Homological Algebra

Summer term 2023

Catherine Meusburger
Department Mathematik
Friedrich-Alexander-Universität Erlangen-Nürnberg

(Date: October 29, 2023)

To prepare this lecture I used the following textbooks that I recommend as reading:

- C. A. Weibel, An introduction to homological algebra, Cambridge Studies in Advanced Mathematics 38.
- S. Mac Lane, Homology, Springer, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen 114.
- P. J. Hilton, U. Stammbach: A Course in Homological Algebra, Springer Graduate Texts in Mathematics 4.
- A. Hatcher: Algebraic Topology, Cambridge University Press.
- K. Brown, Cohomology of Groups, Springer Graduate Texts in Mathematics 87.
- S. Mac Lane, Categories for the working mathematician, Springer Graduate Texts in Mathematics 5.

Acknowledgements

I am grateful to all students who improved this lecture with their questions and comments. I especially thank Miša Aleksić, Frank Alleborn, Martin Doß, Benedikt Fritz, Justin Gassner, Johannes Große, Karin Hoffmann, Nikolay Iakovlev, Finn Klein, Stefan Lippert, Fabio Lischka, Henrik Müller, Daniel Polster, Andreas Räthe, Tobias Simon, Jana Sommerrock, Sven Weiland, Nico Wittrock for questions and pointing out mistakes and typos.

Please send comments and remarks to

`catherine.meusburger@math.uni-erlangen.de`.

Contents

0 Introduction

Homological algebra is a method that plays an important role in many areas of mathematics. It associates to a mathematical object X (such as a topological space, an algebra or a group) a family $(X_n)_{n\in\mathbb{Z}}$ of abelian groups (possibly with additional structure) and a family of group homomorphisms $d_n : X_n \to X_{n-1}$ that satisfy $d_n \circ d_{n+1} = 0$ for all $n \in \mathbb{Z}$. This condition ensures that $\mathrm{im}(d_{n+1}) \subset \ker(d_n)$ is a subgroup. The factor groups $H_n(X) = \ker(d_n)/\mathrm{im}(d_{n+1})$ for $n \in \mathbb{Z}$ are called the *homologies* of X. There is also a dual concept, *cohomology*, where one has a family $(X^n)_{n\in\mathbb{Z}}$ of abelian groups and a family $(d^n)_{n\in\mathbb{Z}}$ of group homomorphisms $d^n : X^n \to X^{n+1}$ with $d^n \circ d^{n-1} = 0$ for all $n \in \mathbb{Z}$. The factor groups $H^n(X) = \ker(d^n)/\mathrm{im}(d^{n-1})$ are called the *cohomologies* of X.

(Co)homologies arise in algebraic topology, including (co)homologies of simplicial complexes, of CW-complexes and singular (co)homologies of topological spaces. They also arise in algebra, where one considers (co)homologies of algebras, groups, bimodules, group representations and representations of Lie algebras, and in geometry, where one considers (co)homologies associated with differential forms on smooth manifolds, with symplectic manifolds and with more specific questions such as intersections of smooth curves on surfaces.

(Co)homologies encode relevant information about the object X, such as the number of connected components of a topological space, the centre of an algebra or the invariants of a group representation. They are often used to establish that two mathematical objects are not isomorphic. Non-trivial (co)homology groups also appear as *obstructions* to certain constructions or to attempts to decompose structures into simpler building blocks. For instance, group cohomologies tell us if a group G can be written as a semidirect product of a normal subgroup $N \subset G$ and the factor group G/N, and (co)homologies of algebras can answer the question if the algebra or a given module is semisimple.

Some of the reasons why homological methods are so common and useful are the following:

1. (Co)homologies are based on *linear* structures, namely modules over rings and certain generalisations thereof. This makes them much more computable and accessible than non-linear structures. Modules over rings include vector spaces, abelian groups and representations of groups and algebras and are a very versatile and general structure.

2. There are infinitely many (co)homologies associated with a mathematical object. This allows them to contain enough information, and this information is organised in an efficient way. (Co)homologies for different values of n can be computed independently from each other. In many cases, all (co)homologies $H_n(X)$ for $n < 0$ vanish, and their complexity grows with increasing n. If one requires only little information, it is often sufficient to compute the first few homologies. In some cases, there are algorithms that compute homologies directly from the data that describes the mathematical structure.

3. Homology theories are general and flexible. They can be formulated abstractly in terms of categories, functors and natural transformations. This allows one to apply them in many situations. There is a good understanding of what data is needed to define a (co)homology theory, and this data can often be characterised by standard constructions. This allows one to easily find new applications and to treat them systematically. One can easily transfer them between different parts of mathematics and adapt methods and insights to other contexts. The wish to have a unified framework for different versions of (co)homology theories was an important motivation in the development of category theory.

1 Algebraic background

1.1 Modules over rings

Modules over unital rings are one of the essential ingredients of (co)homology theories. They are useful because they unify different algebraic structures such as abelian groups, commutative rings, vector spaces over fields and representations of groups and algebras, which are all used to define different versions of (co)homology theories. By working with modules, we can relate these different notions of (co)homologies and treat them in a common framework. It also becomes apparent which properties are general and which depend on the choice of the underlying ring.

In this section we summarise the basic constructions and results for modules over rings. Unless stated otherwise all rings are assumed to be unital in the following, and all ring homomorphisms are assumed to be unital as well, i. e. send multiplicative units to multiplicative units.

Definition 1.1.1: Let R be a ring.

1. A **(left) module** over R or an R**-(left) module** is an abelian group $(M,+)$ together with a map $\rhd : R\times M\to M$, $(r,m)\mapsto r\rhd m$, the **structure map** that satisfies for all $m,m'\in M$ and $r,r'\in R$
$$r\rhd(m+m')=r\rhd m+r\rhd m' \qquad (r+r')\rhd m=r\rhd m+r'\rhd m$$
$$(r\cdot r')\rhd m=r\rhd(r'\rhd m) \qquad 1\rhd m=m.$$
2. A **morphism of** R**-modules** or an R**-linear map** from an R-module $(M,+_M,\rhd_M)$ to an R-module $(N,+_N,\rhd_N)$ is a group homomorphism $\phi:(M,+_M)\to(N,+_N)$ that satisfies
$$\phi(r\rhd_M m)=r\rhd_N\phi(m) \qquad \forall m\in M, r\in R.$$
A bijective R-module morphism $f:M\to N$ is called a R-module **isomorphism**, and one writes $M\cong N$. The set of R-module morphisms $\phi:M\to N$ is denoted $\mathrm{Hom}_R(M,N)$.

Remark 1.1.2: Let R,S be rings.

1. A **right module** over R is a left module over the ring R^{op} with the opposite multiplication $r\cdot_{op}s=s\cdot r$ and a morphism of R-right modules is a morphism of R^{op}-left modules.
Equivalently, we can define a right module over R as an abelian group $(M,+)$ together with a map $\lhd: M\times R\to M$, $(m,r)\mapsto m\lhd r$, such that for all $m,m'\in M$ and $r,r'\in R$:
$$(m+m')\lhd r=m\lhd r+m'\lhd r \qquad m\lhd(r+r')=m\lhd r+m\lhd r'$$
$$m\lhd(r\cdot r')=(m\lhd r)\lhd r' \qquad m\lhd 1=m.$$
If R is commutative, left and right modules over R coincide.
2. An (R,S)-**bimodule** is an abelian group $(M,+)$ with an R-left module structure $\rhd: R\times M\to M$ and an S-right module structure $\lhd: M\times S\to M$ such that $r\rhd(m\lhd s)=(r\rhd m)\lhd s$ for all $r\in R$, $s\in S$ and $m\in M$.

Example 1.1.3: Let R, S be rings.

1. $\mathbb{Z}$-modules are abelian groups and $\mathbb{Z}$-linear maps are group homomorphisms.

   Every abelian group $(M, +)$ has a unique $\mathbb{Z}$-module structure determined by $0 \triangleright m = 0$ and $1 \triangleright m = m$ for all $m \in M$. The additivity of the structure map in the first argument determines the $\mathbb{Z}$-module structure uniquely since for all $m \in M$ and $n \in \mathbb{N}$ one has

   $$\begin{aligned}
   &0 \triangleright m = (0+0) \triangleright m = 0 \triangleright m + 0 \triangleright m \quad \Rightarrow \quad 0 \triangleright m = 0 \\
   &n \triangleright m = (1 + ... + 1) \triangleright m = 1 \triangleright m + ... + 1 \triangleright m = m + ... + m \\
   &0 = 0 \triangleright m = (n-n) \triangleright m = m + ... + m + (-n) \triangleright m \quad \Rightarrow \quad (-n) \triangleright m = -(m + ... + m).
   \end{aligned}$$

2. Modules over a field $\mathbb{F}$ are $\mathbb{F}$-vector spaces and $\mathbb{F}$-module morphisms are $\mathbb{F}$-linear maps.

3. R is a left module over itself with $\triangleright : R \times R \to R$, $r \triangleright r' = r \cdot r'$ and a right module over itself with $\triangleleft : R \times R \to R$, $r' \triangleleft r = r' \cdot r$. This gives R the structure of an (R, R)-bimodule.

4. For any set X and R-module M, the set $\mathrm{Map}(X, M)$ of maps $f : X \to M$ has a canonical R-left module structure given by

   $$(f+g)(x) = f(x) + g(x), \quad (r \triangleright f)(x) = r \triangleright f(x) \qquad \forall x \in X, f, g : X \to M, r \in R.$$

5. For any S-module M and R-module N, the set $\mathrm{Hom}_{\mathbb{Z}}(M, N)$ of group homomorphisms $f : M \to N$ has a canonical R-left module and S-right module structure given by

   $$(f+g)(m) = f(m) + g(m) \qquad (r \triangleright f)(m) = r \triangleright f(m) \qquad (f \triangleleft s)(m) = f(s \triangleright m)$$

   for all $m \in M$, $r \in R$, $s \in S$ and $f, g \in \mathrm{Hom}_{\mathbb{Z}}(M, N)$. This gives $\mathrm{Hom}_{\mathbb{Z}}(M, N)$ the structure of an (R, S)-bimodule.

6. If $\phi : R \to S$ is a ring homomorphism, then every S-module M becomes an R-module with structure map $\triangleright_R : R \times M \to M$, $r \triangleright m = \phi(r) \triangleright_S m$. This is called the **pullback of the module structure** along ϕ.

   In particular, there is a unique ring homomorphism $\phi : \mathbb{Z} \to S$ given by $\phi(0) = 0_S$ and $\phi(1) = 1_S$. The induced $\mathbb{Z}$-module structure on M is precisely its abelian group structure.

7. Every R-module M is a module over the endomorphism ring $\mathrm{End}_R(M) = \mathrm{Hom}_R(M, M)$ with the evaluation map $\triangleright : \mathrm{End}_R(M) \times M \to M$, $(f, m) \mapsto f(m)$.

8. If R is an algebra over a field $\mathbb{F}$, then an R-module M is a **representation** of R:
   an $\mathbb{F}$-vector space M together with an algebra homomorphism $\rho : R \to \mathrm{End}_{\mathbb{F}}(M)$.

   Morphisms of R-modules are **homomorphisms of representations**:
   $\mathbb{F}$-linear maps $\phi : M \to N$ with $\rho_N(r) \circ \phi = \phi \circ \rho_M(r)$ for all $r \in R$.

   The scalar multiplication on M is given by $\lambda m = (\lambda 1_R) \triangleright M$ for $\lambda \in \mathbb{F}$ and the algebra homomorphism $\rho : R \to \mathrm{End}_{\mathbb{F}}(M)$ by $\rho(r)m = r \triangleright m$ for all $r \in R$ and $m \in M$. It is $\mathbb{F}$-linear since $\rho(\lambda r)m = (\lambda r) \triangleright m = (\lambda 1_R) \triangleright (r \triangleright m) = \lambda \rho(r) m$ for $\lambda \in \mathbb{F}$. Conversely, every algebra homomorphism $\rho : R \to \mathrm{End}_{\mathbb{F}}(M)$ defines an R-module structure on M by $r \triangleright m = \rho(r)m$ for all $r \in R$ and $m \in M$.

The concept of algebra representations is important, since it allows one to describe algebras in terms of vector spaces and linear maps and to use techniques from linear algebra to understand their structure. There is an analogous concept of group representations.

Definition 1.1.4: Let G be a group.

1. A **representation** of G over a field $\mathbb{F}$ is a vector space M over $\mathbb{F}$ together with a group homomorphism $\rho : G \to \mathrm{Aut}_{\mathbb{F}}(M)$ into the group of linear automorphisms of M.

2. A **homomorphism of group representations** from (M, ρ_M) to (N, ρ_N) is an $\mathbb{F}$-linear map $\phi : M \to N$ with $\rho_N(g) \circ \phi = \phi \circ \rho_M(g)$ for all $g \in G$.

Group representations are important for the same reasons as representations of algebras, namely that they allow one to investigate groups with methods from linear algebra. It is therefore desirable to also incorporate group representations in the picture and to view them as modules over suitable rings. The relevant rings are the so-called group rings.

Lemma 1.1.5: Let G be a group with unit e, R a ring and $R[G]$ the set of maps $f : G \to R$ with $f(g) = 0$ for almost all $g \in G$. Then $R[G]$ is a ring with the pointwise addition of maps and the **convolution product** $\star : R[G] \times R[G] \to R[G]$

$$f_1 \star f_2(g) = \sum_{g_1 \cdot g_2 = g} f_1(g_1) \cdot f_2(g_2) = \sum_{h \in G} f_1(h) \cdot f_2(h^{-1} \cdot g).$$

The unit element is the map $\delta_e : G \to R$ with $\delta_e(e) = 1_R$ and $\delta_e(g) = 0$ for $g \neq e$.

The ring $R[G]$ is called the **group ring** of G over R. If $R = \mathbb{F}$ is a field, then $\mathbb{F}[G]$ is an algebra over $\mathbb{F}$ with the pointwise scalar multiplication and called the **group algebra** of G.

Proof:
Exercise. $\square$

Remark 1.1.6: Every map $f : G \to R$ with $f(g) = 0$ for almost all $g \in G$ can be expressed uniquely as a finite linear combination $f = \Sigma_{g \in G} f(g)\, \delta_g$, where $\delta_g : G \to R$ are the maps with $\delta_g(g) = 1_R$ and $\delta_g(h) = 0_R$ for $g \neq h$. Their convolution product takes the form $\delta_g \star \delta_h = \delta_{gh}$. With the notation $f = \Sigma_{g \in G}\, r_g\, g$ instead of $f = \Sigma_{g \in G}\, r_g\, \delta_g$ one has

$$(\Sigma_{g \in G}\, r_g\, g) \star (\Sigma_{h \in G}\, s_h\, h) = \Sigma_{g,h \in G}\, r_g s_h\, (gh).$$

The notion of the group ring allows us to view group representations over a field $\mathbb{F}$ as modules over the group algebra $\mathbb{F}[G]$. In this case, the algebra homomorphism $\rho : \mathbb{F}[G] \to \mathrm{End}_{\mathbb{F}}(M)$ from Example 1.1.3, 8. restricts to a group homomorphism $\rho : G \to \mathrm{Aut}_{\mathbb{F}}(M)$. This follows because the elements $g \in \mathbb{F}[G]$ have multiplicative inverses and $\rho(g^{-1}) \circ \rho(g) = \rho(g^{-1} \cdot g) = \rho(e) = \mathrm{id}_M$.

Example 1.1.7: Let G be a group and $\mathbb{F}$ a field. Then modules over the group algebra $\mathbb{F}[G]$ are the representations of G. Homomorphisms of $\mathbb{F}[G]$-modules are the homomorphisms of group representations.

After unifying known algebraic concepts into the notion of a module over a ring, we now generalise the basic constructions for vector spaces - linear subspaces, quotients, direct sums, products and tensor products - to this setting. This leads to the notions of submodules, quotients, direct sums, products and tensor products of modules, which are direct analogues of the corresponding concepts for vector spaces.

Definition 1.1.8: Let R be a ring and M a module over R. A **submodule** of M is a subgroup $N \subset M$ that is closed under the operation of R: $r \rhd n \in N$ for all $r \in R$ and $n \in N$.

Example 1.1.9:

1. For any R-module M, the trivial module $\{0\} \subset M$ and $M \subset M$ are submodules. All other submodules are called **proper submodules**.

2. For any module morphism $\phi : M \to N$, the kernel $\ker(\phi) = \{m \in M \mid \phi(m) = 0\} \subset M$ and the image $\mathrm{im}(\phi) = \{\phi(m) \mid m \in M\} \subset N$ are submodules.

3. If M is an abelian group, i. e. a $\mathbb{Z}$-module, then submodules of M are precisely the subgroups of M.

4. Submodules of modules over a field $\mathbb{F}$ are linear subspaces.

5. Submodules of a ring R as a left (right) module over itself are its left (right) ideals.

With the notion of a submodule, we can also generalise the notion of a quotient to modules. Any submodule $N \subset M$ of an R-module M is a subgroup of the abelian group M. Consequently, the factor group M/N, whose elements are the cosets $mN = \{m + n \mid n \in N\}$, is an abelian group with addition $(mN)+(m'N) = (m+m')N$, and the canonical surjection $\pi : M \to M/N$, $m \mapsto mN$ is a group homomorphism. It is then natural to define an R-module structure on M/N in such a way that the canonical surjection becomes an R-module morphism, i. e. to set $r \rhd (mN) := (r \rhd m)N$ for all $r \in R$ and $m \in M$.

Definition 1.1.10: Let M be a module over a ring R and $N \subset M$ a submodule. The **quotient module** M/N is the factor group M/N with the canonical R-module structure

$$\rhd : R \times M/N \to M/N, \quad r \rhd (mN) \mapsto (r \rhd m)N.$$

Remark 1.1.11:

1. The quotient module structure on M/N is the unique R-module structure on the abelian group M/N with the following **characteristic property**:

   The canonical surjection $\pi : M \to M/N$ is an R-module morphism. For any module morphism $\phi : M \to M'$ with $N \subset \ker(\phi)$, there is a unique module morphism $\tilde{\phi} : M/N \to M'$ such that the following diagram commutes

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[d, "\pi"'] & M' \\ M/N. \arrow[ur, dashed, "\exists! \tilde{\phi}"'] & \end{tikzcd}$$

2. If $\phi : M \to N$ is a morphism of R-modules, then we have a canonical isomorphism of R-modules $\phi : M/\ker(\phi) \to \mathrm{im}(\phi)$, $m + \ker(\phi) \mapsto \phi(m)$.

3. If M is a module over R with submodules $U \subset V \subset M$ then V/U is a submodule of M/U, and there is a canonical R-module isomorphism $(M/U)/(V/U) \to M/V$.

The concepts of a direct sum and a product of vector spaces also have direct generalisations to modules. Their construction neither makes use of the commutativity of the field nor of the existence of multiplicative inverses. By replacing the scalar multiplication by the structure map of a module, we obtain their to modules over rings.

Definition 1.1.12: Let R be a ring and $(M_i)_{i\in I}$ a family of modules over R indexed by a set I. Then the **direct sum** $\oplus_{i\in I} M_i$ and the **product** $\Pi_{i\in I} M_i$ are the sets

$$\begin{aligned}\oplus_{i\in I} M_i &= \{(m_i)_{i\in I} :\ m_i \in M_i, m_i = 0 \text{ for almost all } i \in I\}\\ \Pi_{i\in I} M_i &= \{(m_i)_{i\in I} :\ m_i \in M_i\}\end{aligned}$$

with the R-module structures given by

$$(m_i)_{i\in I} + (m'_i)_{i\in I} := (m_i + m'_i)_{i\in I} \qquad r \rhd (m_i)_{i\in I} := (r \rhd m_i)_{i\in I}.$$

Lemma 1.1.13: The direct product and the direct sum of modules are products and coproducts in the category R-Mod. More precisely:

1. **Universal property of direct sums:**
   The direct sum module structure is the unique R-module structure on $\oplus_{i\in I} M_i$ for which all inclusions $\iota_i : M_i \to M$, $m \mapsto (\delta_{ji} m)_{j\in I}$ are module morphisms.

   For a family $(\phi)_{i\in I}$ of module morphisms $\phi_i : M_i \to N$ there is a unique module morphism $\phi : \oplus_{i\in I} M_i \to N$ such that the following diagram commutes for all $i \in I$

$$\begin{tikzcd} M_i \arrow[r, "\phi_i"] \arrow[d, "\iota_i"'] & N \\ \oplus_{j\in I} M_j. \arrow[ur, dashed, "\exists!\phi"'] & \end{tikzcd}$$

2. **Universal property of products:**
   The product module structure is the unique R-module structure on $\Pi_{i\in I} M_i$ for which all projection maps $\pi_i : \Pi_{i\in I} M_i \to M_i$, $(m_j)_{j\in I} \mapsto m_i$ are module morphisms.

   For a family $(\psi)_{i\in I}$ of module morphisms $\psi_i : L \to M_i$ there is a unique module morphism $\psi : L \to \Pi_{i\in I} M_i$ such that the following diagram commutes for all $i \in I$

$$\begin{tikzcd} M_i & L \arrow[l, "\psi_i"'] \arrow[dl, dashed, "\exists!\psi"] \\ \Pi_{j\in I} M_j. \arrow[u, "\pi_i"] & \end{tikzcd}$$

While the four basic constructions for modules are straightforward generalisations of the corresponding constructions for vector spaces, there is a fundamental difference between vector spaces and modules over general rings, namely the existence of *bases* and of *complements*. While every vector space has a basis and every linear subspace has a complement, this does not hold for modules. Although there are always generating sets, there need not be a *linearly independent* generating set. In contrast to vector spaces general modules can therefore not be described in terms of bases but are characterised by *presentations*.

Definition 1.1.14: Let R be a ring, M an R-module and $A \subset M$ a subset.

1. The **submodule** $\langle A\rangle_M$ **generated by** A is the smallest submodule of M containing A
$$\langle A\rangle_M = \bigcap_{\substack{N\subset M \\ \text{submodule,} \\ A\subset N}} N = \{\Sigma_{a\in A} r_a \rhd a : \; r_a \in R, r_a = 0 \text{ for almost all } a \in A\}.$$
2. The subset $A \subset M$ is called a **generating set** of M if $\langle A\rangle_M = M$. It is called a **basis** of M if it is a generating set and **linearly independent**:

   $\Sigma_{a\in A} r_a \rhd a = 0$ with $r_a \in R$ and $r_a = 0$ for almost all $a \in A$ implies $r_a = 0$ for all $a \in A$.
3. An R-module with a finite generating set is called **finitely generated**. An R-module with a generating set that contains only one element is called **cyclic**. An R-module with a basis is called **free**.
4. The **free** R**-module** generated by a set A is the direct sum $\langle A\rangle_R = \oplus_{a\in A} R$. Equivalently, it can be characterised as the set $\langle A\rangle_R = \{f : A \to R : \; f(a) = 0 \text{ for almost all } a \in A\}$ with the canonical R-module structure
$$(f + g)(a) = f(a) + g(a) \qquad (r \rhd f)(a) = r \cdot f(a) \qquad \forall f, g \in \langle A\rangle_R, r \in R, a \in A.$$
   The maps $\delta_a : A \to R$ with $\delta_a(a) = 1_R$ and $\delta_a(a') = 0_R$ for $a' \neq a$ form a basis of $\langle A\rangle_R$, since every map $f : A \to R$ with $f(a) = 0$ for almost all $a \in A$ can be expressed as a finite R-linear combination $f = \sum_{a\in A} f(a) \rhd \delta_a$.
5. For a subset $B \subset \langle A\rangle_R$, we denote by $\langle A|B\rangle_R$ the quotient module
$$\langle A|B\rangle_R = \langle A\rangle_R / \langle B\rangle_{\langle A\rangle_R}$$
   of the free R-module generated by A with respect to its submodule generated by B. If $M = \langle A|B\rangle_R$, then $\langle A|B\rangle_R$ is called a **presentation** of M, the elements of A are called **generators** and the elements of B **relations**.

Remark 1.1.15:

1. Every module has a presentation $M = \langle A|B\rangle_R$. One can choose $A = M$ and $B = \ker(\tau)$ for the R-module homomorphism $\tau : \langle M\rangle_R \to M$, $\Sigma_{m\in M} r_m \rhd \delta_m \mapsto \Sigma_{m\in M} r_m \rhd m$. However, this presentation is not very useful in practice. One usually looks for presentations that have as few generators and relations as possible.
2. Presentations of modules are characterised by a **universal property**:

   For any R-module M and any map $\phi : A \to M$, there is a unique R-module homomorphism $\phi' : \langle A\rangle_R \to M$ with $\phi'|_A = \phi$. If $B \subset \ker(\phi')$, then by the universal property of the quotient there is a unique map $\phi'' : \langle A|B\rangle_R \to M$ with $\phi'' \circ \pi = \phi'$, where $\pi : \langle A\rangle_R \to \langle A|B\rangle_R$ is the canonical surjection.
3. If R is a commutative ring and M a free R-module, then all bases of M have the same cardinality. This number is called **rank** of M and denoted $\mathrm{rk}(M)$. This notion makes no sense for non-commutative rings since one can have $R^n \cong R^m$ as R-modules for $n \neq m$ (Exercise 4).

Example 1.1.16:

1. Every ring R is a cyclic free module as a left or right module over itself: $R = \langle 1_R \rangle_R$.

2. If $\mathbb{F}$ is a field, then every module over $\mathbb{F}$ is free, since a module over $\mathbb{F}$ is a vector space, and every vector space has a basis. The cyclic $\mathbb{F}$-modules are precisely the one-dimensional vector spaces over $\mathbb{F}$.

3. If M is a free module over a principal ideal domain R, then every submodule $U \subset M$ is free with $\mathrm{rk}(U) \leq \mathrm{rk}(M)$. (For a proof, see [JS]).

4. The $\mathbb{Z}$-module $M = \mathbb{Z}/n\mathbb{Z}$ for $n \in \mathbb{N}$ is cyclic, since $\{\bar{1}\}$ is a generating set, but it is not free. Any generating set of $\mathbb{Z}/n\mathbb{Z}$ must contain at least one element $\bar{k} \neq \bar{0}$, but $n \rhd \bar{k} = \bar{k} + \ldots + \bar{k} = \overline{nk} = \bar{0}$, and hence the generating set cannot be linearly independent. A presentation of the $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$ is given by $\langle A \,|\, B \rangle_{\mathbb{Z}} = \langle 1 | n \rangle$.

For vector spaces, an important consequence of the existence of bases is the existence of a *complement* for any linear subspace $U \subset V$ - a linear subspace $W \subset V$ with $V = U \oplus W$. Such a complement can be constructed by completing a basis of U to a basis of V and taking W as the span of those basis elements that are not contained in the basis of U. As modules over general rings do not need to have bases, this construction does not generalise to rings.

Indeed, there are many examples of submodules without complements. Consider for instance the submodule $n\mathbb{Z} \subset \mathbb{Z}$ of the ring $\mathbb{Z}$ as a module over itself for $n \geq 2$. As $1 \notin n\mathbb{Z}$, any complement M of $n\mathbb{Z}$ would need to contain the element $1 \in \mathbb{Z}$ and hence be equal to $\mathbb{Z}$ since $1 \in M$ implies $n = n \rhd 1 \in M$ for all $n \in \mathbb{Z}$. The same argument shows that a proper submodule of a cyclic module can never have a complement. Sufficient conditions that ensure that a submodule has a complement are the following.

Lemma 1.1.17: Let R be a ring and M a module over R.

1. If $\phi : M \to F$ is a surjective R-module morphism into a free R-module F, then there is an R-module morphism $\psi : F \to M$ with $\phi \circ \psi = \mathrm{id}_F$ and $M \cong \mathrm{im}(\psi) \oplus \ker(\phi)$.
   One says that ψ **splits** the module morphism $\phi : M \to F$.

2. If $N \subset M$ is a submodule such that M/N is free, then there is a submodule $P \subset M$ with $P \cong M/N$ and $M \cong N \oplus P$.

Proof:
1. Choose a basis B of F and for every $b \in B$ an element $m_b \in \phi^{-1}(b) \subset M$. Define the R-module morphism $\psi : F \to M$ by $\psi(b) = m_b$ and R-linear extension to F. Then we have $m = \psi \circ \phi(m) + (m - \psi \circ \phi(m))$ for all $m \in M$ with $\psi \circ \phi(m) \in \mathrm{im}(\psi)$ and $m - \phi \circ \psi(m) \in \ker(\phi)$, since $\phi \circ \psi = \mathrm{id}_F$ implies $\phi(m - \psi \circ \phi(m)) = \phi(m) - (\phi \circ \psi)(\phi(m)) = \phi(m) - \phi(m) = 0$. As $\phi \circ \psi = \mathrm{id}_F$, we have $\ker(\phi) \cap \mathrm{im}(\psi) = \{0\}$ and hence $M = \ker(\phi) \oplus \mathrm{im}(\psi)$.

2. By 1. there is a R-module morphism $\psi : M/N \to M$ which splits the surjective module morphism $\pi : M \to M/N$ and hence $M \cong \ker(\pi) \oplus \mathrm{im}(\psi) \cong N \oplus \mathrm{im}(\psi)$. The R-module morphism $\pi|_{\mathrm{im}(\psi)} : \mathrm{im}(\psi) \to M/N$ is surjective by definition and injective since $\pi \circ \psi = \mathrm{id}_{M/N}$, hence an isomorphism. $\square$

The fact that there are R-modules M without bases is closely related to the presence of elements $m \in M$ for which there is an $r \in R \setminus \{0\}$ with $r \rhd m = 0$, the so-called *torsion elements*. It is clear that an element of a basis can never be a torsion element. Under certain assumptions on the ring, this holds for all non-zero elements of a free module.

Definition 1.1.18: Let R be a ring and M an R-module. An element $m \in M$ is called a **torsion element** if there is an $r \in R \setminus \{0\}$ with $r \rhd m = 0$. The set of torsion elements in M is denoted $\mathrm{Tor}_R(M)$. The R-module M is called **torsion free** if $\mathrm{Tor}_R(M) = 0$.

Example 1.1.19:

1. Any free module M over an integral domain R is torsion free.

   This follows because every torsion element $m \in M$ can be expressed as a finite linear combination $m = \Sigma_{i\in I} r_i \rhd m_i$ of basis elements m_i. The condition $r \rhd m = \Sigma_{i\in I}(rr_i) \rhd m_i = 0$ for $r \in R \setminus \{0\}$ then implies $rr_i = 0$ for all $i \in I$. Because R has no zero divisors and $r \in R \setminus \{0\}$, it follows that $r_i = 0$ for all $i \in I$ and hence $m = 0$.

2. For a commutative ring k as a module over itself, the torsion elements are precisely the zero divisors of k. This implies that every integral domain R as a module over itself is torsion free. In particular, this holds for $\mathbb{Z}$, for any field $\mathbb{F}$ and for the ring $I[X]$ of polynomials over any integral domain I.

3. In the $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$, every element is a torsion element since $n \rhd \overline{k} = \overline{n \cdot k} = \overline{0}$ for all $k \in \mathbb{Z}$. The ring $\mathbb{Z}/n\mathbb{Z}$ as a module over itself is torsion free if and only if n is a prime:

$$\mathrm{Tor}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z} \qquad \mathrm{Tor}_{\mathbb{Z}/n\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}) = \{\bar{k} \mid k \in \mathbb{Z},\, \gcd(k, n) > 1\}.$$

One may expect that the set of torsion elements in an R-module M is a submodule of M. However, this need not hold in general if R is non-commutative or has zero divisors. However, if R is an integral domain, the torsion elements form a submodule, and by quotienting it out, one obtains a module that is torsion free.

Lemma 1.1.20: If M is a module over an integral domain R then $\mathrm{Tor}_R(M) \subset M$ is a submodule and the module $M/\mathrm{Tor}_R(M)$ is torsion free.

Proof:
Let $m, m' \in \mathrm{Tor}_R(M)$ torsion elements and $r, r' \in R \setminus \{0\}$ with $r \rhd m = r' \rhd m' = 0$. Then $(r\cdot r') \rhd (m+m') = r' \rhd (r \rhd m) + r \rhd (r' \rhd m') = 0$. As R is an integral domain, $r \cdot r' \neq 0$ and hence $m + m' \in \mathrm{Tor}_R(M)$. Similarly, for all $s \in R$, one has $r \rhd (s \rhd m) = (r \cdot s) \rhd m = s \rhd (r \rhd m) = 0$, which implies $s \rhd m \in \mathrm{Tor}_R(M)$, and hence $\mathrm{Tor}_R(M) \subset M$ is a submodule. If $[m] \in M/\mathrm{Tor}_R(M)$ is a torsion element, then there is an $r \in R \setminus \{0\}$ with $r \rhd [m] = [r \rhd m] = 0$. This implies $r \rhd m \in \mathrm{Tor}_R(M)$, and there is an $r' \in R \setminus \{0\}$ with $r' \rhd (r \rhd m) = (r \cdot r') \rhd m = 0$. As R is an integral domain, one has $r \cdot r' \neq 0$, which implies $m \in \mathrm{Tor}_R(M)$ and $[m] = 0$. □

Since $\mathrm{Tor}_R(M) \subset M$ is a submodule and $M/\mathrm{Tor}_R(M)$ is torsion free for any integral domain R, it is natural to ask if the torsion submodule $\mathrm{Tor}_R(M)$ has a complement. A sufficient condition for this this is that R is a principal ideal domain and M is finitely generated. In this case, the classification theorem for finitely generated modules over principal ideal domains allows one to identify the torsion elements and their complement. In particular, this applies to finitely generated abelian groups, the finitely generated modules over the principal ideal domain $\mathbb{Z}$.

Lemma 1.1.21: Let R be a principal ideal domain. Then every finitely generated R-module M is of the form $M \cong \mathrm{Tor}_R(M) \oplus R^n$ with a unique $n \in \mathbb{N}_0$. In particular, every finitely generated torsion free R-module is free.

Proof:
The classification theorem for finitely generated modules over principal ideal domain states that every such module is of the form $M \cong R^n \times R/q_1R \times \ldots \times R/q_mR$ with prime powers $q_1, ..., q_m \in R$ and $n \in \mathbb{N}_0$. Every element $m \in R/q_1R \times \ldots \times R/q_lR$ is a torsion element since $(q_1 \cdots q_m) \rhd m = 0$. This shows that $\mathrm{Tor}_R(M) \supset R/q_1R \times \ldots \times R/q_mR$.

Conversely, any element $m \in R^n \times R/q_1R \times \ldots \times R/q_mR$ is a sum $m = \iota_1(m_1) + \iota_2(m_2)$ with $m_1 \in R^n$ and and $m_2 \in R/q_1R \times \ldots \times R/q_mR$, where $\iota_1 : R^n \to R^n \times R/q_1R \times \ldots \times R/q_mR$ and $\iota_2 : R/q_1R \times \ldots \times R/q_mR \to R^n \times R/q_1R \times \ldots \times R/q_mR$ denote the inclusion maps for the direct sum[1]. Then $0 = r \rhd (\iota_1(m_1) + \iota_2(m_2)) = \iota_1(r \rhd m_1) + \iota_2(r \rhd m_2)$ for $r \in R \setminus \{0\}$ implies $r \rhd m_1 = 0$ and $r \rhd m_2 = 0$. As R^n is a free R-module, it is torsion free by Example 4.4.7, 1. and the first condition implies $m_1 = 0$. This shows that $\mathrm{Tor}_R(M) \subset R/q_1R \times \ldots \times R/q_mR$.
□

After generalising four basic constructions for vector spaces to modules over rings and discussing the existence of bases, we will now focus on the last essential construction, namely tensor products. The construction of the tensor product for modules over rings is similar to the one for vector spaces. It is obtained as a quotient of a free module generated by the cartesian products of the underlying sets with respect to certain relations. However, if R is non-commutative, we have to consider a left and a right module over a ring R to form a tensor product, and the result is not an R-module but only an abelian group.

Definition 1.1.22: Let R be a ring, M an R-right module and N an R-left module. The **tensor product** $M \otimes_R N$ is the abelian group generated by the set $M \times N$ with relations

$$\begin{aligned} &(m, n) + (m', n) - (m + m', n), \quad && (m, n) + (m, n') - (m, n + n'), \\ &(m \lhd r, n) - (m, r \rhd n) && \forall m, m' \in M, n, n' \in N, r \in R. \end{aligned}$$

We denote by $m \otimes n = \tau(m, n)$ the images of the elements $(m, n) \in M \times N$ under the map $\tau = \pi \circ \iota : M \times N \to M \otimes_R N$, where $\iota : M \times N \to \langle M \times N \rangle_{\mathbb{Z}}$, $(m, n) \to (m, n)$ is the canonical inclusion and $\pi : \langle M \times N \rangle_{\mathbb{Z}} \to M \otimes_R N$ the canonical surjection.

Remark 1.1.23:

1. The set $\{m \otimes n : \ m \in M, n \in N\}$ generates $M \otimes_R N$, since the elements (m, n) generate the free $\mathbb{Z}$-module $\langle M \times N \rangle_{\mathbb{Z}}$ and the $\mathbb{Z}$-module morphism $\pi : \langle M \times N \rangle_{\mathbb{Z}} \to M \otimes_R N$ is surjective. The relations in Definition 1.1.22 induce the following identities in $M \otimes_R N$:

$$\begin{aligned} &(m + m') \otimes n = m \otimes n + m' \otimes n, \quad && m \otimes (n + n') = m \otimes n + m \otimes n', \\ &(m \lhd r) \otimes n = m \otimes (r \rhd n) && \forall m, m' \in M, n, n' \in N, r \in R. \end{aligned}$$

[1]Note that the product under consideration is a product over a *finite* index set and hence by Definition 1.1.12 it coincides with the direct sum.

2. If M is an R-right-module and N an (R,S)-bimodule, then $M \otimes_R N$ has a canonical S-right module structure given by $(m \otimes n) \lhd s := m \otimes (n \lhd s)$. Similarly, if M is a (Q,R)-bimodule and N an R-left module then $M \otimes_R N$ has a canonical Q-left module structure given by $q \rhd (m \otimes n) := (q \rhd m) \otimes n$.

3. As every left module over a commutative ring R is an (R,R)-bimodule, it follows from 2. that the tensor product $M \otimes_R N$ of modules over a commutative ring R has a canonical (R,R)-bimodule structure, given by

$$r \rhd (m \otimes n) = (r \rhd m) \otimes n = (m \lhd r) \otimes n = m \otimes (r \rhd n) = m \otimes (n \lhd r) = (m \otimes n) \lhd r.$$

4. As every module over a ring R is an abelian group and hence a $(\mathbb{Z},\mathbb{Z})$-bimodule, it is always possible to tensor two R-modules over the ring $\mathbb{Z}$. In this case, the last relation in Definition 1.1.22 is a consequence of the first two.

Example 1.1.24:

1. If $R = \mathbb{F}$ is a field, the tensor product of R-modules is the tensor product of vector spaces.

2. For any ring R and $R^k := R \oplus R \oplus \ldots \oplus R$, one has $R^m \otimes R^n \cong R^{nm}$ (Exercise).

3. If R is commutative and $R[X,Y]$ the polynomial ring over R in two variables X, Y, then $R[X] \otimes_R R[Y] \cong R[X,Y]$.

4. The tensor product of the abelian groups $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}/m\mathbb{Z}$ for $n, m \in \mathbb{N}$ is given by

$$\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/\mathrm{gcd}(m,n)\mathbb{Z}.$$

5. One has $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} \cong 0$. More generally, if R is an integral domain with associated quotient field $Q(R)$ and M an R-module, then $\mathrm{Tor}_R(M) \otimes_R Q(R) \cong 0$. This follows because for every torsion element $m \in M$ there is an $r \in R \setminus \{0\}$ with $r \rhd m = 0$, and this implies

$$m \otimes q = m \otimes (r \cdot q/r) = m \otimes (r \rhd (q/r)) = (m \lhd r) \otimes q/r = 0 \otimes q/r = 0 \quad \forall q \in Q.$$

Just as submodules, quotients, direct sums and products of modules, tensor products of R-modules can be characterised by a universal property. As tensor products are defined in terms of a presentation, this universal property is obtained by applying the one in Remark 1.1.15 to the relations in Definition 1.1.22. It is formulated in terms of bilinear maps $M \times N \to A$ into abelian groups A.

Definition 1.1.25: Let R be a ring, M an R-right module and N an R-left module. A map $f : M \times N \to A$ into an abelian group A is called R-**bilinear** if

$$\begin{aligned} &f(m+m', n) = f(m,n) + f(m',n), && f(m, n+n') = f(m,n) + f(m,n'), \\ &f(m \lhd r, n) = f(m, r \rhd n) && \forall m, m' \in M, n, n' \in N, r \in R. \end{aligned}$$

Lemma 1.1.26: Let R be a ring, M an R-right module and N an R-left module. Then the tensor product $M \otimes_R N$ has the following **universal property**:

The map $\tau : M \times N \to M \otimes_R N$, $(m,n) \mapsto m \otimes n$ is R-bilinear, and for any R-bilinear map $f : M \times N \to A$, there is a unique group homomorphism $f' : M \otimes_R N \to A$ with $f' \circ \tau = f$

$$\begin{tikzcd} M \times N \arrow[r, "f"] \arrow[d, "\tau"'] & A \\ M \otimes_R N. \arrow[ur, dashed, "\exists! f'"'] & \end{tikzcd}$$

Proof:
The first statement holds by definition, since the conditions in Definition 1.1.25 are the defining relations of the tensor product. To define f', note that $\langle M \times N \rangle_{\mathbb{Z}}$ is a free abelian group, and hence there is a unique group homomorphism $f'' : \langle M \times N \rangle_{\mathbb{Z}} \to A$ with $f''|_{M \times N} = f$. This is equivalent to the condition $f'' \circ \iota = f$, where $\iota : M \times N \to \langle M \times N \rangle_{\mathbb{Z}}$ is the canonical inclusion. The submodule $U \subset \langle M \times n \rangle_{\mathbb{Z}}$ spanned by the relations of the tensor product is contained in the kernel of f'' by R-bilinearity of f:

$$\begin{aligned} f''((m+m',n)-(m,n)-(m',n)) &= f(m+m',n)-f(m,n)-f(m',n)=0 \\ f''((m,n+n')-(m,n)-(m,n')) &= f(m,n+n')-f(m,n)-f(m,n')=0 \\ f''((m \lhd r,n)-(m,r \rhd n)) &= f(m \lhd r,n)-f(m,r \rhd n)=0. \end{aligned}$$

Hence, there is a unique group homomorphism $f' : M \otimes_R N \to A$ with $f' \circ \pi = f''$, where $\pi : \langle M \times N \rangle_Z \to M \otimes_R N$ is the canonical surjection. This implies $f' \circ \tau = f' \circ \pi \circ \iota = f'' \circ \iota = f$. The uniqueness of f' follows directly from the fact that τ is surjective. □

As the construction of the tensor product of modules is very similar to the tensor product of vector spaces, it has similar properties. They are direct consequences of its definition and its universal property.

Lemma 1.1.27: Let R, S be rings, I an index set, M, M_i R-right modules, N, N_i R-left modules for all $i \in I$, P a (R,S)-bimodule and Q an S-left module. Then:

1. **tensor products with the trivial module:** $0 \otimes_R N \cong M \otimes_R 0 \cong 0$,
2. **tensor product with the underlying ring:** $M \otimes_R R \cong M$, $R \otimes_R N \cong N$,
3. **direct sums:** $(\oplus_{i \in I} M_i) \otimes_R N \cong \oplus_{i \in I} M_i \otimes_R N$, $M \otimes_R (\oplus_{i \in I} N_i) \cong \oplus_{i \in I} M \otimes_R N_i$,
4. **associativity:** $(M \otimes_R P) \otimes_S Q \cong M \otimes_R (P \otimes_S Q)$.

Proof:
1. This follows directly from the relations of the tensor product in Remark 1.1.23, which imply $0 \otimes n = (0 \lhd 0) \otimes n = 0 \otimes 0 \rhd n = 0 \otimes 0$ for all $n \in N$ and $m \otimes 0 = 0$ for all $m \in M$.

2. We consider the group homomorphism $\phi : M \to M \otimes_R R$, $m \mapsto m \otimes 1$. The group homomorphism $\psi : M \otimes_R R \to M$, $m \otimes r \mapsto m \lhd r$ is an inverse of ϕ, since $\psi \circ \phi(m) = m \lhd 1 = m$ and $\phi \circ \psi(m \otimes r) = (m \lhd r) \otimes 1 = m \otimes (r \rhd 1) = m \otimes r$. The proof for $R \otimes_R N \cong N$ is analogous.

3. Consider the group homomorphisms $\phi_i : M_i \otimes_R N \to (\oplus_{i \in I} M_i) \otimes_R N$, $\phi_i(m_i \otimes n) = \iota_i(m) \otimes n$, where $\iota_i : M_i \to \oplus_{i \in I} M_i$ is the canonical inclusion. By the universal property of the direct sum this defines a unique group homomorphism $\phi : \oplus_{i \in I} M_i \otimes_R N \to (\oplus_{i \in I} M_i) \otimes_R N$ with $\phi \circ \iota'_i = \phi_i$ for the inclusion maps $\iota'_i : M_i \otimes_R N \to \oplus_{i \in I} M_i \otimes_R N$. This group homomorphism has an inverse $\psi : (\oplus_{i \in I} M_i) \otimes_R N \to \oplus_{i \in I} M_i \otimes_R N$ given by $\psi(\iota_i(m_i) \otimes n) = \iota'_i(m_i \otimes n)$ and hence is an isomorphism. The proof for the other identity is analogous.

4. A group isomorphism $\phi : (M \otimes_R P) \otimes_S Q \to M \otimes_R (P \otimes_S Q)$ is given by

$$\phi((m \otimes p) \otimes q) = m \otimes (p \otimes q) \qquad \forall m \in M, p \in P, q \in Q. \qquad \square$$

It remains to investigate the interaction of tensor products over R with R-linear maps. One finds a similar pattern as for the tensor product of vector spaces. The universal property of the tensor product over R allows one to form the product $\phi\otimes\psi : M\otimes_R N \to M'\otimes_R N'$ of an R-left module morphism $\phi : M \to M'$ and an R-right module morphism $\psi : N \to N'$.

Proposition 1.1.28: Let R be a ring, M, M' R-right modules and N, N' R-left modules.

1. For R-linear maps $\phi : M \to M'$ and $\psi : N \to N'$ there is a unique group homomorphism $\phi\otimes\psi : M\otimes_R N \to M'\otimes_R N'$ for which the following diagram commutes

$$\begin{tikzcd} M \times N \arrow[r, "\phi\times\psi"] \arrow[d, "\tau"] & M' \times N' \arrow[d, "\tau'"] \\ M \otimes_R N \arrow[r, dashed, "\exists!\phi\otimes\psi"'] & M' \otimes_R N'. \end{tikzcd}$$

2. The group homomorphisms satisfy $(\phi'\otimes\psi')\circ(\phi\otimes\psi) = (\phi'\circ\phi)\otimes(\psi'\circ\psi)$ for all R-linear maps $\phi : M \to M'$, $\phi' : M' \to M''$ and $\psi : N \to N'$, $\psi' : N' \to N''$ and $\mathrm{id}_M\otimes\mathrm{id}_N = \mathrm{id}_{M\otimes_R N}$.

Proof:
1. The map $\tau' \circ (\phi \times \psi) : M \times N \to M' \otimes_R N'$ is R-bilinear, and by the universal property of the tensor product there is a unique group homomorphism $\phi \otimes \psi : M \otimes_R N \to M' \otimes_R N'$ with $(\phi\otimes\psi) \circ \tau = \tau' \circ (\phi \times \psi)$, or, equivalently, $(\phi \otimes \psi)(m \otimes n) = \phi(m) \otimes \psi(n)$ for all $m \in M, n \in N$.
2. These identities follow from the fact the the following two diagrams commute

$$\begin{tikzcd} M \times N \arrow[r, "\mathrm{id}_M\times\mathrm{id}_N"] \arrow[d, "\tau"] & M \times N \arrow[d, "\tau'"] \\ M \otimes_R N \arrow[r, "\mathrm{id}_{M\otimes_R N}"'] & M \otimes_R N \end{tikzcd} \qquad \begin{tikzcd} M \times N \arrow[r, "\phi\times\psi"] \arrow[d, "\tau"] & M' \times N' \arrow[r, "\phi'\times\psi'"] \arrow[d, "\tau'"] & M'' \times N'' \arrow[d, "\tau''"] \\ M \otimes_R N \arrow[r, "\phi\otimes\psi"'] & M' \otimes_R N' \arrow[r, "\phi'\otimes\psi'"'] & M''\otimes_R N''. \end{tikzcd}$$

$\square$

1.2 Categories, functors and natural transformations

(Co)homology theories relate different mathematical structures. They assign to mathematical structures such as topological spaces, algebras and groups certain modules over rings and to structure preserving maps such as continuous maps, algebra homomorphisms and group homomorphisms certain module homomorphisms. The mathematical concepts that describe these relations are categories and functors. These concepts not only simplify and unify different (co)homology theories, but are required for a systematic investigation of (co)homology theories and for a deeper understanding of their structure.

Definition 1.2.1: A **category** $\mathcal{C}$ consists of:

- a class $\mathrm{Ob}\,\mathcal{C}$ of **objects**,

- for each pair of objects $X, Y \in \mathrm{Ob}\,\mathcal{C}$ a set[2] $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of **morphisms**,
- for each triple of objects X, Y, Z a **composition map**

$$\circ : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z),$$

such that the following axioms are satisfied:

(C1) The sets $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of morphisms are pairwise disjoint,

(C2) The composition is associative: $f \circ (g \circ h) = (f \circ g) \circ h$ for all morphisms $h \in \mathrm{Hom}_{\mathcal{C}}(W, X)$, $g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, $f \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$,

(C3) For every object X there is a morphism $1_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$, the **identity morphism** on X, with $1_X \circ f = f$ and $g \circ 1_X = g$ for all $f \in \mathrm{Hom}_{\mathcal{C}}(W, X)$, $g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

Instead of $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, we also write $f : X \to Y$. The object X is called the **source** of f, and the object Y the **target** of f. A morphism $f : X \to X$ is called an **endomorphism**.

A morphism $f : X \to Y$ is called an **isomorphism**, if there is a morphism $g : Y \to X$ with $g \circ f = 1_X$ and $f \circ g = 1_Y$. In this case, we call the objects X and Y **isomorphic**.

Example 1.2.2:

1. The category Set: the objects of Set are sets, and the morphisms are maps $f : X \to Y$. The composition is the composition of maps and the identity morphisms are the identity maps. Isomorphisms are bijective maps.

   Note that the definition of a category requires that the *morphisms* between any two objects in a category form a set, but not that the *objects* form a set. Requiring that the objects of a category form a set would force one to consider sets of sets when defining the category Set, which leads to a contradiction. A category whose objects form a set is called a **small category**.

2. The category Top of topological spaces. Objects are topological spaces, morphisms $f : X \to Y$ are continuous maps, isomorphisms are homeomorphisms.

3. The category Top^* of **pointed topological spaces**: Objects are pairs (X, x) of a topological space X and a point $x \in X$, morphisms $f : (X, x) \to (Y, y)$ are continuous maps $f : X \to Y$ with $f(x) = y$.

4. The category Top(2) of **pairs of topological spaces**: Objects are pairs (X, A) of a topological space X and a subspace $A \subset X$, morphisms $f : (X, A) \to (Y, B)$ are continuous maps $f : X \to Y$ with $f(A) \subset B$. Isomorphisms are homeomorphisms $f : X \to Y$ with $f(A) = B$.

5. Many examples of categories we will use in the following are categories of algebraic structures. This includes the following:
   - the category $\mathrm{Vect}_{\mathbb{F}}$ of vector spaces over a field $\mathbb{F}$:
     objects: vector spaces over $\mathbb{F}$, morphisms: $\mathbb{F}$-linear maps,

[2]This condition is sometimes relaxed in the literature on category theory. Categories whose morphisms form sets are called *locally small* in these references. All categories considered in the following are locally small.

- the category $\mathrm{Vect}_{\mathbb{F}}^{fin}$ of finite dimensional vector spaces over a field $\mathbb{F}$:
  objects: vector spaces over $\mathbb{F}$, morphisms: $\mathbb{F}$-linear maps,

- the category Grp of groups:
  objects: groups, morphisms: group homomorphisms,

- the category Ab of abelian groups:
  objects: abelian groups, morphisms: group homomorphisms,

- the category Ring of rings:
  objects: rings, morphisms: ring homomorphisms,

- the category URing of unital rings:
  objects: unital rings, morphisms: unital ring homomorphisms,

- the category Field of fields:
  objects: fields, morphisms: field monomorphisms,

- the category $\mathrm{Alg}_{\mathbb{F}}$ of algebras over a field $\mathbb{F}$:
  objects: algebras over $\mathbb{F}$, morphisms: algebra homomorphisms,

- the categories R-Mod and Mod-R of left and right modules over a ring R:
  objects: R-left or right modules, morphisms: R-left or right module homomorphisms.

- the category R-Mod-S of (R, S)-bimodules:
  objects: (R, S)-bimodules, morphisms: (R, S)-bimodule homomorphisms.

In all of the categories in Example 1.2.2 the morphisms are *maps*. A category for which this is the case is called a **concrete category**. A category that is not concrete is given in Exercise 5. Other important examples and basic constructions for categories are the following.

Example 1.2.3:

1. A small category $\mathcal{C}$ in which all morphisms are isomorphisms is called a **groupoid**.

2. A category with a single object X is a **monoid**, and a groupoid $\mathcal{C}$ with a single object X is a group. Group elements are identified with endomorphisms $f : X \to X$ and the composition of morphisms is the group multiplication. More generally, for any object X in a groupoid $\mathcal{C}$, the set $\mathrm{End}_{\mathcal{C}}(X) = \mathrm{Hom}_{\mathcal{C}}(X, X)$ with the composition $\circ : \mathrm{End}_{\mathcal{C}}(X) \times \mathrm{End}_{\mathcal{C}}(X) \to \mathrm{End}_{\mathcal{C}}(X)$ is a group.

3. For every category $\mathcal{C}$, one has an **opposite category** $\mathcal{C}^{op}$, which has the same objects as $\mathcal{C}$, whose morphisms are given by $\mathrm{Hom}_{\mathcal{C}^{op}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$ and in which the order of the composition is reversed.

4. The **cartesian product** of categories $\mathcal{C}, \mathcal{D}$ is the category $\mathcal{C} \times \mathcal{D}$ with pairs (C, D) of objects in $\mathcal{C}$ and $\mathcal{D}$ as objects, with $\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((C, D), (C', D')) = \mathrm{Hom}_{\mathcal{C}}(C, C') \times \mathrm{Hom}_{\mathcal{D}}(D, D')$ and the composition of morphisms $(h, k) \circ (f, g) = (h \circ f, k \circ g)$.

5. A **subcategory** of a category $\mathcal{C}$ is a category $\mathcal{D}$, such that $\mathrm{Ob}(\mathcal{D}) \subset \mathrm{Ob}(\mathcal{C})$ is a subclass, $\mathrm{Hom}_{\mathcal{D}}(D, D') \subset \mathrm{Hom}_{\mathcal{C}}(D, D')$ for all objects D, D' in $\mathcal{D}$ and the composition of morphisms of $\mathcal{D}$ coincides with their composition in $\mathcal{C}$. A subcategory $\mathcal{D}$ of $\mathcal{C}$ is called a **full subcategory** if $\mathrm{Hom}_{\mathcal{D}}(D, D') = \mathrm{Hom}_{\mathcal{C}}(D, D')$ for all objects D, D' in $\mathcal{D}$.

6. **Quotient categories:** Let $\mathcal{C}$ be a category with an equivalence relation $\sim_{X,Y}$ on each morphism set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ that is compatible with the composition of morphisms: $f \sim_{X,Y} g$ and $h \sim_{Y,Z} k$ implies $h \circ f \sim_{X,Z} k \circ g$.

   Then one obtains a category $\mathcal{C}'$, the **quotient category** of $\mathcal{C}$, with the same objects as $\mathcal{C}$ and equivalence classes of morphisms in $\mathcal{C}$ as morphisms.

   The composition of morphisms in $\mathcal{C}'$ is given by $[h] \circ [f] = [h \circ f]$, and the identity morphisms by $[1_X]$. Isomorphisms in $\mathcal{C}'$ are equivalence classes of morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ for which there exists a morphism $g \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$ with $f \circ g \sim_{Y,Y} 1_Y$ and $g \circ f \sim_{X,X} 1_X$.

The construction in the last example plays an important role in classification problems, in particular in the context of topological spaces. Classifying the objects of a category $\mathcal{C}$ usually means classifying them up to isomorphism, i. e. giving a list of objects in $\mathcal{C}$ such that every object in $\mathcal{C}$ is isomorphic to exactly one object in this list.

While this is possible in some contexts - for the category $\mathrm{Vect}_{\mathbb{F}}^{fin}$ of finite dimensional vector spaces over $\mathbb{F}$, the list contains the vector spaces $\mathbb{F}^n$ with $n \in \mathbb{N}_0$ - it is often too difficult to solve this problem in full generality. In this case, it is sometimes simpler to consider instead a quotient category $\mathcal{C}'$ and to attempt a partial classification. If two objects are isomorphic in $\mathcal{C}$, they are by definition isomorphic in $\mathcal{C}'$ since for any objects X, Y in $\mathcal{C}$ and any isomorphism $f : X \to Y$ with inverse $g : Y \to X$, one has $[g] \circ [f] = [g \circ f] = [1_X]$ and $[f] \circ [g] = [f \circ g] = [1_Y]$. However, the converse does not hold - the category $\mathcal{C}'$ yields a weaker classification than $\mathcal{C}$.

To relate different categories, one must not only relate their objects but also their morphisms, in a way that is compatible with source and target objects, the composition of morphisms and the identity morphisms. This leads to the concept of a *functor*.

Definition 1.2.4: Let $\mathcal{C}, \mathcal{D}$ be categories. A **functor** $F : \mathcal{C} \to \mathcal{D}$ consists of:

- an assignment of an object $F(C)$ in $\mathcal{D}$ to every object C in $\mathcal{C}$,
- for each pair of objects C, C' in $\mathcal{C}$, a map

$$\mathrm{Hom}_{\mathcal{C}}(C, C') \to \mathrm{Hom}_{\mathcal{D}}(F(C), F(C')), \quad f \mapsto F(f),$$

that is compatible with the composition of morphisms and with the identity morphisms

$$\begin{aligned} F(g \circ f) &= F(g) \circ F(f) && \forall f \in \mathrm{Hom}_{\mathcal{C}}(C, C'), g \in \mathrm{Hom}_{\mathcal{C}}(C', C'') \\ F(1_C) &= 1_{F(C)} && \forall C \in \mathrm{Ob}\,\mathcal{C}. \end{aligned}$$

A functor $F : \mathcal{C} \to \mathcal{C}$ is called an **endofunctor**. A functor $F : \mathcal{C}^{op} \to \mathcal{D}$ is sometimes called a **contravariant functor** from $\mathcal{C}$ to $\mathcal{D}$. The **composite** of two functors $F : \mathcal{B} \to \mathcal{C}$, $G : \mathcal{C} \to \mathcal{D}$ is the functor $GF : \mathcal{B} \to \mathcal{D}$ given by the assignment $B \mapsto GF(B)$ for all objects B in $\mathcal{B}$ and the maps $\mathrm{Hom}_{\mathcal{B}}(B, B') \to \mathrm{Hom}_{\mathcal{D}}(GF(B), GF(B'))$, $f \mapsto G(F(f))$.

Example 1.2.5:

1. For any category $\mathcal{C}$, identity functor $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$, that assigns each object and morphism in $\mathcal{C}$ to itself is an endofunctor of $\mathcal{C}$.

2. The functor $\mathrm{Vect}_{\mathbb{F}} \to \mathrm{Ab}$ that assigns to each vector space the underlying abelian group and to each linear map the associated group homomorphism, and the functors $\mathrm{Vect}_{\mathbb{F}} \to \mathrm{Set}$, $\mathrm{Ring} \to \mathrm{Set}$, $\mathrm{Grp} \to \mathrm{Set}$, $\mathrm{Top} \to \mathrm{Set}$ etc that assign to each vector space, ring, group, topological space the underlying set and to each morphism the underlying map are functors. A functor of this type is called **forgetful functor**.

3. The functor $* : \mathrm{Vect}_{\mathbb{F}} \to \mathrm{Vect}_{\mathbb{F}}^{op}$, which assigns to a vector space V its dual V^* and to a linear map $f : V \to W$ its adjoint $f^* : W^* \to V^*$, $\alpha \mapsto \alpha \circ f$.

4. For a group G, consider the category BG with a single object, with elements of G as morphisms, and with the multiplication of G as the composition. Then functors $F : BG \to \mathrm{Set}$ correspond to G-sets $X = F(\bullet)$ with the group action $\rhd : G \times X \to X$, $g \rhd x = F(g)(x)$. Functors $F : BG \to \mathrm{Vect}_{\mathbb{F}}$ correspond to representations of G over $\mathbb{F}$, with the representation space $V = F(\bullet)$ and $\rho = F(g) : G \to \mathrm{Aut}_{\mathbb{F}} V$.

5. **Restriction functor**: Let $\phi : R \to S$ a ring homomorphism. The restriction functor $\mathrm{Res} : S\text{-Mod} \to R\text{-Mod}$ sends an S-module $(M, \rhd)$ to the R-module $(M, \rhd_\phi)$ with the pullback module structure $r \rhd_\phi m = \phi(r) \rhd m$ and every S-linear map $f : M \to M'$ to itself.

6. **Tensor products**: Let R be a ring, M an R-right module and N an R-left module.

   • The functor $M \otimes_R - : R\text{-Mod} \to \mathrm{Ab}$ assigns to an R-left module N the abelian group $M \otimes_R N$ and to an R-linear map $f : N \to N'$ the group homomorphism $\mathrm{id}_M \otimes f : M \otimes_R N \to M \otimes_R N'$.

   • The functor $- \otimes_R N : R^{op}\text{-Mod} \to \mathrm{Ab}$ assigns to an R-right module M the abelian group $M \otimes_R N$ and to an R-linear map $f : M \to M'$ the group homomorphism $f \otimes \mathrm{id}_N : M \otimes_R N \to M \otimes_R N'$.

   • The functor $\otimes_R : R^{op}\text{-Mod} \times R\text{-Mod} \to \mathrm{Ab}$ assigns to a pair (M, N) of an R-right module M and an R-left module N the abelian group $M \otimes_R N$ and to a pair of R-linear maps $f : M \to M'$ and $g : N \to N'$ the group homomorphism $f \otimes g : M \otimes_R N \to M' \otimes_R N'$.

   That these are indeed functors follows from Proposition 1.1.28, 2. Note also that for *commutative* rings R, any R-left module is an (R, R)-bimodule and these functors can be defined to take values in R-Mod instead of Ab.

7. The **Hom-functors**: Let $\mathcal{C}$ be a category and C an object in $\mathcal{C}$.

   • The functor $\mathrm{Hom}(C, -) : \mathcal{C} \to \mathrm{Set}$ assigns to an object C' the set $\mathrm{Hom}_{\mathcal{C}}(C, C')$ and to a morphism $f : C' \to C''$ the map $\mathrm{Hom}(C, f) : \mathrm{Hom}_{\mathcal{C}}(C, C') \to \mathrm{Hom}_{\mathcal{C}}(C, C'')$, $g \mapsto f \circ g$.

   • The functor $\mathrm{Hom}(-, C) : \mathcal{C}^{op} \to \mathrm{Set}$ assigns to an object C' the set $\mathrm{Hom}_{\mathcal{C}}(C', C)$ and to a morphism $f : C' \to C''$ the map $\mathrm{Hom}(f, C) : \mathrm{Hom}_{\mathcal{C}}(C'', C) \to \mathrm{Hom}_{\mathcal{C}}(C', C)$, $g \mapsto g \circ f$.

8. The **path component functor** $\pi_0 : \mathrm{Top} \to \mathrm{Set}$ assigns to a topological space X the set $\pi_0(X)$ of its path components $P(x)$ and to a continuous map $f : X \to Y$ the map $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$, $P(x) \mapsto P(f(x))$.

9. The **fundamental group** defines a functor $\pi_1 : \mathrm{Top}^* \to \mathrm{Grp}$ that assigns to a pointed topological space (x, X) its fundamental group $\pi_1(x, X)$ and to a morphism $f : (x, X) \to (y, Y)$ of pointed topological spaces the group homomorphism $\pi_1(f) : \pi_1(x, X) \to \pi_1(y, Y)$, $[\gamma] \mapsto [f \circ \gamma]$.

10. **Abelisation**: The abelisation functor $F : \mathrm{Grp} \to \mathrm{Ab}$ assigns to a group G the abelian group $F(G) = G/[G, G]$, where $[G, G]$ is the normal subgroup generated by the set of all

elements $ghg^{-1}h^{-1}$ for $g, h \in G$, and to a group homomorphism $f : G \to H$ the induced group homomorphism $F(f) : G/[G,G] \to H/[H,H]$, $g + [G,G] \mapsto f(g) + [H,H]$.

When dealing with categories, it is not sufficient to consider functors between different categories. There is another structure that relates different functors. As a functor $F : \mathcal{C} \to \mathcal{D}$ involves maps between the sets $\mathrm{Hom}_{\mathcal{C}}(C, C')$ and $\mathrm{Hom}_{\mathcal{D}}(F(C), F(C'))$, a structure that relates two functors $F, G : \mathcal{C} \to \mathcal{D}$ must in particular relate the sets $\mathrm{Hom}_{\mathcal{D}}(F(C), F(C'))$ and $\mathrm{Hom}_{\mathcal{D}}(G(C), G(C'))$. The simplest way to do this is to assign to each object C in $\mathcal{C}$ a morphism $\eta_C : F(C) \to G(C)$ in $\mathcal{D}$. One then requires compatibility with the morphisms $F(f) : F(C) \to G(C')$ and $G(f) : G(C) \to G(C')$ for all morphisms $f : C \to C'$ in $\mathcal{C}$.

Definition 1.2.6: A **natural transformation** $\eta : F \to G$ between functors $F, G : \mathcal{C} \to \mathcal{D}$ is an assignment of a morphism $\eta_C : F(C) \to G(C)$ in $\mathcal{D}$ to every object C in $\mathcal{C}$ such that the following diagram commutes for all morphisms $f : C \to C'$ in $\mathcal{C}$

$$\begin{tikzcd} F(C) \arrow[r, "\eta_C"] \arrow[d, "F(f)"] & G(C) \arrow[d, "G(f)"] \\ F(C') \arrow[r, "\eta_{C'}"] & G(C'). \end{tikzcd}$$

A **natural isomorphism** is a natural transformation $\eta : F \to G$, for which all morphisms $\eta_X : F(X) \to G(X)$ are isomorphisms. Two functors that are related by a natural isomorphism are called **naturally isomorphic**.

Example 1.2.7:

1. For any functor $F : \mathcal{C} \to \mathcal{D}$ the identity natural transformation $\mathrm{id}_F : F \to F$ with component morphisms $(\mathrm{id}_F)_X = 1_{F(X)} : F(X) \to F(X)$ is a natural isomorphism.

2. Consider the functors $\mathrm{id} : \mathrm{Vect}_{\mathbb{F}} \to \mathrm{Vect}_{\mathbb{F}}$ and $** : \mathrm{Vect}_{\mathbb{F}} \to \mathrm{Vect}_{\mathbb{F}}$. Then there is a canonical natural transformation $\mathrm{can} : \mathrm{id} \to **$, whose component morphisms $\eta_V : V \to V^{**}$ assign to a vector $v \in V$ the unique vector $v^{**} \in V^{**}$ with $v^{**}(\alpha) = \alpha(v)$ for all $\alpha \in V^*$.

3. Consider the category CRing of commutative unital rings and unital ring homomorphisms and the category Grp of groups and group homomorphisms.

   Let $F : \mathrm{CRing} \to \mathrm{Grp}$ the functor that assigns to a commutative unital ring k the group $\mathrm{GL}_n(k)$ of invertible $n \times n$-matrices with entries in k and to a unital ring homomorphism $f : k \to l$ the group homomorphism

   $$\mathrm{GL}_n(f) : \mathrm{GL}_n(k) \to \mathrm{GL}_n(l), \quad M = (m_{ij})_{i,j=1,\ldots,n} \mapsto f(M) = (f(m_{ij}))_{i,j=1,..,n}.$$

   Let $G : \mathrm{CRing} \to \mathrm{Grp}$ be the functor that assigns to a commutative unital ring k the group $G(k) = k^\times$ of units in k and to a unital ring homomorphism $f : k \to l$ the induced group homomorphism $G(f) = f|_{k^\times} : k^\times \to l^\times$.

   The determinant defines a natural transformation $\det : F \to G$ with component morphisms $\det_k : \mathrm{GL}_n(k) \to k^\times$, since the following diagram commutes for every unital ring homomorphism $f : k \to l$

   $$\begin{tikzcd} \mathrm{GL}_n(k) \arrow[r, "\det_k"] \arrow[d, "\mathrm{GL}_n(f)"'] & k^\times \arrow[d, "f|_{k^\times}"] \\ \mathrm{GL}_n(l) \arrow[r, "\det_l"] & l^\times. \end{tikzcd}$$

4. For a group G, denote by BG the groupoid with a fingle object $\bullet$, with group elements $g \in G$ as morphisms and the group multiplication as composition.

   Then by Example 1.2.5, 4. functors $F : BG \to \mathrm{Set}$ are G-sets, and natural transformations between them are G-equivariant maps. Every natural transformation $\eta : F \to F'$ is given by a single component morphism $\eta_\bullet : F(\bullet) \to F'(\bullet)$. The naturality condition states that $\eta_\bullet(g \rhd x) = g \rhd' \eta_\bullet(x)$ for all $g \in G$, $x \in X$.

   Similarly, by Example 1.2.5, 4. functors $F : BG \to \mathrm{Vect}_\mathbb{F}$ are representations of G over $\mathbb{F}$, and natural transformations between them are homomorphisms of representations.

Remark 1.2.8:

1. For any small category $\mathcal{C}$ and category $\mathcal{D}$, the functors $\mathcal{C} \to \mathcal{D}$ and natural transformations between them form a category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, the **functor category**. The composite of two natural transformations $\eta : F \to G$ and $\kappa : G \to H$ is the natural transformation $\kappa \circ \eta : F \to H$ with component morphisms $(\kappa \circ \eta)_X = \kappa_X \circ \eta_X : F(X) \to H(X)$ and the identity morphisms are the identity natural transformations $1_F = \mathrm{id}_F : F \to F$.

2. Natural transformations can be composed with functors.

   If $F, F' : \mathcal{C} \to \mathcal{D}$ are functors and $\eta : F \to F'$ a natural transformation, then for any functor $G : \mathcal{B} \to \mathcal{C}$ one obtains a natural transformation $\eta G : FG \to F'G$ with component morphisms $(\eta G)_B = \eta_{G(B)} : FG(B) \to F'G(B)$. Similarly, any functor $E : \mathcal{D} \to \mathcal{E}$ defines a natural transformation $E\eta : EF \to EF'$ with $(E\eta)_C = E(\eta_C) : EF(C) \to EF'(C)$.

The notions of natural transformations and natural isomorphisms are particularly important as they allow one to generalise the notion of an inverse map and of a bijection to functors. While it is possible to define an *inverse* of a functor $F : \mathcal{C} \to \mathcal{D}$ as a functor $G : \mathcal{D} \to \mathcal{C}$ with $GF = \mathrm{id}_\mathcal{C}$ and $FG = \mathrm{id}_\mathcal{D}$, it turns out that this is too strict. There are very few non-trivial examples of functors with an inverse. A more useful generalisation is obtained by weakening this requirement. Instead of requiring $FG = \mathrm{id}_\mathcal{D}$ and $GF = \mathrm{id}_\mathcal{C}$, one requires only that these functors are *naturally isomorphic* to the identity functors. This leads to the concept of an equivalence of categories.

Definition 1.2.9: A functor $F : \mathcal{C} \to \mathcal{D}$ is called an **equivalence of categories** if there is a functor $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\kappa : GF \to \mathrm{id}_\mathcal{C}$ and $\eta : FG \to \mathrm{id}_D$. In this case, the categories $\mathcal{C}$ and $\mathcal{D}$ are called **equivalent**.

Sometimes it is easier to use a more direct characterisation of an equivalences of categories in terms of its behaviour on objects and morphisms. The proof of the following lemma makes use of the axiom of choice and an be found for instance in [K], Chapter XI, Prop XI.1.5.

Lemma 1.2.10: A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if it is:

1. **essentially surjective**:
   for every object D in $\mathcal{D}$ there is an object C of $\mathcal{C}$ such that D is isomorphic to $F(C)$.
2. **fully faithful**:
   all maps $\mathrm{Hom}_\mathcal{C}(C, C') \to \mathrm{Hom}_\mathcal{D}(F(C), F(C'))$, $f \mapsto F(f)$ are bijections.

Example 1.2.11:

1. The category $\mathrm{Vect}_{\mathbb{F}}^{fin}$ of finite-dimensional vector spaces over $\mathbb{F}$ is equivalent to the category $\mathcal{C}$, whose objects are non-negative integers $n \in \mathbb{N}_0$, whose morphisms $f : n \to m$ are $m \times n$-matrices with entries in $\mathbb{F}$ and with the matrix multiplication as composition of morphisms.

2. The category Set^{fin} of finite sets is equivalent to the category Ord^{fin}, whose objects are finite **ordinal numbers** $\underline{n} = \{0, 1, ..., n-1\}$ for all $n \in \mathbb{N}_0$ and whose morphisms $f : \underline{m} \to \underline{n}$ are maps $f : \{0, 1, ..., m-1\} \to \{0, 1, ..., n-1\}$ with the composition of maps as the composition of morphisms.

Many concepts and constructions from topological or algebraic settings can be generalised straightforwardly to categories. This is true whenever it is possible to characterise them in terms of *universal properties* involving only the *morphisms* in the category. In particular, there is a concept of categorical product and coproduct that generalise cartesian products and disjoint unions of sets and products and sums of topological spaces.

Definition 1.2.12: Let $\mathcal{C}$ be a category and $(C_i)_{i\in I}$ a family of objects in $\mathcal{C}$.

1. A **product** of the family $(C_i)_{i\in I}$ is an object $\Pi_{i\in I} C_i$ in $\mathcal{C}$ together with a family of morphisms $\pi_i : \Pi_{j\in I} C_j \to C_i$, such that for all families of morphisms $f_i : W \to C_i$ there is a unique morphism $f : W \to \Pi_{i\in I} C_i$ such that the diagram

$$\begin{tikzcd} W \arrow[r, dashed, "\exists! f"] \arrow[dr, "f_i"'] & \Pi_{j\in I} C_j \arrow[d, "\pi_i"] \\ & C_i \end{tikzcd} \tag{1}$$

commutes for all $i \in I$. This is called the **universal property** of the product.

2. A **coproduct** of the family $(C_i)_{i\in I}$ is an object $\amalg_{i\in I} C_i$ in $\mathcal{C}$ with a family $(\iota_i)_{i\in I}$ of morphisms $\iota_i : C_i \to \amalg_{j\in I} C_j$, such that for every family $(f_i)_{i\in I}$ of morphisms $f_i : C_i \to Y$ there is a unique morphism $f : \amalg_{i\in I} C_i \to Y$ such that the diagram

$$\begin{tikzcd} Y & \amalg_{j\in I} C_j \arrow[l, dashed, "\exists! f"'] \\ & C_i \arrow[ul, "f_i"] \arrow[u, "\iota_i"'] \end{tikzcd} \tag{2}$$

commutes for all $i \in I$. This is called the **universal property** of the coproduct.

Remark 1.2.13: Products or coproducts do not necessarily exist for a given family of objects $(C_i)_{i\in I}$ in a category $\mathcal{C}$, but if they exist, they are **unique up to unique isomorphism**:

If $(\Pi_{i\in I} C_i, (\pi_i)_{i\in I})$ and $(\Pi'_{i\in I} C_i, (\pi'_i)_{i\in I})$ are two products for a family of objects $(C_i)_{i\in I}$ in $\mathcal{C}$, then there is a unique morphism $\pi' : \Pi'_{i\in I} C_i \to \Pi_{i\in I} C_i$ with $\pi_i \circ \pi' = \pi'_i$ for all $i \in I$, and this morphism is an isomorphism.

This follows directly from the universal property of the products: By the universal property of the product $\Pi_{i\in I} C_i$ applied to the family of morphisms $\pi'_i : \Pi'_{i\in I} C_i \to C_i$, there is a unique morphism $\pi' : \Pi'_{i\in I} C_i \to \Pi_{i\in I} C_i$ such that $\pi_i \circ \pi' = \pi'_i$ for all $i \in I$. Similarly, the universal property of $\Pi'_{i\in I} C_i$ implies that for the family of morphisms $\pi_i : \Pi_{i\in I} C_i \to C_i$ there is a unique morphism $\pi : \Pi_{i\in I} C_i \to \Pi'_{i\in I} C_i$ with $\pi'_i \circ \pi = \pi_i$ for all $i \in I$. It follows that $\pi' \circ \pi : \Pi_{i\in I} C_i \to \Pi_{i\in I} C_i$ is a morphism with $\pi_i \circ \pi \circ \pi' = \pi'_i \circ \pi = \pi_i$ for all $i \in I$. Since the identity morphism on $\Pi_{i\in I} C_i$ is another morphism with this property, the uniqueness implies $\pi' \circ \pi = 1_{\Pi_{i\in I} C_i}$. By the same argument one obtains $\pi \circ \pi' = 1_{\Pi'_{i\in I} C_i}$ and hence π' is an isomorphism with inverse π.

$$\begin{tikzcd}
\Pi'_{i\in I} C_i \arrow[r, "\pi'"] \arrow[rr, bend left, "1_{\Pi'_{i\in I} C_i}"] \arrow[dr, "\pi'_i"'] & \Pi_{i\in I} C_i \arrow[r, "\pi"] \arrow[d, "\pi_i"] & \Pi'_{i\in I} C_i \arrow[dl, "\pi'_i"] \\
& C_i. &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Pi_{i\in I} C_i \arrow[r, "\pi"] \arrow[rr, bend left, "1_{\Pi_{i\in I} C_i}"] \arrow[dr, "\pi_i"'] & \Pi'_{i\in I} C_i \arrow[r, "\pi'"] \arrow[d, "\pi'_i"] & \Pi_{i\in I} C_i \arrow[dl, "\pi_i"] \\
& C_i. &
\end{tikzcd}$$

Example 1.2.14:

1. The cartesian product of sets is a product in Set, and the disjoint union of sets is a coproduct in Set. The product of topological spaces is a product in Top and the topological sum is a coproduct in Top. In Set and Top, products and coproducts exist for all families of objects.

2. The direct sum of vector spaces is a coproduct and the direct product of vector spaces a product in $\mathrm{Vect}_{\mathbb{F}}$. More generally, direct sums and products of R-left (right) modules over a unital ring R are coproducts and products in R-Mod (Mod-R). Again, products and coproducts exist for all families of objects in R-Mod (Mod-R).

3. The wedge sum is a coproduct in the category Top^* of pointed topological spaces. It exists for all families of pointed topological spaces.

4. The direct product of groups is a product in Grp and the free product of groups is a coproduct in Grp. They exist for all families of groups.

In particular, we can consider categorical products and coproducts over empty index sets I. By definition, a categorical product for an empty family of objects is an object $T = \Pi_\emptyset$ such that for every object C in $\mathcal{C}$ there is a unique morphism $t_C : C \to T$. (This is the morphism associated to the empty family of morphisms from C to the objects in the empty family by the universal property of the product). Similarly, a coproduct over an empty index set I is an object $I := \amalg_\emptyset$ in $\mathcal{C}$ such that for every object C in $\mathcal{C}$, there is a unique morphism $i_C : I \to C$. Such objects are called, respectively, *terminal* and *initial* objects in $\mathcal{C}$.

Initial and terminal objects do not exist in every category $\mathcal{C}$, but if they exist they are unique up to unique isomorphism by the universal property of the products and coproducts.

An object that is both, terminal and initial, is called a *zero object*. If it exists, it is unique up to unique isomorphism, and it gives rise to a distinguished morphism, the *zero morphism* $0 = i_{C'} \circ t_C : C \to C'$ between objects C, C' in $\mathcal{C}$.

Definition 1.2.15: Let $\mathcal{C}$ be a category. An object X in a category $\mathcal{C}$ is called:

1. A **final** or **terminal object** in $\mathcal{C}$ is an object T in $\mathcal{C}$ such that for every object C in $\mathcal{C}$ there is a unique morphism $t_C : C \to T$.

2. A **cofinal** or **initial object** in $\mathcal{C}$ is an object I in $\mathcal{C}$ such that for every object C in $\mathcal{C}$ there is a unique morphism $i_C : I \to C$,
3. A **null object** or **zero object** in $\mathcal{C}$ is an object 0 in $\mathcal{C}$ that is both final and initial: for every object C in $\mathcal{C}$ there are a unique morphisms $t_C : C \to 0$ and $i_C : 0 \to C$.
4. If $\mathcal{C}$ has a zero object, then the morphism $0 = i_{C'} \circ t_C : C \to 0 \to C'$ is called the **trivial morphism** or **zero morphism** from C to C'.

Example 1.2.16:

1. The empty set is an initial object in Set and the empty topological space an initial object in Top. Any set with one element is a final object in Set and any one point space an initial object in Top. The categories Set and Top do not have null objects.

2. The null vector space $\{0\}$ is a null object in the category $\mathrm{Vect}_{\mathbb{F}}$. More generally, for any ring R, the trivial R-module $\{0\}$ is a null object in R-Mod (Mod-R).

3. The trivial group $G = \{e\}$ is a null object in Grp and in Ab.

4. The ring $\mathbb{Z}$ is an initial object in the category URing, since for every unital ring R, there is exactly one ring homomorphism $f : \mathbb{Z} \to R$, namely the one determined by $f(0) = 0_R$ and $f(1) = 1_R$. The zero ring $R = \{0\}$ with $0 = 1$ is a final object in URing, but not an initial one. The category URing has no zero object.

5. The category Field does not have initial or final objects. As any ring homomorphism $f : \mathbb{F} \to \mathbb{K}$ between fields is injective, an initial object $\mathbb{F}$ in Field would be a subfield of all other fields, and every field would be a subfield of a final object $\mathbb{F}$. This would imply $\mathrm{char}(\mathbb{F}) = \mathrm{char}(\mathbb{K})$ for all other fields $\mathbb{K}$, a contradiction.

Besides forming an equivalence of categories, there is another important way in which two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ can be related, namely being *adjoints* of each other. Adjoint functors encode universal properties of algebraic constructions such as products and coproducts, freely generated modules or abelisation of groups. The constructions are encoded in the functors and their universal properties in bijections between certain Hom-sets in the categories $\mathcal{C}$ and $\mathcal{D}$. We will see in Section 5.4 that adjoint functors play an essential role in the construction of homology theories.

Definition 1.2.17: A functor $F : \mathcal{C} \to \mathcal{D}$ is called **left adjoint** to a functor $G : \mathcal{D} \to \mathcal{C}$ and G **right adjoint** to F, $F \dashv G$, if the functors $\mathrm{Hom}(F(-), -) : \mathcal{C}^{op} \times \mathcal{D} \to \mathrm{Set}$ and $\mathrm{Hom}(-, G(-)) : \mathcal{C}^{op} \times \mathcal{D} \to \mathrm{Set}$ are naturally isomorphic.

In other words, there is a family of bijections $\phi_{C,D} : \mathrm{Hom}_{\mathcal{C}}(C, G(D)) \to \mathrm{Hom}_{\mathcal{D}}(F(C), D)$, indexed by objects C in $\mathcal{C}$ and D in $\mathcal{D}$, such that the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(C, G(D)) \arrow[r, "\mathrm{Hom}(f,G(g))", "h\mapsto G(g)\circ h\circ f"'] \arrow[d, "\phi_{C,D}"] & \mathrm{Hom}_{\mathcal{C}}(C', G(C')) \arrow[d, "\phi_{C',D'}"] \\ \mathrm{Hom}_{\mathcal{D}}(F(C), D) \arrow[r, "\mathrm{Hom}(F(f),g)", "h\mapsto g\circ h\circ F(f)"'] & \mathrm{Hom}_{\mathcal{D}}(F(C'), D'). \end{tikzcd} \tag{3}$$

commutes for all morphisms $f : C' \to C$ in $\mathcal{C}$ and $g : D \to D'$ in $\mathcal{D}$.

Example 1.2.18:

1. **Forgetful functors and freely generated modules:**
For a ring R, the forgetful functor $G : R\text{-Mod} \to \text{Set}$ is right adjoint to the functor $F : \text{Set} \to R\text{-Mod}$ that assigns to a set A the free R-module $F(A) = \langle A \rangle_R$ generated by A and to a map $f : A \to B$ the R-linear map $F(f) : \langle A \rangle_R \to \langle B \rangle_R$ with $F(f) \circ \iota_A = \iota_B \circ f$.

   By Remark 1.1.15, 2. for every map $f : A \to M$ into an R-module M, there is a unique R-linear map $\langle f \rangle_R : \langle A \rangle_R \to M$ with $\langle f \rangle_R \circ \iota_A = f$ for the inclusion $\iota_A : A \to \langle A \rangle_R$. This defines bijections
$$\phi_{A,M} : \text{Hom}_{\text{Set}}(A, G(M)) \to \text{Hom}_{R-\text{Mod}}(F(A), M), \quad f \mapsto \langle f \rangle_R.$$
   For all maps $f : A' \to A$, $h : A \to M$ and R-linear maps $g : M \to M'$ we have
$$g \circ \langle h \rangle_R \circ F(f) \circ \iota_{A'} = g \circ \langle h \rangle_R \circ \iota_A \circ f = g \circ h \circ f = \langle g \circ h \circ f \rangle_R \circ \iota_{A'}.$$
   By Remark 1.1.15, 2. this implies $\langle g \circ h \circ f \rangle_R = g \circ \langle h \rangle_R \circ F(f)$.

2. **Discrete and indiscrete topology**: The forgetful functor $F : \text{Top} \to \text{Set}$ is left adjoint to the indiscrete topology functor $I : \text{Set} \to \text{Top}$ that assigns to a set X the topological space $(X, \mathcal{O}_{ind})$ with the indiscrete topology and to a map $f : X \to Y$ the continuous map $f : (X, \mathcal{O}_{ind}) \to (Y, \mathcal{O}_{ind})$.
It is right adjoint to the discrete topology functor $D : \text{Set} \to \text{Top}$ that assigns to a set X the topological space $(X, \mathcal{O}_{disc})$ with the discrete topology and to a map $f : X \to Y$ the continuous map $f : (X, \mathcal{O}_{disc}) \to (Y, \mathcal{O}_{disc})$. The bijections between the Hom-Sets are
$$\Phi_{(W,\mathcal{O}),X} : \text{Hom}_{\text{Top}}((W, \mathcal{O}), (X, \mathcal{O}_{ind})) \to \text{Hom}_{\text{Set}}(W, X), \quad f \mapsto f$$
$$\Phi_{X,(W,\mathcal{O})} : \text{Hom}_{\text{Set}}(X, W) \to \text{Hom}_{\text{Top}}((X, \mathcal{O}_{disc}), (W, \mathcal{O})), \quad f \mapsto f.$$
   The statement that these are bijections expresses the fact that any map $f : W \to X$ from a topological space $(W, \mathcal{O})$ into a set X becomes continuous when X is equipped with the indiscrete topology and any map $f : X \to W$ becomes continuous when X is equipped with the discrete topology. The naturality condition in (3) follows directly.

3. **Forgetful functors without left or right adjoints**:
The forgetful functor $V : \text{Field} \to \text{Set}$ has no right or left adjoint. If it had a left adjoint $F : \text{Set} \to \text{Field}$ or a right adjoint $G : \text{Set} \to \text{Field}$ there would be bijections
$$\Phi_{\emptyset,\mathbb{K}} : \text{Hom}_{\text{Set}}(\emptyset, \mathbb{K}) \to \text{Hom}_{\text{Field}}(F(\emptyset), \mathbb{F}). \quad \Phi_{\mathbb{F},\{x\}} : \text{Hom}_{\text{Field}}(\mathbb{F}, G(\{x\})) \to \text{Hom}_{\text{Set}}(\mathbb{F}, \{x\})$$
   for any field $\mathbb{F}$. This would imply that $F(\emptyset)$ is an initial object in Field and hence a subfield of any other field $\mathbb{F}$ and that $G(\{p\})$ is a terminal object in Field and hence contains any field $\mathbb{F}$ as a subfield. It follows that $\text{char}\,\mathbb{F} = \text{char}\,F(\emptyset) = \text{char}\,G(\{x\})$ for all fields $\mathbb{F}$, a contradiction.

4. **Inclusion functor and abelisation**: The inclusion functor $G : \text{Ab} \to \text{Grp}$ is right adjoint to the abelisation functor $F : \text{Grp} \to \text{Ab}$ from Example 1.2.5, 10. (Exercise 7).

5. **Products, coproducts and diagonal functors**:
• Let $\mathcal{C}$ be a category and I a set such that products and coproducts in $\mathcal{C}$ exist for all families of objects indexed by I.

   • Let $\mathcal{C}_I$ be the category with families $(C_i)_{i \in I}$ and $(f_i)_{i \in I} : (C_i)_{i \in i} \to (C'_i)_{i \in I}$ of objects and morphisms in $\mathcal{C}$ as objects and morphisms, with componentwise composition.

• Let $\Delta : \mathcal{C} \to \mathcal{C}_I$ be the diagonal functor that assigns to an object C and a morphism $f : C \to C'$ in $\mathcal{C}$ the constant families $(C)_{i\in I}$ and $(f)_{i\in I}$.

• Let $\Pi_I : \mathcal{C}_I \to \mathcal{C}$ be the product functor that assigns to a family $(C_i)_{i\in I}$ the product $\Pi_{i\in I}C_i$ and to a family $(f_i)_{i\in I} : (C_i)_{i\in I} \to (C'_i)_{i\in I}$ the morphism $\Pi_{i\in I}f_i : \Pi_{i\in I}C_i \to \Pi_{i\in I}C'_i$ with $\pi'_i \circ (\Pi_{i\in I}f_i) = f_i \circ \pi_i$ induced by the universal property of the product.

• Let $\amalg_I : \mathcal{C}_I \to \mathcal{C}$ be the coproduct functor that assigns to a family $(C_i)_{i\in I}$ the coproduct $\amalg_{i\in I}C_i$ and to a family $(f_i)_{i\in I} : (C_i)_{i\in I} \to (C'_i)_{i\in I}$ the morphism $\amalg_{i\in I}f_i : \Pi_{i\in I}C_i \to \Pi_{i\in I}C'_i$ with $(\amalg_{i\in I}f_i) \circ \iota_i = \iota'_i \circ f_i$ induced by the universal property of the coproduct.

Then $\Pi_I : \mathcal{C}_I \to \mathcal{C}$ is right adjoint to Δ and $\amalg_I : \mathcal{C}_I \to \mathcal{C}$ is left adjoint to Δ. The bijections between the Hom-sets are given by

$$\begin{aligned}
&\Phi_{C,(C_i)_{i\in I}} : \mathrm{Hom}_{\mathcal{C}}(C, \Pi_{i\in I}C_i) \to \mathrm{Hom}_{\mathcal{C}_I}((C)_{i\in I}, \Pi_{i\in I}C_i), \quad f \mapsto (\pi_i \circ f)_{i\in I}\\
&\Phi^{-1}_{(C_i)_{i\in I},C} : \mathrm{Hom}_{\mathcal{C}}(\amalg_{i\in I}C_i, C) \to \mathrm{Hom}_{\mathcal{C}_I}((C_i)_{i\in I}, (C)_{i\in I}), \quad f \mapsto (f \circ \iota_i)_{i\in I}.
\end{aligned}$$

The universal property of the (co)product implies that they are bijections, and a short computation shows that they satisfy the naturality condition in (3).

6. **Tensor products and Hom-functors**:
• For any R-right module M, the functor $M\otimes_R- : R\text{-Mod} \to \mathrm{Ab}$ is left adjoint to the functor $\mathrm{Hom}(M, -) : \mathrm{Ab} \to R\text{-Mod}$.
• For any R-left module N the functor $-\otimes_R N : R^{op}\text{-Mod} \to \mathrm{Ab}$ is left adjoint to the functor $\mathrm{Hom}(N, -) : \mathrm{Ab} \to R^{op}\text{-Mod}$.

We prove the claim for R-right modules M. For an abelian group A and R-left module L we equip $\mathrm{Hom}_{\mathrm{Ab}}(M, A)$ with the R-module structure $(r \rhd \phi)(m) = \phi(m \lhd r)$ and define

$$\begin{aligned}
\phi_{L,A} :\ & \mathrm{Hom}_{R\text{-Mod}}(L, \mathrm{Hom}_{\mathrm{Ab}}(M, A)) && \to && \mathrm{Hom}_{\mathrm{Ab}}(M \otimes_R L, A)\\
& \psi : L \to \mathrm{Hom}_{\mathrm{Ab}}(M, A),\ l \mapsto \psi_l && \mapsto && \chi : M\otimes_R L \to A,\ m\otimes l \mapsto \psi_l(m).
\end{aligned}$$

The map $\chi : M\otimes_R L \to A$, $m\otimes l \mapsto \psi_l(m)$ is well defined, since the R-linearity of the map $\psi : L \to \mathrm{Hom}_{\mathrm{Ab}}(M, A)$ implies that $\chi' : M \times L \to A$, $(m, l) \to \psi_l(m)$ is R-bilinear: $\chi'(m, r \rhd l) = \psi_{r\rhd l}(m) = (r \rhd \psi_l)(m) = \psi_l(m \lhd r) = \chi'(m \lhd r, l)$ for all $r \in R$, $l \in L$ and $m \in M$. By the universal property of the tensor product, it induces a unique group homomorphism $\chi : M \otimes_R L \to A$ with $\chi(m\otimes l) = \chi'(m, l)$. The inverse of $\phi_{L,A}$ is given by

$$\begin{aligned}
\phi^{-1}_{L,A} :\ & \mathrm{Hom}_{\mathrm{Ab}}(M \otimes_R L, A) && \to && \mathrm{Hom}_{R\text{-Mod}}(L, \mathrm{Hom}_{\mathrm{Ab}}(M, A))\\
& \chi : M\otimes_R L \to A, && \mapsto && \psi : L \to \mathrm{Hom}_{\mathrm{Ab}}(M, A),\ l \mapsto \psi_l \text{ with } \psi_l(m) = \chi(m \otimes l).
\end{aligned}$$

As we have $\psi_{r\rhd l}(m) = \chi(m \otimes (r \rhd l)) = \chi((m \lhd r) \otimes l) = \psi_l(m \lhd r)$, the map ψ_l is indeed R-linear, and a short computation shows that the diagram (3) commutes for all R-linear maps $f : L' \to L$ and all group homomorphisms $g : A \to A'$.

7. **Restriction, induction and coinduction**:
Let $\phi : R \to S$ be a ring homomorphism and $\mathrm{Res} : S\text{-Mod} \to R\text{-Mod}$ the **restriction functor** from Example 1.2.5, 5. that sends an S-module $(M, \rhd_S)$ to the R-module $(M, \rhd_R)$ with $r \rhd_R m = \phi(r) \rhd_S m$ and every S-linear map $f : M \to M'$ to itself. Then:

• The **induction functor** $\mathrm{Ind} = S\otimes_R- : R\text{-Mod} \to S\text{-Mod}$ is left adjoint to Res. It sends
- an R-module M to the S-module $\mathrm{Ind}(M) = S \otimes_R M$ with $s \rhd (s'\otimes m) = (ss')\otimes m$,
- an R-linear map $f : M \to M'$ to the S-linear map $\mathrm{Ind}(f) = \mathrm{id}_S \otimes f$.

- The **coinduction functor** $\mathrm{Coind} = \mathrm{Hom}_R(S, -) : R\text{-Mod} \to S\text{-Mod}$ is right adjoint to Res. It sends
  - an R-module M to the S-module $\mathrm{Hom}_R(S, M)$ with $(s \rhd f)(s') = f(s' \cdot s)$,
  - an R-linear map $f : M \to M'$ to $\mathrm{Hom}_R(S, f) : g \mapsto f \circ g$.

