bullet(X))\to H_k(C_\bullet(X);M)\to \mathrm{Tor}_R(M, H_{k-1}(C_\bullet(X)))\to 0.$$

for the cellular complex $C_\bullet(X)$ of a CW-complex X. This allows one to compute the relative and cellular homologies with coefficients in an R-module M.

Example 5.4.16: If R is a principal ideal ring and M a free module over R, then the short exact sequence in Theorem 5.4.14 reduces to

$$0\to M\otimes_R H_n(X)\to H_n(X;M)\to 0,$$

and we obtain $H_n(X;M)\cong M\otimes_R H_n(X)$. In particular, this holds for any module M over a field $R=\mathbb{F}$ and for $M=R$ as a module over itself.

Example 5.4.17: We determine $H_k(\mathbb{R}\mathrm{P}^n; M)$ for an abelian group M. By Theorem 5.4.14 there is a short exact sequence

$$0 \to M \otimes_{\mathbb{Z}} H_k(\mathbb{R}\mathrm{P}^n) \to H_k(X; M) \to \mathrm{Tor}_1^R(M, H_{k-1}(\mathbb{R}\mathrm{P}^n)) \to 0.$$

which splits and hence $H_k(\mathbb{R}\mathrm{P}^n; M) \cong (M \otimes_{\mathbb{Z}} H_k(\mathbb{R}\mathrm{P}^n))/\mathrm{Tor}_1^{\mathbb{Z}}(M, H_{k-1}(\mathbb{R}\mathrm{P}^n))$. By Example 5.3.8 the homologies for $R = \mathbb{Z}$ are given by

$$H_k(\mathbb{R}\mathrm{P}^n) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & k \text{ odd},\ 0 < k < n \\ \mathbb{Z} & k = n \text{ odd or } k = 0 \\ 0 & \text{else.} \end{cases}$$

For $k = 0$ we obtain the short exact sequence

$$0 \to M \otimes_{\mathbb{Z}} \mathbb{Z} \cong M \to H_0(\mathbb{R}P^n; M) \to \mathrm{Tor}_1^{\mathbb{Z}}(M, 0) \cong 0 \to 0$$

which implies $H_0(\mathbb{R}\mathrm{P}^n; M) \cong M \otimes_{\mathbb{Z}} \mathbb{Z} \cong M$. For k odd and $0 < k < n$ we have $H_k(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z}/2\mathbb{Z}$, $H_{k+1}(\mathbb{R}\mathrm{P}^n) = 0$ and $H_{k-1}(\mathbb{R}P^n)$ is torsion free. This yields the exact sequence

$$0 \to M \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \to H_k(\mathbb{R}\mathrm{P}^n; M) \to \mathrm{Tor}_1^{\mathbb{Z}}(M, H_{k-1}(\mathbb{R}\mathrm{P}^n)) \cong \mathrm{Tor}^{\mathbb{Z}}(M) \otimes \mathrm{Tor}^{\mathbb{Z}}(H_{k-1}(\mathbb{R}\mathrm{P}^n)) \cong 0 \to 0$$
$$0 \to M \otimes_{\mathbb{Z}} H_{k+1}(\mathbb{R}\mathrm{P}^p) \cong 0 \to H_{k+1}(\mathbb{R}\mathrm{P}^n; M) \to \mathrm{Tor}_1^{\mathbb{Z}}(M, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Tor}^{\mathbb{Z}}(M) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \to 0$$

This implies $H_k(\mathbb{R}P^n; M) \cong M \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z})$ and $H_{k+1}(\mathbb{R}\mathrm{P}^n, M) \cong \mathrm{Tor}^{\mathbb{Z}}(M) \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z})$ for $0 < k < n$ odd. For $k = n$ odd we obtain a short exact sequence

$$0 \to M \otimes_{\mathbb{Z}} \mathbb{Z} \cong M \to H_n(\mathbb{R}\mathrm{P}^n; M) \to \mathrm{Tor}_1^{\mathbb{Z}}(M, 0)(\mathbb{R}\mathrm{P}^n)) \cong 0 \to 0$$

and hence $H_n(\mathbb{R}\mathrm{P}^n; M) \cong M$. By combining these results, we obtain

$$H_k(\mathbb{R}\mathrm{P}^n; M) \cong \begin{cases} M & k = 0 \text{ or } k = n \text{ odd} \\ M \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z}) & 0 < k < n \text{ odd} \\ \mathrm{Tor}^{\mathbb{Z}}(M) \otimes (\mathbb{Z}/2\mathbb{Z}) & 0 < k \leq n \text{ even} \\ 0 & \text{else.} \end{cases}$$