  To see that Ind is left adjoint to Res, note that by Lemma 1.1.27 the (S, R)-bimodule structure on S given by $s \rhd s' = s \cdot s'$ and $s \lhd r = s \cdot \phi(r)$ defines an S-left-module structure on the abelian group $S \otimes_R M$ given by $s \rhd (s' \otimes m) = (s \cdot s') \otimes m$. For all R-modules M and S-modules N the group homomorphisms

  $$\begin{aligned} &\phi_{M,N} : \mathrm{Hom}_R(M, \mathrm{Res}(N)) \to \mathrm{Hom}_S(\mathrm{Ind}(M), N), &&\phi_{M,N}(f)(s \otimes m) = s \rhd f(m) \\ &\psi_{M,N} : \mathrm{Hom}_S(\mathrm{Ind}(M), N) \to \mathrm{Hom}_R(M, \mathrm{Res}(N)), &&\psi_{M,N}(g)(m) = g(1 \otimes m). \end{aligned}$$

  are mutually inverse and hence bjections. To prove that the diagram (3) commutes, we compute for all R-linear maps $f : M' \to M$, $h : M \to N$ and S-linear maps $g : N \to N'$

  $$\begin{aligned} g \circ \phi_{M,N}(h) \circ (\mathrm{id}_S \otimes f)(s \otimes m') &= g \circ \phi_{M,N}(h)(s \otimes f(m')) = g(s \rhd h \circ f(m')) \\ &= s \rhd (g \circ h \circ f(m')) = \phi_{M',N'}(g \circ h \circ f)(s \otimes m'). \end{aligned}$$

  To show that Coind is right adjoint to Res we consider the ring S with the R-left module structure $r \rhd s := \phi(r) \cdot s$ and the abelian group $\mathrm{Hom}_R(S, M)$ with the S-left module structure $(s \rhd f)(s') = f(s' \cdot s)$ and note that the maps

  $$\begin{aligned} &\phi_{M,N} : \mathrm{Hom}_R(\mathrm{Res}(N), M) \to \mathrm{Hom}_S(N, \mathrm{Hom}_R(S, M)), &&\phi_{M,N}(f)(s) = f(s \rhd n) \\ &\psi_{M,N} : \mathrm{Hom}_S(N, \mathrm{Hom}_R(S, M)) \to \mathrm{Hom}_R(\mathrm{Res}(N), M), &&\psi_{M,N}(g)(n) = g(n)(1). \end{aligned}$$

  are mutually inverse and hence bijections. A short computation shows that $\phi_{M,N}$ makes the diagram (3) commute.

8. **Induction, coinduction and forgetful functor**:
   For every ring S, the induction functor $\mathrm{Ind} = S \otimes_{\mathbb{Z}} - : \mathrm{Ab} \to S\text{-Mod}$ is left adjoint and the coinduction functor $\mathrm{Coind} = \mathrm{Hom}_{\mathbb{Z}}(S, -) : \mathrm{Ab} \to S\text{-Mod}$ is right adjoint to the forgetful functor $\mathrm{Res} : S\text{-Mod} \to \mathrm{Ab}$.

   This is Example 1.2.18, 7. for $R = \mathbb{Z}$, where $\mathrm{Res} : S\text{-Mod} \to \mathrm{Ab}$ is the forgetful functor.

These examples show that adjoint functors arise in many contexts in algebra and topology and are often related to certain canonical constructions such as forgetful functors, freely generated modules or tensoring over a ring. Example 1.2.18, 3. shows that a functor need not have left and right adjoints. However, it seems plausible that if they exist, left or right adjoint functors should be unique, at least up to natural isomorphisms. To address this, we work with an alternative characterisation of left and right adjoints in terms of natural transformations.

Proposition 1.2.19: A functor $F : \mathcal{C} \to \mathcal{D}$ is left adjoint to $G : \mathcal{D} \to \mathcal{C}$ if and only if there are natural transformations $\epsilon : FG \to \mathrm{id}_{\mathcal{D}}$ and $\eta : \mathrm{id}_{\mathcal{C}} \to GF$ such that

$$(G\epsilon) \circ (\eta G) = \mathrm{id}_G, \qquad (\epsilon F) \circ (F\eta) = \mathrm{id}_F. \tag{4}$$

Proof:
1. Let $F : \mathcal{C} \to \mathcal{D}$ be left adjoint to $G : \mathcal{D} \to \mathcal{C}$. Then there are bijections

$$\begin{aligned} &\phi_{G(D),D} : \mathrm{Hom}_{\mathcal{C}}(G(D), G(D)) \to \mathrm{Hom}_{\mathcal{D}}(FG(D), D) \\ &\phi^{-1}_{C,F(C)} : \mathrm{Hom}_{\mathcal{D}}(F(C), F(C)) \to \mathrm{Hom}_{\mathcal{C}}(C, GF(C)). \end{aligned}$$

We define the natural transformations $\epsilon : FG \to \mathrm{id}_{\mathcal{D}}$ and $\eta : \mathrm{id}_{\mathcal{C}} \to GF$ by specifying their component morphisms:

$$\epsilon_D := \phi_{G(D),D}(1_{G(D)}) : FG(D) \to D \qquad \eta_C := \phi^{-1}_{C,F(C)}(1_{F(C)}) : C \to GF(C).$$

The commuting diagram (3) in Definition 1.2.17 implies for all morphisms $f : D \to D'$ in $\mathcal{D}$:

$$\begin{aligned} \epsilon_{D'} \circ FG(f) &= \phi_{G(D'),D'}(1_{G(D')}) \circ FG(f) \overset{(3)}{=} \phi_{G(D),D}(1_{G(D')} \circ G(f)) = \phi_{G(D),D}(G(f)) \\ &= \phi_{G(D),D}(G(f) \circ 1_{G(D)}) \overset{(3)}{=} f \circ \phi_{G(D),D}(1_{G(D)}) = f \circ \epsilon_D. \end{aligned}$$

This shows that the morphisms $\epsilon_D : FG(D) \to D$ define a natural transformation $\epsilon : FG \to \mathrm{id}_{\mathcal{D}}$. Diagram (3) then implies for all objects C in $\mathcal{C}$

$$\begin{aligned} \epsilon_{F(C)} \circ F(\eta_C) &= \phi_{GF(C),F(C)}(1_{GF(C)}) \circ F(\phi^{-1}_{C,F(C)}(1_{F(C)})) \\ &\overset{(3)}{=} \phi_{C,F(C)}(1_{GF(C)} \circ \phi^{-1}_{C,F(C)}(1_{F(C)})) = \phi_{C,F(C)} \circ \phi^{-1}_{C,F(C)}(1_{F(C)}) = 1_{F(C)}. \end{aligned}$$

The proofs for $\eta : \mathrm{id}_{\mathcal{C}} \to GF$ and of the identity $G(\epsilon_D) \circ \eta_{G(D)} = 1_{G(D)}$ are analogous.

2. Let $\epsilon : FG \to \mathrm{id}_{\mathcal{D}}$ and $\eta : \mathrm{id}_{\mathcal{C}} \to GF$ be natural transformations that satisfy (4). Consider for all objects C in $\mathcal{C}$ und D in $\mathcal{D}$ the maps

$$\begin{aligned} \phi_{C,D} &= \mathrm{Hom}(1_{F(C)}, \epsilon_D) \circ F : \mathrm{Hom}_{\mathcal{C}}(C, G(D)) \to \mathrm{Hom}_{\mathcal{D}}(F(C), D), \quad f \mapsto \epsilon_D \circ F(f) \\ \psi_{C,D} &= \mathrm{Hom}(\eta_C, 1_{G(D))}) \circ G : \mathrm{Hom}_{\mathcal{D}}(F(C), D) \to \mathrm{Hom}_{\mathcal{C}}(C, G(D)), \quad g \mapsto G(g) \circ \eta_C. \end{aligned}$$

Then we have for all morphisms $f : C \to G(D)$ in $\mathcal{C}$ and $g : F(C) \to D$ in $\mathcal{D}$

$$\begin{aligned} \psi_{C,D} \circ \phi_{C,D}(f) &= G(\epsilon_D) \circ GF(f) \circ \eta_C \overset{\mathrm{nat}}{=} G(\epsilon_D) \circ \eta_{G(D)} \circ f \overset{(4)}{=} f \\ \phi_{C,D} \circ \psi_{C,D}(g) &= \epsilon_D \circ FG(g) \circ F(\eta_C) \overset{\mathrm{nat}}{=} g \circ \epsilon_{F(C)} \circ F(\eta_C) \overset{(4)}{=} g. \end{aligned}$$

This shows that $\psi_{C,D} = \phi^{-1}_{C,D}$ and $\phi_{C,D} : \mathrm{Hom}_{\mathcal{C}}(C, G(D)) \to \mathrm{Hom}_{\mathcal{D}}(F(C), D)$ is a bijection. To verify that the diagram (3) in Definition 1.2.17 commutes, consider morphisms $f : C' \to C$, $h : C \to G(D)$ in $\mathcal{C}$ and $g : D \to D'$ in $\mathcal{D}$ and compute

$$\phi_{C',D'}(G(g) \circ h \circ f) = \epsilon_{D'} \circ FG(g) \circ F(h) \circ F(f) \overset{\mathrm{nat}}{=} g \circ \epsilon_D \circ F(h) \circ F(f) = g \circ \phi_{C,D}(h) \circ F(f).$$

□

Theorem 1.2.20: Left and right adjoint functors are unique up to natural isomorphisms.

Proof:
Let $F, F' : \mathcal{C} \to \mathcal{D}$ be left adjoint to $G : \mathcal{D} \to \mathcal{C}$. Then by Proposition 1.2.19 there are natural transformations $\epsilon : FG \to \mathrm{id}_{\mathcal{D}}$, $\eta : \mathrm{id}_{\mathcal{C}} \to GF$ and $\epsilon' : F'G \to \mathrm{id}_{\mathcal{D}}$, $\eta' : \mathrm{id}_{\mathcal{C}} \to GF'$ satisfying (4). Consider the natural transformations $\kappa = (\epsilon F') \circ (F\eta') : F \to F'$, $\kappa' = (\epsilon' F) \circ (F'\eta) : F' \to F$ with component morphisms $\kappa_C = \epsilon_{F'(C)} \circ F(\eta'_C)$ and $\kappa'_C = \epsilon'_{F(C)} \circ F'(\eta_C)$. Then κ_C und κ'_C are inverse to each other since

$$\begin{aligned} \kappa_C \circ \kappa'_C &\overset{\mathrm{def}\,\kappa}{=} \epsilon_{F'(C)} \circ F(\eta'_C) \circ \kappa'_C \overset{\mathrm{nat}\,\kappa'}{=} \epsilon_{F'(C)} \circ \kappa'_{GF'(C)} \circ F'(\eta'_C) \\ &\overset{\mathrm{def}\,\kappa'}{=} \epsilon_{F'(C)} \circ \epsilon'_{FGF'(C)} \circ F'(\eta_{GF'(C)}) \circ F'(\eta'_C) \overset{\mathrm{nat}\,\epsilon'}{=} \epsilon'_{F'(C)} \circ F'G(\epsilon_{F'(C)}) \circ F'(\eta_{GF'(C)}) \circ F'(\eta'_C) \\ &= \epsilon'_{F'(C)} \circ F'(G(\epsilon_{F'(C)}) \circ \eta_{GF'(C)}) \circ F'(\eta'_C) \overset{(4)}{=} \epsilon'_{F'(C)} \circ F'(\eta'_C) \overset{(4)}{=} 1_{F'(C)}, \end{aligned}$$

and an analogous computation yields $\kappa'_C \circ \kappa_C = 1_{F(C)}$. This shows that κ and κ' are natural isomorphisms and that F is naturally isomorphic to F'. The proof for right adjoints is analogous.
□

2 Examples of (co)homologies

In this section, we introduce examples of homology and cohomology theories and illustrate how they encode information about different mathematical objects such as topological spaces, simplicial complexes, bimodules over algebras, modules over group rings and representations of Lie algebras. A homology theory associates to a mathematical object X a family $(X_n)_{n\in\mathbb{N}_0}$ of modules over a ring and a family $(d_n)_{n\in\mathbb{N}_0}$ of module morphisms $d_n : X_n \to X_{n-1}$ that satisfy the condition $d_n \circ d_{n+1} = 0$ for all $n \in \mathbb{N}_0$. This ensures that $\mathrm{im}(d_{n+1}) \subset \ker(d_n) \subset X_n$ are submodules, and one can form the quotient module $H_n(X) = \ker(d_n)/\mathrm{im}(d_{n+1})$. These quotients are called the homologies of X and encode information about X.

2.1 Singular and simplicial homologies of topological spaces

Historically, the first homology theories were homology theories of topological spaces. The wish to unify different notions of homology for topological spaces was one of the main motivations to develop an abstract formalism. The basic idea is to probe a topological space with certain standard subspaces $\mathbb{R}^n$ that can be described in a mostly combinatorial way. These are the affine simplexes.

Definition 2.1.1: Let $(e_1, ..., e_n)$ be the standard basis of $\mathbb{R}^n$ and $e_0 := 0 \in \mathbb{R}^n$.

1. An **affine m-simplex** $\Delta \subset \mathbb{R}^n$ is the convex hull of $m+1$ points $v_0, ..., v_m \in \mathbb{R}^n$

$$\Delta = \mathrm{conv}(\{v_0, ..., v_m\}) = \{\Sigma_{i=0}^m \lambda_i v_i \mid 0 \leq \lambda_i \leq 1, \Sigma_{i=0}^m \lambda_i = 1\}.$$

   The points $v_0, ..., v_m$ are called the **vertices** of Δ.

2. The k-simplexes $\mathrm{conv}(\{v_{i_0}, ..., v_{i_k}\})$ for subsets $\{v_{i_0}, ..., v_{i_k}\} \subset \{v_0, ..., v_m\}$ with $k+1$ elements are called the **k-faces** of Δ.

3. An **ordered m-simplex** is an affine m-simplex with an ordering of its vertices. We write $[v_0, ..., v_m]$ for $\Delta = \mathrm{conv}(\{v_0, ..., v_m\})$ with ordering $v_0 < v_1 < ... < v_m$.

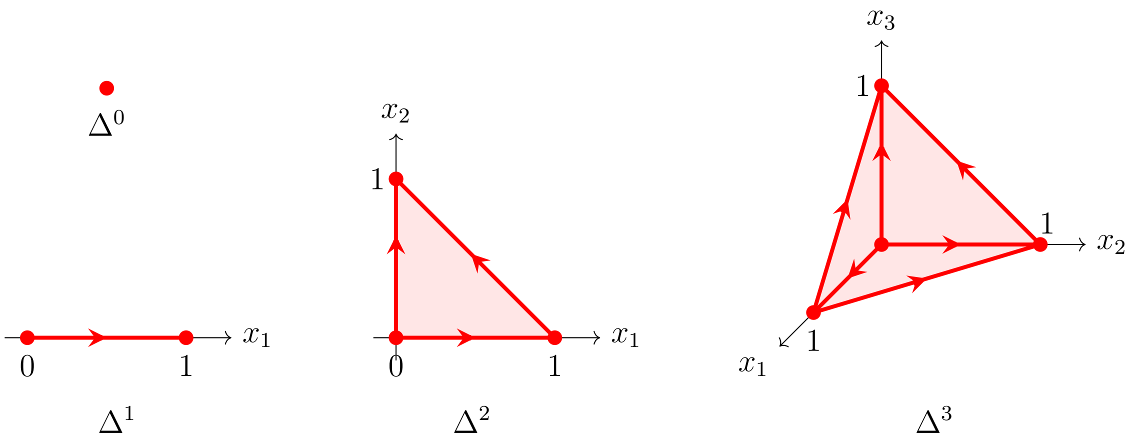
4. For $n \in \mathbb{N}_0$ the **standard n-simplex** $\Delta^n \subset \mathbb{R}^n$ is the ordered n-simplex $[e_0, ..., e_n]$.

5. For $n \in \mathbb{N}$ and $i \in \{0, ..., n\}$ the ith **face map** is the affine linear map $f_i^n : \Delta^{n-1} \to \Delta^n$

$$f_i^n(e_j) = \begin{cases} e_j & j < i \\ e_{j+1} & j \geq i. \end{cases}$$

   that sends $\Delta^{n-1} = [e_0, ..., e_{n-1}]$ to the $(n-1)$-face $[e_0, ...e_{i-1}, e_{i+1}, ..., e_n]$ opposite e_i.

The ordering of an affine m-simplex is pictured by drawing an arrow on each 1-face that points from its vertex of lower order to its vertex of higher order. Note that the face maps respect the ordering of vertices in the standard n-simplexes. They omit vertices but do not change their ordering. Hence, the ordering of the vertices in the $(n-1)$-face $f_i^n(\Delta^{n-1}) \subset \Delta^n$ induced by the ordering of Δ^{n-1} coincides with the one induced by the ordering of Δ^n.



The standard n-simplexes for $n = 0, 1, 2, 3$.

Topological homology theories probe a topological space X by studying continuous maps $\sigma : \Delta^n \to X$ for all $n \in \mathbb{N}_0$. For this, one must decide which continuous maps σ to consider - all of them or only a specific set of continuous maps that satisfy certain compatibility conditions. Different choices lead to different versions of homology. In the following, we focus on two main examples, namely *singular* and *simplicial homology*. The former admits all continuous maps $\sigma : \Delta^n \to X$, even very singular ones that map the entire simplex to a single point. The latter is based on collections of maps that are homeomorphisms onto their image when restricted to the interior of the standard n-simplex and satisfy certain matching conditions.

Definition 2.1.2: Let k be a commutative ring, X a topological space and $n \in \mathbb{N}_0$.

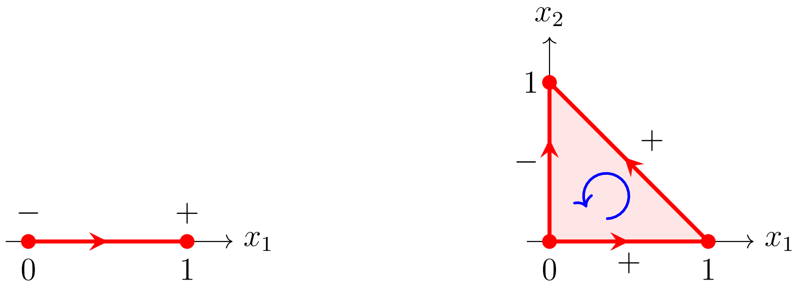
1. A **singular n-simplex** is a continuous map $\sigma : \Delta^n \to X$.
2. The k-module $C_n(X, k)$ of **singular n-chains** is the free k-module generated by the set of singular n-simplexes:

$$C_n(X, k) = \begin{cases} \langle \sigma : \Delta^n \to X \text{ continuous} \rangle_k & n \in \mathbb{N}_0 \\ 0 & n < 0. \end{cases}$$

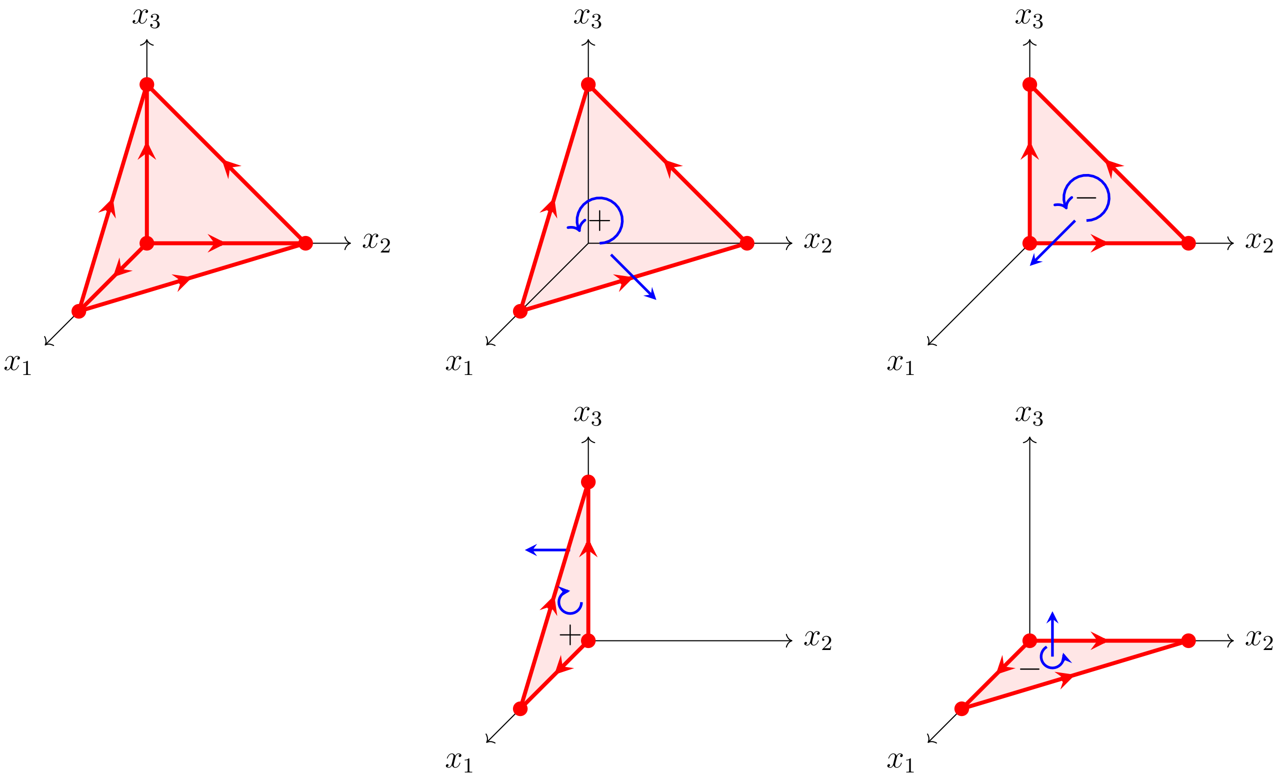
3. The **singular boundary operator** $d_n : C_n(X, k) \to C_{n-1}(X, k)$ is the k-module morphism defined by $d_n = 0$ for $n \leq 0$ and

$$d_n(\sigma) = \sum_{i=0}^{n} (-1)^i \, \sigma \circ f_i^n \qquad \text{for all continuous maps } \sigma : \Delta^n \to X, \; n \in \mathbb{N}.$$

The signs of the boundary operators have a geometrical interpretation and can be visualised easily for $n = 1, 2, 3$. For a 1-simplex $\sigma : \Delta^1 \to X$ the sign in front of the term $\sigma \circ f_i^1$ is $+1$ if the arrow on the ordered 1-simplex $\Delta^1 = [e_0, e_1]$ points towards e_i and -1 if it points away from e_i. For a 2-simplex $\sigma : \Delta^2 \to X$ the sign of the term $\sigma \circ f_i^2$ is given by the orientation of Δ^2. If we orient $\Delta^2 = [e_0, e_1, e_2]$ according to the ordering of the vertices from the vertex of lowest to the vertex of highest order, as indicated by the blue arrow, then the sign is $+1$ if the arrow on the 1-simplex $f_i^2(\Delta^1)$ is oriented parallel to this and -1 if it is oriented against it.



For a 3-simplex $\sigma : \Delta^3 \to X$ the sign in front of the term $\sigma \circ f_i^3$ is given by the *right hand rule*. If one equips each 2-face of Δ^3 with the orientation defined above and the fingers of the right hand follow this orientation, then the sign is $+1$ if the thumb of the right hand points out of Δ^3 and -1 if it points inside Δ^3.



The *boundary operator* is called boundary operator because it assigns to a singular n-simplex $\sigma : \Delta^n \to X$ the alternating sum of the singular $(n-1)$-simplexes $\sigma \circ f_i^n : \Delta^{n-1} \to X$ that are obtained by restricting σ to the $(n-1)$-faces of Δ^n. These $(n-1)$-faces form the boundary $\partial\Delta^n$ of $\Delta^n \subset \mathbb{R}^n$.

The signs in front of the terms $\sigma \circ f_i^n$ ensure that applying the boundary operator twice gives zero. This has a geometrical interpretation. Each $(n-2)$-face f of Δ^n is contained in the boundary of exactly two $(n-1)$-faces. In one of them f is oriented parallel to the orientation of the $(n-1)$-face, in the other against it. Hence, the two contributions have opposite signs and cancel. This encodes the fact that the boundary of the boundary of Δ^n is empty: one has $\partial\Delta^n = \cup_{i=0}^n f_i^n(\Delta^{n-1})$ and $\partial(\partial\Delta^n) = 0$. The algebraic counterpart of this is the following.

Lemma 2.1.3:

1. The face maps satisfy $f_i^n \circ f_{j-1}^{n-1} = f_j^n \circ f_i^{n-1}$ for all $0 \leq i < j \leq n$.
2. The boundary operators $d_n : C_n(X,k) \to C_{n-1}(X,k)$ satisfy $d_{n-1} \circ d_n = 0$ for all $n \in \mathbb{N}$.

Proof:
1. As the face maps are affine linear, they are determined by their values on the vertices $e_0, ..., e_n$ of Δ^n. It is therefore sufficient to check this relation on the vertices.
We have for $0 \leq i \leq j-1 \leq n$

$$f_i^n \circ f_{j-1}^{n-1}(e_k) = \begin{cases} f_i^n(e_k) & k < j-1 \\ f_i^n(e_{k+1}) & k \geq j-1 \end{cases} = \begin{cases} e_k & k < i \\ e_{k+1} & i \leq k < j-1 \\ e_{k+2} & k \geq j-1 \end{cases}$$

$$f_j^n \circ f_i^{n-1}(e_k) = \begin{cases} f_j^n(e_k) & k < i \\ f_j^n(e_{k+1}) & k \geq i \end{cases} = \begin{cases} e_k & k < i \\ e_{k+1} & i \leq k < j-1 \\ e_{k+2} & k \geq j-1 \end{cases}.$$

2. Using these relations, we obtain for all continuous maps $\sigma : \Delta^n \to X$:

$$\begin{aligned} d_{n-1} \circ d_n(\sigma) =& d_{n-1}\left(\Sigma_{j=0}^n (-1)^j \, \sigma \circ f_j^n\right) = \Sigma_{i=0}^{n-1}\Sigma_{j=0}^n (-1)^{i+j} \sigma \circ f_j^n \circ f_i^{n-1} \\ =& \Sigma_{0\leq i<j\leq n}(-1)^{i+j}\sigma \circ f_j^n \circ f_i^{n-1} + \Sigma_{0\leq j\leq i<n}(-1)^{i+j}\sigma \circ f_j^n \circ f_i^{n-1} \\ \overset{1.}{=}& \Sigma_{0\leq i<j\leq n}(-1)^{i+j}\sigma \circ f_i^n \circ f_{j-1}^{n-1} + \Sigma_{0\leq j\leq i<n}(-1)^{i+j}\sigma \circ f_j^n \circ f_i^{n-1} \\ =& \Sigma_{0\leq i\leq j<n}(-1)^{i+j+1}\sigma \circ f_i^n \circ f_j^{n-1} + \Sigma_{0\leq j\leq i<n}(-1)^{i+j}\sigma \circ f_j^n \circ f_i^{n-1} \\ =& \Sigma_{0\leq j\leq i<n}(-1)^{i+j+1}\sigma \circ f_j^n \circ f_i^{n-1} + \Sigma_{0\leq j\leq i<n}(-1)^{i+j}\sigma \circ f_j^n \circ f_i^{n-1} = 0. \end{aligned}$$

As $d_n : C_n(X,k) \to C_{n-1}(X,k)$ is k-linear and the singular n-simplexes $\sigma : \Delta^n \to X$ generate $C_n(X,k)$, this proves the claim. □

Due to the relations $d_n \circ d_{n+1} = 0$ for all $n \in \mathbb{N}_0$, it follows that $\mathrm{im}(d_{n+1}) \subset \ker(d_n)$ is a k-submodule. We can therefore take the quotient $\ker(d_n)/\mathrm{im}(d_{n+1})$, the nth singular homology of the topological space X with values in k.

Definition 2.1.4: Let k be a commutative ring and X a topological space.

1. Elements of $Z_n(X,k) := \ker(d_n) \subset C_n(X,k)$ are called **singular n-cycles**.

2. Elements of $B_n(X,k) := \mathrm{im}(d_{n+1}) \subset Z_n(X,k)$ are called **singular n-boundaries**.

3. The nth **singular homology** of X is the k-module

$$H_n(X,k) = \frac{Z_n(X,k)}{B_n(X,k)}.$$

The nth homology counts the possibilities of combining singular n-simplexes in such a way that there is no boundary, up to those combinations that arise as the boundaries of $(n+1)$-simplexes. This can be viewed as a measure for the number of *holes* in the topological space. For each $(n+1)$-simplex $\sigma : \Delta^{n+1} \to X$, the boundary $d_{n+1}(\sigma)$ is an n-cycle. If we remove a point x in the interior of $\sigma(\Delta^{n+1})$ from X, the continuous map σ is no longer defined, while $d_{n+1}(\sigma)$ still defines an n-cycle in $Z_n(X \setminus \{x\}, k)$. In this way, we have created an n-cycle that is not an n-boundary. To gain more intuition, we determine the first two singular homology groups.

Example 2.1.5: Let X be a topological space and k a commutative ring. Then

$$H_0(X,k) = \frac{\langle X\rangle_k}{\langle \sigma(1)-\sigma(0) \mid \sigma : [0,1]\to X \text{ continuous}\rangle_k} \cong \bigoplus_{\pi_0(X)} k$$

where $\pi_0(X)$ is the set of path components of X.

Proof:
As $\Delta^0 = \{0\}$ all maps $\sigma : \Delta^0 \to X$ are continuous, and they are in bijection with points of X

$$\begin{aligned} &C_0(X,k) = \langle X\rangle_k && d_0 : C_0(X,k) \to \{0\},\ \sigma \mapsto 0 \\ &C_1(X,k) = \langle \sigma : [0,1] \to X \text{ continuous}\rangle_k && d_1 : C_1(X,k) \to C_0(X,k),\ \sigma \mapsto \sigma(1)-\sigma(0). \end{aligned}$$

This yields $Z_0(X,k) = C_0(X,k)$ and $B_0(X,k) = \langle \sigma(1)-\sigma(0) \mid \sigma : [0,1] \to X \text{ continuous}\rangle_k$. Hence, two points $x,y \in X$ are related by a 0-boundary if and only if there is a continuous map $\sigma : [0,1] \to X$ with $\sigma(0) = x$ and $\sigma(1) = y$. Such a map is a path from x to y, and hence $x,y \in X$ are identified if and only if they are in the same path component of X.

By selecting a point x_P in each path component $P \in \pi_0(X)$, we can rewrite any k-linear combination $v = \Sigma_{i=1}^n k_i x_i$ of points $x_i \in X$ uniquely as $v = \Sigma_{i=1}^n k_i x_{P_i} + \sum_{i=1}^n k_i(x_i - x_{P_i})$, where x_{P_i} represents the path component of x_i. This defines an isomorphism $H_0(X,k) \cong \oplus_{\pi_0(x)} k$. □

Given this interpretation of $H_0(X,k)$, it is natural to expect that the first homology group $H_1(X,k)$ should be related to the fundamental group of a topological space X. A 1-chain is a k-linear combination of continuous maps $\sigma : [0,1] \to X$, or, equivalently, of paths in X. The identity $d_1(\sigma) = \sigma(1) - \sigma(0)$ implies that a singular 1-simplex $\sigma : [0,1] \to X$ is a 1-cycle if and only if the path $\sigma : [0,1] \to X$ is closed: $\sigma(0) = \sigma(1)$. One also expects that homotopies between paths with the same endpoints should be related to 2-simplexes.

However, there is an essential difference between the fundamental group $\pi_1(x,X)$ and the first homology group $H_1(X,k)$. The group multiplication of $\pi_1(x,X)$ is induced by the concatenation of paths and in general *not abelian*, whereas the composition of 1-cycles is given by the addition in the *abelian* group $Z_n(X,k)$. For a collection of paths based at x the associated product in the fundamental group $\pi_1(x,X)$ keeps track of the *order* in which the paths are composed, whereas the sum of their homology classes in $H_1(X,k)$ only takes into account *how often* each path in the collection is traversed with or against its orientation. As a consequence, the first homology group and the fundamental group cannot coincide in general.

Instead, for path-connected topological spaces the first homology group $H_1(X,\mathbb{Z})$ is the *abelisation* of the fundamental group $\pi_1(x,X)$. For this, recall that the **commutator subgroup** $[G,G]$ of a group G is the normal subgroup of G generated by the **group commutators** $[g,h] = ghg^{-1}h^{-1}$ of all elements $g,h \in G$ and that the factor group $G/[G,G]$ is abelian. It is called the **abelisation** of G and often denoted $\mathrm{Ab}(G)$. In fact, one can show that abelisation defines a functor $\mathrm{Ab} : \mathrm{Grp} \to \mathrm{Ab}$ from the category Grp of groups to the category Ab of abelian groups (cf. Example 1.2.5 10. and Exercise 7).

Theorem 2.1.6: Let k be a commutative ring and X a path connected topological space.

1. For any $x \in X$ the map $\phi : \pi_1(x,X) \to H_1(X,k)$, $[\gamma]_{\pi_1} \mapsto [\gamma]_{H_1}$ is a group homomorphism.
2. It induces an isomorphism $\phi : \mathrm{Ab}(\pi_1(x,X)) \to H_1(X,\mathbb{Z})$, the **Huréwicz isomorphism**.

Proof:
1. We show that $\phi : \pi_1(x, X) \to H_1(X, k)$, $[\gamma]_{\pi_1} \mapsto [\gamma]_{H_1}$ is well-defined.

Note first that any path $\gamma : [0, 1] \to X$ with $\gamma(0) = \gamma(1)$ is a singular 1-cycle, since we have $\Delta^1 = [0, 1]$ and $d_1(\gamma) = \gamma \circ f_0^1 - \gamma \circ f_1^1 = \gamma(1) - \gamma(0) = 0$.

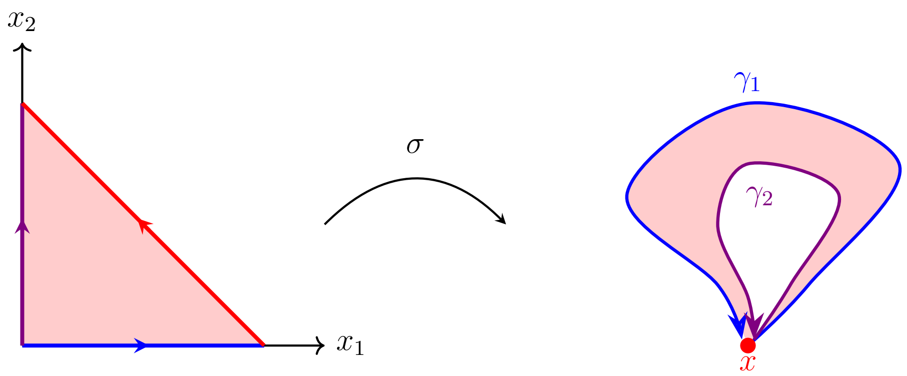
It remains to show that homotopic paths are related by a 1-boundary. Let $\gamma_1, \gamma_2 : [0, 1] \to X$ be paths with $\gamma_i(0) = \gamma_i(1) = x$ and $h : [0, 1] \times [0, 1] \to X$ a homotopy from γ_1 to γ_2. From the homotopy h we construct a map $\sigma : \Delta^2 \to X$ defined by

$$\sigma(s, t) = h(\tfrac{t}{s+t}, s + t) \text{ for } (s, t) \neq (0, 0), \qquad \sigma(0, 0) = x.$$

This map is continuous since $h : [0, 1] \times [0, 1] \to X$ is continuous with $h(s, 0) = x$ for all $s \in [0, 1]$. By applying the boundary operator, we obtain $d_2(\sigma) = \sigma \circ f_0^2 - \sigma \circ f_1^2 + \sigma \circ f_2^2$ with

$$\sigma \circ f_0^2(t) = \sigma(1 - t, t) = x, \qquad \sigma \circ f_1^2(t) = \sigma(0, t) = \gamma_2(t), \qquad \sigma \circ f_2^2(t) = \sigma(t, 0) = \gamma_1(t).$$

Hence, σ sends the face $[e_1, e_2]$ of Δ^2 to x, the face $[e_0, e_2]$ to $\mathrm{im}(\gamma_2)$ and the face $[e_0, e_1]$ to $\mathrm{im}(\gamma_1)$. We have $d_2(\sigma) = \gamma_x - \gamma_2 + \gamma_1$ with the constant 1-cycle $\gamma_x : [0, 1] \to X$, $t \mapsto x$.



As the constant 1-cycle γ_x is a boundary $\gamma_x = d_2(\rho_x)$ of the constant 2-simplex $\rho_x : \Delta^2 \to X$, $(s, t) \mapsto x$, we have $0 = [\gamma_x]_{H_1} = [d_2(\sigma)]_{H_1} + [\gamma_2]_{H_1} - [\gamma_1]_{H_1} = [\gamma_2]_{H_1} - [\gamma_1]_{H_1}$. This shows that $[\gamma]_{H_1}$ depends only on the homotopy class of γ and ϕ is well-defined.

2. We show that $\phi : \pi_1(x, X) \to H_1(X, k)$, $[\gamma]_{\pi_1} \mapsto [\gamma]_{H_1}$ is a group homomorphism. For this, it is sufficient to prove that for all paths $\gamma_1, \gamma_2 : [0, 1] \to X$ with $\gamma_i(0) = \gamma_i(1) = x$ their concatenation $\gamma_2 \star \gamma_1$ agrees with the sum $\gamma_1 + \gamma_2$ of the associated 1-cycles up to a 1-boundary.

By composing the path $\gamma_2 \star \gamma_1$ with the affine map $g : \Delta^2 \to [0, 1]$, $(s, t) \mapsto \frac{1}{2}s + t$, we obtain a 2-simplex $\sigma = (\gamma_2 \star \gamma_1) \circ g : \Delta^2 \to X$, $(s, t) \mapsto (\gamma_2 \star \gamma_1)(\frac{s}{2} + t)$. Its boundary is given by

$$d_2(\sigma) = \gamma_1 + \gamma_2 - \gamma_2 \star \gamma_1,$$

since we have

$$\begin{aligned}
&\sigma \circ f_0^2(t) = \sigma(1 - t, t) = \gamma_2 \star \gamma_1(\tfrac{1}{2} + \tfrac{t}{2}) = \gamma_2(t) \\
&\sigma \circ f_1^2(t) = \sigma(0, t) = \gamma_2 \star \gamma_1(t) \\
&\sigma \circ f_2^2(t) = \sigma(t, 0) = \gamma_2 \star \gamma_1(g(t, 0)) = \gamma_2 \star \gamma_1(\tfrac{t}{2}) = \gamma_1(t).
\end{aligned}$$

This implies

$$\phi([\gamma_1]_{\pi_1}) + \phi([\gamma_2]_{\pi_1}) = [\gamma_1]_{H_1} + [\gamma_2]_{H_1} = [\gamma_2 \star \gamma_1]_{H_1} = \phi([\gamma_2 \star \gamma_1]_{\pi_1}) = \phi([\gamma_2]_{\pi_1} \cdot [\gamma_1]_{\pi_1}).$$

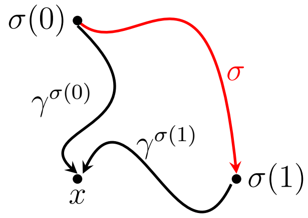
As $H_1(X,k)$ is abelian, we have $[\pi_1(x,X), \pi_1(x,X)] \subset \ker(\phi)$ and obtain a a group homomorphism $\phi : \mathrm{Ab}(\pi_1(x,X)) \to H_1(X,k)$.

3. Suppose now that $k = \mathbb{Z}$. We show that $\phi : \mathrm{Ab}(\pi_1(x,X)) \to H_1(X,\mathbb{Z})$ is a group isomorphism by constructing its inverse.

We choose for every point $y \in X$ a path $\gamma^y : [0,1] \to X$ with $\gamma^y(0) = y$, $\gamma^y(1) = x$ and define

$$K : C_1(X,\mathbb{Z}) \to \mathrm{Ab}(\pi_1(x,X)), \qquad \sigma \mapsto [\gamma^{\sigma(1)} \star \sigma \star \overline{\gamma}^{\sigma(0)}]_{\mathrm{Ab}(\pi_1)}$$

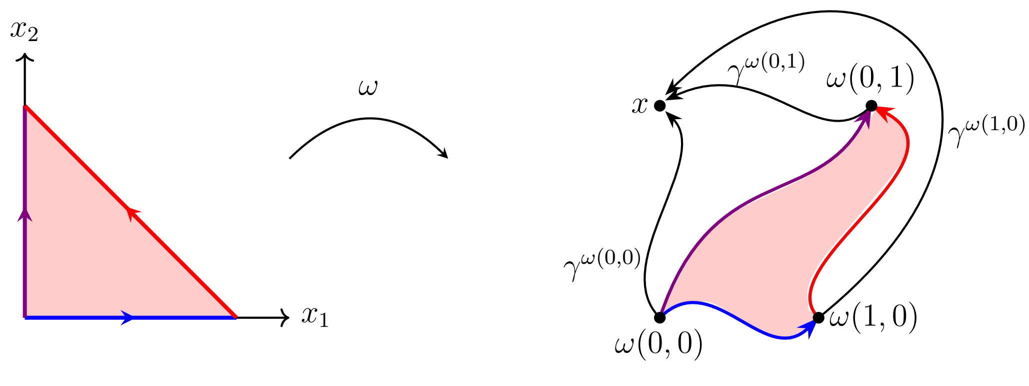
for singular 1-simplexes $\sigma : [0,1] \to X$. As $C_1(X,\mathbb{Z})$ is the free abelian group generated by the singular 1-simplexes, this defines a group homomorphism.



To show that the group homomorphism $K : C_1(X,\mathbb{Z}) \to \mathrm{Ab}(\pi_1(x,X))$ induces a group homomorphism $K : H_1(X,\mathbb{Z}) \to \mathrm{Ab}(\pi_1(x,X))$, we show that $K(d_2(\omega)) = 0$ for every singular 2-simplex $\omega : \Delta^2 \to X$:

$$\begin{aligned} K(d_2\omega) &= K(\omega \circ f_0^2 - \omega \circ f_1^2 + \omega \circ f_2^2) = K(\omega \circ f_0^2) - K(\omega \circ f_1^2) + K(\omega \circ f_2^2) \\ &= [\gamma^{\omega(0,1)} \star (\omega \circ f_0^2) \star \overline{\gamma}^{\omega(1,0)}]_{\mathrm{Ab}(\pi_1)} - [\gamma^{\omega(0,1)} \star (\omega \circ f_1^2) \star \overline{\gamma}^{\omega(0,0)}]_{\mathrm{Ab}(\pi_1)} \\ &+ [\gamma^{\omega(1,0)} \star (\omega \circ f_2^2) \star \overline{\gamma}^{\omega(0,0)}]_{\mathrm{Ab}(\pi_1)} \\ &= [\gamma^{\omega(0,0)} \star (\overline{\omega \circ f_1^2}) \star \overline{\gamma}^{\omega(0,1)} \star \gamma^{\omega(0,1)} \star (\omega \circ f_0^2) \star \overline{\gamma}^{\omega(1,0)} \star \gamma^{\omega(1,0)} \star (\omega \circ f_2^2) \star \overline{\gamma}^{\omega(0,0)}]_{\mathrm{Ab}(\pi_1)} \\ &= [\gamma^{\omega(0,0)} \star (\overline{\omega \circ f_1^2}) \star (\omega \circ f_0^2) \star (\omega \circ f_2^2) \star \overline{\gamma}^{\omega(0,0)}]_{\mathrm{Ab}(\pi_1)} = [\gamma]_{\mathrm{Ab}(\pi_1)}, \end{aligned}$$

where $\gamma : [0,1] \to X$ is a loop with base point x that circles the boundary $\partial\omega(\Delta^2) \subset X$ counterclockwise and we suppress the bracketing in the concatenation of paths. As γ is null homotopic, we have $K(d_2\omega) = 0$. This implies $B_1(X,\mathbb{Z}) \subset \ker(K)$, and K induces a group homomorphism $K : H_1(X,\mathbb{Z}) \to \mathrm{Ab}(\pi_1(x,X))$.



We show that $K : H_1(X,\mathbb{Z}) \to \mathrm{Ab}(\pi_1(x,X))$ is the inverse of $\phi : \mathrm{Ab}(\pi_1(x,X)) \to H_1(X,\mathbb{Z})$. For any path $\delta : [0,1] \to X$ with $\delta(0) = \delta(1) = x$, we have

$$K \circ \phi([\delta]_{\mathrm{Ab}(\pi_1)}) = [\gamma^x \star \delta \star \overline{\gamma}^x]_{\mathrm{Ab}(\pi_1)} = [\gamma^x]_{\mathrm{Ab}(\pi_1)} - [\gamma^x]_{\mathrm{Ab}(\pi_1)} + [\delta]_{\mathrm{Ab}(\pi_1)} = [\delta]_{\mathrm{Ab}(\pi_1)}$$
$$\phi \circ K([\delta]_{H_1}) = [\gamma^x \star \delta \star \overline{\gamma}^x]_{H_1} = [\gamma^x]_{H_1} + [\overline{\gamma}^x]_{H_1} + [\delta]_{H_1} = [\delta]_{H_1},$$

Hence $K = \mathrm{Ab}(\phi)^{-1}$ and $\phi : \mathrm{Ab}(\pi_1(x,X)) \to H_1(X,\mathbb{Z})$ is a group isomorphism. □

Remark 2.1.7: There are analogues of this statement for higher homology and homotopy groups, the **Huréwicz theorem**:

1. For any commutative ring k, path connected topological space X and point $x \in X$ there are group homomorphisms $\phi_n : \pi_n(x,X) \to H_n(X,k)$ for all $n \geq 2$.
2. If $k = \mathbb{Z}$ and X is $(n-1)$**-connected**, that is non-empty and path-connected with $\pi_k(x,X) = \{1\}$ for $1 \leq k \leq n-1$, then ϕ_n is a group isomorphism.

The Huréwicz theorem clarifies the geometrical interpretation of the singular homology groups $H_n(X,\mathbb{Z})$. For $n > 1$ no abelisation is required since the homotopy group $\pi_n(x,X)$ is already abelian. For $(n-1)$-connected topological spaces it reduces the computation of $\pi_n(x,X)$ to the computation of the nth homology group $H_n(X,\mathbb{Z})$.

The Huréwicz theorem is useful for the computation of higher homotopy groups, because it relates them to homology groups, which are computed more easily than homotopy groups. However, the computation of singular homology groups is still difficult without a better understanding of their properties. The nth homology group $H_n(X,k)$ is a quotient of a huge k-module, the k-module of singular n-cycles, by another huge k-module, the k-module of singular n-boundaries, and it is difficult to compute this quotient.

This suggests that one should obtain a simpler and more computable notion of homology by considering a smaller family of continuous maps $\sigma : \Delta^n \to X$. Clearly, the images of the maps in this family should still cover X. To be able to restrict the boundary operator to this family, one must impose that for each simplex $\sigma : \Delta^n \to X$ in this family all simplexes $\sigma \circ f_i^n : \Delta^{n-1} \to X$ are also contained in it. Finally, to work with a family that is as small as possible, it makes sense to impose that the simplexes $\sigma : \Delta^n \to X$ are injective at least in the interior of Δ^n and that the images of different n-simplexes overlap only along the images of k-simplexes for $k < n$. Finally, the topology on X should be compatible with the topology induced by the simplexes in the family, i. e. be the final topology induced by them.

Definition 2.1.8: A (finite) **Δ-complex** or **semisimplicial complex** is a topological space X, together with a (finite) family $\{\sigma_\alpha\}_{\alpha \in I}$ of continuous maps $\sigma_\alpha : \Delta^{n_\alpha} \to X$ such that:

(S1) The maps $\sigma_\alpha|_{\mathring{\Delta}^{n_\alpha}} : \mathring{\Delta}^{n_\alpha} \to X$ are injective for all $\alpha \in I$.

(S2) For every point $x \in X$ there is a unique $\alpha \in I$ with $x \in \sigma_\alpha(\mathring{\Delta}^{n_\alpha})$.

(S3) For every $\alpha \in I$ and $i \in \{0, ..., n_\alpha\}$ there is a $\beta \in I$ with $\sigma_\alpha \circ f_i^{n_\alpha} = \sigma_\beta : \Delta^{n_\alpha - 1} \to X$.

(S4) The topology on X is the final topology induced by the family $\{\sigma_\alpha\}_{\alpha \in I}$:
A subset $A \subset X$ is open if and only if $\sigma_\alpha^{-1}(A) \subset \Delta^{n_\alpha}$ is open for all $\alpha \in I$.

A semisimplicial complex is called a **simplicial complex** if

(S5) For each $\alpha \in I$ the images of the vertices of Δ^{n_α} under σ_α are all distinct: $\sigma_\alpha(e_i) \neq \sigma_\alpha(e_j)$ for all $i \neq j \in \{0, ..., n_\alpha\}$.

(S6) $\{\sigma_\alpha(e_0), ..., \sigma_\alpha(e_{n_\alpha})\} = \{\sigma_\beta(e_0), ..., \sigma_\beta(e_{n_\beta})\}$ implies $\alpha = \beta$.

A **subcomplex** of a (semi)simplicial complex $(X, \{\sigma_\alpha\}_{\alpha \in I})$ is a subspace $A \subset X$ together with a subset $J \subset I$ such that $(A, \{\sigma_\alpha\}_{\alpha \in J})$ is a (semi)simplicial complex.

A **simplicial map** $f : (X, \{\sigma\}_{\alpha \in I}) \to (Y, \{\tau_\beta\}_{\beta \in J})$ between (semi)simplicial complexes is a continuous map $f : X \to Y$ such that for each $\alpha \in I$ there is a $\beta \in J$ with $f \circ \sigma_\alpha = \tau_\beta$.

A given topological space may have many (semi)simplicial complex structures. The notion of a *simplicial complex* is more restrictive than the one of a *semisimplicial complex*. Axiom (S5) forbids that the images of distinct vertices of an n-simplex coincide, and condition (S6) forbids that the vertex sets of different simplexes coincide. This allows one to describe a simplicial complex in a purely combinatorial way. Every k-face in a simplicial complex is determined uniquely by its vertices. The price one pays for this is that simplicial complexes usually require a larger number of simplexes and hence lead to lengthier computations. One can show that every semisimplicial complex can be transformed into a simplicial one by a subdivision procedure.

Simplicial n-chains, boundary operators, n-cycles, n-boundaries and homologies are obtained in the same way as their singular counterparts, by restricting attention to singular simplexes in the chosen family. This works because the axiom (S2) ensures that the boundary operator d_n maps the submodule of $C_n(X, k)$ that is generated by the n-simplexes in the family to the submodule of $C_{n-1}(X, k)$ generated by its $(n-1)$-simplexes.

Definition 2.1.9: Let k be a commutative ring, $\Delta = (X, \{\sigma_\alpha\}_{\alpha \in I})$ a semisimplicial complex.

1. The k-module of **simplicial n-chains** is the trivial k module for $n < 0$ and the free k-module $C_n(\Delta, k) = \langle \{\sigma_\alpha \mid \alpha \in I, n_\alpha = n\} \rangle_k$ for $n \in \mathbb{N}_0$.

2. The **simplicial boundary operator** $d_n : C_n(\Delta, k) \to C_{n-1}(\Delta, k)$ is the k-module morphism defined by $d_n = 0$ for $n \leq 0$ and
$$d_n(\sigma_\alpha) = \sum_{i=0}^{n} (-1)^i \, \sigma_\alpha \circ f_i^n \quad \forall n \in \mathbb{N}, \, \alpha \in I \text{ with } n_\alpha = n.$$
The simplicial boundary operators satisfy $d_{n-1} \circ d_n = 0$ for all $n \in \mathbb{Z}$ by Lemma 2.1.3.

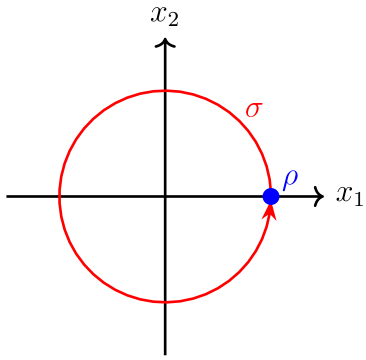
3. The k-modules of **simplicial n-cycles** and **simplicial n-boundaries** are the k-modules $Z_n(\Delta, k) = \ker(d_n) \subset C_n(\Delta, k)$ and $B_n(\Delta, k) = \mathrm{im}(d_{n+1}) \subset Z_n(\Delta, k)$.

4. The nth **simplicial homology** of Δ with values in k is the quotient module
$$H_n(\Delta, k) = \frac{Z_n(\Delta, k)}{B_n(\Delta, k)}.$$

Example 2.1.10:

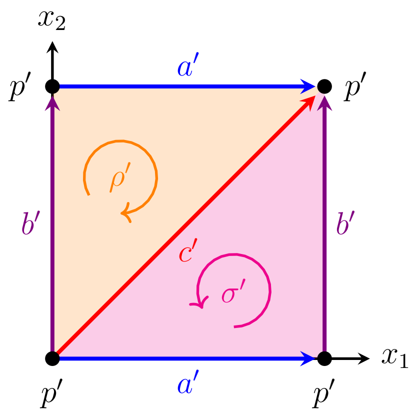
1. A semisimplicial structure on the circle S^1 is given by any continuous map $\sigma : [0, 1] \to S^1$ with $\sigma(0) = \sigma(1) = 1$ and $\sigma|_{(0,1)} : (0, 1) \to S^1$ injective and $\rho : \{0\} \to S^1, 0 \mapsto 1$.

   As $d_1(\sigma) = \sigma(1) - \sigma(0) = 0$ and $d_0(\rho) = 0$, we have $H_0(\Delta, k) \cong Z_0(\Delta, k) = \langle \rho \rangle_k = k$, $H_1(\Delta, k) = Z_1(\Delta, k)/B_1(\Delta, k) = Z_1(\Delta, k) = \langle \sigma \rangle_k = k$ and $H_n(\Delta, k) = \{0\}$ for all $n > 1$.



2. The torus is the quotient $T = [0,1] \times [0,1]/\sim$ with respect to the equivalence relation $(x,0) \sim (x,1)$ and $(0,x) \sim (1,x)$ for all $x \in [0,1]$. It has the structure of a semisimplicial complex with two 2-simplexes, three 1-simplexes and one 0-simplex. They are obtained by composing the canonical surjection $\pi : [0,1] \times [0,1] \to T$ with the affine linear maps

$$\begin{array}{lll} \rho : [e_0, e_1, e_2] \to [e_0, e_2, e_1 + e_2], & \sigma : [e_0, e_1, e_2] \to [e_0, e_1, e_1 + e_2], & p : [e_0] \to [e_0] \\ a : [e_0, e_1] \to [e_0, e_1], & b : [e_0, e_1] \to [e_0, e_2], & c : [e_0, e_1] \to [e_0, e_1 + e_2]. \end{array}$$



Setting $x' = \pi \circ x$ for $x \in \{p, a, b, c, \rho, \sigma\}$, we find that the k-modules of n-chains are given by $C_n(\Delta, k) = 0$ for $n \geq 3$ and

$$C_0(\Delta, k) = \langle p' \rangle \cong k, \quad C_1(\Delta, k) = \langle a', b', c' \rangle_k \cong k \oplus k \oplus k, \quad C_2(\Delta, k) = \langle \rho', \sigma' \rangle_k \cong k \oplus k.$$

The boundary operators are given by

$$d_0(p') = 0, \qquad d_1(a') = d_1(b') = d_1(c') = p' - p' = 0, \qquad d_2(\rho') = d_2(\sigma') = a' + b' - c',$$

and this implies

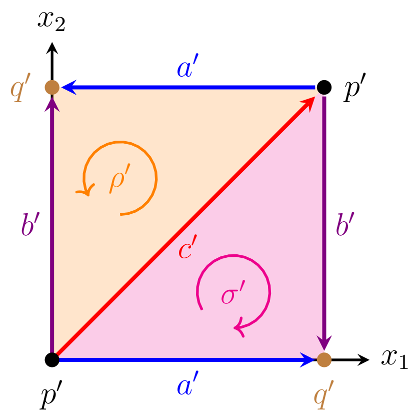
$$\begin{array}{ll} Z_0(\Delta, k) = \langle p' \rangle \cong k & B_0(\Delta, k) = 0 \\ Z_1(\Delta, k) = \langle a', b', c' \rangle \cong k \oplus k \oplus k & B_1(\Delta, k) = \langle a' + b' - c' \rangle \cong k \\ Z_2(\Delta, k) = \langle \rho' - \sigma' \rangle \cong k & B_2(\Delta, k) = 0. \end{array}$$

This yields the simplicial homologies $H_n(\Delta, k) = 0$ for $n > 2$ and

$$\begin{aligned} &H_0(\Delta, k) \cong \langle p' \rangle_k \cong k, \\ &H_1(\Delta, k) \cong \langle a', b', c' \rangle_k / \langle a' + b' - c' \rangle \cong \langle a', b' \rangle_k \cong k \oplus k, \\ &H_2(\Delta, k) \cong \langle \rho' - \sigma' \rangle_k \cong k. \end{aligned}$$

3. Real projective space $\mathbb{R}\mathrm{P}^2$ is the quotient $\mathbb{R}\mathrm{P}^2 = [0,1] \times [0,1]/\sim$ with the equivalence relation $(x,1) \sim (1-x,0)$ and $(0,x) \sim (1,1-x)$ for all $x \in [0,1]$. It has a semisimplicial structure with two 2-simplexes, three 1-simplexes and two 0 -simplexes which are obtained by composing the canonical surjection $\pi : [0,1] \times [0,1] \to \mathbb{R}\mathrm{P}^2$ with the affine simplexes

$$\begin{aligned}
&\rho : [e_0, e_1, e_2] \to [e_0, e_1 + e_2, e_2], \qquad \sigma : [e_0, e_1, e_2] \to [e_0, e_1 + e_2, e_1],\\
&a : [e_0, e_1] \to [e_0, e_1], \quad b : [e_0, e_1] \to [e_0, e_2], \quad c : [e_0, e_1] \to [e_0, e_1 + e_2],\\
&p : [e_0] \to [e_0], \quad q : [e_0] \to [e_1].
\end{aligned}$$



Setting $x' = \pi \circ x$ for $x \in \{p, q, a, b, c, \rho, \sigma\}$ we have

$$C_0(\Delta, k) = \langle p', q' \rangle_k \cong k \oplus k, \quad C_1(\Delta, k) = \langle a', b', c' \rangle_k \cong k \oplus k \oplus k, \quad C_2(\Delta, k) = \langle \rho', \sigma' \rangle_k \cong k \oplus k.$$

and $C_n(\Delta, k) = 0$ for $n \geq 3$. The boundary operators are given by

$$\begin{aligned}
&d_0(p') = d_0(q') = 0, \qquad && d_1(a') = d_1(b') = q' - p', \qquad && d_1(c') = p' - p' = 0,\\
&d_2(\sigma') = b' - a' + c', && d_2(\rho') = a' - b' + c',
\end{aligned}$$

and this implies

$$\begin{aligned}
Z_0(\Delta, k) &= \langle p', q' \rangle_k \cong k \oplus k,\\
B_0(\Delta, k) &= \langle q' - p' \rangle_k \cong k\\
Z_1(\Delta, k) &= \langle a' - b', c' \rangle_k \cong k \oplus k,\\
B_1(\Delta, k) &= \langle b' - a' + c', a' - b' + c' \rangle_k \cong k \oplus 2k,\\
Z_2(\Delta, k) &= \{r(\sigma' + \rho') \mid 2r = 0\} \cong \{r \in k \mid 2r = 0\},\\
B_2(\Delta, k) &= 0.
\end{aligned}$$

The simplicial homologies are then given by $H_n(\Delta, k) = 0$ for $n > 2$ and

$$\begin{aligned}
H_0(\Delta, k) &\cong \langle p', q' \rangle_k / \langle q' - p' \rangle_k \cong \langle p' \rangle_k \cong k,\\
H_1(\Delta, k) &\cong \langle a' - b', c' \rangle_k / \langle b' - a' + c', a' - b' + c' \rangle = \langle c' \rangle_k / \langle 2c' \rangle \cong k/2k\\
H_2(\Delta, k) &\cong \{r \in k \mid 2r = 0\}.
\end{aligned}$$

This shows in particular that homologies of a topological space need not be free modules. The question if a given homology vanishes therefore depends on the choice of the commutative ring k. If k is a field with $\mathrm{char}(\mathbb{F}) \neq 2$, we have $H_2(\Delta, \mathbb{F}) = H_1(\Delta, \mathbb{F}) = 0$. For $k = \mathbb{Z}$ we have $H_1(\Delta, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ and $H_2(\Delta, \mathbb{Z}) = 0$, and for $k = \mathbb{Z}/2\mathbb{Z}$ we obtain $H_2(\Delta, \mathbb{Z}/2\mathbb{Z}) \cong H_1(\Delta, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.

Remark 2.1.11: One can show that for any semisimplicial complex $\Delta = (X, \{\sigma_\alpha\}_{\alpha\in I})$, the simplicial homology of Δ agrees with the singular homology of X: $H_n(X,k) \cong H_n(\Delta, k)$ for all $n \in \mathbb{Z}$. This implies in particular that all semisimplicial complex structures on a topological space X yield the same simplicial homologies. The proof requires methods from algebraic topology, namely the excision theorem for singular homologies.

There is also a dual version of singular and simplicial homology, called singular and simplicial *cohomology*. It is obtained from singular and simplicial homology theory with coefficients in a commutative ring k by applying the functor $\mathrm{Hom}_k(-,k) : k\text{-Mod}^{op} \to k\text{-Mod}$ that assigns to a k-module L the k-module $\mathrm{Hom}_k(L,k)$ of k-module morphisms from L to k and to a k-linear map $f : L \to L'$ the k-module morphism $\mathrm{Hom}_k(f,k) : \mathrm{Hom}_k(L',k) \to \mathrm{Hom}_k(L,k)$, $g \mapsto g \circ f$ As this functor reverses morphisms, the direction of the boundary operators is reversed as well.

Definition 2.1.12: Let k be a commutative ring and X a topological space.

1. The k-module of **singular n-cochains** the k-module $C^n(X,k) = \mathrm{Hom}_k(C_n(X,k),k)$ of k-linear maps $\phi : C_n(X,k) \to k$.
2. The **singular coboundary operator** $d^n : C^n(X,k) \to C^{n+1}(X,k)$ is the k-module morphism defined by $d^n = 0$ for $n < 0$ and
$$d^n(\phi)(\sigma) = \phi(d_{n+1}(\sigma)) = \sum_{i=0}^{n+1}(-1)^i\,\phi(\sigma\circ f_i^{n+1}) \qquad \forall \sigma : \Delta^{n+1} \to X \text{ continuous}, n \in \mathbb{N}_0.$$
The singular coboundary operators satisfy $d^{n+1} \circ d^n = 0$ for all $n \in \mathbb{Z}$ by Lemma 2.1.3.
3. The k-modules of **singular n-cocycles** and **singular n-coboundaries** are the k-modules $Z^n(X,k) = \ker(d^n) \subset C^n(X,k)$ and $B^n(X,k) = \mathrm{im}(d^{n-1}) \subset Z^n(X,k)$.
4. The nth **singular cohomology** of X is the k-module
$$H^n(X,k) = \frac{Z^n(X,k)}{B^n(X,k)}.$$

The simplicial cohomologies are defined analogously. The only difference is that one restricts again attention to the chosen family of simplexes.

Definition 2.1.13: Let k be a commutative ring, $\Delta = (X, \{\sigma_\alpha\}_{\alpha\in I})$ a semisimplicial complex.

1. The k-module of **simplicial n-cochains** is the k-module $C^n(\Delta,k) = \mathrm{Hom}_k(C_n(\Delta,k),k)$ of k-linear maps $\phi : C_n(\Delta,k) \to k$ for $n \in \mathbb{N}_0$.
2. The **simplicial coboundary operator** $d^n : C^n(\Delta,k) \to C^{n+1}(\Delta,k)$ is the k-module morphism defined by $d^n = 0$ for $n < 0$ and
$$d^n(\phi)(\sigma_\alpha) = \phi(d_{n+1}(\sigma_\alpha)) = \sum_{i=0}^{n+1}(-1)^i\,\phi(\sigma_\alpha\circ f_i^{n+1}) \qquad \forall \alpha \in I \text{ with } n_\alpha = n+1, n \in \mathbb{N}_0.$$
The simplicial coboundary operators satisfy $d^{n+1} \circ d^n = 0$ for all $n \in \mathbb{Z}$ by Lemma 2.1.3.

3. The k-modules of **simplicial n-cocycles** and **simplicial n-coboundaries** are $Z^n(\Delta, k) = \ker(d^n) \subset C^n(\Delta, k)$ and $B^n(\Delta, k) = \mathrm{im}(d^{n-1}) \subset Z^n(\Delta, k)$.

4. The nth **simplicial cohomology** of Δ is the k-module
$$H^n(\Delta, k) = \frac{Z^n(\Delta, k)}{B^n(\Delta, k)}.$$

The resulting cohomology theories, singular and simplicial cohomology, have a similar interpretation to the associated homology theories and contain similar information. They are sometimes more convenient, because they are related more directly to smooth geometrical structures such as differential forms, and they are sometimes easier to compute.

2.2 Hochschild homology and cohomology

In this section we consider (co)homologies of algebras. All algebras and algebra homomorphisms are unital unless stated otherwise. To define their (co)homologies in a way that relates them to other (co)homologies later on, we work with a more general notion of algebra, namely an algebra over a commutative ring k. This is analogous to an algebra over a field, only that the scalar multiplication is replaced by a k-module structure.

Definition 2.2.1: Let k be a commutative ring.

1. An **algebra** over k is a ring $(A, +, \cdot)$ with a k-module structure $\rhd : k \times A \to A$, $(\lambda, a) \mapsto \lambda a$ that satisfies $(\lambda a) \cdot b = \lambda(a \cdot b) = a \cdot (\lambda b)$ for all $a, b \in A$ and $\lambda \in k$.

2. A morphism of k-algebras is a ring homomorphism that is also a morphism of k-modules.

Example 2.2.2:

1. An algebra over $\mathbb{Z}$ is a ring, and a homomorphism of $\mathbb{Z}$-algebras is a ring homomorphism.

   This follows because every ring k has a unique $\mathbb{Z}$-module structure, namely its abelian group structure. The compatibility condition between this $\mathbb{Z}$-module structure and the multiplication follows from the distributive law.

2. For any group G and any commutative ring k, the group ring $k[G]$ is an algebra over k with k-module structure $(\lambda f)(g) := \lambda f(g)$ for all $f : G \to k$, $g \in G$ and $\lambda \in k$.

3. The ring $k[X]$ of polynomials with coefficients in a commutative ring k is an algebra over k.

4. The ring $\mathrm{Mat}(n, k)$ of $(n \times n)$-matrices with entries in a commutative ring k is an algebra over k with the matrix multiplication, matrix addition and simultaneous multiplication of all entries with elements of k.

5. For any commutative ring k and any k-module M, the ring $\mathrm{End}_k(M) = \mathrm{Hom}_k(M, M)$ of k-module morphisms $\phi : M \to M$ is an algebra over k with the k-module structure by pointwise multiplication $(\lambda\phi)(m) := \lambda\phi(m) = \phi(\lambda m)$ for all $\lambda \in k$, $m \in M$.

Left and right modules over a k-algebra A are defined as left and right modules over the ring A. Just as in the case of an algebra over a field, every left module M over A inherits a k-module structure given by $\lambda m = (\lambda 1_A) \triangleright m$ for all $m \in M$. The same holds for A-right modules, defined equivalently as A^{op}-modules, and (A, A)-bimodules, defined as $A \otimes_k A^{op}$-modules. It also follows directly that every A-module homomorphism is k-linear.

Definition 2.2.3: Let A be an algebra over a commutative ring k and M an (A, A)-bimodule with structure maps $\triangleright : A \times M \to M$ and $\triangleleft : M \times A \to M$. Denote by $A^{\otimes n} = A \otimes_k ... \otimes_k A$ the n-fold tensor product of A over k with $A^{\otimes 0} := k$.

1. The k-module of n-**chains** is

$$C_n(A, M) = \begin{cases} M \otimes_k A^{\otimes n} & n \in \mathbb{N}_0 \\ 0 & n < 0. \end{cases}$$

2. The **boundary operators** are the k-linear maps $d_n : C_n(A, M) \to C_{n-1}(A, M)$ given by $d_n = 0$ for $n \leq 0$ and $d_n = \Sigma_{i=0}^{n} (-1)^i d_n^i$ for $n \in \mathbb{N}$ with

$$d_n^i(m \otimes a_1 \otimes ... \otimes a_n) = \begin{cases} (m \triangleleft a_1) \otimes a_2 \otimes ... \otimes a_n & i = 0 \\ m \otimes a_1 \otimes ... \otimes a_{i-1} \otimes (a_i a_{i+1}) \otimes a_{i+2} \otimes ... \otimes a_n & 1 \leq i \leq n-1 \\ (a_n \triangleright m) \otimes a_1 \otimes ... \otimes a_{n-1} & i = n. \end{cases} \tag{5}$$

The structures in Definition 2.2.3 will define Hochschild homology. Instead of the k-modules $M \otimes_k A^{\otimes n}$ we can also consider the k-module of k-linear maps $f : A^{\otimes n} \to M$. This leads to a dual version of Definition 2.2.3 that will define Hochschild cohomology.

Definition 2.2.4: Let A be an algebra over a commutative ring k and M an (A, A)-bimodule with structure maps $\triangleright : A \times M \to M$ and $\triangleleft : M \times A \to M$. Denote by $A^{\otimes n} = A \otimes_k ... \otimes_k A$ the n-fold tensor product of A over k with $A^{\otimes 0} := k$.

1. The k-module of n-**cochains** is

$$C^n(A, M) := \begin{cases} \mathrm{Hom}_k(A^{\otimes n}, M) & n \in \mathbb{N}_0 \\ 0 & n < 0. \end{cases}$$

2. The **coboundary operators** are the k-linear maps $d^n : C^n(A, M) \to C^{n+1}(A, M)$ given by $d^n = 0$ for $n < 0$ and $d^n = \Sigma_{i=0}^{n+1} (-1)^i d_i^n$ for $n \in \mathbb{N}_0$ with

$$(d_i^n f)(a_0 \otimes ... \otimes a_n) = \begin{cases} a_0 \triangleright f(a_1 \otimes ... \otimes a_n) & i = 0 \\ f(a_0 \otimes ... \otimes a_{i-2} \otimes (a_{i-1} a_i) \otimes a_{i+1} \otimes ... \otimes a_n) & 1 \leq i \leq n \\ f(a_0 \otimes ... \otimes a_{n-1}) \triangleleft a_n & i = n+1. \end{cases}$$

Just as for the singular and simplicial (co)boundary operators, the composite of two subsequent (co)boundary operators from Definitions 2.2.3 and 2.2.4 is zero. This is a consequence of the combinatorial properties of the maps d_n^i and d_i^n in Definitions 2.2.3 and Definition 2.2.4.

Lemma 2.2.5: Let k be a commutative ring, A an algebra over k and M an (A, A)-bimodule.

1. The k-linear maps $d_n^i : C_n(A, M) \to C_{n-1}(A, M)$ from Definition 2.2.3 satisfy

$$d_{n-1}^i \circ d_n^j = d_{n-1}^{j-1} \circ d_n^i \qquad \forall\, 0 \leq i < j \leq n, \tag{6}$$

and this implies $d_n \circ d_{n+1} = 0$ for all $n \in \mathbb{Z}$.

2. The k-linear maps $d_i^n : C^n(A,M) \to C^{n+1}(A,M)$ from Definition 2.2.4 satisfy

$$d_i^{n+1} \circ d_j^n = d_{j+1}^{n+1} \circ d_i^n \qquad \forall\, 0 \le i \le j \le n, \tag{7}$$

and this implies $d^{n+1} \circ d^n = 0$ for all $n \in \mathbb{Z}$.

Proof:
We prove the second part of the lemma. The proof of the first part is analogous. We compute for $0 < i = j < n$:

$$\begin{aligned}
&d_i^{n+1}(d_i^n f)(a_0 \otimes ... \otimes a_{n+1}) = (d_i^n f)(a_0 \otimes ... \otimes a_{i-2} \otimes (a_{i-1}a_i) \otimes a_{i+1} \otimes ... \otimes a_{n+1}) \\
&= f(a_0 \otimes ... \otimes a_{i-2} \otimes (a_{i-1}a_i a_{i+1}) \otimes a_{i+2} \otimes ... \otimes a_{n+1}) = (d_i^n f)(a_0 \otimes ... \otimes a_{i-1} \otimes (a_i a_{i+1}) \otimes a_{i+2} \otimes ... \otimes a_{n+1}) \\
&= d_{i+1}^{n+1}(d_i^n f)(a_0 \otimes ... \otimes a_{n+1}),
\end{aligned}$$

for $0 < i < j < n$:

$$\begin{aligned}
&d_i^{n+1}(d_j^n f)(a_0 \otimes ... \otimes a_{n+1}) = (d_j^n f)(a_0 \otimes ... \otimes a_{i-2} \otimes (a_{i-1}a_i) \otimes a_{i+1} \otimes ... \otimes a_{n+1}) \\
&= f(a_0 \otimes ... \otimes a_{i-2} \otimes (a_{i-1}a_i) \otimes a_{i+1} \otimes ... \otimes a_{j-1} \otimes (a_j a_{j+1}) \otimes a_{j+2} \otimes ... \otimes a_{n+1}) \\
&= (d_i^n f)(a_0 \otimes ... \otimes a_{j-2} \otimes (a_j a_{j+1}) \otimes a_{i+1} \otimes ... \otimes a_{n+1}) = d_{j+1}^{n+1}(d_i^n f)(a_0 \otimes ... \otimes a_{n+1}),
\end{aligned}$$

for $i = j = 0$:

$$\begin{aligned}
&d_0^{n+1}(d_0^n f)(a_0 \otimes ... \otimes a_{n+1}) = a_0 \rhd (d_0^n f)(a_1 \otimes ... \otimes a_{n+1}) = a_0 \rhd (a_1 \rhd f(a_2 \otimes ... \otimes a_{n+1})) \\
&= (a_0 a_1) \rhd f(a_2 \otimes ... \otimes a_{n+1}) = (d_0^{n+1} f)((a_0 a_1) \otimes ... \otimes a_{n+1}) = d_0^{n+1}(d_1^n f)(a_0 \otimes ... \otimes a_{n+1}),
\end{aligned}$$

and for $i = 0 < j < n$:

$$\begin{aligned}
&d_0^{n+1}(d_j^n f)(a_0 \otimes ... \otimes a_{n+1}) = a_0 \rhd (d_j^n f)(a_1 \otimes ... \otimes a_{n+1}) = a_0 \rhd f(a_1 \otimes ... \otimes a_{j-1} \otimes (a_j a_{j+1}) \otimes ... \otimes a_{n+1}) \\
&= (d_0^n f)(a_0 \otimes ... \otimes a_{j-1} \otimes (a_j a_{j+1}) \otimes ... \otimes a_{n+1}) = d_{j+1}^{n+1}(d_i^n f)(a_0 \otimes ... \otimes a_{n+1}).
\end{aligned}$$

The computations for $i = j = n$ and $0 \le i < j = n$ are analogous. These relations imply

$$\begin{aligned}
d^{n+1} \circ d^n &= \Sigma_{i=0}^{n+2} \Sigma_{j=0}^{n+1} (-1)^{i+j} d_i^{n+1} \circ d_j^n \\
&= \Sigma_{0 \le j < i \le n+2} (-1)^{i+j} d_i^{n+1} \circ d_j^n + \Sigma_{0 \le i \le j \le n+1} (-1)^{i+j} d_i^{n+1} \circ d_j^n \\
&\overset{(7)}{=} \Sigma_{0 \le j < i \le n+2} (-1)^{i+j} d_i^{n+1} \circ d_j^n + \Sigma_{0 \le i \le j \le n+1} (-1)^{i+j} d_{j+1}^{n+1} \circ d_i^n \\
&= \Sigma_{0 \le j < i \le n+2} (-1)^{i+j} d_i^{n+1} \circ d_j^n + \Sigma_{0 \le i < j \le n+2} (-1)^{i+j-1} d_j^{n+1} \circ d_i^n \\
&= \Sigma_{0 \le j < i \le n+2} (-1)^{i+j} (d_i^{n+1} \circ d_j^n - d_i^{n+1} \circ d_j^n) = 0.
\end{aligned}$$

□

As the boundary operators $d_n : C_n(A,M) \to C_{n-1}(A,M)$ satisfy the relations $d_n \circ d_{n+1} = 0$, we have $d_n(\mathrm{im}(d_{n+1})) = 0$ and hence $\mathrm{im}(d_{n+1}) \subset \ker(d_n) \subset C_n(A,M)$. This allows us to consider the quotient modules $\ker(d_n)/\mathrm{im}(d_{n+1})$. Similarly, the relations $d^n \circ d^{n-1} = 0$ for the coboundary operators imply that $\mathrm{im}(d^{n-1}) \subset \ker(d^n) \subset C^n(A,M)$ are submodules and allows us to form the quotient module $\ker(d^n)/\mathrm{im}(d^{n-1})$. These quotients are called, respectively, the Hochschild homologies and cohomologies of A with coefficients in M.

Definition 2.2.6: Let A be an algebra over a commutative ring k and M an (A,A)-bimodule with structure maps $\rhd : A \times M \to M$ and $\lhd : M \times A \to M$.

- The k-module $Z_n(A,M) = \ker(d_n) \subset C_n(A,M)$ is called the k-module of n-**cycles** and the submodule $B_n(A,M) = \mathrm{im}(d_{n+1}) \subset Z_n(A,M)$ the k-module of n-**boundaries**.
- The nth **Hochschild homology** of A with coefficients in M is the quotient module

$$H_n(A,M) = \frac{Z_n(A,M)}{B_n(A,M)} = \frac{\ker(d_n)}{\mathrm{im}(d_{n+1})}.$$

- The k-module $Z^n(A,M) = \ker(d^n) \subset C^n(A,M)$ is called k-module of n-**cocycles** and the submodule $B^n(A,M) = \mathrm{im}(d^{n-1}) \subset Z^n(A,M)$ the k-module of n-**coboundaries**.
- The nth **Hochschild cohomology** of A with coefficients in M is the quotient module

$$H^n(A,M) = \frac{Z^n(A,M)}{B^n(A,M)} = \frac{\ker(d^n)}{\mathrm{im}(d^{n-1})}.$$

Hochschild (co)homologies of A with coefficients in M carry information about the k-algebra A and the (A,A)-bimodule M. As every k-algebra A is an (A,A)-bimodule with its left and right multiplication, we can always consider the (A,A)-bimodule $M = A$ and extract information about the algebra A itself.

We will show that the zeroth Hochschild *cohomology* is the *centre* of an (A,A)-bimodule M, the submodule of elements on which the left and right action of A coincide.

To interpret the first Hochschild *cohomology*, we need the concept of a *derivation*, which generalises derivatives of functions. To see this, consider the algebra $C^n(U)$ of n-times continuously differentiable real functions on an open subset $U \subset \mathbb{R}$ with the pointwise addition, multiplication and multiplication by $\mathbb{R}$. As the product of a C^n-function and a C^{n-1}-function is again a C^{n-1}-function, we can view $C^{n-1}(U)$ as a bimodule over $C^n(U)$ with $f \rhd g = g \lhd f = f \cdot g$ for all $f \in C^n(U)$ and $g \in C^{n-1}(U)$. The derivative $' : C^n(U) \to C^{n-1}(U)$, $f \mapsto f'$ is $\mathbb{R}$-linear and satisfies the *Leibniz identity*: $(f \cdot g)' = f \cdot g' + f' \cdot g = f \rhd g' + f' \lhd g$ for all $f, g \in C^k(U)$. By replacing $C^n(U)$ with an algebra over a commutative ring k and $C^{n-1}(U)$ with a general (A,A)-bimodule M, we obtain the definition of a derivation.

Definition 2.2.7: Let A be an algebra over a commutative ring k and M an (A,A)-bimodule with structure maps $\rhd : A \times M \to M$ and $\lhd : M \times A \to M$.

1. A **derivation** on A with values in M is a k-linear map $f : A \to M$ that satisfies $f(ab) = f(a) \lhd b + a \rhd f(b)$ for all $a, b \in A$. The k-module of derivations $f : A \to M$ is denoted $\mathrm{Der}(A,M)$.
2. A derivation on A with values in M is called an **inner derivation** if it is of the form $f_m : A \to M$, $a \mapsto a \rhd m - m \lhd a$ for some $m \in M$. The submodule of inner derivations is denoted $\mathrm{InnDer}(A,M) \subset \mathrm{Der}(A,M)$.

By computing the first two Hochschild cohomologies from Definition 2.2.4, we can relate them to, respectively, the centre of a bimodule M and to the derivations on A with coefficients in M. An analogous computation can be performed for the Hochschild homologies (Exercise 16).

Lemma 2.2.8: Let A be an algebra over a commutative ring k and M an (A,A)-bimodule with structure maps $\rhd : A \times M \to M$ and $\lhd : M \times A \to M$.

- The first two Hochschild cohomologies of A with coefficients in M are given by

$$H^0(A,M) = Z_A(M) \qquad\qquad H^1(A,M) = \frac{\mathrm{Der}(A,M)}{\mathrm{InnDer}(A,M)}.$$

  where $Z_A(M) = \{m \in M \mid a \rhd m = m \lhd a \, \forall a \in A\}$ is called the **centre** of M.

- For $M = A$ as an (A,A)-bimodule over itself with left and right multiplication we have

$$H^0(A,A) = Z(A) \qquad\qquad H^1(A,A) = \frac{\mathrm{Der}(A,A)}{\mathrm{InnDer}(A,A)},$$

  where $Z(A) = \{a \in A \mid ab = ba \, \forall b \in A\}$ is the **centre** of A.

Proof:
As $\phi : \mathrm{Hom}_k(k,M) \to M$, $f \mapsto f(1)$ is a k-linear isomorphism, $C^0(A,M) = \mathrm{Hom}_k(k,M) \cong M$. With this identification, the first two coboundary operators from Definition 2.2.4 are given by

$$\begin{aligned}
&d^0 : M \to \mathrm{Hom}_k(A,M),\; m \mapsto f_m && f_m(a) = a \rhd m - m \lhd a\\
&d^1 : \mathrm{Hom}_k(A,M) \to \mathrm{Hom}_k(A^{\otimes 2},M), && (d^1 f)(a \otimes b) = a \rhd f(b) - f(ab) + f(a) \lhd b,
\end{aligned}$$

and we obtain

$$\begin{aligned}
&\ker(d^0) = \{m \in M \mid a \rhd m = m \lhd a \, \forall a \in A\} = Z_A(M)\\
&\ker(d^1) = \{f : A \to M \mid a \rhd f(b) - f(ab) + f(a) \lhd b = 0 \, \forall a,b \in A\} = \mathrm{Der}(A,M)\\
&\mathrm{im}(d^0) = \{f_m : A \to M, a \mapsto a \rhd m - m \lhd a \mid m \in M\} = \mathrm{InnDer}(A,M).
\end{aligned}$$
□

This shows that the Hochschild cohomology $H^0(A,M)$ measures the (non-)commutativity of the bimodule M with respect to its left and right A-module structures. In particular, if $M = A$ as a bimodule over itself, it measures the (non-)commutativity of A. If we consider a ring A, viewed as an algebra over $\mathbb{Z}$, and an A-module N, then the abelian group $M = \mathrm{End}_{\mathbb{Z}}(N)$ of $\mathbb{Z}$-module endomorphisms $f : N \to N$ becomes an (A,A)-bimodule by Example 1.1.3, 5 with the bimodule structure $(a \rhd f)(n) = a \rhd f(n)$ and $(f \lhd a)(n) = f(a \rhd n)$. In this case, $H^0(A,M)$ is the subgroup of A-module endomorphisms $f : N \to N$.

The first Hochschild cohomology $H^1(A,M)$ counts the derivations on A with values in M, up to inner derivations. If M is commutative with respect to the actions of A, we have $M = Z_A(M)$ and $\mathrm{InnDer}(A,M) = \{0\}$. In this case, $H^1(A,M)$ counts the derivations on A with values in M. If $M = A$ as an (A,A)-bimodule over itself, then derivations are k-linear maps $f : A \to A$ with $f(ab) = af(b) - f(a)b$ and inner derivations are precisely the *commutator maps* $f_b : A \to A$, $a \mapsto [a,b] = ab - ba$. By the Leibniz rule, every commutator map is a derivation. Hence, the Hochschild cohomology $H^1(A,A)$ counts derivations up to commutator maps.

2.3 Group homology and cohomology

In this section we investigate cohomologies of groups. Given a commutative ring k and a group G, we can consider the group algebra $k[G]$ as an algebra over k. This allows us to define (co)homologies of groups as Hochschild (co)homologies of group algebras $k[G]$. Compared to general case we have the following simplifications:

- The map $\phi : k[G^{\times n}] \to k[G]^{\otimes n}$, $\lambda(g_1, ..., g_n) \mapsto \lambda g_1 \otimes ... \otimes g_n$ is k-module isomorphism for all $n \in \mathbb{N}_0$.

- For all k-modules M the map $\psi : \mathrm{Map}(G, M) \to \mathrm{Hom}_k(k[G], M)$ that extends $f : G \to M$ to a k-linear map $f' : k[G] \to M$ is an isomorphisms of k-modules.

- Every $k[G]$-left module M becomes a $(k[G], k[G])$-bimodule with the **trivial $k[G]$-right module structure** $\lhd : M \times k[G] \to M$, $m \lhd g = m$.

- Similarly, every k-module M becomes a $k[G]$-module with the **trivial $k[G]$-left module structure** $\rhd : k[G] \times M \to M$, $g \rhd m = m$.

The first two points lead to technical simplifications. They allow us to describe Hochschild homologies and cohomologies of group algebras $k[G]$ in terms of products $M \otimes_k k[G^{\times n}]$ and maps $f : G^{\times n} \to M$ instead of tensor products $M \otimes_k k[G]^{\otimes n}$ and k-linear maps $f : k[G]^{\otimes n} \to M$.

The third and fourth point are more fundamental because they allow us to consider bimodules that are (partly) trivial. For a general k-algebra A, a trivial A-module structure on a k-module M is an A-module structure of the form $a \rhd m = \epsilon(a)m$ for all $a \in A$, $m \in M$ with an algebra homomorphism $\epsilon : A \to k$, the **augmentation map**. For $A = k[G]$ the augmentation map is $\epsilon : \Sigma_{g \in G} \lambda_g g \mapsto \Sigma_{g \in G} \lambda_g$. For a general k-algebra A one cannot define augmentation maps in this way, and they need not exist. Hence, there need not be a trivial A-module structure on M.

Defining group (co)homologies as Hochschild (co)homologies of $k[G]$ with coefficients in a $k[G]$-left module M with the trivial $k[G]$-right module structure then yields

Definition 2.3.1: Let k be a commutative ring, G a group, $(M, \rhd)$ a $k[G]$-left module, $(P, \lhd)$ a $k[G]$-right module and $G^{\times n} = G \times ... \times G$ the n-fold product with $G^{\times 0} := \{1\}$.

1. The k-modules of n-**(co)chains** are

$$C_n(G, P) = \begin{cases} P \otimes_k k[G^{\times n}] & n \in \mathbb{N}_0 \\ 0 & n < 0 \end{cases} \qquad C^n(G, M) = \begin{cases} \mathrm{Map}(G^{\times n}, M) & n \in \mathbb{N}_0 \\ 0 & n < 0. \end{cases}$$

2. The **(co)boundary operators** are the k-linear maps

$$d_n = \Sigma_{i=0}^{n} (-1)^i d_n^i : C_n(G, P) \to C_{n-1}(G, P) \quad d^n = \Sigma_{i=0}^{n+1} (-1)^i d_i^n : C^n(G, M) \to C^{n+1}(G, M)$$

given by $d_n^i = 0$, $d_i^n = 0$ for $n < 0$ and for $n \in \mathbb{N}_0$

$$d_n^i(p \otimes (g_1, ..., g_n)) = \begin{cases} (p \lhd g_1) \otimes (g_2, ..., g_n) & i = 0 \\ p \otimes (g_1, ..., g_i g_{i+1}, ..., g_n) & 1 \le i \le n-1 \\ p \otimes (g_1, ..., g_{n-1}) & i = n \end{cases}$$

$$(d_i^n f)(g_0, ..., g_n) = \begin{cases} g_0 \rhd f(g_1, ..., g_n) & i = 0 \\ f(g_0, ..., g_{i-2},\, g_{i-1} g_i\,, g_{i+1}, ..., g_n) & 1 \le i \le n \\ f(g_0, ..., g_{n-1}) & i = n+1. \end{cases}$$

They satisfy $d_n \circ d_{n+1} = 0$ and $d^{n+1} \circ d^n = 0$ for all $n \in \mathbb{Z}$ by Lemma 2.2.5.

3. The k-modules of n-**(co)cycles** and n-**(co)boundaries** are the submodules

$$Z_n(G,M) = \ker(d_n) \quad Z^n(G,M) = \ker(d^n) \quad B_n(G,M) = \mathrm{im}(d_{n+1}) \quad B^n(G,M) = \mathrm{im}(d^{n-1}).$$

4. The nth **group (co)homology** of G with coefficients in M is

$$H_n(G,M) = \frac{Z_n(G,M)}{B_n(G,M)} = \frac{\ker(d_n)}{\mathrm{im}(d_{n+1})} \qquad H^n(G,M) = \frac{Z^n(G,M)}{B^n(G,M)} = \frac{\ker(d^n)}{\mathrm{im}(d^{n-1})}.$$

By adapting the results from Lemma 2.2.8 to the situation at hand, we obtain a characterisation of the group cohomologies $H^0(G,M)$ and $H^1(G,M)$ in terms of the centre $Z_{k[G]}(M)$ and in terms of derivations on $k[G]$. The only difference is that these notions become simpler and have a more direct interpretation. As the $k[G]$-right module structure is chosen to be trivial, the centre $Z_{k[G]}(M)$ is the k-submodule M^G of invariants. There is also a dual notion of coinvariants.

Definition 2.3.2: Let G be a group, k a commutative ring and M a $k[G]$-module with structure map $\rhd : k[G] \times M \to M$. The k-submodule of **invariants** is

$$M^G = \{m \in M \mid g \rhd m = m \,\forall g \in G\}$$

and the k-submodule of **coinvariants**

$$M^{coG} = M/\langle\{m - g \rhd m \mid g \in G, m \in M\}\rangle.$$

Invariants and coinvariants for $k[G]$-right modules are defined analogously. The notions of a derivation and of an inner derivation simplify for group algebras, because the $k[G]$-right module structure is trivial and because we can characterise k-linear maps $k[G] \to M$ in terms of maps $G \to M$. This yields the following definition.

Definition 2.3.3: Let G be a group, k a commutative ring and M a $k[G]$-module with structure map $\rhd : k[G] \times M \to M$.

1. A **derivation** on G with values in M is a map $f : G \to M$ with $f(gh) = f(g) + g \rhd f(h)$ for all $g, h \in G$. The k-module of derivations $f : G \to M$ is denoted $\mathrm{Der}(G,M)$.

2. An **inner derivation** on G with values in M is a derivation of the form $f_m : G \to M$, $g \mapsto g \rhd m - m$ for some $m \in M$. The k-module of inner derivations $f : G \to M$ is denoted $\mathrm{InnDer}(G,M)$.

Using these definitions and specialising Lemma 2.2.8 to group algebras $k[G]$, we obtain the group cohomology counterpart of Lemma 2.2.8. It characterises $H^0(G,M)$ and $H^1(G,M)$ in terms of invariants and derivations. Analogous computations show that the first group homology with coefficicents in a $k[G]$-module M is given by the coinvariants of M and that the group homology $H_1(G,\mathbb{Z})$ with the trivial $\mathbb{Z}[G]$-module structure on $\mathbb{Z}$ is the abelisation of G (Exercise 21).

Corollary 2.3.4: Let k be a commutative ring, G a group, M a $k[G]$-module. Then

$$H^0(G,M) = M^G, \qquad H_0(G,M) = M^{coG}, \qquad H^1(G,M) = \frac{\mathrm{Der}(G,M)}{\mathrm{InnDer}(G,M)}.$$

In particular, for $M = \mathbb{Z}$ equipped with the trivial $\mathbb{Z}[G]$-module structure one has

$$H_1(G,\mathbb{Z}) = \mathrm{Ab}(G) \qquad H^1(G,\mathbb{Z}) = \mathrm{Hom}_{\mathrm{Grp}}(\mathrm{Ab}(G),\mathbb{Z}).$$

As group cohomologies are a special example of Hochschild cohomologies, they contain the same type of information. However, the simplifications for group algebras $k[G]$ make them much simpler to compute and to interpret. We illustrate this with the second cohomology group $H^2(G,M)$. This requires the concept of a group extension.

Definition 2.3.5: Let G, M be groups.

1. An **extension** of G by M is a triple (E, ι, π) of a group E, an injective group homomorphism $\iota : M \to E$ and a surjective group homomorphism $\pi : E \to G$ with $\ker(\pi) = \mathrm{im}(\iota)$. It is called **central** if $\iota(M) \subset Z(E)$: $\iota(m) \cdot e = e \cdot \iota(m)$ for all $m \in M$ and $e \in E$.
2. A **morphism of group extensions** from (E, ι, π) to (E', ι', π') is a group homomorphism $f : E \to E'$ with $f \circ \iota = \iota'$ and $\pi' \circ f = \pi$.

Remark 2.3.6:

1. A group extension (E, ι, π) is also called a **short exact sequence** of groups and denoted
$$0 \to M \xrightarrow{\iota} E \xrightarrow{\pi} G \to 0$$
2. If (E, ι, π) is a group extension of G by M, then $G \cong E/\ker(\pi)$ by the surjectivity of π and $\ker(\pi) = \mathrm{im}(\iota) \cong M$ by injectivity of ι. Hence, a group extension of G by M is a group E that contains M as a normal subgroup and G as the associated factor group.
3. A morphism of group extensions is always bijective and hence an isomorphism. This follows from the group version of the Five-Lemma (cf. Exercise 41, the proof for groups and group homomorphisms is analogous).

Example 2.3.7:

1. Semidirect products are group extensions:

   A group homomorphism $\phi : G \to \mathrm{Aut}(M)$ defines a semidirect product group $M \rtimes_\phi G$. This is the set $M \times G$ with the group multiplication
$$(m_1, g_1) \cdot (m_2, g_2) = (m_1 \phi(g_1)(m_2), g_1 g_2).$$
   The group $E = M \rtimes_\phi G$ is a group extension of G by M with $\iota : M \to M \rtimes_\phi G$, $m \mapsto (m, 1)$ and $\pi : M \rtimes_\phi G \to G$, $(m, g) \mapsto g$.

   For the trivial group homomorphism $\phi : G \to \mathrm{Aut}(M)$, $g \mapsto \mathrm{id}_M$ one obtains the direct product $M \times G$ as an extension of G by M.
2. For any prime p and $n, m \in \mathbb{N}$, the group $E = \mathbb{Z}/p^{m+n}\mathbb{Z}$ is a central extension of the group $G = \mathbb{Z}/p^n\mathbb{Z}$ by $M = \mathbb{Z}/p^m\mathbb{Z}$ with the group homomorphisms
$$\iota : \mathbb{Z}/p^m\mathbb{Z} \to \mathbb{Z}/p^{m+n}\mathbb{Z},\ \bar{k} \mapsto \bar{p}^n \cdot \bar{k} \qquad \pi : \mathbb{Z}/p^{m+n}\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z},\ \bar{k} \mapsto \bar{k}.$$
   Note that $\mathbb{Z}/p^{n+m}\mathbb{Z}$ is *not* a semidirect product of $\mathbb{Z}/p^n\mathbb{Z}$ and $\mathbb{Z}/p^m\mathbb{Z}$ by the classification theorem of abelian groups.
3. Every finite group G is obtained from the trivial group $\{e\}$ by finite sequence of group extensions (E_i, π_i, ι_i) with *simple* groups M_i. Every **solvable group** is obtained from the trivial group $\{e\}$ via a finite sequence of group extensions by *abelian* groups M_i.

We now show that group extensions arise from 2-cocycles $f : G \times G \to M$ and that 2-cocycles that are related by 2-coboundaries define isomorphic group extensions. Central extensions of G by M arise from trivial $\mathbb{Z}[G]$-modules M.

Theorem 2.3.8: Let G be a group and M an abelian group.

1. Isomorphism classes of extensions of G by M are in bijection with pairs $(\rhd, [f])$ of a $\mathbb{Z}[G]$-module structure $\rhd$ on M and an element $[f] \in H^2(G, M)$.
2. Isomorphism classes of central extensions of G by M are in bijection with elements of $H^2(G, M)$, where M is equipped with the trivial $\mathbb{Z}[G]$-module structure.

Proof:
1. Let $\rhd : \mathbb{Z}[G] \times M \to M$ be a $\mathbb{Z}[G]$-module structure on M. By Definition 2.3.1, the coboundary operators $d^1 : C^1(G, M) \to C^2(G, M)$ and $d^2 : C^2(G, M) \to C^3(G, M)$ are given by

$$\begin{aligned} d^1(F)(g, h) &= g \rhd F(h) - F(gh) + F(g) \\ d^2(f)(g, h, k) &= g \rhd f(h, k) - f(gh, k) + f(g, hk) - f(g, h) \end{aligned}$$

for all maps $F : G \to M$ and $f : G^{\times 2} \to M$. Hence, a 2-cocycle is a map $f : G \times G \to M$ with

$$g \rhd f(h, k) - f(gh, k) + f(g, hk) - f(g, h) = 0 \qquad \forall g, h, k \in G, \tag{8}$$

and a 2-coboundary is a map $f : G \times G \to M$ of the form

$$f : G \times G \to M, \quad (g, h) \mapsto g \rhd F(h) - F(gh) + F(g). \tag{9}$$

• 1.1. We show that every 2-cocycle $f : G \times G \to M$ defines a group extension of G by M.

Every 2-cocycle $f : G \times G \to M$ defines a group structure $\cdot_f$ on $M \times G$ with

$$(m, g) \cdot_f (m', g') = (m + g \rhd m' + f(g, g') - g \rhd f(1, 1), gg'). \tag{10}$$

The associativity of $\cdot_f$ follows directly from the 2-cocycle condition (8). To prove that $(0, 1)$ is a unit element and that inverses are given by $(m, g)^{-1} = (-g^{-1} \rhd m - f(g^{-1}, g) + g^{-1} \rhd f(1, 1), g^{-1})$, we note that every 2-cocycle f satisfies the conditions

$$f(g, 1) = g \rhd f(1, 1) \quad f(1, g) = f(1, 1) \quad g \rhd f(g^{-1}, g) = f(g, g^{-1}) - g \rhd f(1, 1) + f(1, 1) \tag{11}$$

for all $g \in G$. A short computation then shows that $\iota : M \to M \times G$, $m \mapsto (m, 1)$ and $\pi : M \times G \to G$, $(m, g) \mapsto g$ are group homomorphisms with respect to $\cdot_f$ and the group structures on M and G and that they satisfy $\ker(\pi) = \mathrm{im}(\iota)$.

• 1.2. We show that the group extensions for 2-cocycles $f_1, f_2 : G \times G \to M$ are isomorphic if $f_1 - f_2$ is a 2-coboundary.

If $f_1 - f_2$ is a 2-coboundary, there is a map $F : G \to M$ with

$$f_1(g, h) - f_2(g, h) = g \rhd F(h) - F(gh) + F(g) \quad \forall g, h \in G. \tag{12}$$

This implies in particular $f_1(1, 1) - f_2(1, 1) = F(1)$. We consider the associated extension groups $E_i = (M \times G, \cdot_{f_i})$ and show that the map

$$\phi : (M \times G, \cdot_{f_1}) \to (M \times G, \cdot_{f_2}), \quad (m, g) \mapsto (m + F(g) - F(1), g)$$

is a group isomorphism. This follows by a direct computation from (10)

$$\begin{aligned}
&\phi(m,g)\cdot_{f_2}\phi(m',g') = (m+F(g)-F(1),g)\cdot_{f_2}(m'+F(g')-F(1),g')\\
&\overset{(10)}{=} (m+F(g)-F(1)+g\rhd(m'+F(g')-F(1))+f_2(g,g')-g\rhd f_2(1,1),gg')\\
&\overset{(12)}{=} (m+g\rhd m'+F(gg')+f_1(g,g')-g\rhd f_1(1,1)-F(1),gg')\\
&=\phi(m+g\rhd m'+f_1(g,g')-g\rhd f_1(1,1),gg')\overset{(10)}{=}\phi((m,g)\cdot_{f_1}(m',g')).
\end{aligned}$$

The group homomorphism ϕ is invertible with inverse $\phi^{-1}:(m,g)\mapsto(m-F(g)+F(1),g)$, and we have for all $g\in G$ and $m\in M$

$$\begin{aligned}
&\phi\circ\iota(m)=\phi(m,1)=(m+F(1)-F(1),1)=(m,1)=\iota(m)\\
&\pi\circ\phi(m,g)=\pi(m+F(g)-F(1),g)=g=\pi(m,g).
\end{aligned}$$

This shows that ϕ is an isomorphism of group extensions from (E_1,ι,π) to (E_2,ι,π). Hence, every element $[f]\in H^2(G,M)$ defines an isomorphism class of extensions of G by M.

• 1.3 We show that every group extension (E,ι,π) of G by M defines a $\mathbb{Z}[G]$-module structure on M and an element of $Z^2(G,M)$.

If (E,ι,π) is an extension of G by M, then $\iota(M)=\ker(\pi)\subset E$ is an abelian normal subgroup isomorphic to M. Because the group homomorphism $\pi:E\to G$ is surjective, we can choose an element $\sigma(g)\in\pi^{-1}(g)$ for each $g\in G$ and obtain a map $\sigma:G\to E$ with $\pi\circ\sigma=\mathrm{id}_G$. Because $\iota(M)=\ker(\pi)\subset E$ is normal, we have $\sigma(g)\cdot\iota(m)\cdot\sigma(g)^{-1}\in\iota(M)$ for all $g\in G$ and $m\in M$. As $\pi:E\to G$ is a group homomorphism with $\pi\circ\sigma=\mathrm{id}_G$, we also have $\pi(\sigma(g)\sigma(h)\sigma(gh)^{-1})=g\cdot h\cdot(gh)^{-1}=1$, and this implies $\sigma(g)\sigma(h)\sigma(gh)^{-1}\in\iota(M)=\ker(\pi)$ for all $g,h\in G$. As $\iota:M\to E$ is injective, this defines maps

$$\begin{aligned}
&\rhd: G\times M\to M \quad\text{with}\quad \iota(g\rhd m)=\sigma(g)\iota(m)\sigma(g)^{-1}\\
&f:G\times G\to M,\quad\text{with}\quad \iota(f(g,h))=\sigma(g)\sigma(h)\sigma(gh)^{-1}.
\end{aligned}\tag{13}$$

Because $\sigma(1)\in\pi^{-1}(1)=\iota(M)$ and $\iota(M)$ is abelian, we have $\iota(1\rhd m)=\sigma(1)\iota(m)\sigma(1)^{-1}=\iota(m)$. By injectivity of ι this implies $1\rhd m=m$ for all $m\in M$. By definition of $\rhd$ and f and because $\iota(f(g,h))$ is contained in the abelian subgroup $\iota(M)\subset E$ we have with (13)

$$\begin{aligned}
\iota(g\rhd(h\rhd m))&=\sigma(g)\iota(h\rhd m)\sigma(g)^{-1}=\sigma(g)\sigma(h)\,\iota(m)\,\sigma(h)^{-1}\sigma(g)^{-1}\\
&=\iota(f(g,h))(\sigma(gh)\iota(m)\sigma(gh)^{-1})\iota(f(g,h))^{-1}\\
&=\iota(f(g,h))\cdot\iota((gh)\rhd m)\cdot\iota(f(g,h))^{-1}=\iota((gh)\rhd m).
\end{aligned}$$

By injectivity of ι it follows that $g\rhd(h\rhd m)=(gh)\rhd m$ for all $m\in M$ and $g,h\in G$, and that $\rhd: G\times M\to M$ defines a $\mathbb{Z}[G]$-module structure on M.

From the associativity of the multiplication in E we obtain for all $g,h,k\in G$

$$\begin{aligned}
\iota(g\rhd f(h,k))&\overset{(13)}{=}\sigma(g)\iota(f(h,k))\sigma(g)^{-1})\overset{(13)}{=}\sigma(g)\sigma(h)\sigma(k)\sigma(hk)^{-1}\sigma(g)^{-1}\\
&\overset{(13)}{=}\iota(f(g,h))\sigma(gh)\sigma(k)\sigma(hk)^{-1}\sigma(g)^{-1}\overset{(13)}{=}\iota(f(g,h))\iota(f(gh,k))\sigma(ghk)\sigma(hk)^{-1}\sigma(g)^{-1}\\
&\overset{(13)}{=}\iota(f(g,h))\iota(f(gh,k))\iota(f(g,hk))^{-1}=\iota(f(g,h)+f(gh,k)-f(g,hk)).
\end{aligned}$$

Using again that ι is injective, we see that f is a 2-cocycle. Hence, we have shown that every extension of G by M defines a $\mathbb{Z}[G]$-module structure $\rhd$ on M and a 2-cocycle $f:G\times G\to M$.

• 1.4. We determine how the $\mathbb{Z}[G]$-module structure $\rhd : \mathbb{Z}[G] \times M \to M$ and the 2-cocycle $f : G \times G \to M$ depend on the choice of the map $\sigma : G \to E$ in 1.3.

Let $\sigma_1, \sigma_2 : G \to E$ be maps with $\pi \circ \sigma_i = \mathrm{id}_G$ and $\rhd_i : G \times M \to M$ and $f_i : G \times G \to M$ the associated $\mathbb{Z}[G]$-module structures on M and 2-cocycles defined in (13). Then we have $\pi(\sigma_2(g)\sigma_1(g)^{-1}) = gg^{-1} = 1$ and $\sigma_2(g)\sigma_1(g)^{-1} \in \iota(M) = \ker(\pi)$ for all $g \in G$. As ι is injective, this defines a map

$$F : G \to M \quad \text{with} \quad \iota(F(g)) = \sigma_2(g)\sigma_1(g)^{-1}. \tag{14}$$

Using this definition and formula (13) for the $\mathbb{Z}[G]$-module structure, we obtain

$$\begin{aligned}\iota(g \rhd_2 m) &\overset{(13)}{=} \sigma_2(g)\iota(m)\sigma_2(g)^{-1} = (\sigma_2(g)\sigma_1(g)^{-1})(\sigma_1(g)\iota(m)\sigma_1(g)^{-1})(\sigma_1(g)\sigma_2(g)^{-1}) \\ &\overset{(14)}{=} \iota(F(g))\iota(g \rhd_1 m)\iota(F(g))^{-1} = \iota(F(g) + g \rhd_1 m - F(g)) = \iota(g \rhd_1 m)\end{aligned}$$

and hence the $\mathbb{Z}[G]$-module structure on M does not depend on the choice of σ. A direct computation using the definitions, the associativity of the multiplication in E and the fact that $\iota(M) \subset E$ is normal then shows that the 2-cocycles $f_i : G \to M$ are related by a 2-coboundary

$$\begin{aligned}\iota(f_2(g,h)) &\overset{(13)}{=} \sigma_2(g)\sigma_2(h)\sigma_2(gh)^{-1} \overset{(14)}{=} \iota(F(g))\sigma_1(g)\sigma_2(h)\sigma_2(gh)^{-1} \\ &\overset{(14)}{=} \iota(F(g))\sigma_1(g)\iota(F(h))\sigma_1(h)\sigma_2(gh)^{-1} \overset{(13)}{=} \iota(F(g))\iota(g \rhd F(h))\sigma_1(g)\sigma_1(h)\sigma_2(gh)^{-1} \\ &\overset{(14)}{=} \iota(F(g))\iota(g \rhd F(h))\sigma_1(g)\sigma_1(h)\sigma_1(gh)^{-1}\iota(F(gh))^{-1} \\ &\overset{(13)}{=} \iota(F(g))\iota(g \rhd F(h))\iota(f_1(g,h))\iota(F(gh))^{-1} = \iota(g \rhd F(h) - F(gh) + F(g) + f_1(g,h)).\end{aligned}$$

As ι is injective, this implies $f_2(g,h) - f_1(g,h) = g \rhd F(h) - F(gh) - F(g)$ for all $g, h \in G$. It follows that different choices of σ define the same cohomology class $[f_1] = [f_2] \in H^2(G, M)$.

• 1.5 We show that isomorphic group extensions (E, π, ι) and (E', π', ι') define the same $\mathbb{Z}[G]$-module structure on M and the same elements of $H^2(G, M)$.

Let (E, π, ι) and (E', π', ι') be isomorphic group extensions. Then there is a group isomorphism $\phi : E \to E'$ with $\phi \circ \iota = \iota'$ and $\pi' \circ \phi = \pi$. For any map $\sigma : G \to E$ with $\pi \circ \sigma = \mathrm{id}_G$, the map $\sigma' = \phi \circ \sigma : G \to E'$ satisfies $\pi' \circ \sigma' = \pi' \circ \phi \circ \sigma = \pi \circ \sigma = \mathrm{id}_G$. This implies for the $\mathbb{Z}[G]$-module structure $\rhd' : G \times M \to M$ and the 2-cocycle $f' : G \times G \to M$ given by σ'

$$\begin{aligned}\iota'(g \rhd' m) &\overset{(13)}{=} \sigma'(g)\iota'(m)\sigma'(g)^{-1} = (\phi \circ \sigma)(g) \cdot (\phi \circ \iota)(m) \cdot (\phi \circ \sigma)(g)^{-1} = \phi(\sigma(g)\iota(m)\sigma(g)^{-1}) \\ &\overset{(13)}{=} \phi \circ \iota(g \rhd m) = \iota'(g \rhd m) \\ \iota'(f'(g,h)) &\overset{(13)}{=} \sigma'(g)\sigma'(h)\sigma'(gh)^{-1} = (\phi \circ \sigma)(g)(\phi \circ \sigma)(h)(\phi \circ \sigma)(gh)^{-1} = \phi(\sigma(g)\sigma(h)\sigma(gh)^{-1}) \\ &\overset{(13)}{=} \phi \circ \iota(f(g,h)) = \iota'(f(g,h))\end{aligned}$$

As $\iota' = \phi \circ \iota$ is the composite of injective maps, it is injective. It follows that the $\mathbb{Z}[G]$-module structures and cocycles defined by σ and σ' are equal, and so are the corresponding elements of $H^2(G, M)$. As $\phi : E \to E'$ is invertible, this proves that the isomorphic group extensions (E, ι, π) and (E', ι', π') define the same element of $H^2(G, M)$.

• 1.6 We show that the map that assigns to a pair $(\rhd, [f])$ the iromorphism class of the a group extension $(M \times G, \iota, \pi)$ defined by (10) and the map that assigns to an isomorphism class of a group extension (E, ι, π) the pair $(\rhd, [f])$ defined by (13) are mutually inverse.

Given the group extension $(M \times G, \iota, \pi)$ from (10) defined by $(\rhd, [f])$, we can choose the map $\sigma : G \to M \times G$, $g \mapsto (0, g)$ that satisfies $\pi \circ \sigma = \mathrm{id}_G$. The associated $\mathbb{Z}[G]$-module structure $\rhd' : \mathbb{Z}[G] \times M \to M$ and 2-cocycle $f' : G \times G \to M$ defined by (13) are the given by $\rhd' = \rhd$ and $f'(g, h) = f(g, h) - g \rhd f(1, 1)$ for all $g, h \in G$. As $f'' : G \times G \to M$, $(g, h) \mapsto g \rhd f(1, 1)$ is a 2-coboundary, it follows that $(\rhd, [f]) = (\rhd', [f'])$.

Conversely, given a group extension (E, ι, π), we choose a map $\sigma : G \to E$ with $\pi \circ \sigma = \mathrm{id}_G$ and consider the associated $\mathbb{Z}[G]$-module structure $\rhd : \mathbb{Z}[G] \times M \to M$ and 2-cocycle $f : G \times G \to M$ defined by (13). The latter define a group extension $M \times G$ with multiplication (10) and group homomorphisms $\iota_{M\times G} : M \to M \times G$, $m \mapsto (m, 0)$ and $\pi_{M\times G} : M \times G \to G$, $(m, g) \mapsto G$. Then the map $\phi : M \times G \to E$, $(m, g) \mapsto \iota(m)\sigma(g)\sigma(1)^{-1}$ satisfies $\phi \circ \iota_{M\times G} = \iota$ and $\pi \circ \phi = \pi_{M\times G}$. A direct computation using (13) and the identity $\sigma(1) = f(1, 1)$ shows that ϕ is a group homomorphism with inverse $\phi^{-1} : E \to M \times G$, $e \mapsto (m, \pi(e))$ with $\iota(m) = e\sigma(1)\sigma(\pi(e))^{-1}$.

2. If $\rhd : \mathbb{Z}[G] \times M \to M$ is the trivial $\mathbb{Z}[G]$-module structure, then the multiplication on $M \times G$ from (10) takes the form

$$(m, g) \cdot_f (m', g') = (m + m' + f(g, g') - f(1, 1), gg'). \tag{15}$$

This implies with (11)

$$\begin{aligned}(m, 1) \cdot_f (m', g') &= (m + m' + f(1, g') - f(1, 1), g') = (m + m', g') \\ &= (m + m' + f(g, 1) - f(1, 1), g') = (m', g') \cdot_f (m, 1)\end{aligned}$$

and hence $M \times \{1\}$ is central in $(M \times G, \cdot_f)$. Conversely, if (E, ι, π) is a central extension of G by M, then $\mathbb{Z}[G]$-module structure on M defined in (13) satisfies

$$\iota(g \rhd m) = \sigma(g)\iota(m)\sigma(g)^{-1} = \sigma(g)\sigma(g)^{-1}\iota(m) = \iota(m)$$

By injectivity of ι, this implies $g \rhd m = m$ for all $g \in G$ and $m \in M$. □

Example 2.3.9: We determine the central extensions of $\mathbb{Z}/2\mathbb{Z}$ by an abelian group M and the cohomologies $H^1(\mathbb{Z}/2\mathbb{Z}, M)$ and $H^2(\mathbb{Z}/2\mathbb{Z}, M)$ for an abelian group M with the trivial $\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$-module structure.

• We compute $H^1(\mathbb{Z}/2\mathbb{Z}, M)$:

By Corollary 2.3.4, the first cohomology $H^1(\mathbb{Z}/2\mathbb{Z}, M)$ is the $\mathbb{Z}$-module of group homomorphisms $f : \mathbb{Z}/2\mathbb{Z} \to M$. This follows because inner derivations on $\mathbb{Z}/2\mathbb{Z}$ with values in a trivial $\mathbb{Z}/2\mathbb{Z}$-module M are trivial and derivations are group homomorphisms from $\mathbb{Z}/2\mathbb{Z}$ to M. Any group homomorphism $f : \mathbb{Z}/2\mathbb{Z} \to M$ is determined uniquely by $f(\bar{1}) \in M$ and must satisfy $2f(\bar{1}) = f(\bar{1}) + f(\bar{1}) = f(\bar{1} + \bar{1}) = f(\bar{0}) = 0$. This implies

$$H^1(\mathbb{Z}/2\mathbb{Z}, M) = \{m \in M \mid 2m = 0\}.$$

• We compute $H^2(\mathbb{Z}/2\mathbb{Z}, M)$:

A map $f : \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to M$ is a 2-cocycle if and only if for all $\bar{p}, \bar{q}, \bar{r} \in \mathbb{Z}/2\mathbb{Z}$

$$f(\bar{q}, \bar{r}) - f(\bar{p} + \bar{q}, \bar{r}) + f(\bar{p}, \bar{q} + \bar{r}) - f(\bar{p}, \bar{q}) = 0.$$

A short computation shows that this holds if and only if $f(\bar{1},\bar{0}) = f(\bar{0},\bar{1}) = f(\bar{0},\bar{0})$. This shows that 2-cocycles are determined by a pair of elements $f(\bar{0},\bar{0}), f(\bar{1},\bar{1}) \in M$, and we have $Z^2(\mathbb{Z}/2\mathbb{Z}, M) \cong M \oplus M$. For a map $F : \mathbb{Z}/2\mathbb{Z} \to M$ we have

$$d^1(F)(\bar{p},\bar{q}) = F(\bar{p}) - F(\bar{p}+\bar{q}) + F(\bar{q}),$$

which implies $d^1(F)(\bar{1},\bar{1}) = 2F(\bar{1}) - F(\bar{0})$ and $d^1(F)(\bar{0},\bar{1}) = d^1(F)(\bar{1},\bar{0}) = d^1(F)(\bar{0},\bar{0}) = F(\bar{0})$. Hence, we have $f = d^1(F)$ for some map $F : \mathbb{Z}/2\mathbb{Z} \to M$ if and only if $f(\bar{1},\bar{1}) + f(\bar{0},\bar{0})$ is contained in the subgroup $2M \subset M$, and we obtain

$$H^2(\mathbb{Z}/2\mathbb{Z}, M) \cong M/2M.$$

- We determine the central extensions of $\mathbb{Z}/2\mathbb{Z}$ by certain abelian groups M:
  - If $M = \mathbb{Z}$, we have $H^1(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}) = 0$ and $H^2(\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. There are two isomorphism classes of central extensions of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}$. From formula (10) one finds that even values of $f(\bar{1},\bar{1}) + f(\bar{0},\bar{0})$ yield the direct product $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. For odd values of $f(\bar{1},\bar{1}) + f(\bar{0},\bar{0})$, formula (10) yields the set $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with group multiplication

$$(z,\bar{p}) \cdot (z',\bar{p}') = \begin{cases} (z+z'+1, \bar{0}) & \bar{p} = \bar{p}' = \bar{1} \\ (z+z', \bar{p}+\bar{p}') & \text{else.} \end{cases}$$

  - If $M = \mathbb{Z}/n\mathbb{Z}$ with odd $n \in \mathbb{N}$, we have again $H^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = 0$. However, in this case $2(\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$ and hence $H^2(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = 0$ as well. Up to isomorphisms, there is only one central extension of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}/n\mathbb{Z}$, namely the abelian group $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

  - If $M = \mathbb{Z}/2^k\mathbb{Z}$ for $k \in \mathbb{N}$ we have $H^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2^k\mathbb{Z}) = H^2(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2^k\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. Up to isomorphisms, there are two central extensions of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}/2^k\mathbb{Z}$. Formula (10) shows that they are the direct product $\mathbb{Z}/2^k\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ for $f(\bar{1},\bar{1}) + f(\bar{0},\bar{0})$ even. For $f(\bar{1},\bar{1}) + f(\bar{0},\bar{0})$ odd, formula (10) yields the set $\mathbb{Z}/2^k\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with multiplication

$$(\bar{q},\bar{p}) \cdot (\bar{q}',\bar{p}') = \begin{cases} (\bar{q}+\bar{q}'+\bar{1}, \bar{0}) & \bar{p} = \bar{p}' = \bar{1} \\ (\bar{q}+\bar{q}', \bar{p}+\bar{p}') & \text{else.} \end{cases}$$

    By the classification theorem for finite abelian groups, up to isomorphisms there are only two abelian groups of order 2^{k+1}, which contain $\mathbb{Z}/2^k\mathbb{Z}$ as a subgroup, namely $\mathbb{Z}/2^k\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/2^{k+1}\mathbb{Z}$. Hence, the non-trivial central extension is isomorphic to $\mathbb{Z}/2^{k+1}\mathbb{Z}$. (Exercise: Find a group isomorphism).

This example illustrates that group cohomologies contain information about groups that would be difficult to access otherwise. However, the procedure to compute the group cohomologies $H^2(\mathbb{Z}/2\mathbb{Z}, M)$ is too pedestrian and not practical for groups with more elements. In Section 4 we derive efficient methods for the computation of group cohomologies and treat other examples.

2.4 Lie algebra cohomology*

In this section, we consider cohomologies of *Lie algebras*. Finite-dimensional Lie algebras can be viewed as a infinitesimal counterparts of *Lie groups*. A Lie group is a smooth manifold with a group structure such that the group multiplication and inversion are smooth maps. The Lie algebra $\mathfrak{g} = \mathrm{Lie}\, G$ of a Lie group G is the tangent space of G in the unit element. Although non-isomorphic Lie groups may have isomorphic Lie algebras, many questions surrounding the classification of Lie groups and their representation theory can be addressed by investigating the associated questions for their Lie algebras, which are more accessible.

Important examples of finite-dimensional Lie groups are *matrix Lie groups*, closed subgroups of the matrix groups $\mathrm{GL}(n, \mathbb{C})$ or $\mathrm{GL}(n, \mathbb{R})$. The associated Lie algebras are called *matrix Lie algebras*. They are linear subspaces of the vector spaces $\mathfrak{gl}(n, \mathbb{R})$ or $\mathfrak{gl}(n, \mathbb{C})$ of $n \times n$-matrices with entries in $\mathbb{R}$ or $\mathbb{C}$. The underlying matrix Lie groups or certain subgroups thereof are obtained by exponentiating the matrix Lie algebras.

Definition 2.4.1: Let $\mathbb{F}$ be a field.

1. A **Lie algebra** over $\mathbb{F}$ is a vector space $\mathfrak{g}$ over $\mathbb{F}$ together with an $\mathbb{F}$-bilinear map $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, the **Lie bracket** that satisfies:
   (L1) **antisymmetry:** $[x, x] = 0$ for all $x \in \mathfrak{g}$.
   (L2) **Jacobi identity:** $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$ for all $x, y, z \in \mathfrak{g}$.
2. A **Lie subalgebra** $\mathfrak{h} \subset \mathfrak{g}$ is a linear subspace $\mathfrak{h} \subset \mathfrak{g}$ that that is a Lie algebra with the restriction of the Lie bracket on $\mathfrak{g}$, i. e. a linear subspace $\mathfrak{h} \subset \mathfrak{g}$ with $[x, y] \in \mathfrak{h}$ for all $x, y \in \mathfrak{h}$.
3. A **morphism of Lie algebras** is a $\mathbb{F}$-linear map $f : \mathfrak{g} \to \mathfrak{h}$ with $[f(x), f(y)]_{\mathfrak{h}} = f([x, y]_{\mathfrak{g}})$ for all $x, y \in \mathfrak{g}$. An **isomorphism of Lie algebras** is a bijective morphism of Lie algebras.

The category of Lie algebras over $\mathbb{F}$ and Lie algebra morphisms is denoted $\mathrm{Liealg}_{\mathbb{F}}$.

The Lie bracket of a Lie algebra $\mathfrak{g}$ can be viewed as the infinitesimal counterpart of the group multiplication of a Lie group G. The antisymmetry of the Lie bracket encodes the fact that $(g \cdot h)^{-1} = h^{-1} \cdot g^{-1}$ for all elements $g, h \in G$, and the Jacobi identity is the infinitesimal counterpart of the associativity of the group multiplication. Note that the Jacobi identity implies that a Lie bracket $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is in general *non-associative*.

Example 2.4.2:

1. Every vector space V over $\mathbb{F}$ becomes a Lie algebra over $\mathbb{F}$ with the trivial Lie bracket $[\ ,\ ] = 0 : V \times V \to V$, $v \mapsto 0$. A Lie algebra with a trivial Lie bracket is called **abelian**.
2. If A is an associative (not necessarily unital) algebra over $\mathbb{F}$, then A is a Lie algebra with the **commutator bracket** $[\ ,\ ] : A \times A \to A$, $(a, b) \mapsto [a, b] = a \cdot b - b \cdot a$. This holds in particular for the algebra $\mathrm{End}_{\mathbb{F}}(V)$ of linear endomorphisms of an $\mathbb{F}$-vector space V.
3. For any algebra A over $\mathbb{F}$ the $\mathbb{F}$-vector space $\mathrm{Der}(A, A) \subset \mathrm{Hom}_{\mathbb{F}}(A, A)$ of derivations on A is a Lie subalgebra of the Lie algebra $\mathrm{End}_{\mathbb{F}}(A)$ with the commutator bracket.
4. Any matrix algebra $\mathfrak{gl}(n, \mathbb{F}) = \mathrm{Mat}(n \times n, \mathbb{F})$ is a Lie algebra with the commutator bracket.

The linear subspaces

$$\mathfrak{sl}(n,\mathbb{F}) = \{M \in \mathrm{Mat}(n,\mathbb{F}) \mid \mathrm{tr}\,(M) = 0\}$$
$$\mathfrak{o}(n,\mathbb{F}) = \{M \in \mathrm{Mat}(n,\mathbb{F}) \mid M^T = -M\}$$
$$\mathfrak{so}(n,\mathbb{F}) = \{M \in \mathrm{Mat}(n,\mathbb{F}) \mid M^T = -M, \mathrm{tr}\,(M) = 0\}$$
$$\mathfrak{c}(n,\mathbb{F}) = \{M \in \mathrm{Mat}(n,\mathbb{F}) \mid M_{ij} = 0 \text{ for } i \neq j\}$$
$$\mathfrak{t}_+(n,\mathbb{F}) = \{M \in \mathrm{Mat}(n,\mathbb{F}) \mid M_{ij} = 0 \text{ for } i > j\}$$
$$\mathfrak{t}_-(n,\mathbb{F}) = \{M \in \mathrm{Mat}(n,\mathbb{F}) \mid M_{ij} = 0 \text{ for } i < j\}$$

of traceless, antisymmetric, diagonal and upper and lower triangular matrices are Lie subalgebras of $\mathfrak{gl}(n,\mathbb{F})$.

5. The matrix algebras

$$\mathfrak{u}(n,\mathbb{C}) = \{M \in \mathrm{Mat}(n,\mathbb{C}) \mid M^\dagger = -M\}$$
$$\mathfrak{su}(n,\mathbb{C}) = \{M \in \mathrm{Mat}(n,\mathbb{C}) \mid M^\dagger = -M, \mathrm{tr}\,(M) = 0\}$$

of antihermitian and traceless antihermitian matrices are Lie subalgebras of $\mathfrak{gl}(n,\mathbb{C})$.

6. **Ado's Theorem** states that any finite-dimensional Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$ of characteristic zero is isomorphic to a Lie subalgebra of a matrix algebra $\mathfrak{gl}(n,\mathbb{F})$.

Just as for groups and algebras, it is advantageous to describe a Lie algebra in terms of *representations*. Lie algebra representations are Lie algebra homomorphisms into the algebra of endomorphisms of a vector space with the commutator bracket. If the vector space is finite-dimensional, this allows for a description in terms of matrices. One can then apply results and methods from linear algebra to describe and classify finite-dimensional Lie algebras.

Definition 2.4.3: Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$.

1. A **representation** of $\mathfrak{g}$ is a vector space M over $\mathbb{F}$ together with a Lie algebra morphism $\rho : \mathfrak{g} \to \mathrm{End}_{\mathbb{F}}(M)$, where $\mathrm{End}_{\mathbb{F}}(M)$ is equipped with the commutator bracket.
2. A **homomorphism of Lie algebra representations** from $\rho : \mathfrak{g} \to \mathrm{End}_{\mathbb{F}}(M)$ to $\rho' : \mathfrak{g} \to \mathrm{End}_{\mathbb{F}}(M')$ is an $\mathbb{F}$-linear map $\phi : M \to M'$ with $\rho'(x) \circ \phi = \phi \circ \rho(x)$ for $x \in \mathfrak{g}$. An **isomorphism of Lie algebra representations** is a bijective morphism of Lie algebra representations.

The category of representations of $\mathfrak{g}$ and morphisms of $\mathfrak{g}$-representations is denoted $\mathrm{Rep}(\mathfrak{g})$.

Example 2.4.4:

1. Every Lie algebra $\mathfrak{g}$ over $\mathbb{F}$ has a **trivial representation** $\rho = 0 : \mathfrak{g} \to \mathrm{End}_{\mathbb{F}}(M)$, $x \mapsto 0$ on any vector space M over $\mathbb{F}$.
2. Every Lie algebra $\mathfrak{g}$ has a representation on itself, the **adjoint representation** $\rho = \mathrm{ad} : \mathfrak{g} \to \mathrm{End}_{\mathbb{F}}(\mathfrak{g})$, $x \mapsto \mathrm{ad}_x$ with $\mathrm{ad}_x(y) = [x, y]$ for all $y \in \mathfrak{g}$.
3. Any Lie subalgebra $\mathfrak{g} \subset \mathfrak{gl}(n,\mathbb{F})$ has a representation $\rho : \mathfrak{h} \to \mathrm{End}_{\mathbb{F}}(\mathbb{F}^n)$, $M \mapsto \phi_M$ with $\phi_M(v) = M \cdot v$.

Just as a representation of a group G on a vector space over $\mathbb{F}$ can be viewed as a module over the group algebra $\mathbb{F}[G]$, we can view a representation of a Lie algebra $\mathfrak{g}$ as a module over the *universal enveloping algebra* $U(\mathfrak{g})$.

Definition 2.4.5: Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$.

1. The **tensor algebra** $T(\mathfrak{g})$ is the $\mathbb{F}$-vector space $T(\mathfrak{g}) = \oplus_{k=0}^{\infty}\mathfrak{g}^{\otimes k}$ with the multiplication
$$(x_1\otimes ... \otimes x_k)\cdot(y_1\otimes ... \otimes y_l) = x_1\otimes ... \otimes x_k\otimes y_1\otimes ... \otimes y_l \qquad \forall x_i, y_j \in \mathfrak{g}.$$
2. The **universal enveloping algebra** $U(\mathfrak{g})$ is the quotient
$$U(\mathfrak{g}) = T(\mathfrak{g})/(x\otimes y - y\otimes x - [x,y])$$
of the tensor algebra $T(\mathfrak{g})$ by the two-sided ideal $I = (x\otimes y - y\otimes x - [x,y])$ generated by the elements $x\otimes y - y\otimes x - [x,y]$ for $x, y \in \mathfrak{g}$.

Proposition 2.4.6: (Universal property of the universal enveloping algebra)

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$. Then the inclusion map $\iota : \mathfrak{g} \to U(\mathfrak{g})$ is a Lie algebra homomorphism, and for every Lie algebra homomorphism $\phi : \mathfrak{g} \to A$ into an algebra A with the commutator bracket, there is a unique algebra homomorphism $\phi' : U(\mathfrak{g}) \to A$ with $\phi' \circ \iota = \phi$.

Proof:
By definition of $U(\mathfrak{g})$ we have
$$\iota(x)\iota(y) - \iota(y)\iota(x) = (x+I)\otimes(y+I) - (y+I)\otimes(x+I) = x\otimes y - y\otimes x + I = [x,y] + I = \iota([x,y]).$$
By the universal property of the tensor algebra $T(\mathfrak{g})$, any $\mathbb{F}$-linear map $\phi : \mathfrak{g} \to A$ induces a unique algebra homomorphism $\phi'' : T(\mathfrak{g}) \to A$, $x_1\otimes ... \otimes x_k \mapsto \phi(x_1)\cdots\phi(x_k)$ with $\phi''|\mathfrak{g} = \phi$. If ϕ is a Lie algebra homomorphism, we have
$$\phi''(x\otimes y - y\otimes x - [x,y]) = \phi''(x)\phi''(y) - \phi''(y)\phi''(x) - \phi''([x,y]) = [\phi(x),\phi(y)] - \phi([x,y]) = 0,$$
which implies $I \subset \ker(\phi'')$. Hence, we obtain a unique algebra homomorphism $\phi' : U(\mathfrak{g}) \to A$ with $\phi' \circ \iota = \phi''|_{\mathfrak{g}} = \phi$. □

As a representation of a Lie algebra $\mathfrak{g}$ over $\mathbb{F}$ on an $\mathbb{F}$-vector space V is a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathrm{End}_{\mathbb{F}}(V)$, it follows that ρ extends to an algebra homomorphism $\rho' : U(\mathfrak{g}) \to \mathrm{End}_{\mathbb{F}}(V)$ with $\rho' \circ \iota = \phi$ or, equivalently, to an $U(\mathfrak{g})$-module structure on V with $\iota(x) \rhd v = \rho(x)v$ for all $x \in \mathfrak{g}$ and $v \in V$. Conversely, for any $U(\mathfrak{g})$-module V, we obtain a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathrm{End}_{\mathbb{F}}(V)$ given by $\rho(x)v = \iota(x) \rhd v$ for all $x \in \mathfrak{g}$ and $v \in V$.

Corollary 2.4.7: For any Lie algebra $\mathfrak{g}$ and vector space V over $\mathbb{F}$, representations of $\mathfrak{g}$ on V are in bijection with $U(\mathfrak{g})$-module structures on V.

With the concept of a Lie algebra and a Lie algebra representation, we can define homologies or cohomologies of Lie algebras. As the latter are often simpler to compute and more well-behaved in the infinite-dimensional case, we focus on cohomologies. The definition is very similar to the one of group cohomology, only that the module over the group algebra $k[G]$ is replaced by a representation of a Lie algebra and the coboundary operators take a different form. The deeper reason for this is that any Lie group representation defines a representation of the associated Lie algebra. Hence, the structures for a Lie algebra can be obtained from the ones for Lie groups by differentiating in the unit element.

Definition 2.4.8: Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$ and $\rho : \mathfrak{g} \to \mathrm{End}_{\mathbb{F}}(M)$ a representation of $\mathfrak{g}$. Denote by $\mathfrak{g}^*$ the dual vector space and by $\mathrm{Hom}_{\mathbb{F}}(\Lambda^n \mathfrak{g}, M)$ the vector space of alternating n-linear maps $\omega : \mathfrak{g}^{\times n} \to M$.

1. The k-module of n-**cochains** is

$$C^n(\mathfrak{g}, M) = \begin{cases} \mathrm{Hom}_{\mathbb{F}}(\Lambda^n \mathfrak{g}^*, M) & n \in \mathbb{N}_0 \\ 0 & n \leq 0. \end{cases}$$

2. The **coboundary operators** are the $\mathbb{F}$-linear maps $d^n : C^n(\mathfrak{g}, M) \to C^{n+1}(\mathfrak{g}, M)$ given by $d^n = 0$ for $n < 0$ and

$$\begin{aligned}(d^n f)(x_0, ..., x_n) = &\Sigma_{i=0}^n (-1)^i \rho(x_i) f(x_0, ..., \widehat{x}_i, ..., x_n) \\ &+ \Sigma_{0 \leq i < j \leq n} (-1)^{i+j} f([x_i, x_j], x_0, ..., \widehat{x}_i, ..., \widehat{x}_j, ..., x_n)\end{aligned}$$

   for $n \in \mathbb{N}_0$. They satisfy $d^{n+1} \circ d^n = 0$ for all $n \in \mathbb{Z}$ (Exercise).

3. The $\mathbb{F}$-vector spaces of n-**cocycles** and of n-**coboundaries** are the linear subspaces $Z^n(\mathfrak{g}, M) = \ker(d^n) \subset C^n(\mathfrak{g}, M)$ and $B^n(\mathfrak{g}, M) = \mathrm{im}(d^{n-1}) \subset Z^n(\mathfrak{g}, M)$.

4. The nth **Lie algebra cohomology** of $\mathfrak{g}$ with coefficients in M is the quotient space

$$H^n(\mathfrak{g}, M) = \frac{Z^n(\mathfrak{g}, M)}{B^n(\mathfrak{g}, M)} = \frac{\ker(d^n)}{\mathrm{im}(d^{n-1})}.$$

The interpretation of the first two cohomologies for Lie algebras are similar to the ones for groups and algebras. The Lie algebra cohomology $H^0(\mathfrak{g}, M)$ describes the *invariants* of the representation of $\mathfrak{g}$, and the first cohomology $H^1(\mathfrak{g}, M)$ the derivations modulo inner derivations. The only difference is that the concepts of an invariant and of a derivation are the infinitesimal version of the ones for a group.

Definition 2.4.9:
Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$ and $\rho : \mathfrak{g} \to \mathrm{End}_{\mathbb{F}}(M)$ a representation of $\mathfrak{g}$ on M.

1. An **invariant** of the representation ρ is an element $m \in M$ with $\rho(x)m = 0$ for all $x \in \mathfrak{g}$. The vector space of invariants of ρ is denoted $M^{\mathfrak{g}}$.

2. A **derivation** on $\mathfrak{g}$ with values in M is a linear map $f : \mathfrak{g} \to M$ that satisfies $f([x, y]) = \rho(x)f(y) - \rho(y)f(x)$ for all $x, y \in \mathfrak{g}$. The vector space of derivations $f : \mathfrak{g} \to M$ is denoted $\mathrm{Der}(\mathfrak{g}, M)$.

3. An **inner derivation** on $\mathfrak{g}$ with values in M is a derivation of the form $f_m : \mathfrak{g} \to M$, $x \mapsto \rho(x)m$ for some $m \in M$. The vector space of inner derivations $f : \mathfrak{g} \to M$ is denoted $\mathrm{InnDer}(\mathfrak{g}, M)$.

Given the concepts of an invariant and an (inner) derivation, we can derive and interpret the first two cohomology groups for a Lie algebra $\mathfrak{g}$ and a representation $\rho : \mathfrak{g} \to \mathrm{End}_{\mathbb{F}}(M)$. The computation and the result is fully analogous to the one for groups.

Lemma 2.4.10: Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$ and $\rho : \mathfrak{g} \to \mathrm{End}_{\mathbb{F}}(M)$ a representation of $\mathfrak{g}$. The first two Lie algebra cohomologies of $\mathfrak{g}$ with coefficients in M are given by

$$H^0(\mathfrak{g}, M) = M^{\mathfrak{g}} \qquad\qquad H^1(\mathfrak{g}, M) = \frac{\mathrm{Der}(\mathfrak{g}, M)}{\mathrm{InnDer}(\mathfrak{g}, M)}.$$

Proof:
The first non-trivial coboundary operators are given by

$$\begin{aligned} &d^0 : C^0(\mathfrak{g}, M) = M \to C^1(\mathfrak{g}, M),\ m \mapsto f_m \quad \text{with} \quad f_m(x) = \rho(x)m \\ &d^1 : C^1(\mathfrak{g}, M) \to C^2(\mathfrak{g}, M) \quad d^1(f)(x, y) = \rho(x)f(y) - \rho(y)f(x) - f([x, y]), \end{aligned}$$

and this implies

$$\begin{aligned} &\ker(d^0) = \{m \in M \mid \rho(x)m = 0\ \forall x \in \mathfrak{g}\} = M^{\mathfrak{g}} \\ &\mathrm{im}(d^0) = \{f : \mathfrak{g} \to M \mid \exists m \in M : f(x) = \rho(x)m\} = \mathrm{InnDer}(\mathfrak{g}, M) \\ &\ker(d^1) = \{f : \mathfrak{g} \to M \mid f([x, y]) = \rho(x)f(y) - \rho(y)f(x)\ \forall x, y \in \mathfrak{g}\} = \mathrm{Der}(\mathfrak{g}, M). \end{aligned}$$

□

The similarities between group and Lie algebra cohomologies extend also to higher cohomologies. The only difference is that the concepts that describe these group cohomologies have to be adapted to Lie algebras by replacing modules over group rings by Lie algebra representations, group multiplications by Lie brackets and group homomorphisms by Lie algebra homomorphisms. If one applies this procedure to the concept of a group extension from Definition 2.3.5, one obtains the following definition.

Definition 2.4.11: Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras over $\mathbb{F}$.

1. A **Lie algebra extension** of $\mathfrak{g}$ by $\mathfrak{h}$ is a triple $(\mathfrak{e}, \iota, \pi)$ of a Lie algebra $\mathfrak{e}$ over $\mathbb{F}$ together with an injective Lie algebra morphism $\iota : \mathfrak{h} \to \mathfrak{e}$ and a surjective Lie algebra morphism $\pi : \mathfrak{e} \to \mathfrak{g}$ such that $\ker(\pi) = \mathrm{im}(\iota)$.

2. An extension $(\mathfrak{e}, \iota, \pi)$ of $\mathfrak{g}$ by $\mathfrak{h}$ is called **central** if $[x, y] = 0$ for all $x \in \mathfrak{h}$ and $y \in \mathfrak{e}$.

3. A **morphism of Lie algebra extensions** from $(\mathfrak{e}, \iota, \pi)$ to $(\mathfrak{e}', \iota', \pi')$ is a Lie algebra morphism $f : \mathfrak{e} \to \mathfrak{e}'$ with $\phi \circ \iota = \iota'$ and $\pi' \circ f = \pi$. An **isomorphism of Lie algebra extensions** is a bijective morphism of Lie algebra extensions.

Given the concept of a Lie algebra extension, we can show that the cohomology $H^2(\mathfrak{g}, M)$ classifies isomorphism classes of extensions of $\mathfrak{g}$ by M with the trivial Lie algebra structure. The result and its proof are completely analogous to the one for groups, only that some of the computations simplify because tof linearity in the Lie algebra case.

Theorem 2.4.12: Let $\mathfrak{g}$ be a Lie algebra and M a vector space over $\mathbb{F}$.

1. Isomorphism classes of extensions of $\mathfrak{g}$ by the abelian Lie algebra M are in bijection with pairs $(\rho, [f])$ of a representation $\rho : \mathfrak{g} \to \mathrm{End}_{\mathbb{F}}(M)$ and element $[f] \in H^2(\mathfrak{g}, M)$.

2. Isomorphism classes of central extensions of $\mathfrak{g}$ by M are in bijection with elements of $H^2(\mathfrak{g}, M)$, where M is equipped with the abelian Lie algebra structure.

Proof:

1. Let $\rho : \mathfrak{g} \to \mathrm{End}_{\mathbb{F}}(M)$ be a representation of $\mathfrak{g}$ on M. By Definition 2.4.8, the coboundary operators $d^1 : C^1(\mathfrak{g}, M) \to C^2(\mathfrak{g}, M)$ and $d^2 : C^2(\mathfrak{g}, M) \to C^3(\mathfrak{g}, M)$ are given by

$$d^1(F)(x, y) = \rho(x)F(y) - \rho(y)F(x) - F([x, y])$$
$$d^2(f)(x, y, x) = \rho(x)f(y, z) - \rho(y)f(x, z) + \rho(z)f(x, y) - f([x, y], z) + f([x, z], y) - f([y, z], x)$$

for all linear maps $F : \mathfrak{g} \to M$ and alternating linear maps $f : \mathfrak{g} \times \mathfrak{g} \to M$. Hence, a 2-cocycle is an alternating bilinear map $f : \mathfrak{g} \times \mathfrak{g} \to M$ with

$$\rho(x)f(y, z) - \rho(y)f(x, z) + \rho(z)f(x, y) = -f([x, y], z) + f([x, z], y) - f([y, z], x) \qquad (16)$$

for all $x, y, z \in \mathfrak{g}$, and a 2-boundary is an alternating bilinear map $f : \mathfrak{g} \times \mathfrak{g} \to M$ of the form

$$f : \mathfrak{g} \times \mathfrak{g} \to M, \quad (x, y) \mapsto \rho(x) \rhd F(y) - \rho(y)F(x) - F([x, y]). \qquad (17)$$

• 1.1 We show that every 2-cocycle $f : \mathfrak{g} \times \mathfrak{g} \to M$ gives rise to a group extension of $\mathfrak{g}$ by M.

Every 2-cocycle $f : \mathfrak{g} \times \mathfrak{g} \to M$ defines a Lie bracket $[\ ,\ ]_f$ on $M \oplus \mathfrak{g}$ given by

$$[(m, x), (m', x')]_f = (\rho(x)m' - \rho(x')m + f(x, x'), [x, x']). \qquad (18)$$

The bilinearity and antisymmetry is obvious, and the Jacobi identity follows directly from the 2-cocycle condition (16). A short computation shows that the inclusion map $\iota : M \to M \oplus \mathfrak{g}$, $m \mapsto (m, 0)$ and projection map $\pi : M \oplus \mathfrak{g} \to \mathfrak{g}$, $(m, x) \mapsto x$ are Lie algebra homomorphisms with respect to $[\ ,\ ]_f$ and the Lie algebra structures on M and $\mathfrak{g}$. As we have $\ker(\pi) = \mathrm{im}(\iota)$, it follows that $(M \oplus \mathfrak{g}, [\ ,\ ]_f)$ is an extension of $\mathfrak{g}$ by the abelian Lie algebra M.

• 1.2. We show that 2-cocycles that are related by a 2-coboundary define isomorphic extensions.

Suppose that $f_1, f_2 : \mathfrak{g} \times \mathfrak{g} \to M$ are 2-cocycles such that $f_2 - f_1$ is a 2-coboundary. Then

$$f_2(x, y) - f_1(x, y) = \rho(x)F(y) - \rho(y)F(x) - F([x, y]) \qquad (19)$$

for some linear map $F : \mathfrak{g} \to M$ and $x, y \in \mathfrak{g}$. We consider the associated extension Lie algebras $\mathfrak{e}_i = (M \oplus \mathfrak{g}, [\ ,\ ]_{f_i})$ and show that

$$\phi : (M \oplus \mathfrak{g}, [\ ,\ ]_{f_1}) \to (M \oplus \mathfrak{g}, [\ ,\ ]_{f_2}), \quad (m, x) \mapsto (m + F(x), x)$$

is a Lie algebra isomorphism. This follows by a direct computation from (18)

$$\begin{aligned}
[\phi(m, x), \phi(m', x')]_{f_2} &= [(m + F(x), x), (m' + F(x'), x')]_{f_2} \\
&\overset{(18)}{=} (\rho(x)m' + \rho(x)F(x') - \rho(x')m - \rho(x')F(x) + f_2(x, x'), [x, x']) \\
&\overset{(19)}{=} (\rho(x)m' - \rho(x')m + F([x, x']) + f_1(x, x'), [x, x']) = \phi([(m, x), (m', x')]_{f_1}).
\end{aligned}$$

As the Lie algebra homomorphism ϕ is invertible with inverse

$$\phi^{-1} : M \oplus \mathfrak{g} \to M \oplus \mathfrak{g},\ (m, x) \mapsto (m - F(x), x)$$

and we have for all $x \in \mathfrak{g}$ and $m \in M$

$$\phi \circ \iota(m) = \phi(m, 0) = (m, 0) = \iota(m) \qquad \pi \circ \phi(m, x) = \pi(m + F(x), x) = x = \pi(m, x),$$

this shows that the ϕ is an isomorphism of Lie algebra extensions from $(\mathfrak{e}_1, \iota, \pi)$ to $(\mathfrak{e}_2, \iota, \pi)$. Hence, every element $[f] \in H^2(\mathfrak{g}, M)$ defines an isomorphism class of extensions of $\mathfrak{g}$ by M.

• 1.3. We show that an extension $(\mathfrak{e}, \iota, \pi)$ be an extension of $\mathfrak{g}$ by M defines a representation $\rho : \mathfrak{g} \to \mathrm{End}_{\mathbb{F}}(M)$ and a 2-cocycle $f : \mathfrak{g} \times \mathfrak{g} \to M$.

If $(\mathfrak{e}, \iota, \pi)$ is an extension of $\mathfrak{g}$ by M, then $\iota(M) = \ker(\pi) \subset \mathfrak{e}$ is an abelian Lie subalgebra isomorphic to M. Because the Lie algebra homomorphism $\pi : \mathfrak{e} \to \mathfrak{g}$ is surjective, we can choose an element $\sigma(x) \in \pi^{-1}(x)$ for each $x \in \mathfrak{g}$ and obtain a map $\sigma : \mathfrak{g} \to \mathfrak{e}$ with $\pi \circ \sigma = \mathrm{id}_{\mathfrak{g}}$. For all $m \in M$ and $x \in \mathfrak{e}$, we have $\pi([x, \iota(m)]) = [\pi(x), \pi \circ \iota(m)] = [\pi(x), 0] = 0$ and hence $[x, \iota(m)] \in \iota(M)$ for all $x \in \mathfrak{e}$, $m \in M$. As $\pi : \mathfrak{e} \to \mathfrak{g}$ is a Lie algebra morphism with $\pi \circ \sigma = \mathrm{id}_{\mathfrak{g}}$, we also have $\pi([\sigma(x), \sigma(y)] - \sigma([x, y])) = [x, y] - [x, y] = 0$ and hence $[\sigma(x), \sigma(y)] - \sigma([x, y]) \in \iota(M) = \ker(\pi)$ for all $x, y \in \mathfrak{g}$. As $\iota : M \to \mathfrak{e}$ is injective, this defines two maps

$$\begin{aligned} &\rho : \mathfrak{g} \to \mathrm{End}_{\mathbb{F}}(M) && \text{with} \quad \iota(\rho(x)m) = [\sigma(x), \iota(m)] \\ &f : \mathfrak{g} \times \mathfrak{g} \to M, && \text{with} \quad \iota(f(x, y)) = [\sigma(x), \sigma(y)] - \sigma([x, y]). \end{aligned} \tag{20}$$

Clearly, ρ is linear and f is alternating and bilinear. To show that ρ is a representation of $\mathfrak{g}$ on M, we compute

$$\begin{aligned} &\iota(\rho(x)\rho(y)m - \rho(y)\rho(x)m) \overset{(20)}{=} [\sigma(x), \iota(\rho(y)m)] - [\sigma(y), \iota(\rho(x)m)] \\ &\overset{(20)}{=} [\sigma(x), [\sigma(y), \iota(m)]] - [\sigma(y), [\sigma(x), \iota(m)]] = [[\sigma(x), \sigma(y)], \iota(m)] \\ &\overset{(20)}{=} [\sigma([x, y]), \iota(m)] + [\iota(f(x, x')), \iota(m)] \overset{(20)}{=} \iota(\rho([x, y]), m) + \iota([f(x, x'), m]) = \iota(\rho([x, x']m), \end{aligned}$$

where we used first the definition of ρ, then the Jacobi identity in $\mathfrak{e}$ and the definition of f to pass to the third line and finally the fact that M is abelian. As ι is injective, this shows that $\rho : \mathfrak{g} \to \mathrm{End}_{\mathbb{F}}(M)$ is a representation of $\mathfrak{g}$ on M. From the Jacobi identity in $\mathfrak{e}$, we obtain

$$\begin{aligned} &\iota(\rho(x)f(y, z) - \rho(y)f(x, z) + \rho(z)f(x, y)) \\ &\overset{(20)}{=} [\sigma(x), \iota \circ f(y, z)] - [\sigma(y), \iota \circ f(x, z)] + [\sigma(z), \iota \circ f(x, y)] \\ &\overset{(20)}{=} [\sigma(x), [\sigma(y), \sigma(z)]] - [\sigma(x), \sigma([y, z])] - [\sigma(y), [\sigma(x), \sigma(z)]] \\ &\quad + [\sigma(y), \sigma([x, z])] + [\sigma(z), [\sigma(x), \sigma(y)]] - [\sigma(z), \sigma([x, y])] \\ &\overset{\text{Jac}}{=} -[\sigma(x), \sigma([y, z])] + [\sigma(y), \sigma([x, z])] - [\sigma(z), \sigma([x, y])] \\ &\overset{\text{Jac}}{=} -[\sigma(x), \sigma([y, z])] + [\sigma(y), \sigma([x, z])] - \sigma([y, [x, z]]) + \sigma([x, [y, z]] + [y, [z, x]] + [z, [x, y]]) \\ &\overset{(20)}{=} -f([x, y], z) + f([x, z], y) - f([y, z], x) \end{aligned}$$

Using again that ι is injective and comparing with (16), we see that f is a 2-cocycle. Hence, we have shown that every extension of $\mathfrak{g}$ by M defines a representation of $\mathfrak{g}$ on M and a 2-cocycle $f : \mathfrak{g} \times \mathfrak{g} \to M$.

• 1.4. We determine how the representation and the 2-cocycle in 1.3 depend on $\sigma : \mathfrak{g} \to \mathfrak{e}$.

Let $\sigma_1, \sigma_2 : \mathfrak{g} \to \mathfrak{e}$ two maps with $\pi \circ \sigma_i = \mathrm{id}_{\mathfrak{g}}$, let $\rho_i : \mathfrak{g} \to \mathrm{End}_{\mathbb{F}}(M)$ by the associated representations and $f_i : \mathfrak{g} \times \mathfrak{g} \to M$ the associated 2-cocycles defined by (20). Then we have $\pi(\sigma_2(x) - \sigma_1(x)) = x - x = 0$, which implies $\sigma_2(x) - \sigma_1(x) \in \iota(M) = \ker(\pi)$ for all $x \in \mathfrak{g}$. As ι is injective, this defines a map

$$F : \mathfrak{g} \to M, \quad \text{with} \quad \iota(F(x)) = \sigma_2(x) - \sigma_1(x). \tag{21}$$

Then we have from (20)

$$\iota(\rho_2(x)m) \stackrel{(20)}{=} [\sigma_2(x), \iota(m)] \stackrel{(21)}{=} [\sigma_1(x) + \iota(F(x)), \iota(m)] = [\sigma_1(x), \iota(m)] \stackrel{(20)}{=} \iota(\rho_1(x)m),$$

where we used that $F(x) \in M$ and M is abelian. As ι is injective, this implies $\rho_1 = \rho_2 =: \rho$. By a direct computation using the definitions, the fact that M is abelian and the Jacobi identity in $\mathfrak{e}$ for all $x \in \mathfrak{e}$ and $m \in M$, we can relate the map F to the 2-cocycles f_1, f_2

$$\begin{aligned}\iota(f_2(x,y)) &\stackrel{(20)}{=} [\sigma_2(x), \sigma_2(y)] - \sigma_2([x,y]) \\ &\stackrel{(21)}{=} [\sigma_1(x) + \iota\circ F(x), \sigma_1(y) + \iota\circ F(y)] - \sigma_1([x,y]) - \iota\circ F([x,y]) \\ &= [\sigma_1(x), \sigma_1(y)] - \sigma_1([x,y]) + [\sigma_1(x), \iota\circ F(y)] - [\sigma_1(y), \iota\circ F(x)] - \iota\circ F([x,y]) \\ &\stackrel{(20)}{=} \iota(f_1(x,y) + \rho(x)F(y) - \rho(y)F(x) - F([x,y]))\end{aligned}$$

As ι is injective, this shows that the 2-cocycles $f_i : \mathfrak{g}\times\mathfrak{g} \to M$ are related by a coboundary: $f_2(x,y) - f_1(x,y) = \rho(x)F(y) - \rho(y)F(x) - F([x,y])$ for all $x, y \in \mathfrak{g}$. It follows that different choices of σ define the same cohomology class $[f_1] = [f_2] \in H^2(\mathfrak{g}, M)$.

• 1.5 We show that isomorphic extensions define the same representations of $\mathfrak{g}$ and the same elements in $H^2(\mathfrak{g}, M)$.

If $(\mathfrak{e}, \iota, \pi)$ and $(\mathfrak{e}', \iota', \pi')$ are isomorphic extensions, then there is a Lie algebra isomorphism $\phi : \mathfrak{e} \to \mathfrak{e}'$ with $\phi\circ\iota = \iota'$ and $\pi'\circ\phi = \pi$. For any map $\sigma : \mathfrak{g} \to \mathfrak{e}$ with $\pi\circ\sigma = \mathrm{id}_{\mathfrak{g}}$, the map $\sigma' = \phi\circ\sigma : \mathfrak{g} \to \mathfrak{e}'$ satisfies $\pi'\circ\sigma' = \pi'\circ\phi\circ\sigma = \pi\circ\sigma = \mathrm{id}_{\mathfrak{g}}$. This implies for the representation $\rho' : \mathfrak{g} \to \mathrm{End}_{\mathbb{F}}(M)$ and the 2-cocycle $f' : \mathfrak{g}\times\mathfrak{g} \to M$ defined by σ'

$$\iota'(\rho'(x)m) \stackrel{(20)}{=} [\sigma'(x), \iota'(m)] = [\phi\circ\sigma(x), \phi\circ\iota(m)] = \phi([\sigma(x), \iota(m)]) \stackrel{(20)}{=} \phi\circ\iota(\rho(x)m) = \iota'(\rho(x)m)$$

$$\begin{aligned}\iota'(f'(x,y)) &\stackrel{(20)}{=} [\sigma'(x), \sigma'(y)] - \sigma'([x,y]) = [\phi\circ\sigma(x), \phi\circ\sigma(y)] - \phi\circ\sigma([x,y]) \\ &= \phi([\sigma(x), \sigma(y)] - \sigma([x,y])) \stackrel{(20)}{=} \phi\circ\iota(f(x,y)) = \iota'(f(x,y))\end{aligned}$$

As $\iota' = \phi\circ\iota$ is the composite of injective maps, it is injective. It follows that the representations and cocycles defined by σ and σ' are equal, and so are the corresponding elements of $H^2(\mathfrak{g}, M)$. As $\phi : \mathfrak{e} \to \mathfrak{e}'$ is invertible, this proves that the isomorphic Lie algebra extensions $(\mathfrak{e}, \iota, \pi)$ and $(\mathfrak{e}', \iota', \pi')$ define the same element of $H^2(\mathfrak{g}, M)$.

• 1.6 We show that the map that assigns to a pair $(\rho, [f])$ the iromorphism class of the an extension $(M\oplus\mathfrak{g}, \iota, \pi)$ defined by (18) and the map that assigns to an isomorphism class of a Lie algebra extension $(\mathfrak{e}, \iota, \pi)$ the pair $(\rho, [f])$ defined by (20) are mutually inverse.

For the extension $(M\oplus\mathfrak{g}, \iota, \pi)$ from (18) defined by $(\rho, [f])$, we choose $\sigma : \mathfrak{g} \to M\oplus\mathfrak{g}$, $x \mapsto (0, x)$ with $\pi\circ\sigma = \mathrm{id}_{\mathfrak{g}}$. A direct computation then shows that the representation $\rho' : \mathfrak{g} \to \mathrm{End}_{\mathbb{F}}(M)$ and 2-cocycle $f' : \mathfrak{g}\times\mathfrak{g} \to M$ defined by (20) are $\rho' = \rho$ and $f' = f$.

Conversely, given an extension $(\mathfrak{e}, \iota, \pi)$, we choose $\sigma : \mathfrak{g} \to E$ with $\pi\circ\sigma = \mathrm{id}_G$ and consider the associated representation $\rho : \mathfrak{g} \to \mathrm{End}_{\mathbb{F}}(M)$ and 2-cocycle $f : \mathfrak{g}\times\mathfrak{g} \to M$ defined by (20). The latter define an extension $M\oplus\mathfrak{g}$ with Lie bracket (18) and Lie algebra homomorphisms $\iota_{M\oplus\mathfrak{g}} : M \to M\oplus\mathfrak{g}$, $m \mapsto (m, 0)$ and $\pi_{M\oplus\mathfrak{g}} : M\oplus\mathfrak{g} \to \mathfrak{g}$, $(m, x) \mapsto x$. Then the map $\phi : M\oplus\mathfrak{g} \to \mathfrak{e}$, $(m, x) \mapsto \iota(m) + \sigma(x)$ satisfies $\phi\circ\iota_{M\oplus\mathfrak{g}} = \iota$ and $\pi\circ\phi = \pi_{M\oplus\mathfrak{g}}$. A direct computation using (20) shows that ϕ is a Lie algebra homomorphism with inverse $\phi^{-1} : \mathfrak{e} \to M\oplus\mathfrak{g}$, $e \mapsto (m, \pi(e))$ with $\iota(m) = e - \sigma\circ\pi(e)$.

2. If $\rho : \mathfrak{g} \to \mathrm{End}_F(M)$, $x \mapsto 0$ is a trivial representation, then the Lie bracket on $M \oplus \mathfrak{g}$ from (18) takes the form $[(m, x), (m', x')]_f = (f(x, x'), [x, x'])$ for all $x, x' \in \mathfrak{g}$ and $m, m' \in M$. This implies $[(m, 0), (m', x')]_f = (f(0, x'), [0, x']) = f(0, 0) = 0$, and hence the extension $M \oplus \mathfrak{g}$ is central. Conversely, if $(\mathfrak{e}, \iota, \pi)$ is a central extension of $\mathfrak{g}$ by M, then the representation $\rho : \mathfrak{g} \to \mathrm{End}_{\mathbb{F}}(M)$ from (20) satisfies $\iota(\rho(x)m) = [\sigma(x), \iota(m)] = 0$, and by injectivity of ι, this implies $\rho(x)m = 0$ for all $x \in \mathfrak{g}$ and $m \in M$. □

2.5 Summary and questions

In this section, we encountered homologies and cohomologies for different mathematical objects, namely topological spaces, algebras and bimodules over algebras, groups and group representations as well as Lie algebras and Lie algebra representations. Although the mathematical objects under consideration were very different, the associated (co)homology theories have structural similarities and in many cases also similar interpretations.

There are many other examples of homology and cohomology theories such as deRham-cohomology of smooth manifolds, symplectic homology, intersection cohomology on surfaces and homologies in more advanced settings such as braided tensor categories. While the mathematical objects under consideration are different, the general pattern is the same as for the examples in this section. This makes (co)homologies a useful tool in many areas of mathematics.

While the examples treated so far illustrate the versatility and usefulness of (co)homologies, the treatment of this examples was too pedestrian and has its limitations. In particular, the examples considered so far raise the following questions that call for a more systematic and abstract investigation:

- Although we assigned homologies to *objects* in certain categories (topological spaces, bimodules over algebras, modules over group rings and representations of Lie algebras) we did not consider the *morphisms* in these categories so far. Do morphisms in these categories (continuous maps between topological spaces, morphisms of bimodules, morphisms of modules or morphisms of representations) induce maps between the homologies of their source and target objects? Is there a systematic way of including morphisms in the picture?

- What is the origin of the modules of (co)chains and the (co)boundary operators in the concrete examples? Is there a general construction or formalism that allows one to formulate (co)homology theories for objects in any category that satisfies certain assumptions? Are (co)chains necessarily realised as modules over certain rings and (co)boundary operators as module morphisms, or is there a more general framework?

- In all examples considered so far, the (co)boundary operators were obtained as alternating sums over certain module homomorphisms. These module homomorphisms were largely combinatorial, such as the face maps in singular (co)homology and the maps that multiply two adjacent factors in a tensor product of algebras or a product of groups. Is this a general pattern oder a coincidence? What is the appropriate mathematical framework to formulate this question more precisely?

- How much arbitrariness is there in the definition of the (co)chains and (co)boundary operators? Are the (co)boundary operators introduced so far essentially the only way of defining these structures, or are there many other formulations that lead to equivalent definitions of (co)homologies? How much does the concrete choice of (co)chains and (co)boundary operators matter? Is there a way to define (co)homologies that relies less on the choice of modules of n-(co)chains and (co)boundary operators and more on the objects under consideration?

- What is the algebraic framework to compare and relate different (co)homology theories?

We will answer these questions in the next sections. This requires a more systematic and abstract approach that uses the language of categories and functors.

3 Chain complexes, chain maps and chain homotopies

3.1 Abelian categories

In this section, we determine the general mathematical setting for (co)chains, (co)boundary operators and (co)homologies. Although the examples in Section 2 were concerned with very different mathematical data, we always associated to this data a family of modules over a ring R and a family of R-linear maps between them such that the composite of two subsequent maps vanishes. This allowed us to define (co)homologies as quotients of their kernels and images.

This suggests that the appropriate setting for a general (co)homology theory could be categories of modules over rings. However, it turns out that this is neither the most general possibility nor an efficient viewpoint. Instead, we determine the most general setting for (co)homology theories abstractly, in terms of categories. We start with a category $\mathcal{C}$ and investigate which additional structures are needed in order to formulate (co)homology theories as in Section 2.

- All of the examples in Section 2 made use of direct sums of R-modules and the fact that there is a trivial R-module $\{0\}$, which can be viewed as a direct sum of R-modules over an empty index set. As the direct sum of R-modules is an example of a categorical coproduct, one should at least impose that *coproducts* in $\mathcal{C}$ exist for all *finite* families $(C_i)_{i\in I}$ of objects in $\mathcal{C}$. For symmetry and because this will follow automatically from the next condition, we also impose that *products* in $\mathcal{C}$ exist for all *finite* families of objects in $\mathcal{C}$.
- The (co)boundary operators in the examples from in Section 2 were defined as an alternating sum of certain R-module morphisms. To generalise this construction to a category $\mathcal{C}$, we need to be able to take sums of morphisms in $\mathcal{C}$ and to ensure that this is compatible with their composition. Hence, we have to impose that all morphism sets $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ have the structure of *abelian groups* and that the group addition is compatible with the composition of morphisms in $\mathcal{C}$.
- To define (co)homologies in the examples from in Section 2, we considered *kernels* of R-module morphisms and took *quotients by their images*. Hence, the category $\mathcal{C}$ needs to be equipped with a concept of *kernels* and *images* that mimics the kernels and images of R-linear maps and gives rise to a sensible notion of homology.

The first two conditions lead to the concept of an *additive category*. Functors between *additive categories* that respect these conditions are called *additive functors*.

Definition 3.1.1: A category $\mathcal{C}$ is called **additive** if

(Add1) For all objects C, C' of $\mathcal{C}$ the set of morphisms $\mathrm{Hom}_{\mathcal{C}}(C, C')$ has the structure of an abelian group, and the composition of morphisms is $\mathbb{Z}$-bilinear: $g\circ(f+f') = g\circ f+g\circ f'$ and $(g + g') \circ f = g \circ f + g' \circ f$ for all morphisms $f, f' : C \to C'$ and $g, g' : C' \to C''$.

(Add2) Products and coproducts exist for all finite families of objects in $\mathcal{C}$.

A functor $F : \mathcal{C} \to \mathcal{D}$ between additive categories $\mathcal{C}$, $\mathcal{D}$ is called **additive** if for all objects C, C' in $\mathcal{C}$ the map $F : \mathrm{Hom}_{\mathcal{C}}(C, C') \to \mathrm{Hom}_{\mathcal{D}}(F(C), F(C'))$ is a group homomorphism.

Remark 3.1.2:

1. In particular, Definition 3.1.1 requires the existence of an *empty* product and an *empty coproduct*, a terminal object $T = \Pi_\emptyset$ and an initial object $I = \amalg_\emptyset$ (see Definition 1.2.15). In an additive category $\mathcal{C}$, these objects are isomorphic and hence zero objects: $I \cong T \cong 0$.

   This follows because one has $\mathrm{Hom}_{\mathcal{C}}(I, I) = \{1_I\} = \{0\}$ by definition of an initial object, where 0 denotes the neutral element of the abelian group $\mathrm{Hom}_{\mathcal{C}}(I, I)$. If $\mathcal{C}$ is additive, this implies $f = 1_I \circ f = 0 \circ f = 0 : C \to I$ for any morphism $f : C \to I$, since the composition of morphisms is $\mathbb{Z}$-bilinear. It follows that $\mathrm{Hom}_{\mathcal{C}}(C, I) = \{0\}$ and hence I is terminal.

2. It follows that for any two objects C, C' in an additive category $\mathcal{C}$, the neutral element of the abelian group $\mathrm{Hom}_{\mathcal{C}}(C, C')$ is $0 = i_{C'} \circ t_C : C \to 0 \to C'$.

3. Finite products and coproducts in additive categories are canonically isomorphic: $\Pi_{i\in I} C_i \cong \amalg_{i\in I} C_i$ for all finite index sets $i \in I$ and objects C_i in $\mathcal{C}$.

   The isomorphism is induced by the family $(f_{ij})_{i,j\in I}$ of morphisms $f_{ij} = \delta_{ij} 1_{C_i} : C_i \to C_j$ with $f_{ij} = 0$ for $i \neq j$ and $f_{ii} = 1_{C_i}$. By the universal property of the (co)product there is a unique morphism $f : \amalg_{k\in I} C_k \to \Pi_{k\in I} C_k$ with $\pi_j \circ f \circ \iota_i = \delta_{ij} 1_{C_i}$. The inverse of this morphism is $f^{-1} = \Sigma_{i\in I}\, \iota_i \circ \pi_i : \Pi_{k\in I} C_k \to \amalg_{k\in I} C_k$, since

$$\begin{aligned} &\pi_k \circ f \circ f^{-1} = \Sigma_{i\in I}\, \pi_k \circ f \circ \iota_i \circ \pi_i = \Sigma_{i\in I}\, \delta_{ik} 1_{C_i} \circ \pi_i = \pi_k \\ &f^{-1} \circ f \circ \iota_k = \Sigma_{i\in I}\, \iota_i \circ \pi_i \circ f \circ \iota_k = \Sigma_{i\in I}\, \iota_i \circ \delta_{ik} 1_{C_i} = \iota_k \qquad \forall k \in I, \end{aligned}$$

   and the universal property of the (co)product implies $f \circ f^{-1} = 1_{\Pi_{i\in I} C_i}$, $f^{-1} \circ f = 1_{\amalg_{i\in I} C_i}$.

4. The abelian group structure on the sets $\mathrm{Hom}_{\mathcal{C}}(C, C')$ in an additive category $\mathcal{C}$ is determined uniquely by its products and coproducts.

   For a finite index set I and an object C in $\mathcal{C}$ we denote by $\phi_C : \amalg_{i\in I} C \to \Pi_{i\in I} C$ the unique morphism with $\pi_i \circ \phi_C \circ \iota_j = \delta_{ij}\, 1_C$ from 3. with inverse $\phi_C^{-1} = \Sigma_{i\in I}\, \iota_i \circ \pi_i : \Pi_{i\in I} C \to \amalg_{i\in I} C$. We also consider the unique morphism $\Delta_C : C \to \Pi_{i\in I} C$ with $\pi_i \circ \Delta_C = 1_C$ for all $i \in I$ and the unique morphism $\nabla_C : \amalg_{i\in I} C \to C$ with $\nabla_C \circ \iota_i = 1_C$ for all $i \in I$. For a finite family $(f_i)_{i\in I}$ of morphisms $f_i : C \to D$, we consider the unique morphism $f : \amalg_{i\in I} C \to \Pi_{i\in I} D$ with $\pi_j \circ f \circ \iota_i = \delta_{ij} f_i$ from 3. Then we have

$$\begin{aligned} \nabla_D \circ \phi_D^{-1} \circ f \circ \phi_C^{-1} \circ \Delta_C &= \Sigma_{i,j\in I} \nabla_D \circ \iota_i \circ \pi_i \circ f \circ \iota_j \circ \pi_j \circ \Delta_C \\ &= \Sigma_{i,j\in I} 1_D \circ (\pi_i \circ f \circ \iota_j) \circ 1_C = \Sigma_{i,j\in I} \delta_{ij}\, 1_D \circ f_i \circ 1_C = \Sigma_{i\in I} f_i. \end{aligned}$$

   Hence, we expressed the sum of the morphisms f_i in terms of quantities that are defined in terms of the product and coproduct in an additive category. (This includes the morphism f, since the zero object that enters its definition is the empty coproduct). As products and coproducts are unique up to unique isomorphisms, a given category $\mathcal{C}$ has at most one additive structure. Additivity is a *property*, not a choice of *structure*.

5. An object X in an additive category $\mathcal{C}$ is a product or coproduct of a finite family of objects $(C_i)_{i\in I}$ if and only if there are families $(i_j)_{j\in I}$ and $(p_j)_{j\in I}$ of morphisms $i_j : C_j \to X$ and $p_j : X \to C_j$ with $p_j \circ i_k = \delta_{jk} 1_{C_j}$ and $1_X = \Sigma_{j\in I} \iota_j \circ p_j$ (Exercise 25).

6. A functor $F : \mathcal{C} \to \mathcal{D}$ between additive categories $\mathcal{C}, \mathcal{D}$ is additive if and only if it preserves finite products or finite coproducts (Exercise 26):

   $F(\Pi_{i\in I} C_i) \cong \Pi_{i\in I} F(C_i)$, $F(\amalg_{i\in I} C_i) \cong \amalg_{i\in I} F(C_i)$ for all finite families of objects $(C_i)_{i\in I}$.

Example 3.1.3:

1. For any ring R the category R-Mod of R-modules und R-linear maps is additive.

   Products and coproducts are products and direct sums of modules and exist for all families of modules. The set $\mathrm{Hom}_R(M,N)$ of R-linear maps $f : M \to N$ is an abelian group with the pointwise addition, and this is compatible with their composition.

2. For any ring homomorphism $\phi : R \to S$, the functor $F_\phi : S\text{-Mod} \to R\text{-Mod}$ that sends an S-module $(M, \rhd)$ to the R-module $(M, \rhd_R)$ with $r \rhd_R m = \phi(r) \rhd m$ and an S-linear map $f : (M, \rhd) \to (M', \rhd')$ to the associated R-linear map $f : (M, \rhd_R) \to (M', \rhd'_R)$ is additive.

3. Every full subcategory of an additive category $\mathcal{C}$ in which finite products and coproducts exist, is an additive category as well.

4. For every small category $\mathcal{C}$ and additive category $\mathcal{A}$, the category $\mathrm{Fun}(\mathcal{C}, \mathcal{A})$ of functors $F : \mathcal{C} \to \mathcal{A}$ and natural transformations between them is an additive category.

   • The product of a family of functors $(F_i)_{i\in I}$ is the functor $\Pi_{i\in I} F_i : \mathcal{C} \to \mathcal{A}$ that assigns to an object C the product $\Pi_{i\in I} F_i(C)$ and to a morphism $\alpha : C \to C'$ the unique morphism $\Pi_{i\in I} F_i(\alpha) : \Pi_{i\in I} F_i(C) \to \Pi_{i\in I} F_i(C')$ with $\pi_{iC'} \circ \Pi_{i\in I} F_i(\alpha) = F_i(\alpha) \circ \pi_{iC}$, where $\pi_{iC} : \Pi_{i\in I} F_i(C) \to F_i(C)$ are the projection morphisms for the product in $\mathcal{A}$.

   • The projection morphisms for $\Pi_{i\in I} F_i$ are the natural transformations $\pi_i : \Pi_{i\in I} F_i \to F_i$ with component morphisms $\pi_{iC} : \Pi_{i\in I} F_i(C) \to F_j(C)$.

   • Coproducts of functors are defined analogously, and the sum of two natural transformations $\eta, \kappa : F \to G$ is the natural transformation $\eta + \kappa : F \to G$ with component morphisms $(\eta + \kappa)_C = \eta_C + \kappa_C : F(C) \to G(C)$.

In any additive category $\mathcal{A}$ we can consider a generalisation of chains, families $(C_n)_{n\in\mathbb{Z}}$ of objects in $\mathcal{C}$, and boundary operators between them, families $(d_n)_{n\in\mathbb{Z}}$ of morphisms $d_n : C_n \to C_{n-1}$ with $d_{n-1} \circ d_n = 0 : C_n \to C_{n-2}$ for all $n \in \mathbb{Z}$. An analogous definition is possible for cochains and coboundary operators.

To define homologies we also require kernels and quotients of kernels by images. In contrast to the standard definition of a kernel and image of an R-module morphism $f : M \to N$, as *subsets* of the modules M and N, a sensible categorical notion of a kernel and image must be formulated purely in terms of morphisms and universal properties. It does not require additivity, but a zero object 0 in $\mathcal{C}$ and the associated zero morphisms $0 = i_{C'} \circ t_C : C \to 0 \to C'$.

Definition 3.1.4: Let $\mathcal{C}$ be a category with a zero object and $f : X \to Y$ a morphism in $\mathcal{C}$.

1. A **kernel** of f is a morphism $\iota : \ker(f) \to X$ with the following **universal property**: $f \circ \iota = 0 : \ker(f) \to Y$, and for every morphism $g : W \to X$ with $f \circ g = 0 : W \to Y$ there is a unique morphism $g' : W \to \ker(f)$ with $\iota \circ g' = g$.

$$\begin{tikzcd}
\ker(f) \arrow[r, "\iota"] \arrow[rr, bend left, "0"] & X \arrow[r, "f"] & Y \\
& W \arrow[ul, dashed, "\exists! g'"] \arrow[u, "g"] \arrow[ur, "0"'] &
\end{tikzcd}$$

2. A **cokernel** of f is a morphism $\pi : Y \to \mathrm{coker}(f)$ with the following **universal property**: $\pi \circ f = 0 : X \to \mathrm{coker}(f)$, and for every morphism $g : Y \to W$ with $g \circ f = 0 : X \to W$ there is a unique morphism $g' : \mathrm{coker}(f) \to W$ with $g' \circ \pi = g$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend left, "0"] \arrow[dr, "0"'] & Y \arrow[r, "\pi"] \arrow[d, "g"] & \mathrm{coker}(f) \arrow[dl, dashed, "\exists! g'"] \\
& W &
\end{tikzcd}$$

3. A kernel of a cokernel of f is called an **image** of f and denoted $\iota' : \mathrm{im}(f) \to Y$. A cokernel of a kernel of f is called a **coimage** of f and denoted $\pi' : X \to \mathrm{coim}(f)$.

Remark 3.1.5: As (co)kernels and (co)images are defined by a universal property, they are unique up to unique isomorphism: If $\iota : \ker(f) \to X$, $\eta : \ker(f)' \to X$ are two kernels for $f : X \to Y$, then there is a unique morphism $\phi : \ker(f) \to \ker(f)'$ with $\eta \circ \phi = \iota$, and this morphism is an isomorphism. Analogous statements hold for cokernels, images and coimages.

Example 3.1.6: Let R be a ring and $f : M \to N$ an R-linear map.

- The inclusion map $\iota : \ker(f) \to M$ is a kernel of f in R-Mod.
- The canonical surjection $\pi : N \to N/\mathrm{im}(f)$ is a cokernel of f in R-Mod.
- The canonical inclusion $\iota' : \mathrm{im}(f) \to N$ is an image of f in R-Mod.
- The canonical surjection $\pi' : M \to M/\ker(f)$ is a coimage of f in R-Mod.

That $\iota : \ker(f) \to M$ is a kernel of f follows, because $f \circ \iota = 0$ and for any R-linear map $\phi : L \to M$ with $f \circ \phi = 0$, one has $\mathrm{im}(\phi) \subset \ker(f)$. The corestriction $\phi' : L \to \ker(f)$, $l \mapsto \phi(l)$ is an R-linear map with $\iota \circ \phi' = \phi$. As ι is injective, it is the only one.

That $\pi : N \to N/\mathrm{im}(f)$ is a cokernel of f follows, because $\pi \circ f = 0$ and for any R-linear map $\psi : N \to P$ with $\psi \circ f = 0$ one has $\mathrm{im}(f) \subset \ker(\psi)$. By the characteristic property of the quotient, there is a unique R-linear map $\psi' : N/\mathrm{im}(f) \to P$, $[n] \mapsto \psi(n)$ with $\psi' \circ \pi = \psi$.

That the inclusion map $\iota' : \mathrm{im}(f) \to N$ is a kernel of $\pi : N \to N/\mathrm{im}(f)$ follows, because $\pi \circ \iota' = 0$, and for any R-linear map $\chi : L \to N$ with $\pi \circ \chi = 0$, one has $\mathrm{im}(\chi) \subset \ker(\pi) = \mathrm{im}(f)$. The corestriction $\chi' : L \to \mathrm{im}(f)$, $l \mapsto \chi(l)$ satisfies $\chi' \circ \iota' = \chi$ and is the only R-linear map with this property, since ι' is injective.

That the canonical surjection $\pi' : M \to M/\ker(f)$ is a cokernel of $\iota : \ker(f) \to M$ follows because $\pi' \circ \iota = 0$ and because any R-linear map $\xi : M \to P$ with $\xi \circ \iota = 0$ satisfies $\mathrm{im}(\iota) = \ker(f) \subset \ker(\xi)$. By the characteristic property of the quotient, there is a unique R-linear map $\xi' : M/\ker(f) \to P$ with $\xi' \circ \pi' = \xi$.

In addition to kernels and cokernels, we also require appropriate concepts of injectivity and surjectivity and relate them to kernels and cokernels. Again, they must be formulated purely in terms of morphisms and universal properties. They are obtained from the observation that a map $\iota : X \to Y$ is injective (a map $\pi : X \to Y$ is surjective) if and only if $\iota \circ f = \iota \circ g$ ($f \circ \pi = g \circ \pi$) implies $f = g$ for all maps $f, g : W \to X$ (for all maps $f, g : Y \to Z$). This notion of injectivity and surjectivity in Set generalises to any category.

Definition 3.1.7: Let $\mathcal{C}$ be a category.

1. A morphism $\iota : X \to Y$ in $\mathcal{C}$ is called a **monomorphism**, if $\iota \circ f = \iota \circ g$ for morphisms $f, g : W \to X$ implies $f = g$.
2. A morphism $\pi : X \to Y$ in $\mathcal{C}$ is called an **epimorphism**, if $f \circ \pi = g \circ \pi$ for morphisms $f, g : Y \to Z$ implies $f = g$.

In diagrams, monomorphisms $\iota : X \to Y$ are denoted $X \overset{\iota}{\hookrightarrow} Y$ and epimorphisms $\pi : X \to Y$ are denoted $X \overset{\pi}{\twoheadrightarrow} Y$.

Remark 3.1.8: Clearly, every isomorphism is a monomorphism and an epimorphism. However, a morphism that is a monomorphism and an epimorphism need not be an isomorphism. A counterexample is the inclusion morphism $\iota : \mathbb{Z} \to \mathbb{Q}$ in the category of unital rings.

We now relate epimorphisms and monomorphisms to (co)kernels and (co)images. Example 3.1.6 shows that in the category R-Mod the kernel $\iota : \ker(f) \to M$ of an R-linear map $f : M \to N$ is injective and its cokernel $\pi : N \to N/\mathrm{im}(f)$ is surjective. Moreover, the module morphism $0 \to M$ is a kernel of f if and only if f is injective and the module morphism $N \to 0$ is a cokernel of f if and only if f is surjective. Analogues of this hold in all *additive* categories.

Lemma 3.1.9: Let $\mathcal{C}$ be an additive category.

1. All kernels of morphisms in $\mathcal{C}$ are monomorphisms. A morphism $f : X \to Y$ is a monomorphism if and only if the morphism $i_X : 0 \to X$ is a kernel of f.
2. All cokernels of morphisms in $\mathcal{C}$ are epimorphisms. A morphism $f : X \to Y$ is an epimorphism if and only if the morphism $t_Y : Y \to 0$ is a cokernel of f.

Proof:
We prove the first statement. The proof of the second one is analogous. Let $\iota : \ker(f) \to X$ be a kernel of $f : X \to Y$ and $g_1, g_2 : W \to \ker(f)$ morphisms with $\iota \circ g_1 = \iota \circ g_2$. Then we have $f \circ (\iota \circ g_i) = (f \circ \iota) \circ g_i = 0 \circ g_i = 0 : W \to Y$, and by the universal property of the kernel, there is a *unique* morphism $g' : W \to \ker(f)$ with $\iota \circ g' = \iota \circ g_1 = \iota \circ g_2$. The uniqueness implies $g' = g_1 = g_2$, and hence $\iota : W \to \ker(f)$ is a monomorphism.

Let now $f : X \to Y$ be a monomorphism. We have $f \circ i_X = i_Y = 0 : 0 \to Y$. If $g : W \to X$ is a morphism with $f \circ g = 0 : W \to X$ then $f \circ i_X \circ t_W = 0 : W \to X$ as well, and because f is a monomorphism, it follows that $g = i_X \circ t_W$. Hence, $i_X : 0 \to X$ is a kernel of f.

Conversely, if $i_X : 0 \to X$ is a kernel of f and $g_1, g_2 : W \to X$ are morphisms with $f \circ g_1 = f \circ g_2$, then $f \circ (g_1 - g_2) = 0$ and by the universal property of the kernel, there is a unique morphism $g' : W \to 0$ with $i_X \circ g' = g_1 - g_2 = 0$. Since $g' = t_W : W \to 0$ is the only morphism from W to 0, we have $g_1 - g_2 = i_X \circ t_W = 0 : W \to X$ and $g_1 = g_2$. This shows that f is a monomorphism
$\square$

This lemma shows that in any additive category, kernels are monomorphisms and cokernels epimorphisms, as expected from the corresponding statement for R-Mod. However, in R-Mod, the converse also holds. Every injective R-linear map $f : M \to N$ is a kernel, namely of its cokernel $\pi : N \to N/\mathrm{im}(f)$. This follows because $\pi \circ f = 0$, and for every R-linear map $g : L \to N$ with $\pi \circ g = 0$ one has $\mathrm{im}(g) \subset \ker(\pi) = \mathrm{im}(f)$. Hence, by injectivity of f there is a unique R-linear map $g' : L \to M$ with $f \circ g' = g$.

Similarly, every surjective R-linear map $f : M \to N$ is a cokernel of its kernel $\iota : \ker(f) \to M$. One has $f \circ \iota = 0$ and $\ker(f) = \mathrm{im}(\iota) \subset \ker(g)$ for every R-linear map $g : M \to L$ with $g \circ \iota = 0$. As f is surjective, there is a unique R-linear map $g' : N \to L$, $f(m) \mapsto g(m)$ with $g' \circ f = g$.

In contrast to the claims in Lemma 3.1.9, these statements do not hold automatically in an additive category. They require in particular that every monomorphism has a cokernel and every epimorphism has a kernel, which is not guaranteed in an additive category. If we impose these conditions and the existence of kernels and cokernels for all morphisms, we obtain the notion of an *abelian category*, which we will use later as the framework for (co)homology.

We also consider functors between abelian categories that preserve these structures. Clearly, such functors need to be additive and to map kernels to kernels and cokernels to cokernels. We will see in the following that there are many additive functors that satisfy only one the last two conditions and that these functors play an important role in (co)homology.

Definition 3.1.10:

1. An additive category is called **abelian** if it satisfies the following additional conditions:
   (Ab1) Every morphism has a kernel and a cokernel.
   (Ab2) Every monomorphism is a kernel of its cokernel or, equivalently, an image of itself.
   (Ab3) Every epimorphism is a cokernel of its kernel or, equivalently, a coimage of itself.
2. A functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories $\mathcal{A}$, $\mathcal{B}$ is called
   - **left exact** if it is additive and preserves kernels:
     if $\iota : \ker(f) \to X$ is a kernel of $f : X \to Y$, then $F(\iota) : F(\ker(f)) \to F(X)$ is a kernel of $F(f) : F(X) \to F(Y)$.
   - **right exact** if it is additive and preserves cokernels:
     if $\pi : Y \to \mathrm{coker}(f)$ is a cokernel of $f : X \to Y$, then $F(\pi) : F(Y) \to F(\mathrm{coker}(f))$ is a cokernel of $F(f) : F(X) \to F(Y)$.
   - **exact** if it is left exact and right exact.

Example 3.1.11:

1. For any ring R, the category R-Mod is abelian.

   By Example 3.1.3, 1. it is additive, and by Example 3.1.6, every R-linear map $f : M \to N$ has a kernel $\iota : \ker(f) \to M$ and a cokernel $\pi : N \to N/\mathrm{im}(f)$. As shown above, every monomorphism in R-Mod is a kernel of its cokernel and every epimorphism a cokernel of its kernel.

2. The full category of Ab of finitely generated free abelian groups is additive, but not abelian. (Exercise 32).

3. The full subcategory of $\mathrm{Vect}_{\mathbb{F}}^{fd}$ with even-dimensional $\mathbb{F}$-vector spaces as objects is additive, but not abelian. (Exercise 33).

4. For any abelian category $\mathcal{A}$, the category $\mathcal{A}^{op}$ is abelian. Kernels and cokernels in $\mathcal{A}^{op}$ correspond to cokernels and kernels in $\mathcal{A}$, respectively (Exercise 31).

5. For any small category $\mathcal{C}$ and any abelian category $\mathcal{A}$ the category $\mathrm{Fun}(\mathcal{C}, \mathcal{A})$ of functors $F : \mathcal{C} \to \mathcal{A}$ and natural transformations between them is abelian (Exercise 34).

Remark 3.1.12:

1. One can show that in an abelian category $\mathcal{A}$ a morphism that is both a monomorphism and an epimorphism is an isomorphism (Exercise 30).

2. Like additivity, being abelian is a *property* of a category and *not a choice of structure*. If all objects in an additive category have kernels and cokernels that satisfy the conditions in Definition 3.1.10, these (co)kernels are unique up to unique isomorphism and determined by the additive structure.

3. The **Feyd-Mitchell embedding theorem** states that any *small* abelian category $\mathcal{A}$ is equivalent to a full subcategory of the abelian category R-Mod for some ring R, with an exact equivalence of categories. For a proof, see [Mi, p 151].

Although the embedding theorem allows one to interpret any *small* abelian category as a subcategory of the abelian category R-Mod for a suitable ring R, it is still advantageous to work with general abelian categories. Firstly, there are also non-small abelian categories. Secondly, the construction of the associated ring R and the subcategory of R-Mod for an abelian category $\mathcal{A}$ in the embedding theorem is implicit and not very useful in concrete computations.

However, we will sometimes use the embedding theorem to conduct proofs in R-Mod that become too cumbersome and technical in general abelian categories. This does not restrict generality of the proofs if the claims involve only a small full subcategory of $\mathcal{A}$ that is again an abelian category.

In an abelian category kernels and cokernels exist for all morphisms and generalise the inclusion maps $\iota : \ker(f) \to X$ and the canonical surjections $\pi : Y \to Y/\mathrm{im}(f)$ for R-linear maps $f : X \to Y$. To define homologies we require one additional ingredient.

Recall that for R-linear maps $d_{n+1} : X_{n+1} \to X_n$ and $d_n : X_n \to X_{n-1}$ with $d_n \circ d_{n+1} = 0$ one has $\mathrm{im}(d_{n+1}) \subset \ker(d_n)$, and there is an inclusion map $\iota : \mathrm{im}(d_{n+1}) \to \ker(d_n)$, which is a monomorphism in R-Mod. This allows one to define the homologies as the quotients $\ker(d_n)/\mathrm{im}(d_{n+1})$ or, equivalently, as the cokernels of the inclusion $\iota : \mathrm{im}(d_{n+1}) \to \ker(d_n)$. Note also that for any R-linear map $f : M \to N$ one has $\mathrm{coim}(f) = M/\ker(f) \cong \mathrm{im}(f)$ so there is no ambiguity when dealing with images and coimages. To generalise this to abelian categories, we need a monomorphism $\phi : \mathrm{im}(f) \to \ker(g)$ for all morphisms $f : X \to Y$ and $g : Y \to Z$ with $g \circ f = 0$ and $\mathrm{im}(f) \cong \mathrm{coim}(f)$ for all morphisms $f : X \to Y$.

Lemma 3.1.13: Let $\mathcal{A}$ be an abelian category.

1. Every morphism $f : X \to Y$ in $\mathcal{A}$ factorises as $f = \iota'_f \circ \pi'_f$ where $\iota'_f : \mathrm{im}(f) \to Y$ is an image of f and $\pi'_f : X \to \mathrm{im}(f)$ a coimage of f. This is called the **canonical factorisation** of f and implies $\mathrm{coim}(f) \cong \mathrm{im}(f)$

2. If $f : X \to Y$, $g : Y \to Z$ are morphisms in $\mathcal{A}$ with $g \circ f = 0$, there is a unique monomorphism $\phi : \mathrm{im}(f) \to \ker(g)$ such that the following diagram commutes

$$\begin{tikzcd}
\mathrm{im}(f) \arrow[rr, hook, dashed, "\exists!\phi"] \arrow[dr, "\iota'_f"'] & & \ker(g) \arrow[dl, "\iota_g"] \\
X \arrow[u, two heads, "\pi'_f"] \arrow[r, "f"] \arrow[rr, bend right, "0"'] & Y \arrow[r, "g"] & Z.
\end{tikzcd} \tag{22}$$

Proof:
1. For any morphism $f : X \to Y$ in $\mathcal{A}$, we have $\pi_f \circ f = 0$ for the cokernel $\pi_f : Y \to \mathrm{coker}(f)$. By the universal property of the image $\iota'_f : \mathrm{im}(f) \to Y$, there is a unique morphism $\pi'_f : X \to \mathrm{im}(f)$ with $\iota'_f \circ \pi'_f = f$. We show that $\pi'_f : X \to \mathrm{im}(f)$ is an epimorphism. The first claim then follows because every epimorphism is its own coimage, or, equivalently, a cokernel of its kernel. By Exercise 27 the morphisms π'_f and $f = \iota'_f \circ \pi'_f$ have the same kernel and hence the same coimage $\pi'_f : X \to \mathrm{im}(f)$.

To show that $\pi'_f : X \to \mathrm{im}(f)$ is an epimorphism, we show that $\phi = 0$ for every morphism $\phi : \mathrm{im}(f) \to U$ with $\phi \circ \pi'_f = 0$. By the universal property of the kernel $\iota_\phi : \ker(\phi) \to \mathrm{im}(f)$, there is a unique morphism $f' : X \to \ker(\phi)$ with $\iota_\phi \circ f' = \pi'_f$:

$$\begin{tikzcd}
\ker(\phi) \arrow[r, hook, "\iota_\phi"] & \mathrm{im}(f) \arrow[r, "\phi"] \arrow[d, hook, "\iota'_f"] & U \\
X \arrow[u, dashed, "\exists! f'"] \arrow[ur, "\pi'_f"'] \arrow[r, "f"'] & Y.
\end{tikzcd}$$

The morphism $\iota'_f \circ \iota_\phi : \ker(\phi) \to Y$ is a monomorphism as a composite of two monomorphisms. Hence, it is a kernel of its cokernel $\pi' : Y \to \mathrm{coker}(\iota'_f \circ \iota_\phi)$. This implies

$$\pi' \circ f = \pi' \circ (\iota'_f \circ \iota_\phi \circ f') = (\pi' \circ \iota'_f \circ \iota_\phi) \circ f' = 0 \circ f' = 0,$$

and by the universal property of the cokernel $\pi_f : Y \to \mathrm{coker}(f)$ there is a unique morphism $\pi'' : \mathrm{coker}(f) \to \mathrm{coker}(\iota'_f \circ \iota_\phi)$ with $\pi'' \circ \pi_f = \pi'$

$$\begin{tikzcd}
\ker(\phi) \arrow[r, hook, "\iota_\phi"] & \mathrm{im}(f) \arrow[r, "\phi"] \arrow[d, hook, "\iota'_f"] & U \\
X \arrow[u, "f'"] \arrow[ur, "\pi'_f"'] \arrow[r, "f"'] & Y \arrow[r, two heads, "\pi'"] \arrow[d, "\pi_f"] & \mathrm{coker}(\iota'_f \circ \iota_\phi) \\
& \mathrm{coker}(f). \arrow[ur, dashed, "\exists! \pi''"'] &
\end{tikzcd}$$

This implies $\pi' \circ \iota'_f = (\pi'' \circ \pi_f) \circ \iota'_f = \pi'' \circ (\pi_f \circ \iota'_f) = \pi'' \circ 0 = 0$, since $\iota'_f : \mathrm{im}(f) \to Y$ is a kernel of $\pi_f : Y \to \mathrm{coker}(f)$. As $\iota'_f \circ \iota_\phi$ is a kernel of π' and $\pi' \circ \iota'_f = 0$, the universal property of the kernel $\iota'_f \circ \iota_\phi$ implies that there is a unique morphism $\iota'' : \mathrm{im}(f) \to \ker(\phi)$ with $\iota'_f \circ \iota_\phi \circ \iota'' = \iota'_f$. Because ι'_f is a monomorphism, it follows that $\iota_\phi \circ \iota'' = 1_{\mathrm{im}(f)}$. As ι_ϕ is a kernel of ϕ, this implies $\phi = \phi \circ 1_{\mathrm{im}_f} = \phi \circ (\iota_\phi \circ \iota'') = (\phi \circ \iota_\phi) \circ \iota'' = 0 \circ \iota'' = 0$.