For $M \cong \mathbb{Z}/2p\mathbb{Z}$ with $p \in \mathbb{N}$, this yields

$$H_k(\mathbb{R}\mathrm{P}^n; M) \cong \begin{cases} M & k = 0 \text{ or } k = n \text{ odd} \\ \mathbb{Z}/2\mathbb{Z} & 0 < k < n \text{ odd} \\ \mathbb{Z}/2\mathbb{Z} & 0 < k \leq n \text{ even,} \\ 0 & \text{else} \end{cases}$$

and for $M \cong \mathbb{Z}/(2p+1)\mathbb{Z}$ with $p \in \mathbb{N}$, we obtain

$$H_k(\mathbb{R}\mathrm{P}^n; M) \cong \begin{cases} M & k = 0 \text{ or } k = n \text{ odd} \\ 0 & \text{else.} \end{cases}$$

5.5 Exercises for Section 5

Exercise 1: Let $f:(X,A)\to(Y,B)$ be a homotopy equivalence between pairs of topological spaces such that $f|_A: A\to B$ is a homotopy equivalence. Show that $H_n(f): H_n(X,A)\to H_n(Y,B)$ is an isomorphism.

Exercise 2: Show that every continuous map $f:\mathbb{R}P^{2m}\to\mathbb{R}P^{2m}$, $m\in\mathbb{N}$, has a fix point.

Exercise 3: Show that for any continuous map $f: S^n\to S^n$ the map $f': D^{n+1}\to D^{n+1}$

$$f'(x)=\begin{cases}0 & x=0\\ ||x||\cdot f\left(\frac{x}{||x||}\right) & x\neq 0\end{cases}$$

is continuous and induces a continuous map $\tilde f: S^{n+1}\approx D^{n+1}/\partial D^{n+1}\to S^{n+1}\approx D^{n+1}/\partial D^{n+1}$ with $\deg(\tilde f)=\deg(f)$.

Exercise 4: A group action $\rhd: G\times X\to X$ of a group G on a topological space X is called **free** if the map $g\rhd - : X\to X$, $x\mapsto g\rhd x$ has no fix points for $g\in G\setminus\{e\}$.

(a) Show that a group action of a topological group G on S^n, $n\geq 1$, induces a group homomorphism $\phi: G\to\mathbb{Z}/2\mathbb{Z}$.
(b) Show that for $n\geq 1$ even the group $\mathbb{Z}_2$ is the only non-trivial group that acts freely on S^n.

Exercise 5: A continuous map $f: S^n\to S^n$ is called **even** if $f(-x)=f(x)$ for all $x\in S^n$. Prove the following claims:

(a) For n even: if f is even then $\deg(f)=0$.
(b) For n odd: if f is even then $\deg(f)$ even.
(c) For n odd: for any even $k\in\mathbb{Z}$ there is an even map $f: S^n\to S^n$ with $\deg(f)=k$.

Hint: Use that any even map $f: S^n\to S^n$ can be expressed as $f=h\circ\bar f$ with continuous maps $\bar f: S^n\to\mathbb{R}\mathrm{P}^n$ and $h:\mathbb{R}\mathrm{P}^n\to S^n$. Consider the associated morphism of abelian groups $H_n(f): H_n(S^n)\to H_n(S^n)$.

Exercise 6: Let $f: S^n\to S^n$ be a continuous map of degree $\deg(f)=m$ and $X=D^{n+1}\cup_f S^n$ the topological space obtained by attaching an $(n+1)$-cell to S^n with f. Compute the homologies of X.

Exercise 7: Let $(X,\cup_{n\geq 0}X^n)$ and $(Y,\cup_{n\geq 0}Y^n)$ be CW-complexes and $f: X\to Y$ a cellular map, a continuous map $f: X\to Y$ with $f(X^n)\subset Y^n$ for all $n\in\mathbb{N}_0$. Show that f induces a chain map $C_\bullet(f): C_\bullet(X)\to C_\bullet(Y)$ between the cellular complexes $C_\bullet(X)$ and $C_\bullet(Y)$ such that the following diagram commutes for all $n\in\mathbb{N}_0$

$$\begin{tikzcd} H_n(C_\bullet(X)) \arrow[r, "H_n(C_\bullet(f))"] \arrow[d, "\cong"] & H_n(C_\bullet(Y)) \arrow[d, "\cong"] \\ H_n(X) \arrow[r, "H_n(f)"'] & H_n(Y). \end{tikzcd}$$

Exercise 8: Compute the Euler-Poincaré characteristic of an oriented genus g surface with n points removed for $g, n \in \mathbb{N}_0$.

Exercise 9: Let $(X, \cup_{n\geq 0} X^n)$ be a CW-complex and R a principal ideal ring. Show that all homologies $H_n(X^n)$ are free modules over R.

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