2. We consider the commuting diagram

$$\begin{tikzcd}
\mathrm{im}(f) \arrow[dr, hook, "\iota'_f"] & & \ker(g) \arrow[dl, hook', "\iota_g"'] \\
X \arrow[u, "\pi'_f"] \arrow[r, "f"] \arrow[rr, bend right, "0"'] & Y \arrow[r, "g"] & Z.
\end{tikzcd}$$

As $g \circ f = g \circ \iota'_f \circ \pi'_f = 0$ and π'_f is an epimorphism, we have $g \circ \iota'_f = 0$. By the universal property of the kernel $\iota_g : \ker(g) \to Y$ there is a unique morphism $\phi : \mathrm{im}(f) \to \ker(g)$ with $\iota_g \circ \phi = \iota'_f$. If $\phi \circ h = 0$ for some morphism $h : U \to \mathrm{im}(f)$, then $0 = \iota_g \circ \phi \circ h = \iota'_f \circ h = 0 = \iota'_f \circ 0$, and because ι'_f is a monomorphism, we obtain $h = 0$. Hence, ϕ is a monomorphism. $\square$

After clarifying the properties of abelian categories, we now focus on functors that are compatible with abelian categories - *exact functors* - and on functors that are partially compatible with it - *left or right exact functors*. It turns out that there are few exact functors, and most of them arise from certain standard constructions. Important examples are the following.

Example 3.1.14:

1. For any abelian category $\mathcal{A}$, the cartesian product $\mathcal{A} \times \mathcal{A}$ is abelian and the functors $\Pi : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and $\amalg : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ are exact. (Exercise 35).

2. For any abelian category $\mathcal{A}$, small category $\mathcal{C}$ and object C in $\mathcal{C}$, the evaluation functor $\mathrm{ev}_C : \mathrm{Fun}(\mathcal{C}, \mathcal{A}) \to \mathcal{A}$ that sends a functor $F : \mathcal{C} \to \mathcal{A}$ to the object $F(C)$ and a natural transformation $\eta : F \to G$ to the component morphism $\eta_C : F(C) \to G(C)$ is exact (Exercise 34).

Most functors that are relevant for homology are only left exact or right exact, but not exact. In fact, we will see in Section 4 that we can view homologies as a measure of the non-exactness of a left or right exact functor. One reason why so many functors of interest are left or right exact is that one typically considers functors related to certain constructions, such as tensoring, abelisation or Hom-functors, and such functors tend to have adjoints. The existence of a left (right) adjoint to a given functor is sufficient to ensure its left (right) exactness.

Lemma 3.1.15: Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ additive functors. If F is left adjoint to G, then F is right exact and G left exact.

Proof:
If F is left adjoint to G, by Proposition 1.2.19 there are natural transformations $\epsilon : FG \to \mathrm{id}_{\mathcal{B}}$ and $\eta : \mathrm{id}_{\mathcal{A}} \to GF$ with $(\epsilon F) \circ (F\eta) = \mathrm{id}_F$ and $(G\epsilon) \circ (\eta G) = \mathrm{id}_G$. We show that F is right exact by proving that it sends cokernels to cokernels. The proof that G is left exact is analogous.

Let $\pi : A' \to \mathrm{coker}(f)$ be a cokernel of $f : A \to A'$. Then $\pi \circ f = 0$, and for every morphism $g : A' \to A''$ with $g \circ f = 0$ there is a unique morphism $g' : \mathrm{coker}(f) \to A''$ with $g' \circ \pi = g$.

To show that $F(\pi) : F(A') \to F(\mathrm{coker}(f))$ is a cokernel of $F(f) : F(A) \to F(A')$, note first that the additivity of F implies $F(\pi) \circ F(f) = F(\pi \circ f) = F(0) = 0$.

We show that for every morphism $h : F(A') \to B$ with $h \circ F(f) = 0$, there is a unique morphism $h' : F(\mathrm{coker}(f)) \to B$ with $h' \circ F(\pi) = h$. The morphism $G(h) \circ \eta_{A'} : A' \to GF(A) \to G(B)$ can be pre-composed with f, and by additivity of G and naturality of η we have

$$G(h) \circ \eta_{A'} \circ f = G(h) \circ GF(f) \circ \eta_A = G(h \circ F(f)) \circ \eta_A = G(0) \circ \eta_A = 0.$$

By the universal property of the cokernel π there is a unique morphism $k : \mathrm{coker}(f) \to G(B)$ with $G(h) \circ \eta_{A'} = k \circ \pi$. The morphism $h' = \epsilon_B \circ F(k) : F(\mathrm{coker}(f)) \to FG(B) \to B$ satisfies

$$h' \circ F(\pi) = \epsilon_B \circ F(k) \circ F(\pi) = \epsilon_B \circ F(k \circ \pi) = \epsilon_B \circ FG(h) \circ F(\eta_{A'}) = h \circ \epsilon_{F(A')} \circ F(\eta_{A'}) = h.$$

If $h'' : F(\mathrm{coker}(f)) \to B$ is another morphism with $h'' \circ F(\pi) = h = h' \circ F(\pi)$, then we have $(h'' - h') \circ F(\pi) = 0$. This implies $0 = G(h'' - h') \circ GF(\pi) \circ \eta_{A'} = G(h'' - h') \circ \eta_{\mathrm{coker}(f)} \circ \pi$ and hence $G(h'' - h') \circ \eta_{\mathrm{coker}(f)} = 0$, because π is an epimorphism. Using the naturality of ϵ and the condition $(\epsilon F) \circ (F\eta) = \mathrm{id}_F$ for the adjunction, we obtain

$$h'' - h' = (h'' - h') \circ \epsilon_{F(\mathrm{coker}(f))} \circ F(\eta_{\mathrm{coker}(f)}) = \epsilon_B \circ FG(h'' - h') \circ F(\eta_{\mathrm{coker}(f)}) = \epsilon_B \circ F(0) = 0.$$

This shows that $h'' = h'$ and that h' is the *unique* morphism with $h' \circ F(\pi) = h$. Hence, $F(\pi) : F(A') \to F(\mathrm{coker}(f))$ has the universal property of the cokernel and F is right exact. □

Two of the most important right exact functors are the functors $M\otimes_R - : R\text{-Mod} \to \mathrm{Ab}$ and $-\otimes_R N : R^{op}\text{-Mod} \to \mathrm{Ab}$ for an R-right module M and an R-left module N. They are left adjoint to $\mathrm{Hom}(M,-) : \mathrm{Ab} \to R\text{-Mod}$ and $\mathrm{Hom}(N,-) : \mathrm{Ab} \to R^{op}\text{-Mod}$ by Example 1.2.18, 6.

Corollary 3.1.16: Let R be a ring, M an R-right module and N an R-left module. Then

1. The functors $M\otimes_R - : R\text{-Mod} \to \mathrm{Ab}$ and $-\otimes_R N : R^{op}\text{-Mod} \to \mathrm{Ab}$ are right exact.
2. The functors $\mathrm{Hom}(M,-) : \mathrm{Ab} \to R\text{-Mod}$ and $\mathrm{Hom}(N,-) : \mathrm{Ab} \to R^{op}\text{-Mod}$ are left exact.

While there is no direct analogue of the tensor product for general abelian categories, the functors $\mathrm{Hom}(A,-) : \mathcal{A} \to \mathrm{Ab}$, $\mathrm{Hom}(-,A) : \mathcal{A}^{op} \to \mathrm{Ab}$ are defined for any abelian category $\mathcal{A}$.

- The functor $\mathrm{Hom}(A,-)$ sends an object A' to the abelian group $\mathrm{Hom}_{\mathcal{A}}(A,A')$ and a morphism $f : A' \to A''$ to the group homomorphism
$$\mathrm{Hom}(A,f) : \mathrm{Hom}_{\mathcal{A}}(A,A') \to \mathrm{Hom}_{\mathcal{A}}(A,A''), \quad g \mapsto f \circ g.$$
- The functor $\mathrm{Hom}(-,A)$ sends an object A' in $\mathcal{A}$ to the abelian group $\mathrm{Hom}_{\mathcal{A}}(A',A)$ and a morphism $f : A' \to A''$ to the group homomorphism
$$\mathrm{Hom}(f,A) : \mathrm{Hom}_{\mathcal{A}}(A'',A) \to \mathrm{Hom}_{\mathcal{A}}(A',A), \quad g \mapsto g \circ f.$$

This raises the question if left exactness holds for these generalisations as well. Indeed, it is possible to prove this without Lemma 3.1.15.

Lemma 3.1.17: Let $\mathcal{A}$ be an abelian category and $f : X \to Y$ a morphism in $\mathcal{A}$.

1. For any object A in $\mathcal{A}$ the functor $\mathrm{Hom}(A,-) : \mathcal{A} \to \mathrm{Ab}$ is left exact:
   A morphism $\iota : W \to X$ is a kernel of $f : X \to Y$ in $\mathcal{A}$ if and only if for all objects A in $\mathcal{A}$ the morphism $\iota_* = \mathrm{Hom}(A,\iota)$ in Ab is a kernel of $f_* = \mathrm{Hom}(A,f)$.
2. For any object A in $\mathcal{A}$ the functor $\mathrm{Hom}(-,A) : \mathcal{A}^{op} \to \mathrm{Ab}$ is left exact:
   A morphism $\pi : Y \to Z$ is a cokernel of $f : X \to Y$ in $\mathcal{A}$ if and only if for all objects A in $\mathcal{A}$ the morphism $\pi^* = \mathrm{Hom}(\pi,A)$ in Ab is a kernel of $f^* = \mathrm{Hom}(f,A)$.

Proof:
We prove the first claim. The proof of the second claim is analogous if one takes into account that kernels and cokernels in $\mathcal{A}$ are cokernels and kernels in $\mathcal{A}^{op}$, respectively.

As we work in $\mathrm{Ab} = \mathbb{Z}\text{-Mod}$, Example 3.1.6 implies that the group homomorphism ι_* is a kernel of the group homomorphism f_* if and only if (i) ι_* is injective and (ii) $\mathrm{im}(\iota_*) = \ker(f_*)$.

Condition (i) is satisfied if and only if $\iota \circ g = \iota_*(g) = \iota_*(g') = \iota \circ g'$ implies $g = g'$ for all $g, g' \in \mathrm{Hom}_{\mathcal{A}}(A,W)$, and this is equivalent to the statement that ι is a monomorphism. Condition (ii) is satisfied if and only if (iia) $f_*(\iota_*(h)) = f \circ \iota \circ h = 0$ for all morphisms $h : A \to W$ and (iib) for every morphism $g : A \to X$ with $f_*(g) = f \circ g = 0$ there is a morphism $g' : A \to X$ with $g = \iota_*(g') = \iota \circ g'$. Condition (iia) is satisfied iff $f \circ \iota = 0$. Condition (iib) then states that ι is a kernel of f. □

Remark 3.1.18:

1. In general the functor $A\otimes_R - : R\text{-Mod} \to \text{Ab}$ is not left exact.
2. The functors $\text{Hom}(A, -) : \mathcal{A} \to \text{Ab}$, $\text{Hom}(-, A) : \mathcal{A}^{op} \to \text{Ab}$ are in general not right exact.

   A counterexample is $R\text{-Mod} = \mathcal{A} = \text{Ab}$ and $A = \mathbb{Z}/n\mathbb{Z}$ for $n \geq 2$. Then $\iota : \mathbb{Z} \to \mathbb{Z}$, $z \mapsto nz$ is a kernel of $\pi : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, $z \mapsto \bar{z}$ and π a cokernel of ι.

   However, $\text{id}\otimes\iota = 0 : \mathbb{Z}/n\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}$, since $(\text{id}\otimes\iota)(\bar{k}\otimes z) = \bar{k}\otimes(nz) = \overline{nk}\otimes z = 0$ for all $\bar{k} \in \mathbb{Z}/n\mathbb{Z}$ and $z \in \mathbb{Z}$. Hence, $\text{id}\otimes\iota = 0$ is not injective and not a kernel of $\text{id}\otimes\pi$.

   Similarly, $\text{Hom}(\mathbb{Z}/n\mathbb{Z}, \pi) = 0 : \text{Hom}_{\text{Ab}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \to \text{Hom}_{\text{Ab}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$, $g \mapsto \pi \circ g$ is not surjective and hence not a cokernel of $\text{Hom}(\mathbb{Z}/n\mathbb{Z}, \iota)$.

   Likewise, $\text{Hom}(\iota, \mathbb{Z}/n\mathbb{Z}) = 0 : \text{Hom}_{\text{Ab}}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \to \text{Hom}_{\text{Ab}}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$, $g \mapsto g \circ \iota$, since $g \circ \iota(z) = g(nz) = ng(z) = 0$ for all $z \in \mathbb{Z}$ and group homomorphisms $g : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. Hence, $\text{Hom}(\iota, \mathbb{Z}/n\mathbb{Z}) = 0$ is not surjective and not a cokernel of $\text{Hom}(\pi, \mathbb{Z}/n\mathbb{Z})$.

R-right modules A for which the associated functor $A\otimes_R - : R\text{-Mod} \to \text{Ab}$ is not only right exact but exact and objects in an abelian category $\mathcal{A}$ for which the functor $\text{Hom}(A, -)$ or $\text{Hom}(-, A)$ are exact play a special role in representation theory and homology theories.

Definition 3.1.19:
A right module A over a ring R is called **flat** if the functor $A\otimes_R - : R\text{-Mod} \to \text{Ab}$ is exact.

Definition 3.1.20: An object A in an abelian category $\mathcal{A}$ is called

- **projective** if the functor $\text{Hom}(A, -) : \mathcal{A} \to \text{Ab}$ is exact,
- **injective** if the functor $\text{Hom}(-, A) : \mathcal{A}^{op} \to \text{Ab}$ is exact.

One can show (Exercise 55) that for $\mathcal{A} = R^{op}\text{-Mod}$ any projective R^{op}-module is flat. Hence, projectivity and injectivity are not only more general concepts, but also stronger conditions. There is an alternative characterisations of projectivity and injectivity that is easier to handle and generalises to non-abelian categories.

Lemma 3.1.21: Let $\mathcal{A}$ be an abelian category.

1. An object A in $\mathcal{A}$ is projective if and only if for every epimorphism $\pi : X \to Y$ and every morphism $f : A \to Y$ there is a morphism $f' : A \to X$ with $\pi \circ f' = f$

$$\begin{tikzcd} & A \arrow[dl, dashed, "\exists f'"'] \arrow[d, "f"] & \\ X \arrow[r, two heads, "\pi"] & Y \arrow[r] & 0 \end{tikzcd}$$

2. An object A in $\mathcal{A}$ is injective if and only if for every monomorphism $\iota : Y \to X$ and every morphism $f : Y \to A$ there is a morphism $f' : X \to A$ with $f' \circ \iota = f$

$$\begin{tikzcd} & A & \\ X \arrow[ur, dashed, "\exists f'"] & Y \arrow[l, hook', "\iota"'] \arrow[u, "f"'] & 0 \arrow[l] \end{tikzcd}$$

Proof:
We prove the first statement. The proof of the second one is analogous.

$\Rightarrow$ Let A be projective. Then $\mathrm{Hom}(A,-)$ is exact and maps kernels to kernels and cokernels to cokernels. As every epimorphism $\pi : X \to Y$ in $\mathcal{A}$ is a cokernel of its kernel, the morphism $\mathrm{Hom}(A,\pi) : \mathrm{Hom}_{\mathcal{A}}(A,X) \to \mathrm{Hom}_{\mathcal{A}}(A,Y)$ is a cokernel as well and hence an epimorphism in Ab by Lemma 3.1.9. This means that for every morphism $f : A \to Y$, there is a morphism $f' : A \to X$ with $\mathrm{Hom}(A,\pi)(f') = \pi \circ f' = f$.

$\Leftarrow$ Suppose that for every morphism $f : A \to Y$ and epimorphism $\pi : X \to Y$ there is a morphism $f' : A \to X$ with $\pi \circ f' = f$. Then $\mathrm{Hom}(A,\pi) : \mathrm{Hom}_{\mathcal{A}}(A,X) \to \mathrm{Hom}_{\mathcal{A}}(A,Y)$ is an epimorphism for every epimorphism $\pi : X \to Y$.

We show that $\mathrm{Hom}(A,-) : \mathcal{A} \to \mathrm{Ab}$ is right exact, i. e. preserves cokernels. For this, let $f : A \to X$ be a morphism with cokernel $\pi : X \to Y$. By Lemma 3.1.13 f has a canonical factorisation $f = \iota' \circ \pi'$ with a monomorphism ι' and an epimorphism π'. By Exercise 27 the morphism $\pi : X \to Y$ is also a cokernel of ι'. As ι' is a monomorphism, it is a kernel of its cokernel $\pi : X \to Y$. By left-exactness of $\mathrm{Hom}(A,-)$, it follows that $\mathrm{Hom}(A,\iota')$ is a kernel of $\mathrm{Hom}(A,\pi)$. As every epimorphism is a cokernel of its kernel, it follows that $\mathrm{Hom}(A,\pi)$ is a cokernel of $\mathrm{Hom}(A,\iota')$. As $\mathrm{Hom}(A,f) = \mathrm{Hom}(A,\iota' \circ \pi') = \mathrm{Hom}(A,\iota') \circ \mathrm{Hom}(A,\pi')$ and $\mathrm{Hom}(A,\pi')$ is an epimorphism, Exercise 27 implies that $\mathrm{Hom}(A,\pi)$ is also a cokernel of $\mathrm{Hom}(A,f)$ and hence $\mathrm{Hom}(A,-)$ is right exact. $\square$

Example 3.1.22:

1. By Remark 3.1.18 the objects $\mathbb{Z}/n\mathbb{Z}$ in Ab for $n \geq 2$ are neither projective nor injective.

2. For every ring R, any free R-module is projective.

   If A is a free R-module with basis B, $\pi : X \to Y$ R-linear and surjective and $f : A \to Y$ R-linear, then we can choose for every element $b \in B$ an element $f'(b) \in \pi^{-1}(f(b))$ and obtain an R-linear map $f' : A \to X$, $b \mapsto f'(b)$ with $\pi \circ f' = f$.

3. The object $\mathbb{Z}$ in Ab is projective, but not injective.

   The projectivity of $\mathbb{Z}$ follows from 2. However, $\mathbb{Z}$ is not injective, because for the monomorphism $\iota : \mathbb{Z} \to \mathbb{Z}$, $z \mapsto nz$ with $n \geq 2$, and the group homomorphism $f = \mathrm{id}_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}$ there is no morphism $f' : \mathbb{Z} \to \mathbb{Z}$ with $f' \circ \iota = f = \mathrm{id}_{\mathbb{Z}}$.

3.2 Chain complexes and homology

We are now ready to define and investigate homology theories in general abelian categories. The fundamental concept is that of a chain complex, which generalises the (co)chains and (co)boundary operators in the examples from Section 2. It is obtained by replacing the modules of n-(co)chains by objects in an abelian category and the (co)boundary operators by morphisms, such that subsequent morphisms compose to zero.

Definition 3.2.1: Let $\mathcal{A}$ be an abelian category.

1. A **chain complex** $(X_\bullet, d_\bullet)$ in $\mathcal{A}$ is a family $X_\bullet = (X_n)_{n\in\mathbb{Z}}$ of objects and a family $d_\bullet = (d_n)_{n\in\mathbb{Z}}$ of morphisms $d_n : X_n \to X_{n-1}$ in $\mathcal{A}$ with $d_{n-1} \circ d_n = 0$ for all $n \in \mathbb{Z}$.
$$\ldots \xrightarrow{d_{n+2}} X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} \ldots .$$

2. A **chain map** $f_\bullet : (X_\bullet, d_\bullet) \to (X'_\bullet, d'_\bullet)$ is a family $(f_n)_{n\in\mathbb{Z}}$ of morphisms $f_n : X_n \to X'_n$ such that $d'_n \circ f_n = f_{n-1} \circ d_n$ for all $n \in \mathbb{Z}$.
$$\begin{tikzcd}
\ldots \arrow[r, "d_{n+2}"] & X_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] & X_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & X_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "f_{n-1}"'] & \ldots \\
\ldots \arrow[r, "d'_{n+2}"] & X'_{n+1} \arrow[r, "d'_{n+1}"] & X'_n \arrow[r, "d'_n"] & X'_{n-1} \arrow[r, "d'_{n-1}"] & \ldots
\end{tikzcd}$$

Notation 3.2.2: It is standard to omit subsequences of zero objects and morphisms between them from chain complexes:

- $0 \to X_m \xrightarrow{d_m} X_{m-1} \xrightarrow{d_{m-1}} ...$ stands for a chain complex with $X_k = 0$ for all $k > m$. Such a chain complex is called **bounded above**. It is called **negative** if $m = 0$.
- $... \xrightarrow{d_{m+2}} X_{m+1} \xrightarrow{d_{m+1}} X_m \to 0$ stands for a chain complex with $X_k = 0$ for all $k < m$. Such a chain complex is called **bounded below**. It is called **positive** if $m = 0$.
- If $X_k = 0$ for all $k < m$ and $k > l > m$, the chain complex is called **finite** or **bounded** and denoted
$$0 \to X_l \xrightarrow{d_l} X_{l-1} \xrightarrow{d_{l-1}} ... \xrightarrow{d_{m+2}} X_{m+1} \xrightarrow{d_{m+1}} X_m \to 0.$$
- A chain complex $... \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} 0 \to 0 \to ... \to 0 \xrightarrow{d_{n-m}} X_{n-m-1} \xrightarrow{d_{n-m-1}} ...$ is viewed as two chain complexes, one with $X_k = 0$ for $k < n$ and one with $X_k = 0$ for $k \geq n - m$.

We also denote a chain complex $(X_\bullet, d_\bullet)$ simply by $X_\bullet$ when this causes no ambiguity.

Remark 3.2.3: Analogously, one defines a **cochain complex** $(X^\bullet, d^\bullet)$ in an abelian category $\mathcal{A}$ as a family $X^\bullet = (X^n)_{n\in\mathbb{Z}}$ of objects X^n in $\mathcal{A}$ together with a family $d^\bullet = (d^n)_{n\in\mathbb{Z}}$ of morphisms $d^n : X^n \to X^{n+1}$ with $d^{n+1} \circ d^n = 0$ for all $n \in \mathbb{Z}$.

In this section we restrict attention to chain complexes, since a cochain complex $(X^\bullet, d^\bullet)$ can be transformed into a chain complex $(X_\bullet, d_\bullet)$ by setting $X_n = X^{-n}$ and $d_n = d^{-n} : X_n \to X_{n-1}$ for all $n \in \mathbb{Z}$. Nevertheless, sometimes it is necessary to consider both structures.

Remark 3.2.4:

1. Chain complexes and chain maps in an abelian category $\mathcal{A}$ form a category $\mathrm{Ch}_{\mathcal{A}}$ with the composition of morphisms $g_\bullet \circ f_\bullet = (g_n \circ f_n)_{n\in\mathbb{Z}}$ and the identity chain maps $1_{X_\bullet} = (1_{X_n})_{n\in\mathbb{Z}}$ as identity morphisms.
$$\begin{tikzcd}
\ldots \arrow[r, "d_{n+2}"] & X_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] & X_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & X_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "f_{n-1}"'] & \ldots \\
\ldots \arrow[r, "d'_{n+2}"] & X'_{n+1} \arrow[r, "d'_{n+1}"] \arrow[d, "g_{n+1}"'] & X'_n \arrow[r, "d'_n"] \arrow[d, "g_n"'] & X'_{n-1} \arrow[r, "d'_{n-1}"] \arrow[d, "g_{n-1}"'] & \ldots \\
\ldots \arrow[r, "d''_{n+2}"] & X''_{n+1} \arrow[r, "d''_{n+1}"] & X''_n \arrow[r, "d''_n"] & X''_{n-1} \arrow[r, "d''_{n-1}"] & \ldots
\end{tikzcd}$$

2. The category $\mathrm{Ch}_{\mathcal{A}}$ is abelian:
   - Coproducts of chain complexes are given by $\amalg_{i\in I}X^i_\bullet = (\amalg_{i\in I}X^i_n)_{n\in\mathbb{Z}}$ and the chain maps $\amalg_{i\in I}d^i_\bullet = (\amalg_{i\in I}d^i_n)_{n\in\mathbb{Z}}$, where $\amalg_{i\in I}d^i_n : \amalg_{i\in I}X^i_n \to \amalg_{i\in I}X^i_{n-1}$ is induced by the morphisms $d^i_n : X^i_n \to X^i_{n-1}$ and the universal property of the coproduct.
   - Products are defined analogously.
   - The addition of chain maps is given by $f_\bullet + g_\bullet = (f_n + g_n)_{n\in\mathbb{Z}}$.
   - Kernels and cokernels of a chain map $f_\bullet = (f_n)_{n\in\mathbb{Z}} : X_\bullet \to X'_\bullet$ are given by $\iota_\bullet = (\iota_n : \ker(f_n) \to X_n)_{n\in\mathbb{Z}}$ and $\pi_\bullet = (\pi_n : X'_n \to \mathrm{coker}(f_n))_{n\in\mathbb{Z}}$.
3. Bounded chain complexes, chain complexes that are bounded below, positive chain complexes, chain complexes that are bounded above and negative chains complexes in $\mathcal{A}$ form full abelian subcategories $\mathrm{Ch}_{\mathcal{A}\,\mathrm{fin}}$, $\mathrm{Ch}_{\mathcal{A}+}$ $\mathrm{Ch}_{\mathcal{A}\geq 0}$, $\mathrm{Ch}_{\mathcal{A}-}$ and $\mathrm{Ch}_{\mathcal{A}\leq 0}$ of $\mathrm{Ch}_{\mathcal{A}}$.

This remark has important implications. It allows one to consider chain complexes in the abelian category $\mathrm{Ch}_{\mathcal{A}}$ and to relate their homologies to the homologies of certain chain complexes in $\mathcal{A}$. This leads to techniques that are useful for the computation of homologies. We will see basic examples of these techniques in Section 4.5.

All the examples of (co)homologies from Section 2 define (co)chain complexes in the abelian category $\mathcal{A} = k$-Mod for some commutative ring k. The objects X_n of the (co)chain complexes are the modules of n-(co)chains and the morphisms d_n the (co)boundary operators. They are associated with objects in a certain category $\mathcal{C}$, such as $\mathcal{C} = \mathrm{Top}$, $\mathcal{C} = A$-Mod-A for a k-algebra A, $\mathcal{C} = k[G]$-Mod for a group G or $\mathcal{C} = \mathrm{Rep}(\mathfrak{g})$ for a Lie algebra $\mathfrak{g}$.

It turns out that morphisms in $\mathcal{C}$ define (co)chain maps between these (co)chain complexes. As the assignments of (co)chain maps to morphisms in $\mathcal{C}$ is compatible with the composition of morphisms and the identity morphisms, we can view them as functors $F : \mathcal{C} \to \mathrm{Ch}_{\mathcal{A}}$ or $F : \mathcal{C}^{op} \to \mathrm{Ch}_{\mathcal{A}}$ from the category $\mathcal{C}$ under investigation or its opposite into the category $\mathrm{Ch}_{\mathcal{A}}$ of chain complexes in $\mathcal{A} = k$-Mod.

Example 3.2.5:

1. The chain complex $(C_\bullet(X,k), d_\bullet)$ in $\mathcal{A} = k$-Mod from Definition 2.1.2 is called the **singular chain complex** of X with coefficients in k and given by

$$C_n(X,k) = \langle \sigma : \Delta^n \to X \text{ continuous}\rangle_k \qquad\qquad d_n(\sigma) = \Sigma_{i=0}^n (-1)^i \sigma \circ f_i^n.$$

   A continuous map $f : X \to Y$ induces a chain map $C_\bullet(f,k) : C_\bullet(X,k) \to C_\bullet(Y,k)$ with $C_n(f,k)(\sigma) = f \circ \sigma : \Delta^n \to Y$ for all singular n-simplexes $\sigma : \Delta^n \to X$. This defines a functor $C_\bullet(-,k) : \mathrm{Top} \to \mathrm{Ch}_{k\text{-Mod}}$.

2. The cochain complex $(C^\bullet(X,k), d^\bullet)$ in $\mathcal{A} = k$-Mod from Definition 2.1.12 is called the **singular cochain complex** of X with coefficients in k and given by

$$C^n(X,k) = \mathrm{Hom}_k(C_n(X,k), k)$$
$$d^n(\phi)(\sigma) = \phi(d_{n+1}\sigma) \text{ for all continuous } \sigma : \Delta^{n+1} \to X.$$

   A continuous map $f : X \to Y$ induces a cochain map $C^\bullet(f,k) : C^\bullet(Y,k) \to C^\bullet(X,k)$ with $C^n(f,k)(\phi) = \phi \circ f : C_n(X,k) \to k$ for all k-linear maps $\phi : C_n(Y,k) \to k$. This defines a functor $C^\bullet(-,k) : \mathrm{Top}^{op} \to \mathrm{Ch}_{k\text{-Mod}}$.

3. Analogous statements hold for the **simplicial chain complex** $(C_\bullet(\Delta, k), d_\bullet)$ from Definition 2.1.9 and the **simplicial cochain complex** $(C^\bullet(\Delta), d^\bullet)$ from Definition 2.1.13 and for simplicial maps $f : \Delta \to \Delta'$.

4. The chain complex $(C_\bullet(A, M), d_\bullet)$ in $\mathcal{A} = k$-Mod from Definition 2.2.3 is called the **Hochschild complex** of A with coefficients in M and given by

$$\begin{aligned} C_n(A,M) &= M \otimes_k A^{\otimes n} \\ d_n(m \otimes a_1 \otimes ... \otimes a_n) &= m \lhd a_1 \otimes a_2 \otimes ... \otimes a_n - m \otimes (a_1 a_2) \otimes ... \otimes m + m \otimes a_1 \otimes (a_2 a_3) \otimes ... \otimes a_n \\ &\quad \pm \ldots + (-1)^{n-1} m \otimes a_1 \otimes ... \otimes (a_{n-1} a_n) + (-1)^n a_n \rhd m \otimes a_1 \otimes ... \otimes a_{n-1} \end{aligned}$$

The cochain complex $(C^\bullet(M, A), d^\bullet)$ in $\mathcal{A} = k$-Mod from Definition 2.2.4 is called the **Hochschild cocomplex** of A with coefficients in M and given by

$$\begin{aligned} C^n(A,M) &= \mathrm{Hom}_k(A^{\otimes n}, M) \\ d^n(\phi)(a_0 \otimes ... \otimes a_n) &= a_0 \rhd \phi(a_1 \otimes ... \otimes a_n) - \phi(a_0 a_1 \otimes a_2 \otimes ... \otimes a_n) + \phi(a_0 \otimes (a_1 a_2) \otimes a_3 \otimes ... \otimes a_n) \\ &\quad \pm \ldots + (-1)^{n-1} \phi(a_0 \otimes ... \otimes a_{n-2} \otimes a_{n-1} a_n) + (-1)^{n+1} \phi(a_0 \otimes ... \otimes a_{n-1}) \lhd a_n. \end{aligned}$$

Every (A, A)-bimodule morphism $f : M \to N$ defines (co)chain maps

$$\begin{aligned} &C_\bullet(A,f) : C_\bullet(A,M) \to C_\bullet(A,N) \quad \text{with} \quad C_n(A,f) = f \otimes \mathrm{id}_A^{\otimes n} : C_n(A,M) \to C_n(A,N) \\ &C^\bullet(A,f) : C^\bullet(A,M) \to C^\bullet(A,N) \quad \text{with} \quad C^n(A,f) : C^n(A,M) \to C^n(A,N),\ \phi \mapsto f \circ \phi. \end{aligned}$$

This defines functors $C_\bullet(A, -), C^\bullet(A, -) : A\text{-Mod-}A \to \mathrm{Ch}_{k\text{-mod}}$.

5. The chain complex $(C_\bullet(G, M), d_\bullet)$ and the cochain complex $(C^\bullet(G, M), d^\bullet)$ in $\mathcal{A} = k$-Mod from Definition 2.3.1 are called the **chain** and **cochain complex of group cohomology**.

   Every morphism $f : M \to N$ of $k[G]$-modules induces a chain and a cochain map analogously to 4., and this defines functors $C_\bullet(G, -) : k[G]^{op}\text{-Mod} \to \mathrm{Ch}_{k\text{-Mod}}$ and $C^\bullet(G, -) : k[G]\text{-Mod} \to \mathrm{Ch}_{k\text{-Mod}}$.

6. The cochain complex $(C^\bullet(\mathfrak{g}, M), d^\bullet)$ in $\mathcal{A} = \mathrm{Vect}_\mathbb{F}$ from Definition 2.4.8 is called the **Chevalley-Eilenberg complex** and given by

$$\begin{aligned} C^n(\mathfrak{g}, M) &= \mathrm{Hom}_\mathbb{F}(\Lambda^n g^*, M) \\ (d^n f)(x_0, ..., x_n) &= \Sigma_{i=0}^n (-1)^i \rho(x_i) f(x_0, ..., \widehat{x}_i, ..., x_n) \\ &\quad + \Sigma_{0 \le i < j \le n} (-1)^{i+j} f([x_i, x_j], x_0, ..., \widehat{x}_i, ..., \widehat{x}_j, ..., x_n). \end{aligned}$$

Every morphism $f : M \to N$ of $\mathfrak{g}$-representations induces a cochain map

$$C^\bullet(\mathfrak{g}, f) : C^\bullet(\mathfrak{g}, M) \to C^\bullet(\mathfrak{g}, N) \quad \text{with} \quad C^n(\mathfrak{g}, f) : C^n(\mathfrak{g}, M) \to C^n(\mathfrak{g}, N),\ \phi \mapsto f \circ \phi,$$

and this defines a functor $C^\bullet(\mathfrak{g}, -) : \mathrm{Rep}(\mathfrak{g}) \to \mathrm{Ch}_{\mathrm{Vect}_\mathbb{F}}$.

The (co)chain maps in these examples are structural or canonical in the sense that they are images of morphisms in the category $\mathcal{C}$ under a functor $F : \mathcal{C} \to \mathrm{Ch}_\mathcal{A}$. We will see later that this is a general pattern and not specific to these examples. However, there are also less obvious chain maps that do not arise from such morphisms.

Example 3.2.6:
Let k be a commutative ring, G a group and $\langle M\rangle_k$ the free k-module generated by a set M. Consider the chain complexes

- $X_\bullet$ with $X_n = \langle G^{\times(n+1)}\rangle_k$, the $k[G]$-module structure $g \rhd (g_0, ..., g_n) = (gg_0, ..., gg_n)$ on X_n and boundary operator $d_n = \Sigma_{i=0}^n (-1)^i d_n^i : X_n \to X_{n-1}$ for $n \in \mathbb{N}_0$ with
$$d_n^i(g_0, ..., g_n) = (g_0, ..., \widehat{g}_i, ..., g_n).$$
- $X'_\bullet$ with $X'_n = \langle G^{\times n}\rangle_{k[G]}$, $k[G]$-linear boundary operator $d'_n = \Sigma_{i=0}^{n+1} (-1)^i d_n'^i : X'_n \to X'_{n-1}$ for $n \in \mathbb{N}_0$ with
$$d_n'^i(g_1, ..., g_n) = \begin{cases} g_1 \rhd (g_2, ..., g_n) & i = 0 \\ (..., g_i g_{i+1}, ...) & 1 \le i \le n-1 \\ (g_1, ..., g_{n-1}) & i = n. \end{cases}$$

Then the $k[G]$-linear maps $f_n : X'_n \to X_n$, $(g_1, ..., g_n) \mapsto (1, g_1, g_1g_2, \ldots, g_1g_2 \cdots g_{n-1}g_n)$ define an invertible chain map $f_\bullet : X'_\bullet \to X_\bullet$ (Exercise 37).

Example 3.2.6 hints at a relation between group (co)homology and singular and simplicial (co)homology of topological spaces. The chain complex $X'_\bullet$ in Example 3.2.6 is similar to the (co)chain complexes of group (co)homology, whereas the boundary operator of chain complex $X_\bullet$ recalls the face maps $f_i^n : \Delta^{n-1} \to \Delta^n$ that send the standard simplex Δ^{n-1} to the $(n-1)$-face of Δ^n opposite the vertex e_i. We will see later that this is not a coincidence.

After defining chain complexes and chain maps in general abelian categories $\mathcal{A}$, we can now construct their homologies. For this, we generalise the definitions in Section 2 as follows:

- the inclusion maps $\iota_n : \ker(d_n) \to X_n$ for the morphisms $d_n : X_n \to X_{n-1}$ in R-Mod are replaced by the kernels $\iota_n : \ker(d_n) \to X_n$ of the boundary morphisms,
- the images of the morphisms $d_n : X_n \to X_{n-1}$ in R-Mod are replaced by the images $\iota'_n : \mathrm{im}(d_{n+1}) \to X_{n-1}$ and coimages $\pi'_n : X_n \to \mathrm{im}(d_{n+1})$ from Lemma 3.1.13,
- the inclusion morphisms $\phi_n : \mathrm{im}(d_{n+1}) \to \ker(d_n)$ for morphisms $d_n : X_n \to X_{n-1}$ and $d_{n+1} : X_{n+1} \to X_n$ in R-Mod with $d_n \circ d_{n+1} = 0$ are replaced by the monomorphisms $\phi_n : \mathrm{im}(d_{n+1}) \to \ker(d_n)$ from Lemma 3.1.13,
- the canonical surjections $p_n : \ker(d_n) \to \ker(d_n)/\mathrm{im}(d_{n+1})$ in R-Mod are replaced by the cokernels $p_n : \ker(d_n) \to \mathrm{coker}(\phi_n)$ of the monomorphisms ϕ_n

$$\begin{tikzcd}
 & & \mathrm{coker}(\phi_{n+1}) & & \mathrm{coker}(\phi_n) & & \mathrm{coker}(\phi_{n-1}) & \\
 & \mathrm{im}(d_{n+2}) \arrow[r, hook, "\phi_{n+1}"] \arrow[dr, hook, "\iota'_{n+2}"'] & \ker(d_{n+1}) \arrow[u, two heads, "p_{n+1}"] \arrow[d, hook, "\iota_{n+1}"] & \mathrm{im}(d_{n+1}) \arrow[r, hook, "\phi_n"] \arrow[dr, hook, "\iota'_{n+1}"'] & \ker(d_n) \arrow[u, two heads, "p_n"] \arrow[d, hook, "\iota_n"] & \mathrm{im}(d_n) \arrow[r, hook, "\phi_{n-1}"] \arrow[dr, hook, "\iota'_n"'] & \ker(d_{n-1}) \arrow[u, two heads, "p_{n-1}"] \arrow[d, hook, "\iota_{n-1}"] & \\
\ldots \arrow[rr, "d_{n+2}"'] & & X_{n+1} \arrow[rr, "d_{n+1}"'] \arrow[ur, two heads, "\pi'_{n+1}"'] & & X_n \arrow[rr, "d_n"'] \arrow[ur, two heads, "\pi'_n"'] & & X_{n-1} \arrow[r, "d_{n-1}"'] & \ldots
\end{tikzcd} \tag{23}$$

Definition 3.2.7: Let $X_\bullet$ be a chain complex in an abelian category $\mathcal{A}$

1. The n-**cycle object** of $X_\bullet$ is the kernel object $Z_n(X_\bullet) := \ker(d_n)$ of the morphism $d_n : X_n \to X_{n-1}$.
2. The n-**boundary object** of $X_\bullet$ is the image object $B_n(X_\bullet) := \mathrm{im}(d_{n+1})$ of the morphism $d_{n+1} : X_{n+1} \to X_n$.
3. The nth **homology** of $X_\bullet$ is the cokernel object $H_n(X_\bullet) = \mathrm{coker}(\phi_n)$ of the monomorphism $\phi_n : \mathrm{im}(d_{n+1}) \to \ker(d_n)$ from Lemma 3.1.13 and (23).
4. The chain complex $X_\bullet$ is called **exact in** X_n if $H_n(X_\bullet) = 0$ or, equivalently, if the monomorphism $\phi_n : \mathrm{im}(d_{n+1}) \to \ker(d_n)$ is an isomorphism. It is called **exact**, or an **exact sequence** if it is exact in X_n for all $n \in \mathbb{Z}$.

That $H_n(X_\bullet) = 0$ if and only if the monomorphism $\phi_n : \mathrm{im}(d_{n+1}) \to \ker(d_n)$ is an isomorphism follows, because any isomorphism $\phi : X \to Y$ in $\mathcal{A}$ is an epimorphism and hence has cokernel $0 : Y \to 0$ by Lemma 3.1.9. Conversely, if a monomorphism $\phi : X \to Y$ in $\mathcal{A}$ has cokernel $0 : Y \to 0$, then it is also an epimorphism and hence an isomorphism by Exercise 30. This shows that the homologies of a chain complex measure its failure to be exact.

It remains to investigate how chain maps between chain complexes affect their homologies. As the nth homology assigns to every chain complex $X_\bullet$ in $\mathcal{A}$ an object $H_n(X_\bullet)$ in $\mathcal{A}$, it is plausible that a chain map $f_\bullet : X_\bullet \to Y_\bullet$ should induce morphisms $H_n(f_\bullet) : H_n(X_\bullet) \to H_n(Y_\bullet)$. This should be compatible with the composition of chain maps and with identity chain maps. In other words, homologies should define functors from the category $\mathrm{Ch}_\mathcal{A}$ of chain complexes in $\mathcal{A}$ to the underlying abelian category $\mathcal{A}$.

Proposition 3.2.8: Let $\mathcal{A}$ be an abelian category. Then the nth homology defines an additive functor $H_n : \mathrm{Ch}_\mathcal{A} \to \mathcal{A}$ that assigns to a chain complex $X_\bullet$ its nth homology $H_n(X_\bullet)$ and to a chain map $f_\bullet : X_\bullet \to X'_\bullet$ the unique morphism $H_n(f_\bullet) : H_n(X_\bullet) \to H_n(X'_\bullet)$ for which the following diagram commutes

$$\begin{tikzcd}
\mathrm{im}(d_{n+1}) \arrow[r, hook, "\phi_n"] & \ker(d_n) \arrow[r, two heads, "p_n"] \arrow[d, hook, "\iota_n"'] \arrow[dddd, dashed, bend left, "\exists!\bar f_n"] & H_n(X_\bullet) = \mathrm{coker}(\phi_n) \arrow[dddd, dashed, "\exists! H_n(f_\bullet)"] \\
X_{n+1} \arrow[u, two heads, "\pi_{n+1}"] \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] & X_n \arrow[d, "f_n"'] & \\
X'_{n+1} \arrow[r, "d'_{n+1}"] \arrow[d, two heads, "\pi'_{n+1}"'] & X'_n & \\
\mathrm{im}(d'_{n+1}) \arrow[r, hook, "\phi'_n"'] & \ker(d'_n) \arrow[r, two heads, "p'_n"'] \arrow[u, hook, "\iota'_n"'] & H_n(X'_\bullet) = \mathrm{coker}(\phi'_n).
\end{tikzcd} \tag{24}$$

Proof:
1. We show that $H_n(f_\bullet)$ is well-defined:
As $f_\bullet$ is a chain map, $d'_n \circ f_n \circ \iota_n = f_{n-1} \circ d_n \circ \iota_n = 0$. By the universal property of the kernel $\iota'_n : \ker(d'_n) \to X'_n$ there is a unique morphism $\bar f_n : \ker(d_n) \to \ker(d'_n)$ with $\iota'_n \circ \bar f_n = f_n \circ \iota_n$. From the diagram we have

$$\iota'_n \circ \bar f_n \circ \phi_n \circ \pi_{n+1} = f_n \circ \iota_n \circ \phi_n \circ \pi_{n+1} = f_n \circ d_{n+1} = d'_{n+1} \circ f_{n+1} = \iota'_n \circ \phi'_n \circ \pi'_{n+1} \circ f_{n+1}.$$

As ι'_n is a monomorphism, it follows that $\bar{f}_n \circ \phi_n \circ \pi_{n+1} = \phi'_n \circ \pi'_{n+1} \circ f_{n+1}$. The morphism $p'_n : \ker(d'_n) \to \mathrm{coker}(\phi'_n)$ is a cokernel of ϕ'_n, and this implies

$$p'_n \circ \bar{f}_n \circ \phi_n \circ \pi_{n+1} = p'_n \circ \phi'_n \circ \pi'_{n+1} \circ f_{n+1} = 0 \circ \pi'_{n+1} \circ f_{n+1} = 0.$$

Because π_{n+1} is an epimorphism, this implies $p'_n \circ \bar{f}_n \circ \phi_n = 0$. By the universal property of the cokernel $p_n : \ker(d_n) \to \mathrm{coker}(\phi_n)$, there is a unique morphism $H_n(f_\bullet) : H_n(X_\bullet) \to H_n(X'_\bullet)$ with $H_n(f_\bullet) \circ p_n = p'_n \circ \bar{f}_n$.

2. We prove that this defines a functor:
To show that $H_n(1_{X_\bullet}) = 1_{H_n(X_\bullet)}$ it is sufficient to note that diagram (24) commutes if we set $f_k = 1_{X_k}$, $d'_k = d_k$, $\pi'_k = \pi_k$, $\iota'_k = \iota_k$, $p'_k = p_k$, $\bar{f}_n = 1_{\ker(d_n)}$ and $H_n(f_\bullet) = 1_{H_n(X_\bullet)}$. To show that $H_n(g_\bullet \circ f_\bullet) = H_n(g_\bullet) \circ H_n(f_\bullet)$ for all chain maps $f_\bullet : X_\bullet \to X'_\bullet$ and $g_\bullet : X'_\bullet \to X''_\bullet$ we consider the commuting diagram obtained by composing diagrams (24) for $f_\bullet$ and $g_\bullet$

$$\begin{tikzcd}
X_{n+1} \arrow[r, "\psi_n"] \arrow[d, "f_{n+1}"] \arrow[dd, bend right=60, "g_{n+1}\circ f_{n+1}"'] & \ker(d_n) \arrow[r, two heads, "p_n"] \arrow[d, "\bar{f}_n"] & H_n(X_\bullet) \arrow[d, "H_n(f_\bullet)"'] \arrow[dd, bend left=60, "H_n(g_\bullet\circ f_\bullet)"] \\
X'_{n+1} \arrow[r, "\psi'_n"] \arrow[d, "g_{n+1}"] & \ker(d'_n) \arrow[r, two heads, "p'_n"] \arrow[d, "\bar{g}_n"] & H_n(X'_\bullet) \arrow[d, "H_n(g_\bullet)"'] \\
X''_{n+1} \arrow[r, "\psi''_n"] & \ker(d''_n) \arrow[r, two heads, "p''_n"] & H_n(X''_\bullet).
\end{tikzcd}$$

with $\psi_n = \phi_n \circ \pi_{n+1}$, $\psi'_n = \phi'_n \circ \pi'_{n+1}$ and $\psi''_n = \phi_n \circ \pi''_{n+1}$. As the morphism $H_n(f_\bullet) \circ H_n(g_\bullet)$ is defined uniquely by the commutativity of (24), this shows that $H_n(g_\bullet \circ f_\bullet) = H_n(g_\bullet) \circ H_n(f_\bullet)$.

3. That $H_n : \mathrm{Ch}_{\mathcal{A}} \to \mathcal{A}$ is additive follows because the diagram (24) commutes if we set $f_n = f'_n + f''_n$, $\bar{f}_n = \bar{f}'_n + \bar{f}''_n$ and $H_n(f_\bullet) = H_n(f'_\bullet) + H_n(f''_\bullet)$. As the diagram defines $H_n(f_\bullet)$ uniquely, the claim follows. $\square$

Remark 3.2.9: If $\mathcal{A} = R$-Mod for a ring R, the morphisms in diagram (24) are the following:

- $\pi_n : X_n \to \mathrm{im}(d_n)$, $x \mapsto d_n(x)$ is the corestriction of $d_n : X_n \to X_{n-1}$,
- $\phi_n : \mathrm{im}(d_{n+1}) \to \ker(d_n)$, $x \mapsto x$ is the inclusion map,
- $\iota_n : \ker(d_n) \to X_n$, $x \mapsto x$ is the inclusion map,
- $p_n : \ker(d_n) \to \ker(d_n)/\mathrm{im}(d_{n+1})$, $x \mapsto [x]$ is the canonical surjection,
- $\bar{f}_n : \ker(d_n) \to \ker(d'_n)$, $x \mapsto f_n(x)$ is the restriction and corestriction $f_n : X_n \to X'_n$,
- $H_n(f_\bullet) : \ker(d_n)/\mathrm{im}(d_{n+1}) \to \ker(d'_n)/\mathrm{im}(d'_{n+1})$, $[x] \mapsto [f_n(x)]$ is the induced map between the quotient modules.

3.3 Chain homotopies

One reason why homologies are powerful is that there is another layer of structure beyond chain complexes and chain maps, namely chain homotopies between chain maps. The role of chain homotopies in homology theories is similar to the role of homotopies between continuous maps in homotopy theory. They define an equivalence relation on the set of chain maps between given chain complexes that is compatible with the composition of chain maps.

This allows one to form a new category whose objects are chain complexes and whose morphisms chain homotopy classes of chain maps between them. The isomorphisms in this category are chain homotopy classes of chain homotopy equivalences. They play a similar role as homotopy equivalences for topological spaces. In fact, the first examples of chain homotopies were constructed from homotopies between continuous maps.

Definition 3.3.1: Let $\mathcal{A}$ be an abelian category.

1. A **chain homotopy** $h_\bullet : f_\bullet \Rightarrow f'_\bullet$ from a chain map $f_\bullet : X_\bullet \to X'_\bullet$ to $f'_\bullet : X_\bullet \to X'_\bullet$ in $\mathcal{A}$ is a family $(h_n)_{n\in\mathbb{Z}}$ of morphisms $h_n : X_n \to X'_{n+1}$ with
$$f_n - f'_n = h_{n-1} \circ d_n + d'_{n+1} \circ h_n \qquad \forall n \in \mathbb{Z}.$$
If there is a chain homotopy $h_\bullet : f_\bullet \Rightarrow f'_\bullet$, then $f_\bullet$ und $f'_\bullet$ are called **chain homotopic**, and one writes $f_\bullet \sim f'_\bullet$.

2. A chain map $f_\bullet : X_\bullet \to X'_\bullet$ is called a **chain homotopy equivalence** if there is a chain map $g : X'_\bullet \to X_\bullet$ with $g_\bullet \circ f_\bullet \sim 1_{X_\bullet}$ and $f_\bullet \circ g_\bullet \sim 1_{X'_\bullet}$. In this case the chain complexes $X_\bullet$ und $X'_\bullet$ are called **chain homotopy equivalent** and one writes $X_\bullet \simeq X'_\bullet$.

Remark 3.3.2:

1. For given chain complexes $X_\bullet$, $X'_\bullet$ in an abelian category $\mathcal{A}$, the chain maps $f_\bullet : X_\bullet \to X'_\bullet$ and chain homotopies between them form an abelian groupoid.

   The composite of two chain homotopies $h : f_\bullet \Rightarrow f'_\bullet$ and $h'_\bullet : f'_\bullet \Rightarrow f''_\bullet$ is the chain homotopy $h'_\bullet \circ h_\bullet = (h_n + h'_n)_{n\in\mathbb{Z}} : f_\bullet \Rightarrow f''_\bullet$, the identity morphisms are trivial chain homotopies $1_{f_\bullet} = (0)_{n\in\mathbb{Z}}$ and the inverse of $h_\bullet : f_\bullet \Rightarrow f'_\bullet$ is $h_\bullet^{-1} = (-h_n)_{n\in\mathbb{Z}} : f'_\bullet \Rightarrow f_\bullet$.

2. By 1. being chain homotopic defines an equivalence relation on the abelian groups $\mathrm{Hom}_{\mathrm{Ch}_\mathcal{A}}(X_\bullet, X'_\bullet)$. It is compatible with the composition of morphisms:

   For all chain maps $f_\bullet, f'_\bullet : X_\bullet \to X'_\bullet$ and $g_\bullet, g'_\bullet : X'_\bullet \to X''_\bullet$ and chain homotopies $h_\bullet : f_\bullet \Rightarrow f'_\bullet$ and $h'_\bullet : g_\bullet \Rightarrow g'_\bullet$, the family of morphisms $k_\bullet = (g'_{n+1} \circ h_n + h'_n \circ f_n)_{n\in\mathbb{Z}}$ is a chain homotopy $k_\bullet : g_\bullet \circ f_\bullet \Rightarrow g'_\bullet \circ f'_\bullet$ since
$$\begin{aligned}&g_n \circ f_n - g'_n \circ f'_n = (g_n - g'_n) \circ f_n + g'_n \circ (f_n - f'_n)\\&= (h'_{n-1} \circ d'_n + d''_{n+1} \circ h'_n) \circ f_n + g'_n \circ (h_{n-1} \circ d_n + d'_{n+1} \circ h_n)\\&= (g'_n \circ h_{n-1} + h'_{n-1} \circ f_{n-1}) \circ d_n + d''_{n+1} \circ (g'_{n+1} \circ h_n + h'_n \circ f_n) = k_{n-1} \circ d_n + d''_{n+1} \circ k_n.\end{aligned}$$

3. We obtain a category $K(\mathcal{A})$, called the **homotopy category of chain complexes** in $\mathcal{A}$, whose objects are chain complexes in $\mathcal{A}$ and whose morphisms are chain homotopy classes of chain maps in $\mathcal{A}$. The isomorphisms in $K(\mathcal{A})$ are chain homotopy classes of chain homotopy equivalences.

Although the definition of a chain homotopy looks very different from that of a chain map, chain homotopies are in fact chain maps. This is analogous to homotopies between continuous maps $f, g : X \to X'$, which are defined as continuous maps $h : [0,1] \times X \to X'$ with $h \circ \iota^0 = f$ and $h \circ \iota^1 = g$ for the inclusions $\iota^i : X \to [0,1] \times X$, $x \mapsto (i, x)$.

Similarly, chain homotopies between chain maps $f_\bullet, g_\bullet : X_\bullet \to X'_\bullet$ are chain maps $k_\bullet : Y_\bullet \to X'_\bullet$ from a certain chain complex $Y_\bullet$ constructed from $X_\bullet$. The chain complex $Y_\bullet$ is equipped with inclusions $\iota^0_\bullet, \iota^1_\bullet : X_\bullet \to Y_\bullet$ such that $k_\bullet \circ \iota^0_\bullet = f_\bullet$ and $k_\bullet \circ \iota^1_\bullet = g_\bullet$. We illustrate this for the abelian category R-Mod. In Section 4.6 we will see that $Y_\bullet$ can be viewed as a tensor product of the chain complex $X_\bullet$ with a chain complex that represents the unit interval $[0, 1]$.

Remark 3.3.3: Let $(X_\bullet, d_\bullet)$, $(X'_\bullet, d'_\bullet)$ be chain complexes in R-Mod and $(Y_\bullet, d^Y_\bullet)$ given by

$$Y_n = X_n \oplus X_n \oplus X_{n-1}, \qquad d^Y_n : Y_n \to Y_{n-1},\ \ (x, x', x'') \mapsto (d_n(x) + x'', d_n(x') - x'', -d_{n-1}(x'')).$$

Then the inclusions $\iota^0_n : X_n \to Y_n$, $x \mapsto (x, 0, 0)$ and $\iota^1_n : X_n \to Y_n$, $x \mapsto (0, x, 0)$ define chain maps $\iota^0_\bullet, \iota^1_\bullet : X_\bullet \to Y_\bullet$.

Chain homotopies $h_\bullet : f_\bullet \Rightarrow g_\bullet$ between chain maps $f_\bullet, g_\bullet : X_\bullet \to X'_\bullet$ are in bijection with chain maps $k_\bullet : Y_\bullet \to X'_\bullet$ such that $k_\bullet \circ \iota^0_\bullet = f_\bullet$ and $k_\bullet \circ \iota^1_\bullet = g_\bullet$.

Proof:
A direct computation shows that $(Y_\bullet, d^Y_\bullet)$ is indeed a chain complex and the inclusions ι^0_n, ι^1_n define chain maps $\iota^0_\bullet, \iota^1_\bullet : X_\bullet \to Y_\bullet$.

By the universal property of direct sums, an R-linear map $k_n : Y_n \to X'_n$ is given by a triple (f_n, g_n, h_{n-1}) of R-linear maps $f_n, g_n : X_n \to X'_n$ and $h_{n-1} : X_{n-1} \to X'_n$ as

$$k_n(x, x', x'') = f_n(x) + g_n(x') + h_{n-1}(x'') \qquad x, x' \in X_n, x'' \in X_{n-1}.$$

This states that $k_n \circ \iota^0_n = f_n$ and $k_n \circ \iota^1_n = g_n$ for all $n \in \mathbb{Z}$.

The R-linear maps k_n define a chain map if and only if

$$\begin{aligned} &d'_n \circ f_n(x) + d'_n \circ g_n(x') + d'_n \circ h_{n-1}(x'') = d'_n \circ k_n(x, x', x'') \\ =&k_{n-1} \circ d^Y_n(x, x', x'') = f_{n-1} \circ d_n(x) + f_{n-1}(x'') + g_{n-1} \circ d_n(x') - g_{n-1}(x'') - h_{n-2} \circ d_{n-1}(x''). \end{aligned}$$

By setting $x' = x'' = 0$, $x = x'' = 0$ or $x = x' = 0$, one finds that is the case if and only if $f_\bullet$ and $g_\bullet$ are chain maps and $h_\bullet : f_\bullet \Rightarrow g_\bullet$ is a chain homotopy. $\square$

The analogy between chain complexes, chain maps and chain homotopies and topological spaces, continuous maps and homotopies also manifests itself in their homologies. Just as homotopic maps between topological spaces induce the same group homomorphisms between the homotopy groups, chain homotopic chain maps induce the same morphisms between homologies. As a consequence, chain homotopy equivalent chain complexes have isomorphic homologies. This allows one to view the homologies as functors $H_n : K(\mathcal{A}) \to \mathcal{A}$, where $K(\mathcal{A})$ is the homotopy category of chain complexes from Remark 3.3.2.

Proposition 3.3.4: Let $\mathcal{A}$ be an abelian category.

1. Chain homotopic chain maps in $\mathcal{A}$ induce the same morphisms on the homologies: if $f_\bullet \sim g_\bullet$ then $H_n(f_\bullet) = H_n(g_\bullet)$ for all $n \in \mathbb{Z}$.
2. The nth homology induces a functor $H_n : K(\mathcal{A}) \to \mathcal{A}$ for all $n \in \mathbb{Z}$.
3. Chain homotopy equivalences induce isomorphisms on the homologies: if $X_\bullet \simeq X'_\bullet$, then $H_n(X_\bullet) \cong H_n(X'_\bullet)$ for all $n \in \mathbb{Z}$.

Proof:
To prove the first claim, let $f_\bullet, g_\bullet : X_\bullet \to X'_\bullet$ be chain maps and $h_\bullet : f_\bullet \Rightarrow g_\bullet$ a chain homotopy. The morphism $H_n(f_\bullet) : H_n(X_\bullet) \to H_n(X'_\bullet)$ is defined by diagram (24) as the unique morphism

with $H_n(f_\bullet) \circ p_n = p'_n \circ \bar{f}_n$.

$$\begin{tikzcd}
\mathrm{im}(d_{n+1}) \arrow[r, hook, "\phi_n"] & \ker(d_n) \arrow[r, two heads, "p_n"] \arrow[d, hook, "\iota_n"] \arrow[ddd, dashed, bend left=60, "\exists! \bar{f}_n"] & H_n(X_\bullet) = \mathrm{coker}(\phi_n) \arrow[ddd, dashed, "\exists! H_n(f_\bullet)"] \\
X_{n+1} \arrow[u, two heads, "\pi_{n+1}"] \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] & X_n \arrow[d, "f_n"] & \\
X'_{n+1} \arrow[r, "d'_{n+1}"] \arrow[d, two heads, "\pi'_{n+1}"'] & X'_n & \\
\mathrm{im}(d'_{n+1}) \arrow[r, hook, "\phi'_n"'] & \ker(d'_n) \arrow[r, two heads, "p'_n"'] \arrow[u, hook, "\iota'_n"] & H_n(X'_\bullet) = \mathrm{coker}(\phi'_n).
\end{tikzcd}$$

As p_n is an epimorphism, it is sufficient to show that $H_n(f_\bullet - g_\bullet) \circ p_n = p'_n \circ (\bar{f}_n - \bar{g}_n) = 0$. As $h_\bullet : f_\bullet \Rightarrow g_\bullet$ is a chain homotopy, we have

$$\begin{aligned}
\iota'_n \circ (\bar{f}_n - \bar{g}_n) &= (f_n - g_n) \circ \iota_n = h_{n-1} \circ d_n \circ \iota_n + d'_{n+1} \circ h_n \circ \iota_n = h_{n-1} \circ 0 + d'_{n+1} \circ h_n \circ \iota_n \\
&= \iota'_n \circ \phi'_n \circ \pi'_{n+1} \circ h_n \circ \iota_n.
\end{aligned}$$

As ι'_n is a monomorphism, this implies $\bar{f}_n - \bar{g}_n = \phi'_n \circ \pi'_{n+1} \circ h_n \circ \iota_n$. As $p'_n : \ker(d'_n) \to H_n(X'_\bullet)$ is the cokernel of ϕ'_n, it follows that $p'_n \circ (\bar{f}_n - \bar{g}_n) = p'_n \circ \phi'_n \circ \pi'_{n+1} \circ h_n \circ \iota_n = 0 \circ \pi'_{n+1} \circ h_n \circ \iota_n = 0$.

The second claim follow from the first, and so does the third: if $f_\bullet : X_\bullet \to X'_\bullet$ and $g_\bullet : X'_\bullet \to X_\bullet$ are chain maps with $f_\bullet \circ g_\bullet \sim 1_{X'_\bullet}$ and $g_\bullet \circ f_\bullet \sim 1_{X_\bullet}$, then one has

$$\begin{aligned}
H_n(f_\bullet) \circ H_n(g_\bullet) &= H_n(f_\bullet \circ g_\bullet) = H_n(1_{X'_\bullet}) = 1_{H_n(X'_\bullet)} \\
H_n(g_\bullet) \circ H_n(f_\bullet) &= H_n(g_\bullet \circ f_\bullet) = H_n(1_{X_\bullet}) = 1_{H_n(X_\bullet)}.
\end{aligned}$$

This shows that $H_n(f_\bullet) : H_n(X_\bullet) \to H_n(X'_\bullet)$ is an isomorphism with inverse $H_n(f_\bullet)^{-1} = H_n(g_\bullet)$.
□

With Proposition 3.3.4 one can show that a chain complex $X_\bullet$ has trivial homologies without computing them explicitly. For this, it is sufficient to construct a chain homotopy between the chain maps $1_{X_\bullet} : X_\bullet \to X_\bullet$ and $0_\bullet : X_\bullet \to X_\bullet$. The chain maps $f : X_\bullet \to 0_\bullet$ and $g : 0_\bullet \to X_\bullet$ then form a chain homotopy equivalence between $X_\bullet$ and the trivial chain complex $0_\bullet$, since they satisfy $f_\bullet \circ g_\bullet = 1_{0_\bullet}$ and $g_\bullet \circ f_\bullet = 0_{X_\bullet} \sim 1_{X_\bullet}$. This implies $H_n(X_\bullet) \cong H_n(0_\bullet) = 0$ for all $n \in \mathbb{Z}$ by Proposition 3.3.4, 3.

Example 3.3.5: Let k be a commutative ring, G a group and A an algebra over k.

1. The chain complex $(X_\bullet, d_\bullet)$ in $\mathcal{A} = k$-Mod given by

$$\begin{aligned}
&X_n = \begin{cases} \langle G^{\times(n+1)} \rangle_k & n \in \mathbb{N}_0 \\ k & n = -1 \\ 0 & n < -1 \end{cases} \\
&d_n : X_n \to X_{n-1}, \quad (g_0, ..., g_n) \mapsto \Sigma_{i=0}^n (-1)^i (g_0, ..., \widehat{g_i}, ..., g_n) \quad n > 0 \\
&d_0 : X_0 \to X_{-1}, \quad (g_0) \mapsto 1
\end{aligned}$$

is exact, because the k-linear maps $h_n : X_n \to X_{n+1}$, $(g_0, ..., g_n) \mapsto (1, g_0, ..., g_n)$ define a chain homotopy $h_\bullet : 1_{X_\bullet} \Rightarrow 0_{X_\bullet}$:

$$\begin{aligned}
&(h_{n-1} \circ d_n + d_{n+1} \circ h_n)(g_0, ..., g_n) = \Sigma_{i=0}^n (-1)^i h_{n-1}(g_0, ..., \widehat{g_i}, ..., g_n) + d_{n+1}(1, g_0, ..., g_n) \\
&= \Sigma_{i=0}^n (-1)^i (1, g_0, ..., \widehat{g_i}, ..., g_n) + (g_0, ..., g_n) + \Sigma_{i=0}^n (-1)^{i+1} (1, g_0, ..., \widehat{g_i}, ..., g_n) \\
&= (g_0, ..., g_n)
\end{aligned}$$

$$(h_{-1} \circ d_0 + d_1 \circ h_0)(g_0) = h_{-1}(1) + d_1(1, g_0) = (1) + (g_0) - (1) = (g_0)$$

It is an exact chain complex in $k[G]$-Mod with the $k[G]$-module structures $g \rhd (g_0, ..., g_n) = (gg_0, ..., gg_n)$ on X_n for $n > -1$ and the trivial $k[G]$-module structure on k.

2. The chain complex $(X_\bullet, d_\bullet)$ in $\mathcal{A} = k$-Mod given by

$$X_n = \begin{cases} A^{\otimes(n+2)} & n \geq -1 \\ 0 & n < -1 \end{cases}$$
$$d_n : X_n \to X_{n-1}, \; a_0 \otimes ... \otimes a_{n+1} \mapsto \Sigma_{i=0}^n (-1)^i a_0 \otimes ... \otimes (a_i a_{i+1}) \otimes ... \otimes a_{n+1}$$

is exact, because the k-linear maps $h_n : X_n \to X_{n+1}$, $(a_0 \otimes ... \otimes a_{n+1}) \mapsto 1 \otimes a_0 \otimes ... \otimes a_{n+1}$ define a chain homotopy $h_\bullet : 1_{X_\bullet} \Rightarrow 0_{X_\bullet}$:

$$\begin{aligned}
&(h_{n-1} \circ d_n + d_{n+1} \circ h_n)(a_0 \otimes ... \otimes a_{n+1}) \\
&= \Sigma_{i=0}^n (-1)^i h_{n-1}(a_0 \otimes ... \otimes (a_i a_{i+1}) \otimes ... \otimes a_{n+1}) + d_{n+1}(1 \otimes a_0 \otimes ... \otimes a_{n+1}) \\
&= \Sigma_{i=0}^n (-1)^i 1 \otimes a_0 \otimes ... \otimes (a_i a_{i+1}) \otimes ... \otimes a_{n+1} + a_0 \otimes ... \otimes a_{n+1} \\
&+ \Sigma_{i=0}^n (-1)^{i+1} 1 \otimes a_0 \otimes ... \otimes (a_i a_{i+1}) \otimes ... \otimes a_{n+1} = a_0 \otimes ... \otimes a_{n+1}.
\end{aligned}$$

This chain complex becomes an exact chain complex in A-Mod-A with the (A, A)-bimodule structure $b \rhd (a_0 \otimes ... \otimes a_{n+1}) \lhd c = (ba_0) \otimes a_1 \otimes ... \otimes a_n \otimes (a_{n+1} c)$ on $A^{\otimes(n+2)}$.

Although these examples of chain complexes look similar to the cochain complex of group cohomology from Example 3.2.6 and to the Hochschild complex from Example 3.2.5, 4. there is a fundamental difference, namely the absence of the $k[G]$-module or (A, A)-bimodule M. In fact, the chain complexes of group cohomology and Hochschild (co)homology are obtained by applying the functor $\mathrm{Hom}(-, M)$ to this chain complex or tensoring with the module M. This will allow us to view group cohomology and Hochschild (co)homologies as the homologies of certain functors rather than homologies of chain complexes.

Important examples of chain homotopies originate in topology, and this motivates the name *chain homotopy*. Every homotopy between continuous maps induces a chain homotopy between the associated chain maps in singular homology.

Proposition 3.3.6: Let k be a commutative ring and $C_\bullet(-, k) : \mathrm{Top} \to \mathrm{Ch}_{k\text{-Mod}}$ the functor from Example 3.2.5 that assigns to a topological space X the singular chain complex $C_\bullet(X, k)$

$$C_n(X, k) = \langle \sigma : \Delta^n \to X \text{ continuous} \rangle_k \qquad d_n(\sigma) = \Sigma_{i=0}^n (-1)^i \sigma \circ f_i^n$$

and to a continuous map $f : X \to Y$ the chain map

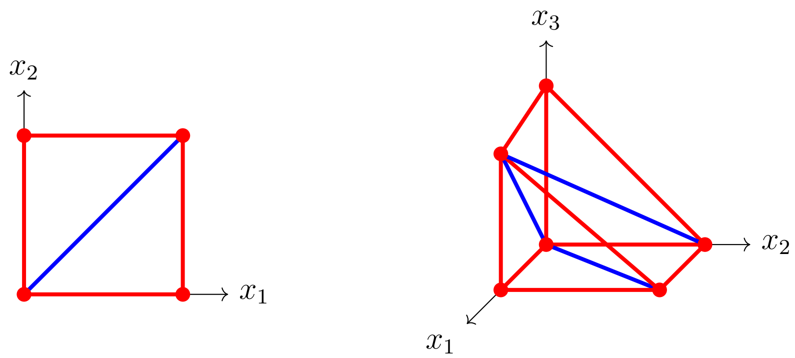
$$C_\bullet(f, k) : C_\bullet(X, k) \to C_\bullet(Y, k), \quad C_n(f, k)(\sigma) = f \circ \sigma : \Delta^n \to Y.$$

1. Every homotopy $h : f \Rightarrow g$ between continuous maps $f, g : X \to Y$ induces a chain homotopy $C_\bullet(h, k) : C_\bullet(f, k) \Rightarrow C_\bullet(g, k)$.
2. Homotopic maps $f, g : X \to Y$ induce the same morphisms between the singular homologies: $f \sim g \Rightarrow H_n(f, k) = H_n(g, k) : H_n(X, k) \to H_n(Y, k)$ for all $n \in \mathbb{Z}$.
3. Homotopy equivalent topological spaces have isomorphic singular homologies: $X \simeq Y \Rightarrow H_n(X, k) \cong H_n(Y, k)$ for all $n \in \mathbb{Z}$.

Proof:
The second and the third claim follow from the first and from Proposition 3.3.4. The chain homotopy $C_\bullet(h, k) : C_\bullet(f, k) \Rightarrow C_\bullet(g, k)$ induced by a homotopy $h : [0, 1] \times X \to Y$ from $f : X \to Y$ to $g : X \to Y$ is given by the **prism maps**, the affine linear maps

$$T_n^j : \Delta^{n+1} \to [0,1] \times \Delta^n, \quad T_n^j(e_k) = \begin{cases} (0, e_k) & 0 \leq k \leq j \leq n \\ (1, e_{k-1}) & 0 \leq j < k \leq n+1. \end{cases}$$

The prism maps have a direct geometric interpretation. They decompose the set $[0,1] \times \Delta^n$ into $(n+1)$ different $(n+1)$-simplexes.



The prism maps T_n^j for $n = 1, 2$.

A direct computation shows that the prism maps satisfy the relations

$$\begin{array}{llll} T_n^j \circ f_i^{n+1} = (\mathrm{id}_{[0,1]} \times f_i^n) \circ T_{n-1}^{j-1} & \forall j > i & T_n^j \circ f_i^{n+1} = (\mathrm{id}_{[0,1]} \times f_{i-1}^n) \circ T_{n-1}^j & \forall j < i-1 \\ T_n^i \circ f_i^{n+1} = T_n^{i-1} \circ f_i^{n+1} & \forall i \in \{1, ..., n\} & T_n^0 \circ f_0^{n+1} = i_1, \; T_n^n \circ f_{n+1}^{n+1} = i_0, & \end{array} \tag{25}$$

where $i_t : \Delta^n \to [0,1] \times \Delta^n$, $x \mapsto (t, x)$ is the inclusion map and $f_j^{n+1} : \Delta^n \to \Delta^{n+1}$ the face map from Definition 2.1.1. By composing the prism maps with the homotopy $h : [0,1] \times X \to Y$ one obtains k-linear maps

$$C_n(h, k) : C_n(X, k) \to C_{n+1}(Y, k), \quad \sigma \mapsto \Sigma_{j=0}^n (-1)^j h \circ (\mathrm{id}_{[0,1]} \times \sigma) \circ T_n^j. \tag{26}$$

A direct computation using (25) and the identities $h(0, x) = f(x)$ and $h(1, x) = g(x)$ for all $x \in X$ shows that the maps $C_n(h, k)$ define a chain homotopy $C_\bullet(h, k) : C_\bullet(f, k) \Rightarrow C_\bullet(g, k)$ (Exercise 45). □

Proposition 3.3.6 relates homotopy classes of continuous maps to chain homotopy classes of chain maps. Hence, we can view singular homology as a functor $hC_\bullet(-, k) : \mathrm{hTop} \to K(k\text{-Mod})$ from the category hTop with topological spaces as objects and homotopy classes of continuous maps as morphisms to the homotopy category $K(k\text{-Mod})$ from Remark 3.3.2, 3. with chain complexes in k-Mod as objects and chain homotopy classes of chain maps as morphisms. Denoting by $C_\bullet(-,k) : \mathrm{Top} \to \mathrm{Ch}_{k\text{-Mod}}$ the singular homology functor and by $P_{\mathrm{Ch}} : \mathrm{Ch}_{k\text{-Mod}} \to K(k\text{-Mod})$ and $P_{\mathrm{Top}} : \mathrm{Top} \to \mathrm{hTop}$ the projection functors that send each object to itself and each morphism to its homotopy class, we obtain a commuting diagram

$$\begin{tikzcd} \mathrm{Top} \arrow[r, "C_\bullet(-{,}k)"] \arrow[d, "P_{\mathrm{Top}}"] & \mathrm{Ch}_{k\text{-Mod}} \arrow[d, "P_{\mathrm{Ch}}"] \\ \mathrm{hTop} \arrow[r, "hC_\bullet(-{,}k)"'] & K(k\text{-Mod}). \end{tikzcd}$$

3.4 The long exact homology sequence

As chain complexes and chain maps in an abelian category $\mathcal{A}$ form an abelian category $\mathrm{Ch}_{\mathcal{A}}$ and the homologies are functors $H_n : \mathrm{Ch}_{\mathcal{A}} \to \mathcal{A}$, it is natural to ask if these functors are exact or, more generally, how they behave with respect to exact sequences in $\mathrm{Ch}_{\mathcal{A}}$. We will see in the following that the homology functors $H_n : \mathrm{Ch}_{\mathcal{A}} \to \mathcal{A}$ are in general *not exact*.

Instead, there is an exact sequence, the *long exact homology sequence*, that relates the homologies H_n for different $n \in \mathbb{Z}$ of an exact sequence of chain complexes. This long exact homology sequence appears in many applications in algebra and topology and is one of the most useful tools for computing homologies. To derive this result, we require the concept of a *short exact sequence* in an abelian category $\mathcal{A}$, which can be viewed as an alternative description of kernels and cokernels and the relation between them.

Definition 3.4.1: Let $\mathcal{A}$ be an abelian category. A **short exact sequence** in $\mathcal{A}$ is an exact chain complex of the form $0 \to X \xrightarrow{\iota} Y \xrightarrow{\pi} Z \to 0$.

Short exact sequences are the shortest exact sequences that carry information that cannot be stated in a much simpler way. A chain complex of the form $0 \to X \to 0$ is exact if and only if the object X is isomorphic to the zero object in $\mathcal{A}$, and a chain complex of the form $0 \to X \to Y \to 0$ is exact if and only if the morphism in the middle is an isomorphism. The information contained in short exact sequences is less trivial. The following example shows that they are generalisations of quotient modules.

Example 3.4.2: A short exact sequence $0 \to L \xrightarrow{\iota} M \xrightarrow{\pi} N \to 0$ in the abelian category $\mathcal{A} = R\text{-Mod}$ corresponds to a triple (L, M, N) such that $L \subset M$ is a submodule and $N = M/L$ the associated quotient module.

This follows because for any submodule $L \subset M$, the inclusion map $\iota : L \to M$ is a monomorphism in R-Mod and the canonical surjection $\pi : M \to M/L$ is an epimorphism in R-Mod with $\ker(\pi) = \mathrm{im}(\iota)$. Conversely, given a monomorphism $\iota : L \to M$ and an epimorphism $\pi : M \to N$ with $\ker(\pi) = \mathrm{im}(\iota)$, one has $L \cong \mathrm{im}(\iota) \subset M$ and $N \cong M/\ker(\pi) \cong M/\mathrm{im}(\iota) \cong M/L$.

Short exact sequences are important because they are the basic building blocks of abelian categories. Instead of kernels and cokernels, we can take short exact sequences as the fundamental structures that characterise abelian categories and exact functors. A functor between abelian categories is exact if and only if it sends short exact sequences to short exact sequences. Left and right exact functors satisfy similar but weaker conditions.

Lemma 3.4.3:

1. A chain complex $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in an abelian category $\mathcal{A}$ is a short exact sequence if and only if one of the following equivalent conditions holds:
   (i) f is a monomorphism and g a cokernel of f.
   (ii) g is an epimorphism and f a kernel of g.

2. An additive functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories $\mathcal{A}$, $\mathcal{B}$ is
   - left exact if and only if $0 \to F(X) \xrightarrow{F(\iota)} F(Y) \xrightarrow{F(\pi)} F(Z)$ is an exact sequence in $\mathcal{B}$ for all exact sequences $0 \to X \xrightarrow{\iota} Y \xrightarrow{\pi} Z$ in $\mathcal{A}$,
   - right exact if and only if $F(X) \xrightarrow{F(\iota)} F(Y) \xrightarrow{F(\pi)} F(Z) \to 0$ is an exact sequence in $\mathcal{B}$ for all exact sequences $X \xrightarrow{\iota} Y \xrightarrow{\pi} Z \to 0$ in $\mathcal{A}$,
   - exact if and only if $0 \to F(X) \xrightarrow{F(\iota)} F(Y) \xrightarrow{F(\pi)} F(Z) \to 0$ is an exact sequence in $\mathcal{B}$ for all short exact sequences $0 \to X \xrightarrow{\iota} Y \xrightarrow{\pi} Z \to 0$ in $\mathcal{A}$.

Proof:
1. As $i_X : 0 \to X$ is a monomorphism and hence an image of itself by Definition 3.1.10 and $t_Z : Z \to 0$ has the kernel $1_Z : Z \to Z$, we obtain the commuting diagram

$$\begin{tikzcd}
0 \arrow[rr, hook, "\phi_0"] \arrow[dr, hook] & & \ker(f) \arrow[dl, hook, "\iota_f"'] & \mathrm{im}(f) \arrow[rr, hook, "\phi_1"] \arrow[dr, hook, "\iota'_f"] & & \ker(g) \arrow[dl, hook, "\iota_g"'] & \mathrm{im}(g) \arrow[rr, hook, "\phi_2"] \arrow[dr, hook, "\iota'_g"] & & Z \arrow[dl, hook, "1_Z"'] \\
0 \arrow[r] & X \arrow[urr, two heads, "\pi'_f"'] \arrow[rrr, "f"'] & & & Y \arrow[urr, two heads, "\pi'_g"'] \arrow[rrr, "g"'] & & & Z \arrow[r] & 0
\end{tikzcd}$$

• The commuting triangle on the left shows that ϕ_0 is an isomorphism if and only if $0 \to X$ is a kernel of f, which by Lemma 3.1.9 is the case if and only if f is a monomorphism.

• The commuting triangle on the right shows that ϕ_2 is an isomorphism if and only if $1_Z : Z \to Z$ is an image of g, which is equivalent to the statement that $Z \to 0$ is a cokernel of g. By Lemma 3.1.9 this is the case if and only if $g : Y \to Z$ is an epimorphism.

• The commuting triangle in the middle states that ϕ_1 is an isomorphism if and only if the image $\iota'_f : \mathrm{im}(f) \to Y$ is a kernel of g.

If f is a monomorphism, it is an image of itself and ϕ_1 is an isomorphism if and only if $f : X \to Y$ is a kernel of $g : Y \to Z$.

Similarly, if g is an epimorphism, it is a cokernel of its kernel, and ι'_f is a kernel of g if and only if g is a cokernel of ι'_f. As $f = \iota'_f \circ \pi_f$ and π'_f is an epimorphism by Lemma 3.1.13, the morphisms ι'_f and f have the same cokernel by Exercise 27. Hence, if g is an epimorphism, then ϕ_1 is an isomorphism if and only if g is a cokernel of f.

This shows that (i) and (ii) are satisfied, if the sequence in 1. is exact, and that (i) and (ii) guarantee the exactness of the sequence.

2. The proof of 1. shows that the exactness of a sequence $0 \to X \xrightarrow{\iota} Y \xrightarrow{\pi} Z$ is equivalent to the statement that ι is a kernel of π. Thus, an additive functor F is left exact, i. e. preserves kernels, if and only if it preserves the exactness of such sequences.

Similarly, the proof of 1. shows that a sequence $X \xrightarrow{\iota} Y \xrightarrow{\pi} Z \to 0$ is exact if and only if π is a cokernel of ι. Thus, an additive functor F is right exact, i. e. preserves cokernels, if and only if it preserves the exactness of such sequences.

This also shows that an exact functor $F : \mathcal{A} \to \mathcal{B}$ preserves short exact sequences. To show the converse, suppose that F preserves short exact sequences and consider a morphism $f : X \to Y$ in $\mathcal{A}$, its kernel $\iota : \ker(f) \to X$ and its cokernel $\pi : Y \to \mathrm{coker}(f)$. By Lemma 3.1.13 we can factorise f as $f = \iota'_f \circ \pi'_f$ with a monomorphism $\iota'_f : \mathrm{im}(f) \to Y$ and an epimorphism $\pi'_f : X \to \mathrm{im}(f)$. By Exercise 27 the morphisms f and ι'_f have the same cokernel and the morphisms f and π'_f the same kernel. This yields two short exact sequences

$$0 \to \ker(f) \xrightarrow{\iota} X \xrightarrow{\pi'_f} \mathrm{im}(f) \to 0 \qquad 0 \to \mathrm{im}(f) \xrightarrow{\iota'_f} Y \xrightarrow{\pi} \mathrm{coker}(f) \to 0,$$

whose images under F are again short exact sequences. The exactness of the image of the first sequence states that $F(\pi'_f)$ is an epimorphism and $F(\iota)$ a kernel of $F(\pi'_f)$. The exactness of the image of the second sequence states that $F(\iota'_f)$ is a monomorphism and $F(\pi)$ a cokernel of $F(\iota'_f)$. This implies that $F(f) = F(\iota'_f) \circ F(\pi'_f)$ is the canonical factorisation of $F(f)$ and with Exercise 27 that $F(\iota)$ is a kernel of $F(f)$ and $F(\pi)$ a cokernel of $F(f)$. Thus, F is exact. □

Lemma 3.4.3 motivates the names left exact functor, right exact functor and exact functor. A *left exact* functor preserves exactness of a short exact sequence only in the first two objects *on the left*, whereas a *right exact* functor preserves it only it the first two objects *on the right*. The former belong to kernels and the latter to cokernels.

The alternative definition of left exact, right exact and exact functors in terms of short exact sequences has many advantages. One of them is that it is easier to combine with natural transformations than the criteria in Definition 3.1.10. In particular, it shows directly that left exactness, right exactness and exactness are preserved under natural isomorphisms.

Corollary 3.4.4: Let $F, G : \mathcal{A} \to \mathcal{B}$ be additive functors between abelian categories. If F and G are naturally isomorphic, then F is left exact, right exact or exact if and only if G is.

Proof:
Let $\eta : F \to G$ be a natural isomorphism. For every short exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in $\mathcal{A}$ the following diagram commutes by naturality of η

$$\begin{tikzcd}
0 \arrow[r] & F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X", "\cong"'] & F(Y) \arrow[r, "F(g)"] \arrow[d, "\eta_Y", "\cong"'] & F(Z) \arrow[r] \arrow[d, "\eta_Z", "\cong"'] & 0 \\
0 \arrow[r] & G(X) \arrow[r, "G(f)"] & G(Y) \arrow[r, "G(g)"] & G(Z) \arrow[r] & 0.
\end{tikzcd} \tag{27}$$

As the vertical arrows are isomorphisms, the first row is exact in $F(X)$, $F(Y)$ or $F(Z)$, respectively, if and only if the second row is exact in $G(X)$, $G(Y)$ or $G(Z)$. Hence, by Lemma 3.4.3 F is left exact, right exact or exact if and only if this holds for G. □

Lemma 3.4.3 is a strong motivation to investigate how the homology functors $H_n : \mathrm{Ch}_{\mathcal{A}} \to \mathcal{A}$ interact with short exact sequences in the category $\mathrm{Ch}_{\mathcal{A}}$. Another morivation is that short exact sequences of chain complexes arise in many applications in group and Lie algebra (co)homology, Hochschild (co)homology and singular (co)homology. Examples are group (co)homologies with coefficients in submodules and singular homologies of subspaces.

Example 3.4.5: Let k be a commutative ring.

1. Let G be a group and $0 \to L \xrightarrow{\iota} M \xrightarrow{\pi} N \to 0$ a short exact sequence of $k[G]$-modules. Then we obtain a short exact sequence of chain complexes
$$0 \to C^\bullet(G, L) \xrightarrow{C^\bullet(G,\iota)} C^\bullet(G, M) \xrightarrow{C^\bullet(G,\pi)} C^\bullet(G, N) \to 0$$
where $C^\bullet(G, L)$ and $C^\bullet(G, M)$ are the cochain complexes for group cohomology from Definition 2.3.1 and $C^\bullet(G, \iota)$ and $C^\bullet(G, \pi)$ the chain maps from Example 3.2.5, 5.

2. Let X be a topological space and $A \subset X$ a subspace. Then the inclusion map $\iota : A \to X$ defines a short exact sequence of chain complexes
$$0 \to C_\bullet(A, k) \xrightarrow{C_\bullet(\iota,k)} C_\bullet(X, k) \xrightarrow{\pi} C_\bullet(X, A, k) \to 0$$
where $C_\bullet(A, k)$ and $C_\bullet(X, k)$ are the singular chain complexes from Definition 2.1.4, $C_\bullet(\iota, k)$ the chain map from Example 3.2.5, 1. and $C_\bullet(X, A, k)$ the chain complex with $C_n(X, A, k) = C_n(X, k)/C_n(A, k)$ and $d_n : C_n(X, A, k) \to C_{n-1}(X, A, k)$, $[\sigma] \mapsto [d_n(\sigma)]$ for all singular n-simplexes $\sigma : \Delta^n \to X$.

The first example can be used to compute group (co)homologies with coefficients in submodules or quotient modules, once one is able to relate the homologies for a short exact sequence of chain complexes. This is useful since group cohomologies with coefficients in free $k[G]$-modules are often particularly simple to compute and every $k[G]$-module can be written as quotient of a free $k[G]$-module by an appropriate submodule.

The second example is relevant for the computation of singular homologies of quotient spaces. One can show that if $\emptyset \neq A \subset X$ is closed and a deformation retract of a neighbourhood of a point $x \in X$, then $H_n C_\bullet(X, A, k) \cong H_n(X/A, k)$, where X/A is the topological space obtained from X by collapsing A, see for instance [H, Theorem 2.13]. Once one is able to relate the associated homologies for a short exact sequence of chain complexes, one can then compute the homologies of the space X/A from the homologies of X and A.

We now derive the exact sequence, the *long exact homology sequence*, that relates the homologies of a short exact sequence of chain complexes. To do so, we need the following technical lemma known as the *snake lemma*. It is called *snake lemma*, because it involves a morphism that is represented by a snakelike arrow in a commuting diagram.

Lemma 3.4.6 (snake lemma): Let $\mathcal{A}$ be an abelian category and

$$\begin{tikzcd} & L \arrow[r, "f"] \arrow[d, "\alpha"] & M \arrow[r, "g"] \arrow[d, "\beta"] & N \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & L' \arrow[r, "f'"'] & M' \arrow[r, "g'"'] & N'. \end{tikzcd}$$

a commuting diagram in $\mathcal{A}$ with exact rows. Then there are unique morphisms $\bar{f}, \bar{g}$, $\bar{f}'$, $\bar{g}'$ that make the following diagram commute

$$
\begin{tikzcd}
 & \ker(\alpha) \arrow[r, dashed, "\exists!\bar{f}"] \arrow[d, hook, "\iota_\alpha"] & \ker(\beta) \arrow[r, dashed, "\exists!\bar{g}"] \arrow[d, hook, "\iota_\beta"] & \ker(\gamma) \arrow[d, hook, "\iota_\gamma"] \arrow[dddll, rounded corners, "\partial"] & \\
 & L \arrow[r, "f"] \arrow[d, "\alpha"] & M \arrow[r, "g"] \arrow[d, "\beta"] & N \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & L' \arrow[r, "f'"] \arrow[d, "\pi_\alpha"] & M' \arrow[r, "g'"] \arrow[d, "\pi_\beta"] & N' \arrow[d, "\pi_\gamma"] & \\
 & \mathrm{coker}(\alpha) \arrow[r, dashed, "\exists!\bar{f}'"'] & \mathrm{coker}(\beta) \arrow[r, dashed, "\exists!\bar{g}'"'] & \mathrm{coker}(\gamma) &
\end{tikzcd}
\tag{28}
$$

and a unique morphism $\partial : \ker(\gamma) \to \mathrm{coker}(\alpha)$, the **connecting morphism**, such that

$$\ker(\alpha) \xrightarrow{\bar{f}} \ker(\beta) \xrightarrow{\bar{g}} \ker(\gamma) \xrightarrow{\partial} \mathrm{coker}(\alpha) \xrightarrow{\bar{f}'} \mathrm{coker}(\beta) \xrightarrow{\bar{g}'} \mathrm{coker}(\gamma).$$

is exact. If $f : L \to M$ is a monomorphism, then $\bar{f} : \ker(\alpha) \to \ker(\beta)$ is a monomorphism. If $g' : M' \to N'$ is an epimorphism, then $\bar{g}' : \mathrm{coker}(\beta) \to \mathrm{coker}(\gamma)$ is an epimorphism.

Proof:
We prove the lemma for $\mathcal{A} = R$-Mod. The general claim then follows from the embedding theorem, since we can restrict attention to a small full abelian subcategory of $\mathcal{A}$. A proof that does not use the embedding theorem is given in [McL2, VIII.4, Lemma 5].

1. As the commutativity of the diagram (28) implies $f' \circ \alpha = \beta \circ f$, we have

$$\beta \circ f \circ \iota_\alpha = f' \circ \alpha \circ \iota_\alpha = f' \circ 0 = 0 \qquad \pi_\beta \circ f' \circ \alpha = \pi_\beta \circ \beta \circ f = 0 \circ f = 0,$$

and by the universal properties of $\iota_\beta : \ker(\beta) \to M$ and $\pi_\alpha : L' \to \mathrm{coker}(\alpha)$, there are unique morphisms $\bar{f} : \ker(\alpha) \to \ker(\beta)$ and $\bar{f}' : \mathrm{coker}(\alpha) \to \mathrm{coker}(\beta)$ with $\iota_\beta \circ \bar{f} = f \circ \iota_\alpha$ and $\bar{f}' \circ \pi_\alpha = \pi_\beta \circ f'$. Similarly, the commutativity of the diagram implies $g' \circ \beta = \gamma \circ g$, which yields

$$\gamma \circ g \circ \iota_\beta = g' \circ \beta \circ \iota_\beta = g' \circ 0 = 0 \qquad \pi_\gamma \circ g' \circ \beta = \pi_\gamma \circ \gamma \circ g = 0 \circ g = 0.$$

By the universal property of $\iota_\gamma : \ker(\gamma) \to N$ and $\pi_\beta : M' \to \mathrm{coker}(\beta)$, there are unique $\bar{g} : \ker(\beta) \to \ker(\gamma)$ and $\bar{g}' : \mathrm{coker}(\beta) \to \mathrm{coker}(\gamma)$ with $\iota_\gamma \circ \bar{g} = g \circ \iota_\beta$ and $\bar{g}' \circ \pi_\beta = \pi_\gamma \circ g$.

If $f : L \to M$ is a monomorphism, then for all morphisms $k : X \to \ker(\alpha)$ with $\bar{f} \circ k = 0$ we have $\iota_\beta \circ \bar{f} \circ k = f \circ \iota_\alpha \circ k = 0$. Because $f \circ \iota_\alpha$ is a monomorphism, this implies $k = 0$ and shows that $\bar{f}$ is a monomorphism as well. If $g' : M' \to N'$ is an epimorphism, then for all morphisms $k : \mathrm{coker}(\gamma) \to X$ with $k \circ \bar{g}' = 0$ we have $0 = k \circ \bar{g}' \circ \pi_\beta = k \circ \pi_\gamma \circ g$. Because $\pi_\gamma \circ g$ is an epimorphism, this implies $k = 0$ and that $\bar{g}'$ is an epimorphism.

2. We show that the first and fourth row of the diagram (28) are exact:
The commutativity of the diagram (28) implies $\iota_\gamma \circ \bar{g} \circ \bar{f} = g \circ \iota_\beta \circ \bar{f} = g \circ f \circ \iota_\alpha = 0 \circ \iota_\alpha = 0$, and because ι_γ is injective, this implies $\bar{g} \circ \bar{f} = 0$ and $\mathrm{im}(\bar{f}) \subset \ker(\bar{g})$. Conversely, for any $m \in \ker(\bar{g}) = \ker(\beta) \cap \ker(g) \subset \ker(g)$, by the exactness of the second row there is an $l \in L$ with $m = f(l)$. The commutativity of (28) implies $f' \circ \alpha(l) = \beta \circ f(l) = \beta(m) = 0$. As $f' : L' \to M'$ is a monomorphism, it follows that $\alpha(l) = 0$ and $l \in \ker(\alpha)$. Then we have $\bar{f}'(l) = f(l) = m$ and $m \in \mathrm{im}(\bar{f})$, which shows that $\ker(\bar{g}) = \mathrm{im}(\bar{f})$.

Similarly, $\bar{g}'\circ\bar{f}'\circ\pi_\alpha = \bar{g}'\circ\pi_\beta\circ f' = \pi_\gamma\circ g'\circ f' = 0$ implies $\bar{g}'\circ\bar{f}' = 0$ and $\mathrm{im}(\bar{f}') \subset \ker(\bar{g}')$ because π_α is an epimorphism. If $x \in \ker(\bar{g}') \subset \mathrm{coker}(\beta)$, there is an element $m' \in M'$ with $\pi_\beta(m') = x$, since π_β is an epimorphism, and it follows that $\pi_\gamma\circ g'(m') = \bar{g}'\circ\pi_\beta(m') = \bar{g}'(x) = 0$. This implies $g'(m') \in \ker(\pi_\gamma) = \mathrm{im}(\gamma)$. Hence, there is an element $n \in N$ with $\gamma(n) = g'(m')$, and because g is an epimorphism, an element $m \in M$ with $g(m) = n$. This implies $\gamma\circ g(m) = g'\circ\beta(m) = g'(m')$ and $m' - \beta(m) \in \ker(g') = \mathrm{im}(f')$. Hence, there is an element $l' \in L'$ with $f'(l') = m' - \beta(m)$, and $\bar{f}'(\pi_\alpha(l')) = \pi_\beta(f(l')) = \pi_\beta(m' - \beta(m)) = \pi_\beta(m') = x$. This shows that $\ker(\bar{g}') = \mathrm{im}(\bar{f}')$.

3. We construct the morphism $\partial : \ker(\gamma) \to L'/\mathrm{im}(\alpha)$:
Consider an element $n \in \ker(\gamma) \subset N$. Then by surjectivity of g there is an $m \in M$ with $g(m) = n$. For any other element $m' \in M$ with $g(m') = n$, the exactness of the second row implies $m - m' \in \ker(g) = \mathrm{im}(f)$, and there is an $l \in L$ with $m' = m + f(l)$. By commutativity of (28) we have $g'\circ\beta(m) = \gamma\circ g(m) = \gamma(n) = 0$. This shows that $\beta(m) \in \ker(g')$, and analogously $\beta(m') \in \ker(g')$. By exactness of the third row one has $\ker(g') = \mathrm{im}(f')$, and there is an $l' \in L'$ with $f'(l') = \beta(m)$. As f' is injective, this element $l' \in L$ is unique. Similarly, we obtain a unique element $l'' \in L'$ with $f(l'') = \beta(m') = \beta(m) + \beta\circ f(l) = f'(l') + f'\circ\alpha(l) = f'(l' + \alpha(l))$. As f' is injective, this implies $l'' = l' + \alpha(l)$, and $\pi_\alpha(l) = \pi_\alpha(l')$. We obtain a well-defined map

$$\partial : \ker(\gamma) \to L'/\mathrm{im}(\alpha), \quad n \mapsto \pi_\alpha(l') \quad \text{where} \quad n = g(m), f'(l') = \beta(m). \tag{29}$$

It is R-linear by construction, since $n = g(m)$, $f(l') = \beta(m)$ and $n' = g(m')$, $f(l'') = \beta(m')$ imply $n + n' = g(m + m')$ and $f(l' + l'') = \beta(m + m')$ and hence $\partial(n + n') = \pi_\alpha(l' + l'') = \pi_\alpha(l') + \pi_\alpha(l'') = \partial(n) + \partial(n')$.

4. The connecting homomorphism in (29) yields a sequence

$$\ker(\alpha) \xrightarrow{\bar{f}} \ker(\beta) \xrightarrow{\bar{g}} \ker(\gamma) \xrightarrow{\partial} L'/\mathrm{im}(\alpha) \xrightarrow{\bar{f}'} M'/\mathrm{im}(\beta) \xrightarrow{\bar{g}'} N'/\mathrm{im}(\gamma).$$

which is already exact in all entries except $\ker(\gamma)$ and $L'/\mathrm{im}(\alpha)$. To show the exactness in $\ker(\gamma)$, consider an element $n \in \mathrm{im}(\bar{g})$. Then there is an element $m \in \ker(\beta)$ with $n = g(m)$, and (29) yields an element $l' \in L$ with $f'(l') = \beta(m) = 0$. The injectivity of f' implies $l' = 0$, and by definition of the connecting homomorphism in (29) we have $\partial(n) = 0$. Hence, $\mathrm{im}(\bar{g}) \subset \ker(\partial)$.

Conversely, if $n \in \ker(\partial)$, then by (29) there are elements $m \in M$, $l' \in L'$ with $n = g(m)$, $\beta(m) = f'(l')$ and $\pi_\alpha(l') = 0$. This implies $l' \in \ker(\pi_\alpha) = \mathrm{im}(\alpha)$, and there is an $l \in L$ with $l' = \alpha(l)$. This yields $\beta(m) = f'\circ\alpha(l) = \beta\circ f(l)$, and hence $m - f(l) \in \ker(\beta)$. By exactness of the second row of (28), this yields $g(m - f(l)) = g(m) - g\circ f(l) = g(m) = n$, and we have $n \in g(\ker(\beta)) = \mathrm{im}(\bar{g})$. This shows that $\ker(\partial) \subset \mathrm{im}(\bar{g})$ and proves exactness in $\ker(\gamma)$.

The proof of the exactness in $\mathrm{coker}(\alpha) = L'/\mathrm{im}(\alpha)$ is analogous. If $x \in \mathrm{im}(\partial)$, then by definition of ∂ in (29), there are elements $m \in M$ and $l' \in L'$ with $x = \pi_\alpha(l')$ and $f'(l') = \beta(m)$. This implies $\bar{f}'(x) = \bar{f}'\circ\pi_\alpha(l') = \pi_\beta\circ f'(l') = \pi_\beta\circ\beta(m) = 0$ and $x \in \ker(\bar{f}')$. Conversely, if $x \in \ker(\bar{f}')$, then there is an $l' \in L'$ with $x = \pi_\alpha(l')$ and $\pi_\beta\circ f'(l') = \bar{f}'\circ\pi_\alpha(l') = \bar{f}'(x) = 0$. Hence, we have $f'(l') \in \ker(\pi_\beta) = \mathrm{im}(\beta)$, and there is an $m \in M$ with $f'(l') = \beta(m)$. This implies $\partial(g(m)) = \pi_\alpha(l') = x$ by definition of the connecting morphism in (29), and hence $x \in \mathrm{im}(\partial)$. This shows that $\mathrm{im}(\partial) = \ker(\bar{f}')$ and proves the exactness in $\mathrm{coker}(\alpha) = L'/\mathrm{im}(\alpha)$.
□

By applying the snake lemma, we can determine the image of a short exact sequence of chain complexes in $\mathcal{A}$ under the homology functors $H_n : \mathrm{Ch}_\mathcal{A} \to \mathcal{A}$. Although the individual homology functors H_n are not exact, it turns out that the homologies for different $n \in \mathbb{Z}$ can be combined into an exact sequence in $\mathcal{A}$.

Theorem 3.4.7: Every short exact sequence of chain complexes $0 \to L_\bullet \xrightarrow{f_\bullet} M_\bullet \xrightarrow{g_\bullet} N_\bullet \to 0$ in an abelian category $\mathcal{A}$ induces an exact sequence

$$\ldots \xrightarrow{H_{n+1}(g_\bullet)} H_{n+1}(N_\bullet) \xrightarrow{\partial_{n+1}} H_n(L_\bullet) \xrightarrow{H_n(f_\bullet)} H_n(M_\bullet) \xrightarrow{H_n(g_\bullet)} H_n(N_\bullet) \xrightarrow{\partial_n} H_{n-1}(L_\bullet) \xrightarrow{H_{n-1}(f_\bullet)} \ldots,$$

the **long exact homology sequence**. The morphisms $\partial_n : H_n(N_\bullet) \to H_{n-1}(L_\bullet)$ are called **connecting morphisms**.

Proof:
We prove the theorem for $\mathcal{A} = R$-Mod. The general case then follows with the embedding theorem. For all $n \in \mathbb{Z}$ the short exact sequence defines a commuting diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & L_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "d^L_{n+1}"] & M_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "d^M_{n+1}"] & N_{n+1} \arrow[r] \arrow[d, "d^N_{n+1}"] & 0 \\ 0 \arrow[r] & L_n \arrow[r, "f_n"'] & M_n \arrow[r, "g_n"'] & N_n \arrow[r] & 0. \end{tikzcd}$$

From the snake lemma, we obtain unique morphisms

$\bar{f}_{n+1} : \ker(d^L_{n+1}) \to \ker(d^M_{n+1}),\ l \mapsto f_{n+1}(l),$ $\qquad \bar{g}_{n+1} : \ker(d^M_{n+1}) \to \ker(d^N_{n+1}),\ m \mapsto g_{n+1}(m),$
$\bar{f}'_n : \mathrm{coker}(d^L_{n+1}) \to \mathrm{coker}(d^M_{n+1}),\ [l] \mapsto [f_n(l)],$ $\qquad \bar{g}'_n : \mathrm{coker}(d^M_{n+1}) \to \mathrm{coker}(d^N_{n+1}),\ [m] \mapsto [g_n(m)]$

for which the diagram

$$\begin{tikzcd} & \ker(d^L_{n+1}) \arrow[r, dashed, "\exists! \bar{f}_{n+1}"] \arrow[d, hook, "\iota^L_{n+1}"] & \ker(d^M_{n+1}) \arrow[r, dashed, "\exists! \bar{g}_{n+1}"] \arrow[d, hook, "\iota^M_{n+1}"] & \ker(d^N_{n+1}) \arrow[d, hook, "\iota^N_{n+1}"] & \\ 0 \arrow[r] & L_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "d^L_{n+1}"] & M_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "d^M_{n+1}"] & N_{n+1} \arrow[r] \arrow[d, "d^N_{n+1}"] & 0 \\ 0 \arrow[r] & L_n \arrow[r, "f_n"'] \arrow[d, "\pi^L_n"] & M_n \arrow[r, "g_n"'] \arrow[d, "\pi^M_n"] & N_n \arrow[r] \arrow[d, "\pi^N_n"] & 0 \\ & \mathrm{coker}(d^L_{n+1}) \arrow[r, dashed, "\exists! \bar{f}'_n"'] & \mathrm{coker}(d^M_{n+1}) \arrow[r, dashed, "\exists! \bar{g}'_n"'] & \mathrm{coker}(d^N_{n+1}) & \end{tikzcd}$$

commutes and a unique morphism $\partial_{n+1} : \ker(d^N_{n+1}) \to \mathrm{coker}(d^L_{n+1})$ that makes the sequence

$$\ker(d^L_{n+1}) \xrightarrow{\bar{f}_{n+1}} \ker(d^M_{n+1}) \xrightarrow{\bar{g}_{n+1}} \ker(d^N_{n+1}) \xrightarrow{\partial_{n+1}} \mathrm{coker}(d^L_{n+1}) \xrightarrow{\bar{f}'_n} \mathrm{coker}(d^M_{n+1}) \xrightarrow{\bar{g}'_n} \mathrm{coker}(d^N_{n+1})$$

exact. As f_{n+1} is a monomorphism and g_n an epimorphism, the morphism $\bar{f}_{n+1}$ is a monomorphism and the morphism $\bar{g}'_n$ an epimorphism for all $n \in \mathbb{Z}$.

The morphisms $\bar{d}^X_n : \mathrm{coker}(d^X_{n+1}) \to \ker(d^X_{n-1}),\ [x] \mapsto d_n(x)$ induced by $d^X_n : X_n \to X_{n-1}$ and the universal property of the quotient $\mathrm{coker}(d^X_{n+1}) = X_n/\mathrm{im}(d^X_{n+1})$ satisfy the identities

$$\bar{f}_{n-1} \circ \bar{d}^L_n([l]) = \bar{f}_{n-1} \circ d^L_n(l) = d^M_n \circ f_n(l) = \bar{d}^M_n([f_n(l)]) = \bar{d}^M_n \circ \bar{f}'_n([l])$$
$$\bar{g}_{n-1} \circ \bar{d}^M_n([m]) = \bar{g}_{n-1} \circ d^M_n(m) = d^N_n \circ g_n(m) = \bar{d}^N_n([g_n(m)]) = \bar{d}^N_n \circ \bar{g}'_n([m])$$

for all $l \in L_n$, $m \in M_n$, and hence we obtain the following commuting diagram with exact rows

$$\begin{tikzcd} & \mathrm{coker}(d^L_{n+1}) \arrow[r, "\bar{f}'_n"] \arrow[d, "\bar{d}^L_n"] & \mathrm{coker}(d^M_{n+1}) \arrow[r, "\bar{g}'_n"] \arrow[d, "\bar{d}^M_n"] & \mathrm{coker}(d^N_{n+1}) \arrow[r] \arrow[d, "\bar{d}^N_n"] & 0 \\ 0 \arrow[r] & \ker(d^L_{n-1}) \arrow[r, "\bar{f}_{n-1}"] & \ker(d^M_{n-1}) \arrow[r, "\bar{g}_{n-1}"] & \ker(d^N_{n-1}). & \end{tikzcd}$$

We construct the long exact homology sequence by applying the snake lemma to this diagram for all $n \in \mathbb{Z}$. For this, note that the homologies of the three chain complexes are given by

$$H_n(X_\bullet) = \ker(d_n^X)/\mathrm{im}(d_{n+1}^X) = \ker(\bar{d}_n^X) = \mathrm{coker}(\bar{d}_{n+1}^X)$$

and satisfy the conditions

$$\begin{aligned} &\iota_n^M \circ H_n(f_\bullet) = \bar{f}_n' \circ \iota_n^L && \iota_n^N \circ H_n(g_\bullet) = \bar{g}_n' \circ \iota_n^M \\ &H_{n-1}(f_\bullet) \circ \pi_{n-1}^L = \pi_{n-1}^M \circ \bar{f}_{n-1} && H_{n-1}(g_\bullet) \circ \pi_{n-1}^M = \pi_{n-1}^N \circ \bar{g}_{n-1} \end{aligned}$$

where $\iota_n^X : \ker(d_n^X)/\mathrm{im}(d_{n+1}^X) \to X_n/\mathrm{im}(d_{n+1}^X)$ and $\pi_{n-1}^X : \ker(d_{n-1}^X) \to \ker(d_{n-1}^X)/\mathrm{im}(d_n^X)$ are the inclusions and canonical surjections. The snake lemma then yields for all $n \in \mathbb{Z}$ commuting diagrams (31) and unique morphisms $\partial_n : H_n(N_\bullet) \to H_{n-1}(L_\bullet)$ such that the sequences

$$H_n(L_\bullet) \xrightarrow{H_n(f_\bullet)} H_n(M_\bullet) \xrightarrow{H_n(g_\bullet)} H_n(N_\bullet) \xrightarrow{\partial_n} H_{n-1}(L_\bullet) \xrightarrow{H_{n-1}(f_\bullet)} H_{n-1}(M_\bullet) \xrightarrow{H_{n-1}(g_\bullet)} H_{n-1}(N_\bullet)$$

are exact. Combining these exact sequences for different $n \in \mathbb{Z}$ yields the long exact homology sequence. Explicitly, the connecting morphism $\partial_k : H_k(N_\bullet) \to H_{k-1}(L_\bullet)$ is given by

$$\partial_k([n]) = [l] \ \text{ where } \ n = g_k(m), \ f_{k-1}(l) = d_k^M(m) \ \text{ for some } \ m \in M_k, l \in Z_{k-1}(L_\bullet). \tag{30}$$

$$\begin{tikzcd} & H_n(L_\bullet) \arrow[r, "H_n(f_\bullet)"] \arrow[d, hook, "\iota_n^L"] & H_n(M_\bullet) \arrow[r, "H_n(g_\bullet)"] \arrow[d, hook, "\iota_n^M"] & H_n(N_\bullet) \arrow[d, hook, "\iota_n^N"] \arrow[dddll, rounded corners, "\partial_n"] & \\ & \mathrm{coker}(d_{n+1}^L) \arrow[r, "\bar{f}_n'"] \arrow[d, "\bar{d}_n^L"] & \mathrm{coker}(\bar{d}_n^M) \arrow[r, "\bar{g}_n'"] \arrow[d, "\bar{d}_n^M"] & \mathrm{coker}(\bar{d}_n^N) \arrow[r] \arrow[d, "\bar{d}_n^N"] & 0 \\ 0 \arrow[r] & \ker(d_{n-1}^L) \arrow[r, "\bar{f}_{n-1}"] \arrow[d, "\pi_{n-1}^L"] & \ker(d_{n-1}^M) \arrow[r, "\bar{g}_{n-1}"] \arrow[d, "\pi_{n-1}^M"] & \ker(d_{n-1}^L) \arrow[d, "\pi_{n-1}^N"] & \\ & H_{n-1}(L_\bullet) \arrow[r, "H_{n-1}(f_\bullet)"'] & H_{n-1}(M_\bullet) \arrow[r, "H_{n-1}(g_\bullet)"'] & H_{n-1}(N_\bullet) & \end{tikzcd} \tag{31}$$

□

As the connecting morphism is constructed implicitly by a diagram chase, it does not appear very intuitive at first. Nevertheless, it has nice properties and a conceptual interpretation. In particular, it is compatible with chain maps between short exact sequences of chain complexes. As a consequence, every chain map between short exact sequences in $\mathrm{Ch}_\mathcal{A}$ defines a chain map between the associated long exact homology sequences in $\mathcal{A}$.

Theorem 3.4.8: Let $0 \to L_\bullet \xrightarrow{f_\bullet} M_\bullet \xrightarrow{g_\bullet} N_\bullet \to 0$ und $0 \to L_\bullet' \xrightarrow{f_\bullet'} M_\bullet' \xrightarrow{g_\bullet'} N_\bullet' \to 0$ be short exact sequences of chain complexes in an abelian category $\mathcal{A}$ and $\alpha_\bullet : L_\bullet \to L_\bullet'$, $\beta_\bullet : M_\bullet \to M_\bullet'$ and $\gamma_\bullet : N_\bullet \to N_\bullet'$ chain maps such that the following diagram in $\mathrm{Ch}_\mathcal{A}$ commutes

$$\begin{tikzcd} 0 \arrow[r] & L_\bullet \arrow[r, "f_\bullet"] \arrow[d, "\alpha_\bullet"] & M_\bullet \arrow[r, "g_\bullet"] \arrow[d, "\beta_\bullet"] & N_\bullet \arrow[r] \arrow[d, "\gamma_\bullet"] & 0 \\ 0 \arrow[r] & L_\bullet' \arrow[r, "f_\bullet'"] & M_\bullet' \arrow[r, "g_\bullet'"] & N_\bullet' \arrow[r] & 0. \end{tikzcd} \tag{32}$$

Then we obtain the following commuting diagram with exact rows

$$\begin{tikzcd}[column sep=small]
\ldots \arrow[r, "\partial_{n+1}"] & H_n(L_\bullet) \arrow[r, "H_n(f_\bullet)"] \arrow[d, "H_n(\alpha_\bullet)"] & H_n(M_\bullet) \arrow[r, "H_n(g_\bullet)"] \arrow[d, "H_n(\beta_\bullet)"] & H_n(N_\bullet) \arrow[r, "\partial_n"] \arrow[d, "H_n(\gamma_\bullet)"] & H_{n-1}(L_\bullet) \arrow[r, "H_{n-1}(f_\bullet)"] \arrow[d, "H_{n-1}(\alpha_\bullet)"] & H_{n-1}(M_\bullet) \arrow[r, "H_{n-1}(g_\bullet)"] \arrow[d, "H_{n-1}(\beta_\bullet)"] & H_{n-1}(N_\bullet) \arrow[r, "\partial_{n-1}"] \arrow[d, "H_{n-2}(\gamma_\bullet)"] & \ldots \\
\ldots \arrow[r, "\partial'_{n+1}"] & H_n(L'_\bullet) \arrow[r, "H_n(f'_\bullet)"] & H_n(M'_\bullet) \arrow[r, "H_n(g'_\bullet)"] & H_n(N'_\bullet) \arrow[r, "\partial'_n"] & H_{n-1}(L'_\bullet) \arrow[r, "H_{n-1}(f'_\bullet)"] & H_{n-1}(M'_\bullet) \arrow[r, "H_{n-1}(g'_\bullet)"] & H_{n-1}(N'_\bullet) \arrow[r, "\partial'_{n-1}"] & \ldots
\end{tikzcd}$$

where $\partial_n : H_n(N_\bullet) \to H_{n-1}(L_\bullet)$ and $\partial'_n : H_n(N'_\bullet) \to H_{n-1}(L'_\bullet)$ are the connecting morphisms.

Proof:
The squares in the diagram that do not involve the connecting morphisms commute by the commutativity of (32) and because the homologies are functors. It is sufficient to show that

$$\begin{tikzcd}
H_k(N_\bullet) \arrow[r, "\partial_k"] \arrow[d, "H_k(\gamma_\bullet)"'] & H_{k-1}(L_\bullet) \arrow[d, "H_{k-1}(\alpha_\bullet)"] \\
H_k(N'_\bullet) \arrow[r, "\partial'_k"'] & H_{k-1}(L'_\bullet)
\end{tikzcd}$$

commutes for all $k \in \mathbb{Z}$. We prove this for $\mathcal{A} = R$-mod. The general proof follows from the embedding theorem. By (30) the connecting morphisms $\partial_k : H_k(N_\bullet) \to H_{k-1}(L_\bullet)$ are characterised by the condition

$$\partial_k([n]) = [l] \;\text{ where }\; n = g_k(m),\; f_{k-1}(l) = d_k^M(m) \;\text{ for some }\; m \in M_k, l \in Z_{k-1}(L_\bullet). \tag{33}$$

Together with the commutativity of (32) and the fact that $\beta_\bullet$ is a chain map this implies

$$\begin{aligned}
&\gamma_k(n) \overset{(33)}{=} \gamma_k \circ g_k(m) = g'_k \circ \beta_k(m) \\
&d_k^{M'} \circ \beta_k(m) = \beta_{k-1} \circ d_k^M(m) \overset{(33)}{=} \beta_{k-1} \circ f_{k-1}(l) = f'_{k-1} \circ \alpha_{k-1}(l).
\end{aligned} \tag{34}$$

By definition of the connecting morphism ∂'_k we then have for all $[n] \in H_k(N_\bullet)$ and $k \in \mathbb{Z}$

$$\partial'_k \circ H_k(\gamma_\bullet)([n]) = \partial'_k([\gamma_k(n)]) \overset{(34)}{=} [\alpha_{k-1}(l)] \overset{(33)}{=} H_{k-1}(\alpha_\bullet) \circ \partial_k([n]). \qquad \square$$

Remark 3.4.9: The result in Theorem 3.4.8 is sometimes called the **naturality of the connecting morphism** and allows one to view the connecting morphisms as natural transformations between certain functors.

Denote by $\mathrm{Sh}_{\mathrm{Ch}_\mathcal{A}}$ the full subcategory of $\mathrm{Ch}_{\mathrm{Ch}_\mathcal{A}}$ whose objects are short exact sequences in $\mathrm{Ch}_\mathcal{A}$. Then Theorem 3.4.8 states that the connecting morphisms define natural transformations $\partial_n : H_n^3 \to H_{n-1}^1$ between the functors $H_n^3 : \mathrm{Sh}_{\mathrm{Ch}_\mathcal{A}} \to \mathcal{A}$ and $H_{n-1}^1 : \mathrm{Sh}_{\mathrm{Ch}_\mathcal{A}} \to \mathcal{A}$ that assign to a short exact sequence $0 \to L_\bullet \to M_\bullet \to N_\bullet \to 0$ in $\mathrm{Ch}_\mathcal{A}$ the homologies $H_n(N_\bullet)$ and $H_{n-1}(L_\bullet)$, respectively, and to a triple of chain maps as in (32) the morphisms $H_n(\gamma_\bullet) : H_n(N_\bullet) \to H_n(N'_\bullet)$ and $H_{n-1}(\alpha_\bullet) : H_{n-1}(L_\bullet) \to H_{n-1}(L'_\bullet)$.

4 Derived functors and (co)homologies

By comparing the results from Section 3 with the examples of (co)homologies in Section 2, we see that the concrete approach from Section 2 has certain drawbacks and difficulties. Although it was shown in Proposition 3.3.4 that the homologies of a chain complex depend only on its *chain homotopy equivalence class*, all definitions of (co)homology theories in Section 2 involved *concrete choices of chain complexes*. As it is difficult to see if two given chain complexes are chain homotopy equivalent, the consequences of these choices are hard to control.

There could be other chain complexes that are chain homotopy equivalent to the singular chain complex, the Hochschild (co)complex or the chain complexes for group and Lie algebra cohomology in Section 2 and give rise to much simpler computations of (co)homologies. In fact, it is nearly impossible to compute Hochschild (co)homologies from their definition in Section 2 for any non-trivial examples. With a better understanding of the possible choices of chain complexes one can compute (co)homologies more efficiently and treat more examples.

The concrete definitions in Section 2 are also conceptually unpleasant because they involve arbitrary choices. The (co)homologies in Section 2 encoded properties of an object in a certain category, namely a topological space, a bimodule over an algebra, a module over a group ring or a Lie algebra representation. However, these (co)homologies were defined as (co)homologies of certain (co)chain complexes constructed from this object. It is not clear if there are alternative choices of chain complexes for each of these objects that would define other (co)homologies.

Moreover, the definitions of (co)homology theories in Section 2 did not systematically incorporate *morphisms* into the picture. We showed in Example 3.2.5 that morphisms such as continuous maps between topological spaces, morphisms of bimodules over an algebra or morphisms of modules over group rings or morphisms of Lie algebra representations give rise to chain maps between the associated chain complexes. It was also shown in Example 3.2.5 that these assignments define functors. However, it is not clear if this is a general pattern. We do not have a general formalism that systematically assigns chain complexes to objects in a category $\mathcal{C}$ and chain maps to morphisms between them.

In this section, we develop a more systematic approach that associates homologies not to chain complexes in abelian categories but to certain *additive functors* between abelian categories. The idea is the following:

- Associate to each object A in an abelian category $\mathcal{A}$ an *exact* chain complex $A_\bullet$, a so-called *resolution* of A. Suppose that the resolutions $A_\bullet$ are unique up to chain homotopy equivalence and constructed in such a way that any morphism $f : A \to A'$ in $\mathcal{A}$ extends to a chain map $f_\bullet : A_\bullet \to A'_\bullet$ that is unique up to chain homotopy.

- Apply an additive functor $F : \mathcal{A} \to \mathcal{B}$ into an abelian category $\mathcal{B}$ to the resolutions of objects in $\mathcal{A}$ and chain maps between them. This associates to each object A in $\mathcal{A}$ a chain complex $F(A_\bullet)$ that is again unique up to chain homotopy equivalence. To each morphism $f : A \to A'$ it associates a chain map $F(f_\bullet) : F(A_\bullet) \to F(A'_\bullet)$ that is unique up to chain homotopy.

- The homologies $H_nF(A_\bullet)$ depend only on the chain homotopy equivalence class of $F(A_\bullet)$ by Proposition 3.3.4 and Exercise 47. Hence, they are independent of the choice of the resolution $A_\bullet$ and depend only on the object A.

- The chain maps $F(f_\bullet) : F(A_\bullet) \to F(A'_\bullet)$ that extend morphisms $f : A \to A'$ in $\mathcal{A}$ induce morphisms $H_n F(f_\bullet) : H_n F(A_\bullet) \to H_n F(A'_\bullet)$ in $\mathcal{B}$ that depend only on the chain homotopy class of $F(f_\bullet)$ by Proposition 3.3.4. Hence, they depend only on the morphism $f : A \to A'$ and not on the choice of the chain map $f_\bullet : A_\bullet \to A'_\bullet$.
- If this construction is compatible with the composition of morphisms in $\mathcal{A}$, it defines a collection of functors $H_n F : \mathcal{A} \to \mathcal{B}$ that send an object A to the homology $H_n F(A_\bullet)$ and a morphism $f : A \to A'$ to the morphism $H_n F(f_\bullet) : H_n F(A_\bullet) \to H_n F(A'_\bullet)$.

We will see in the following that most examples from Section 2, namely Hochschild (co)homologies, group cohomologies and cohomologies of Lie algebras can be realised as homologies of functors in this way. This viewpoint has several advantages:

- It does not depend on non-canonical choices of chain complexes to define (co)homologies but defines them *intrinsically*, as homologies of functors.
- It brings different notions of homology and cohomology into a common framework and hence allows one to investigate them more systematically.
- It allows one to compute (co)homologies more efficiently through the choice of appropriate resolutions. This is a major advantage since the definitions in Section 2 can lead to very cumbersome computations.

4.1 Resolutions

To determine under which assumptions this idea can be implemented, we investigate the existence and uniqueness of resolutions and the extension of morphisms to chain maps between them. In this, we restrict attention to (co)chain complexes that are bounded below, like the (co)chain complexes from Section 2. More specifically, in agreement with the standard conventions in the literature, we require that all objects with index < -1 are zero objects and that the objects under consideration occur in the (co)chain complex with index -1.

Definition 4.1.1: Let $\mathcal{A}$ be an abelian category and A an object in $\mathcal{A}$.

1. A **left resolution** of A is an exact sequence in $\mathcal{A}$ of the form

$$\ldots \xrightarrow{d_3} A_2 \xrightarrow{d_2} A_1 \xrightarrow{d_1} A_0 \xrightarrow{d_0} A \to 0$$

2. A **right resolution** of A is an exact sequence in $\mathcal{A}$ of the form

$$0 \to A \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \xrightarrow{d^2} \cdots$$

A left or right resolution is called **projective** or **injective** if A_n or A^n is projective or injective for all $n \in \mathbb{N}_0$. If $\mathcal{A} = R$-Mod, then a left or right resolution is called **free** if A_n or A^n is a free R-module for all $n \in \mathbb{N}_0$ and **flat** if A_n or A^n is a flat R-module for all $n \in \mathbb{N}_0$.

Example 4.1.2: (**Bar resolution for group (co)homology**)

Let k be a commutative ring and G a group. Consider the chain complex $X_\bullet$ in $k[G]$-Mod with

$$X_n = \begin{cases} \langle G^{\times n}\rangle_{k[G]} & n \geq 0 \\ k & n = -1 \end{cases}$$
$$d_n(g_1, ..., g_n) = g_1 \rhd (g_2, ..., g_n) + \Sigma_{i=1}^{n-1}(-1)^i(g_1, ..., g_i g_{i+1}, ..., g_n) + (-1)^n(g_1, ..., g_{n-1})$$
$$d_0(g_1) = 1,$$

where k carries the trivial $k[G]$-module structure and $X_n = 0$ for $n < -1$.

This chain complex is exact, since the k-linear (but not $k[G]$-linear) maps

$$h_{-1} : k \to k[G], \quad 1 \mapsto 1, \qquad h_n : X_n \to X_{n+1}, \quad g_0 \rhd (g_1, ..., g_n) \mapsto (g_0, ..., g_n) \quad \text{for} \quad n \in \mathbb{N}_0$$

define a chain homotopy $h_\bullet : 1_{X_\bullet} \Rightarrow 0_{X_\bullet}$ in k-Mod. As the homologies of $X_\bullet$ as a chain complex in k-Mod and $k[G]$-Mod are the same, $X_\bullet$ is also exact in $k[G]$-Mod.

This shows that $X_\bullet$ is a free and hence projective left resolution of the trivial $k[G]$-module k in $k[G]$-Mod, the **bar resolution** of k.

Example 4.1.3: (**Hochschild resolution**)

Let k be a commutative ring and A an algebra over k. We consider the chain complex $X_\bullet$ in A-Mod-A from Example 3.3.5, 2. with

$$X_n = A^{\otimes(n+2)} \qquad n \geq -1$$
$$d_n : X_n \to X_{n-1}, \quad a_0 \otimes .... \otimes a_{n+1} \mapsto \Sigma_{i=0}^{n}(-1)^i a_0 \otimes ... \otimes (a_i a_{i+1}) \otimes ... \otimes a_{n+1},$$

$X_n = 0$ for $n < -1$ and the (A, A)-bimodule or $A \otimes A^{op}$-module structure

$$\rhd : A \otimes A^{op} \times X_n \to X_n, \qquad (b \otimes c) \rhd a_0 \otimes .... \otimes a_{n+1} = (ba_0) \otimes a_1 \otimes ... \otimes a_n \otimes (a_{n+1}c)$$
$$\rhd : A \otimes A^{op} \times X_{-1} \to X_{-1}, \quad (b \otimes c) \rhd a_0 = ba_0c.$$

This chain complex is exact, because the k-linear maps

$$h_n : X_n \to X_{n+1}, \quad a_0 \otimes ... \otimes a_{n+1} \mapsto 1 \otimes a_0 \otimes ... \otimes a_{n+1}$$

define a chain homotopy $1_{X_\bullet} \Rightarrow 0_{X_\bullet}$ in k-Mod by Example 3.3.5. As the homologies of $X_\bullet$ in k-Mod and in A-Mod-A are the same, $X_\bullet$ is also exact as a chain complex in A-Mod-A.

This shows that $X_\bullet$ is a resolution of A in A-Mod-A, the **Hochschild resolution** of A. As (A, A)-bimodule structures on a k-module M are in bijection with $A \otimes A^{op}$-module structures via $a \rhd m \lhd b = (a \otimes b) \rhd m$, we can also view this as a resolution of A in $A \otimes A^{op}$-Mod.

Example 4.1.4: (**Chevalley-Eilenberg resolution for Lie algebra cohomology**)

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$. Then $\Lambda^n\mathfrak{g}$ is a free $U(\mathfrak{g})$-module with the module structure induced by the adjoint representation $\mathrm{ad}: \mathfrak{g} \to \mathrm{End}_{\mathbb{F}}(\Lambda^n\mathfrak{g})$, $\mathrm{ad}(x)y = [x, y]$. The chain complex $X_\bullet$ in $U(\mathfrak{g})$-Mod with

$$\begin{aligned} &X_n = U(\mathfrak{g})\otimes\Lambda^n\mathfrak{g}, \quad X_{-1} = \mathbb{F} \\ &d_n(y\otimes x_1 \wedge ... \wedge x_n) = \Sigma_{i=1}^n (-1)^{i+1}\, yx_i \otimes x_1 \wedge ... \wedge \widehat{x_i} \wedge ... \wedge x_n \\ &\qquad\qquad + \Sigma_{1\leq i<j\leq n}(-1)^{i+j}\, y\otimes[x_i, x_j] \wedge x_1 \wedge ... \wedge \widehat{x_i} \wedge ... \wedge \widehat{x_j} \wedge ... \wedge x_n \end{aligned}$$

and $U(\mathfrak{g})$-module structure $\rhd : U(\mathfrak{g}) \times X_n \to X_n$, $z \rhd (y\otimes x_1 \wedge ... \wedge x_n) = (zy)\otimes x_1 \wedge ... \wedge x_n$ is a projective left resolution of the trivial $U(\mathfrak{g})$-module $\mathbb{F}$ in $U(\mathfrak{g})$-Mod. The proof of this statement can be found in [HS, VII.4].

Example 4.1.5: If R is a principal ideal domain, every finitely generated R-module N has a free left resolution that is a short exact sequence $0 \to L \xrightarrow{\iota} M \xrightarrow{\pi} N \to 0$.

This follows because the classification theorem for finitely generated modules over principal ideal domains states that N is of the form $N \cong R^n \times R/q_1R \times ... \times R/q_kR$ with $n, k \in \mathbb{N}_0$ and prime powers $q_i \in R$. We can choose the free R-modules $L = R^k$ and $M = R^{n+k}$ and

$$\begin{aligned} &\iota : L \to M,\ (r_1, ..., r_k) \mapsto (0, ..., 0, q_1r_1, .., q_kr_k) \\ &\pi : R^{n+k} \to N,\ (r_1, ..., r_{n+k}) \mapsto (r_1, ..., r_n, \bar{r}_{n+1}, ..., \bar{r}_{n+k}). \end{aligned}$$

To implement the idea outlined at the beginning of this section, we need resolutions that allow us to extend every morphism between objects to a chain map between their resolutions. We determine for which resolutions this is possible.

For this we consider two objects A, A' in $\mathcal{A}$ with left resolutions $A_\bullet$, $A'_\bullet$ and a morphism $f : A \to A'$. The aim is to extend $f : A \to A'$ to a chain map $f_\bullet : A_\bullet \to A'_\bullet$ with $f_{-1} = f$

$$\begin{tikzcd} \ldots \arrow[r, "d_3"] & A_2 \arrow[r, "d_2"] \arrow[d, dashed, "\exists f_2"] & A_1 \arrow[r, "d_1"] \arrow[d, dashed, "\exists f_1"] & A_0 \arrow[r, "d_0"] \arrow[d, dashed, "\exists f_0"] & A \arrow[r] \arrow[d, "f"] & 0 \\ \ldots \arrow[r, "d'_3"] & A'_2 \arrow[r, "d'_2"] & A'_1 \arrow[r, "d'_1"] & A'_0 \arrow[r, "d'_0"] & A' \arrow[r] & 0. \end{tikzcd}$$

This should be done step by step from the right. In the first step, we require that for any morphism $f : A \to A'$ and epimorphisms $d_0 : A_0 \to A$ and $d'_0 : A'_0 \to A'$ there is a morphism $f_0 : A_0 \to A'_0$ with $d'_0 \circ f_0 = f \circ d_0$.

If we replace the morphism $f \circ d_0 : A_0 \to A'$ in this condition by a general morphism $g : A_0 \to A'$ in $\mathcal{A}$, this is equivalent to the projectivity of A_0 by Lemma 3.1.21. Hence, we should consider left resolutions with *projective* objects A_0. We then attempt to extend $f : A \to A'$ to a chain map $f_\bullet : A_\bullet \to A'_\bullet$ by iterating the construction of $f_0 : A_0 \to A'_0$. For this, we should impose that *each* object A_n in the left resolution $A_\bullet$ except $A_{-1} = A$ is *projective*.

Indeed, we find that these conditions are sufficient to extend $f : A \to A'$ to a chain map $f_\bullet : A_\bullet \to A'_\bullet$ and to ensure that $f_\bullet$ is unique up to chain homotopy. The corresponding conditions and results for *right resolutions* are obtained by identifying right resolutions and injective objects in $\mathcal{A}$ with left resolutions and projective objects in $\mathcal{A}^{op}$.

Theorem 4.1.6: (**fundamental lemma of homological algebra**)

Let $\mathcal{A}$ be an abelian category.

1. If $A'_\bullet = ... \xrightarrow{d'_1} A'_0 \xrightarrow{d'_0} A' \to 0$ is exact and $A_\bullet = ... \xrightarrow{d_1} A_0 \xrightarrow{d_0} A \to 0$ a chain complex in $\mathcal{A}$ with A_n projective for all $n \in \mathbb{N}_0$, then for every morphism $f : A \to A'$ there is a chain map $f_\bullet : A_\bullet \to A'_\bullet$ with $f_{-1} = f$, and this chain map is unique up to chain homotopy.
2. If $A'^\bullet = 0 \to A' \xrightarrow{d'^{-1}} A'^0 \xrightarrow{d'^0} ...$ is exact and $A^\bullet = 0 \to A \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} ...$ a chain complex in $\mathcal{A}$ with A^n injective for all $n \in \mathbb{N}_0$, then for every morphism $f : A \to A'$ there is a chain map $f^\bullet : A^\bullet \to A'^\bullet$ with $f^{-1} = f$, and this chain map is unique up to chain homotopy.

Proof:
We prove the first statement, since the second statement is the first one for $\mathcal{A}^{op}$.

1. Construction of $f_\bullet$:

As A_0 is projective and $d'_0 : A'_0 \to A'$ an epimorphism, by Lemma 3.1.21 there is a morphism $f_0 : A_0 \to A'_0$ for which the following diagram commutes

$$\begin{tikzcd}
A_0 \arrow[r, "d_0"] \arrow[d, dashed, "\exists f_0"] & A \arrow[r] \arrow[d, "f"] & 0 \\
A'_0 \arrow[r, "d'_0"'] & A' \arrow[r] & 0.
\end{tikzcd}$$

As $d_0 \circ d_1 = 0$ and $d'_0 \circ d'_1 = 0$, by the universal property of the kernels $\iota_0 : \ker(d_0) \to A_0$ and $\iota'_0 : \ker(d'_0) \to A'_0$ there are unique morphisms $\bar{d}_1 : A_1 \to \ker(d_0)$ and $\bar{d}'_1 : A'_1 \to \ker(d'_0)$ with $\iota_0 \circ \bar{d}_1 = d_1$ and $\iota'_0 \circ \bar{d}'_1 = d'_1$. As $A'_\bullet$ is exact, we have $\ker(d'_0) \cong \mathrm{im}(d'_1)$ and $d'_1 = \iota'_0 \circ \bar{d}'_1$ is the canonical factorisation of d'_1 from Lemma 3.1.13. It follows that $\bar{d}'_1$ is an epimorphism. As we have $d'_0 \circ f_0 \circ \iota_0 = f \circ d_0 \circ \iota_0 = 0$, by the universal property of the kernel $\iota'_0 : \ker(d'_0) \to A'_0$, there is a unique morphism $f'_0 : \ker(d_0) \to \ker(d'_0)$ with $\iota'_0 \circ f'_0 = f_0 \circ \iota_0$. As A_1 is projective and $\bar{d}'_1 : A'_1 \to \ker(d'_0)$ an epimorphism, there is a morphism $f_1 : A_1 \to A'_1$ with $\bar{d}'_1 \circ f_1 = f'_0 \circ \bar{d}_1$.

$$\begin{tikzcd}
A_1 \arrow[rr, "d_1"] \arrow[dr, dashed, "\exists ! \bar{d}_1"'] \arrow[ddd, dashed, "\exists f_1"'] & & A_0 \arrow[r, "d_0"] \arrow[ddd, "f_0"] & A \arrow[r] \arrow[ddd, "f"] & 0 \\
& \ker(d_0) \arrow[ur, hook, "\iota_0"'] \arrow[d, dashed, "\exists ! f'_0"] & & & \\
& \ker(d'_0) \arrow[dr, hook, "\iota'_0"] & & & \\
A'_1 \arrow[rr, "d'_1"'] \arrow[ur, dashed, two heads, "\exists ! \bar{d}'_1"] & & A'_0 \arrow[r, "d'_0"'] & A' \arrow[r] & 0
\end{tikzcd}$$

Iterating this procedure yields a chain map $f_\bullet : A_\bullet \to A'_\bullet$ with $f_{-1} = f$.

2. Uniqueness of $f_\bullet$:

It is sufficient to show that any chain map $f_\bullet : A_\bullet \to A'_\bullet$ with $f_{-1} = 0 : A \to A'$ is chain homotopic to $0_\bullet : A_\bullet \to A'_\bullet$. We iteratively construct morphisms $h_n : A_n \to A'_{n+1}$ for $n \geq -1$ with $f_n = h_{n-1} \circ d_n + d'_{n+1} \circ h_n$. For this, we set $h_{-1} = 0 : A \to A'_0$ and consider the commuting diagram from 1:

$$
\begin{tikzcd}
A_1 \arrow[rr, "d_1"] & & A_0 \arrow[r, "d_0"] \arrow[dd, "f_0"] & A \arrow[r] \arrow[dd, "0"] & 0 \\
& \ker(d'_0) \arrow[dr, hook, "\iota'_0"] & & & \\
A'_1 \arrow[ur, two heads, "\bar{d}'_1"] \arrow[rr, "d'_1"'] & & A'_0 \arrow[r, "d'_0"'] & A' \arrow[r] & 0.
\end{tikzcd}
$$

As $d'_0 \circ f_0 = 0 \circ d_0 = 0$, by the universal property of the kernel $\iota'_0 : A_0 \to \ker(d'_0)$ there is a unique morphism $f''_0 : A_0 \to \ker(d'_0)$ with $\iota'_0 \circ f''_0 = f_0$. As A_0 is projective and $\bar{d}'_1 : A'_1 \to \ker(d'_0)$ an epimorphism, by Lemma 3.1.21 there is a morphism $h_0 : A_0 \to A'_1$ with $h_0 \circ \bar{d}'_1 = f''_0$

$$
\begin{tikzcd}
A_1 \arrow[rr, "d_1"] & & A_0 \arrow[r, "d_0"] \arrow[dd, "f_0"] \arrow[dll, dashed, bend right, "\exists h_0"'] \arrow[dl, dashed, "\exists ! f''_0"] & A \arrow[r] \arrow[dd, "0"] & 0 \\
& \ker(d'_0) \arrow[dr, hook, "\iota'_0"] & & & \\
A'_1 \arrow[ur, two heads, "\bar{d}'_1"] \arrow[rr, "d'_1"'] & & A'_0 \arrow[r, "d'_0"'] & A' \arrow[r] & 0.
\end{tikzcd}
$$

This implies $d'_1 \circ h_0 + h_{-1} \circ d_1 = d'_1 \circ h_1 + 0 \circ d_1 = d'_1 \circ h_0 = \iota'_0 \circ \bar{d}'_1 \circ h_0 = \iota'_0 \circ f''_0 = f_0$ and $d'_1 \circ (f_1 - h_0 \circ d_1) = d'_1 \circ f_1 - f_0 \circ d_1 = 0 = 0 \circ d_1$. We can apply the same argument again and obtain the commuting diagram

$$
\begin{tikzcd}
A_2 \arrow[rr, "d_2"] & & A_1 \arrow[r, "d_1"] \arrow[dd, "f_1 - h_0 \circ d_1"] \arrow[dll, dashed, bend right, "\exists h_1"'] \arrow[dl, dashed, "\exists ! f''_1"] & A_0 \arrow[r] \arrow[dd, "0"] & 0 \\
& \ker(d'_1) \arrow[dr, hook, "\iota'_1"] & & & \\
A'_2 \arrow[ur, two heads, "\bar{d}'_2"] \arrow[rr, "d'_2"'] & & A'_1 \arrow[r, "d'_1"'] & A'_0 \arrow[r] & 0.
\end{tikzcd}
$$

It follows that $d'_2 \circ h_1 = \iota'_1 \circ \bar{d}'_2 \circ h_1 = \iota'_1 \circ f''_1 = f_1 - h_0 \circ d_1$ and $f_1 = d'_2 \circ h_1 + h_0 \circ d_1$. Iterating this procedure yields a chain homotopy $h : f_\bullet \Rightarrow 0_\bullet$. □

Theorem 4.1.6 shows that projective resolutions of objects in an abelian category $\mathcal{A}$ have the required extension properties for morphisms. They allow one to extend every morphism between objects in $\mathcal{A}$ to a chain map between their resolutions that is unique up to chain homotopy.

We now determine under which conditions each object A in $\mathcal{A}$ has a projective left or injective right resolution

$$A_\bullet = \ldots \xrightarrow{d_3} A_2 \xrightarrow{d_2} A_1 \xrightarrow{d_1} A_0 \xrightarrow{d_0} A \to 0 \qquad \text{or} \qquad A^\bullet = 0 \to A \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \xrightarrow{d^2} \ldots$$

For this, note that the exactness of these chain complexes in A is equivalent to the statements that d_0 is an epimorphism and d^{-1} a monomorphism. We therefore must require at least that for each object A in $\mathcal{A}$, there is a *projective* object A_0 and an *epimorphism* $d_0 : A_0 \to A$ or an *injective* object A^0 and a *monomorphism* $d^{-1} : A \to A^0$.

Definition 4.1.7:

1. An abelian category $\mathcal{A}$ has **enough projectives** if for every object A in $\mathcal{A}$ there is a projective object P in $\mathcal{A}$ and an epimorphism $\pi : P \to A$.

2. An abelian category $\mathcal{A}$ has **enough injectives** if for every object A in $\mathcal{A}$ there is an injective object I in $\mathcal{A}$ and a monomorphism $\iota : A \to I$.

It turns out that the conditions in Definition 4.1.7 are not only necessary for the existence of projective and injective resolutions but also sufficient. They guarantee the existence of projective or injective resolutions for all objects in $\mathcal{A}$. Their uniqueness up to chain homotopy equivalence then follows directly from Theorem 4.1.6.

Theorem 4.1.8: If an abelian category $\mathcal{A}$ has enough projectives (injectives), then every object in $\mathcal{A}$ has a projective left (injective right) resolution, unique up to chain homotopy equivalence.

Proof:
We prove the claim for projective left resolutions. The claim for injective right resolutions follows, since injective right resolutions in $\mathcal{A}$ are projective left resolutions in $\mathcal{A}^{op}$.

Let A be an object in $\mathcal{A}$. As $\mathcal{A}$ has enough projectives, there is a projective object A_0 and an epimorphism $d_0 : A_0 \to A$. For the kernel $\iota_0 : \ker(d_0) \to A_0$ there is a projective object A_1 and an epimorphism $d_1' : A_1 \to \ker(d_0)$. The morphism $d_1 = \iota_0 \circ d_1' : A_1 \to A_0$ satisfies $\mathrm{im}(d_1) = \ker(\mathrm{coker}(\iota_0 \circ d_1')) \cong \ker(\mathrm{coker}(\iota_0)) \cong \mathrm{im}(\iota_0) \cong \ker(d_0)$ since d_1' is an epimorphism and ι_0 a monomorphism and hence its own image.

This shows that the sequence $A_1 \xrightarrow{d_1} A_0 \xrightarrow{d_0} A \to 0$ is exact. Iterating this procedure yields a projective resolution of A

$$\begin{tikzcd}
\ldots \arrow[r, "d_{n+1}"] & A_n \arrow[rr, "d_n"] \arrow[drr, "d_n'"] & & A_{n-1} \arrow[r, "d_{n-1}"] & \ldots \arrow[r, "d_3"] & A_2 \arrow[r, "d_2"] \arrow[dr, "d_2'"] & A_1 \arrow[r, "d_1"] \arrow[dr, "d_1'"] & A_0 \arrow[r, two heads, "d_0"] & A \to 0 \\
\ldots & \ker(d_n) \arrow[u, hook, "\iota_n"] & & \ker(d_{n-1}) \arrow[u, hook, "\iota_{n-1}"] & \ldots & & \ker(d_1) \arrow[u, hook, "\iota_1"] & \ker(d_0). \arrow[u, hook, "\iota_0"] &
\end{tikzcd}$$

Let now $A_\bullet$, $A_\bullet'$ be two projective resolutions of A. Then by Theorem 4.1.6 there are chain maps $f_\bullet : A_\bullet \to A_\bullet'$ and $f_\bullet' : A_\bullet' \to A_\bullet$ with $f_{-1} = 1_A = f_{-1}'$. Their composites $g_\bullet = f_\bullet' \circ f_\bullet : A_\bullet \to A_\bullet$ and $g_\bullet' = f_\bullet \circ f_\bullet' : A_\bullet' \to A_\bullet'$ are chain maps with $g_{-1} = 1_A = g_{-1}'$. As the identity chain maps $1_{A_\bullet} : A_\bullet \to A_\bullet$ and $1_{A_\bullet'} : A_\bullet' \to A_\bullet'$ also satisfy this condition, we have $f_\bullet' \circ f_\bullet \sim 1_{A_\bullet}$ and $f_\bullet \circ f_\bullet' \sim 1_{A_\bullet'}$ by Theorem 4.1.6. Hence, $f_\bullet : A_\bullet \to A_\bullet'$ is a chain homotopy equivalence. □

Theorems 4.1.6 and 4.1.8 guarantee that in an abelian category $\mathcal{A}$ with enough projectives (injectives) every object has a projective (injective) resolution, unique up to chain homotopy equivalence, and every morphism lifts to a chain map between resolutions, unique up to chain homotopy. We can thus implement the idea at the beginning of this section in any abelian category with enough projectives (injectives).

4.2 Projectivity and injectivity criteria

Our main examples of abelian categories are the categories R-Mod of modules over a ring R. To apply the formalism from the last section to R-Mod, we need to check that R-Mod has enough projectives or injectives. We also need to verify that our standard resolutions, the bar resolution for group cohomology, the Hochschild resolution and the Chevalley-Eilenberg resolution for Lie algebra cohomology, are indeed projective resolutions and to derive derive general criteria for projectivity and injectivity in abelian categories.

To do so, we recall the results on projectivity and injectivity from Section 3.1.

- By Definition 3.1.20 an object A in an abelian category $\mathcal{A}$ is projective and injective, respectively, if the functors $\mathrm{Hom}(A, -) : \mathcal{A} \to \mathrm{Ab}$ and $\mathrm{Hom}(-, A) : \mathcal{A}^{op} \to \mathrm{Ab}$ are exact.
- By Lemma 3.1.21 projectivity of A is equivalent to the existence of a morphism $f' : A \to X$ with $\pi \circ f' = f$ for every morphism $f : A \to Y$ and epimorphism $\pi : X \to Y$. Injectivity of A is equivalent to the existence of a morphism $f' : Y \to A$ with $f' \circ \iota = f$ for every morphism $f : X \to A$ and monomorphism $\iota : X \to Y$.
- By Example 3.1.22 every *free module* over a ring R is projective. The abelian group $\mathbb{Z}/n\mathbb{Z}$ is neither injective nor projective, while the abelian group $\mathbb{Z}$ is a projective, but not an injective $\mathbb{Z}$-module.

The fact that every free module in R-Mod is projective implies immediately that the category R-Mod has enough projectives. By Remark 1.1.15, 1. every R-module M is a quotient $M = F/L$ of a free and hence projective R-module F by a submodule $L \subset F$, and the canonical surjection $\pi : F \to M$ is an epimorphism. The proof that R-Mod has enough injectives is more involved, see for instance [JS, Satz E.9] or [W, pp 39 –41].

Example 4.2.1:

1. For any ring R, the category R-Mod has enough projectives.
2. For any ring R, the category R-Mod has enough injectives.

The fact that free R-modules are projective implies directly that the bar resolution of group cohomology from Example 4.1.2 is a projective resolution, since it is a resolution by free $k[G]$-modules. However, for the Hochschild resolution and the Chevalley-Eilenberg resolution from Examples 4.1.3 and 4.1.4, the situation is more complicated. It is a priori not guaranteed that the $A \otimes A^{op}$-modules and $U(\mathfrak{g})$-modules in these resolutions are free. In fact, finite tensor products $A^{\otimes n} = A \otimes_k ... \otimes_k A$ of a k-algebra A with itself need not even be projective k-modules. In order to derive good criteria for the projectivity of these standard resolutions, we need criteria for the projectivity of products, direct sums and tensor products in R-Mod.

Criteria for the projectivity of coproducts and the injectivity of products can be derived in more generality from the original definition of projective and injective objects in terms of exactness of the functors $\mathrm{Hom}(A, -) : \mathcal{A} \to \mathrm{Ab}$ and $\mathrm{Hom}(-, A) : \mathcal{A}^{op} \to \mathrm{Ab}$. For this it is sufficient to notice that the functors $\mathrm{Hom}(\amalg_{i \in I} X_i, -), \Pi_{i \in I} \mathrm{Hom}(X_i, -) : \mathcal{A} \to \mathrm{Ab}$ are naturally isomorphic, and so are the functors $\mathrm{Hom}(-, \Pi_{i \in I} X_i), \Pi_{i \in I} \mathrm{Hom}(-, X_i) : \mathcal{A}^{op} \to \mathrm{Ab}$. The exactness of $\mathrm{Hom}(\amalg_{i \in I} X_i, -)$ and $\mathrm{Hom}(-, \Pi_{i \in I} X_i)$ amounts to the projectivity of the coproduct and the injectivity of the product, whereas the exactness of the functors $\Pi_{i \in I} \mathrm{Hom}(X_i, -)$ and $\Pi_{i \in I} \mathrm{Hom}(-, X_i)$ amounts to the projectivity and injectivity of all objects X_i for $i \in I$.

Lemma 4.2.2: Let $\mathcal{A}$ be an abelian category and $(X_i)_{i\in I}$ a family of objects in $\mathcal{A}$ whose (co)product exists. Then:

1. The coproduct $\amalg_{i\in I}X_i$ is projective if and only if X_i is projective for all $i \in I$.
2. The product $\Pi_{i\in I}X_i$ is injective if and only if X_i is injective for all $i \in I$.

Proof:
We prove the second claim by considering

- the functor $F = \mathrm{Hom}(-, \Pi_{i\in I}X_i) : \mathcal{A}^{op} \to \mathrm{Ab}$ that assigns to an object A in $\mathcal{A}$ the abelian group $\mathrm{Hom}_{\mathcal{A}}(A, \Pi_{i\in I}X_i)$, and to a morphism $f : A \to B$ in $\mathcal{A}$ the group homomorphism

$$F(f) = \mathrm{Hom}(f, \Pi_{i\in I}X_i) : \mathrm{Hom}_{\mathcal{A}}(B, \Pi_{i\in I}X_i) \to \mathrm{Hom}_{\mathcal{A}}(A, \Pi_{i\in I}X_i),\ g \mapsto g \circ f,$$

- the functor $G = \Pi_{i\in I}\mathrm{Hom}(-, X_i) : \mathcal{A}^{op} \to \mathrm{Ab}$ that assigns to an object A the abelian group $\Pi_{i\in I}\mathrm{Hom}_{\mathcal{A}}(A, X_i)$ and to a morphism $f : A \to B$ the unique group homomorphism

$$G(f) = \Pi_{i\in I}\mathrm{Hom}(f, X_i) : \Pi_{i\in I}\mathrm{Hom}_{\mathcal{A}}(B, X_i) \to \Pi_{i\in I}\mathrm{Hom}_{\mathcal{A}}(A, X_i).$$

  with $\pi_j \circ \Pi_{i\in I}\mathrm{Hom}(f, X_i) = \mathrm{Hom}(f, X_j) \circ \pi_j$ for all $j \in I$.

By the universal property of the product we have isomorphisms of abelian groups

$$\eta_A : \mathrm{Hom}_{\mathcal{A}}(A, \Pi_{i\in I}X_i) \to \Pi_{i\in I}\mathrm{Hom}_{\mathcal{A}}(A, X_i), \quad g \mapsto (\pi_i \circ g)_{i\in I}$$

for each object A in $\mathcal{A}$. The isomorphisms η_A define a natural isomorphism $\eta : F \to G$, since for any morphism $f : A \to B$ the following diagram commutes

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{A}}(B, \Pi_{i\in I}X_i) \arrow[r, "\eta_B", "g\mapsto(\pi_i\circ g)_{i\in I}"'] \arrow[d, "\mathrm{Hom}(f,\Pi_{i\in I}X_i):g\mapsto g\circ f"'] & \Pi_{i\in I}\mathrm{Hom}_{\mathcal{A}}(B, X_i) \arrow[d, "\Pi_{i\in I}\mathrm{Hom}(f,X_i):(g_i)_{i\in I}\mapsto(g_i\circ f)_{i\in I}"] \\
\mathrm{Hom}_{\mathcal{A}}(A, \Pi_{i\in I}X_i) \arrow[r, "\eta_A", "g\mapsto(\pi_i\circ g)_{i\in I}"'] & \Pi_{i\in I}\mathrm{Hom}_{\mathcal{A}}(A, X_i).
\end{tikzcd}$$

As F and G are naturally isomorphic, F is exact if and only if G is by Corollary 3.4.4. By Definition 3.1.20 the exactness of F is equivalent to the injectivity of $\Pi_{i\in I}X_i$. Exactness of G is equivalent to the statement that for each short exact sequence $0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0$ in $\mathcal{A}$

$$0 \to \Pi_{i\in I}\mathrm{Hom}(C, X_i) \xrightarrow{\Pi_{i\in I}\mathrm{Hom}(\pi,X_i)} \Pi_{i\in I}\mathrm{Hom}(B, X_i) \xrightarrow{\Pi_{i\in I}\mathrm{Hom}(\iota,X_i)} \Pi_{i\in I}\mathrm{Hom}(A, X_i) \to 0$$

is an exact sequence in $\mathrm{Ab} = \mathbb{Z}$-Mod. From the concrete definition of the product in $\mathbb{Z}$-Mod in Definition 1.1.12, one finds that this is equivalent to the exactness of the sequence

$$0 \to \mathrm{Hom}(C, X_i) \xrightarrow{\mathrm{Hom}(\pi,X_i)} \mathrm{Hom}(B, X_i) \xrightarrow{\mathrm{Hom}(\iota,X_i)} \mathrm{Hom}(A, X_i) \to 0$$

for all $i \in I$ and hence to the injectivity of X_i for all $i \in I$. □

By applying Lemma 4.2.2 to finite coproducts in the abelian category $\mathcal{A} = R$-Mod, we obtain a useful projectivity criterion in terms of direct sums. The compatibility between tensor products and direct sums then yields a criterion for the projectivity of tensor products over commutative rings that can be applied to the Hochschild resolution and Chevalley-Eilenberg resolution.

Corollary 4.2.3: Let R be a ring.

1. An R-module A is projective if and only if there is an R-module B with $A \oplus B$ free.
2. If R is commutative and A_1 and A_2 are projective, then $A_1 \otimes_R A_2$ is projective.

Proof:
1. By Example 3.1.22, 2. free R-modules are projective and by Lemma 4.2.2 a direct sum $A \oplus B$ is projective if and only if A and B are projective. Hence, if $A \oplus B$ is free, it is projective and A and B are projective by Lemma 4.2.2.

Conversely, every R-module A is a quotient $A = F/L$ of a free module F by a submodule $L \subset F$, and the canonical surjection $\pi : F \to A$ is an epimorphism. If A is projective, there is an R-linear map $f : A \to F$ with $\pi \circ f = \mathrm{id}_A$ by Lemma 3.1.21. As id_A is injective, f is injective as well and hence an isomorphism onto its image $A \cong \mathrm{im}(f) \subset F$. Hence $A \oplus \ker(\pi) \cong \mathrm{im}(f) \oplus \ker(\pi) \cong F$.

2. If A_1 and A_2 are projective, then by 1. there are R-modules B_i and free R-modules $F_i = \oplus_{I_i} R$ with $A_i \oplus B_i = F_i$. The R-module $A_1 \otimes_R A_2$ is then projective by 1, since the compatibility of tensor products and direct sums implies

$$
\begin{aligned}
&(A_1 \otimes_R A_2) \oplus (A_1 \otimes_R B_2 \oplus B_1 \otimes_R A_2 \oplus B_1 \otimes_R B_2) \\
&\cong (A_1 \oplus B_1) \otimes_R (A_2 \oplus B_2) = F_1 \otimes_R F_2 \cong (\oplus_{I_1} R) \otimes_R (\oplus_{I_2} R) \cong \oplus_{I_1 \times I_2} R \otimes_R R \cong \oplus_{I_1 \times I_2} R. \qquad \square
\end{aligned}
$$

A similar reasoning can be used for flat modules. With an argument analogous to the proof of Lemma 4.2.2, one can show that a direct sum of R-modules is flat if and only if each summand is flat. By combining this with Corollary 4.2.3 one finds that every projective R-module is flat (Exercise 55).

Corollary 4.2.4: Let R be a ring and $(M_i)_{i \in I}$ a family of R-modules.

1. The direct sum $\oplus_{i \in I} M_i$ is flat if and only if M_i is flat for all $i \in I$.
2. Every projective R-module is flat.

Inductively, the second claim in Corollary 4.2.3 implies that all finite tensor products of projective modules over a commutative ring are projective. In particular, if A is an algebra over a commutative ring that is projective as a k-module, then all finite tensor products $A^{\otimes n} = A \otimes_k A \otimes_k \ldots \otimes_k A$ are projective k-modules. Hence, under the assumption that A is projective as a k-module, the Hochschild resolution is a projective resolution of A in k-Mod. We show that it is a projective resolution in $A \otimes_k A^{op}$-Mod $=$ A-Mod-A. By specialising to $A = U(\mathfrak{g})$ we obtain an analogous result for the Chevalley-Eilenberg resolution.

Corollary 4.2.5: Let k be a commutative ring and $\mathbb{F}$ a field.

1. For any k-algebra A that is a projective k-module, the Hochschild resolution from Example 4.1.3 is a projective resolution of A in $A \otimes_k A^{op}$-Mod.
2. For any Lie algebra $\mathfrak{g}$ over $\mathbb{F}$, the Chevalley-Eilenberg resolution from Example 4.1.4 is a projective resolution of $\mathbb{F}$ in $U(\mathfrak{g})$-Mod.

Proof:
1. We show that for any projective k-module M the module $A \otimes_k M \otimes_k A$ is a projective $A \otimes_k A^{op}$-module with the canonical $A \otimes_k A^{op}$-module structure $(b \otimes c) \rhd (a \otimes m \otimes a') = (ba) \otimes m \otimes (a'c)$. The

claim then follows because the $A\otimes_k A^{op}$-modules in the Hochschild resolution for $n \in \mathbb{N}_0$ take this form with $M = A^{\otimes n}$, and $A^{\otimes n}$ is a projective k-module by Corollary 4.2.3.

Projectivity of the k-module M implies that the functor $\mathrm{Hom}_k(M, -) : k\text{-Mod} \to \mathrm{Ab}$ is exact. By applying the forgetful functor $F : A\otimes_k A^{op}\text{-Mod} \to k\text{-Mod}$, it follows that the functor $\mathrm{Hom}_k(M, F(-)) : A\otimes_k A^{op}\text{-Mod} \to \mathrm{Ab}$ is exact as well, because forgetting the $A\otimes_k A^{op}$-module structures does not change the kernels or cokernels.

Projectivity of the $A\otimes_k A^{op}$-module $A\otimes_k M\otimes_k A$ is equivalent to the exactness of the functor $\mathrm{Hom}_{A\otimes A^{op}}(A\otimes_k M\otimes_k A, -) : A\otimes_k A^{op}\text{-Mod} \to \mathrm{Ab}$. By Corollary 3.4.4 this follows, if the functors $\mathrm{Hom}_{A\otimes A^{op}}(A\otimes_k M\otimes_k A, -)$ and $\mathrm{Hom}_k(M, F(-))$ are naturally isomorphic.

For this, note that for any $A\otimes_k A^{op}$-module N, the map

$$\eta_N : \mathrm{Hom}_{A\otimes_k A^{op}}(A\otimes_k M\otimes_k A, N) \to \mathrm{Hom}_k(M, F(N)),\ f \mapsto f' \qquad f'(m) = f(1\otimes m\otimes 1)$$

is an isomorphism of k-modules with inverse

$$\eta_N^{-1} : \mathrm{Hom}_k(M, F(N)) \to \mathrm{Hom}_{A\otimes_k A^{op}}(A\otimes_k M\otimes_k A, N),\ g \mapsto g' \qquad g'(a\otimes m\otimes a') = (a\otimes a')\rhd g(m),$$

and for every $A\otimes A^{op}$-linear map $f : N \to N'$, the following diagram commutes

$$\begin{tikzcd}
\mathrm{Hom}_{A\otimes_k A^{op}}(A\otimes_k M\otimes_k A, N) \arrow[r, "\eta_N : g\mapsto g'"] \arrow[d, "g\mapsto f\circ g"'] & \mathrm{Hom}_k(M, N) \arrow[d, "g'\mapsto f\circ g'"] \\
\mathrm{Hom}_{A\otimes_k A^{op}}(A\otimes_k M\otimes_k A, N') \arrow[r, "\eta_{N'} : g\mapsto g'"'] & \mathrm{Hom}_k(M, N')
\end{tikzcd}$$

This shows that the group isomorphisms η_N define a natural isomorphism

$$\eta : \mathrm{Hom}_{A\otimes A^{op}}(A\otimes_k M\otimes_k A, -) \to \mathrm{Hom}_k(M, F(-)).$$

2. The claim follows from 1. by specialising to $k = \mathbb{F}$, $A = U(\mathfrak{g})$ for a Lie algebra $\mathfrak{g}$ over $\mathbb{F}$ and to trivial $U(\mathfrak{g})$-right module structures. The results in 1. imply that the $U(\mathfrak{g})$-module $U(\mathfrak{g})\otimes\Lambda^n\mathfrak{g}$ is a projective $U(\mathfrak{g})$-module, since $\Lambda^n\mathfrak{g}$ is a free $\mathbb{F}$-module. $\square$

We have thus established that projective and injective resolutions exist for all objects in the abelian category R-Mod and that the bar resolution and Chevalley-Eilenberg resolution are indeed projective resolutions. For the Hochschild resolution, this holds if the k-algebra A is a projective k-module, which is always the case if k is a field. In contrast, the bar resolution of group cohomology is projective for any commutative ring k since it is a free resolution and free modules are projective.

4.3 Derived functors

Given an abelian category $\mathcal{A}$ with enough projectives, we assign to each object A a projective left resolution $A_\bullet$, unique up to chain homotopy equivalence, and to each morphism $f : A \to A'$ a chain map $f_\bullet : A_\bullet \to A'_\bullet$ between the resolutions, unique up to chain homotopy.

An additive functor $F : \mathcal{A} \to \mathcal{B}$ sends chain complexes $A_\bullet$ to chain complexes $F(A_\bullet)$, chain maps $f_\bullet : A_\bullet \to A'_\bullet$ to chain maps $F(f_\bullet) : F(A_\bullet) \to F(A'_\bullet)$ and chain homotopies $h_\bullet : f_\bullet \Rightarrow g_\bullet$ to chain homotopies $F(h_\bullet) : F(f_\bullet) \Rightarrow F(g_\bullet)$ (Exercise 47). It follows that for each object A in $\mathcal{A}$ and projective resolution $A_\bullet$ with $A_{-1} = A$, the homologies $H_n F(A_\bullet)$ do not depend on the choice of $A_\bullet$. Similarly, for a morphism $f : A \to A'$ the associated morphism $H_n F(f_\bullet) : F(A_\bullet) \to F(A'_\bullet)$ depends only on f and not on the choice of $f_\bullet$. Hence, we can view the objects $H_n(A_\bullet)$ and morphisms $H_n(f_\bullet) : H_n(A_\bullet) \to H_n(A'_\bullet)$ as quantities assigned to A and $f : A \to A'$.

The assignment of the morphisms $H_n F(f_\bullet) : H_n F(A_\bullet) \to H_n F(A'_\bullet)$ to morphisms $f : A \to A'$ is compatible with the composition of morphisms and the identity morphisms. If $f_\bullet : A_\bullet \to A'_\bullet$ and $f'_\bullet : A'_\bullet \to A''_\bullet$ are chain maps with $f_{-1} = f : A \to A'$ and $f'_{-1} = f' : A' \to A''$, then their composite $f'_\bullet \circ f_\bullet : A_\bullet \to A''_\bullet$ is a chain map with $(f'_\bullet \circ f_\bullet)_{-1} = f'_{-1} \circ f_{-1} : A \to A''$. As $F : \mathcal{A} \to \mathcal{B}$ is an additive functor and $H_n : \mathrm{Ch}_{\mathcal{B}} \to \mathcal{B}$ a functor, one has $H_n F(f'_\bullet \circ f_\bullet) = H_n F(f'_\bullet) \circ H_n F(f_\bullet)$. Similarly, the identity chain map $1_{A_\bullet} : A_\bullet \to A_\bullet$ extends $1_A : A \to A$, and the functor $H_n F : \mathcal{A} \to \mathcal{B}$ sends it to $H_n F(1_{A_\bullet}) = 1_{H_n F(A_\bullet)}$.

To ensure that the 0th homology is the object $F(A)$, one modifies the chain complex $F(A_\bullet)$ by removing the object at index -1 and considers the homologies of the resulting chain complex $F(A_\bullet)_{\geq 0}$. That this indeed ensures that $H_0(F(A_\bullet)) \cong F(A)$ is shown in Lemma 4.3.3 below.

The construction then defines functors $L_n F : \mathcal{A} \to \mathcal{B}$, the *left derived functors* of F. An analogous construction for injective resolutions yields the *right-derived functors* $R^n F : \mathcal{A} \to \mathcal{B}$. As they are useful and of interest mainly for functors $F : \mathcal{A} \to \mathcal{B}$ that are left or right exact, we restrict attention to these cases.

Definition 4.3.1: Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories.

1. If $\mathcal{A}$ has enough projectives and $F : \mathcal{A} \to \mathcal{B}$ is a right exact functor, the **left derived functors** $L_n F : \mathcal{A} \to \mathcal{B}$ for $n \in \mathbb{N}_0$ are defined by:

   - $L_n F(A) = H_n F(A_\bullet)_{\geq 0}$, for a projective left resolution $A_\bullet$ of $A \in \mathrm{Ob}\,\mathcal{A}$, where $F(A_\bullet)_{\geq 0}$ is the chain complex
   $$F(A_\bullet)_{\geq 0} = \ldots \xrightarrow{F(d_{n+1})} F(A_n) \xrightarrow{F(d_n)} \ldots \xrightarrow{F(d_2)} F(A_1) \xrightarrow{F(d_1)} F(A_0) \to 0.$$
   - $L_n F(f) = H_n F(f_\bullet)_{\geq 0}$ for a morphism $f : A \to A'$, where $f_\bullet : A_\bullet \to A'_\bullet$ is a chain map between projective resolutions $A_\bullet$ of A and $A'_\bullet$ of A' with $f_{-1} = f$.

2. If $\mathcal{A}$ has enough injectives and $F : \mathcal{A} \to \mathcal{B}$ is a left exact functor, the **right derived functors** $R^n F : \mathcal{A} \to \mathcal{B}$ for $n \in \mathbb{N}_0$ are defined by:

   - $R^n F(A) = H^n F(A^\bullet)_{\geq 0}$, for an injective right resolution $A^\bullet$ of $A \in \mathrm{Ob}\,\mathcal{A}$, where $F(A^\bullet)_{\geq 0}$ is the chain complex
   $$F(A^\bullet)_{\geq 0} = 0 \to F(A^0) \xrightarrow{F(d^0)} F(A^1) \xrightarrow{F(d^1)} \ldots \xrightarrow{F(d^{n-1})} F(A^n) \xrightarrow{F(d^n)} \ldots$$
   - $R^n F(f) = H^n F(f^\bullet)_{\geq 0}$ for a morphism $f : A \to A'$, where $f^\bullet : A^\bullet \to A'^\bullet$ is a chain map between injective right resolutions $A^\bullet$ of A and $A'^\bullet$ of A' with $f^{-1} = f$.

Remark 4.3.2:

1. The left (right) derived functors of a right (left) exact functor F are additive (Exercise 59).
2. If $F : \mathcal{A} \to \mathcal{B}$ is *exact*, then $R^nF = 0$ and $L_nF = 0$ for all $n > 0$. In this case, $F(P_\bullet)$ is exact for all projective resolutions $P_\bullet$ of A and hence $L_nF(A) = H_n(F(P_\bullet)) = 0$ for all $n > 0$ and $A \in \mathcal{A}$. The reasoning for the right derived functors is similar.
3. More generally, for any *exact* functor $G : \mathcal{B} \to \mathcal{C}$ one has $L_n(GF) = G(L_nF)$ for all $n \in \mathbb{N}_0$ and right exact functors $F : \mathcal{A} \to \mathcal{B}$ and $R^n(GH) = G(R^nH)$ for all $n \in \mathbb{N}_0$ and left exact functors $H : \mathcal{A} \to \mathcal{B}$ (Exercise 60).
4. If A is a projective (injective) object in $\mathcal{A}$, then one has $L_nF(A) = 0$ for all $n > 0$ ($R^nF(A) = 0$ for all $n > 0$) for all right (left) exact functors $F : \mathcal{A} \to \mathcal{B}$.

   This follows because $A_\bullet = 0 \to A \xrightarrow{1_A} A \to 0$ is a projective (injective) resolution of A. As $F(A_\bullet)$ is exact for any additive functor $F : \mathcal{A} \to \mathcal{B}$ all (co)homologies of the chain complex $F(A_\bullet)_{\geq 0}$ except the 0th (co)homology vanish.
5. Any natural transformation $\eta : F \to F'$ between right (left) exact functors $F : \mathcal{A} \to \mathcal{B}$ induces a family $(L_n\eta)_{n\in\mathbb{N}_0}$ of natural transformations $L_n\eta : L_nF \to L_nF'$ (a family $(R^n\eta)_{n\in\mathbb{N}_0}$ of natural transformations $R^n\eta : R^nF \to R^nF'$) (Exercise 58).

We now compute the 0th left (right) derived functors for a right (left) exact functor $F : \mathcal{A} \to \mathcal{B}$ and show that they coincide with F. This is the main motivation for defining the left and right derived functors with the chain complexes $F(A_\bullet)_{\geq 0}$ and $F(A^\bullet)_{\geq 0}$ instead of $F(A_\bullet)$ and $F(A^\bullet)$.

Lemma 4.3.3: Let $\mathcal{A}, \mathcal{B}$ be abelian categories such that $\mathcal{A}$ has enough projectives (injectives) and $F : \mathcal{A} \to \mathcal{B}$ right (left) exact. Then there is a natural isomorphism $L_0F \to F$ ($R^0F \to F$).

Proof:
We prove the claim for right exact functors. Let $A_\bullet$ be a projective resolution of A. Then the homology $H_0(F(A_\bullet)_{\geq 0}) = L_0F(A)$ is defined by the diagram

$$\begin{tikzcd}
\mathrm{im}(F(d_1)) \arrow[rr, hook, dashed, "\exists!\phi_0"] \arrow[dr, hook, "\iota'_1"] & & \ker(0) \cong F(A_0) \arrow[r, two heads, "p_0"] \arrow[dl, "\iota_0", "\cong"'] & \mathrm{coker}(\phi_0) = H_0(F(A_\bullet)_{\geq 0}) \\
F(A_1) \arrow[u, two heads, "\pi'_1"] \arrow[r, "F(d_1)"'] & F(A_0) \arrow[rr] & & 0.
\end{tikzcd}$$

As ι_0 is an isomorphism and π'_1 an epimorphism, we have

$$L_0F(A) = \mathrm{coker}(\phi_0) \cong \mathrm{coker}(\iota_0 \circ \phi_0) = \mathrm{coker}(\iota'_1) \cong \mathrm{coker}(\iota'_1 \circ \pi'_1) = \mathrm{coker}(F(d_1)).$$

As $F : \mathcal{C} \to \mathcal{D}$ is right exact, it preserves cokernels, which implies $\mathrm{coker}(F(d_1)) \cong F(\mathrm{coker}(d_1))$. As $A_\bullet$ is a projective resolution of A, we have $\mathrm{coker}(d_1) \cong A$ and $\mathrm{coker}(F(d_1)) \cong F(A)$.

The naturality of the isomorphism $L_0F(A) \to F(A)$ follows, because the diagram

$$\begin{tikzcd}
\mathrm{im}(F(d_1)) \arrow[rr, hook, dashed, "\exists! \phi_0"] \arrow[dr, hook, "\iota_1"] & & \ker(0) \arrow[r, two heads, "p_0"] \arrow[dl, "\cong"] & \mathrm{coker}(\phi_0) = H_0(F(A_\bullet)_{\geq 0}) = L_0F(A) \arrow[ddd, "L_0F(f)"', "=H_0(F(f_\bullet)_{\geq 0})"] \\
F(A_1) \arrow[r, "F(d_1)"'] \arrow[u, "\pi_1"] \arrow[d, "F(f_1)"'] & F(A_0) \arrow[r, two heads, "F(d_0)"'] \arrow[d, "F(f_0)"] & F(A) \cong \mathrm{coker}(F(d_1)) \arrow[ur, "\cong"] \arrow[d, "F(f)"] & \\
F(A_1') \arrow[r, "F(d_1')"] \arrow[d, "\pi_1'"'] & F(A_0') \arrow[r, two heads, "F(d_0')"] & F(A') \cong \mathrm{coker}(F(d_1')) \arrow[dr, "\cong"] & \\
\mathrm{im}(F(d_1')) \arrow[rr, hook, dashed, "\exists! \phi_0'"] \arrow[ur, hook, "\iota_1'"] & & \ker(0) \arrow[r, two heads, "p_0'"] \arrow[ul, "\cong"'] & \mathrm{coker}(\phi_0') = H_0(F(A'_\bullet)_{\geq 0}) = L_0F(A')
\end{tikzcd}$$

commutes for any morphism $f : A \to A'$, projective resolutions $A_\bullet$ of A and $A'_\bullet$ of A' and chain map $f_\bullet : A_\bullet \to A'_\bullet$ with $f_{-1} = f$. □

Short exact sequences of chain complexes and the associated long exact homology sequences are important tools for the computation of homologies. As the left and right derived functors are defined as the homologies of certain chain complexes, it is natural to relate their values on short exact sequences in $\mathcal{A}$. The first step is then to extend short exact sequences of objects in $\mathcal{A}$ to short exact sequences of projective (injective) resolutions. As injective resolutions in $\mathcal{A}$ correspond to projective resolutions in $\mathcal{A}^{op}$, it is sufficient to consider projective resolutions.

Lemma 4.3.4: (Horseshoe Lemma)
Let $\mathcal{A}$ be an abelian category with enough projectives, $0 \to L \xrightarrow{\iota} M \xrightarrow{\pi} N \to 0$ a short exact sequence in $\mathcal{A}$ and $L_\bullet$ and $N_\bullet$ projective resolutions of L and N. Then there is a projective resolution $M_\bullet$ of M and chain maps $\iota_\bullet : L_\bullet \to M_\bullet$ and $\pi_\bullet : M_\bullet \to N_\bullet$ with $\iota_{-1} = \iota$ and $\pi_{-1} = \pi$ such that $0 \to L_\bullet \xrightarrow{\iota_\bullet} M_\bullet \xrightarrow{\pi_\bullet} N_\bullet \to 0$ is a short exact sequence in $\mathrm{Ch}_{\mathcal{A}}$.

Proof:
1. We construct $M_\bullet$ inductively. For this, we set $M_{-1} = M$ and $M_0 = L_0 \amalg N_0$. As coproducts of projective objects are projective by Lemma 4.2.2, the object M_0 is projective. Denote by $\iota_0^1 : L_0 \to M_0$, $\iota_0^2 : N_0 \to M_0$ the canonical injections and by $\pi_0^1 : M_0 \to L_0$ and $\pi_0^2 : M_0 \to N_0$ the canonical surjections for the factors in $M_0 = L_0 \amalg N_0$. Then we have $\ker(\pi_0^i) = \mathrm{im}(\iota_0^j)$ for $i \neq j$ and obtain a short exact sequence

$$0 \to L_0 \xrightarrow{\iota_0^1} L_0 \amalg N_0 \xrightarrow{\pi_0^2} N_0 \to 0.$$

We now construct the boundary morphism $d_0^M : M_0 \to M$. As N_0 is projective and $\pi : M \to N$ an epimorphism, there is a morphism $d_0'^N : N_0 \to M$ with $\pi \circ d_0'^N = d_0^N$. By the universal property of the coproduct, there is a unique morphism $d_0^M : L_0 \amalg N_0 \to M$ with $d_0^M \circ \iota_0^1 = \iota \circ d_0^L$ and $d_0^M \circ \iota_0^2 = d_0'^N$. This implies

$$\pi \circ d_0^M \circ \iota_0^1 = \pi \circ \iota \circ d_0^L = 0 = d_0^N \circ \pi_0^2 \circ \iota_0^1 \qquad \pi \circ d_0^M \circ \iota_0^2 = \pi \circ d_0'^N = d_0^N = d_0^N \circ \pi_0^2 \circ \iota_0^2$$

and with the universal property of the coproduct $\pi \circ d_0^M = d_0^N \circ \pi_0^2$. We have the following commuting diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & L_0 \arrow[r, hook, "\iota_0^1"] \arrow[d, "d_0^L"] & L_0 \amalg N_0 \arrow[r, two heads, "\pi_0^2"] \arrow[d, dashed, "\exists! d_0^M"'] & N_0 \arrow[r] \arrow[d, "d_0^N"] \arrow[dl, dashed, "\exists d_0'^N"] & 0 \\
0 \arrow[r] & L \arrow[r, hook, "\iota"'] & M \arrow[r, two heads, "\pi"'] & N \arrow[r] & 0.
\end{tikzcd}$$

By applying the snake lemma to this diagram we obtain an exact sequence

$$0 \to \ker(d_0^L) \xrightarrow{\bar{\iota}_0^1} \ker(d_0^M) \xrightarrow{\bar{\pi}_0^2} \ker(d_0^N) \xrightarrow{\partial} \mathrm{coker}(d_0^L) \to \mathrm{coker}(d_0^M) \to \mathrm{coker}(d_0^N) \to 0. \tag{35}$$

As $L_\bullet$ and $N_\bullet$ are exact, we have $\mathrm{coker}(d_0^L) = 0 = \mathrm{coker}(d_0^N)$. This yields a short exact sequence

$$0 \to \ker(d_0^L) \xrightarrow{\bar{\iota}_0^1} \ker(d_0^M) \xrightarrow{\bar{\pi}_0^2} \ker(d_0^N) \to 0 \tag{36}$$

and implies $\mathrm{coker}(d_0^M) = 0$. Hence $d_0^M : M_0 \to M$ is an epimorphism.

2. We now consider the short exact sequence (36) and for $X = L, N$ the chain complexes $X'_\bullet$ with $X'_{-1} = \ker(d_0^X)$, $X'_n = X_{n+1}$ for $n \in \mathbb{N}_0$ and boundary operators given by $d'^X_n = d^X_{n+1}$ for $n \in \mathbb{N}$ and $d'^X_0 : X_1 \to \ker(d_0^X)$ induced by $d_1^X : X_1 \to X_0$ via exactness and the universal property of the kernel. Then $L'_\bullet$ and $N'_\bullet$ form projective resolutions of $\ker(d_0^L)$ and $\ker(d_0^N)$.

By applying step 1. to these projective resolutions and the short exact sequence (36), we obtain a projective object $M_1 = L_1 \amalg N_1$ and an epimorphism $d'^M_1 : M_1 \to \ker(d_0^M)$. By composing it with the inclusion morphism $\iota : \ker(d_0^M) \to M_0$, we obtain a morphism $d_1^M = \iota \circ d'^M_1 : M_1 \to M_0$ and the following commuting diagram with exact rows and columns

$$\begin{tikzcd}
0 \arrow[r] & L_1 \arrow[r, hook, "\iota_1^1"] \arrow[d, "d_1^L"] & L_1 \amalg N_1 \arrow[r, two heads, "\pi_1^2"] \arrow[d, "d_1^M"'] & N_1 \arrow[r] \arrow[d, "d_1^N"] & 0 \\
0 \arrow[r] & L_0 \arrow[r, hook, "\iota_0^1"] \arrow[d, "d_0^L"] & L_0 \amalg N_0 \arrow[r, two heads, "\pi_0^2"] \arrow[d, "d_0^M"'] & N_0 \arrow[r] \arrow[d, "d_0^N"] & 0 \\
0 \arrow[r] & L \arrow[r, hook, "\iota"'] & M \arrow[r, two heads, "\pi"'] & N \arrow[r] & 0.
\end{tikzcd}$$

As in step 1, applying the snake lemma to its upper two rows yields a short exact sequence

$$0 \to \ker(d_1^L) \xrightarrow{\bar{\iota}_1^1} \ker(d_1^M) \xrightarrow{\bar{\pi}_1^2} \ker(d_1^N) \to 0. \tag{37}$$

Iterating step 2 then yields a short exact sequence of chain complexes $0 \to L_\bullet \xrightarrow{\iota_\bullet} M_\bullet \xrightarrow{\pi_\bullet} N_\bullet \to 0$ such that $M_\bullet$ is a projective resolution of M. □

Using this lemma and the long exact homology sequence from Theorem 3.4.7, we can relate the values of left and right derived functors on a short exact sequence of objects in $\mathcal{A}$. This yields a long exact sequence of derived functors. Morphisms of short exact sequences in $\mathcal{A}$ induce chain maps between these long exact sequences.

Theorem 4.3.5: Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories, $F : \mathcal{A} \to \mathcal{B}$ an additive functor.

1. If $\mathcal{A}$ has enough projectives and F is right exact, then for every short exact sequence $0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0$ in $\mathcal{A}$ one has a **long exact sequence of left derived functors**

$$\ldots \xrightarrow{L_1F(\iota)} L_1F(B) \xrightarrow{L_1F(\pi)} L_1F(C) \xrightarrow{\partial_1} L_0F(A) \xrightarrow{L_0F(\iota)} L_0F(B) \xrightarrow{L_0F(\pi)} L_0F(C) \to 0,$$

and for every chain map between short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\iota"] \arrow[d, "\alpha"] & B \arrow[r, "\pi"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "\iota'"] & B' \arrow[r, "\pi'"] & C' \arrow[r] & 0
\end{tikzcd} \tag{38}$$

one has a commuting diagram

$$\begin{tikzcd}
\ldots \arrow[r, "\partial_{n+1}"] & L_nF(A) \arrow[r, "L_nF(\iota)"] \arrow[d, "L_nF(\alpha)"] & L_nF(B) \arrow[r, "L_nF(\pi)"] \arrow[d, "L_nF(\beta)"] & L_nF(C) \arrow[r, "\partial_n"] \arrow[d, "L_nF(\gamma)"] & L_{n-1}F(A) \arrow[r, "L_{n-1}F(\iota)"] \arrow[d, "L_{n-1}F(\alpha)"] & \ldots \\
\ldots \arrow[r, "\partial'_{n+1}"] & L_nF(A') \arrow[r, "L_nF(\iota')"] & L_nF(B') \arrow[r, "L_nF(\pi')"] & L_nF(C') \arrow[r, "\partial'_n"] & L_{n-1}F(C') \arrow[r, "L_{n-1}F(\iota')"] & \ldots
\end{tikzcd} \tag{39}$$

2. If $\mathcal{A}$ has enough injectives and F is left exact, then for every short exact sequence $0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0$ one has a **long exact sequence of right derived functors**

$$0 \to R^0F(A) \xrightarrow{R^0F(\iota)} R^0F(B) \xrightarrow{R^0F(\pi)} R^0F(C) \xrightarrow{\partial^0} R^1F(A) \xrightarrow{R^1F(\iota)} R^1F(B) \xrightarrow{R^1F(\pi)} \ldots$$

and for every chain map between short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\iota"] \arrow[d, "\alpha"] & B \arrow[r, "\pi"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "\iota'"] & B' \arrow[r, "\pi'"] & C' \arrow[r] & 0
\end{tikzcd}$$

one has a commuting diagram

$$\begin{tikzcd}
\ldots \arrow[r, "\partial^{n-1}"] & R^nF(A) \arrow[r, "R^nF(\iota)"] \arrow[d, "R^nF(\alpha)"] & R^nF(B) \arrow[r, "R^nF(\pi)"] \arrow[d, "R^nF(\beta)"] & R^nF(C) \arrow[r, "\partial^n"] \arrow[d, "R^nF(\gamma)"] & R^{n+1}F(A) \arrow[r, "R^{n+1}F(\iota)"] \arrow[d, "R^{n+1}F(\alpha)"] & \ldots \\
\ldots \arrow[r, "\partial'^{n-1}"] & R^nF(A') \arrow[r, "R^nF(\iota')"] & R^nF(B') \arrow[r, "R^nF(\pi')"] & R^nF(C') \arrow[r, "\partial'^n"] & R^{n+1}F(C') \arrow[r, "R^{n+1}F(\iota')"] & \ldots
\end{tikzcd}$$

Proof:
By Remark 4.3.2, left and right derived functors are additive. We prove the remaining claims for right exact functors. The claim for left exact functors then follows by considering $\mathcal{A}^{op}$.

By Lemma 4.3.4 there are projective resolutions $A_\bullet$, $B_\bullet$ and $C_\bullet$ of A, B, C that form a short exact sequence of chain complexes $0 \to A_\bullet \xrightarrow{\iota_\bullet} B_\bullet \xrightarrow{\pi_\bullet} C_\bullet \to 0$. By applying the functor F and removing the terms in degree -1 in this sequence, we obtain an exact sequence

$$0 \to F(A_\bullet)_{\geq 0} \xrightarrow{F(\iota_\bullet)} F(B_\bullet)_{\geq 0} \xrightarrow{F(\pi_\bullet)} F(C_\bullet)_{\geq 0} \to 0.$$

The exactness of this sequence follows from the construction of $0 \to A_\bullet \xrightarrow{\iota_\bullet} B_\bullet \xrightarrow{\pi_\bullet} C_\bullet \to 0$ in Lemma 4.3.4. For $n \geq 0$, we have $B_n = A_n \amalg C_n$ with the inclusion and the projection morphisms $\iota_n = \iota_n^1 : A_n \to A_n \amalg C_n$ and $\pi_n = \pi_n^2 : A_n \amalg C_n \to C_n$ of the (co)product. As F is additive, it preserves finite (co)products and hence the sequence

$$0 \to F(A_n) \xrightarrow{F(\iota_n^1)=\iota_n^1} F(A_n \amalg C_n) \cong F(A_n) \amalg F(C_n) \xrightarrow{F(\pi_n^2)=\pi_n^2} F(C_n) \to 0$$

is exact for all $n \in \mathbb{N}_0$. The first statement then follows from Theorem 3.4.7 about the long exact homology sequence, since the left derived functors are given by $L_nF(X) = H_n(F(X_\bullet)_{n\geq 0})$ for $X = A, B, C$.

Given two short exact sequences $0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0$ and $0 \to A' \xrightarrow{\iota'} B' \xrightarrow{\pi'} C' \to 0$ and morphisms $\alpha : A \to A'$, $\beta : B \to B'$ and $\gamma : C \to C'$ as in (38), we can choose projective resolutions $A_\bullet$, $B_\bullet$, $C_\bullet$ and $A'_\bullet$, $B'_\bullet$, $C'_\bullet$ that form two short exact sequences of chain complexes by Lemma 4.3.4. Theorem 4.1.6 then yields chain maps $\alpha_\bullet : A_\bullet \to A'_\bullet$, $\beta_\bullet : B_\bullet \to B'_\bullet$ and

$\gamma_\bullet : C_\bullet \to C'_\bullet$ with $\alpha_{-1} = \alpha$, $\beta_{-1} = \beta$ and $\gamma_{-1} = \gamma$. Applying F and omitting the lowest terms gives the following commuting diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & F(A_\bullet)_{\geq 0} \arrow[r, "F(\iota_\bullet)"] \arrow[d, "F(\alpha_\bullet)_{\geq 0}"] & F(B_\bullet)_{\geq 0} \arrow[r, "F(\pi_\bullet)"] \arrow[d, "F(\beta_\bullet)_{\geq 0}"] & F(C_\bullet)_{\geq 0} \arrow[r] \arrow[d, "F(\gamma_\bullet)_{\geq 0}"] & 0 \\
0 \arrow[r] & F(A'_\bullet)_{\geq 0} \arrow[r, "F(\iota'_\bullet)"] & F(B'_\bullet)_{\geq 0} \arrow[r, "F(\pi'_\bullet)"] & F(C'_\bullet)_{\geq 0} \arrow[r] & 0,
\end{tikzcd}$$

and Theorem 3.4.8 yields the commuting diagram (39). □

Remark 4.3.6: Theorem 4.3.5 implies that $L_nF = 0$ for all $n > 0$ ($R^nF = 0$ for all $n > 0$) of a right (left) exact functor $F : \mathcal{A} \to \mathcal{B}$ if and only if F is exact.

The only if-statement follows direct from the definition. Theorem 4.3.5 implies that if all left derived functors L_nF vanish, we have short exact sequence

$$0 \to L_0F(A) \xrightarrow{L_0F(\iota)} L_0F(B) \xrightarrow{L_0F(\pi)} L_0F(C) \to 0.$$

As $L_0F \cong F$, this implies the exactness of $0 \to F(A) \xrightarrow{F(\iota)} F(B) \xrightarrow{F(\pi)} F(C) \to 0$ for each short exact sequence $0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0$ in $\mathcal{A}$ and hence the exactness of F by Lemma 3.4.3.

4.4 The functors Tor and Ext

The results from the last subsection allow us to consider the left (right) derived functors of right (left) exact functors $F : \mathcal{A} \to \mathcal{B}$ whenever the category $\mathcal{A}$ has enough projectives (injectives). In particular, these conditions are satisfied for the category R-Mod for any ring R. The most important examples of right and left exact functors are the functors $M \otimes_R - : R\text{-Mod} \to \text{Ab}$ and $\text{Hom}_R(-, N) : R\text{-Mod}^{op} \to \text{Ab}$ for a fixed R-right module M and R-left module N, which are right and left exact by Corollary 3.1.16 and Lemma 3.1.17.

Definition 4.4.1: Let R be a ring, M an R-right module and N an R-left module.

1. The left derived functors of the right exact functor $M \otimes_R - : R\text{-Mod} \to \text{Ab}$ are denoted $\text{Tor}_n^R(M, -) := L_n(M \otimes_R -)$.
2. The right derived functors of the left exact functor $\text{Hom}_R(-, N) : R\text{-Mod}^{op} \to \text{Ab}$ are denoted $\text{Ext}_R^n(-, N) := R^n\text{Hom}_R(-, N)$.

Remark 4.4.2:

1. To compute the value of the functors Tor and Ext on an R-module A, one uses *projective resolutions* of A. For Tor, this holds by definition of the left derived functors. For Ext this holds because an injective resolution of A in $R\text{-Mod}^{op}$ is the same as a projective resolution of A in R-Mod.

2. One has $\text{Ext}_R^n(M, N) = 0$ for all $n > 0$ and R-modules M if and only if N is injective and $\text{Tor}_n^R(M, N) = 0$ for all $n > 0$ and R-modules N if and only if M is flat.

3. All R-linear maps $f: M \to M'$ and $g: N \to N'$ define natural transformations $f \otimes_R - : M \otimes_R - \to M' \otimes_R -$ and $\mathrm{Hom}(-, g) : \mathrm{Hom}(-, N) \to \mathrm{Hom}(-, N')$. By Remark 4.3.2, 5. they induce natural transformations $\mathrm{Tor}_n^R(f, -) : \mathrm{Tor}_n^R(M, -) \to \mathrm{Tor}_n^R(M', -)$ and natural transformations $\mathrm{Ext}^n_R(-, g) : \mathrm{Ext}^n_R(-, N) \to \mathrm{Ext}^n_R(-, N')$.

It turns out that most of the (co)homology theories in Section 2 are nothing but the functors Tor and Ext for specific choices of rings. This gives a simpler description of these cohomology theories and places them in a common framework.

Example 4.4.3: (**Group (co)homology**)

Let k be a commutative ring and G a group. The bar resolution from Example 4.1.2

$$X_n = \begin{cases} \langle G^{\times n} \rangle_{k[G]} & n \geq 0 \\ k & n = -1 \end{cases}$$

$$d_n(g_1, ..., g_n) = g_1 \rhd (g_2, ..., g_n) + \Sigma_{i=1}^{n-1} (-1)^i (g_1, ..., g_i g_{i+1}, ..., g_n) + (-1)^n (g_1, ..., g_{n-1})$$

$$d_0(g_1) = 1.$$

is a free and hence projective resolution of the trivial $k[G]$-module k.

Applying the functor $P \otimes_{k[G]} - : k[G]\text{-Mod} \to k\text{-Mod}$ for a $k[G]$-right module P to the chain complex $(X_\bullet)_{\geq 0}$ with X_{-1} replaced by 0 yields the chain complex $C_\bullet(G, P)$ for group homology

$$C(G, P)_n = P \otimes_{k[G]} \langle G^{\times n} \rangle_{k[G]} \cong P \otimes_k \langle G^{\times n} \rangle_k$$

$$d_n(p \otimes (g_1, ..., g_n)) = (p \lhd g_1) \otimes (g_2, ..., g_n) - p \otimes (g_1 g_2, ..., g_n) \pm \ldots + (-1)^n p \otimes (g_1, ..., g_{n-1}).$$

Applying the functor $\mathrm{Hom}(-, M) : k[G]\text{-Mod}^{op} \to k\text{-Mod}$ for a $k[G]$-module M to the chain complex $(X_\bullet)_{\geq 0}$ with X_{-1} replaced by 0 yields a cochain complex $C^\bullet(G, M)$ of group cohomology

$$C^n(G, M) = \mathrm{Hom}_{k[G]}(\langle G^{\times n} \rangle_{k[G]}, M) \cong \mathrm{Map}(G^{\times n}, M)$$

$$d^n(\phi)(g_0, ..., g_n) = g_0 \rhd \phi(g_1, ..., g_n) + \Sigma_{i=1}^n (-1)^i \phi(g_0, ..., g_{i-1} g_i, ..., g_n) + (-1)^{n+1} \phi(g_0, ..., g_{n-1}).$$

Hence, we have

$$H_n(G, P) = \mathrm{Tor}_n^{k[G]}(P, k) \qquad\qquad H^n(G, M) = \mathrm{Ext}^n_{k[G]}(k, M).$$

Example 4.4.4: (**Hochschild (co)homology**)

Let k be a commutative ring, A an algebra over k that is projective as a k-module. Then by Example 4.1.3 and Corollary 4.2.5 the chain complex $X_\bullet$ with

$$X_n = A^{\otimes(n+2)} \qquad n \geq -1$$

$$d_n : X_n \to X_{n-1}, \quad a_0 \otimes .... \otimes a_{n+1} \mapsto \Sigma_{i=0}^n (-1)^i \, a_0 \otimes ... \otimes (a_i a_{i+1}) \otimes ... \otimes a_{n+1}.$$

is a projective resolution of A in $A \otimes A^{op}$-Mod.

Applying the functor $M \otimes_{A \otimes A^{op}} - : A \otimes A^{op}\text{-Mod} \to k\text{-Mod}$ for an (A, A)-bimodule M to the associated chain complex $(X_\bullet)_{\geq 0}$ with X_{-1} removed yields a chain complex $Y_\bullet$ in k-Mod with

$$Y_n = M \otimes_{A \otimes A^{op}} A^{\otimes(n+2)} \qquad n \in \mathbb{N}_0$$

$$d_n(m \otimes a_0 \otimes ... \otimes a_{n+1}) = \Sigma_{i=0}^n (-1)^i m \otimes a_0 \otimes ... \otimes (a_i a_{i+1}) \otimes ... \otimes a_{n+1},$$

and the k-linear maps

$$f_n : M \otimes_{A\otimes A^{op}} A^{\otimes(n+2)} \to M \otimes_k A^{\otimes n}, \quad m\otimes a_0 \otimes ... \otimes a_{n+1} \mapsto (a_{n+1} \rhd m \lhd a_0)\otimes a_1 \otimes ... \otimes a_n$$

define an invertible chain map in k-Mod from $Y_\bullet$ to the Hochschild complex $C_\bullet(A, M)$ from Definition 2.2.3. If we let the Tor functors for $A\otimes A^{op}$ take values in k-Mod, we obtain

$$H_n(A, M) = \mathrm{Tor}_n^{A\otimes A^{op}}(M, A).$$

Applying the functor $\mathrm{Hom}_{A\otimes A^{op}}(-, M) : A\otimes A^{op}\text{-Mod}^{op} \to k\text{-Mod}$ to $(X_\bullet)_{\geq 0}$ yields a cochain complex $Z^\bullet$ in k-Mod with

$$\begin{aligned} &Z^n = \mathrm{Hom}_{A\otimes A^{op}}(A^{\otimes(n+2)}, M) \qquad n \in \mathbb{N}_0 \\ &d^n(\phi)(a_0\otimes ... \otimes a_{n+1}\otimes a_{n+2}) = \Sigma_{i=0}^n (-1)^i \phi(a_0 \otimes ... \otimes (a_i a_{i+1}) \otimes ... \otimes a_{n+1}), \end{aligned}$$

and the k-linear maps

$$f_n : \mathrm{Hom}_{A\otimes A^{op}}(A^{\otimes(n+2)}, M) \to \mathrm{Hom}_k(A^{\otimes n}, M), \quad f_n(\phi)(a_1\otimes ... \otimes a_n) = \phi(1\otimes a_1 \otimes ... \otimes a_n \otimes 1)$$

define an invertible cochain map in k-Mod from $Z^\bullet$ to the Hochschild cocomplex $C^\bullet(A, M)$ from Definition 2.2.4. This implies

$$H^n(A, M) = \mathrm{Ext}^n_{A\otimes A^{op}}(A, M).$$

Example 4.4.5: (**Lie algebra cohomology**)

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$. Then the Chevalley-Eilenberg complex $X_\bullet$ from Example 4.1.4

$$\begin{aligned} &X_n = U(\mathfrak{g})\otimes\Lambda^n\mathfrak{g}, \quad X_{-1} = \mathbb{F} \\ &d_n(y\otimes x_1 \wedge ... \wedge x_n) = \Sigma_{i=1}^n (-1)^{i+1}\, yx_i \otimes x_1 \wedge ... \wedge \widehat{x_i} \wedge ... \wedge x_n \\ &\qquad\qquad + \Sigma_{1\leq i<j\leq n} (-1)^{i+j}\, y\otimes [x_i, x_j] \wedge x_1 \wedge ... \wedge \widehat{x_i} \wedge ... \wedge \widehat{x_j} \wedge ... \wedge x_n \end{aligned}$$

with $U(\mathfrak{g})$-module structure $\rhd : U(\mathfrak{g}) \times X_n \to X_n$, $z \rhd (y\otimes x_1 \wedge ... \wedge x_n) = (zy)\otimes x_1 \wedge ... \wedge x_n$ is a projective resolution of the trivial $U(\mathfrak{g})$-module $\mathbb{F}$ in $U(\mathfrak{g})$-Mod.

Applying the functor $\mathrm{Hom}(-, M) : U(\mathfrak{g})\text{-Mod} \to \mathrm{Vect}_{\mathbb{F}}$ to $X_\bullet$ for an $U(\mathfrak{g})$-module M and omitting the term X_{-1} yields the chain complex $W_\bullet$ in $\mathrm{Vect}_{\mathbb{F}}$ with

$$\begin{aligned} &W_n = \mathrm{Hom}_{U(\mathfrak{g})}(U(\mathfrak{g})\otimes\Lambda^n\mathfrak{g}, M) \cong \mathrm{Hom}_{\mathbb{F}}(\Lambda^n\mathfrak{g}, M) \\ &d_n(f)(x_0, ..., x_n) = \Sigma_{i=0}^n (-1)^i\, x_i \rhd f(x_1, ..., \widehat{x_i}, ..., x_n) \\ &\qquad\qquad + \Sigma_{0\leq i<j\leq n} (-1)^{i+j}\, f([x_i, x_j], x_0, ..., \widehat{x_i}, ..., \widehat{x_j}, ...x_n). \end{aligned}$$

This is the cochain complex of Lie algebra cohomology, and we have

$$H^n(\mathfrak{g}, M) = \mathrm{Ext}^n_{U(\mathfrak{g})}(\mathbb{F}, M).$$

This allows us to give an alternative definition of the (co)homology theories from Section 2 in terms of the functors Tor and Ext. This is more conceptual as it does not rely on concrete choices of chain complexes. It is also much better for computations, because it allows one to compute (co)homologies from *any* projective resolution of the objects under consideration. In many cases, there are much simpler projective resolutions than the standard resolutions.

Definition 4.4.6: Let k be a commutative ring and $\mathbb{F}$ a field.

1. Group homology of a group G with coefficients in a $k[G]$-right module M and group cohomology of G with coefficients in a $k[G]$-left module M are given by

$$H_n(G,M) = \mathrm{Tor}_n^{k[G]}(M,k) \qquad\qquad H^n(G,M) = \mathrm{Ext}^n_{k[G]}(k,M).$$

2. Hochschild homology and cohomology of an algebra A that is a projective k-module with coefficients in an (A,A)-bimodule M are given by

$$H_n(A,M) = \mathrm{Tor}_n^{A\otimes A^{op}}(M,A) \qquad\qquad H^n(A,M) = \mathrm{Ext}^n_{A\otimes A^{op}}(A,M).$$

3. Lie algebra homology of a Lie algebra $\mathfrak{g}$ over $\mathbb{F}$ with coefficients in a right $U(\mathfrak{g})$-module M and Lie algebra cohomology of $\mathfrak{g}$ with coefficients in a $U(\mathfrak{g})$-module M are given by

$$H_n(\mathfrak{g},M) = \mathrm{Tor}_n^{U(\mathfrak{g})}(M,\mathbb{F}) \qquad\qquad H^n(\mathfrak{g},M) = \mathrm{Ext}^n_{U(\mathfrak{g})}(\mathbb{F},M).$$

Example 4.4.7: (**Group (co)homologies of cyclic groups**)

We compute $H_n(\mathbb{Z}/m\mathbb{Z},\mathbb{Z})$ and $H^n(\mathbb{Z}/m\mathbb{Z},\mathbb{Z})$ for the trivial $\mathbb{Z}[\mathbb{Z}/m\mathbb{Z}]$-module $\mathbb{Z}$.

To distinguish the group multiplication in $\mathbb{Z}/m\mathbb{Z}$ from the addition in the group ring $R := \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}]$ we identify $\mathbb{Z}/m\mathbb{Z}$ with the subgroup $\mathbb{Z}/m\mathbb{Z} = \{e^{2\pi\mathrm{i}k/m} \mid k = 0,1,...,m-1\} \subset \mathbb{C}^\times$.

• We consider the following chain complex in R-Mod

$$\ldots \xrightarrow{d_4} \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}] \xrightarrow{d_3} \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}] \xrightarrow{d_2} \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}] \xrightarrow{d_1} \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}] \xrightarrow{d_0} \mathbb{Z} \to 0$$
$$d_n(e^{2\pi\mathrm{i}k/m}) = \begin{cases} 1 & n = 0 \\ e^{2\pi\mathrm{i}k/m} - e^{2\pi\mathrm{i}(k+1)/m} & n \text{ odd} \\ 1 + e^{2\pi\mathrm{i}/m} + ... + e^{2\pi\mathrm{i}(m-1)/m} & n > 0 \text{ even,} \end{cases} \tag{40}$$

where $+$ stands for the addition in $R = \mathbb{Z}[\mathbb{Z}/m\mathbb{Z}]$. This is a free (and hence projective) resolution of $\mathbb{Z}$ in R-Mod since $\mathrm{im}(d_0) = \mathbb{Z}$ and

$$\ker(d_{2n+1}) = \{\lambda(1 + e^{2\pi\mathrm{i}/m} + ... + e^{2\pi\mathrm{i}(m-1)/m}) \mid \lambda \in \mathbb{Z}\} = \mathrm{im}(d_{2n+2})$$
$$\ker(d_{2n}) = \{\lambda_0\, 1 + \lambda_1\, e^{2\pi\mathrm{i}/m} + ... + \lambda_{m-1}\, e^{2\pi\mathrm{i}(m-1)/m} \mid \lambda_i \in \mathbb{Z}, \lambda_0 + ... + \lambda_m = 0\} = \mathrm{im}(d_{2n+1}).$$

• To compute $H_n(\mathbb{Z}/m\mathbb{Z},\mathbb{Z})$, we apply the functor $\mathbb{Z}\otimes_R - : R\text{-Mod} \to \mathrm{Ab}$ to the free resolution (40) and omit the last entry on the right. As $\phi : \mathbb{Z}\otimes_R R \to \mathbb{Z}$, $z\otimes r \mapsto z \lhd r$ is an isomorphism in Ab with inverse $\phi^{-1} : \mathbb{Z} \to \mathbb{Z}\otimes_R R$, $z \mapsto z\otimes 1$ and

$$(\mathrm{id}_\mathbb{Z}\otimes d_{2n+1})(z\otimes e^{2\pi\mathrm{i}k/m}) = z\otimes(e^{2\pi\mathrm{i}k/m} - e^{2\pi\mathrm{i}(k+1)/m}) = z\otimes 1 - z\otimes 1 = 0$$
$$(\mathrm{id}_\mathbb{Z}\otimes d_{2n})(z\otimes e^{2\pi\mathrm{i}k/m}) = z\otimes(1 + e^{2\pi\mathrm{i}/m} + ... + e^{2\pi\mathrm{i}(m-1)/m}) = mz\otimes 1 = \phi^{-1}(m\cdot\phi(z\otimes e^{2\pi\mathrm{i}k/m})),$$

we have an isomorphism of chain complexes

$$\begin{tikzcd}
\ldots \arrow[r] & \mathbb{Z}\otimes_R R \arrow[r, "\mathrm{id}\otimes d_4"] \arrow[d, "\cong"', "\phi"] & \mathbb{Z}\otimes_R R \arrow[r, "\mathrm{id}\otimes d_3"] \arrow[d, "\cong"', "\phi"] & \mathbb{Z}\otimes_R R \arrow[r, "\mathrm{id}\otimes d_2"] \arrow[d, "\cong"', "\phi"] & \mathbb{Z}\otimes_R R \arrow[r, "\mathrm{id}\otimes d_1"] \arrow[d, "\cong"', "\phi"] & \mathbb{Z}\otimes_R R \arrow[r] \arrow[d, "\cong"', "\phi"] & 0 \\
\ldots \arrow[r] & \mathbb{Z} \arrow[r, "z\mapsto mz"'] & \mathbb{Z} \arrow[r, "0"'] & \mathbb{Z} \arrow[r, "z\mapsto mz"'] & \mathbb{Z} \arrow[r, "0"'] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

and the group homologies are given by

$$H_n(\mathbb{Z}/m\mathbb{Z},\mathbb{Z}) = \mathrm{Tor}_n^{\mathbb{Z}[\mathbb{Z}/m\mathbb{Z}]}(\mathbb{Z},\mathbb{Z}) = \begin{cases} \ker(0)/\mathrm{im}(0) = \mathbb{Z} & n=0 \\ \ker(z\mapsto mz)/\mathrm{im}(0) = 0 & n>0 \text{ even} \\ \ker(0)/\mathrm{im}(z\mapsto mz) = \mathbb{Z}/m\mathbb{Z} & n \text{ odd.} \end{cases}$$

• For the cohomologies $H^n(\mathbb{Z}/m\mathbb{Z},\mathbb{Z})$, we apply $\mathrm{Hom}_R(-,\mathbb{Z}) : R\text{-Mod}^{op} \to \mathrm{Ab}$ to the free resolution (40) and omit the first term on the left. As the R-module R is cyclic with generator 1 and $\mathbb{Z}$ is equipped with the trivial R-module structure, the map $\phi : \mathrm{Hom}_R(R,\mathbb{Z}) \to \mathbb{Z}$, $f \mapsto f(1)$ is an isomorphism in Ab with inverse $\phi^{-1} : \mathbb{Z} \to \mathrm{Hom}_R(R,\mathbb{Z})$, $z \mapsto f_z$ with the constant map $f_z : R \to \mathbb{Z}$, $r \mapsto z$. The coboundary operators satisfy

$$\begin{aligned} &\phi\circ\mathrm{Hom}_R(d_{2n+1},\mathbb{Z})(f) = f(d_{2n+1}(1)) = f(1) - f(e^{2\pi\mathrm{i}/m}) = 0 \\ &\phi\circ\mathrm{Hom}_R(d_{2n+2},\mathbb{Z})(f) = f(d_{2n+2}(1)) = f(1+e^{2\pi\mathrm{i}/m}+...+e^{2\pi\mathrm{i}(m-1)/m}) = mf(1) = m\cdot\phi(f), \end{aligned}$$

and we obtain an isomorphism of cochain complexes

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_R(R,\mathbb{Z}) \arrow[r, "\mathrm{Hom}(d_1{,}\mathbb{Z})"] \arrow[d, "\cong"', "\phi"] & \mathrm{Hom}_R(R,\mathbb{Z}) \arrow[r, "\mathrm{Hom}(d_2{,}\mathbb{Z})"] \arrow[d, "\cong"', "\phi"] & \mathrm{Hom}_R(R,\mathbb{Z}) \arrow[r, "\mathrm{Hom}(d_3{,}\mathbb{Z})"] \arrow[d, "\cong"', "\phi"] & \mathrm{Hom}_R(R,\mathbb{Z}) \arrow[r, "\mathrm{Hom}(d_4{,}\mathbb{Z})"] \arrow[d, "\cong"', "\phi"] & \ldots \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "0"'] & \mathbb{Z} \arrow[r, "z\mapsto mz"'] & \mathbb{Z} \arrow[r, "0"'] & \mathbb{Z} \arrow[r, "z\mapsto mz"'] & \ldots \end{tikzcd}$$

The group cohomologies $H^n(\mathbb{Z}/m\mathbb{Z},\mathbb{Z})$ are the cohomologies of this cochain complex:

$$H^n(\mathbb{Z}/m\mathbb{Z},\mathbb{Z}) = \mathrm{Ext}^n_{\mathbb{Z}[\mathbb{Z}/m\mathbb{Z}]}(\mathbb{Z},\mathbb{Z}) = \begin{cases} \ker(0)/\mathrm{im}(0) = \mathbb{Z} & n=0 \\ \ker(z\mapsto mz)/\mathrm{im}(0) = 0 & n \text{ odd} \\ \ker(0)/\mathrm{im}(z\mapsto mz) = \mathbb{Z}/m\mathbb{Z} & n \text{ even.} \end{cases}$$

We now consider Hochschild (co)homologies. We already established in Exercise 19 that for any separable algebra A over a field k and any (A,A)-bimodule M all Hochschild homologies $H_n(A,M)$ and cohomologies $H^n(A,M)$ for $n\in\mathbb{N}$ vanish. This applies in particular to any finite-dimensional semisimple algebra A over an algebraically closed field k such as matrix algebras and group algebras $k[G]$ with $\mathrm{char}(k) \nmid |G|$. Hence, Hochschild (co)homologies are of interest mainly for non-semisimple algebras with a more complicated structure. They are rather difficult to compute in practice, so we treat only a simple example.

Example 4.4.8: (**Hochschild homologies of the tensor algebra**)

Let V be a vector space over a field $\mathbb{F}$. The **tensor algebra** $T(V)$ of V is the vector space $T(V) = \oplus_{n=0}^{\infty} V^{\otimes n}$ with $V^{\otimes 0} := \mathbb{F}$ and the multiplication given by concatenation

$$(v_1\otimes...\otimes v_n)\cdot(v_{n+1}\otimes...\otimes v_{n+k}) = v_1\otimes...\otimes v_{n+k} \qquad \forall n,k\in\mathbb{N}, v_i\in V.$$

• We consider the chain complex $X_\bullet$ given by

$$\begin{aligned} &0 \to T(V)\otimes_{\mathbb{F}} V\otimes_{\mathbb{F}} T(V) \xrightarrow{d_1} T(V)\otimes_{\mathbb{F}} T(V) \xrightarrow{d_0} T(V) \to 0 \\ &d_1(x\otimes v\otimes y) = (x\cdot v)\otimes y - x\otimes(v\cdot y),\ d_0(x\otimes y) = x\cdot y. \end{aligned}$$

It is a free resolution of the $T(V)\otimes T(V)^{op}$-module $T(V)$ in $T(V)\otimes T(V)^{op}$-Mod, since we have $d_0 \circ d_1 = 0$ and the $\mathbb{F}$-linear maps

$$\begin{aligned}
&h_{-1} : T(V) \to T(V)\otimes_{\mathbb{F}} T(V),\ x \mapsto x\otimes 1\\
&h_0 : T(V)\otimes_{\mathbb{F}} T(V) \to T(V)\otimes_{\mathbb{F}} V\otimes_{\mathbb{F}} T(V),\ x\otimes(v_1\cdots v_n) \mapsto -\Sigma_{i=1}^n xv_1\cdots v_{i-1}\otimes v_i\otimes v_{i+1}\cdots v_n
\end{aligned}$$

define a chain homotopy from $1_{X_\bullet} : X_\bullet \to X_\bullet$ to $0_{X_\bullet} : X_\bullet \to X_\bullet$.

• To compute the Hochschild homologies, we omit the first term on the right and apply the functor $T(V)\otimes_R - : R\text{-Mod} \to \mathrm{Vect}_{\mathbb{F}}$ for $R := T(V)\otimes T(V)^{op}$. To simplify the resulting chain complex, we consider the $\mathbb{F}$-linear isomorphisms

$$\begin{aligned}
&\phi : T(V)\otimes_R(T(V)\otimes_{\mathbb{F}} V\otimes_{\mathbb{F}} T(V)) \to T(V)\otimes_{\mathbb{F}} V, && w\otimes(x\otimes v\otimes y) \mapsto ywx\otimes v\\
&\psi : T(V)\otimes_R(T(V)\otimes_{\mathbb{F}} T(V)) \to T(V), && w\otimes(x\otimes y) \mapsto ywx.
\end{aligned}$$

with inverses

$$\begin{aligned}
&\phi^{-1} : T(V)\otimes_{\mathbb{F}} V \to T(V)\otimes_R(T(V)\otimes_{\mathbb{F}} V\otimes_{\mathbb{F}} T(V)), && w\otimes v \mapsto w\otimes(1\otimes v\otimes 1)\\
&\psi^{-1} : T(V) \to T(V)\otimes_R(T(V)\otimes_{\mathbb{F}} T(V)), && w \mapsto w\otimes(1\otimes 1).
\end{aligned}$$

As we have

$$\psi \circ (\mathrm{id}\otimes d_1)(w\otimes x\otimes v\otimes y) = \psi(w\otimes(xv)\otimes y - w\otimes x\otimes(vy)) = ywx\cdot v - v\cdot ywx,$$

we obtain an isomorphism of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & T(V)\otimes_R(T(V)\otimes_{\mathbb{F}} V\otimes_{\mathbb{F}} T(V)) \arrow[r, "\mathrm{id}\otimes d_1"] \arrow[d, "\phi", "\cong"'] & T(V)\otimes_R(T(V)\otimes_{\mathbb{F}} T(V)) \arrow[r] \arrow[d, "\psi", "\cong"'] & 0\\
0 \arrow[r] & T(V)\otimes_{\mathbb{F}} V \arrow[r, "d_1' :\, x\otimes v\mapsto x\cdot v - v\cdot x"'] & T(V) \arrow[r] & 0.
\end{tikzcd}$$

The Hochschild homologies of $T(V)$ are the homologies of this chain complex. Denoting by $\tau_n : V^{\otimes n} \to V^{\otimes n}$, $v_1\otimes ...\otimes v_n \mapsto v_n\otimes v_1\otimes ...\otimes v_{n-1}$ the linear map that cyclically permutes the factors in the tensor product $V^{\otimes n}$ we obtain

$$H_n(T(V),T(V)) = \begin{cases} T(V)/\mathrm{im}(d_1') & n=0\\ \ker(d_1') & n=1\\ 0 & n>1 \end{cases} \cong \begin{cases} \mathbb{F}\oplus V\oplus \oplus_{k\geq 2} V^{\otimes n}/(\mathrm{id}_{V^{\otimes k}} - \tau_k)V^{\otimes n} & n=0\\ V\oplus\oplus_{k\geq 2}\{x\in V^{\otimes k} \mid \tau_k(x)=x\} & n=1\\ 0 & n>1. \end{cases}$$

For $V = \mathbb{F}$ we obtain the polynomial algebra $\mathbb{F}[x] = T(\mathbb{F})$ and its Hochschild homologies

$$H_n(\mathbb{F}[x],\mathbb{F}[x]) = \begin{cases} \mathbb{F}[x] & n=0\\ x\mathbb{F}[x] & n=1\\ 0 & n>1. \end{cases}$$

While we clarified the interpretation of Tor and Ext in the context of Hochschild (co)homology, group (co)homology and Lie algebra (co)homology in Section 2, we do not know what properties of the R-modules are encoded in Tor and Ext for a general rings R.

By considering the functors $\mathrm{Tor}_n^R : R\text{-Mod} \to \mathrm{Ab}$ for finitely generated modules over a principal ideal domain, we find that they are related to the *torsion* of the modules, which motivates the name Tor. For this, recall that every finitely generated module over a principal ideal domain R is isomorphic to a module $M = R^n \times R/p_1^{n_1} \times ... \times R/p_k^{n_k}$ with $n \in \mathbb{N}_0$, $n_i \in \mathbb{N}$ and prime elements $p_i \in R$. By Lemma 1.1.21 we have $M \cong R^n \oplus \mathrm{Tor}_R(M)$ with $\mathrm{Tor}_R(M) = R/p_1^{n_1}R \times ... \times R/p_k^{n_k}R$, and by Lemma 1.1.17 every submodule of a free R-module is free.

Example 4.4.9: (Tor$_n^R$ for finitely generated modules over principal ideal domains)

1. Let R be a principal ideal domain. We compute $\mathrm{Tor}_n^R(R/qR, R/pR)$ for $p, q \in R$.
   By Lemma 4.3.3 we have

   $$\mathrm{Tor}_0^R(R/qR, R/pR) = L_0(R/qR \otimes_R -)(R/pR) = R/qR \otimes_R R/pR = R/\gcd(p,q)R.$$

   To compute the higher homologies, we use the free (and therefore projective) resolution

   $$0 \to R \xrightarrow{d_1 : r \mapsto pr} R \xrightarrow{d_0 = \pi} R/pR \to 0. \tag{41}$$

   By applying the functor $R/qR \otimes_R - : \mathrm{Ab} \to \mathrm{Ab}$, omitting the last term on the right and using the isomorphism $R/qR \otimes_R R \cong R/qR$, we obtain the chain complex

   $$0 \to R/qR \xrightarrow{d_1 : \bar{r} \mapsto \overline{pr}} R/qR \to 0.$$

   Its first homology is given by

   $$\mathrm{Tor}_1^R(R/qR, R/pR) = \ker(d_1) = \{\bar{x} \in R/qR \mid q \,|\, p \cdot x\} \cong R/\gcd(p,q)R,$$

   while all higher homologies vanish. This yields

   $$\mathrm{Tor}_k^R(R/qR, R/pR) \cong \begin{cases} R/\gcd(p,q)R & k = 0, 1 \\ 0 & k \geq 2. \end{cases}$$

2. We compute $\mathrm{Tor}_k^R(R/qR, R)$. By Lemma 4.3.3 $\mathrm{Tor}_0^R(R/qR, R) \cong R/qR \otimes_R R \cong R/qR$. As R is a free R-module, it is projective by Example 3.1.22, and Remark 4.3.2, 4. implies

   $$\mathrm{Tor}_k^R(R/qR, R) \cong \begin{cases} R/qR & k = 0 \\ 0 & k \geq 1. \end{cases}$$

3. We compute $\mathrm{Tor}_k^R(R, R/pR)$ and $\mathrm{Tor}_k^R(R, R)$. By Lemma 4.3.3 we have

   $$\mathrm{Tor}_0^R(R, R/pR) \cong R \otimes_R R/pR \cong R/pR \qquad \mathrm{Tor}_0^R(R, R) \cong R \otimes_R R \cong R.$$

   As R is a projective R-module, it is flat by Corollary 4.2.4, and Remark 4.4.2, 2. implies

   $$\mathrm{Tor}_k^R(R, R/pR) \cong \begin{cases} R/pR & k = 0 \\ 0 & k \geq 1, \end{cases} \qquad \mathrm{Tor}_k^R(R, R) \cong \begin{cases} R & k = 0 \\ 0 & k \geq 1. \end{cases}$$

As the left derived functors are additive and every finitely generated R-module is of the form $M = R^n \times \mathrm{Tor}_R(M)$ with the torsion submodule given by $\mathrm{Tor}_R(M) = R/q_1R \times ... \times R/q_kR$ for prime powers $q_i \in R$, Example 4.4.9 shows that $\mathrm{Tor}_n^R(M,N) = \{0\}$ for all $n > 0$ if M or N are torsion free. Hence, Tor_n^R is related to the torsion of the modules M, N.

This example also shows a more general pattern. For any ring R and R-left module N that has a short exact sequence as a projective resolution, all torsion functors $\mathrm{Tor}_n^R(M,N)$ for $n \geq 2$ and arbitrary R-right modules M vanish. If N is projective, then $\mathrm{Tor}_1^R(M,N) = 0$ as well.

By a similar computation, we can determine the functors $\mathrm{Ext}^n_R : R\text{-Mod} \to R\text{-Mod}$ for principal ideal domains R (Exercise 63). To motivate its name, we show that it classifies *extensions of modules*. This holds more generally for abelian categories $\mathcal{A}$, since $\mathrm{Hom}(-,A) : \mathcal{A} \to \mathrm{Ab}$ and $\mathrm{Ext}^n_{\mathcal{A}} = R^n\mathrm{Hom}(-,A) : \mathcal{A} \to \mathrm{Ab}$ are defined for any abelian category $\mathcal{A}$. For our purposes, it is sufficient to consider the case $\mathcal{A} = R\text{-Mod}$ for some ring R and the functor $\mathrm{Ext}^1_{R\text{-Mod}}$.

Definition 4.4.10: Let R be a ring and N an R-module.

1. An **extension** of N by an R-module L is a short exact sequence $0 \to L \xrightarrow{\iota} M \xrightarrow{\pi} N \to 0$.

2. Two extensions $0 \to L \xrightarrow{\iota} M \xrightarrow{\pi} N \to 0$ und $0 \to L \xrightarrow{\iota'} M' \xrightarrow{\pi'} N \to 0$ are called **equivalent** if there is an isomorphism $f : M \to M'$ for which the following diagram commutes
$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "\iota"] \arrow[d, "\mathrm{id}_L"] & M \arrow[r, "\pi"] \arrow[d, "f", "\cong"'] & N \arrow[r] \arrow[d, "\mathrm{id}_N"] & 0 \\ 0 \arrow[r] & L \arrow[r, "\iota'"] & M' \arrow[r, "\pi'"] & N \arrow[r] & 0. \end{tikzcd}$$

3. One says an extension **splits**, if it is equivalent to an extension of the form
$$0 \to L \xrightarrow{\iota_1} L \oplus N \xrightarrow{\pi_2} N \to 0,$$
where $\iota_1 : L \to L \oplus N$ and $\pi_2 : L \oplus N \to N$ denote the inclusion map for the first and the projection map for the second factor in the direct sum.

Proposition 4.4.11: Let L, N be modules over a ring R. Then equivalence classes of extensions of N by L are in bijection with elements of $\mathrm{Ext}^1_R(N,L)$. Extensions that split correspond to the element $0 \in \mathrm{Ext}^1_R(N,L)$.

Proof:
1. We define a map $\phi : \mathrm{Ex}(N,L)/\!\sim\, \to \mathrm{Ext}^1_R(N,L)$, where $\mathrm{Ex}(N,L)/\!\sim$ is the set of equivalence classes of extensions of N by L.

For every extension $0 \to L \xrightarrow{\iota} M \xrightarrow{\pi} N \to 0$ the long exact sequence of derived functors from Theorem 4.3.5 yields an exact sequence
$$0 \to \mathrm{Hom}_R(N,L) \xrightarrow{f\mapsto f\circ\pi} \mathrm{Hom}_R(M,L) \xrightarrow{f\mapsto f\circ\iota} \mathrm{Hom}_R(L,L) \xrightarrow{\partial^0} \mathrm{Ext}^1_R(N,L) \to \dots \tag{42}$$
By assigning to the extension $0 \to L \xrightarrow{\iota} M \xrightarrow{\pi} N \to 0$ the element $\partial^0(\mathrm{id}_L) \in \mathrm{Ext}^1_R(N,L)$ we obtain a map $\phi : \mathrm{Ex}(N,L) \to \mathrm{Ext}^1_R(N,L)$ from the set of extensions of N by L to $\mathrm{Ext}^1_R(N,L)$.

If $0 \to L \xrightarrow{\iota'} M' \xrightarrow{\pi'} N \to 0$ is another extension equivalent to $0 \to L \xrightarrow{\iota} M \xrightarrow{\pi} N \to 0$, then there is an R-linear isomorphism $f : M \to M'$ with $f \circ \iota = \iota'$ and $\pi' \circ f = \pi$. The naturality of the connecting morphism then yields the following commuting diagram with exact rows

$$\begin{tikzcd}
\ldots \arrow[r] & \mathrm{Hom}_R(M,L) \arrow[r, "g\mapsto g\circ\iota"] & \mathrm{Hom}_R(L,L) \arrow[r, "\partial^0"] & \mathrm{Ext}^1_R(N,L) \arrow[r] & \ldots \\
\ldots \arrow[r] & \mathrm{Hom}_R(M',L) \arrow[r, "g\mapsto g\circ\iota'"] \arrow[u, "g\mapsto g\circ f", "\cong"'] & \mathrm{Hom}_R(L,L) \arrow[r, "\partial'^0"] \arrow[u, "\mathrm{id}"] & \mathrm{Ext}^1_R(N,L) \arrow[r] \arrow[u, "\mathrm{id}"'] & \ldots,
\end{tikzcd}$$

which implies $\partial^0(\mathrm{id}_L) = \partial'^0(\mathrm{id}_L)$. This shows that $\phi : \mathrm{Ex}(N,L) \to \mathrm{Ext}^1_R(N,L)$ is constant on equivalence classes of extensions and induces a map $\phi : \mathrm{Ex}(N,L)/\sim\, \to \mathrm{Ext}^1_R(N,L)$.

2. We show that ϕ is surjective by constructing for each element $m \in \mathrm{Ext}^1_R(N,L)$ an extension $m_\bullet = 0 \to L \xrightarrow{\iota} M \xrightarrow{\pi} N \to 0$ with $\phi(m_\bullet) = m$.

2.(a) We construct an auxiliary exact sequence $x_\bullet = 0 \to Y \xrightarrow{I} X \xrightarrow{P} N \to 0$ with projective X and surjective connecting homomorphism $\partial^0_x : \mathrm{Hom}_R(Y,L) \to \mathrm{Ext}^1_R(N,L)$:

As R-Mod has enough projectives, there is a projective R-module X and an epimorphism $P : X \to N$. With $Y = \ker(P)$ and the inclusion $I : \ker(P) \to X$, we obtain an exact sequence

$$0 \to Y = \ker(P) \xrightarrow{I} X \xrightarrow{P} N \to 0.$$

As X is projective, we have $\mathrm{Ext}^1_R(X,L) = 0$ for all R-modules L. As $\mathrm{Ext}^0_R(A,L) = \mathrm{Hom}_R(A,L)$ for all R-modules A Theorem 4.3.5 yields the long exact sequence of right derived functors

$$0 \to \mathrm{Hom}_R(N,L) \xrightarrow{f\mapsto f\circ P} \mathrm{Hom}_R(X,L) \xrightarrow{f\mapsto f\circ I} \mathrm{Hom}_R(Y,L) \xrightarrow{\partial^0_x} \mathrm{Ext}^1_R(N,L) \to 0 \to ...$$

This implies that $\partial^0_x : \mathrm{Hom}_R(Y,L) \to \mathrm{Ext}^1_R(N,L)$ is surjective: for every $m \in \mathrm{Ext}^1_R(N,L)$ there is an R-linear map $f : Y \to L$ with $m = \partial^0_x(f)$.

2.(b) We construct an extension $m_\bullet = 0 \to L \xrightarrow{\iota} M \xrightarrow{\pi} N \to 0$ with $\phi(m_\bullet) = m$ by setting

$$M = (X \oplus L)/\mathrm{im}(g) \qquad \text{with} \qquad g = i_1 \circ I - i_2 \circ f : Y \to X \oplus L, \tag{43}$$

where $i_1 : X \to X \oplus L$ and $i_2 : L \to X \oplus L$ are the inclusions. We denote by $p_1 : X \oplus L \to X$ and $p_2 : X \oplus L \to L$ its projections and by $p : X \oplus L \to M$ the canonical surjection.

By the universal property of the direct sum, there is a unique morphism $\pi'' : X \oplus L \to N$ with $\pi'' \circ i_1 = P$ and $\pi'' \circ i_2 = 0$. As P is an epimorphism, π'' is an epimorphism as well, and it satisfies $\pi'' \circ g = \pi'' \circ \iota_1 \circ I = P \circ I = 0$. By the universal property of the cokernel p, there is a unique morphism $\pi : M \to N$ with $\pi \circ p = \pi''$, and π is an epimorphism

$$\begin{tikzcd}
 & L \arrow[r, "i_2"] & X \oplus L \arrow[r, "p"] \arrow[dr, "\pi''"] & M \arrow[d, dashed, "\exists!\pi"] & \\
0 \arrow[r] & Y \arrow[r, "I"] & X \arrow[r, "P"] \arrow[u, "i_1"] & N \arrow[r] & 0.
\end{tikzcd}$$

Composing the inclusion $i_2 : L \to X \oplus L$ with the canonical surjection p yields an injection $\iota = p \circ i_2 : L \to M$ due to the definition of g in (43). By definition of ι and of g, p in (43), we have $\iota \circ f = p \circ i_2 \circ f = p \circ i_1 \circ I$. By definition of π'' and π, we have $\pi \circ p \circ i_1 = \pi'' \circ i_1 = P$. Hence, we have a commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "I"] \arrow[d, "f"] & X \arrow[r, "P"] \arrow[d, "p\circ i_1"] & N \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & L \arrow[r, "\iota"] & M \arrow[r, "\pi"] & N \arrow[r] & 0.
\end{tikzcd}$$

Its first row is exact by construction, and the second, because π is surjective, ι is injective and

$$\begin{aligned}\ker(\pi) &= \{[(x,l)] \mid x \in X, l \in L, \pi(x) = 0\} \overset{\ker(\pi)=\mathrm{im}(\iota)}{=} \{[(\iota(y), l)] :\, y \in Y, l \in L\} \\ &\overset{(43)}{=} \{[(0, l + f(y))] :\, l \in L, y \in Y\} = \{[(0,l)] :\, l \in L\} = \mathrm{im}(\iota).\end{aligned}$$

The naturality of the connecting morphism then yields the commuting diagram

$$\begin{tikzcd}
\mathrm{Hom}_R(Y,L) \arrow[r, "\partial^0_x"] & \mathrm{Ext}^1_R(N,L) \\
\mathrm{Hom}_R(L,L) \arrow[r, "\partial^0"] \arrow[u, "h\mapsto h\circ f"] & \mathrm{Ext}^1_R(N,L), \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

which implies $m = \partial^0_x(f) = \partial^0_x(\mathrm{id}_L \circ f) = \partial^0(\mathrm{id}_L)$. Hence, $m_\bullet = \ 0 \to L \xrightarrow{\iota} M \xrightarrow{\pi} N \to 0$ is an extension of N by L with $\phi(m_\bullet) = m$, and ϕ is surjective.

3. We prove injectivity of ϕ. For this, we show that every extension $0 \to L \xrightarrow{\iota'} M' \xrightarrow{\pi'} N \to 0$ with $\partial'^0(\mathrm{id}_L) = m$ is equivalent to $m_\bullet$.

As X is projective and P an epimorphism, there is an R-linear map $h' : X \to M'$ with $\pi' \circ h' = P$. This implies $\pi' \circ h' \circ I = P \circ I = 0$. By the universal property of the kernel ι' there is a unique R-linear map $f' : Y \to L$ with $\iota' \circ f' = h' \circ I$. We obtain the following commuting diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "\iota'"] & M' \arrow[r, "\pi'"] & N \arrow[r] & 0 \\
0 \arrow[r] & Y \arrow[r, "I"] \arrow[u, "f'"] \arrow[d, "f"'] & X \arrow[r, "P"] \arrow[u, "h'"] \arrow[d, "p\circ i_1"'] & N \arrow[r] \arrow[u, "\mathrm{id}_N"'] \arrow[d, "\mathrm{id}_N"'] & 0 \\
0 \arrow[r] & L \arrow[r, "\iota"] & M \arrow[r, "\pi"] & N \arrow[r] & 0.
\end{tikzcd}$$

As $\partial^0(\mathrm{id}_L) = \partial^0_x(f) = \partial^0_x(f') = m = \partial'^0(\mathrm{id}_L)$, we have $f - f' \in \ker(\partial^0_x)$. By the exactness of the long exact sequence (42) of derived functors for $x_\bullet$, there is an R-linear map $k : X \to L$ with $f - f' = k \circ I$. We consider the R-linear map

$$r = (h' + \iota' \circ k) \circ p_1 + \iota' \circ p_2 : X \oplus L \to M', \ x + l \mapsto h'(x) + \iota' \circ k(x) + \iota'(l). \qquad (44)$$

By definition of g in (43) it satisfies

$$\begin{aligned} r \circ g &\overset{(44)}{=} h' \circ p_1 \circ g + \iota' \circ k \circ p_1 \circ g + \iota' \circ p_2 \circ g \overset{(43)}{=} h' \circ I + \iota' \circ k \circ I - \iota' \circ f \\ &= \iota' \circ (f' + k \circ I - f) = \iota' \circ 0 = 0.\end{aligned}$$

By the universal property of the cokernel $p : X \oplus L \to M$, it induces a unique R-linear map $r' : M \to M'$ with $r' \circ p = r$. By definition of r' and by definition of ι, p in step 2. we have

$$\begin{aligned}&r' \circ \iota = r' \circ p \circ i_2 = r \circ i_2 = \iota' \\ &\pi' \circ r' \circ p = \pi' \circ r = \pi' \circ h' \circ p_1 + \pi' \circ \iota' \circ k \circ p_1 + \pi' \circ \iota' \circ p_2 = \pi' \circ h' \circ p_1 = P \circ p_1 = \pi \circ p.\end{aligned}$$

As p is an epimorphism, the second identity implies $\pi' \circ r' = \pi$, and we obtain a commuting diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "\cong"] & L \arrow[r, "\iota"] \arrow[d, "\mathrm{id}_L"] & M \arrow[r, "\pi"] \arrow[d, "r'"] & N \arrow[r] \arrow[d, "\mathrm{id}_N"'] & 0 \arrow[d, "\cong"] \\
0 \arrow[r] & L \arrow[r, "\iota'"] & M' \arrow[r, "\pi'"] & N \arrow[r] & 0.
\end{tikzcd}$$

By the 5-Lemma (Exercise 41) $r' : M \to M'$ is an isomorphism. This shows that the extension $0 \to L \xrightarrow{\iota''} M' \xrightarrow{\pi''} N \to 0$ is equivalent to $m_\bullet = 0 \to L \xrightarrow{\iota'} M \xrightarrow{\pi'} N \to 0$ and ϕ is injective.

4. We prove that an extension $X_\bullet = 0 \to L \xrightarrow{\iota} M \xrightarrow{\pi} N \to 0$ splits if and only if $\phi(X_\bullet) = 0$.

If $X_\bullet$ splits, then it is equivalent to an extension of the form $0 \to L \xrightarrow{\iota_1} L \oplus N \xrightarrow{\pi_2} N \to 0$. The map $\mathrm{Hom}(\iota_1, L) : \mathrm{Hom}_R(L \oplus N, L) \to \mathrm{Hom}_R(L, L)$, $f \mapsto f \circ \iota_1$ is surjective, and this implies $\partial^0 = 0$ in the exact sequence (42). Hence, we have $\phi(X_\bullet) = \partial^0(\mathrm{id}_L) = 0$. Conversely, if $\phi(X_\bullet) = \partial^0(\mathrm{id}_L) = 0$, we have $\mathrm{id}_L \in \ker(\partial^0) = \mathrm{im}(\mathrm{Hom}(\iota, L))$ by the exactness of (42). Hence, there is an R-linear map $f : M \to L$ with $f \circ \iota = \mathrm{id}_L$, and $X_\bullet$ splits by Exercise 52. □

4.5 Tor and Ext as bifunctors

To get a full understanding of the functors Tor and Ext, it remains to clarify one issue. The functors Tor and Ext were introduced in Definition 4.4.1 as, respectively, the left derived functors $\mathrm{Tor}_n^R(L, -) = L_n(L \otimes_R -) : R\text{-Mod} \to \mathrm{Ab}$ for an R-right module L and the right derived functors $\mathrm{Ext}^n_R(-, M) = R^n\mathrm{Hom}_R(-, M) : R\text{-Mod}^{op} \to \mathrm{Ab}$ for an R-left module M.

This involved arbitrary choices, namely tensoring with L on the *left* and considering R-linear maps *into* the module M. Instead of the right exact functor $L \otimes_R - : R\text{-Mod} \to \mathrm{Ab}$ and the left exact functor $\mathrm{Hom}_R(-, M) : R\text{-Mod}^{op} \to \mathrm{Ab}$, we could have considered the right exact functor $- \otimes_R M : R^{op}\text{-Mod} \to \mathrm{Ab}$ and the left exact functor $\mathrm{Hom}_R(M, -) : R\text{-Mod} \to \mathrm{Ab}$, with their left and right derived functors $\mathrm{Tor}'^R_n(-, M) = L_n(- \otimes_R M)$ and $\mathrm{Ext}'^n_R(M, -) = R^n\mathrm{Hom}_R(M, -)$.

To determine how Tor' and Ext' are related to Tor and Ext, we consider the functors $\otimes_R : R^{op}\text{-Mod} \times R\text{-Mod} \to \mathrm{Ab}$ and $\mathrm{Hom}(-, -) : R\text{-Mod}^{op} \times R\text{-Mod} \to \mathrm{Ab}$. It turns out that they send a pair of chain complexes to a chain complex in the abelian category $\mathrm{Ch}_{R\text{-Mod}}$ of chain complexes in R-Mod. However, working in $\mathrm{Ch}_{R\text{-Mod}}$ breaks the symmetry between the two arguments and obscures some of the structures. For this reason, one works with an equivalent concept, *double complexes* in R-Mod, in which both arguments appear on an equal footing.

Definition 4.5.1: Let $\mathcal{A}$ be an abelian category.

1. A **double complex** in $\mathcal{A}$ is a family $X_{\bullet\bullet} = (X_{i,j})_{i,j\in\mathbb{Z}}$ of objects in $\mathcal{A}$ together with two families $(d^h_{i,j})_{i,j\in\mathbb{Z}}$ and $(d^v_{i,j})_{i,j\in\mathbb{Z}}$ of morphisms $d^h_{i,j} : X_{i,j} \to X_{i-1,j}$ and $d^v_{i,j} : X_{i,j} \to X_{i,j-1}$, the **horizontal** and **vertical differentials**, such that for all $i, j \in \mathbb{Z}$:
$$d^h_{i,j-1} \circ d^v_{i,j} = -d^v_{i-1,j} \circ d^h_{i,j} : X_{i,j} \to X_{i-1,j-1} \qquad d^h_{i-1,j} \circ d^h_{i,j} = 0 \qquad d^v_{i,j-1} \circ d^v_{i,j} = 0.$$
2. A **morphism of double complexes** $f_{\bullet\bullet} : X_{\bullet\bullet} \to Y_{\bullet\bullet}$ is a family $(f_{ij})_{i,j\in\mathbb{Z}}$ of morphisms $f_{i,j} : X_{i,j} \to Y_{i,j}$ in $\mathcal{A}$ that satisfy for all $i, j \in \mathbb{Z}$
$$d^{h,Y}_{i,j} \circ f_{i,j} = f_{i-1,j} \circ d^{h,X}_{i,j} \qquad d^{v,Y}_{i,j} \circ f_{i,j} = f_{i,j-1} \circ d^{v,Y}_{i,j}.$$

A double complex $X_{\bullet\bullet}$ is called **bounded on the left (bounded below)** if there is an $n \in \mathbb{Z}$ with $X_{i,j} = 0$ for all $i < n$ and $j \in \mathbb{Z}$ (with $X_{i,j} = 0$ for all $j < n$ and $i \in \mathbb{Z}$).

Remark 4.5.2:

1. Double complexes and morphisms of double complexes in $\mathcal{A}$ form a category $\mathrm{DCh}_{\mathcal{A}}$.

2. We can regard double complexes $X_{\bullet\bullet}$ in $\mathcal{A}$ as chain complexes in $\mathrm{Ch}_{\mathcal{A}}$ and vice versa.

   Every double complex $X_{\bullet\bullet}$ in $\mathcal{A}$ defines a family of chain complexes $X^{h,j}_\bullet = (X_{i,j})_{i\in\mathbb{Z}}$ with differentials $d^{h,j}_\bullet = (d^h_{i,j})_{i\in\mathbb{Z}}$. The morphisms $d^{v,j}_i = (-1)^i d^v_{i,j} : X^{h,j}_i \to X^{h,j-1}_i$ define chain maps $d^{v,j}_\bullet : X^{h,j}_\bullet \to X^{h,j-1}_\bullet$:

   $$d^{v,j}_{i-1} \circ d^{j,h}_i = (-1)^{i-1} d^v_{i-1,j} \circ d^h_{i,j} = (-1)^i d^h_{i,j-1} \circ d^v_{i,j} = d^{j-1,h}_i \circ d^{v,j}_i \qquad \forall i,j \in \mathbb{Z}.$$

   Hence, we have a chain complex $(X_\bullet, d^v_\bullet)$ in $\mathrm{Ch}_{\mathcal{A}}$ with $X_j = X^{h,j}_\bullet$ and boundary operators $d^v_j = d^{v,j}_\bullet : X^{h,j}_\bullet \to X^{h,j-1}_\bullet$. A morphism $f_{\bullet\bullet} : X_{\bullet\bullet} \to Y_{\bullet\bullet}$ of double complexes yields chain maps $f^j_\bullet = (f_{i,j})_{i,j\in\mathbb{Z}} : X^{h,j}_\bullet \to X^{h,j-1}_\bullet$ that define a chain map $f_\bullet : X_\bullet \to Y_\bullet$.

   Conversely, every chain complex $(X_\bullet, d^v_\bullet) = (X^j_\bullet, d^{vj}_\bullet)_{j\in\mathbb{Z}}$ in $\mathrm{Ch}_{\mathcal{A}}$ with $X^j_\bullet = (X^j_i, d^{hj}_i)_{i\in\mathbb{Z}}$ defines a double complex $X_{\bullet\bullet} = (X_{i,j})_{i,j\in\mathbb{Z}}$ in $\mathcal{A}$ with $X_{i,j} = X^j_i$, $d^h_{ij} = d^{hj}_i$, $d^v_{ij} = (-1)^i d^{vj}_i$. The minus sign in the vertical differential is sometimes called the **Koszul sign trick**.

$$\begin{tikzcd}
 & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \\
\cdots \arrow[r, "d^h_{i+2,j+1}"] & X_{i+1,j+1} \arrow[r, "d^h_{i+1,j+1}"] \arrow[d, "d^v_{i+1,j+1}"] & X_{i,j+1} \arrow[r, "d^h_{i,j+1}"] \arrow[d, "d^v_{i,j+1}"] & X_{i-1,j+1} \arrow[r, "d^h_{i-1,j+1}"] \arrow[d, "d^v_{i-1,j+1}"] & \cdots \\
\cdots \arrow[r, "d^h_{i+2,j}"] & X_{i+1,j} \arrow[r, "d^h_{i+1,j}"] \arrow[d, "d^v_{i+1,j}"] & X_{i,j} \arrow[r, "d^h_{i,j}"] \arrow[d, "d^v_{i,j}"] & X_{i-1,j} \arrow[r, "d^h_{i-1,j}"] \arrow[d, "d^v_{i-1,j}"] & \cdots \\
\cdots \arrow[r, "d^h_{i+2,j-1}"] & X_{i+1,j-1} \arrow[r, "d^h_{i+1,j-1}"] \arrow[d] & X_{i,j-1} \arrow[r, "d^h_{i,j-1}"] \arrow[d] & X_{i-1,j-1} \arrow[r, "d^h_{i-1,j-1}"] \arrow[d] & \cdots \\
 & \cdots & \cdots & \cdots &
\end{tikzcd}$$

Double complexes are relevant for our question, because tensoring a chain complex $L_\bullet$ in R^{op}-Mod and a chain complex $M_\bullet$ in R-Mod yields a double complex in Ab. The same holds if we apply the functor $\mathrm{Hom}(-,-) : R\text{-Mod}^{op} \times R\text{-Mod} \to \mathrm{Ab}$ to chain complexes $M_\bullet$ in R-Mod^{op} and $N_\bullet = (N_i)_{i\in I}$ in R-Mod. For the latter, we can also view $M_\bullet$ as a chain complex in R-Mod by replacing $M_i \to M_{-i}$ and $d_i \to d_{-i}$.

Example 4.5.3: Let R be a ring.

1. If $L_\bullet$ is a chain complex in R^{op}-Mod and $M_\bullet$ a chain complex in R-Mod, then we obtain a double complex $X_{\bullet\bullet}$ in Ab with

   $$X_{i,j} = L_i \otimes_R M_j, \qquad d^h_{i,j} = d^L_i \otimes \mathrm{id}_{M_j}, \qquad d^v_{i,j} = (-1)^i\, \mathrm{id}_{L_i} \otimes d^M_j.$$

2. If $M_\bullet, N_\bullet$ are chain complexes in R-Mod, we obtain a double complex $Y_{\bullet\bullet}$ in Ab with

   $$Y_{i,j} = \mathrm{Hom}_R(M_{-i}, N_j), \qquad d^h_{i,j} : f \mapsto f \circ d^M_{-i+1}, \qquad d^v_{i,j} : f \mapsto (-1)^i d^N_j \circ f.$$

In constrast to chain complexes in $\mathrm{Ch}_{\mathcal{A}}$, double complexes in $\mathcal{A}$ clearly exhibit the symmetries between their rows and columns. They also allow one to construct chain complexes in $\mathcal{A}$ by taking diagonal complexes. For this, one combines all objects X_{ij} with fixed $i+j$ into a product or coproduct and their horizontal and vertical differentials into a new chain map. This yields the so-called *total complexes.* To define them for general double complexes, one has to require that countable products and coproducts exist for all chain complexes in $\mathcal{A}$. If one restricts attention to double complexes bounded on the left and below, this is not necessary.

Lemma 4.5.4: Let $\mathcal{A}$ be an abelian category in which products and coproducts exist for all countable families of objects in $\mathcal{A}$.

1. Every double complex $X_{\bullet\bullet}$ in $\mathcal{A}$ defines two chain complexes $\mathrm{Tot}^{\amalg}_\bullet(X_{\bullet\bullet})$ and $\mathrm{Tot}^{\Pi}_\bullet(X_{\bullet\bullet})$ in $\mathcal{A}$, the **total complexes** of $X_{\bullet\bullet}$, given by

$$\begin{aligned}
&\mathrm{Tot}^{\amalg}_n(X_{\bullet\bullet}) = \amalg_{i+j=n} X_{i,j}, \qquad && d^{\amalg}_n \circ \iota^n_{i,j} = \iota^{n-1}_{i-1,j} \circ d^h_{i,j} + \iota^{n-1}_{i,j-1} \circ d^v_{i,j},\\
&\mathrm{Tot}^{\Pi}_n(X_{\bullet\bullet}) = \Pi_{i+j=n} X_{i,j}, && \pi^{n-1}_{i,j} \circ d^{\Pi}_n = d^h_{i+1,j} \circ \pi^n_{i+1,j} + d^v_{i,j+1} \circ \pi^n_{i,j+1},
\end{aligned}$$

   where $\iota^n_{i,j} : X_{i,j} \to \mathrm{Tot}^{\amalg}_n(X_{\bullet\bullet})$ and $\pi^n_{i,j} : \mathrm{Tot}^{\Pi}_n(X_{\bullet\bullet}) \to X_{i,j}$ are the inclusion and projection morphisms for the coproduct and product.

2. Every morphism $f_{\bullet\bullet} : X_{\bullet\bullet} \to Y_{\bullet\bullet}$ of double complexes induces chain maps

$$\begin{aligned}
&f^{\amalg}_\bullet : \mathrm{Tot}^{\amalg}_\bullet(X_{\bullet\bullet}) \to \mathrm{Tot}^{\amalg}_\bullet(Y_{\bullet\bullet}) \qquad && f^{\amalg}_n \circ \iota^{Xn}_{i,j} = \iota^{Yn}_{i,j} \circ f_{i,j}\\
&f^{\Pi}_\bullet : \mathrm{Tot}^{\Pi}_\bullet(X_{\bullet\bullet}) \to \mathrm{Tot}^{\Pi}_\bullet(Y_{\bullet\bullet}) && \pi^{nY}_{i,j} \circ f^{\Pi}_n = f_{i,j} \circ \pi^{nX}_{i,j}.
\end{aligned}$$

3. This defines functors $\mathrm{Tot}^{\amalg}, \mathrm{Tot}^{\Pi} : \mathrm{DCh}_{\mathcal{A}} \to \mathrm{Ch}_{\mathcal{A}}$.

Proof:
We prove the claims for the chain complex $\mathrm{Tot}^{\amalg}_\bullet(X_{\bullet\bullet})$. The proof for $\mathrm{Tot}^{\Pi}_\bullet(X_{\bullet\bullet})$ is analogous.

1. From the definition of $d^{\amalg}_n : \mathrm{Tot}^{\amalg}_n \to \mathrm{Tot}^{\amalg}_{n-1}$ we obtain for all $i, j \in \mathbb{Z}$ and $n = i + j$

$$\begin{aligned}
&d^{\amalg}_{n+1} \circ d^{\amalg}_{n+2} \circ \iota^{n+2}_{i+1,j+1} = d^{\amalg}_{n+1} \circ \iota^{n+1}_{i,j+1} \circ d^h_{i+1,j+1} + d^{\amalg}_{n+1} \circ \iota^{n+1}_{i+1,j} \circ d^v_{i+1,j+1}\\
&= \iota^n_{i-1,j+1} \circ d^h_{i,j+1} \circ d^h_{i+1,j+1} + \iota^n_{i,j} \circ (d^v_{i,j+1} \circ d^h_{i+1,j+1} + d^h_{i+1,j} \circ d^v_{i+1,j+1}) + \iota^n_{i+1,j-1} \circ d^v_{i+1,j} \circ d^v_{i+1,j+1} = 0.
\end{aligned}$$

By the universal property of the coproduct, this implies $d^{\amalg}_n \circ d^{\amalg}_{n+1} = 0$ for all $n \in \mathbb{Z}$.

2. If $f_{\bullet\bullet} : X_{\bullet\bullet} \to Y_{\bullet\bullet}$ is a morphism of double complexes, then we have

$$\begin{aligned}
d^{Y\amalg}_n \circ f^{\amalg}_n \circ \iota^{Xn}_{i,j} &= d^{Y\amalg}_n \circ \iota^{Yn}_{i,j} \circ f_{i,j} = \iota^{Y\,n-1}_{i-1,j} \circ d^{Yh}_{i,j} \circ f_{i,j} + \iota^{Yn-1}_{i,j-1} \circ d^{Yv}_{i,j} \circ f_{i,j}\\
&= \iota^{Yn-1}_{i-1,j} \circ f_{i-1,j} \circ d^{Xh}_{i,j} + \iota^{Yn-1}_{i,j-1} \circ f_{i,j-1} \circ d^{Xv}_{i,j} = f^{\amalg}_{n-1} \circ (\iota^{Xn-1}_{i-1,j} \circ d^{Xh}_{i,j} + \iota^{Xn-1}_{i,j-1} \circ d^{Xh}_{i,j})\\
&= f^{\amalg}_{n-1} \circ d^{X\amalg}_n \circ \iota^{Xn}_{i,j}
\end{aligned}$$

for $i, j \in \mathbb{Z}$ and $n = i + j$. This implies $d^{Y\amalg}_n \circ f^{\amalg}_n = f^{\amalg}_{n-1} \circ d^{X\amalg}_n$ by the universal property of the coproduct and shows that $f^{\amalg}_\bullet : \mathrm{Tot}^{\amalg}_\bullet(X_{\bullet\bullet}) \to \mathrm{Tot}^{\amalg}_\bullet(Y_{\bullet\bullet})$ is a chain map. As this is compatible with the composition and the identity morphisms, we obtain a functor $\mathrm{Tot}^{\amalg} : \mathrm{DCh}_{\mathcal{A}} \to \mathrm{Ch}_{\mathcal{A}}$.
□

The total complexes of a double complex allow one to describe double complexes in an abelian category $\mathcal{A}$ in terms of chain complexes in $\mathcal{A}$. Moreover, one has sufficient conditions on the double complexes that imply the exactness of the associated total complexes in $\mathcal{A}$. Whenever a double complex is bounded from below or from the left, the exactness of the row or column complexes guarantees that its total complexes are also exact.

Lemma 4.5.5: Let $X_{\bullet\bullet} = (X_{i,j})_{i,j\in\mathbb{Z}}$ be a double complex in an abelian category $\mathcal{A}$ in which products and coproducts exist for all countable families of objects. Then:

1. The total complex $\mathrm{Tot}^{\amalg}_\bullet(X_{\bullet\bullet})$ is exact if all row complexes $X^j_\bullet = (X_{i,j}, d^h_{i,j})_{i\in\mathbb{Z}}$ are exact and $X_{\bullet\bullet}$ is bounded below or if all column complexes $X^i_\bullet = (X_{i,j}, d^v_{i,j})_{j\in\mathbb{Z}}$ are exact and $X_{\bullet\bullet}$ is bounded on the left.

2. The total complex $\mathrm{Tot}^{\Pi}_\bullet(X_{\bullet\bullet})$ is exact if all row complexes $X^j_\bullet = (X_{i,j}, d^h_{i,j})_{i\in\mathbb{Z}}$ are exact and $X_{\bullet\bullet}$ is bounded on the left or if all column complexes $X^i_\bullet = (X_{i,j}, d^v_{i,j})_{j\in\mathbb{Z}}$ are exact and $X_{\bullet\bullet}$ is bounded below.

Proof:
We prove the claims for $\mathcal{A} = R$-Mod and for exact row complexes. The proof for exact column complexes is analogous.

1. It is sufficient to prove exactness in $\mathrm{Tot}^{\amalg}_0(X_{\bullet\bullet}) = \amalg_{n\in\mathbb{Z}} X_{-n,n} = \amalg_{n\in\mathbb{N}_0} X_{-n,n}$ for a double complex $X_{\bullet\bullet}$ with $X_{i,j} = 0$ for all $j < 0$. All other cases can be obtained from this by renumbering the rows and columns. Let $x = (x_n)_{n\in\mathbb{N}_0}$ with $x_n \in X_{-n,n}$ be in $\ker(d^{\amalg}_0) \subset \mathrm{Tot}^{\amalg}_0(X_{\bullet\bullet})$. Then $d^{\amalg}_0(x) = (d^h_{-n,n}(x_n) + d^v_{-n-1,n+1}(x_{n+1}))_{n\in\mathbb{N}_0} = 0$, and by definition of the coproduct there is an $i \in \mathbb{N}_0$ with $x_n = 0$ for all $n > i$. This implies

$$d^h_{-i,i}(x_i) = 0, \qquad d^h_{-n,n}(x_n) + d^v_{-n-1,n+1}(x_{n+1}) = 0 \text{ for } 0 < n < i, \qquad d^v_{0,0}(x_0) = 0.$$

We construct an element $y = (y_n)_{n\in\mathbb{N}_0} \in \mathrm{Tot}^{\amalg}_1(X_{\bullet\bullet})$ with $y_n \in X_{-n+1,n}$ and $d^{\amalg}_1(y) = x$. For this, we set $y_n = 0$ for $n > i$ and inductively construct elements $y_{i-m} \in X_{-i+m+1,i-m}$ for $m \in \mathbb{N}_0$.

For $m = 0$, the exactness of the row complexes and the identity $d^h_{-i,i}(x_i) = 0$ imply that there is an element $y_i \in X_{-i+1,i}$ with $d^h_{-i+1,i}(y_i) = x_i$. Suppose we constructed for $k \in \{i, i-1, ..., j+1\}$ elements $y_k \in X_{-k+1,k}$ that satisfy $x_k = d^h_{-k+1,k}(y_k) + d^v_{-k,k+1}(y_{k+1})$. Then we have

$$\begin{aligned}
&d^h_{-j,j}(x_j - d^v_{-j,j+1}(y_{j+1})) = d^h_{-j,-j}(x_j) + d^v_{-j-1,j+1} \circ d^h_{-j,j+1}(y_{j+1})\\
=&d^h_{-j,-j}(x_j) + d^v_{-j-1,j+1}(x_{j+1} - d^v_{-j-1,j+2}(y_{j+2})) = d^h_{-j,j}(x_j) + d^v_{-j-1,j+1}(x_{j+1}) = 0,
\end{aligned}$$

and by exactness of the row complex there is a $y_j \in X_{-j+1,j}$ with $d^h_{-j+1,j}(y_j) = x_j - d^v_{-j,j+1}(y_{j+1})$. For $j = -1$, we obtain $x_{-1} = 0$ and $d^h_{1,-1} \circ d^v_{1,0}(y_0) = -d^v_{0,0}(x_0) = 0$. Hence we can choose $y_n = 0$ for $n \leq -1$ and obtain an element $y = (y_n)_{n\in\mathbb{N}_0} \in \mathrm{Tot}^{\amalg}_1$ with $d^{\amalg}_1(y) = x$.

2. It is sufficient to prove exactness in $\mathrm{Tot}^{\Pi}_0(X_{\bullet\bullet}) = \Pi_{n\in\mathbb{Z}} X_{n,-n} = \Pi_{n\in\mathbb{N}_0} X_{n,-n}$ for double complexes $X_{\bullet\bullet}$ with $X_{i,j} = 0$ for $i < 0$. All other cases are obtained from this by renumbering the rows and columns. Let $x = (x_n)_{n\in\mathbb{N}_0} \in \ker(d^{\Pi}_0) \subset \mathrm{Tot}^{\Pi}_0(X_{\bullet\bullet})$ with $x_n \in X_{n,-n}$. Then

$$d^h_{n,-n}(x_n) + d^v_{n-1,-n+1}(x_{n-1}) = 0 \ \text{ for } \ n \geq 1 \qquad\qquad d^h_{0,0}(x_0) = 0.$$

We construct an element $y = (y_n)_{n\in\mathbb{N}_0} \in \mathrm{Tot}^{\Pi}_1(X_{\bullet\bullet})$ with $y_n \in X_{n+1,-n}$ and $d^{\Pi}_1(y) = x$ by induction over $n \in \mathbb{N}_0$. For $n = 0$ the exactness of the row complex and the identity $d^h_{0,0}(x) = 0$ imply that there is an element $y_0 \in X_{1,0}$ with $d^h_{1,0}(y_0) = x_0$. Suppose we constructed elements $y_k \in X_{k+1,-k}$ with $x_k = d^h_{k+1,-k}(y_k) + d^v_{k,-k+1}(y_{k-1})$ for all $0 \leq k < n$. Then

$$\begin{aligned}
&d^h_{n,-n}(x_n - d^v_{n,-n+1}(y_{n-1})) = d^h_{n,-n}(x_n) + d^v_{n-1,-n+1} \circ d^h_{n,-n+1}(y_{n-1})\\
&= d^h_{n,-n}(x_n) + d^v_{n-1,-n+1}(x_{n-1} - d^v_{n-1,-n+2}(y_{n-2})) = d^h_{n,-n}(x_n) + d^v_{n-1,-n+1}(x_{n-1}) = 0.
\end{aligned}$$

As the row complexes are exact, there is a $y_n \in X_{n+1,-n}$ with $d^h_{n+1,-n}(y_n) = x_n - d^v_{n,-n+1}(y_{n-1})$. Hence we constructed an element $y = (y_n)_{n\in\mathbb{N}_0} \in \mathrm{Tot}^{\Pi}_1(X_{\bullet\bullet})$ with $d^{\Pi}_1(y) = x$. □

We can now apply these results to the double complexes from Example 4.5.3. By tensoring a projective resolution $L_\bullet$ of an R-right module L and a projective resolution $M_\bullet$ of an R-left module M and omitting the terms with $L_{-1} = L$ or $M_{-1} = M$, we obtain bounded double complexes with, respectively, exact rows or columns. By Lemma 4.5.5 the associated total complexes are exact. Tensoring the resolution $L_\bullet$ with M or the resolution $M_\bullet$ with L over R yields a short exact sequence of chain complexes. The associated long exact homology sequence then relates $\mathrm{Tor}_n^R(L,M) = L_n(L\otimes_R -)(M)$ to $\mathrm{Tor}_n'^R(L,M) = L_n(-\otimes_R M)(L)$. An analogous procedure for R-left modules M,N relates $\mathrm{Ext}^n_R(M,N) = R^n\mathrm{Hom}_R(-,N)(M)$ and $\mathrm{Ext}'^n_R(M,N) = R^n\mathrm{Hom}_R(M,-)(N)$.

Theorem 4.5.6: Let R be a ring. Then for all R-left modules M,N and R-right modules L

$$\mathrm{Tor}_n^R(L,M)\cong \mathrm{Tor}_n'^R(L,M)\qquad \mathrm{Ext}^n_R(M,N)\cong\mathrm{Ext}'^n_R(M,N)\qquad\qquad \forall n\in\mathbb{N}_0.$$

and for all R-linear maps $f: L\to L'$, $g: M\to M'$ and $h: N\to N'$

$$\begin{aligned}&\mathrm{Tor}_n'^R(f,g) = L_n(-\otimes g)(f) = L_n(f\otimes -)(g) = \mathrm{Tor}_n^R(f,g)\\ &\mathrm{Ext}'^n_R(g,h) = R^n\mathrm{Hom}_R(g,-)(h) = R^n\mathrm{Hom}(-,h)(g) = \mathrm{Ext}^n_R(g,h).\end{aligned}$$

Proof:
We prove $\mathrm{Tor}_n^R(L,M) = \mathrm{Tor}_n'^R(L,M)$. The proof of $\mathrm{Ext}^n_R(M,N) = \mathrm{Ext}'^n_R(M,N)$ is analogous.

We choose projective resolutions $L_\bullet$ of L and $M_\bullet$ of M in, respectively, R^{op}-Mod and R-Mod and denote by $\bar L_\bullet = L_{\bullet\geq 0}$ and $\bar M_\bullet = M_{\bullet\geq 0}$ the associated chain complexes with $L_{-1} = L$ and $M_{-1} = M$ replaced by 0. Let $X_{\bullet\bullet}$ and $Y_{\bullet\bullet}$ be the double complexes in Ab from Example 4.5.3

$$X_{i,j} = L_i\otimes_R \bar M_j,\qquad Y_{i,j} = \bar L_i\otimes_R M_j,\qquad d^h_{i,j} = d^L_i\otimes \mathrm{id}_{M_j},\qquad d^v_{i,j} = (-1)^i\,\mathrm{id}_{L_i}\otimes d^M_j.\qquad (45)$$

As $\bar L_i$ and $\bar M_i$ are projective for all $i\geq -1$, they are flat by Corollary 4.2.4 and Exercise 55. As $L_\bullet$ and $M_\bullet$ are exact, it follows that all column complexes $Y^i_\bullet = \bar L_i\otimes M_\bullet$ and row complexes $X^j_\bullet = L_\bullet\otimes_R \bar M_j$ are exact as well. As $X_{\bullet\bullet}$ and $Y_{\bullet\bullet}$ are bounded from the left and from below, it follows that the total complexes $\mathrm{Tot}^{\mathrm{II}}_\bullet(X_{\bullet\bullet})$ and $\mathrm{Tot}^{\mathrm{II}}_\bullet(Y_{\bullet\bullet})$ are exact by Lemma 4.5.5.

Denoting by $L'_\bullet$ and $M'_\bullet$ the chain complexes with $L'_{-1} = L$, $M'_{-1} = M$ and $L'_i = M'_i = 0$ for $i\neq -1$, we have short exact sequences of chain complexes

$$0\to L'_\bullet\xrightarrow{\iota^L_\bullet} L_\bullet\xrightarrow{\pi^L_\bullet}\bar L_\bullet\to 0\qquad\qquad 0\to M'_\bullet\xrightarrow{\iota^M_\bullet} M_\bullet\xrightarrow{\pi^M_\bullet}\bar M_\bullet\to 0.$$

As L_j and M_j are projective for all $j\in\mathbb{N}_0$, this defines short exact sequences of chain complexes

$$\begin{aligned}&0\to L'_\bullet\otimes_R M_j\xrightarrow{\iota^L_\bullet\otimes\mathrm{id}_{M_j}} L_\bullet\otimes_R M_j\xrightarrow{\pi^L_\bullet\otimes\mathrm{id}_{M_j}}\bar L_\bullet\otimes_R M_j\to 0\\ &0\to L_j\otimes_R M'_\bullet\xrightarrow{\mathrm{id}_{L_j}\otimes\iota^M_\bullet} L_j\otimes_R M_\bullet\xrightarrow{\mathrm{id}_{L_j}\otimes\pi^M_\bullet} L_j\otimes_R\bar M_\bullet\to 0\end{aligned}$$

in Ab for all $j\in\mathbb{N}_0$ and short exact sequences of double complexes

$$0\to L'_{\bullet\bullet}\xrightarrow{\iota^L_{\bullet\bullet}} X_{\bullet\bullet}\xrightarrow{\pi^L_{\bullet\bullet}} Z_{\bullet\bullet}\to 0\qquad\qquad 0\to M'_{\bullet\bullet}\xrightarrow{\iota^M_{\bullet\bullet}} Y_{\bullet\bullet}\xrightarrow{\pi^M_{\bullet\bullet}} Z_{\bullet\bullet}\to 0,\qquad (46)$$

where $L'_{\bullet\bullet}$, $M'_{\bullet\bullet}$ and $Z_{\bullet\bullet}$ are the double complexes with

$$L'_{-1,j} = L\otimes_R\bar M_j,\ L'_{i,j} = 0 \text{ for } i\neq -1,\quad M'_{i,-1} = \bar L_i\otimes_R M,\ M'_{i,j} = 0\text{ for } i\neq -1,\quad Z_{i,j} = \bar L_i\otimes_R\bar M_j.$$

The differentials of these double complexes are given by the differentials of $L'_\bullet$, $M'_\bullet$, $\bar{L}_\bullet$ and $\bar{M}_\bullet$, as in (45). Lemma 4.5.4 yields short exact sequences of total complexes

$$\begin{aligned} &0 \to \mathrm{Tot}^{\mathrm{II}}_\bullet(L'_{\bullet\bullet}) \xrightarrow{\mathrm{Tot}^{\mathrm{II}}_\bullet(\iota^L_{\bullet\bullet})} \mathrm{Tot}^{\mathrm{II}}_\bullet(X_{\bullet\bullet}) \xrightarrow{\mathrm{Tot}^{\mathrm{II}}_\bullet(\pi^L_{\bullet\bullet})} \mathrm{Tot}^{\mathrm{II}}_\bullet(Z_{\bullet\bullet}) \to 0 \\ &0 \to \mathrm{Tot}^{\mathrm{II}}_\bullet(M'_{\bullet\bullet}) \xrightarrow{\mathrm{Tot}^{\mathrm{II}}_\bullet(\iota^M_{\bullet\bullet})} \mathrm{Tot}^{\mathrm{II}}_\bullet(Y_{\bullet\bullet}) \xrightarrow{\mathrm{Tot}^{\mathrm{II}}_\bullet(\pi^M_{\bullet\bullet})} \mathrm{Tot}^{\mathrm{II}}_\bullet(Z_{\bullet\bullet}) \to 0. \end{aligned} \tag{47}$$

As $X_\bullet, Y_\bullet$ are bounded below and on the left, $\mathrm{Tot}^{\mathrm{II}}_\bullet(X_{\bullet\bullet})$ and $\mathrm{Tot}^{\mathrm{II}}_\bullet(Y_{\bullet\bullet})$ are exact by Lemma 4.5.5, all homologies $H_n(\mathrm{Tot}^{\mathrm{II}}_\bullet(X_{\bullet\bullet}))$ and $H_n(\mathrm{Tot}^{\mathrm{II}}_\bullet(Y_{\bullet\bullet}))$ vanish. Their long exact homology sequences from Theorem 3.4.7 take the form

$$...0 \to H_{n+1}(\mathrm{Tot}^{\mathrm{II}}_\bullet(Z_{\bullet\bullet})) \xrightarrow{\partial^L_{n+1}} H_n(\mathrm{Tot}^{\mathrm{II}}_\bullet(L'_{\bullet\bullet})) \to 0 \to H_n(\mathrm{Tot}^{\mathrm{II}}_\bullet(Z_{\bullet\bullet})) \xrightarrow{\partial^L_n} H_{n-1}(\mathrm{Tot}^{\mathrm{II}}_\bullet(L'_{\bullet\bullet})) \to 0...$$
$$...0 \to H_{n+1}(\mathrm{Tot}^{\mathrm{II}}_\bullet(Z_{\bullet\bullet})) \xrightarrow{\partial^M_{n+1}} H_n(\mathrm{Tot}^{\mathrm{II}}_\bullet(M'_{\bullet\bullet})) \to 0 \to H_n(\mathrm{Tot}^{\mathrm{II}}_\bullet(Z_{\bullet\bullet})) \xrightarrow{\partial^M_n} H_{n-1}(\mathrm{Tot}^{\mathrm{II}}_\bullet(M'_{\bullet\bullet})) \to 0...$$

and $\partial^L_n : H_n(\mathrm{Tot}^{\mathrm{II}}_\bullet(Z_{\bullet\bullet})) \to H_{n-1}(\mathrm{Tot}^{\mathrm{II}}_\bullet(L'_{\bullet\bullet}))$, $\partial^M_n : H_n(\mathrm{Tot}^{\mathrm{II}}_\bullet(Z_{\bullet\bullet})) \to H_{n-1}(\mathrm{Tot}^{\mathrm{II}}_\bullet(M'_{\bullet\bullet}))$ are isomorphisms for all $n \in \mathbb{N}$. By definition, we have

$$\mathrm{Tot}^{\mathrm{II}}_{n-1}(L'_{\bullet\bullet}) = L\otimes_R\bar{M}_n = (L\otimes_R-)(\bar{M}_n) \qquad \mathrm{Tot}^{\mathrm{II}}_{n-1}(M'_{\bullet\bullet}) = \bar{L}_n\otimes_R M = (-\otimes_R M)(\bar{L}_n)$$

and hence

$$\begin{aligned} \mathrm{Tor}'^R_n(L,M) &= L_n(-\otimes_R M)(L) = H_{n-1}(\mathrm{Tot}^{\mathrm{II}}_\bullet(M'_{\bullet\bullet})) \cong H_n(\mathrm{Tot}^{\mathrm{II}}_\bullet(Z_{\bullet\bullet})) \\ &\cong H_{n-1}(\mathrm{Tot}^{\mathrm{II}}_\bullet(L'_{\bullet\bullet})) = L_n(L\otimes_R-)(M) = \mathrm{Tor}^R_n(L,M). \end{aligned}$$

All R-linear maps $f : L \to L'$ and $g : M \to M'$ extend to chain maps $f_\bullet : L_\bullet \to L'_\bullet$ and $g_\bullet : M_\bullet \to M'_\bullet$ between the projective resolutions, to chain maps between the associated short exact sequences of double complexes in (46) and to chain maps between the associated short exact sequences of total complexes in (47). By Proposition 3.4.8 they induce chain maps between the associated long exact homology sequences, and this implies

$$\mathrm{Tor}'^R_n(f,g) = L_n(-\otimes g)(f) = L_n(f\otimes-)(g) = \mathrm{Tor}^R_n(f,g),$$

where $\mathrm{Tor}^R_n(f,-) = L_n(f\otimes-) : \mathrm{Tor}^R_n(L,-) \to \mathrm{Tor}^R_n(L',-)$ is the natural transformation from Remark 4.4.2, 3. and $\mathrm{Tor}'^R_n(-,g) = L_n(-\otimes g)$ its counterpart for Tor'^R_n. □

Theorem 4.5.6 shows that the choices involved in the definition of Tor and Ext, namely the decision to tensor on the *left* and to take R-linear maps *into* a given module are of no consequence, since tensoring on the *right* and taking R-linear maps *from* a module lead to the same functors. Moreover, it shows that Tor and Ext are functors in both arguments and that applying morphisms in the first and in the second argument commutes. In other words, they define functors from the product categories R^{op}-Mod × R-Mod and R-Mod^{op} × R-Mod. Such functors are also called **bifunctors**.

Corollary 4.5.7: For any ring R, the functors Tor and Ext define a family of functors

$$\mathrm{Tor}^R_n : R^{op}\text{-Mod} \times R\text{-Mod} \to \mathrm{Ab} \qquad \mathrm{Ext}^n_R : R\text{-Mod}^{op} \times R\text{-Mod} \to \mathrm{Ab}.$$

4.6 Tensor products of chain complexes

Total complexes of double complexes have other interesting applications beyond Tor and Ext. In particular, they allow one to define tensor products of chain complexes in R-Mod. In this section, we investigate the properties of such tensor products and derive a general formula that relates the homologies of two chain complexes to the homologies of their tensor product, the *Künneth formula*. This formula has important applications in singular homology. It allows one to compute the homologies of product spaces and characterises the homologies $H_n(X,k)$ of a topological space X for a commutative ring k in terms of the homologies $H_n(X,\mathbb{Z})$.

Definition 4.6.1: Let $X_\bullet$ be a chain complex in R^{op}-Mod and $X'_\bullet$ a chain complex in R-Mod. The **tensor product** $X_\bullet \otimes X'_\bullet$ is the total complex of the double complex in Example 4.5.3, 1:

$$\begin{aligned}
&(X_\bullet \otimes X'_\bullet)_n = \oplus_{k\in\mathbb{Z}}\, X_k \otimes_R X'_{n-k},\\
&d_n^\otimes : (X_\bullet \otimes X'_\bullet)_n \to (X_\bullet \otimes X'_\bullet)_{n-1},\\
&d_n^\otimes \circ \iota_{k,n-k} = \iota_{k-1,n-k} \circ (d_k \otimes \mathrm{id}_{X'_{n-k}}) + (-1)^k \iota_{k,n-k-1} \circ (\mathrm{id}_{X_k} \otimes d'_{n-k})
\end{aligned}$$

where $\iota_{k,n-k} : X_k \otimes_R X'_{n-k} \to (X_\bullet \otimes X'_\bullet)_n$ are the inclusions for the direct sums.

Tensor products of chain complexes give a more intuitive description of chain homotopies that is closer to the definition of homotopies between continuous maps. Although chain homotopies were defined by a technical condition in Definition 3.3.1, it was already shown in Remark 3.3.3 that a chain homotopy $h_\bullet : f_\bullet \Rightarrow g_\bullet$ between chain maps $f_\bullet, g_\bullet : X_\bullet \to X'_\bullet$ in R-Mod can be viewed as a chain map $h_\bullet : Y_\bullet \to X'_\bullet$ for a certain chain complex $Y_\bullet$ constructed from $X_\bullet$ with $h_\bullet \circ \iota^0_\bullet = f_\bullet$ and $h_\bullet \circ \iota^1_\bullet = g_\bullet$ for inclusion chain maps $\iota^0_\bullet, \iota^1_\bullet : X_\bullet \to Y_\bullet$. By using the tensor product of chain complexes, we can show that the chain complex $Y_\bullet$ in Remark 3.3.3 is just the tensor product of $X_\bullet$ with a standard chain complex in R-Mod-R.

Example 4.6.2: Let R be a commutative ring and consider the chain complex

$$\Delta^1_\bullet = 0 \to R \xrightarrow{(\mathrm{id},-\mathrm{id})} R \oplus R \to 0$$

in R-Mod-R with the module structures given by left and right multiplication. Then for any chain complex $X_\bullet$ in R-Mod the tensor product $\Delta^1_\bullet \otimes X_\bullet$ is given by

$$(\Delta^1_\bullet \otimes X_\bullet)_n = \Delta^1_0 \otimes_R X_n \oplus \Delta^1_1 \otimes_R X_{n-1} = (R \oplus R) \otimes_R X_n \oplus R \otimes_R X_{n-1} \cong X_n \oplus X_n \oplus X_{n-1}$$

and the boundary morphism $d_n^\otimes : (\Delta^1_\bullet \otimes X_\bullet)_n \to (\Delta^1_\bullet \otimes X_\bullet)_{n-1}$ take the form

$$d_n^\otimes : X_n \oplus X_n \oplus X_{n-1} \to X_{n-1} \oplus X_{n-1} \oplus X_{n-2}, (x, x', x'') \mapsto (d_n(x) + x'', d_n(x') - x'', -d_{n-1}(x'')).$$

This is the chain complex $Y_\bullet$ from Remark 3.3.3.

Together with Remark 3.3.3, this shows that a chain homotopy from a chain map $f_\bullet : X_\bullet \to X'_\bullet$ to $g_\bullet : X_\bullet \to X'_\bullet$ can be viewed as a chain map $h_\bullet : \Delta^1_\bullet \otimes_R X_\bullet \to X'_\bullet$ with $h_\bullet \circ \iota^0_\bullet = f_\bullet$ and $h_\bullet \circ \iota^1_\bullet = g_\bullet$ for the canonical inclusions $\iota^0_\bullet, \iota^1_\bullet : X_\bullet \to \Delta^1_\bullet \otimes X_\bullet$ from Remark 3.3.3. This is the $\mathrm{Ch}_{R\text{-Mod}}$-counterpart of the definition of a homotopy in Top, where a homotopy $h : f \Rightarrow g$ between continuous maps $f, g : X \to X'$ is defined as a continuous map $h : [0,1] \times X \to X'$ with $h \circ \iota^0 = f$ and $h \circ \iota^1 = g$ for the inclusions $\iota^i : X \to [0,1] \times X$, $x \mapsto (i, x)$.

The chain complex $\Delta^1_\bullet$ in $\mathrm{Ch}_{R\text{-Mod}}$ plays the role of the unit interval $[0,1]$ in Top. This is more than just an analogy. We can view unit interval $[0,1]$ as a simplicial complex that consists of two 0-simplices representing its endpoints and a single 1-simplex. Then the associated simplicial chain complex from Definition 2.1.9 becomes precisely the chain complex $\Delta^1_\bullet$.

We will now derive a formula for the homologies of the tensor product $X_\bullet \otimes X'_\bullet$ in terms of the homologies of $X_\bullet$ and $X'_\bullet$. For this we need the assumption that all R-modules X_n and also their images $d_n(X_n)$ are flat. Note that the first assumption is satisfied if all objects in $X_\bullet$ are projective and, in particular, if all modules X_n are free. If the underlying ring R is a prinicpal ideal domain, the condition that each module X_n is free also ensures $d_n(X_n) \subset X_n$ is free by Example 1.1.16, 3 and hence flat. Note that this holds in particular for the chain complexes $C_\bullet(X, k)$ in singular homology if k is a principal ideal domain.

The key idea is to view the flat chain complex $X_\bullet$ as the middle term in a short exact sequence of chain complexes. This short exact sequence is obtained by taking the modules $Z_n = \ker(d_n)$ and $B_n = d_n(X_n)$ and combining them into two chain complexes $Z_\bullet$ and $B_\bullet$ with trivial boundary morphisms. The inclusions and the boundary morphisms then define a short exact sequence $0 \to Z_\bullet \to X_\bullet \to B_\bullet \to 0$. By tensoring this short exact sequence of chain complexes with $X'_\bullet$ one obtains a short exact sequence of double complexes and a short exact sequence of their total complexes. The result then follows by computing its long exact sequence of homologies.

Theorem 4.6.3: (**Künneth formula for chain complexes**)
Let $X_\bullet$ be a chain complex in R^{op}-Mod and $X'_\bullet$ a chain complex in R-Mod. If X_n and $d_n(X_n)$ are flat for all $n \in \mathbb{Z}$, then there is a short exact sequence

$$0 \to \bigoplus_{k\in\mathbb{Z}} H_k(X_\bullet) \otimes_R H_{n-k}(X'_\bullet) \xrightarrow{i_n} H_n(X_\bullet \otimes X'_\bullet) \xrightarrow{p_n} \bigoplus_{k\in\mathbb{Z}} \mathrm{Tor}_1^R(H_k(X_\bullet), H_{n-k-1}(X'_\bullet)) \to 0$$

with $i_n : \oplus_{k\in\mathbb{Z}} H_k(X_\bullet) \otimes_R H_{n-k}(X'_\bullet) \to H_n(X_\bullet \otimes X'_\bullet)$, $[x_k] \otimes [x'_{n-k}] \mapsto [x_k \otimes x'_{n-k}]$.

Proof:
1. To compute the homologies $H_n(X_\bullet \otimes X'_\bullet)$, we consider the chain complexes $Z_\bullet$, $B_\bullet$ with $Z_n = \ker(d_n) \subset X_n$, $B_n = d_n(X_n) \subset Z_{n-1}$ and with zero boundary morphisms. Then the inclusions $\iota_n : \ker(d_n) \to X_n$ and the corestrictions $d_n : X_n \to d_n(X_n)$ define a short exact sequence of chain complexes

$$0 \to Z_\bullet \xrightarrow{\iota_\bullet} X_\bullet \xrightarrow{d_\bullet} B_\bullet \to 0. \tag{48}$$

For any R-module M, the functor $-\otimes_R M : R^{op}\text{-Mod} \to \mathrm{Ab}$ is right exact with left derived functors $\mathrm{Tor}'^R_n(-, M) = L_n(-\otimes_R M) : R^{op}\text{-Mod} \to \mathrm{Ab}$, and by Theorem 4.5.6 we have $\mathrm{Tor}^R_n(L, M) \cong \mathrm{Tor}'^R_n(L, M)$ for any R^{op}-module L. With this, one finds that the long exact sequence of left derived functors from Theorem 4.3.5 for the short exact sequence $0 \to Z_n \xrightarrow{\iota_n} X_n \xrightarrow{d_n} B_n \to 0$ in R^{op}-Mod takes the form

$$\begin{aligned} &\ldots \to \mathrm{Tor}^R_{k+1}(B_n, M) \xrightarrow{\partial_{k+1}} \mathrm{Tor}^R_k(Z_n, M) \to \mathrm{Tor}^R_k(X_n, M) \to \mathrm{Tor}^R_k(B_n, M) \xrightarrow{\partial_k} \ldots \\ &\ldots \to \mathrm{Tor}^R_1(B_n, M) \xrightarrow{\partial_1} Z_n \otimes_R M \xrightarrow{\iota_n \otimes \mathrm{id}_M} X_n \otimes_R M \xrightarrow{d_n \otimes \mathrm{id}_M} B_n \otimes_R M \to 0. \end{aligned}$$

As the R^{op}-modules X_n and B_n are flat, we have $\mathrm{Tor}^R_k(X_n, M) = \mathrm{Tor}^R_k(B_n, M) = 0$ for all $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$ by Remark 4.4.2, 2. The long exact sequence of left derived functors simplifies to

$$\ldots \to 0 \xrightarrow{\partial_{k+1}} \mathrm{Tor}^R_k(Z_n, M) \to 0 \to \ldots \to 0 \xrightarrow{\partial_1} Z_n \otimes_R M \xrightarrow{\iota_n \otimes_R M} X_n \otimes_R M \xrightarrow{d_n \otimes \mathrm{id}_M} B_n \otimes_R M \to 0.$$

The exactness of this sequence implies that $\mathrm{Tor}_k^R(Z_n, M) = 0$ for all $k \in \mathbb{N}$ and hence Z_n is flat for all $n \in \mathbb{N}_0$ by Remark 4.4.2, 2. The last terms on the right form the short exact sequence

$$0 \to Z_n \otimes_R M \xrightarrow{\iota_n \otimes_R \mathrm{id}} X_n \otimes_R M \xrightarrow{d_n \otimes \mathrm{id}} B_n \otimes_R M \to 0. \tag{49}$$

2. To compute the homologies $H_n(X_\bullet \otimes X'_\bullet)$ we tensor the short exact sequence (48) with the chain complex $X'_\bullet$. This yields a short exact sequence of double complexes

$$0 \to Z_{\bullet\bullet} \xrightarrow{\iota_{\bullet\bullet}} X_{\bullet\bullet} \xrightarrow{d_{\bullet\bullet}} B_{\bullet\bullet} \to 0$$

with $Z_{k,l} = Z_k \otimes_R X'_l$, $X_{k,l} = X_k \otimes_R X'_l$, $B_{k,l} = B_k \otimes_R X'_l$ and the morphisms $\iota_{\bullet\bullet}$ and $d_{\bullet\bullet}$ given by $\iota_{k,l} = \iota_k \otimes \mathrm{id}_{X'_l}$, $d_{k,l} = d_k \otimes \mathrm{id}_{X'_l}$. Its exactness follows by setting $M = X'_l$ in the short exact sequence (49). It induces a short exact sequence of the associated total complexes

$$0 \to Z_\bullet \otimes X'_\bullet \xrightarrow{\iota^\otimes_\bullet} X_\bullet \otimes X'_\bullet \xrightarrow{d^\otimes_\bullet} B_\bullet \otimes X'_\bullet \to 0$$

with $\iota^\otimes_\bullet$ and $d^\otimes_\bullet$ given by $\iota^\otimes_n(z_k \otimes x'_{n-k}) = \iota_k(z_k) \otimes x'_{n-k}$ and $d^\otimes_n(x_k \otimes x'_{n-k}) = d_k(x_k) \otimes x'_{n-k}$ for all $x_k \in X_k$, $z_k \in Z_k$ and $x'_{n-k} \in X'_{n-k}$. Because the chain complexes $B_\bullet$ and $Z_\bullet$ have trivial boundary morphisms, the differentials of $Z_\bullet \otimes X'_\bullet$ and $B_\bullet \otimes X'_\bullet$ are given by

$$d_n(z_k \otimes x'_{n-k}) = (-1)^k z_k \otimes d'_{n-k}(x'_{n-k}) \qquad d_n(b_k \otimes x'_{n-k}) = (-1)^k b_k \otimes d'_{n-k}(x'_{n-k}) \tag{50}$$

for all $z_k \in Z_k$, $b_k \in B_k$, $x'_{n-k} \in X'_{n-k}$. The associated long exact homology sequence reads

$$\ldots \xrightarrow{\partial^\otimes_{n+1}} H_n(Z_\bullet \otimes X'_\bullet) \xrightarrow{H_n(\iota^\otimes_\bullet)} H_n(X_\bullet \otimes X'_\bullet) \xrightarrow{H_n(d^\otimes_\bullet)} H_n(B_\bullet \otimes X'_\bullet) \xrightarrow{\partial^\otimes_n} H_{n-1}(Z_\bullet \otimes X'_\bullet) \xrightarrow{H_{n-1}(\iota^\otimes_\bullet)} \ldots \tag{51}$$

Because B_k is flat by assumption, Z_k is flat by 1. and due to (50) we have

$$H_n(B_\bullet \otimes X'_\bullet) = (B_\bullet \otimes H_\bullet(X'_\bullet))_n \qquad H_n(Z_\bullet \otimes X'_\bullet) = (Z_\bullet \otimes H_\bullet(X'_\bullet))_n.$$

and the R-linear maps $H_n(\iota_\bullet)$ and $H_n(d_\bullet)$ are given by

$$\begin{aligned} H_n(\iota_\bullet) &: \oplus_{k\in\mathbb{Z}} Z_k \otimes_R H_{n-k}(X'_\bullet) \to H_n(X_\bullet \otimes X'_\bullet), \quad z_k \otimes [z'_{n-k}] \mapsto [z_k \otimes z'_{n-k}] \\ H_n(d_\bullet) &: H_n(X_\bullet \otimes X'_\bullet) \to \oplus_{k\in\mathbb{Z}} B_k \otimes_R H_{n-k}(X'_\bullet), \quad [x_k \otimes z'_{n-k}] \mapsto d_k(x_k) \otimes [z'_{n-k}]. \end{aligned} \tag{52}$$

The connecting homomorphisms $\partial^\otimes_n$ in (51) are determined by (30) and given by the inclusions

$$\partial^\otimes_n : \oplus_{k\in\mathbb{Z}} B_k \otimes_R H_{n-k}(X'_\bullet) \to \oplus_{k\in\mathbb{Z}} Z_{k-1} \otimes_R H_{n-k}(X'_\bullet), \quad b_k \otimes [z'_{n-k-1}] \mapsto b_k \otimes [z'_{n-k-1}]. \tag{53}$$

Using that $\mathrm{coker}(\partial^\otimes_{n+1}) \cong H_n(Z_\bullet \otimes X'_\bullet)/\mathrm{im}(\partial^\otimes_{n+1})$ and $\mathrm{im}(H_n(\iota^\otimes_\bullet)) \cong H_n(Z_\bullet \otimes X'_\bullet)/\ker(H_n(\iota^\otimes_\bullet))$ we see that (51) is exact if and only if for all $n \in \mathbb{N}_0$ we have exact sequences

$$0 \to \mathrm{coker}(\partial^\otimes_{n+1}) \xrightarrow{\iota'_n} H_n(X_\bullet \otimes X'_\bullet) \xrightarrow{d'_n} \ker(\partial^\otimes_n) \to 0, \tag{54}$$

where ι'_n and d'_n are induced by $H_n(\iota^\otimes_\bullet)$ and $H_n(d^\otimes_\bullet)$.

3. To compute $\mathrm{Tor}_1^R(H_k(X_\bullet), H_{n-k-1}(X'_\bullet))$ we consider the short exact sequences

$$0 \to B_{k+1} \xrightarrow{i_k} Z_k \xrightarrow{p_k} H_k(X_\bullet) \to 0 \tag{55}$$

with the inclusions $i_k : d_{k+1}(X_{k+1}) \to \ker(d_k)$ and canonical surjections $p_k : Z_k \to H_k(X_\bullet)$. As $\mathrm{Tor}_n^R(M,L) = \mathrm{Tor}'^R_n(M,L)$ by Theorem 4.5.6, the long exact sequence of left derived functors for (55) and $-\otimes_R H_m(X'_\bullet) : R^{op}\text{-Mod} \to \mathrm{Ab}$ reads

$$\ldots \xrightarrow{\partial'_{n+1}} \mathrm{Tor}_n^R(B_{k+1}, H_m(X'_\bullet)) \to \mathrm{Tor}_n^R(Z_k, H_m(X'_\bullet)) \to \mathrm{Tor}_n^R(H_k(X_\bullet), H_m(X'_\bullet)) \xrightarrow{\partial'_n} \ldots$$
$$\ldots \to \mathrm{Tor}_1^R(H_k(X_\bullet), H_m(X'_\bullet)) \xrightarrow{\partial'_1} B_{k+1}\otimes_R H_m(X'_\bullet) \xrightarrow{i_k\otimes\mathrm{id}} Z_k\otimes_R H_m(X'_\bullet) \xrightarrow{p_k\otimes\mathrm{id}} H_k(X_\bullet)\otimes_R H_m(X'_\bullet) \to 0.$$

As B_{k+1} and Z_k are flat, we have $\mathrm{Tor}_n^R(B_{k+1}, H_m(X'_\bullet)) = \mathrm{Tor}_n^R(Z_k, H_m(X'_\bullet)) = 0$ for all $k \in \mathbb{N}$. The exactness of the sequence then implies $\mathrm{Tor}_n^R(H_k(X_\bullet), H_m(X'_\bullet)) = 0$ for $n > 1$, and its last six terms on the right form an exact squence

$$0 \to \mathrm{Tor}_1^R(H_k(X_\bullet), H_m(X'_\bullet)) \xrightarrow{\partial'_1} B_{k+1}\otimes_R H_m(X'_\bullet) \xrightarrow{i_k\otimes\mathrm{id}_M} Z_k\otimes_R H_m(X'_\bullet) \xrightarrow{p_k\otimes\mathrm{id}_M} H_k(X_\bullet)\otimes_R H_m(X'_\bullet) \to 0.$$

Setting $m = n-k$, summing over k and comparing with the expression (53) for the connection morphism, we obtain an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \oplus_{k\in\mathbb{Z}}\mathrm{Tor}_1^R(H_k(X_\bullet), H_{n-k}(X'_\bullet)) \arrow[r, "\alpha"] & \bigoplus_{k\in\mathbb{Z}} B_{k+1}\otimes_R H_{n-k}(X'_\bullet) \arrow[d, "\partial^\otimes_{n+1}"] \\ 0 & \oplus_{k\in\mathbb{Z}} H_k(X_\bullet)\otimes_R H_{n-k}(X'_\bullet) \arrow[l] & \bigoplus_{k\in\mathbb{Z}} Z_k\otimes_R H_{n-k}(X'_\bullet) \arrow[l, "\beta"'] \end{tikzcd} \qquad (56)$$

The exactness of (56) implies

$$\begin{aligned} \ker(\partial^\otimes_{n+1}) &= \mathrm{im}(\alpha) \cong \oplus_{k\in\mathbb{Z}}\mathrm{Tor}_1^R(H_k(X_\bullet), H_{n-k}(X'_\bullet)) \\ \mathrm{coker}(\partial^\otimes_{n+1}) &\cong \mathrm{im}(\beta) = \oplus_{k\in\mathbb{Z}} H_k(X_\bullet)\otimes_R H_{n-k}(X'_\bullet), \end{aligned} \qquad (57)$$

and inserting this into (54) yields the exact sequence in the theorem

$$0 \to \oplus_{k\in\mathbb{Z}} H_k(X_\bullet)\otimes H_{n-k}(X'_\bullet) \xrightarrow{\iota'_n} H_n(X_\bullet\otimes X'_\bullet) \xrightarrow{\pi'_n} \oplus_{k\in\mathbb{Z}}\mathrm{Tor}_1^R(H_k(X_\bullet), H_{n-k-1}(X'_\bullet)) \to 0.$$

From the expression (52) for $H_n(\iota_\bullet)$, we find that ι'_n is given by

$$\iota'_n : \oplus_{k\in\mathbb{Z}} H_k(X_\bullet)\otimes_R H_{n-k}(X'_\bullet) \to H_n(X_\bullet\otimes X'_\bullet), \quad [x_k]\otimes[x'_{n-k}] \mapsto [x_k\otimes x'_{n-k}]. \qquad \square$$

The Künneth formula in Theorem 4.6.3 also exists in a reduced version where the chain complex $X'_\bullet$ is replaced by an R-module M. This is obtained by taking for the chain complex $X'_\bullet$ in Theorem 4.6.3 a trivial chain complex of the form $X'_\bullet = 0 \to M \to 0$. As its only non-trivial entry is $X'_0 = M$, we have $H_0(X'_\bullet) = M$ and all other homologies vanish. Inserting this into Theorem 4.6.3 yields the following corollary.

Corollary 4.6.4: (**Künneth formula for modules**)

Let $X_\bullet$ be a chain complex in R^{op}-Mod such that X_n and $d_n(X_n)$ are flat for all $n \in \mathbb{Z}$. Then for every R-module M, there is a short exact sequence

$$0 \to H_n(X_\bullet)\otimes_R M \xrightarrow{i_n} H_n(X_\bullet\otimes_R M) \xrightarrow{p_n} \mathrm{Tor}_1^R(H_{n-1}(X_\bullet), M) \to 0$$

with $i_n : H_n(X_\bullet)\otimes_R M \to H_n(X_\bullet\otimes_R M)$, $[x]\otimes m \mapsto [x\otimes m]$.

This reduced Künneth formula can be used to compute singular and simplicial homologies of a topological space from its homologies with coefficients in $\mathbb{Z}$. By Definitions 2.1.2, 2.1.4 and 2.1.9, these homologies depend on the choice of a commutative ring k. This was also apparent in Example 2.1.10, 3. where it was shown that the second simplicial homology $H_2(\mathbb{R}\mathrm{P}^2, k)$ vanishes if $2 \nmid \mathrm{char}(k)$, whereas $H_2(\mathbb{R}\mathrm{P}^2, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. Corollary 4.6.4 not only establishes that all singular and simplicial homologies with coefficients in a ring k can be reduced to singular and simplicial homologies with coefficients in $\mathbb{Z}$, but also gives a way to compute them. For singular homologies this is known as the *universal coefficient theorem.*

Theorem 4.6.5: (**universal coefficient theorem for singular homology**)

Let X be a topological space, k a commutative ring and $H_n(X,k)$ the nth singular homology of X with coefficients in k from Definition 2.1.4. Then one has a short exact sequence

$$0 \to H_n(X,\mathbb{Z}) \otimes_{\mathbb{Z}} k \xrightarrow{i_n} H_n(X,k) \xrightarrow{p_n} \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(X,\mathbb{Z}),k) \to 0.$$

Proof:
We consider the singular chain complexes $C_\bullet(X,k)$ from Definition 2.1.2 and Example 3.2.5. As $C_n(X,k)$ is a free k-module, $k \cong \mathbb{Z}\otimes_{\mathbb{Z}} k$ for any ring k and due to the compatibility between tensor products and direct sums, we have $C_n(X,k) \cong C_n(X,\mathbb{Z})\otimes_{\mathbb{Z}} k$.

The chain complex $X_\bullet = C_\bullet(X,\mathbb{Z})$ satisfies the assumptions of Corollary 4.6.4. Because all modules $C_n(X,\mathbb{Z})$ are free, they are projective by Example 3.1.22, 1. and hence flat by Corollary 4.2.4 and Exercise 55. As $\mathbb{Z}$ is a principal ideal domain, the submodules $d_n(C_n(X,\mathbb{Z})) \subset C_{n-1}(X,\mathbb{Z})$ are also free by Example 1.1.16, 3. and hence flat as well. As we have $C_\bullet(X,k) \cong C_\bullet(X,\mathbb{Z})\otimes_{\mathbb{Z}} k$, the result then follows from Corollary 4.6.4 with $R = \mathbb{Z}$, $M = k$ and $X_\bullet = C_\bullet(X,\mathbb{Z})$. □

The Künneth formula for chain complexes in Theorem 4.6.3 also allows one to compute singular homologies of product spaces $X \times Y$ from the homologies of X and Y. The main ingredient is the *Eilenberg-Zilber Theorem*, which states that the singular chain complexes $C_\bullet(X \times Y, k)$ and $C_\bullet(X,k)\otimes_k C_\bullet(Y,k)$ are chain homotopy equivalent for any commutative ring k and topological spaces X, Y. We prove it in Section 5.5 following [McL1, VIII.8], [tD, 9.7] and [D, VI.12].

By restricting attention to principal ideal domains and combining the Eilenberg-Zilber Theorem with Theorem 4.6.3 we obtain a short exact sequence that relates the homologies of two topological spaces to the homologies of their products.

Theorem 4.6.6: (**Künneth theorem**)

Let X, Y be topological spaces and k a principal ideal domain. Then for all $n \in \mathbb{Z}$ there is a short exact sequence relating the singular homologies

$$0 \to \bigoplus_{j=0}^{n} H_j(X,k)\otimes_k H_{n-j}(Y,k) \to H_n(X \times Y, k) \to \bigoplus_{j=0}^{n-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_j(X,k), H_{n-j-1}(Y,k)) \to 0.$$

Proof. This follows by applying the Künneth formula from Theorem 4.6.3 to the chain complexes $X_\bullet = C_\bullet(X_\bullet, k)$ and $X'_\bullet = C_\bullet(Y,k)$ for singular homology. As $C_n(X,k)$ is a free module, it is projective and hence flat. As k is a principal ideal domain, $d_n(C_n(X,k)) \subset C_{n-1}(X,k)$ is free as a submodule of a free module and hence flat as well. The Eilenberg-Zilber Theorem ensures that $H_n(X \times Y, k) \cong H_n(X_\bullet \otimes X'_\bullet)$. □

5 Simplicial methods

In the last section, we derived a unified description of certain (co)homologies in terms of the functors Tor and Ext. These functors are obtained as the left derived functors of the functor $L\otimes_R- : R\text{-Mod} \to \text{Ab}$ for an R-right module L and the right derived functors of the functor $\text{Hom}(-, M) : R - \text{Mod}^{op} \to \text{Ab}$ for an R-left module M. This description of (co)homologies in terms of Tor and Ext includes Hochschild (co)homologies, group (co)homologies and (co)homologies of Lie algebras and does not rely on specific choices of chain complexes. The standard chain complexes that were used to define these (co)homologies in Section 2 were just specific examples of projective resolutions.

That the resulting description is independent of the choice of resolutions is conceptually nice and very helpful in computations. However, it does not explain the origin and mathematical structure of the standard resolutions such as the Hochschild resolution, the bar resolution for group cohomology and the Chevalley-Eilenberg resolution for Lie algebra cohomology. Although they are just specific choices of resolutions, they are distinguished by the fact that they work globally, for all possible algebras, groups or Lie algebras under consideration.

Another interesting feature of the standard chain complexes from Section 2 is that they have a very similar combinatorial structure. In all examples from Section 2, the boundary operators of the relevant chain complexes $C_\bullet$ are alternating sums $d_n = \Sigma_{i=0}^n (-1)^i d_n^i$ for certain R-linear maps $d_n^i : C_n \to C_{n-1}$. In all cases, these R-linear maps $d_n^i : C_n \to C_{n-1}$ exhibit similar commutation relations, derived in Lemma 2.1.3 and 2.2.5. These commutation relations ensure that the boundary operators satisfy the identities $d_n \circ d_{n+1} = 0$ and define a chain complex.

In the case of singular and simplicial homology, the boundary operators have a geometrical interpretation. They are defined by the *face maps*, that send the standard $(n-1)$-simplex Δ^{n-1} to the face opposite the vertex e_i in the standard n-simplex Δ^n. However, in the end the description is purely combinatorial and relies only on the ordering of the $n+1$ vertices in Δ^n. The geometrical interpretation of the face maps in singular and simplicial homology also does not explain why similar combinatorial structures arise in the context of Hochschild (co)homology, group (co)homology and (co)homology of Lie algebras.

This suggests that the combinatorial structure and the associated commutation relations between the maps d_n^i could be a global pattern that characterises chain complexes. One could then define chain complexes in any abelian category $\mathcal{A}$ by identifying a collection of morphisms $d_n^i : X_n \to X_{n-1}$ in $\mathcal{A}$ with similar commutation relations and defining the boundary operators as alternating sums $d_n = \Sigma_{i=0}^n (-1)^i d_n^i$. The question is how to find such collections of morphisms d_n^i and which chain complexes in $\mathcal{A}$ can be obtained in this way.

In this section, we investigate this construction systematically and show that up to chain homotopy equivalence *all* positive chain complexes in an abelian category $\mathcal{A}$ can be obtained from this construction. This is the famous *Dold-Kan correspondence*. It is not an isolated result, but the foundation of a general combinatorial approach to homologies that also incorporates chain maps and chain homotopies into the picture. Among others, it leads to a systematic construction of resolutions from (co)monads and adjoint functors.

5.1 The simplex category

The first step to understand the common patterns in the boundary operators from Section 2 is to understand their combinatorics. The suitable framework is a category. This category must encode the combinatorics of the standard n-simplexes Δ^n for all $n \in \mathbb{N}_0$, and of all maps between them that are obtained by composing face maps. Hence, we require a category with one object for each $n \in \mathbb{N}_0$ that describes the standard n-Simplex Δ^n. To account for the $n+1$ ordered vertices of Δ^n we choose for this object the finite ordinal $[n+1] = \{0, 1, ..., n\}$. Face maps between standard n-simplexes are maps $f_n : \Delta^{n-1} \to \Delta^n$ that are determined by their behaviour on the vertices and respect their ordering. Hence, we can describe them as strictly monotonic maps $f : [n] \to [n+1]$ that skip exactly one element of $[n+1]$ in their image.

For reasons that will become apparent later, it makes sense to introduce an additional object $[0] = \emptyset$ and to consider all *weakly monotonic* maps between finite ordinals. This yields the *augmented simplex category* or *algebraist's simplex category*. The *simplex category* or *topologist's simplex category* is the full subcategory obtained by omitting the object $[0] = \emptyset$.

Definition 5.1.1:

1. The **augmented simplex category** Δ has as objects the finite **ordinal numbers** $[n] = \{0, 1, ..., n-1\}$ for $n \in \mathbb{N}_0$ with $[0] = \emptyset$. The morphisms $f : [n] \to [m]$ are monotonic maps $f : \{0, ..., n-1\} \to \{0, ..., m-1\}$, and their composition is the composition of maps.
2. The **simplex category** Δ^+ is the full subcategory of Δ with objects $[n]$ for $n \in \mathbb{N}$.

To understand the simplex category and apply it to homological algebra, we need a more detailed understanding of its morphisms, in particular the morphisms that generalise face maps between standard n-simplexes. Clearly, the *face maps* correspond to the injective morphisms $\delta^i_n : [n] \to [n+1]$ that skip the element $i \in [n+1]$. There are also surjective counterparts of the face maps, the *degeneracies* $\sigma^j_n : [n+1] \to [n]$ that send j and $j+1$ to j. It turns out that any morphism in Δ can be expressed uniquely a product of these maps, with $\delta^0_0 : [0] \to [1]$ corresponding to the empty map.

Proposition 5.1.2: (**factorisation in the simplex category**)

1. Every morphism $f : [m] \to [n]$ in Δ can be expressed uniquely as a composite

$$\begin{aligned} &f = \delta^{i_1}_{n-1} \circ ... \circ \delta^{i_k}_{m-l} \circ \sigma^{j_1}_{m-l} \circ ... \circ \sigma^{j_l}_{m-1} \\ &n = m - l + k, \quad 0 \leq i_k < ... < i_1 < n, \quad 0 \leq j_1 < ... < j_l < m-1 \end{aligned} \tag{58}$$

   of the **face maps** $\delta^i_n : [n] \to [n+1]$ and the **degeneracies** $\sigma^j_n : [n+1] \to [n]$ for $i \in \{0, ..., n\}$ and $j \in \{0, ..., n-1\}$

$$\delta^i_n(k) = \begin{cases} k & 0 \leq k < i \\ k+1 & i \leq k < n \end{cases} \qquad \sigma^j_n(k) = \begin{cases} k & 0 \leq k \leq j \\ k-1 & j < k \leq n. \end{cases}$$

2. The morphisms $\delta^i_n : [n] \to [n+1]$ and $\sigma^j_n : [n+1] \to [n]$ satisfy the relations

$$\begin{aligned} &\delta^i_{n+1} \circ \delta^j_n = \delta^{j+1}_{n+1} \circ \delta^i_n && \text{for} \quad i \leq j \\ &\sigma^j_n \circ \sigma^i_{n+1} = \sigma^i_n \circ \sigma^{j+1}_{n+1} && \text{for} \quad i \leq j \\ &\sigma^j_n \circ \delta^i_n = \begin{cases} \delta^i_{n-1} \circ \sigma^{j-1}_{n-1} & i < j \\ 1_{[n]} & i \in \{j, j+1\} \\ \delta^{i-1}_{n-1} \circ \sigma^j_{n-1} & i > j+1. \end{cases} \end{aligned} \tag{59}$$

Proof:
1. Every monotonic map $f : [m] \to [n]$ is determined uniquely by the sets

$$M_\delta = \{i_1, ..., i_k\} = [n] \setminus \mathrm{im}(f) \qquad M_\sigma = \{j_1, ..., j_l\} = \{x \in [m-1] \mid f(x) = f(x+1)\}$$

with $n - k = m - l$. If $0 \leq i_k < ... < i_1 < n$, $0 \leq j_1 < ... < j_l < m-1$ and $\mathrm{im}(f) = \{l_1, ..., l_{n-k}\}$ with $0 \leq l_1 < ... < l_{n-k}$, then f factorises uniquely as $f = g \circ h$ with an injective monotonic map $g : [m-l] \to [n]$ and a surjective monotonic map $h : [m] \to [m-l]$ given by

$$g(r) = l_{r+1} \qquad\qquad h(r) = \begin{cases} r & r \leq j_1 \\ r - s & j_s < r \leq j_{s+1} \\ r - l & j_l < r. \end{cases}$$

This implies $g = \delta^{i_1}_{n-1} \circ ... \circ \delta^{i_k}_{m-l}$ and $h = \sigma^{j_1}_{m-l} \circ ... \circ \sigma^{j_l}_{m-1}$.

2. The relations between the maps $\delta^i_n : [n] \to [n+1]$ and $\sigma^i_n : [n+1] \to [n]$ follow by a direct computation. For $0 \leq i \leq j \leq n-1$, we have

$$\delta^i_{n+1} \circ \delta^j_n(k) = \begin{cases} \delta^i_{n+1}(k) & 0 \leq k < j \\ \delta^i_{n+1}(k+1) & j \leq k \leq n-1 \end{cases} = \begin{cases} k & 0 \leq k < i \\ k+1 & i \leq k < j \\ k+2 & j \leq k \leq n-1 \end{cases}$$

$$\delta^{j+1}_{n+1} \circ \delta^i_n(k) = \begin{cases} \delta^{j+1}_{n+1}(k) & 0 \leq k < i \\ \delta^{j+1}_{n+1}(k+1) & i \leq k \leq n-1 \end{cases} = \begin{cases} k & 0 \leq k < i \\ k+1 & i \leq k < j \\ k+2 & j \leq k \leq n-1. \end{cases}$$

The computations for the other relations are similar. □

Remark 5.1.3: As the relations (59) allow one to transform any composite of the morphisms δ^i_n and σ^j_n into the form (58) and the factorisation in (58) is unique, there can be no further relations between the morphisms δ^i_n and σ^j_n. All relations between them are obtained by composing (59) with other morphisms in Δ. One says that Δ is **generated as a category** or **presented as a category** by the morphisms δ^i_n and σ^j_n with the **relations** (59).

The relations between the face maps in (59) resemble the relations between the face maps f^n_i from singular and simplicial (co)homology from Lemma 2.1.3 and the relations between the maps d^i_n and d^n_i from Hochschild (co)homology in Lemma 2.2.5. More precisely, the relations for the *cohomologies* coincide with the relations (59) while the composition of the face maps is reversed for homologies.

Any functor $F : \Delta^+ \to \mathcal{C}$ into a category $\mathcal{C}$ preserves the relations between the face maps, and any functor $F : \Delta^{+op} \to \mathcal{C}$ reverses them. This suggests to construct chain complexes in an abelian category $\mathcal{C}$ from functors $F : \Delta^+ \to \mathcal{C}$ and $F : \Delta^{+op} \to \mathcal{C}$, respectively. Such functors are called *cosimplicial* and *simplicial objects* in $\mathcal{C}$. It will be useful in the following to consider such functors for general categories $\mathcal{C}$, not just abelian ones.

Definition 5.1.4: Let $\mathcal{C}$ be a category.

1. A **simplicial object** in $\mathcal{C}$ is a functor $F : \Delta^{+op} \to \mathcal{C}$. An **augmented simplicial object** in $\mathcal{C}$ is a functor $F : \Delta^{op} \to \mathcal{C}$.
2. A **morphism of simplicial objects** from $F : \Delta^{+op} \to \mathcal{C}$ to $G : \Delta^{+op} \to \mathcal{C}$ is a natural transformation $\eta : F \to G$. A **morphism of augmented simplicial objects** from $F : \Delta^{op} \to \mathcal{C}$ to $G : \Delta^{op} \to \mathcal{C}$ is a natural transformation $\eta : F \to G$.
3. A **cosimplicial object** in $\mathcal{C}$ is a functor $F : \Delta^+ \to \mathcal{C}$. An **augmented cosimplicial object** in $\mathcal{C}$ is a functor $F : \Delta \to \mathcal{C}$.
4. A **morphism of cosimplicial objects** from $F : \Delta^+ \to \mathcal{C}$ to $G : \Delta^+ \to \mathcal{C}$ is a natural transformation $\eta : F \to G$. A **morphism of augmented cosimplicial objects** from $F : \Delta \to \mathcal{C}$ to $G : \Delta \to \mathcal{C}$ is a natural transformation $\eta : F \to G$.

Remark 5.1.5:

1. Simplicial objects in a category $\mathcal{D}$ and morphisms of simplicial objects form a category, namely the category $\mathrm{Fun}(\Delta^{+op}, \mathcal{D})$. Cosimplicial objects and morphisms of cosimplicial objects in $\mathcal{D}$ form the category $\mathrm{Fun}(\Delta^+, \mathcal{D})$. Analogous statements hold for augmented (co)simplicial objects and morphisms of augmented (co)simplicial objects.
2. Let $G : \mathcal{C} \to \mathcal{D}$ be a functor. Then for any simplicial object $F : \Delta^{+op} \to \mathcal{C}$ the functor $GF : \Delta^{+op} \to \mathcal{D}$ is a simplicial object in $\mathcal{D}$, and for any simplicial morphism $\eta : F \to F'$ the natural transformation $G\eta : GF \to GF'$ with component morphisms $G\eta_{[n]} = G(\eta_{[n]}) : GF([n]) \to GF'([n])$ is a simplicial morphism in $\mathcal{D}$. Analogous statements hold in the cosimplicial and in the augmented case.

Remark 5.1.6:

1. As the morphisms $\delta^i_n : [n] \to [n+1]$ and $\sigma^j_n : [n+1] \to [n]$ from Proposition 5.1.2 generate the simplex category Δ^+ subject to the relations (59), a (co)simplicial object is determined uniquely by the images of the objects $[n]$ for $n \in \mathbb{N}_0$ and the images of the morphisms δ^i_n and σ^j_n, which must satisfy relations analogous to (59).

   Hence, a simplicial object in $\mathcal{C}$ can be defined equivalently as a family $(C_n)_{n\in\mathbb{N}_0}$ of objects in C together with morphisms $d^i_n : C_n \to C_{n-1}$, the **face operators**, and $s^i_n : C_n \to C_{n+1}$, the **degeneracies**, for $0 \leq i \leq n$ that satisfy the **simplicial identities**

$$
\begin{aligned}
&d^j_n \circ d^i_{n+1} = d^i_n \circ d^{j+1}_{n+1} \qquad \text{for} \quad i \leq j\\
&s^i_{n+1} \circ s^j_n = s^{j+1}_{n+1} \circ s^i_n \qquad \text{for} \quad i \leq j\\
&d^i_{n+1} \circ s^j_n = \begin{cases} s^{j-1}_{n-1} \circ d^i_n & i < j \\ 1_{C_n} & i \in \{j, j+1\} \\ s^j_{n-1} \circ d^{i-1}_n & j+1 < i \leq n+1. \end{cases}
\end{aligned}
\tag{60}
$$

They correspond to a functor $F : \Delta^{+op} \to \mathcal{C}$ with $C_n = F([n+1])$, $d_n^i = F(\delta_n^i)$ and $s_n^i = F(\sigma_{n+1}^i)$. The shift in indices is a standard convention.

2. Similarly, one can define a cosimplicial object in $\mathcal{C}$ as a family $(C^n)_{n\in\mathbb{N}_0}$ of objects in $\mathcal{C}$ together with morphisms $d_i^n : C^{n-1} \to C^n$ and $s_i^n : C^{n+1} \to C^n$ for $0 \le i \le n$ that satisfy the **cosimplicial identities**

$$\begin{aligned} d_i^{n+1} \circ d_j^n &= d_{j+1}^{n+1} \circ d_i^n && \text{for} \quad i \le j \\ s_j^{n-1} \circ s_i^n &= s_i^{n-1} \circ s_{j+1}^n && \text{for} \quad i \le j \\ s_j^n \circ d_i^{n+1} &= \begin{cases} d_i^n \circ s_{j-1}^{n-1} & i < j \\ 1_{C^n} & i \in \{j, j+1\} \\ d_{i-1}^n \circ s_j^{n-1} & i > j+1. \end{cases} \end{aligned} \tag{61}$$

They define a functor $F : \Delta^+ \to \mathcal{C}$ with $C^n = F([n+1])$, $d_i^n = F(\delta_n^i)$, $s_i^n = F(\sigma_{n+1}^i)$.

As expected, the (co)chain complexes for singular and simplicial (co)homology, for Hochschild (co)homology, group cohomology and cohomology of Lie algebras from Section 2 all arise from (co)simplicial objects in k-Mod, where k is a commutative ring. The additional information contained in an *augmented* (co)simplicial object defines the associated standard resolution.

Example 5.1.7: (**Hochschild homology and cohomology**)
Let A be an algebra over a commutative ring k and M an (A,A)-bimodule.

1. The functor $F : \Delta^{op} \to A\otimes A^{op}$-Mod with

$$\begin{aligned} C_n &= F([n+1]) = A^{\otimes(n+2)} && n \ge -1 \\ d_n^i &= F(\delta_n^i) : A^{\otimes(n+2)} \to A^{\otimes(n+1)}, && a_0\otimes\ldots\otimes a_{n+1} \mapsto a_0\otimes\ldots\otimes(a_i a_{i+1})\otimes\ldots\otimes a_{n+1} \\ s_n^i &= F(\sigma_{n+1}^i) : A^{\otimes(n+2)} \to A^{\otimes(n+3)}, && a_0\otimes\ldots\otimes a_{n+1} \mapsto a_0\otimes\ldots\otimes a_i\otimes 1\otimes a_{i+1}\otimes\ldots\otimes a_{n+1}. \end{aligned}$$

is an augmented simplicial object in $A\otimes A^{op}$-Mod. The (A,A)-bimodules C_n and the morphisms $d_n^i : C_n \to C_{n-1}$ are the ones of the Hochschild resolution in Example 4.1.3.

2. Composing its restriction $F^+ : \Delta^{+op} \to A\otimes A^{op}$-Mod with $M\otimes_{A\otimes A^{op}}- : A\otimes A^{op}$-Mod $\to$ k-Mod yields a simplicial object $(M\otimes_{A\otimes A^{op}}-)F^+$ in k-Mod that defines the chain complex of Hochschild homology from Definition 2.2.3.

3. Composing $F^+ : \Delta^{+op} \to A\otimes A^{op}$-Mod with $\mathrm{Hom}_{A\otimes A^{op}}(-,M) : A\otimes A^{op}\text{-Mod}^{op} \to k$-Mod, yields a cosimplicial object $\mathrm{Hom}_{A\otimes A^{op}}(-,M)F^+$ in k-Mod that defines the cochain complex of Hochschild cohomology from Definition 2.2.4.

4. By specialising to the case $A = k[G]$ for a group G and bimodules with trivial $k[G]$-right module structures, we obtain the corresponding statements for group (co)homology. By considering the algebra $A = U(\mathfrak{g})$ for a Lie algebra $\mathfrak{g}$ and bimodules with the trivial $U(\mathfrak{g})$-right module structures, we obtain Lie algebra (co)homology.

The categories k-Mod and $A\otimes A^{op}$-Mod in Example 5.1.7 are abelian, and we will see later that any (co)simplicial object in an abelian category gives rise to a (co)chain complex. To obtain (co)chain complexes from (co)simplicial objects in non-abelian categories such as Top or Set, one has to compose the (co)simplicial objects with functors into an abelian category. These functors are often very simple and obtained from a functor into the category Set.

Example 5.1.8: (**Singular homology and cohomology**)

1. The family of standard n-simplexes $(\Delta^n)_{n\in\mathbb{N}_0}$ with the face maps and degeneracy maps

$$d_i^n = f_i^n : \Delta^{n-1} \to \Delta^n \qquad\qquad s_i^n : \Delta^{n+1} \mapsto \Delta^n$$
$$f_i^n(e_k) = \begin{cases} e_k & 0 \leq k < i \\ e_{k+1} & i \leq k \leq n \end{cases} \qquad\qquad s_i^n(e_k) = \begin{cases} e_k & 0 \leq k < i \\ e_{k-1} & i \leq k \leq n+1 \end{cases}$$

form a cosimplicial object $F_\Delta : \Delta^+ \to \mathrm{Top}$ with

$$F_\Delta([n]) = \Delta^{n-1}, \qquad F_\Delta(\delta_n^i) = f_i^n, \qquad F_\Delta(\sigma_n^i) = s_i^{n-1}.$$

More generally, for a morphism $\alpha : [m+1] \to [n+1]$ in Δ^+ the morphism $F_\Delta(\alpha)$ is the affine-linear map $F_\Delta(\alpha) : \Delta^m \to \Delta^n$ with $F_\Delta(\alpha)(e_k) = e_{\alpha(k)}$ for all $k \in \{0, ..., m\}$.

2. Let X be a topological space and $\mathrm{Hom}_{\mathrm{Top}}(-, X) : \mathrm{Top} \to \mathrm{Set}^{op}$ the functor that assigns
   - to a topological space Y the set $\mathrm{Hom}_{\mathrm{Top}}(Y, X)$ of continuous maps $f : Y \to X$
   - to a continuous map $\sigma : Y \to Z$ the map

$$\mathrm{Hom}(\sigma, X) : \mathrm{Hom}_{\mathrm{Top}}(Z, X) \to \mathrm{Hom}_{\mathrm{Top}}(Y, X),\ g \mapsto g \circ \sigma.$$

By composing $\mathrm{Hom}_{\mathrm{Top}}(-, X)$ with the functor F_Δ from 1., we obtain a simplicial object $C^X = \mathrm{Hom}(-, X) \circ F_\Delta : \Delta^{+op} \to \mathrm{Set}$ given by

$$C^X([n+1]) = C_n^X = \mathrm{Hom}_{\mathrm{Top}}(\Delta^n, X),$$
$$C^X(\delta_n^i) = d_n^{Xi} : C_n^X \to C_{n-1}^X,\ \sigma \mapsto \sigma \circ f_i^n, \quad C^X(\sigma_{n+1}^i) = s_n^{Xi} : C_n^X \to C_{n+1}^X,\ \sigma \mapsto \sigma \circ s_i^n.$$

3. This defines a functor $\mathrm{Sing} : \mathrm{Top} \to \mathrm{Fun}(\Delta^{+op}, \mathrm{Set})$ that assigns
   - to a topological space X the simplicial object $\mathrm{Sing}(X) = C^X : \Delta^{+op} \to X$ in Set
   - to a continuous map $f : X \to Y$ the simplicial morphism $\mathrm{Sing}(f) : C^X \to C^Y$ with component morphisms $\mathrm{Sing}(f)_{[n+1]} : C_n^X \to C_n^Y$, $\sigma \mapsto f \circ \sigma$.

4. Let k be a commutative ring and $\langle\ \rangle_k : \mathrm{Set} \to k\text{-Mod}$ the functor that assigns
   - to a set X the free k-module $\langle X \rangle_k$
   - to a map $f : X \to Y$ the induced k-module homomorphism $\langle f \rangle_k : \langle X \rangle_k \to \langle Y \rangle_k$.

   By composing this functor with the functor C^X from 2. we obtain a simplicial object $C^{X,k} = \langle\ \rangle_k \circ \mathrm{Hom}(-, X) \circ F^\Delta : \Delta^{+op} \to k\text{-Mod}$. This simplicial object defines the chain complex of singular homology from Definition 2.1.2:

$$C^{X,k}([n+1]) = C_n(X, k) = \langle \mathrm{Hom}_{\mathrm{Top}}(\Delta^n, X) \rangle_k,$$
$$C^{X,k}(\delta_n^i) = d_n^i : C_n(X, k) \to C_{n-1}(X, k),\ \sigma \mapsto \sigma \circ f_i^n,$$
$$C^{X,k}(\sigma_{n+1}^i) = s_n^i : C_n(X, k) \to C_{n+1}(X, k),\ \sigma \mapsto \sigma \circ s_i^n.$$

5. By combining the functors $\mathrm{Sing} : \mathrm{Top} \to \mathrm{Fun}(\Delta^{+op}, \mathrm{Set})$ from 3. and $\langle\ \rangle_k : \mathrm{Set} \to k\text{-Mod}$, from 4. we obtain a functor $\mathrm{Top} \to \mathrm{Fun}(\Delta^{+op}, k\text{-Mod})$ that assigns to a topological space X the functor $C^{X,k} : \Delta^{+op} \to k\text{-Mod}$ and to a continuous map $f : X \to Y$ the natural transformation $f^{X,k} : C^{X,k} \to C^{Y,k}$, $\sigma \mapsto f \circ \sigma$. This is the functor that defines the chain complex of singular homology from Definition 2.1.2.

6. By composing $C^{X,k} : \Delta^{+op} \to k\text{-Mod}$ with $\mathrm{Hom}_k(-, k) : k\text{-Mod} \to k\text{-Mod}$, we obtain a functor $\mathrm{Hom}_k(-, k) \circ C^{X,k} : \Delta^+ \to k\text{-Mod}$ for each topological space X and a functor $\mathrm{Top} \to \mathrm{Fun}(\Delta^+, k\text{-Mod})$ that defines the chain complexes of singular cohomology from Definition 2.1.12.

This example motivates the shift in indices in Remark 5.1.6 and the topologist's version of the simplex category without the object $[0] = \emptyset$. There is no need to include the empty topological space as a standard (-1)-simplex. However, the algebraist' s version of the simplex category has other advantages that will become apparent in the next section.

Simplicial objects in the category Set are called **simplicial sets** and natural transformations between them are called **simplicial maps**. They play an important role in modern approaches to topology. In particular, they allow one to systematically construct semisimplicial complexes, for an accessible introduction see [F]. The information in a simplicial set $S : \Delta^{+op} \to \mathrm{Set}$ is precisely the data needed to construct a semisimplicial complex by gluing n-simplexes.

Example 5.1.9: (**Geometric realisation**)

- The **geometric realisation** of a simplicial set $S : \Delta^{+op} \to \mathrm{Set}$ is the topological space $\mathrm{Geom}(S)$ obtained as follows. One equips all sets $S_n = S([n + 1])$ with the discrete topology and forms the quotient space
$$\mathrm{Geom}(S) = (\amalg_{n\in\mathbb{N}_0} S_n \times \Delta^n)/\sim$$
with the equivalence relation $(S(\alpha)x, p) \sim (x, F_\Delta(\alpha)p)$ for all $\alpha \in \mathrm{Hom}_{\Delta^{+op}}([n+1], [m+1])$, where $F_\Delta(\alpha) : \Delta^m \to \Delta^n$, $e_k \to e_{\alpha(k)}$ is the affine map from Example 5.1.8.

- The topological space $\mathrm{Geom}(S)$ is a semisimplicial complex (Exercise 73).

  The simplicial set $S : \Delta^{+op} \to \mathrm{Set}$ describes the construction of $\mathrm{Geom}(S)$ by gluing standard simplexes. The elements of the sets S_n label the n-simplexes in the semisimplicial complex, and the maps $S(\alpha) : S_n \to S_m$ for a morphism $\alpha : [m + 1] \to [n + 1]$ in Δ^+ specify the gluing pattern, as shown in Figure 1.

- For any simplicial map $\eta : S \to S'$ with component morphisms $\eta_{[n+1]} : S_n \to S'_n$, one obtains a continuous map $\mathrm{Geom}(\eta) : \mathrm{Geom}(S) \to \mathrm{Geom}(S')$ given by
$$\mathrm{Geom}(\eta)[(x, p)] = [(\eta_{[n+1]}(x), p)] \quad \forall (x, p) \in S_n \times \Delta^n.$$
It is a simplicial map between the semisimplicial complexes $\mathrm{Geom}(S)$ and $\mathrm{Geom}(S')$ in the sense of Definition 2.1.8.

- As these assignments are compatible with the composition of morphisms and unit morphisms in $\mathrm{Fun}(\Delta^+, \mathrm{Set})$, they define a functor $\mathrm{Geom} : \mathrm{Fun}(\Delta^{+op}, \mathrm{Set}) \to \mathrm{Top}$.

It should be noted that (co)simplicial objects and (co)simplicial morphisms are not the only structures investigated in simplicial approaches to homological algebra. There is also a notion of a **simplicial homotopy** that defines an equivalence relation on the set of simplicial morphisms between fixed simplicial objects $S, S' : \Delta^{+op} \to \mathcal{A}$ and generalises the notion of chain homotopy. Moreover, there is a concept of *simplicial homotopy groups* for simplicial sets that satisfy certain additional conditions. These simplicial homotopy groups behave like the homotopy groups of topological spaces and are related to them by the geometric realisation functor. Details on these constructions can be found in [W, Chapter 8.3].

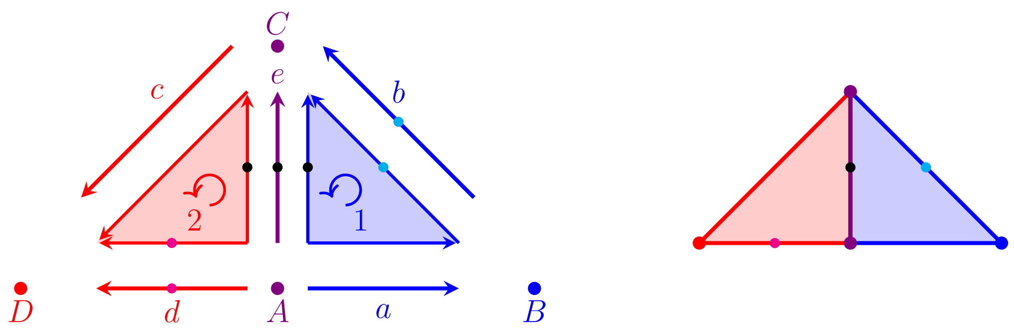


Figure 1: The gluing pattern for a simplicial set $S : \Delta^{+op} \to \mathrm{Set}$.

5.2 Dold-Kan correspondence

Our main motivation to consider simplicial objects is that simplicial objects in *abelian categories* $\mathcal{A}$ define chain complexes in $\mathcal{A}$. As suggested by the simplicial relations (60), the chain complex for a simplicial object $S : \Delta^{+op} \to \mathcal{A}$ is obtained by taking an alternating sum over the face maps $d_n^i = S(\delta_n^i)$. The degeneracies do not enter the definition of this chain complex, but they also carry relevant information. Their images define a subcomplex with trivial homologies that can be removed to obtain a more efficient description.

To describe this construction in a general abelian category $\mathcal{A}$, we consider for each finite family $(f_i)_{i\in I}$ of morphisms $f_i : X \to Y$ the morphisms $f : \amalg_{i\in I} X \to Y$ with $f \circ \iota_i = f_i : X \to Y$ and $f' : X \to \amalg_{i\in I} Y$ with $\pi_i \circ f' = f_i$ for all $i \in I$ induced by the universal property of the (co)product. We define the objects $+_{i\in I}\mathrm{im}(f_i) := \mathrm{im}(f)$ and $\cap_{i\in I}\ker(f_i) := \ker(f')$ as their image and kernel object. For $\mathcal{A} = R$-Mod, they are the sum $+_{i\in I}\mathrm{im}(f_i) \subset Y$ and the intersection $\cap_{i\in I}\ker(f_i) \subset X$ of the submodules $\mathrm{im}(f_i) \subseteq Y$ and $\ker(f_i) \subset X$, as suggested by the notation.

Proposition 5.2.1: Let $S : \Delta^{+op} \to \mathcal{A}$ be a simplicial object in an abelian category $\mathcal{A}$ and

$$S_n := S([n+1]), \qquad d_n^i := S(\delta_n^i) : S_n \to S_{n-1}, \qquad s_n^i := S(\sigma_{n+1}^i) : S_n \to S_{n+1}$$

for $n \in \mathbb{N}_0$ and $0 \leq i \leq n$.

1. The following are positive chain complexes in $\mathcal{A}$:
   - The **standard chain complex** $S_\bullet$ with
   $$S_n = S([n+1]) \qquad\qquad d_n = \Sigma_{i=0}^n (-1)^i d_n^i : S_n \to S_{n-1}.$$
   - The **normalised chain complex** $NS_\bullet$ with
   $$NS_n = \cap_{i=0}^{n-1} \ker(d_n^i) \subset S_n \qquad d_n = (-1)^n d_n^n : NS_n \to NS_{n-1}.$$
   - The **degenerate chain complex** $DS_\bullet$ with
   $$DS_n = +_{i=0}^{n-1} \mathrm{im}(s_{n-1}^i) \subset S_n \qquad d_n = \Sigma_{i=0}^n (-1)^i d_n^i : DS_n \to DS_{n-1}.$$
2. They are related by the identity $S_\bullet = NS_\bullet \amalg DS_\bullet$.
3. The chain complexes $S_\bullet$ and $NS_\bullet$ are chain homotopy equivalent, and the chain complex $DS_\bullet$ is chain homotopy equivalent to the trivial chain complex. This implies for all $n \in \mathbb{N}_0$
$$H_n(S_\bullet) = H_n(NS_\bullet) \qquad\qquad H_n(DS_\bullet) = 0.$$

Proof:
We prove the claims for $\mathcal{A} = R$-Mod.

1. That $S_\bullet$ is a chain complex in R-Mod follows directly from the simplicial relations (60). They imply for $n \in \mathbb{N}$

$$\begin{aligned}
d_{n-1} \circ d_n &= \Sigma_{i=0}^n \Sigma_{j=0}^{n-1} (-1)^{i+j} d_{n-1}^j \circ d_n^i \\
&= \Sigma_{0 \le i \le j \le n-1} (-1)^{i+j} d_{n-1}^j \circ d_n^i + \Sigma_{0 \le j < i \le n} (-1)^{i+j} d_{n-1}^j \circ d_n^i \\
&\overset{(60)}{=} \Sigma_{0 \le i \le j \le n-1} (-1)^{i+j} d_{n-1}^i \circ d_n^{j+1} + \Sigma_{0 \le j < i \le n} (-1)^{i+j} d_{n-1}^j \circ d_n^i \\
&= \Sigma_{0 \le i < j \le n} (-1)^{i+j+1} d_{n-1}^i \circ d_n^j + \Sigma_{0 \le j < i \le n} (-1)^{i+j} d_{n-1}^j \circ d_n^i = 0.
\end{aligned}$$

2. We show that $NS_\bullet$ and $DS_\bullet$ are chain complexes in R-Mod. From the simplicial relations (60) we have $d_{n-1}^i \circ d_n^n(x) = d_{n-1}^{n-1} \circ d_n^i(x) = 0$ for all $0 \le i < n$ and $x \in NS_n$, and this implies $d_n(NS_n) \subset NS_{n-1}$. This shows that $NS_\bullet$ is a chain complex. Similarly, we obtain for all $0 \le i \le n-1$ and $x \in S_{n-1}$

$$\begin{aligned}
d_n(s_{n-1}^i(x)) &= \Sigma_{k=0}^n (-1)^k d_n^k \circ s_{n-1}^i(x) \\
&\overset{(60)}{=} \Sigma_{k=0}^{i-1} (-1)^k s_{n-2}^{i-1} \circ d_{n-1}^k(x) + (-1)^i x + (-1)^{i+1} x + \Sigma_{k=i+2}^n (-1)^k s_{n-2}^i \circ d_n^{k-1}(x) \\
&= \Sigma_{k=0}^{i-1} (-1)^k s_{n-2}^{i-1} \circ d_{n-1}^k(x) + \Sigma_{k=i+2}^n (-1)^k s_{n-2}^i \circ d_n^{k-1}(x) \in +_{i=0}^{n-2} \mathrm{im}(s_{n-2}^i)
\end{aligned}$$

This shows that $d_n(DS_n) \subset DS_{n-1}$ and $DS_\bullet$ is a chain complex in R-Mod.

3. We show that $S_n = NS_n \oplus DS_n$ for all $n \in \mathbb{N}_0$. To see that $DS_n \cap NS_n = \{0\}$, let $0 \neq x \in DS_n \cap NS_n$ and set $j = \max\{k \in \{0, ..., n-1\} \mid x \in +_{i=k}^{n-1} \mathrm{im}(s_{n-1}^i)\}$. Then we have $x = \Sigma_{i=j}^{n-1} s_{n-1}^i(x_i)$ with $s_{n-1}^j(x_j) \neq 0$. As $x \in NS_n$, we have

$$0 = d_n^j(x) = \Sigma_{i=j}^{n-1} d_n^j \circ s_{n-1}^i(x_i) = x_j + \Sigma_{i=j+1}^{n-1} s_{n-2}^{i-1} \circ d_n^j(x_i).$$

If $j = n-1$, it follows that $x_{n-1} = 0$ and $x = s_{n-1}^{n-1}(x_{n-1}) = 0$. If $j < n-1$ it follows that

$$s_{n-1}^j(x_j) = -\Sigma_{i=j+1}^{n-1} s_{n-1}^j \circ s_{n-2}^{i-1} \circ d_n^j(x_i) = -\Sigma_{i=j+1}^{n-1} s_{n-1}^i \circ s_{n-2}^j \circ d_n^j(x_i) \in +_{i=j+1}^{n-1} \mathrm{im}(s_{n-1}^i)$$

and $x \in +_{k=j+1}^{n-1} \mathrm{im}(s_{n-1}^i)$, in contradiction to the maximality of j. Hence, $NS_n \cap DS_n = \{0\}$.

To show that $S_n = NS_n + DS_n$, let $x \in S_n$ and $j_x = \min\{k \in \{0, ..., n\} \mid d_n^k(x) \neq 0\}$. If $j_x = n$, then $x \in NS_n$. If $j_x = j < n$, we have $x = x_1 + y_1 = (x - s_{n-1}^j \circ d_n^j(x)) + s_{n-1}^j \circ d_n^j(x)$

$$\begin{aligned}
d_n^j(x_1) &= d_n^j(x - s_{n-1}^j \circ d_n^j(x)) = d_n^j(x) - d_n^j(x) = 0 \qquad (62)\\
d_n^k(x_1) &= d_n^k(x - s_{n-1}^j \circ d_n^j(x)) = d_n^k(x) - s_{n-2}^{j-1} \circ d_{n-1}^k \circ d_n^j(x) = -s_{n-2}^{j-1} \circ d_{n-1}^{j-1} \circ d_n^k(x) = 0
\end{aligned}$$

for $k \in \{0, ..., j-1\}$. We thus decomposed x as $x = x_1 + y_1$ with $y_1 \in DS_n$ and an element $x_1 \in S_n$ with $j_{x_1} \ge j_x + 1$. By iterating this procedure, we obtain elements $y_1, ..., y_k \in DS_n$ and $x_1, ..., x_k \in S_n$ with $x_i = x_{i+1} + y_{i+1}$ and $x_k \in NS_n$. This shows that $x = x_k + \Sigma_{i=1}^k y_i \in NS_n + DS_n$ and hence $S_n = NS_n + DS_n$.

4. We show that the chain complex $NS_\bullet$ is chain homotopy equivalent to $S_\bullet$. For this, we set $NS_\bullet^{-1} = S_\bullet$ and consider for $j \in \mathbb{N}_0$ the subcomplexes $NS_\bullet^j = (NS_n^j)_{n \in \mathbb{N}_0} \subset S_\bullet$ in R-Mod with

$$NS_n^j = \begin{cases} \cap_{i=0}^j \ker(d_n^i) & n \ge j+2 \\ NS_n = \cap_{i=0}^{n-1} \ker(d_n^i) & 0 \le n \le j+1. \end{cases}$$

That $NS^j_\bullet$ is indeed a chain complex follows because $d_n(NS^j_n) \subset NS^j_{n-1}$ for $n \leq j+1$ by 2., and the simplicial relations imply for $j+2 \leq n$, $x \in NS^j_n$ and $0 \leq i \leq j$

$$d^i_{n-1} \circ d_n(x) = \Sigma^n_{k=0}(-1)^k d^i_{n-1} \circ d^k_n(x) = \Sigma^n_{k=j+1}(-1)^k d^i_{n-1} \circ d^k_n(x) = \Sigma^n_{k=j+1}(-1)^k d^{k-1}_{n-1} \circ d^i_n(x) = 0.$$

This shows that $d_n(NS^j_n) \subset NS^j_{n-1}$ for all $n, j \in \mathbb{N}_0$ and $NS^j_\bullet \subset S_\bullet$ is a subcomplex.

As $NS^{j+1}_n \subset NS^j_n$ for all $j \geq -1$, the inclusion maps $\iota^j_n : NS^{j+1}_n \to NS^j_n$ define chain maps $\iota^j_\bullet : NS^{j+1}_\bullet \to NS^j_\bullet$. We show that $\iota^j_\bullet : NS^{j+1}_\bullet \to NS^j_\bullet$ is a chain homotopy equivalence by constructing chain maps $f^j_\bullet : NS^j_\bullet \to NS^{j+1}_\bullet$ with $f^j_\bullet \circ \iota^j_\bullet = 1_{NS^{j+1}_\bullet}$ and a chain homotopies $t^j_\bullet : 1_{NS^j_\bullet} \Rightarrow \iota^j_\bullet \circ f^j_\bullet$. For this, we consider for $j \geq -1$ and $n \in \mathbb{N}_0$ the R-linear maps

$$f^j_n : NS^j_n \to NS^{j+1}_n, \quad x \mapsto \begin{cases} x - s^{j+1}_{n-1} \circ d^{j+1}_n(x) & n \geq j+2 \\ x & n \leq j+1 \end{cases}$$

that take values in NS^{j+1}_n by (62). They define chain maps $f^j_\bullet : NS^j_\bullet \to NS^{j+1}_\bullet$, since

$$\begin{aligned}
d_n \circ f^j_n(x) &= (-1)^n d^n_n(x) = (-1)^n f^j_{n-1} \circ d^n_n(x) = f^j_{n-1} \circ d_n(x) && n \leq j+1 \\
d_n \circ f^j_n(x) &= (-1)^n d^n_n(x) - (-1)^n d^n_n \circ s^{n-1}_{n-1} \circ d^{n-1}_n(x) = (-1)^n d^n_n(x) + (-1)^{n-1} d^{n-1}_n(x) \\
&= d_n(x) = f^j_{n-1} \circ d_n(x) && n = j+2 \\
d_n \circ f^j_n(x) &= \Sigma^{n+1}_{k=j+2}(-1)^k d^k_n \circ f^j_n(x) \\
&= \Sigma^{n+1}_{k=j+2}(-1)^k d^k_n(x) - \Sigma^n_{k=j+2}(-1)^k d^k_n \circ s^{j+1}_{n-1} \circ d^{j+1}_n(x) \\
&= \Sigma^n_{k=j+1}(-1)^k d^k_n(x) - \Sigma^n_{k=j+1}(-1)^k s^{j+1}_{n-2} \circ d^{j+1}_{n-1} \circ d^k_n(x) = f^j_{n-1} \circ d_n(x) && n > j+2
\end{aligned}$$

and satisfy $1_{NS^{j+1}_\bullet} = f^j_\bullet \circ \iota^j_\bullet : NS^{j+1}_\bullet \to NS^{j+1}_\bullet$ by definition. For $j \geq -1$ the R-linear maps

$$t^j_n : NS^j_n \to NS^j_{n+1}, \quad x \mapsto \begin{cases} (-1)^{j+1} s^{j+1}_n(x) & n \geq j+1 \\ 0 & n < j+1 \end{cases}$$

define chain homotopies $t^j_\bullet : 1_{NS^j_\bullet} \Rightarrow \iota^j_\bullet \circ f^j_\bullet$, since one has

$$\begin{aligned}
& d_{n+1} \circ t^j_n(x) + t^j_{n-1} \circ d_n(x) = 0 = x - \iota^j_n \circ f^j_n(x) && n < j+1, \\
& d_{n+1} \circ t^j_n(x) + t^j_{n-1} \circ d_n(x) = (-1)^{n+j+1} d^n_{n+1} \circ s^{j+1}_n(x) + (-1)^{n+j} d^{n+1}_{n+1} \circ s^{j+1}_n(x) \\
& \quad = x - x = 0 = x - \iota^j_n \circ f^j_n(x) && n = j+1, \\
& d_{n+1} \circ t^j_n(x) + t^j_{n-1} \circ d_n(x) \\
& \quad = \Sigma^{n+1}_{k=j+1}(-1)^{k+j+1} d^k_{n+1} \circ s^{j+1}_n(x) + \Sigma^n_{k=j+1}(-1)^{k+j+1} s^{j+1}_{n-1} \circ d^k_n(x) \\
& \quad = x - x + \Sigma^{n+1}_{k=j+3}(-1)^{k+j+1} s^{j+1}_{n-1} \circ d^{k-1}_n(x) + \Sigma^n_{k=j+1}(-1)^{k+j+1} s^{j+1}_{n-1} \circ d^k_n(x) \\
& \quad = s^{j+1}_{n-1} \circ d^{j+1}_n(x) = x - \iota^j_n \circ f^j_n(x) && n \geq j+2.
\end{aligned}$$

We now show that the inclusion map $\iota_\bullet : NS_\bullet \to S_\bullet$ is a chain homotopy equivalence. For this, we note that $\iota_n = \iota^{-1}_n \circ ... \circ \iota^{n-2}_n : NS_n \to S_n$ and consider for $n \in \mathbb{N}_0$ the R-linear maps

$$\begin{aligned}
g_n &= f^{n-2}_n \circ f^{n-3}_n \circ ... \circ f^0_n \circ f^{-1}_n : S_n \to NS_n \\
h_n &= \Sigma^{n-2}_{k=-2}\, \iota^{-1}_{n+1} \circ ... \circ \iota^k_{n+1} \circ t^{k+1}_n \circ f^k_n \circ ... \circ f^{-1}_n : S_n \to S_{n+1}.
\end{aligned}$$

The R-linear maps g_n define a chain map $g_\bullet : S_\bullet \to NS_\bullet$ because the maps $f^j_\bullet : N^j_\bullet \to N^{j+1}_\bullet$ are chain maps for all $j \geq -1$:

$$\begin{aligned}
d_n \circ g_n &= d_n \circ f^{n-2}_n \circ f^{n-3}_n \circ ... \circ f^0_n \circ f^{-1}_n = f^{n-2}_{n-1} \circ d_n \circ f^{n-3}_n \circ ... \circ f^0_n \circ f^{-1}_n \\
&= ... = f^{n-2}_{n-1} \circ f^{n-3}_{n-1} \circ ... \circ f^0_{n-1} \circ d_n \circ f^{-1}_n = f^{n-2}_{n-1} \circ f^{n-3}_{n-1} \circ ... \circ f^0_{n-1} \circ f^{-1}_{n-1} \circ d_n = g_{n-1} \circ d_n.
\end{aligned}$$

They also satisfy $1_{NS_\bullet} = g_\bullet \circ \iota_\bullet : NS_\bullet \to NS_\bullet$ since $1_{NS^{j+1}_\bullet} = f^j_\bullet \circ \iota^j_\bullet$ for all $j \geq -1$:

$$\begin{aligned} g_n \circ \iota_n &= f_n^{n-2} \circ f_n^{n-3} \circ ... \circ f_n^0 \circ f_n^{-1} \circ \iota_n^{-1} \circ ... \circ \iota_n^{n-2} = f_n^{n-2} \circ f_n^{n-3} \circ ... \circ f_n^0 \circ \iota_n^0 \circ ... \circ \iota_n^{n-2} \\ &= ... = f_n^{n-2} \circ \iota_n^{n-2} = \mathrm{id}_{NS_n}. \end{aligned}$$

The maps $h_n : S_n \to S_{n+1}$ define a chain homotopy $h_\bullet : 1_{S_\bullet} \Rightarrow \iota_\bullet \circ g_\bullet$ since $t^j_\bullet : 1_{N^j_\bullet} \Rightarrow \iota^j_\bullet \circ f^j_\bullet$ is a chain homotopy for all $j \geq -1$:

$$\begin{aligned} &d_{n+1} \circ h_n + h_{n-1} \circ d_n \\ &= \Sigma_{k=-2}^{n-2}\, d_{n+1} \circ \iota_{n+1}^{-1} \circ ... \circ \iota_{n+1}^k \circ t_n^{k+1} \circ f_n^k \circ ... \circ f_n^{-1} + \Sigma_{k=-2}^{n-3} \iota_n^{-1} \circ ... \circ \iota_n^k \circ t_{n-1}^{k+1} \circ f_{n-1}^k \circ ... \circ f_{n-1}^{-1} \circ d_n \\ &= \Sigma_{k=-2}^{n-2} \iota_n^{-1} \circ ... \circ \iota_n^k \circ d_{n+1} \circ t_n^{k+1} \circ f_n^k \circ ... \circ f_n^{-1} + \Sigma_{k=-2}^{n-2} \iota_n^{-1} \circ ... \circ \iota_n^k \circ t_{n-1}^{k+1} \circ d_n \circ f_n^k \circ ... \circ f_n^{-1} \\ &= \Sigma_{k=-2}^{n-2} \iota_n^{-1} \circ ... \circ \iota_n^k \circ (d_{n+1} \circ t_n^{k+1} + t_{n-1}^{k+1} \circ d_n) \circ f_n^k \circ ... \circ f_n^{-1} \\ &= \Sigma_{k=-2}^{n-2} \iota_n^{-1} \circ ... \circ \iota_n^k \circ (\mathrm{id}_{N_n^{k+1}} - \iota_n^{k+1} \circ f_n^{k+1}) \circ f_n^k \circ ... \circ f_n^{-1} \\ &= \Sigma_{k=-2}^{n-2} \iota_n^{-1} \circ ... \circ \iota_n^k \circ f_n^k \circ ... \circ f_n^{-1} - \Sigma_{k=-1}^{n-1} \iota_n^{-1} \circ ... \circ \iota_n^k \circ \iota_n^{k+1} \circ f_n^{k+1} \circ f_n^k \circ ... \circ f_n^{-1} \\ &= \mathrm{id}_{S_n} - \iota_n^{-1} \circ ... \circ \iota_n^{n-1} \circ f_n^{n-1} \circ ... \circ f_n^{-1} = \mathrm{id}_{S_n} - \iota_n^{-1} \circ ... \circ \iota_n^{n-2} \circ f_n^{n-2} \circ ... \circ f_n^{-1} = \mathrm{id}_{S_n} - \iota_n \circ g_n. \end{aligned}$$

This shows that $g_\bullet : S_\bullet \to NS_\bullet$ and $\iota_\bullet : NS_\bullet \to S_\bullet$ are chain homotopy equivalences. By Proposition 3.3.4 this implies $H_n(S_\bullet) = H_n(NS_\bullet)$ for all $n \in \mathbb{N}_0$.

5. That $DS_\bullet$ is chain homotopy equivalent to $0_\bullet$ follows directly from 4. Let $\iota'_\bullet : DS_\bullet \to S_\bullet$ be the chain map induced by the inclusion morphisms $\iota'_n : DS_n \to NS_n \amalg DS_n$ and $\pi'_\bullet : S_\bullet \to DS_\bullet$ the chain map induced by the projection morphisms $\pi'_n : NS_n \amalg DS_n \to DS_n$. Then we have

$$\pi'_\bullet \circ \iota'_\bullet = \mathrm{id}_\bullet : DS_\bullet \to DS_\bullet \qquad \pi'_\bullet \circ \iota_\bullet \circ g_\bullet \circ \iota'_\bullet = 0_\bullet : DS_\bullet \to DS_\bullet,$$

where $\iota_\bullet : NS_\bullet \to S_\bullet$ and $g_\bullet : S_\bullet \to NS_\bullet$ are the chain maps from 4. The chain homotopy $h_\bullet$ from 4. then define a chain homotopy $k_\bullet : \mathrm{id}_\bullet \Rightarrow 0_\bullet$ with $k_n = \pi'_{n+1} \circ h_n \circ \iota'_n : DS_n \to DS_{n+1}$

$$\begin{aligned} d_{n+1} \circ k_n + k_{n-1} \circ d_n &= d_{n+1} \circ \pi'_{n+1} \circ h_n \circ \iota'_n + \pi'_n \circ h_{n-1} \circ \iota'_{n-1} \circ d_n \\ =\pi'_n \circ (d_{n+1} \circ h_n + h_{n-1} \circ d_n) \circ \iota'_n &= \pi'_n \circ (\mathrm{id}_{S_n} - \iota_n \circ g_n) \circ \iota'_n = \mathrm{id}_{DS_n}. \end{aligned}$$

□

The fact that the degenerate chain complex $DS_\bullet$ is chain homotopic to the trivial chain complex has a geometrical interpretation in simplicial and singular homology. It states that degenerate n-simplexes that are of the form $\sigma = \tau \circ s_i^{n-1} : \Delta^n \to X$ with an $(n-1)$-simplex $\tau : \Delta^{n-1} \to X$ do not contribute to the homologies. This motivates the restriction to n-simplexes $\sigma : \Delta^n \to X$ with $\sigma|_{\mathring{\Delta}^n} : \mathring{\Delta}^n \to X$ injective in the definition of a (semi)simplicial complex.

The construction of chain complexes from simplicial objects $S : \Delta^{+op} \to \mathcal{A}$ in an abelian category $\mathcal{A}$ via Proposition 5.2.1 can be extended to simplicial morphisms. As expected, every simplicial morphism $\eta : S \to T$ defines a chain map $\eta_\bullet : S_\bullet \to T_\bullet$ between the associated standard chain complexes. It restricts to chain maps between the degenerate complexes and the normalised chain complexes. As these are compatible with the composition of simplicial morphisms and the unit morphisms, they define functors from the category $\mathrm{Fun}(\Delta^{+op}, \mathcal{A})$ of simplicial objects in $\mathcal{A}$ to the category $\mathrm{Ch}_{\mathcal{A}\geq 0}$ of positive chain complexes in $\mathcal{A}$.

Proposition 5.2.2: Let $\mathrm{S}(\mathcal{A}) = \mathrm{Fun}(\Delta^{+op}, \mathcal{A})$ be the category of simplicial objects in an abelian category $\mathcal{A}$. Then the following are functors:

1. The standard chain complex functor $\bullet : \mathrm{S}(\mathcal{A}) \to \mathrm{Ch}_{\mathcal{A}\geq 0}$ that sends
   - a simplicial object $S : \Delta^{+op} \to \mathcal{A}$ to the chain complex $S_\bullet$
   - a simplicial morphism $\eta : S \to T$ to the chain map $\eta_\bullet : S_\bullet \to T_\bullet$ with $\eta_n = \eta_{[n+1]} : S_n \to T_n$.
2. The normalised chain complex functor $N : S(\mathcal{A}) \to \mathrm{Ch}_{\mathcal{A}\geq 0}$ that sends
   - a simplicial object $S : \Delta^{+op} \to \mathcal{A}$ to the chain complex $NS_\bullet$
   - a simplicial morphism $\eta : S \to T$ to the chain map $N\eta_\bullet : NS_\bullet \to NT_\bullet$ induced by $\eta_\bullet : S_\bullet \to T_\bullet$.

Proof:
1. Let $S, T : \Delta^{+op} \to \mathcal{A}$ be simplicial objects in $\mathcal{A}$ with associated standard chain complexes $S_\bullet$ and $T_\bullet$ in $\mathcal{A}$ and $\eta : S \to T$ a natural transformation. We show that its component morphisms $\eta_n = \eta_{[n+1]} : S_n \to T_n$ define a chain map $\eta_\bullet : S_\bullet \to T_\bullet$. This follows from the naturality of η, which implies that for all morphisms $\alpha : [m] \to [n]$ in Δ^+ one has

$$\eta_{m-1} \circ S(\alpha) = \eta_{[m]} \circ S(\alpha) \overset{\text{nat}}{=} T(\alpha) \circ \eta_{[n]} = T(\alpha) \circ \eta_{n-1}. \tag{63}$$

In particular, this holds for the morphisms $\delta_n^i : [n] \to [n+1]$, and we obtain

$$d_n^T \circ \eta_n = \Sigma_{i=0}^n (-1)^i T(\delta_n^i) \circ \eta_{[n+1]} \overset{\text{nat}}{=} \Sigma_{i=0}^n (-1)^i \eta_{[n]} \circ S(\delta_n^i) = \eta_{n-1} \circ d_n^S.$$

2. We show that the assignments of chain complexes to simplicial objects and chain maps to simplicial morphisms respects the composition of morphisms and the identity morphisms. For $\eta = \mathrm{id}_S : S \to S$ we have $(\mathrm{id}_S)_n = 1_{S([n+1])} = 1_{S_n} : S_n \to S_n$, and we obtain the identity chain map $\mathrm{id}_{S_\bullet} : S_\bullet \to S_\bullet$. For simplicial objects $R, S, T : \Delta^{+op} \to \mathcal{A}$ and natural transformations $\eta : R \to S$ and $\kappa : S \to T$, the composite natural transformation $\kappa\eta : R \to T$ has component morphisms $(\kappa\eta)_{[n]} = \kappa_{[n]} \circ \eta_{[n]}$. This shows that the morphisms $\kappa\eta_n : R_n \to T_n$ are given by $\kappa\eta_n = (\kappa\eta)_{[n+1]} = \kappa_{[n+1]} \circ \eta_{[n+1]} = \kappa_n \circ \eta_n$, and the chain maps satisfy $\kappa\eta_\bullet = \kappa_\bullet \circ \eta_\bullet$.

3. To prove the statement for the normalised chain complex functor, it is sufficient to show that for every natural transformation $\eta : S \to T$ the chain map $\eta_\bullet : S_\bullet \to T_\bullet$ between the associated standard chain complexes restricts to a chain map $N\eta_\bullet : NS_\bullet \to NT_\bullet$. By applying (63) to the morphisms $\delta_n^i : [n] \to [n+1]$, we obtain $d_n^{Ti} \circ \eta_n(x) = \eta_{n-1} \circ d_n^{Si}(x)$ for all $i \in \{0, ..., n\}$. As the chain complex $NS_\bullet$ is given by $NS_n = \cap_{j=0}^{n-1} \ker(d_n^i)$, this implies $\eta_n(NS_n) \subset NT_n$, and hence $\eta_\bullet$ induces a chain map $N\eta_\bullet^\eta : NS_\bullet \to NT_\bullet$. □

The standard chain complex functor explains the similarities between the (co)chain complexes in Section 2. These examples are obtained from the simplicial object for Hochschild homology in Example 5.1.7 and the simplicial object for singular homology in Example 5.1.8 by applying the standard chain complex functor. This is a nice explanation for the specific form of the chain complexes in Section 2. This makes it natural to ask if *all* positive chain complexes in an abelian category $\mathcal{A}$ are obtained from simplicial objects in $\mathcal{A}$, possibly up to chain homotopy equivalence. Surprisingly, the answer to this question is *yes*.

Theorem 5.2.3: (**Dold-Kan correspondence**)
For any abelian category $\mathcal{A}$ the normalised chain complex functor $N : \mathrm{Fun}(\Delta^{+op}, \mathcal{A}) \to \mathrm{Ch}_{\mathcal{A}\geq 0}$ is an equivalence of categories between the category $\mathrm{Fun}(\Delta^{+op}, \mathcal{A})$ of simplicial objects and the category $\mathrm{Ch}_{\mathcal{A}\geq 0}$ of positive chain complexes in $\mathcal{A}$.

Proof:
Instead of showing that the normalised chain complex functor $N : \mathrm{Fun}(\Delta^{+op}, \mathcal{A}) \to \mathrm{Ch}_{\mathcal{A}\geq 0}$ from Propositions 5.2.1 and 5.2.2 is an equivalence of categories, we show this for the functor $N' : \mathrm{Fun}(\Delta^{+op}, \mathcal{A}) \to \mathrm{Ch}_{\mathcal{A}\geq 0}$ that associates to a simplicial object $S : \Delta^{+op} \to \mathcal{A}$ the chain complex $N'S_\bullet$ with $N'S_n = NS_n$ and $d'_n = (-1)^n d^N_n = S(\delta^n_n) : NS_n \to NS_{n-1}$. The morphisms

$$\tau^S_{2k+1} = (-1)^{k+1} 1_{NS_{2k+1}} : NS_{2k+1} \to N'S_{2k+1}, \quad \tau^S_{2k} = (-1)^k 1_{NS_{2k}} : NS_{2k} \to N'S_{2k} \qquad k \in \mathbb{N}_0$$

form an isomorphism $\tau^S_\bullet : NS_\bullet \to N'S_\bullet$. As they satisfy $\tau^T_n \circ \eta_{[n+1]} = \eta_{[n+1]} \circ \tau^S_n$ for all natural transformations $\eta : S \to T$, the chain maps $\tau^S_\bullet : NS_\bullet \to N'S_\bullet$ define a natural isomorphism $\tau : N \to N'$. It follows that N is an equivalence of categories if and only if N' is.

To show that $N' : \mathrm{Fun}(\Delta^{+op}, \mathcal{A}) \to \mathrm{Ch}_{\mathcal{A}\geq 0}$ is an equivalence of categories, we construct a functor $K : \mathrm{Ch}_{\mathcal{A}\geq 0} \to \mathrm{Fun}(\Delta^{+op}, \mathcal{A})$ such that $N'K = \mathrm{id}_{\mathrm{Ch}_{\mathcal{A}\geq 0}}$ and KN' is naturally isomorphic to the identity functor $\mathrm{id}_{\mathrm{Fun}(\Delta^{+op},\mathcal{A})}$.

• **step 1:** We define $K : \mathrm{Ch}_{\mathcal{A}\geq 0} \to \mathrm{Fun}(\Delta^{+op}, \mathcal{A})$ on the objects of $\mathrm{Ch}_{\mathcal{A}\geq 0}$:

To a positive chain complex $C_\bullet$ in $\mathcal{A}$ we assign the functor $K^C := K(C_\bullet) : \Delta^{+op} \to \mathcal{A}$ that sends an ordinal $[n+1]$ to the object

$$K^C([n+1]) = K^C_n = \coprod_{\substack{0 \leq p \leq n,\\ \sigma:[n+1] \twoheadrightarrow [p+1]}} C_p, \tag{64}$$

where the coproduct runs over all monotonic surjections $\sigma : [n+1] \to [p+1]$ with $0 \leq p \leq n$. We denote by $\iota_\sigma : C_p \to K^C_n$ the inclusion morphism for the factor associated with σ.

For a morphism $\alpha : [m+1] \to [n+1]$ in Δ^+ we define $K^C(\alpha) : K^C_n \to K^C_m$ via the canonical factorisation and the universal property of the coproduct. Proposition 5.1.2 implies that for every monotonic map $\alpha : [m+1] \to [n+1]$ and every monotonic surjection $\sigma : [n+1] \to [p+1]$ in Δ^+, there is a unique $q \leq \min(m, p)$, a unique monotonic surjection $\sigma_\alpha : [m+1] \to [q+1]$ and monotonic injection $\alpha_\sigma : [q+1] \to [p+1]$ with $\sigma \circ \alpha = \alpha_\sigma \circ \sigma_\alpha$

$$\begin{tikzcd} {[m+1]} \arrow[r, "\alpha"] \arrow[d, "\sigma_\alpha"] & {[n+1]} \arrow[d, "\sigma"] \\ {[q+1]} \arrow[r, "\alpha_\sigma"'] & {[p+1].} \end{tikzcd}$$

Define $K^C(\alpha) : K^C_n \to K^C_m$ as the unique morphism for which the following diagram commutes

$$\begin{tikzcd} K^C_n \arrow[r, "K^C(\alpha)"] & K^C_m \\ C_p \arrow[u, "\iota_\sigma"] \arrow[r, "K^C(\alpha)^\tau_\sigma"'] & C_q \arrow[u, "\iota_\tau"'] \end{tikzcd}, \tag{65}$$

with

$$K^C(\alpha)^\tau_\sigma = \begin{cases} \delta^\tau_{\sigma_\alpha} 1_{C_p} : C_p \to C_p & \alpha_\sigma = 1_{[p+1]} \\ \delta^\tau_{\sigma_\alpha} d_p : C_p \to C_{p-1} & \alpha_\sigma = \delta^p_p \\ 0 : C_p \to C_q & \text{else.} \end{cases} \tag{66}$$

Note that $K^C(\alpha)^\tau_\sigma$ is trivial unless $\tau = \sigma_\alpha$. It is also trivial if the reordering of the face maps and degeneracies in $\sigma \circ \alpha$ into the standard form (58) destroys more than one face map and degeneracy, which implies $q < p-1$ and $\alpha_\sigma \notin \{1_p, \delta^p_p\}$.

If $\alpha : [m+1] \to [n+1]$ is a monotonic *surjection*, we have $\sigma_\alpha = \sigma \circ \alpha : [m+1] \to [p+1] = [q+1]$ and $\alpha_\sigma = 1_{[p+1]}$, which implies $K^C(\alpha)^\tau_\sigma = \delta^\tau_{\sigma\circ\alpha} 1_{C_p}$. In this case, $K^C(\alpha) : K^C_n \to K^C_m$ is the morphism in $\mathcal{A}$ that sends the copy of C_p in K^C_n associated with σ to the copy of C_p in K^C_m associated with $\sigma \circ \alpha$. In particular, we have $K^C(1_{[n+1]}) = 1_{K^C_n} : K^C_n \to K^C_n$.

To show that this defines a functor $K^C : \Delta^{+op} \to \mathcal{A}$, it remains to show compatibility with the composition of morphisms in Δ^+. We consider monotonic maps $\beta : [l+1] \to [m+1]$ and $\alpha : [m+1] \to [n+1]$. To show that $K^C(\alpha \circ \beta) = K^C(\beta) \circ K^C(\alpha)$, it is sufficient by (65) to show that for all monotonic surjections σ with source $[n+1]$ and ρ with source $[l+1]$ one has

$$\Sigma_\nu\, K^C(\beta)^\rho_\nu \circ K^C(\alpha)^\nu_\sigma = K^C(\alpha \circ \beta)^\rho_\sigma, \tag{67}$$

where the sum runs over all monotonic surjections ν with source $[m+1]$.

To prove this, we have to express the factorisation of $\alpha \circ \beta$ in terms of the factorisations for α and β. Any monotonic surjection $\sigma : [n+1] \to [p+1]$ yields numbers $0 \leq r \leq q \leq p$, monotonic surjections $\sigma_\alpha : [m+1] \to [q+1]$ and $(\sigma_\alpha)_\beta : [r+1] \to [q+1]$ and monotonic injections $\alpha_\sigma : [q+1] \to [p+1]$, $\beta_{\sigma_\alpha} : [r+1] \to [q+1]$ such that the following diagram commutes

$$\begin{tikzcd}
{[l+1]} \arrow[r, "\beta"'] \arrow[rr, bend left, "\alpha\circ\beta"] \arrow[d, "(\sigma_\alpha)_\beta"'] & {[m+1]} \arrow[r, "\alpha"'] \arrow[d, "\sigma_\alpha"'] & {[n+1]} \arrow[d, "\sigma"'] \\
{[r+1]} \arrow[r, hook, "\beta_{\sigma_\alpha}"] \arrow[rr, bend right, "(\alpha\circ\beta)_\sigma"'] & {[q+1]} \arrow[r, hook, "\alpha_\sigma"] & {[p+1]}.
\end{tikzcd}$$

This implies $\sigma_{\alpha\circ\beta} = (\sigma_\alpha)_\beta$. We now distinguish three cases:

- If $r < p-1$, the right-hand side of (67) vanishes by (66). The left hand side can only give a non-trivial contribution for $r = q-1 = p-2$, $\alpha_\sigma = \delta^p_p$ and $\beta_{\sigma_\alpha} = \delta^{p-1}_{p-1}$. In this case, it is proportional to $d_{p-1} \circ d_p$, which vanishes as well.

- For $r = p$ we obtain from (66)

$$K^C(\alpha \circ \beta)^\rho_\sigma = \delta^\rho_{\sigma_{\alpha\circ\beta}}\, 1_{C_p} = \delta^\rho_{(\sigma_\alpha)_\beta}\, 1_{C_p} = \Sigma_\nu\, \delta^\rho_{\nu_\beta} \delta^\nu_{\sigma_\alpha} 1_{C_p} = \Sigma_\nu\, K^C(\beta)^\rho_\nu \circ K^C(\alpha)^\nu_\sigma.$$

- For $r = p-1$ we either have $\alpha_\sigma = \delta^i_p$ and $\beta_{\sigma_\alpha} = 1_{[p-1]}$ or $\alpha_\sigma = 1_{[p+1]}$ and $\beta_{\sigma_\alpha} = \delta^i_p$. If $i \neq p$ both sides of the equation vanish. For $i = p$ we obtain in both cases

$$K^C(\alpha \circ \beta)^\rho_\sigma = \delta^\rho_{\sigma_{\alpha\circ\beta}}\, d_p = \delta^\rho_{(\sigma_\alpha)_\beta} d_p = \Sigma_\nu\, \delta^\rho_{\nu_\beta} \delta^\nu_{\sigma_\alpha} d_p = \Sigma_\nu\, K^C(\beta)^\rho_\nu \circ K^C(\alpha)^\nu_\sigma.$$

This proves (67) and shows that $K^C : \Delta^{+op} \to \mathcal{A}$ is a simplicial object.

• **step 2:** We define $K : \mathrm{Ch}_{\mathcal{A}\geq 0} \to \mathrm{Fun}(\Delta^{+op}, \mathcal{A})$ on the morphisms of $\mathrm{Ch}_{\mathcal{A}\geq 0}$:

Let $C_\bullet$ and $C'_\bullet$ be positive chain complexes in $\mathcal{A}$ with simplicial objects $K^C, K^{C'} : \Delta^{+op} \to \mathcal{A}$ and $f_\bullet : C_\bullet \to C'_\bullet$ a chain map. We define a natural transformation $K^f := K(f_\bullet) : K^C \to K^{C'}$ by its component morphisms $K^f_{[n+1]} : K^C_n \to K^{C'}_n$.

For this we use the universal property of the coproduct in (64) and define $K^f_{[n+1]}$ as the unique morphism for which the diagram

$$\begin{tikzcd} K^C_n \arrow[r, "K^f_{[n+1]}"] & K^{C'}_n \\ C_p \arrow[u, "\iota_\sigma"] \arrow[r, "f_p"'] & C'_p \arrow[u, "\iota'_\sigma"'] \end{tikzcd} \tag{68}$$

commutes for all $0 \leq p \leq n$ and monotonic surjections $\sigma : [n+1] \to [p+1]$. That this defines a natural transformation $K^f : K^C \to K^{C'}$ follows from the diagram

$$\begin{tikzcd} C_p \arrow[rrr, "f_p"] \arrow[dr, "\iota_\sigma"] \arrow[ddd, "K^C(\alpha)^\tau_\sigma"'] & & & C'_p \arrow[dl, "\iota'_\sigma"'] \arrow[ddd, "K^{C'}(\alpha)^\tau_\sigma"] \\ & K^C_n \arrow[r, "K^f_{[n+1]}"] \arrow[d, "K^C(\alpha)"'] & K^{C'}_n \arrow[d, "K^{C'}(\alpha)"] & \\ & K^C_m \arrow[r, "K^f_{[m+1]}"'] & K^{C'}_m & \\ C_q \arrow[rrr, "f_q"'] \arrow[ur, "\iota_\tau"] & & & C'_q, \arrow[ul, "\iota'_\tau"'] \end{tikzcd}$$

in which the top and bottom quadrilaterals commute by definition of K^f and the left and right quadrilateral commute by definition of K^C. The outer square commutes by definition of $K^C(\alpha)^\tau_\sigma$ in (66) and because $f_\bullet$ is a chain map. This implies

$$\begin{aligned} K^{C'}(\alpha) \circ K^f_{[n+1]} \circ \iota_\sigma &= K^{C'}(\alpha) \circ \iota'_\sigma \circ f_p = \iota'_\tau \circ K^{C'}(\alpha)^\tau_\sigma \circ f_p = \iota'_\tau \circ f_q \circ K^C(\alpha)^\tau_\sigma \\ &= K^f_{[m+1]} \circ \iota_\tau \circ K^C(\alpha)^\tau_\sigma = K^f_{[m+1]} \circ K^C(\alpha) \circ \iota_\sigma. \end{aligned}$$

The universal property of the coproduct then implies $K^{C'}(\alpha) \circ K^f_{[n+1]} = K^f_{[m+1]} \circ K^C(\alpha)$. This proves that the morphisms $K^f_{[n+1]} : K^C_n \to K^{C'}_n$ form a natural transformation $K^f : K^C \to K^{C'}$.

• **step 3:** We show that $K : \mathrm{Ch}_{\mathcal{A}\geq 0} \to \mathrm{Fun}(\Delta^{+op}, \mathcal{A})$ is a functor:

Setting $K^C = K^{C'}$, $C_p = C'_p$, $\iota_\sigma = \iota'_\sigma$, $f_p = 1_{C_p}$ and $K^f_{[n+1]} = 1_{K^n_C}$ in (68), we find that the diagram commutes for all $p \leq n \in \mathbb{N}_0$ and monotonic surjections $\sigma : [n+1] \to [p+1]$. This shows that $K(1_{C_\bullet}) = \mathrm{id}_{K^C} : K^C \to K^C$ is the identity natural transformation.

If $C_\bullet$, $C'_\bullet$, $C''_\bullet$ are positive chain complexes in $\mathcal{A}$ and $f_\bullet : C_\bullet \to C'_\bullet$, $f'_\bullet : C'_\bullet \to C''_\bullet$ chain maps, then composing the commutative diagrams (68) for $f_\bullet$ and $f'_\bullet$ yields the commuting diagram

$$\begin{tikzcd} K^C_n \arrow[r, "K^f_{[n+1]}"'] \arrow[rr, bend left, "K^{f'\circ f}_{[n+1]}"] & K^{C'}_n \arrow[r, "K^{f'}_{[n+1]}"'] & K^{C''}_n \\ C_p \arrow[u, "\iota_\sigma"] \arrow[r, "f_p"] \arrow[rr, bend right, "f'_p \circ f_p"'] & C'_p \arrow[u, "\iota'_\sigma"] \arrow[r, "f'_p"] & C''_p. \arrow[u, "\iota''_\sigma"] \end{tikzcd}$$

This shows that $K^{f'\circ f}_{[n+1]} = K^{f'}_{[n+1]} \circ K^{f}_{[n+1]}$ for all $n \in \mathbb{N}_0$ and $K(f'_\bullet \circ f_\bullet) = K^{f'} \circ K^{f} = K(f'_\bullet) \circ K(f_\bullet)$.

• **step 4:** We show that $N'K = \mathrm{id}_{\mathrm{Ch}_{\mathcal{A}\geq 0}}$:

The functor $N'K : \mathrm{Ch}_{\mathcal{A}\geq 0} \to \mathrm{Ch}_{\mathcal{A}\geq 0}$ sends a chain complex $C_\bullet$ in $\mathcal{A}$ to the chain complex $N'(K^C)_\bullet$ for the simplicial object $K^C : \Delta^{+op} \to \mathcal{A}$ from step 1 and a chain map $f_\bullet : C_\bullet \to C'_\bullet$ to the chain map $N'K^f$ for the simplicial morphism $K^f : K^C \to K^{C'}$ from step 2. we show that $N'(K^C)_\bullet = C_\bullet$ and $N'(K^f)_\bullet = f_\bullet : C_\bullet \to C'_\bullet$.

By (66), we have $K_C(\alpha)^\tau_\sigma = \delta^\tau_{\sigma\circ\alpha} 1_{C_p}$ for each monotonic surjection $\alpha : [m+1] \to [n+1]$ and monotonic surjection $\sigma : [n+1] \to [p+1]$. It follows that all components of C_p in (64) except the one for $n = p$ and $\sigma = 1_{[n+1]}$ are in the images of the maps $s^j_{n-1} = K^C(\sigma^j_n)$. This shows that the R-modules of the degenerate chain complex and the chain complex $N'K$ are given by

$$DK^C_n = +_{j=0}^{n-1} \mathrm{im}(s^j_{n-1}) = \coprod_{\substack{0\leq p<n,\\ \sigma:[n+1]\twoheadrightarrow[p+1]}} C_p \qquad\qquad N'K^C_n = C_n.$$

The morphisms $d'_n = K^C(\delta^n_n) : C_n = N'K^C_n \to N'K^C_{n-1} = C_{n-1}$ can be computed from (66). In this case, we have the factorisation $1_{[n+1]} \circ \delta^n_n = \delta^n_n \circ 1_{[n]}$ with $p = q+1 = n+1$, and (66) yields $K^C(\delta^n_n) = d_n : C_n \to C_{n-1}$. Hence, we have $N'K(C_\bullet) = N'(K^C) = C_\bullet$ for all positive chain complexes $C_\bullet$ in $\mathcal{A}$.

For every chain map $f_\bullet : C_\bullet \to C'_\bullet$, we obtain a natural transformation $K^f = K(f_\bullet) : K^C \to K^{C'}$ given by diagram (68). As only the summand for $\sigma = 1_{[n+1]}$ in (64) contributes to $N'K^C_n$, we have $N'K(f_\bullet)_n = N'(K^f)_n = f_n : C_n \to C'_n$ and $N'K = \mathrm{id}_{\mathrm{Ch}_{\mathcal{A}\geq 0}}$.

• **step 5:** We construct a natural transformation $\eta : KN' \to \mathrm{id}_{\mathrm{Fun}(\Delta^{+op},\mathcal{A})}$:

A natural transformation $\eta : KN' \to \mathrm{id}_{\mathrm{Fun}(\Delta^{+op},\mathcal{A})}$ assigns to simplicial objects $S : \Delta^{+op} \to \mathcal{A}$ natural transformations $\eta^S : KN'(S) \to S$, such that $\eta^S \circ KN'(\tau) = \tau \circ \eta^T$ for every natural transformation $\tau : S \to T$. Each natural transformation $\eta^S : KN'(S) \to S$ is determined by its component morphisms $\eta^S_{[n+1]} : K^{N'S}_n \to S_n$ in $\mathcal{A}$. From (64) we have

$$K^{N'S}_n = KN'(S)([n+1]) = \coprod_{\substack{0\leq p\leq n,\\ \sigma:[n+1]\twoheadrightarrow[p+1]}} NS_p. \tag{69}$$

We denote by $i_p : NS_p \to S_p$ the inclusions and define $\eta^S_{[n+1]} : K^{N'S}_n \to S_n$ as the unique morphism induced by the universal property of the coproduct and the morphisms $S(\sigma) \circ i_p : NS_p \to S_n$ for each monotonic surjection $\sigma : [n+1] \to [p+1]$:

$$\begin{tikzcd} K^{N'S}_n \arrow[r, "\eta^S_{[n+1]}"] & S_n \\ NS_p \arrow[u, "\iota_\sigma"] \arrow[r, "i_p"'] & S_p, \arrow[u, "S(\sigma)"'] \end{tikzcd} \tag{70}$$

To prove that this defines a natural transformation $\eta^S : KN'(S) \to S$, we have to show that

$$\eta^S_{[m+1]} \circ K^{N'S}(\alpha) \circ \iota_\sigma = S(\alpha) \circ \eta^S_{[n+1]} \circ \iota_\sigma$$

for all monotonic surjections $\sigma : [n+1] \to [p+1]$ and monotonic maps $\alpha : [m+1] \to [n+1]$. The two sides of this equation are given by

$$\eta^S_{[m+1]} \circ K^{N'S}(\alpha) \circ \iota_\sigma \overset{(65)}{=} \eta^S_{[m+1]} \circ \iota_{\sigma_\alpha} \circ K^{N'S}(\alpha)^{\sigma_\alpha}_\sigma \overset{(70)}{=} S(\sigma_\alpha) \circ i_q \circ K^{N'S}(\alpha)^{\sigma_\alpha}_\sigma \tag{71}$$
$$S(\alpha) \circ \eta^S_{[n+1]} \circ \iota_\sigma \overset{(70)}{=} S(\alpha) \circ S(\sigma) \circ i_p = S(\sigma \circ \alpha) \circ i_p,$$

where $\sigma \circ \alpha = \alpha_\sigma \circ \sigma_\alpha$ with a monotonic injection $\alpha_\sigma : [q+1] \to [p+1]$, a monotonic surjection $\sigma_\alpha : [m+1] \to [q+1]$. As we have $NS_n = \cap_{j=0}^{n-1} \ker(d^j_n) = \cap_{j=0}^{n-1} \ker(S(\delta^j_n))$, $S_n = NS_n \amalg DS_n$ and $\alpha : [m+1] \to [n+1]$ can be factorised as in Proposition 5.1.2, it is sufficient to prove that the expressions in (71) are equal for monotonic surjections α and for $\alpha = \delta^n_n : [n] \to [n+1]$.

If α is a monotonic surjection, we have $p = q$, $\sigma_\alpha = \sigma \circ \alpha$ and $K^{N'S}(\alpha)^{\sigma_\alpha}_\sigma = 1_{NS_p}$ by (66). If $\alpha = \delta^n_n$ we have $p = q+1$, $\alpha_\sigma = \delta^n_n$, $\delta^n_n \circ \sigma_\alpha = \sigma \circ \delta^n_n$ and $i_q \circ K^{N'S}(\delta^n_n)^{\sigma_\alpha}_\sigma = d'^n_n = S(\delta^n_n) \circ i_p$. In both cases, the two sides of (71) agree. Hence we assigned to each simplicial object $S : \Delta^{+op} \to \mathcal{A}$ a natural transformation $\eta^S : KN'(S) \to S$.

To prove that this defines a natural transformation $\eta : KN \to \mathrm{id}_{\mathrm{Fun}(\Delta^{+op},\mathcal{A})}$ we show that for each natural transformation $\tau : S \to T$ we have $\eta^T \circ KN(\tau) = \tau \circ \eta^S$. With the definition of $KN(\tau) = K^{N(\tau)}$ in (68) and the definition of η^S in (70) we compute

$$\eta^T_{[n+1]} \circ KN(\tau_{[n+1]}) \circ \iota_\sigma \overset{(68)}{=} \eta^T_{[n+1]} \circ \iota'_\sigma \circ N(\tau_{[n+1]}) \overset{(70)}{=} T(\sigma) \circ i^T_p \circ N(\tau_{[n+1]}) = T(\sigma) \circ \tau_{[n+1]} \circ i^S_p$$
$$\overset{\text{nat } \tau}{=} \tau_{[n+1]} \circ S(\sigma) \circ i^S_p \overset{(70)}{=} \tau_{[n+1]} \circ \eta^S_{[n+1]} \circ \iota_\sigma,$$

where $i^S_p : NS_p \to S_p$ and $i^T_p : NT_p \to T_p$ denote the inclusion morphisms for the subcomplexes $NS_\bullet \subset S_\bullet$ and $NT_\bullet \subset T_\bullet$. With the universal property of the coproduct, this implies $\eta^T_{[n+1]} \circ KN(\tau_{[n+1]}) = \tau_{[n+1]} \circ \eta^S_{[n+1]}$ for all $n \in \mathbb{N}_0$ and $\eta^T \circ KN(\tau) = \tau \circ \eta^S$. This shows that the natural isomorphisms $\eta^S : KN'(S) \to S$ define a natural transformation $\eta : KN' \to S$.

• **step 6:** We show that $\eta : KN' \to \mathrm{id}_{\mathrm{Fun}(\Delta^{+op},\mathcal{A})}$ is a natural isomorphism:

We prove that the morphisms $\eta^S_{[n+1]} : K^{N'S}_n \to S_n$ are isomorphisms for all simplicial objects $S : \Delta^{+op} \to \mathcal{A}$ and $n \in \mathbb{N}_0$ by induction over n. For simplicity we assume $\mathcal{A} = R$-Mod.

$n = 0$: We have $K^{N'S}_0 = S_0 = S([1])$ since $\sigma = 1_{[1]} : [1] \to [1]$ is the only surjective morphism in the coproduct in (69) and $\eta^S_{[1]} = \mathrm{id}_{S_0} : S_0 \to S_0$ by (70).

$n-1 \Rightarrow n$: Suppose we established that $\eta^S_{[k+1]} : K^{N'S}_k \to S_k$ is an isomorphism for all $k \leq n-1$. As $\eta^S : KN'(S) \to S$ is a natural transformation, we obtain with the induction hypothesis

$$s^j_{n-1} = S(\sigma^j_n) = \eta^S_{[n+1]} \circ KN'(\sigma^j_n) \circ (\eta^S_{[n]})^{-1} \quad \Rightarrow \quad \mathrm{im}(s^j_{n-1}) \subset \mathrm{im}(\eta^S_{[n+1]})\ \forall j \in \{0, ..., n-1\}.$$

By choosing $\sigma = 1_{[n+1]}$ in (70) we obtain $\eta^S_{[n+1]} \circ \iota_{1_{[n+1]}} = i_n$, and this implies $NS_n \subset \mathrm{im}(\eta^S_{[n+1]})$. As we have $S_n = NS_n \amalg DS_n$ with $DS_n = +_{j=0}^{n-1} \mathrm{im}(s^j_{n-1})$ for all $n \in \mathbb{N}_0$ by Proposition 5.2.1, this shows that $\eta^S_{[n+1]} : K^{N'S}_n \to S_n$ is an epimorphism.

To show that $\eta^S_{[n+1]} : K^{N'S}_n \to S_n$ is a monomorphism, we note that for each monotonic surjection $\sigma : [n+1] \to [p+1]$ we can use the relations (59) in Δ^+ to construct a monotonic injection $\delta : [p+1] \to [n+1]$ with $\sigma \circ \delta = 1_{[p+1]}$. With the definition of η^S in (70) we then obtain

$$S(\delta) \circ \eta^S_{[n+1]} \circ \iota_\sigma \overset{(70)}{=} S(\delta) \circ S(\sigma) \circ i_p = S(\sigma \circ \delta) \circ i_p = S(1_{[p+1]}) \circ i_p = i_p.$$

As $i_p : NS_p \to S_p$ is a monomorphism, it follows that $\eta^S_{[n+1]} \circ \iota_\sigma$ is a monomorphism for each monotonic surjection $\sigma : [n+1] \to [p+1]$. With the definition of η^S in (70) via the universal property of the coproduct, it follows that $\eta^S_{[n+1]} : K_n^{N'S} \to S_n$ is a monomorphism. □

The proof of Theorem 5.2.3 appears rather lengthy and technical, but this is because all computations are carried out in detail. The essential idea in the proof is the construction of the functor $K : \mathrm{Ch}_{\mathcal{A}\geq 0} \to \mathrm{Fun}(\Delta^{+op}, \mathcal{A})$ in (64), (65) and (66). Once this is done, all other constructions in the proof are essentially determined and can be verified by routine computations.

The construction of K addresses the problem that the boundary operator of a positive chain complex in $\mathcal{A}$ involves only the face maps, whereas a simplicial object also requires information about the degeneracies. The functor K can be motivated by considering **semisimplicial objects** in an abelian category $\mathcal{A}$.

Semisimplicial objects in $\mathcal{A}$ are functors $S : \Delta^{+op}_{inj} \to \mathcal{A}$, where $\Delta^+_{inj} \subset \Delta^+$ is the subcategory with the same objects and only *injective* monotonic maps as morphisms. Morphisms of semisimplicial objects are natural transformations between them. They form the category $\mathrm{Fun}(\Delta^{+op}_{inj}, \mathcal{A})$. As all injective monotonic maps in Δ^+_{inj} are composites of face maps, it is plausible that a positive chain complex in $\mathcal{A}$ defines a semisimplicial object $S : \Delta^{+op}_{inj} \to \mathcal{A}$.

Exercise 74 shows that the construction of K in the proof of Theorem 5.2.3 defines a functor $L : \mathrm{Fun}(\Delta^{+op}_{inj}, \mathcal{A}) \to \mathrm{Fun}(\Delta^{+op}, \mathcal{A})$ and that K factorises as $K = LG$ for a simple and obvious functor $G : \mathrm{Ch}_{\mathcal{A}\geq 0} \to \mathrm{Fun}(\Delta^{+op}_{inj}, \mathcal{A})$. Exercise 75 shows that the functor L is left adjoint to the restriction functor $R : \mathrm{Fun}(\Delta^{+op}, \mathcal{A}) \to \mathrm{Fun}(\Delta^{+op}_{inj}, \mathcal{A})$. It also shows that the functor G is left adjoint to a functor $N'' : \mathrm{Fun}(\Delta^{+op}_{inj}, \mathcal{A}) \to \mathrm{Ch}_{\mathcal{A}\geq 0}$ with $N''R = N'$ that is defined analogously to the functor N' in the proof of Theorem 5.2.3. Consequently, $K : \mathrm{Ch}_{\mathcal{A}\geq 0} \to \mathrm{Fun}(\Delta^{+op}, \mathcal{A})$ is left adjoint to the normalised chain complex functor $N : \mathrm{Fun}(\Delta^{+op}, \mathcal{A}) \to \mathrm{Ch}_{\mathcal{A}\geq 0}$.

Exercise 76 shows that for every semisimplicial object $S : \Delta^{+op}_{inj} \to \mathcal{A}$ the associated simplicial object $LS : \Delta^{+op} \to \mathcal{A}$ is a left **Kan extension** of S along the inclusion functor $\iota : \Delta^{+op}_{inj} \to \Delta^{+op}$. Kan extensions are standard constructions in category theory that encompass many constructions such as induced representations, geometric realisation of simplicial sets, (co)products, (co)equalisers and (co)limits. The construction of K is obvious from this perspective and follows from a standard formula for Kan extensions.

It should also be mentioned that Dold-Kan correspondence extends to simplicial homotopies. As explained at the end of Section 5.1, there is a notion of simplicial homotopy $h : f \Rightarrow g$ that relates simplicial morphisms $f, g : S \to S'$ in an abelian category $\mathcal{A}$. By Proposition 5.2.2 the simplicial morphisms f, g define chain maps $Nf_\bullet, Ng_\bullet : NS_\bullet \to NS'_\bullet$ between the normalised complexes associated with S and S'. Similarly, one can show that every simplicial homotopy $h : f \Rightarrow g$ defines a chain homotopy $Nh_\bullet : Nf_\bullet \Rightarrow Ng_\bullet$ [W, Lemma 8.3.13].

Conversely, every chain homotopy $h_\bullet : f_\bullet \Rightarrow g_\bullet$ between chain maps $f_\bullet, g_\bullet : C_\bullet \to C'_\bullet$ defines a simplicial homotopy $K(h) : K(f) \Rightarrow K(g)$, where $K : \mathrm{Ch}_{\mathcal{A}\geq 0} \to \mathrm{Fun}(\Delta^{+op}, \mathcal{A})$ is the functor from the Dold-Kan correspondence [W, Section 8.4, p273ff]. This allows one to formulate Dold-Kan correspondence as an equivalence of categories between the homotopy category of chain complexes from Remark 3.3.2 and the homotopy category of simplicial objects.

5.3 Simplicial objects from comonoids in monoidal categories

The Dold-Kan correspondence shows that positive chain complexes in an abelian category $\mathcal{A}$ and chain maps between them are obtained from simplicial objects and simplicial morphisms in $\mathcal{A}$ via the standard and normalised chain complex functor. Hence, we can investigate positive chain complexes in $\mathcal{A}$ by studying simplicial objects in $\mathcal{A}$ and vice versa.

In this section, we derive a general method that allows one to *construct* simplicial objects and morphisms from much simpler data. This method works for categories that are equipped with additional structure, namely a categorical *tensor product* that generalises the tensor product over a commutative ring k.

To generalise the tensor product in k-Mod from Definition 1.1.22 to other categories, we have to formulate it in such a way that it involves only objects, morphisms, functors and natural transformations. We already established in Example 1.2.5, 6. that the tensor product of modules over a commutative ring k defines a functor $\otimes : k\text{-Mod} \times k\text{-Mod} \to k\text{-Mod}$. This suggests that one should view a tensor product in a general category $\mathcal{C}$ as a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ that satisfies certain additional conditions.

Of the properties of the tensor product in Lemma 1.1.27, only the second and the fourth can be formulated in general categories $\mathcal{C}$ without additional structures. They state, respectively, that k acts as a unit for the tensor product and that the tensor product over k is associative. The fact that k acts as a unit for the tensor product is encoded in the k-module isomorphisms

$$l_M : k\otimes_k M \to M,\ \lambda\otimes m \mapsto \lambda m \qquad\qquad r_M : M\otimes_k k \to M,\ m\otimes\lambda \mapsto \lambda m$$

from Lemma 1.1.27. If we denote by $k \times \mathrm{id} : k\text{-Mod} \to k\text{-Mod} \times k\text{-Mod}$ the functor that assigns to a k-module M the pair (k, M) and to a k-linear map $f : M \to M'$ the pair (id_k, f), then the k-module isomorphisms $l_M : k\otimes_k M \to M$ and $r_M : M\otimes_k k \to M$ relate the functors $\otimes(k \times \mathrm{id}) : k\text{-Mod} \to k\text{-Mod}$ and $\otimes(\mathrm{id} \times k) : k\text{-Mod} \to k\text{-Mod}$ to the identity functor $\mathrm{id}_{k\text{-Mod}}$.

Similarly, the associativity of the tensor product is encoded in the k-module isomorphisms

$$a_{M,N,P} : (M\otimes_k N)\otimes_k P \to M\otimes_k(N\otimes_k P),\ (m\otimes n)\otimes p \mapsto m\otimes(n\otimes p)$$

from Lemma 1.1.27 that relate the value of the functors $\otimes(\otimes \times \mathrm{id})$ and $\otimes(\mathrm{id} \times \otimes)$ on the triple (M, N, P) of objects in k-Mod.

The k-linear isomorphisms l_M, r_M and $a_{M,N,P}$ commute with k-linear maps. For all k-linear maps $f : M \to M'$, $g : N \to N'$ and $h : P \to P'$ we have

$$a_{M',N',P'} \circ ((f\otimes g)\otimes h) = (f\otimes(g\otimes h)) \circ a_{M,N,P},\ \ l_{M'} \circ (\mathrm{id}_k\otimes f) = f \circ l_M,\ \ r_{M'} \circ (f\otimes \mathrm{id}_k) = f \circ r_M.$$

We can therefore interpret $a_{M,N,P}$, l_M and r_M as component morphisms of natural isomorphisms $a : \otimes(\otimes \times \mathrm{id}) \to \otimes(\mathrm{id} \times \otimes)$, $l : \otimes(k \times \mathrm{id}) \to \mathrm{id}$ and $r : \otimes(\mathrm{id} \times k) \to \mathrm{id}$. Note also that there are identities between composites of the maps l_M, r_M and $a_{M,N,P}$ that allow us to omit tensoring with k and the brackets in iterated tensor products.

The existence of a special object e that generalises the commutative ring k and of natural isomorphisms $a : \otimes(\otimes \times \mathrm{id}) \to \otimes(\mathrm{id} \times \otimes)$, $l : \otimes(e \times \mathrm{id}) \to \mathrm{id}$ and $r : \otimes(\mathrm{id} \times e) \to \mathrm{id}$ can be imposed in any category $\mathcal{C}$ with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. If we also take into account the identities between multiple composites of the natural isomorphisms a, l and r, we obtain the following definition that generalises tensor products over commutative rings.

Definition 5.3.1:

A **monoidal category** is a sextuple $(\mathcal{C}, \otimes, e, a, l, r)$ consisting of

- a category $\mathcal{C}$,
- a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, the **tensor product**,
- an object e in $\mathcal{C}$, the **tensor unit**,
- a natural isomorphism $a : \otimes(\otimes \times \mathrm{id}_\mathcal{C}) \to \otimes(\mathrm{id}_\mathcal{C} \times \otimes)$, the **associator**,
- natural isomorphisms $r : \otimes(\mathrm{id}_\mathcal{C} \times e) \to \mathrm{id}_\mathcal{C}$ and $l : \otimes(e \times \mathrm{id}_\mathcal{C}) \to \mathrm{id}_\mathcal{C}$, the **unit constraints**,

subject to the following two conditions:

1. **pentagon axiom**: for all objects U, V, W, X of $\mathcal{C}$ the following diagram commutes

$$\begin{tikzcd}
((U\otimes V)\otimes W)\otimes X \arrow[r, "a_{U\otimes V,W,X}"] \arrow[d, "a_{U,V,W}\otimes 1_X"'] & (U\otimes V)\otimes(W\otimes X) \arrow[r, "a_{U,V,W\otimes X}"] & U\otimes(V\otimes(W\otimes X)) \\
(U\otimes(V\otimes W))\otimes X \arrow[r, "a_{U,V\otimes W,X}"'] & U\otimes((V\otimes W)\otimes X). \arrow[ur, "1_U\otimes a_{V,W,X}"'] &
\end{tikzcd}$$

2. **triangle axiom**: for all objects V, W of $\mathcal{C}$ the following diagram commutes

$$\begin{tikzcd}
(V\otimes e)\otimes W \arrow[rr, "a_{V,e,W}"] \arrow[dr, "r_V\otimes 1_W"'] & & V\otimes(e\otimes W) \arrow[dl, "1_V\otimes l_W"] \\
& V\otimes W. &
\end{tikzcd}$$

It is called **strict** if a, r and l are identity natural transformations.

Remark 5.3.2:

1. The tensor unit and the unit constraints are determined by $\otimes$ uniquely up to unique isomorphism:

   If there are natural isomorphisms $r' : \otimes(\mathrm{id}_\mathcal{C} \times e') \to \mathrm{id}_\mathcal{C}$ and $l' : \otimes(e' \times \mathrm{id}_\mathcal{C}) \to \mathrm{id}_\mathcal{C}$ for an object e' in $\mathcal{C}$, then there is a unique isomorphism $\phi : e \to e'$ with $r'_X \circ (1_X \otimes \phi) = r_X$ and $l'_X \circ (\phi \otimes 1_X) = l_X$ for all objects X in $\mathcal{C}$. (Exercise).

   However, the functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and associator $a : \otimes(\otimes \times \mathrm{id}) \to \otimes(\mathrm{id} \times a)$ are in general not unique. It is a choice of *structure*, not a property. A category $\mathcal{C}$ may have several different monoidal structures.

2. One can show that if C, D are objects of a monoidal category $(\mathcal{C}, \otimes, e, a, l, r)$ and $f, g : C \to D$ morphisms in $\mathcal{C}$ that are obtained by composing identity morphisms, component morphisms of the associator a and component morphisms of the left and right unit constraints l, r with the composition of morphisms and the tensor product, then f and g are equal. This is MacLane's famous **coherence theorem**. A proof of this statement can be found in [McL2, Chapter VI.2] and [K, Chapter XI.5].

3. The name *monoidal category* is motivated by the fact that for a monoidal category $(\mathcal{C}, \otimes, e, a, l, r)$ the endomorphisms of the tensor unit form a commutative monoid $(\mathrm{End}_\mathcal{C}(e), \circ)$. This is a consequence of the coherence theorem.

Mac Lane's coherence theorem allows one to omit brackets in iterated tensor products and tensor products with the tensor unit. If one computes a composite of certain morphisms in a monoidal category with two different bracketings or with different insertions of units, then by MacLane's coherence theorem the resulting expressions are related by a unique composite of the associators and left and right unit constraints. This means that any bracketings or units omitted or changed in a computation can be reconstructed at the end, and one can compute without brackets. This is already assumed implicitly for tensor products of vector spaces.

Many of the categories from algebra or topology considered so far have the structure of a monoidal category, some of them even several non-equivalent ones.

Example 5.3.3:

1. For any commutative ring k, the category k-Mod is a monoidal category with:
   - the functor $\otimes : k\text{-Mod} \times k\text{-Mod} \to k\text{-Mod}$ that assigns to a pair (M, N) of k-modules the k-module $M \otimes_k N$ and to a pair (f, g) of k-linear maps $f : M \to M'$, $g : N \to N'$ the linear map $f \otimes g : M \otimes_k N \to M' \otimes_k N'$, $m \otimes n \mapsto f(m) \otimes g(n)$,
   - the tensor unit $e = k$,
   - the associator with component isomorphisms $a_{M,N,P} : (M \otimes N) \otimes P \to M \otimes (N \otimes P)$, $(m \otimes n) \otimes p \mapsto m \otimes (n \otimes p)$,
   - the unit constraints with component morphisms $r_M : M \otimes_k k \to M$, $m \otimes \lambda \mapsto \lambda m$ and $l_M : k \otimes_k M \to M$, $\lambda \otimes m \mapsto \lambda m$.

   This includes the category $\mathbb{F}$-Mod $=$ $\text{Vect}_{\mathbb{F}}$ for a field $\mathbb{F}$, $\mathbb{Z}$-Mod $=$ Ab and also the category of modules over the polynomial ring $k[X]$.

2. For any small category $\mathcal{C}$, the category $\text{End}(\mathcal{C})$ of endofunctors $F : \mathcal{C} \to \mathcal{C}$ and natural transformations between them is a *strict* monoidal category with:
   - the functor $\otimes : \text{End}(\mathcal{C}) \times \text{End}(\mathcal{C}) \to \text{End}(\mathcal{C})$ that assigns to a pair (F, G) of functors $F, G : \mathcal{C} \to \mathcal{C}$ the functor $FG : \mathcal{C} \to \mathcal{C}$ and to a pair (μ, η) of natural transformations $\mu : F \to F'$, $\eta : G \to G'$ the natural transformation $\mu \otimes \eta : FG \to F'G'$ with component morphisms $(\mu \otimes \eta)_C = \mu_{G'(C)} \circ F(\eta_C) = F'(\eta_C) \circ \mu_{G(C)} : FG(C) \to F'G'(C)$,
   - the identity functor as the tensor unit: $e = \text{id}_{\mathcal{C}}$.

3. The categories Set and Top are monoidal categories with:
   - the functor $\otimes : \text{Set} \times \text{Set} \to \text{Set}$ that assigns to a pair of sets (X, Y) their cartesian product $X \times Y$ and to a pair (f, g) of maps $f : X \to X'$, $g : Y \to Y'$ the product map $f \times g : X \times Y \to X' \times Y'$,
   - the functor $\otimes : \text{Top} \times \text{Top} \to \text{Top}$ that sends a pair (X, Y) of topological spaces the product space $X \times Y$ and a pair of continuous maps $f : X \to X'$, $g : Y \to Y'$ to the product map $f \times g : X \times Y \to X' \times Y'$,
   - the one-point set $\{p\}$ and the one-point space $\{p\}$ as the tensor unit,
   - the associators with component morphisms $a_{X,Y,Z} : (X \times Y) \times Z \to X \times (Y \times Z)$, $((x, y), z) \mapsto (x, (y, z))$,
   - the unit constraints with component morphisms $r_X : X \times \{p\} \to X$, $(x, p) \mapsto x$ and $l_X : \{p\} \times X \to X$, $(p, x) \mapsto x$.

4. More generally, any category $\mathcal{C}$ with finite (co)products is a monoidal category with:
   - the functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ that sends a pair of objects to their (co)product and a

pair of morphisms to the induced morphism between (co)products,

- the empty (co)product, i. e. the final (initial) object in $\mathcal{C}$ as the tensor unit,
- the associators induced by the universal properties of the (co)products,
- the unit constraints induced by the universal properties of the (co)products.

Such monoidal categories are called **(co)cartesian** monoidal categories. They include:

- any abelian category $\mathcal{A}$,
- the category Set with the disjoint union of sets and the empty set, or with the Cartesian product of sets and the 1-point set,
- the category Top with the sum of topological spaces and the empty space, or with the product of topological spaces and the 1-point space,
- the category Top^1 of pointed topological spaces with wedge sums and the one-point space or with products of pointed spaces and the one-point space,
- the category Grp with the direct product of groups and the trivial group or with the free product of groups and the trivial group.

5. For any commutative ring k, the category $\mathrm{Ch}_{k\text{-Mod}}$ of chain complexes in k-Mod is a monoidal category with the tensor product of chain complexes from Definition 4.6.1

$$(A_\bullet \otimes B_\bullet)_n = \oplus_{j=0}^n A_j \otimes_k B_{n-j},\; d_n^{A\otimes B}(a\otimes b) = d_j^A(a)\otimes b + (-1)^k a \otimes d_{n-j}^B(b) \text{ for } a\in A_j, b\in B_{n-j}$$

and with the tensor product of chain maps given by

$$(f_\bullet \otimes g_\bullet)_n(a\otimes b) = f_j(a)\otimes g_{n-j}(b) \qquad \text{for} \quad a\in A_j, b\in B_{n-j}.$$

The tensor unit is the chain complex $0\to k\to 0$ and the associators and unit constraints are induced by the ones in k via the universal property of direct sums.

6. For any monoidal category $\mathcal{C}$ and small category $\mathcal{B}$, the category $\mathrm{Fun}(\mathcal{B},\mathcal{C})$ is a monoidal category with

- the tensor product of two functors $F, G : \mathcal{B}\to\mathcal{C}$ given by $(F\otimes G)(B) = F(B)\otimes G(B)$ and $(F\otimes G)(f) = F(f)\otimes G(f)$ for all objects $B\in \mathrm{Ob}\,\mathcal{B}$ and morphisms $f : B\to B'$, and the tensor product of natural transformations η,κ given by $(\eta\otimes\kappa)_B = \eta_B\otimes\kappa_B$,
- the constant functor $I:\mathcal{B}\to\mathcal{C}$ with $I(B)=e$ and $I(\beta)=1_e$ for all objects B and morphisms β in $\mathcal{B}$ as the tensor unit,
- the associator and the unit constraints induced by the associators and unit constraints in $\mathcal{C}$.

7. In particular, 4. and 6. imply that for any abelian category $\mathcal{A}$, the category $\mathrm{Fun}(\Delta^{+op},\mathcal{A})$ of simplicial objects and simplicial morphisms in $\mathcal{A}$ is a monoidal category.

When considering functors between monoidal categories and natural transformations between them, it makes sense to ask that these functors and natural transformations respect the monoidal structures - up to isomorphisms. This leads to the concept of a *monoidal functor* or *tensor functor* and of a monoidal *natural transformation*.

Definition 5.3.4: Let $(\mathcal{C}, \otimes_\mathcal{C}, e_\mathcal{C}, a^\mathcal{C}, l^\mathcal{C}, r^\mathcal{C})$ and $(\mathcal{D}, \otimes_\mathcal{D}, e_\mathcal{D}, a^\mathcal{D}, l^\mathcal{D}, r^\mathcal{D})$ be monoidal categories.

1. A **monoidal functor** or **tensor functor** from $\mathcal{C}$ to $\mathcal{D}$ is a triple $(F, \phi^e, \phi^\otimes)$ of
   - a functor $F : \mathcal{C} \to \mathcal{D}$,
   - an isomorphism $\phi^e : e_\mathcal{D} \to F(e_\mathcal{C})$ in $\mathcal{D}$,
   - a natural isomorphism $\phi^\otimes : \otimes_\mathcal{D}(F \times F) \to F\otimes_\mathcal{C}$,

   that satisfy the following axioms:

   (a) **compatibility with the associativity constraint**:
   for all objects U, V, W of $\mathcal{C}$ the following diagram commutes

$$\begin{tikzcd}
(F(U)\otimes F(V))\otimes F(W) \arrow[r, "a^\mathcal{D}_{F(U),F(V),F(W)}"] \arrow[d, "\phi^\otimes_{U,V}\otimes 1_{F(W)}"'] & F(U)\otimes(F(V)\otimes F(W)) \arrow[d, "1_U\otimes\phi^\otimes_{V,W}"] \\
F(U\otimes V)\otimes F(W) \arrow[d, "\phi^\otimes_{U\otimes V,W}"'] & F(U)\otimes F(V\otimes W) \arrow[d, "\phi^\otimes_{U,V\otimes W}"] \\
F((U\otimes V)\otimes W) \arrow[r, "F(a^\mathcal{C}_{U,V,W})"'] & F(U\otimes(V\otimes W)).
\end{tikzcd}$$

   (b) **compatibility with the unit constraints**:
   for all objects V of $\mathcal{C}$ the following diagrams commute

$$\begin{tikzcd}
e_\mathcal{D}\otimes F(V) \arrow[r, "\phi^e\otimes 1_{F(V)}"] \arrow[d, "l^\mathcal{D}_{F(V)}"'] & F(e_\mathcal{C})\otimes F(V) \arrow[d, "\phi^\otimes_{e_\mathcal{C},V}"] \\
F(V) & F(e_\mathcal{C}\otimes V) \arrow[l, "F(l^\mathcal{C}_V)"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F(V)\otimes e_\mathcal{D} \arrow[r, "1_{F(V)}\otimes\phi^e"] \arrow[d, "r^\mathcal{D}_{F(V)}"'] & F(V)\otimes F(e_\mathcal{C}) \arrow[d, "\phi^\otimes_{V,e_\mathcal{C}}"] \\
F(V) & F(V\otimes e_\mathcal{C}). \arrow[l, "F(r^\mathcal{C}_V)"]
\end{tikzcd}$$

   A monoidal functor $(F, \phi^e, \phi^\otimes)$ is called **strict** if $\phi^e = 1_{e_\mathcal{D}}$ and $\phi^\otimes = \mathrm{id}_{F\otimes_\mathcal{C}}$ is the identity natural transformation. It is called a **monoidal equivalence** if $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories.

2. Let $(F, \phi^e, \phi^\otimes), (F', \phi'^e, \phi'^\otimes) : \mathcal{C} \to \mathcal{D}$ be monoidal functors. A **monoidal natural transformation** is a natural transformation $\eta : F \to F'$ for which the following diagrams commute:

   (a) **compatibility with ϕ^e and ϕ'^e**:

$$\begin{tikzcd}
F(e_\mathcal{C}) \arrow[rr, "\eta_{e_\mathcal{C}}"] & & F'(e_\mathcal{C}) \\
& e_\mathcal{D}. \arrow[ul, "\phi^e"] \arrow[ur, "\phi'^e"'] &
\end{tikzcd}$$

   (b) **compatibility with $\phi^\otimes$ and $\phi'^\otimes$**: for all objects C, C' in $\mathcal{C}$

$$\begin{tikzcd}
F(C)\otimes F(C') \arrow[r, "\eta_C\otimes\eta_{C'}"] \arrow[d, "\phi^\otimes_{C,C'}"'] & F'(C)\otimes F'(C') \arrow[d, "\phi'^\otimes_{C,C'}"] \\
F(C\otimes C') \arrow[r, "\eta_{C\otimes C'}"'] & F'(C\otimes C').
\end{tikzcd}$$

   A monoidal natural transformation $\eta : F \to F'$ is called a **monoidal isomorphism** if $\eta_C : F(C) \to F'(C)$ is an isomorphism for all objects C in $\mathcal{C}$.

Remark 5.3.5: There are also two weaker concepts of monoidal functors:

1. The definition of a **lax monoidal functor** is obtained from Definition 5.3.4 by replacing the *isomorphism* $\phi^e : e_{\mathcal{D}} \to F(e_{\mathcal{C}})$ by a *morphism* $\phi^e : e_{\mathcal{D}} \to F(e_{\mathcal{C}})$, and the natural *isomorphism* $\phi^\otimes : \otimes_{\mathcal{D}}(F \times F) \to F\otimes_{\mathcal{C}}$ by a natural *transformation* $\phi^\otimes : \otimes_{\mathcal{D}}(F \times F) \to F\otimes_{\mathcal{C}}$, while all other conditions are unchanged.

2. The definition of an **op-lax monoidal functor** replaces the *isomorphism* $\phi^e : e_{\mathcal{D}} \to F(e_{\mathcal{C}})$ by a *morphism* $\phi^e : F(e_{\mathcal{C}}) \to e_{\mathcal{D}}$ and the natural *isomorphism* $\phi^\otimes : \otimes_{\mathcal{D}}(F \times F) \to F\otimes_{\mathcal{C}}$ by a natural *transformation* $\phi^\otimes : F\otimes_{\mathcal{C}} \to \otimes_{\mathcal{D}}(F \times F)$ and reverses all arrows labelled ϕ^e or $\phi^\otimes$ in Definition 5.3.4.

Monoidal natural transformations for lax and op-lax monoidal functors are defined analogously.

Example 5.3.6:

1. For any ring isomorphism $\phi : k \to l$, the functor $F_\phi : l\text{-Mod} \to k\text{-Mod}$ that sends an l-module $(M, \rhd_l)$ to the k-Module $(M, \rhd_k)$ with $\lambda \rhd_k m := \phi(\lambda) \rhd_l m$ and every l-linear map to itself is a monoidal equivalence. Its coherence data is given by $\phi^e = \phi : k \to l$ and $\phi^\otimes_{(M,N)} : F_\phi(M)\otimes_k F_\phi(N) \to F_\phi(M\otimes_l N)$, $m\otimes_k n \mapsto m\otimes_l n$. If $\phi : k \to l$ is only a ring homomorphism, this functor becomes lax monoidal.

2. The forgetful functor $F : \text{Top} \to \text{Set}$ is a strict monoidal functor, when Top and Set are equipped with the monoidal structures defined by their products or coproducts.

3. The functor $F : \text{Set} \to k\text{-Mod}$ that assigns to a set X the free k-module $F(X) = \langle X\rangle_k$ and to a map $f : X \to Y$ the unique k-linear map $F(f) : \langle X\rangle_k \to \langle Y\rangle_k$ with $F(f) \circ \iota_X = \iota_Y \circ f$ is a monoidal functor, when Set is equipped with the product monoidal structure. Its coherence data is given by the maps $\phi^e : k \to \langle p\rangle_k$, $\lambda \mapsto \lambda p$ and $\phi^\otimes_{X,Y} : \langle X\rangle_k\otimes_k\langle Y\rangle_k \to \langle X \times Y\rangle_k$, $x\otimes y \mapsto (x, y)$.

4. Let $\mathcal{D}, \mathcal{E}$ be small categories, $\mathcal{C}$ a monoidal category and equip $\text{Fun}(\mathcal{D}, \mathcal{C})$ and $\text{Fun}(\mathcal{E}, \mathcal{C})$ with the monoidal structures from Example 5.3.3, 6.

   • Pre-composition with a functor $F : \mathcal{E} \to \mathcal{D}$ defines a monoidal functor $F^* : \text{Fun}(\mathcal{D}, \mathcal{C}) \to \text{Fun}(\mathcal{E}, \mathcal{C})$ that sends a functor $G : \mathcal{D} \to \mathcal{C}$ to GF and a natural transformation $\eta : G \to G'$ to $\eta F : GF \to G'F$.

   • Pre-composition with a natural transformation $\eta : F \to F'$ defines a monoidal natural transformation $\eta^* : F^* \to F'^*$ with component morphisms $\eta^*_G = G\eta : GF \to GF'$ for all functors $G : \mathcal{D} \to \mathcal{C}$ (Exercise).

The reason why monoidal categories are relevant in homological algebra is that the *augmented* simplex category is a monoidal category with a particularly simple structure. This does not hold for the simplex category, and this is the reason why the *augmented simplex category* is preferable from the algebraic viewpoint.

Example 5.3.7:
The augmented simplex category Δ from Definition 5.1.1 is a strict monoidal category with:

- the functor $\otimes : \Delta \times \Delta \to \Delta$ that assigns to a pair $([m], [n])$ of ordinals the ordinal $[m]\otimes[n] = [m+n]$ and to a pair (f, g) of monotonic maps $f : [m] \to [m']$ and $g : [n] \to [n']$

the monotonic map $f\otimes g : [m+n] \to [m'+n']$ given by concatenation of f and g

$$(f\otimes g)(i) = \begin{cases} f(i) & 0 \leq i < m \\ m' + g(i-m) & m \leq i < n+m, \end{cases}$$

- the ordinal $[0] = \emptyset$ as the tensor unit.

In Proposition 5.1.2 and Remark 5.1.3 we presented the simplex category *as a category.* We specified certain generating morphisms, the face maps and degeneracies, and relations between them. This allowed us to describe every morphism uniquely as an ordered product of the generating morphisms.

A similar procedure can be applied to monoidal categories, but in this case, one generates via *composition and tensor products.* One specifies a collection of generating objects such that every object is isomorphic to a tensor product of the generating objects and a collection of generating morphisms and certain relations between them. One requires that any morphism can be expressed in terms of the generating morphisms via composition and tensor products. All relations between morphisms are obtained from the generating relations and the coherence data of the tensor product via tensor products and composition. For details on this procedure see for instance [K, XII.1].

It turns out that the presentation of the augmented simplex category *as a monoidal category* is much simpler than its presentation *as a category* from Proposition 5.1.2.

Lemma 5.3.8: The augmented simplex category Δ is **presented as a strict monoidal category** by the object $[1]$ and the morphisms $\sigma_1^0 : [2] \to [1]$, $\delta_0^0 : [0] \to [1]$, subject to the **relations**

$$\sigma_1^0 \circ (\sigma_1^0 \otimes 1_{[1]}) = \sigma_1^0 \circ (1_{[1]} \otimes \sigma_1^0) \qquad \sigma_1^0 \circ (1_{[1]} \otimes \delta_0^0) = 1_{[1]} = \sigma_1^0 \circ (\delta_0^0 \otimes 1_{[1]}). \tag{72}$$

In other words:

1. Every object $[n]$ is a multiple tensor product of the object $[1]$ with itself.
2. Every morphism in Δ is given as a multiple composite and tensor product of the morphisms σ_1^0, δ_0^0 and identity morphisms.
3. All relations between morphisms in Δ arise either from the properties of a monoidal category or from (72) via the tensor product and the composition of morphisms.

Proof:
By definition of the tensor product in Δ we have $[m]\otimes[n] = [m+n]$ for all $m, n \in \mathbb{N}_0$, and this implies inductively $[n] = [n-1]\otimes[1] = [n-2]\otimes[1]\otimes[1] = ... = [1]^{\otimes n}$ for all $n \in \mathbb{N}$. For $n = 0$ we have the tensor unit $[0] = [1]^{\otimes 0}$. To show that every morphism in Δ is a composite of the morphisms σ_1^0 and δ_0^0, it is sufficient to prove this for the morphisms $\delta_n^i : [n] \to [n+1]$ and $\sigma_n^i : [n+1] \to [n]$, since these morphisms generate Δ as a category by Proposition 5.1.2. These morphisms are given by

$$\delta_n^i = 1_{[i]} \otimes \delta_0^0 \otimes 1_{[n-i]} : [n] \to [n+1], \qquad \sigma_n^i = 1_{[i]} \otimes \sigma_1^0 \otimes 1_{[n-i-1]} : [n+1] \to [n]. \tag{73}$$

The defining relations between the morphisms δ_n^i and σ_m^j in (59) follow from the definition of the tensor product in Δ and from the relations (72). For instance, we have

$$\begin{aligned}\sigma_{n-1}^i \circ \sigma_n^i &= (1_{[i]}\otimes\sigma_1^0\otimes 1_{[n-i-2]}) \circ (1_{[i]}\otimes\sigma_1^0\otimes 1_{[n-i-1]})\\ &= 1_{[i]}\otimes(\sigma_1^0 \circ (\sigma_1^0\otimes 1_{[1]}))\otimes 1_{[n-i-2]} = 1_{[i]}\otimes(\sigma_1^0 \circ (1_{[1]}\otimes\sigma_1^0))\otimes 1_{[n-i-2]}\\ &= (1_{[i]}\otimes\sigma_1^0\otimes 1_{[n-i-2]}) \circ (1_{[i+1]}\otimes\sigma_1^0\otimes 1_{[n-i-2]})\\ &= \sigma_{n-1}^i \circ \sigma_n^{i+1},\end{aligned}$$

and the proofs of the other relations in (59) are similar (Exercise). As the relations (59) are the defining relations of the augmented simplex category Δ (see Remark 5.1.3), the claim follows. □

Lemma 5.3.8 gives a more efficient description of the augmented simplex category Δ with fewer generators and relations than the description in terms of face maps and degeneracies in Proposition 5.1.2. It involves only two morphisms and the two relations in (72). The first relation resembles an associativity condition and the second resembles a unit law for a monoid or an algebra. This can be made precise by generalising the concept of an algebra over a commutative ring to a monoidal category.

Recall from Definition 2.2.1 that an algebra over commutative ring k is a k-module A together with an associative k-bilinear map $\cdot : A \times A \to A$ and a unit $1_A \in A$ with $1_A \cdot a = a \cdot 1_A = a$ for all $a \in A$. By the universal property of the tensor product over k, we can also view the multiplication as a k-linear map $\mu : A\otimes_k A \to A$, $a\otimes a' \mapsto a \cdot a'$. The unit element of A can be encoded in a k-linear map $\eta : k \to A$, $\lambda \mapsto \lambda 1_A$. The conditions that μ is associative and that 1_A is a unit for A then read

$$\mu \circ (\mu\otimes\mathrm{id}) = \mu \circ (\mathrm{id}\otimes\mu) \circ a_{A,A,A} \qquad \mu \circ (\mathrm{id}\otimes\eta) \circ r_A^{-1} = \mu \circ (\eta\otimes\mathrm{id}) \circ l_A^{-1} = \mathrm{id}_A,$$

where $a_{A,A,A} : (A\otimes_k A)\otimes_k A \to A\otimes_k(A\otimes_k A)$ is the associator, $r_A : A\otimes_k k \to A$, $l_A : k\otimes_k A \to A$ are its left and right unit constraints. This definition generalises to any monoidal category. Moreover, one obtains a dual concept by reversing the directions of the multiplication and unit morphism.

Definition 5.3.9: Let $(\mathcal{C}, \otimes, e, a, l, r)$ be a monoidal category.

1. A **monoid** or **algebra object** in $\mathcal{C}$ is a triple (A, μ, η) of an object A in $\mathcal{C}$ and morphisms $\mu : A\otimes A \to A$, $\eta : e \to A$, the **multiplication** and **unit morphism**, such that the following diagrams commute

$$\begin{tikzcd} (A\otimes A)\otimes A \arrow[rr, "a_{A,A,A}", "\cong"'] \arrow[d, "\mu\otimes 1_A"'] & & A\otimes(A\otimes A) \arrow[d, "1_A\otimes\mu"] \\ A\otimes A \arrow[r, "\mu"'] & A & A\otimes A \arrow[l, "\mu"] \end{tikzcd} \qquad \begin{tikzcd} A\otimes e \arrow[r, "1_A\otimes\eta"] \arrow[dr, "r_A"', "\cong"] & A\otimes A \arrow[d, "\mu"] & e\otimes A \arrow[l, "\eta\otimes 1_A"'] \arrow[dl, "l_A", "\cong"'] \\ & A. & \end{tikzcd}$$

2. A **comonoid** or **coalgebra object** in $\mathcal{C}$ is a triple (C, Δ, ϵ) of an object C in $\mathcal{C}$ and morphisms $\Delta : C \to C\otimes C$, $\epsilon : C \to e$, the **comultiplication** and **counit morphism**, such that the following diagrams commute

$$\begin{tikzcd} (C\otimes C)\otimes C \arrow[rr, "a_{C,C,C}", "\cong"'] & & C\otimes(C\otimes C) \\ C\otimes C \arrow[u, "\Delta\otimes 1_C"] & C \arrow[l, "\Delta"] \arrow[r, "\Delta"'] & C\otimes C \arrow[u, "1_C\otimes\Delta"'] \end{tikzcd} \qquad \begin{tikzcd} C\otimes e & C\otimes C \arrow[l, "1_C\otimes\epsilon"'] \arrow[r, "\epsilon\otimes 1_C"] & e\otimes C \\ & C. \arrow[ul, "r_C^{-1}", "\cong"'] \arrow[u, "\Delta"'] \arrow[ur, "l_C^{-1}"', "\cong"] & \end{tikzcd}$$

Example 5.3.10:

1. A monoid in the monoidal category $(\mathrm{Set}, \times, \{p\})$ is simply a monoid $(M, \cdot, e)$ in the usual sense (cf. Example 1.2.3, 2.). The multiplication morphism is the monoid multiplication $\mu : M \times M \to M$, $(m, m') \mapsto m \cdot m'$ and the unit morphism is given by $\eta : \{p\} \to M$, $p \mapsto e$.

2. A monoid in the monoidal category $(k\text{-Mod}, \otimes, k)$ for a commutative ring k is precisely a k-algebra. In particular, monoids in $(\mathrm{Vect}_{\mathbb{F}}, \otimes, \mathbb{F})$ are algebras over $\mathbb{F}$.

3. Every set X is a comonoid in $(\mathrm{Set}, \times, \{p\})$ and every topological space X is a comonoid in $(\mathrm{Top}, \times, \{p\})$ with the diagonal map $\Delta : X \to X \times X$, $x \mapsto (x, x)$ as the comultiplication and the counit $\epsilon : X \to \{p\}$, $x \mapsto p$.

4. More generally, every object C of a cartesian monoidal category $(\mathcal{C}, \times, t)$ is a comonoid in $\mathcal{C}$ with $\Delta : C \to C \times C$ induced by the universal property of the product via $\pi_i \circ \Delta = 1_C$ for $i = 1, 2$ and the terminal morphism $\epsilon : C \to t$.

   Dually, every object A of a cocartesian monoidal category $(\mathcal{C}, \amalg, i)$ is a monoid in $\mathcal{C}$ with $\mu : A \amalg A \to A$ induced by the universal property of the coproduct and the initial morphism $\eta : i \to A$.

5. We consider the strict monoidal category $\mathrm{End}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}, \mathcal{C})$ for a small category $\mathcal{C}$ from Example 5.3.3, 2. A monoid in $\mathrm{End}(\mathcal{C})$ is called a **monad** in $\mathcal{C}$. It is a triple (T, μ, η) of a functor $T : \mathcal{C} \to \mathcal{C}$ and natural transformations $\mu : T^2 \to T$ and $\eta : \mathrm{id}_{\mathcal{C}} \to T$ such that the following diagrams commute

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "\eta T"] \arrow[dr, "\mathrm{id}_T"'] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, "\mathrm{id}_T"] \\ & T. & \end{tikzcd} \tag{74}$$

   A comonoid in $\mathrm{End}(\mathcal{C})$ is called a **comonad** in $\mathcal{C}$. It is a triple (C, Δ, ϵ) of a functor $C : \mathcal{C} \to \mathcal{C}$ and natural transformations $\Delta : C \to C^2$ and $\epsilon : C \to \mathrm{id}_{\mathcal{C}}$ such that the following diagrams commute

$$\begin{tikzcd} C^3 & C^2 \arrow[l, "C\Delta"'] \\ C^2 \arrow[u, "\Delta C"] & C \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \end{tikzcd} \qquad \begin{tikzcd} C \arrow[dr, "\mathrm{id}_C"'] & C^2 \arrow[l, "\epsilon C"'] \arrow[r, "C\epsilon"] & C \arrow[dl, "\mathrm{id}_C"] \\ & C. \arrow[u, "\Delta"] & \end{tikzcd} \tag{75}$$

6. The triple $([1], \sigma_1^0, \delta_0^0)$ of the ordinal $[1] = \{0\}$, the map $\sigma_1^0 : [2] \to [1]$ and the empty map $\delta_0^0 : \emptyset \to [1]$ from Lemma 5.3.8 is a monoid in the strict monoidal category $(\Delta, \otimes, \emptyset)$. This follows from the definition of the tensor product in Δ in Example 5.3.7 and the relations (73) in the augmented simplex category Δ.

The monoid $([1], \sigma_1^0, \delta_0^0)$ in the augmented simplex category Δ plays a special role in homology. By Lemma 5.3.8 the augmented simplex category is generated as a monoidal category by the object $[1]$ and the morphisms $\sigma_1^0 : [2] \to [1]$, $x \mapsto 0$ and the empty map $\delta_0^0 = \emptyset : [0] \to [1]$. Its defining relations (72) are precisely the defining relations for monoid in a monoidal category. This allows one to characterise monoidal functors from Δ to a strict monoidal category $\mathcal{C}$ by monoids in $\mathcal{C}$ and monoidal functors from Δ^{op} to $\mathcal{C}$ by comonoids. Every comonoid in $\mathcal{C}$ defines a simplicial object in $\mathcal{C}$ and every monoid in $\mathcal{C}$ a cosimplicial object in $\mathcal{C}$. This is sometimes called the *universality* of the augmented simplex category.

Proposition 5.3.11: **(Universality of Δ)**

Let $(\mathcal{C}, \otimes, e)$ be a strict monoidal category.

1. For every monoid (A, μ, η) in $\mathcal{C}$, there is a unique strict monoidal functor $F : \Delta \to \mathcal{C}$ with $F([1]) = A$, $F(\sigma_1^0) = \mu$ and $F(\delta_0^0) = \eta$.
2. For every comonoid (C, Δ, ϵ) in $\mathcal{C}$, there is a unique strict monoidal functor $F : \Delta^{op} \to \mathcal{C}$ with $F([1]) = C$, $F(\sigma_1^0) = \Delta$ and $F(\delta_0^0) = \epsilon$.

Proof:
We prove 1., since 2. is obtained from 1. by reversing the direction of morphisms. By Lemma 5.3.8 the augmented simplex category Δ is generated as a strict monoidal category by the object [1] and the morphisms $\sigma_1^0 : [2] \to [1]$ and $\delta_0^0 : [0] \to [1]$ subject to the associativity and the unit relations (72).

If $F : \Delta \to \mathcal{C}$ is a strict monoidal functor, then F is determined on the objects by $F([1])$ since it satisfies $F([0]) = e$ and $F([n]) = F([1]^{\otimes n}) = F([1])^{\otimes n}$ for all $n \in \mathbb{N}_0$. Similarly, F is determined on the morphisms of Δ by $F(\sigma_1^0) : F([1]) \otimes F([1]) \to F([1])$ and $F(\delta_0^0) : e \to F([1])$, since every morphism in Δ is a composite of σ_0^0 and δ_0^0 via the composition $\circ$ and the tensor product $\otimes$.

The generating relations (72) of Δ are equivalent to the statement that $(A, \mu, \eta) = (F([1], F(\sigma_1^0), F(\delta_0^0))$ is a monoid in $\mathcal{C}$. Given a monoid (A, μ, η) in $\mathcal{C}$, we can thus define a strict monoidal functor $F : \Delta \to \mathcal{C}$ by setting $F([1]) = A$, $F(\sigma_1^0) = \mu$ and $F(\delta_0^0) = \eta$. □

Corollary 5.3.12: Let $(\mathcal{C}, \otimes, e)$ be a strict monoidal category. Then:

1. Every monoid (A, μ, η) in $\mathcal{C}$ defines an augmented cosimplicial object $F : \Delta \to \mathcal{C}$ in $\mathcal{C}$ with $F([n]) = A^{\otimes n}$ and
$$F(\sigma_n^i) = 1_A^{\otimes i} \otimes \mu \otimes 1_A^{\otimes(n-i-1)} : A^{\otimes(n+1)} \to A^{\otimes n} \quad F(\delta_i^n) = 1_A^{\otimes i} \otimes \eta \otimes 1_A^{\otimes(n-i)} : A^{\otimes n} \to A^{\otimes(n+1)}.$$
2. Every comonoid (C, Δ, ϵ) in $\mathcal{C}$ defines an augmented simplicial object $F : \Delta^{op} \to \mathcal{C}$ in $\mathcal{C}$ with $F([n]) = C^{\otimes n}$ and
$$F(\sigma_n^i) = 1_C^{\otimes i} \otimes \Delta \otimes 1_C^{\otimes(n-i-1)} : C^{\otimes n} \to C^{\otimes(n+1)} \quad F(\delta_n^i) = 1_C^{\otimes i} \otimes \epsilon \otimes 1_C^{\otimes(n-i)} : C^{\otimes(n+1)} \to C^{\otimes n}.$$

In fact, the restriction to strict tensor categories in Proposition 5.3.11 and Corollary 5.3.12 is not necessary. It just simplifies the statement of the result. Mac Lane's coherence theorem (see Remark 5.3.2, 2) allows one to extend this result to non-strict monoidal categories $\mathcal{C}$ by replacing the strict monoidal functor in Proposition 5.3.11 by a monoidal functor and including associators and left and right unit constraints into the formulas. The resulting functor is then unique up to natural isomorphisms constructed from associators and unit constraints in $\mathcal{C}$. We will make use of this in the following and also consider monoids and comonoids in non-strict monoidal categories and the associated augmented cosimplicial and simplicial objects.

Example 5.3.13: Every group G is a comonoid in $(\mathrm{Set}, \times, \{x\})$ with $\Delta : G \to G \times G$, $g \mapsto (g, g)$ and $\epsilon : G \to \{x\}$, $g \mapsto x$. The associated augmented simplicial object $F : \Delta^{op} \to \mathrm{Set}$ is given by $F([n]) = G^{\times n}$ for all $n \in \mathbb{N}_0$ and

$$F(\sigma_n^i) : G^{\times n} \to G^{\times(n+1)}, \quad (g_1, ..., g_n) \mapsto (g_1, ..., g_i, g_{i+1}, g_{i+1}, g_{i+2} ..., g_n)$$
$$F(\delta_n^i) : G^{\times(n+1)} \to G^{\times n}, \quad (g_1, ..., g_{n+1}) \mapsto (g_1, ...g_{i-1}, g_{i+1}, ..., g_{n+1}).$$

By comparing this example to the chain complexes in Examples 3.2.6, 3.3.5 and 4.1.2, we see that it is related to the bar resolution of group cohomology. Note, however, that this construction for group cohomology cannot be generalised to algebras over a commutative ring k since the diagonal map $\Delta : A \to A \otimes_k A$, $a \mapsto a \otimes a$ and the map $\epsilon : A \to k$, $a \mapsto 1$ are *not k-linear*.

Note also that this is not sufficiently general for our purposes. We do not want to associate augmented simplicial objects to specific groups or algebras, Lie algebras or topological spaces but to associate them *systematically* to *all* groups, algebras, Lie algebras and topological spaces at once. This suggests that the relevant comonoids should be given by *functors*. For this reason, we consider comonoids in a category $\mathrm{End}(\mathcal{D}) = \mathrm{Fun}(\mathcal{D}, \mathcal{D})$ from Example 5.3.10, 5.

Example 5.3.14: Let (C, Δ, ϵ) be a comonad in a small category $\mathcal{D}$.

• By Proposition 5.3.11 and Corollary 5.3.12 the comonad (C, Δ, ϵ) determines a unique augmented simplicial object $S_C : \Delta^{op} \to \mathrm{End}(\mathcal{D})$ given by

$$S_C([n]) = C^n : \mathcal{D} \to \mathcal{D}, \quad S_C(\sigma_n^i) = C^i \Delta C^{n-i-1} : C^n \to C^{n+1}, \quad S_C(\delta_n^i) = C^i \epsilon C^{n-i} : C^{n+1} \to C^n.$$

• This defines a functor $F_C : \mathcal{D} \to \mathrm{Fun}(\Delta^{op}, \mathcal{D})$ that assigns

- to an object D in $\mathcal{D}$ the functor $F_C(D) : \Delta^{op} \to \mathcal{D}$ with $F_C(D)([n]) = C^n(D)$ and $F_C(D)(\alpha) = S_C(\alpha)_D : C^n(D) \to C^m(D)$ for all monotonic maps $\alpha : [m] \to [n]$,
- to a morphism $f : D \to D'$ in $\mathcal{D}$ the natural transformation $F_C(f) : F_C(D) \to F_C(D')$ with component morphisms $F_C(f)_{[n]} = C^n(f) : C^n(D) \to C^n(D')$.

• Post-composition with a functor $H : \mathcal{D} \to \mathcal{A}$ into an abelian category $\mathcal{A}$ defines a functor $HF_C : \mathcal{D} \to \mathrm{Fun}(\Delta^{op}, \mathcal{A})$ that assigns

- to an object D the functor $HF_C(D) : \Delta^{op} \to \mathcal{A}$ with $F_C(D)([n]) = HC^n(D)$ and $HF_C(D)(\alpha) = HS_C(\alpha)_D : HC^n(D) \to HC^m(D)$ for all monotonic maps $\alpha : [m] \to [n]$,
- to a morphism $f : D \to D'$ in $\mathcal{D}$ the natural transformation $HF_C(f) : HF_C(D) \to HF_C(D')$ with component morphisms $HF_C(f)_{[n]} = HC^n(f) : HC^n(D) \to HC^n(D')$.

• Restricting to the full subcategory $\Delta^{+op} \subset \Delta^{op}$ yields a functor $HF_C : \mathcal{D} \to \mathrm{Fun}(\Delta^{+op}, \mathcal{A})$ from $\mathcal{D}$ into the category of simplicial objects and morphisms in the abelian category $\mathcal{A}$.

• Applying the standard chain complex functor $\bullet : \mathrm{Fun}(\Delta^{op+}, \mathcal{A}) \to \mathrm{Ch}_{\mathcal{A}\geq 0}$ from Proposition 5.2.2, we obtain a functor $G : \mathcal{D} \to \mathrm{Ch}_{\mathcal{A}\geq 0}$ that assigns

- to an object D in $\mathcal{D}$ the standard chain complex $HF_C(D)_\bullet$ in $\mathcal{A}$,
- to a morphism $f : D \to D'$ the chain map $HF_C(f)_\bullet : HF_C(D)_\bullet \to HF_C(D')_\bullet$.

• Applying the normalised chain complex functor $N : \mathrm{Fun}(\Delta^{op+}, \mathcal{A}) \to \mathrm{Ch}_{\mathcal{A}\geq 0}$ from Proposition 5.2.2, we obtain a functor $G' : \mathcal{D} \to \mathrm{Ch}_{\mathcal{A}\geq 0}$ that assigns

- to an object D in $\mathcal{D}$ the normalised chain complex $NHF_C(D)_\bullet$ in $\mathcal{A}$,
- to a morphism $f : D \to D'$ in $\mathcal{C}$ the chain map $NHF_C(f)_\bullet : NHF_C(D)_\bullet \to NHF_C(D')_\bullet$.

Example 5.3.14 gives a general formalism that allows one to construct chain complexes in an abelian category $\mathcal{A}$ from a comonad in a small category $\mathcal{C}$ and a functor $H : \mathcal{C} \to \mathcal{A}$. If $\mathcal{C}$ is already abelian, we can choose $H = \mathrm{id}_{\mathcal{C}}$ and apply the standard chain complex functor directly to the functor F_C in Example 5.3.14. We can also drop the requirement that $\mathcal{C}$ is small, since we can always restrict attention to a small full subcategory of $\mathcal{C}$.

We now focus on the case, where $\mathcal{C}$ is abelian and investigate the chain complexes defined by comonads in $\mathcal{C}$. It turns out that under a mild additional assumption, the resulting chain complexes generalise the pattern observed in Example 5.3.13. Via the construction in Example 5.3.14, comonads in an abelian category $\mathcal{A}$ define *resolutions* of objects in $\mathcal{A}$.

Definition 5.3.15: Let (C, Δ, ϵ) be a comonad in an abelian category $\mathcal{A}$. An object A in $\mathcal{A}$ is called C-**projective** if there is a morphism $f : A \to C(A)$ with $\epsilon_A \circ f = 1_A$.

Proposition 5.3.16: Let (C, Δ, ϵ) be a comonad in an abelian category $\mathcal{A}$. Then for any C-projective object A in $\mathcal{A}$ the chain complex

$$C(A)_\bullet = \ldots \xrightarrow{d_2} C^2(A) \xrightarrow{d_1} C(A) \xrightarrow{\epsilon_A} A \to 0 \quad \text{with} \quad d_n = \Sigma_{i=0}^n (-1)^i C^i \epsilon C_A^{n-i} : C^{n+1}(A) \to C^n(A)$$

is exact. It is called the **canonical resolution** of A defined by C.

Proof:
Let $f : A \to C(A)$ a morphism with $\epsilon_A \circ f = 1_A$. We consider for $-1 \leq n \in \mathbb{Z}$ the morphisms

$$h_n = (-1)^{n+1} C^{n+1}(f) : C^{n+1}(A) \to C^{n+2}(A)$$

and show that they define a chain homotopy $h_\bullet : 1_{C(A)_\bullet} \Rightarrow 0_{C(A)_\bullet}$. By Proposition 3.3.4 we then have $H_n(C(A)_\bullet) = 0$ for all $n \in \mathbb{N}_0$.

The boundary operator of $C(A)_\bullet$ is given by $d_n = \Sigma_{i=0}^n (-1)^i d_n^i : C^{n+1}(A) \to C^n(A)$, where $d_n^i = C^i \epsilon C_A^{n-i} : C^{n+1}(A) \to C^n(A)$ are the component morphisms of the natural transformation $C^i \epsilon C^{n-i} : C^{n+1} \to C^n$. This implies for $0 \leq i \leq n$

$$\begin{aligned} d_{n+1}^{n+1} \circ h_n &= (-1)^{n+1} C^{n+1}(\epsilon_A) \circ C^{n+1}(f) = (-1)^{n+1} C^{n+1}(\epsilon_A \circ f) = (-1)^{n+1} 1_{C^{n+1}(A)} \\ d_{n+1}^i \circ h_n &= (-1)^{n+1} C^i(\epsilon_{C^{n+1-i}(A)}) \circ C^{n+1}(f) \overset{\text{nat}}{=} (-1)^{n+1} C^n(f) \circ C^i(\epsilon_{C^{n-i}(A)}) = -h_{n-1} \circ d_n^i. \end{aligned}$$

Combining these expressions and taking an alternating sum over the morphisms d_n^i we obtain

$$\begin{aligned} & d_{n+1} \circ h_n + h_{n-1} \circ d_n = \Sigma_{i=0}^{n+1} (-1)^i d_{n+1}^i \circ h_n + \Sigma_{i=0}^n (-1)^i h_{n-1} \circ d_n^i \\ & = 1_{C^{n+1}(A)} + \Sigma_{i=0}^n (-1)^{i+1} h_{n-1} \circ d_n^i + \Sigma_{i=0}^n (-1)^i h_{n-1} \circ d_n^i = 1_{C^{n+1}(A)} \qquad \text{for } n \in \mathbb{N}_0. \end{aligned}$$

This shows that $h_\bullet : 1_{C(A)_\bullet} \Rightarrow 0_{C(A)_\bullet}$ is a chain homotopy and $H_n(C(A)_\bullet) = 0$ for all $n \in \mathbb{N}_0$. □

Proposition 5.3.16 gives a systematic procedure to construct resolutions all objects in an abelian category $\mathcal{A}$ from a comonad in $\mathcal{A}$. Moreover, these resolutions are *canonical* and not based on the specific properties of certain objects in $\mathcal{A}$. We will see in the next section that most of the standard resolutions considered so far, namely the Hochschild resolution, the bar resolution of group cohomology and free resolutions of R-modules can be obtained in this way.

5.4 Resolutions from adjoint functors

The results of the last subsection are a strong motivation to investigate monads and comonads and to construct them systematically. It turns out that monads and comonads are rather common, since they arise from pairs of adjoint functors. Pairs of adjoint functors occur in many situations and are directly related to universal properties of certain standard constructions, as discussed in Section 1.2. To see that a pair of adjoint functors defines a monad and a comonad, we work with the characterisation of adjoint functors in Proposition 1.2.19.

Proposition 5.4.1: Let $F:\mathcal{C}\to\mathcal{D}$ be left adjoint to $G:\mathcal{D}\to\mathcal{C}$ with natural transformations $\epsilon: FG\to \mathrm{id}_{\mathcal{D}}$ and $\eta:\mathrm{id}_{\mathcal{C}}\to GF$ satisfying $(\epsilon F)\circ(F\eta)=\mathrm{id}_F$ and $(G\epsilon)\circ(\eta G)=\mathrm{id}_G$. Then $(GF, G\epsilon F,\eta)$ is a monad in $\mathcal{C}$ and $(FG, F\eta G,\epsilon)$ is a comonad in $\mathcal{D}$.

Proof:
We prove the claim for the comonad by verifying that $(FG, F\eta G,\epsilon)$ satisfies the conditions in Example 5.3.10, 5. The component morphisms of $\Delta=F\eta G: FG\to FGFG$ are given by $\Delta_D=F(\eta_{G(D)}): FG(D)\to FGFG(D)$. From the naturality of $\eta:\mathrm{id}_{\mathcal{C}}\to GF$ we then obtain the first condition in (74)

$$\begin{aligned}&FG(\Delta_D)\circ\Delta_D=FGF(\eta_{G(D)})\circ F(\eta_{G(D)})=F(GF(\eta_{G(D)})\circ\eta_{G(D)})\\ &\overset{\text{nat}}{=}F(\eta_{GFG(D)}\circ\eta_{G(D)})=F(\eta_{GFG(D)})\circ F(\eta_{G(D)})=\Delta_{FG(D)}\circ\Delta_D.\end{aligned}$$

The identities $(\epsilon F)\circ(F\eta)=\mathrm{id}_F$ and $(G\epsilon)\circ(\eta G)=\mathrm{id}_G$ from (4) imply the second condition

$$\begin{aligned}&\epsilon_{FG(D)}\circ\Delta_D=\epsilon_{FG(D)}\circ F(\eta_{G(D)})\overset{(4)}{=}1_{FG(D)}\\ &FG(\epsilon_D)\circ\Delta_D=FG(\epsilon_D)\circ F(\eta_{G(D)})=F(G(\epsilon_D)\circ\eta_{G(D)})\overset{(4)}{=}F(1_{G(D)})=1_{FG(D)}.\end{aligned}$$ □

By combining this result with Proposition 5.3.16, we find that every pair of adjoint functors $F:\mathcal{C}\to\mathcal{A}$ and $G:\mathcal{A}\to\mathcal{C}$ defines standard resolutions of certain objects in an abelian category $\mathcal{A}$, namely of those that are isomorphic to objects in the image of F. On one hand, this motivates why monads and comonads play an important role in homological algebra. They arise from simple and canonical constructions that define the underlying adjoint functors.

If we take the opposite viewpoint and take a comonad as the fundamental structure that defines canonical resolutions, it explains why the resolutions for the homology and cohomology theories from Section 2 are all obtained by iterating certain simple constructions such as tensor products or direct products of groups.

Corollary 5.4.2: Let $\mathcal{A}$ be an abelian category and $\mathcal{D}$ a category. If a functor $G:\mathcal{A}\to\mathcal{D}$ has a left adjoint $F:\mathcal{D}\to\mathcal{A}$, then the comonad $(FG, F\eta G,\epsilon)$ in $\mathcal{A}$ defines a resolution

$$C(A)_\bullet=\ldots\xrightarrow{d_2}C^2(A)\xrightarrow{d_1}C(A)\xrightarrow{\epsilon_A}A\to 0\quad\text{with}\quad d_n=\Sigma_{i=0}^n(-1)^iC^i\epsilon C_A^{n-i}:C^{n+1}(A)\to C^n(A)$$

of every object A in $\mathcal{A}$ that is isomorphic to an object in the image of F.

Proof:
By Propositions 5.3.16 and 5.4.1 it is sufficient to show that every object A in $\mathcal{A}$ for which there is an isomorphism $\phi: A\to F(D)$ is C-projective. This follows because the morphism $f=FG(\phi^{-1})\circ F(\eta_D)\circ\phi: A\to FG(A)$ satisfies

$$\epsilon_A\circ f=\epsilon_A\circ FG(\phi^{-1})\circ F(\eta_D)\circ\phi\overset{\text{nat}}{=}\phi^{-1}\circ\epsilon_{F(D)}\circ F(\eta_D)\circ\phi\overset{(4)}{=}\phi^{-1}\circ\phi=1_A.$$

□

To see how the standard resolutions considered so far arise from this construction, we need to identify the pairs of adjoint functors that define the comonad underlying their standard resolutions. It turns out that these are the restriction functor and the induction functor from Example 1.2.18, 7. for the Hochschild resolution and the bar-resolution of group cohomology. Free resolutions of R-modules are obtained from the forgetful functor R-Mod $\to$ Set and the freely generated module functors from Example 1.2.18, 1.

Example 5.4.3: (**The Hochschild resolution from a comonad**)

Let k be a commutative ring and $\phi : k \to R$ a ring homomorphism.

• By Example 1.2.18, 7. the restriction functor $G = \mathrm{Res} : R\text{-Mod} \to k\text{-Mod}$ is right adjoint to the induction functor $F = \mathrm{Ind} = R\otimes_k - : k\text{-Mod} \to R\text{-Mod}$.

The natural transformations $\epsilon : FG \to \mathrm{id}_{R\text{-Mod}}$ and $\eta : \mathrm{id}_{k\text{-Mod}} \to GF$ from Proposition 5.4.1 are given by their component morphisms $\epsilon_M : R\otimes_k M \to M$, $r\otimes m \mapsto r \rhd m$ and $\eta_N : N \to R\otimes_k N$, $n \mapsto 1\otimes n$ for each R-module M and k-module N.

• The associated comonad $FG = R\otimes_k - : R\text{-Mod} \to R\text{-Mod}$ sends an R-module M to the R-module $FG(M) = R\otimes_k M$ with $r \rhd (r'\otimes m) = (rr')\otimes m$ and an R-linear map $f : M \to M'$ to the R-linear map $\mathrm{id}_R\otimes f : R\otimes_k M \to R\otimes_k M'$.

• By Proposition 5.4.1 $(FG, F\eta G, \epsilon)$ is a comonad in R-Mod and Corollary 5.4.2 yields a resolution $C(M)_\bullet$ in R-Mod for every FG-projective R-module M. It is given by

$$C(M)_n = (FG)^{n+1}(M) = R^{\otimes_k (n+1)}\otimes_k M$$
$$d_n = \Sigma_{i=0}^n (-1)^i\,(FG)^i\epsilon(FG)^{n-i}_M : R^{\otimes_k(n+1)}\otimes_k M \to R^{\otimes_k n}\otimes_k M$$
$$r_0\otimes ...\otimes r_n\otimes m \mapsto (r_0r_1)\otimes ...\otimes r_n\otimes m \pm ... + (-1)^{n-1}r_0\otimes ...\otimes(r_{n-1}r_n)\otimes m + (-1)^n r_0\otimes ...\otimes(r_n \rhd m)$$

If M is an (R,S)-bimodule, this resolution is also a resolution in $R\otimes S^{op}$-Mod.

• By Corollary 5.4.2 the R-module $M = R$ is FG-projective since $R \cong R\otimes_k k = F(k)$. By setting $M = R$, we obtain a resolution of R in $R\otimes R^{op}$-Mod

$$... \xrightarrow{d_4} R^{\otimes_k 5} \xrightarrow{d_3} R^{\otimes_k 4} \xrightarrow{d_2} R^{\otimes_k 3} \xrightarrow{d_1} R^{\otimes_k 2} \xrightarrow{\epsilon_R} R \to 0$$

$$d_n = \Sigma_{i=0}^n (-1)^i d_n^i : R^{\otimes_k(n+2)} \to R^{\otimes_k(n+1)}$$
$$r_0\otimes ...\otimes r_{n+1} \mapsto (r_0r_1)\otimes r_2 ...\otimes r_{n+1} - r_0\otimes(r_1r_2)\otimes r_3 ...\otimes r_{n+1} \pm ... + (-1)^n r_0\otimes ...\otimes r_{n-1}\otimes(r_nr_{n+1}).$$

• If A is an algebra over k, then we can choose $R = A$ and $\phi : k \to A$, $\lambda \mapsto \lambda 1_A$. In this case, A is an (A,A)-bimodule with $a \rhd b \lhd c = abc$, and the resulting resolution of A in $A\otimes A^{op}$-mod is the Hochschild resolution from Example 4.1.3.

Example 5.4.4: (**The bar resolution from a comonad**)

Let k be a commutative ring and G a group.

• By setting $R = k[G]$ in Example 5.4.3, we obtain a comonad $(FG, \epsilon, F\eta G)$ from the forgetful functor $G : k[G]\text{-Mod} \to k\text{-Mod}$ and its left adjoint $F = k[G]\otimes_k - : k\text{-Mod} \to k[G]\text{-Mod}$.

• As $k[G]^{\otimes n} \cong \langle G^{\times n}\rangle_k$ as a k-module, the associated chain complex for a $k[G]$-module M is

$$... \xrightarrow{d_3} \langle G^{\times 3}\rangle_k \otimes_k M \xrightarrow{d_2} \langle G^{\times 2}\rangle_k \otimes_k M \xrightarrow{d_1} \langle G\rangle_k \otimes_k M \xrightarrow{\epsilon_M} M \to 0$$

$$d_n : \langle G^{\times (n+1)}\rangle_k \otimes_k M \to \langle G^{\times n}\rangle_k \otimes_k M$$
$$(g_0, ..., g_n)\otimes m \mapsto (g_0 g_1, g_2, ..., g_n)\otimes m - (g_0, g_1 g_2, ...g_n)\otimes m \pm ... + (-1)^n (g_0, ..., g_{n-1})\otimes(g_n \rhd m).$$

• The trivial $k[G]$-module $M = k$ is FG-projective since $FG(k) = k[G]\otimes_k k \cong k[G]$. and the $k[G]$-linear maps $f : k \to k[G]$, $\lambda \mapsto \lambda e$ and $\epsilon_k : k[G] \to k$, $\lambda g \mapsto \lambda$ satisfy $\epsilon_k \circ f = \mathrm{id}_k$.

• Setting $M = k$ and noting that $\langle G^{\times(n+1)}\rangle_k \otimes_k k \cong \langle G^{\times (n+1)}\rangle_k \cong \langle G^{\times n}\rangle_{k[G]}$ we obtain the bar-resolution from Example 4.1.2

$$\ldots \xrightarrow{d_5} \langle G^{\times 4}\rangle_{k[G]} \xrightarrow{d_4} \langle G^{\times 3}\rangle_{k[G]} \xrightarrow{d_3} \langle G^{\times 2}\rangle_{k[G]} \xrightarrow{d_2} \langle G\rangle_{k[G]} \xrightarrow{d_1} k[G] \xrightarrow{\epsilon_k} k \to 0$$

$$d_n : \langle G^{\times n}\rangle_{k[G]} \to \langle G^{\times (n-1)}\rangle_{k[G]}$$
$$(g_1, ..., g_n) \mapsto g_1 \rhd (g_2, ..., g_n) + (-1)^1 (g_1 g_2, ..., g_n) + ... + (-1)^{n-1}(g_1, ..., g_{n-1} g_n) + (-1)^n (g_1, ..., g_{n-1}).$$

Example 5.4.5: (**Free resolutions from a comonad**)

• For any ring R the functor $F = \langle\ \rangle_R : \mathrm{Set} \to R\text{-Mod}$ is left adjoint to the forgetful functor $G : R\text{-Mod} \to \mathrm{Set}$ by Example 1.2.18, 1.

• The associated comonad $FG : R\text{-Mod} \to R\text{-Mod}$ assigns to an R-module M the free R-module $\langle M\rangle_R$ generated by the set M and to an R-linear map $f : M \to M'$ the R-linear map $\langle f\rangle_R : \langle M\rangle_R \to \langle M'\rangle_R$ with $\langle f\rangle_R(m) = f(m)$. The natural transformations $\epsilon : FG \to \mathrm{id}_{R\text{-Mod}}$ and $\eta : \mathrm{id}_{\mathrm{Set}} \to GF$ are given by their component morphisms $\epsilon_M : \langle M\rangle_R \to M$, $\Sigma_{m\in M} r_m\, m \mapsto \Sigma_{m\in M} r_m \rhd m$ and $\eta_X : X \to \langle X\rangle_R$, $x \mapsto x$ for an R-module M and a set X.

• If we set $\langle M\rangle_R^0 := M$ and $\langle M\rangle_R^{n+1} := \langle\langle M\rangle_R^n\rangle_R$ for all $n \in \mathbb{N}_0$, we have $FG^n(M) = \langle M\rangle_R^n$. An element of $\langle M\rangle_R^n$ is a finite sum of elements of the form $(r_1, ..., r_n, m)$ with $r_i \in R$ and $m \in M$. The R-module structure on $\langle M\rangle_R^n$ is given by $r \rhd (r_1, ..., r_n, m) = (rr_1, r_2, ..., r_n, m)$.

• In this case, every projective R-module M is FG-projective, since $\epsilon_M : \langle M\rangle_R \to M$ is surjective. By Lemma 3.1.21 for every projective R-module M there is a morphism $f : M \to \langle M\rangle_R$ with $\epsilon_M \circ f = \mathrm{id}_M$.

• By Proposition 5.3.16 the comonad $(FG, F\eta G, \epsilon)$ defines a free resolution for every projective object M in R-Mod given by

$$... \xrightarrow{d_4} \langle M\rangle_R^4 \xrightarrow{d_3} \langle M\rangle_R^3 \xrightarrow{d_2} \langle M\rangle_R^2 \xrightarrow{d_1} \langle M\rangle_R \xrightarrow{\epsilon_M} M \to 0$$

$$d_n : \langle M\rangle_R^{n+1} \to \langle M\rangle_R^n,$$
$$(r_0, ..., r_n, m) \mapsto (r_0 r_1, ..., r_n, m) \pm ... + (-1)^{n-1}(r_0, ..., r_{n-1} r_n, m) + (-1)^n (r_0, ..., r_{n-1}, r \rhd m).$$

As the category Top is not abelian, singular (co)homology does not directly fit into this pattern. However, it is still related to a pair of adjoint functors. It was shown in Example 5.1.8 that singular homology is obtained from a functor $\mathrm{Sing} : \mathrm{Top} \to \mathrm{Fun}(\Delta^{+op}, \mathrm{Set})$ by composing it with the functor $\langle\ \rangle_k : \mathrm{Set} \to k\text{-Mod}$ for a commutative ring k. Exercise 78 shows that the singular functor Sing : Top $\to$ Fun(Δ^{+op}, Set) is right adjoint to the geometric realisation functor Geom : Fun(Δ^{+op}, Set) $\to$ Top from Example 5.1.9.

Examples 5.4.3 to 5.4.5 show that Corollary 5.4.2 gives rise to the standard resolutions for Hochschild (co)homology and group cohomology from Example 4.1.3 and Example 4.1.2 and to a free standard resolution of every R-module M. All that is required is a pair of adjoint functors for the structures under consideration.

As already noted for the Hochschild resolution and the bar resolution, the standard resolutions are often not very practical for computations. This also holds for the free standard resolution in Example 5.4.5. Nevertheless, the resolutions defined by a comonad (C, Δ, ϵ) in an abelian category $\mathcal{A}$ are of conceptual importance. They allow one to view homologies as *homologies of functors*, even for abelian categories without enough projectives or injectives and for additive functors that are not right or left exact.

If $\mathcal{A}$ and $\mathcal{B}$ are abelian categories and $K : \mathcal{A} \to \mathcal{B}$ is additive, we can define the homologies of objects in $\mathcal{A}$ by choosing a comonad (C, Δ, ϵ) in $\mathcal{A}$ and setting $H_n(A) = H_n(KC(A)_{\bullet\geq 0})$, where $C(A)_\bullet$ is the chain complex from Proposition 5.3.16 and $KC(A)_{\bullet\geq 0}$ its image under $K : \mathcal{A} \to \mathcal{B}$ with the last entry on the right removed. This is called the **Barr-Beck (co)homology** or **cotriple (cohomology)**.

Definition 5.4.6: (Barr-Beck (co)homology, cotriple (co)homology)

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and (C, Δ, ϵ) a comonad in $\mathcal{A}$.

1. The **comonad homology** $H_n^C(A, K)$ of an object A in $\mathcal{A}$ with coefficients in an additive functor $K : \mathcal{A} \to \mathcal{B}$ is the homology $H_n(KC(A)_{\bullet\geq 0})$, where $C(A)_\bullet$ is the chain complex from Proposition 5.3.16.

2. The **comonad cohomology** $H^n_C(A, K)$ of an object A in $\mathcal{A}$ with coefficients in an additive functor $K : \mathcal{A}^{op} \to \mathcal{B}$ is the cohomology $H^n(KC(A)^{\bullet\geq 0})$, where $C(A)_\bullet$ is the chain complex from Proposition 5.3.16.

Remark 5.4.7:

1. The comonad homologies and cohomologies for a fixed comonad (C, Δ, ϵ) in $\mathcal{A}$ define functors $H_n^C : \mathcal{A} \times \mathrm{Fun}^{\mathrm{add}}(\mathcal{A}, \mathcal{B}) \to \mathcal{B}$ and functors $H^n_C : \mathcal{A} \times \mathrm{Fun}^{\mathrm{add}}(\mathcal{A}^{op}, \mathcal{B}) \to \mathcal{B}$, where $\mathrm{Fun}^{\mathrm{add}}(\mathcal{A}, \mathcal{B}) \subset \mathrm{Fun}(\mathcal{A}, \mathcal{B})$ and $\mathrm{Fun}^{\mathrm{add}}(\mathcal{A}^{op}, \mathcal{B}) \subset \mathrm{Fun}(\mathcal{A}^{op}, \mathcal{B})$ denote the full subcategories with additive functors as objects.
2. There are analogous definitions of monad (co)homologies, where the chain complex $C(A)_\bullet$ is replaced by a cochain complex.

Note that for C-projective objects $A \in \mathrm{Ob}\,\mathcal{A}$ the chain complex $C(A)_\bullet$ in Definition 5.4.6 is a resolution of A. This applies in particular to the standard resolution of an R-module M from Example 5.4.5, which is a free resolution of M. As the category R-Mod has enough projectives by Corollary 4.2.1 and any free resolution is a projective resolution by Corollary 4.2.3, this resolution is unique up to chain homotopy equivalence by Theorem 4.1.8.

It follows that for any right exact functor $K : R\text{-Mod} \to \mathcal{B}$, the homologies of the resulting chain complex $KC(A)_\bullet$ are precisely the left derived functors of K. Similarly, for any left exact functor $K : R\text{-Mod}^{op} \to \mathcal{B}$ one obtains the right derived functors of K. This allows one to realise the functors Tor and Ext as comonad homologies of objects in R-Mod with coefficients in the functors $K = M\otimes_R- : R\text{-Mod} \to \text{Ab}$ and $K = \text{Hom}(-, M) : R\text{-Mod}^{op} \to \text{Ab}$.

Example 5.4.8: (Tor and Ext)

Let R be a ring and (C, Δ, ϵ) the comonad from Example 5.4.5 in R-Mod. Then the comonad homology of an R-left module N with coefficients in $K = M\otimes_R- : R\text{-Mod} \to \text{Ab}$ is given by

$$H_n^C(N, M\otimes_R-) = L_n(M\otimes_R-)(N) = \text{Tor}_n^R(M, N)$$

and its comonad cohomology with coefficients in $K = \text{Hom}_R(-, M) : R\text{-Mod}^{op} \to \text{Ab}$ by

$$H^n_C(N, \text{Hom}_R(-, M)) = R^n\text{Hom}_R(-, M)(N) = \text{Ext}^n_R(N, M).$$

This final example includes Hochschild (co)homologies, group (co)homologies and (co)homologies of Lie algebras and identifies them as comonad (co)homologies, since they are obtained as functors Tor and Ext for specific choices of the underlying ring R (see Examples 4.4.3, 4.4.4 and 4.4.5 and Definition 4.4.6). It also motivates the functors Tor and Ext and, more generally, left and right derived functors of right exact functors $K : R\text{-Mod} \to \mathcal{B}$ or left exact functors $K : R\text{-Mod}^{op} \to \mathcal{B}$ from the viewpoint of simplicial homology. A left or right derived functor is nothing but the comonad homology for the free resolution from Example 5.4.5 with coefficients in a right or left exact functor $K : R\text{-Mod} \to \mathcal{B}$ or $K : R\text{-Mod}^{op} \to \mathcal{B}$.

5.5 The Eilenberg-Zilber Theorem*

To conclude our investigation of monoidal structures in homological algebra, we relate the tensor product of chain complexes from Definition 4.6.1 to a tensor product in the category of simplicial objects in R-Mod for each commutative ring R.

This has applications in many homology theories. Together with the Künneth formula, it allows one to compute the simplicial and singular (co)homologies of product spaces. In group (co)homology it allows one to compute (co)homologies of product groups and in Hochschild (co)homology, the (co)homologies of tensor products of algebras.

Recall from Example 5.3.3, 5. that for any commutative ring R the tensor product of chain complexes from Definition 4.6.1 equips the category $\text{Ch}_{R\text{-Mod}\geq 0}$ with the structure of a monoidal category. The tensor product of two chain complexes $X_\bullet$, $Y_\bullet$ is the chain complex $X_\bullet \otimes Y_\bullet$ with

$$(X_\bullet\otimes Y_\bullet)_n = \oplus_{k=0}^n X_k \otimes_R Y_{n-k},$$
$$d_n : (X_\bullet\otimes Y_\bullet)_n \to (X_\bullet\otimes Y_\bullet)_{n-1}, \quad d_n(x\otimes y) = d_k(x)\otimes y + (-1)^k x\otimes d_{n-k}(y) \text{ for } x\otimes y \in X_k\otimes_R Y_{n-k}.$$

The tensor product of two chain maps $f_\bullet : X_\bullet \to X'_\bullet$ and $g_\bullet : Y_\bullet \to Y'_\bullet$ is the chain map

$$f_\bullet\otimes g_\bullet : X_\bullet\otimes Y_\bullet \to X'_\bullet\otimes Y'_\bullet, \quad (f_\bullet\otimes g_\bullet)_n(x\otimes y) = f_k(x)\otimes g_{n-k}(y) \text{ for } x\otimes y \in X_k\otimes_R Y_{n-k}.$$

The coherence data for the monoidal structure is induced by the coherence data for R-Mod via the universal property of the tensor product and the direct sum.

On the other hand, Exercise 80 shows that for any monoidal category $\mathcal{A}$ the category $\mathrm{Fun}(\Delta^{+op}, \mathcal{A})$ becomes monoidal with the objectwise tensor product of functors and natural transformations. The tensor product of simplicial objects $X, Y : \Delta^{+op} \to \mathcal{A}$ is thus given by

$$(X\otimes' Y)_n = X_n \otimes_{\mathcal{A}} Y_n \qquad (X\otimes' Y)(\alpha) = X(\alpha)\otimes_{\mathcal{A}} Y(\alpha) : X_n\otimes_{\mathcal{A}} Y_n \to X_m \otimes_{\mathcal{A}} Y_m$$

for all $n \in \mathbb{N}_0$ and morphisms $\alpha : [m+1] \to [n+1]$ in Δ^+. The tensor product of simplicial morphisms $\alpha : X \to X'$ and $\beta : Y \to Y'$ is the simplicial morphism

$$(\alpha\otimes'\beta) : X\otimes' Y \to X'\otimes' Y' \quad (\alpha\otimes'\beta)_n = \alpha_n \otimes_{\mathcal{A}} \beta_n : X_n \otimes_{\mathcal{A}} Y_n \to X'_n \otimes_{\mathcal{A}} Y'_n.$$

In particular, if $\mathcal{A} = R$-Mod for a commutative ring R, this defines a monoidal structure on the category $\mathrm{Fun}(\Delta^{+op}, R\text{-Mod})$ of simplicial objects and simplicial morphisms in R-Mod.

In Section 5.2 we found that each simplicial object $X : \Delta^{+op} \to \mathcal{A}$ in an abelian category $\mathcal{A}$ defines a chain complex $X_\bullet$ in $\mathcal{A}$, its *standard chain complex*, with

$$X_n = X([n+1]) \qquad d_n = \Sigma_{i=0}^n (-1)^i X(\delta_n^i) : X_n \to X_{n-1}.$$

Each simplicial morphism $\alpha : X \to X'$ defines a chain map $\alpha_\bullet : X_\bullet \to X'_\bullet$ given by $\alpha_n = \alpha_{[n+1]} : X_n \to X'_n$. In Proposition 5.2.2 we established that this defines a functor, the *standard chain complex functor* $\bullet : \mathrm{Fun}(\Delta^{+op}, \mathcal{A}) \to \mathrm{Ch}_{\mathcal{A}\geq 0}$.

We now investigate the interaction of this functor with the monoidal structures in the categories $\mathrm{Ch}_{\mathcal{A}\geq 0}$ and $\mathrm{Fun}(\Delta^{+op}, \mathcal{A})$ when $\mathcal{A} = R$-Mod for a commutative ring R. As both, the category $\mathrm{Fun}(\Delta^{+op}, R\text{-Mod})$ and the category $\mathrm{Ch}_{R\text{-Mod}}$ are monoidal, the simplest guess is that the standard chain complex functor $\bullet : \mathrm{Fun}(\Delta^{+op}, R\text{-Mod}) \to \mathrm{Ch}_{R\text{-Mod}}$ could be a *monoidal functor* in the sense of Definition 5.3.4. However, we will find that this is not the case. Instead, it is both *lax monoidal* and *op-lax monoidal* in the sense of Remark 5.3.5. The morphisms that describe its lax and op-lax monoidal structures are chain homotopy equivalences.

The proof proceeds in two steps. We first establish that there are natural transformations $f : \bullet\otimes' \to \otimes(\bullet \times \bullet)$ and $g : \otimes(\bullet \times \bullet) \to \otimes'\bullet$ whose component morphisms form chain homotopy equivalences. This already ensures that the homologies of the two chain complexes are isomorphic: $H_n(X_\bullet \otimes Y_\bullet) \cong H_n((X\otimes' Y)_\bullet)$ for all simplicial objects $X, Y : \Delta^{+op} \to R$-Mod. In a second step, we derive expressions for these natural transformations and prove that they give the standard chain complex functor the structure of a lax and op-lax monoidal functor.

The first step is largely abstract and reminiscent of the constructions with projective and injective resolutions in Section 4. The only difference is that the exactness and projectivity or injectivity requirements from Section 4 are replaced by the requirement that the chain complexes are *acyclic* and by naturality conditions. This is known as the method of **acyclic models**.

Definition 5.5.1: Let R be a ring. A chain complex $X_\bullet$ in R-Mod is called

- **augmented**, if $X_n = 0$ for $n < 0$, and there is an R-linear map $\epsilon : X_0 \to R$ with $\epsilon \circ d_1 = 0$,
- **acyclic**, if it is augmented, $H_n(X_\bullet) = 0$ for $n \neq 0$ and the induced map $\epsilon : H_0(X_\bullet) \to R$ is an isomorphism.

An augmented chain complex $X_\bullet$ is called *augmented* because it can be prolonged to a chain complex $X^\epsilon_\bullet$ with the ring R in degree -1

$$X^\epsilon_\bullet = \ldots \to X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \xrightarrow{\epsilon} R \to 0.$$

Exactness of the chain complex $X^\epsilon_\bullet$ is equivalent to the conditions $H_n(X^\epsilon_\bullet) = H_n(X_\bullet) = 0$ for $n \in \mathbb{N}$, to surjectivity of ϵ and to the condition $\ker(\epsilon) = \mathrm{im}(d_1)$. The last two conditions state that the induced map $\epsilon : H_0(X_\bullet) = X_0/\mathrm{im}(d_1) = X_0/\ker(\epsilon) \to R$ is an isomorphism. Hence $X^\epsilon_\bullet$ is exact if and only if $X_\bullet$ is acyclic.

To relate the chain complexes $X_\bullet \otimes Y_\bullet$ and $(X\otimes' Y)_\bullet$ for functors $X, Y : \Delta^{+op} \to R$-Mod, we consider a special simplicial object $A^n : \Delta^{+op} \to R$-Mod. Just as all R-linear maps $f : R \to M$ into an R-module M are of the form $f : R \to M$, $r \mapsto r \rhd f(1)$ and hence in bijection with elements of M, simplicial morphisms from A^n to a simplicial object $X : \Delta^{+op} \to R$-Mod are in bijection with elements of X_n.

This simplicial object A^n is obtained from the simplicial set $\mathrm{Hom}(-, [n+1]) : \Delta^{+op} \to \mathrm{Set}$. The Yoneda-Lemma states that natural transformations $\eta : \mathrm{Hom}(-, [n+1]) \to X$ into a simplicial set $X : \Delta^{+op} \to \mathrm{Set}$ are in bijection with elements of $X([n+1]) = X_n$. By composing it with the functor $\langle\ \rangle_R : \mathrm{Set} \to R$-Mod that assigns to a set Y the free R-module $\langle Y\rangle_R$ we obtain a simplicial object $A^n : \Delta^{+op} \to R$-Mod. It is given by

$$\begin{aligned} &A^n([p+1]) = \langle \mathrm{Hom}_{\Delta^+}([p+1], [n+1])\rangle_R && (76)\\ &A^n(\alpha) : \langle \mathrm{Hom}_{\Delta^+}([p+1], [n+1])\rangle_R \to \langle \mathrm{Hom}_{\Delta^+}([q+1], [n+1])\rangle_R, \quad \beta \mapsto \beta\circ\alpha. \end{aligned}$$

for all $p \in \mathbb{N}_0$ and morphisms $\alpha : [q+1] \to [p+1]$ in Δ^+. Its standard chain complex $A^n_\bullet$ from Proposition 5.2.1 takes the form

$$A^n_p = \langle \mathrm{Hom}_{\Delta^+}([p+1], [n+1])\rangle_R \qquad d_p : A^n_p \to A^n_{p-1},\ \beta \mapsto \Sigma_{i=0}^p (-1)^i \beta\circ\delta^i_p \tag{77}$$

and turns out to be acyclic. It can be viewed as a model for all other acyclic complexes arising from simplicial objects in R-Mod, just as the ring R can be viewed as a model of all R-modules.

Proposition 5.5.2: Let R be a commutative ring and $n \in \mathbb{N}_0$. Then:

1. The chain complex $A^n_\bullet$ from (77) is acyclic.
2. For each simplicial object $X : \Delta^{+op} \to R$-Mod and each $x \in X_n = X([n+1])$, there is a unique simplicial morphism $a^{n,x} : A^n \to X$ with $a^{n,x}_{[n+1]}(1_{[n+1]}) = x$.

Proof. 1. We define an R-linear map $\epsilon : A^n_0 \to R$ by $\epsilon(\alpha) = 1$ for all morphisms $\alpha : [1] \to [n+1]$ in Δ^+. This yields $\epsilon\circ d_1(\alpha) = \epsilon\circ\alpha\circ\delta^0_1 - \epsilon\circ\alpha\circ\delta^1_1 = 0$ and gives $A^n_\bullet$ the structure of an augmented chain complex. To prove that $A^n_\bullet$ is acyclic, we show that the augmented chain complex

$$A^{\epsilon n}_\bullet = \ldots \xrightarrow{d_3} A^n_2 \xrightarrow{d_2} A^n_1 \xrightarrow{d_1} A^n_0 \xrightarrow{\epsilon} R \to 0$$

is exact. We define a chain homotopy between $0_\bullet, \mathrm{id}_\bullet : A^{\epsilon n}_\bullet \to A^{\epsilon n}_\bullet$ by setting

$$\begin{aligned} &h_{-1} : R \to A^n_0,\ r \mapsto r \rhd \alpha_0 && \text{with } \alpha_0 : [1] \mapsto [n+1], 0 \to 0, && (78)\\ &h_p : A^n_p \to A^n_{p+1}, \alpha \mapsto h_p(\alpha) && \text{with } h_p(\alpha)(0) = 0, h_p(\alpha)(k) = \alpha(k-1)\ \forall k \in \{1, ..., p+1\}, p \in \mathbb{N}_0. \end{aligned}$$

With the definition of the face morphisms δ^i_n in Proposition 5.1.2 we then obtain for all $p \in \mathbb{N}_0$ and morphisms $\alpha : [p+1] \to [n+1]$

$$\begin{aligned} &A^n(\delta^0_{p+1}) \circ h_p(\alpha) = h_p(\alpha)\circ\delta^0_{p+1} = \alpha, && (79)\\ &A^n(\delta^i_{p+1}) \circ h_p(\alpha) = h_p(\alpha)\circ\delta^i_{p+1} = h_p(\alpha\circ\delta^{i-1}_p) = h_{p-1}\circ A^n(\delta^{i-1}_p)(\alpha) \qquad i \in \{1, ..., p+1\}. \end{aligned}$$

This implies for all $p \in \mathbb{N}_0$

$$d_{p+1} \circ h_p + h_{p-1} \circ d_p \overset{(77)}{=} \Sigma_{i=0}^{p+1} (-1)^i A^n(\delta^i_{p+1}) \circ h_p + \Sigma_{i=0}^{p} (-1)^i h_{p-1} \circ A^n(\delta^i_p)$$
$$\overset{(79)}{=} \mathrm{id}_{A^n_p} + \Sigma_{i=1}^{p+1} (-1)^i h_{p-1} \circ A^n(\delta^{i-1}_p) + \Sigma_{i=0}^{p} (-1)^i h_{p-1} \circ A^n(\delta^i_p) = \mathrm{id}_{A^n_p}.$$

As we also have $\epsilon \circ h_{-1} = \mathrm{id}_R$, the maps h_p define a chain homotopy $h_\bullet : 0_\bullet \Rightarrow \mathrm{id}_\bullet$.

2. Let now $X : \Delta^{+op} \to R$-Mod be a simplicial object in R-Mod. Then a simplicial morphism $a : A^n \to X$ is a natural transformation from A^n to X, given by its component morphisms $a_{[p+1]} : A^n_p \to X_n$. The naturality implies for each morphism $\alpha : [p+1] \to [n+1]$

$$a_{[p+1]}(\alpha) = a_{[p+1]}(1_{[n+1]} \circ \alpha) = a_{[p+1]} \circ A^n(\alpha)(1_{[n+1]}) \overset{\text{nat}}{=} X(\alpha) \circ a_{[n+1]}(1_{[n+1]}).$$

This shows that a is determined uniquely by $a_{[n+1]}(1_{[n+1]}) \in X_n$ and every element $x \in X_n$ yields a unique simplicial morphism $a^{n,x} : A^n \to X$ with $a^{n,x}_{[n+1]}(1_{[n+1]}) = x$. □

Remark 5.5.3: The chain complexes $A^p_\bullet \otimes A^q_\bullet$ and $(A^p \otimes' A^q)_\bullet$ are also acyclic for all $p, q \in \mathbb{N}_0$.

For $A^p_\bullet \otimes A^q_\bullet$ this follows with the Künneth formula in Theorem 4.6.3. For $(A^p \otimes' A^q)_\bullet$ this follows, because the maps $h_k \otimes h_k : A^{\epsilon p}_k \otimes A^{\epsilon q}_k \to A^{\epsilon p}_{k+1} \otimes A^{\epsilon q}_{k+1}$ with $h_k : A^{\epsilon n}_k \to A^{\epsilon n}_{k-1}$ as in (78) form a chain homotopy between $0_\bullet, \mathrm{id}_\bullet : (A^{\epsilon p} \otimes' A^{\epsilon q})_\bullet \to (A^{\epsilon p} \otimes' A^{\epsilon q})_\bullet$. This follows by a direct computation analogous to the one in the proof of Proposition 5.5.2 (Exercise).

Using the acyclic chain complexes $A^p_\bullet \otimes A^q_\bullet$ and $(A^p \otimes' A^q)_\bullet$ from Remark 5.5.3, we can now construct natural transformations $f : \bullet \otimes' \to \otimes(\bullet \times \bullet)$ and $g : \otimes(\bullet \times \bullet) \to \bullet \otimes'$ that form chain homotopy equivalences between the complexes $(X \otimes' Y)_\bullet$ and $X_\bullet \otimes Y_\bullet$ for all simplicial objects $X, Y : \Delta^{+op} \to R$-Mod.

For this, we proceed similarly to the construction of the resolutions in Section 4. We construct their component morphisms $f^{X,Y}_\bullet : (X \otimes' Y)_\bullet \to X_\bullet \otimes Y_\bullet$ and $g^{X,Y}_\bullet : X_\bullet \otimes Y_\bullet \to (X \otimes' Y)_\bullet$ by induction over the degree. The naturality condition and model property of the chain complexes $A^n_\bullet$ in Proposition 5.5.2, 2. reduce this construction to the cases $(A^n \otimes' A^n)_\bullet$ and $A^p_\bullet \otimes A^q_\bullet$, where we can use the acyclicity. The construction of the chain homotopies is analogous.

Theorem 5.5.4: (**Eilenberg-Zilber Theorem**)

Let R be a commutative ring and $\bullet : \mathrm{Fun}(\Delta^{+op}, R\text{-Mod}) \to \mathrm{Ch}_{R\text{-Mod}\geq 0}$ the standard chain complex functor from Proposition 5.2.2.

1. There are natural transformations $f : \bullet \otimes' \to \otimes(\bullet \times \bullet)$ and $g : \otimes(\bullet \times \bullet) \to \otimes' \bullet$ with $f^{X,Y}_0 = g^{X,Y}_0 = \mathrm{id}_{X_0 \otimes Y_0}$ for all simplicial objects $X, Y : \Delta^{+op} \to R$-Mod.

2. Their component morphisms $f^{X,Y}_\bullet$ and $g^{X,Y}_\bullet$ are unique up to chain homotopies that are natural in X and Y.

3. Their component morphisms $f^{X,Y}_\bullet$ and $g^{X,Y}_\bullet$ form a chain homotopy equivalence with chain homotopies that are natural in X and Y.

Proof. 1. We define the natural transformations f and g by inductively constructing their component morphisms. We set $f^{X,Y}_0 = g^{X,Y}_0 = \mathrm{id} : X_0 \otimes Y_0 \to X_0 \otimes Y_0$ for all simplicial objects X, Y in R-Mod. Suppose we constructed R-linear maps $f^{X,Y}_k : (X \otimes' Y)_k \to (X_\bullet \otimes Y_\bullet)_k$ and $g^{X,Y}_k : (X_\bullet \otimes Y_\bullet)_k \to (X \otimes' Y)_k$ for all $k < n$ and simplicial objects X, Y such that:

(i) $d_k \circ f_k^{X,Y} = f_{k-1}^{X,Y} \circ d_k$ and $d_k \circ g_k^{X,Y} = g_{k-1}^{X,Y} \circ d_k$,

(ii) $f_k^{X',Y'} \circ (\alpha \otimes' \beta)_k = (\alpha_\bullet \otimes \beta_\bullet)_k \circ f_k^{X,Y}$ and $g_k^{X',Y'} \circ (\alpha_\bullet \otimes \beta_\bullet)_k = (\alpha \otimes' \beta)_k \circ g_k^{X,Y}$

for all $k < n$ and simplicial morphisms $\alpha : X \to X'$ and $\beta : Y \to Y'$.

To define R-linear maps $f_n^{X,Y} : (X \otimes' Y)_n \to (X_\bullet \otimes Y_\bullet)_n$ and $g_n^{X,Y} : (X_\bullet \otimes Y_\bullet)_n \to (X \otimes' Y)_n$ that satisfy (i) and (ii) for $k = n$, we consider the acyclic complex $A^n_\bullet$ from (77) and Proposition 5.5.2 and recall that the chain complexes $A^p_\bullet \otimes A^q_\bullet$ and $(A^p \otimes' A^q)_\bullet$ are acyclic for all $p, q \in \mathbb{N}_0$ by Remark 5.5.3. Because they are acyclic and we have

$$d_{n-1} \circ f_{n-1}^{A^n,A^n} \circ d_n = f_{n-1}^{A^n,A^n} \circ d_{n-1} \circ d_n = 0 \quad n > 1, \qquad (\epsilon \otimes \epsilon) \circ f_0^{A^n,A^n} \circ d_1 = (\epsilon \otimes \epsilon) \circ d_1 = 0,$$
$$d_{n-1} \circ g_{n-1}^{A^p,A^{n-p}} \circ d_n = g_{n-1}^{A^p,A^{n-p}} \circ d_{n-1} \circ d_n = 0 \quad n > 1, \quad (\epsilon \otimes \epsilon) \circ g_0^{A^p,A^{n-p}} \circ d_1 = (\epsilon \otimes \epsilon) \circ d_1 = 0,$$

for the map $\epsilon : A^n_0 \to R$, $\alpha \mapsto 1$ from the proof of Proposition 5.5.2, it follows that there are elements $z \in (A^n_\bullet \otimes A^n_\bullet)_n$, $z_p \in (A^p \otimes' A^{n-p})_n$ with

$$f_{n-1}^{A^n,A^n} \circ d_n(1_{[n+1]} \otimes 1_{[n+1]}) = d_n(z) \qquad g_{n-1}^{A^p,A^{n-p}} \circ d_n(1_{[p+1]} \otimes 1_{[n-p+1]}) = d_n(z_p). \tag{80}$$

By Proposition 5.5.2 for any simplicial object $X : \Delta^{+op} \to R$-Mod and any element $x \in X_n$, there is a unique simplicial morphism $a^{n,x} : A^n \to X$ with $a^{n,x}_{[n+1]}(1_{[n+1]}) = x$. We define

$$\begin{aligned} f_n^{X,Y}(x \otimes y) &:= (a^{n,x}_\bullet \otimes a^{n,y}_\bullet)_n(z) && x \in X_n, y \in Y_n \\ g_n^{X,Y}(x \otimes y) &:= (a^{p,x} \otimes' a^{n-p,y})_n(z_p) && x \in X_p, y \in Y_{n-p}. \end{aligned} \tag{81}$$

This defines the R-linear maps $f_n^{X,Y}$ and $g_n^{X,Y}$ uniquely, and we have

$$\begin{aligned}
& d_n \circ f_n^{X,Y}(x \otimes y) \overset{(81)}{=} d_n \circ (a^{n,x}_\bullet \otimes a^{n,y}_\bullet)_n(z) \overset{*}{=} (a^{n,x}_\bullet \otimes a^{n,y}_\bullet)_{n-1} \circ d_n(z) \\
& \overset{(80)}{=} (a^{n,x}_\bullet \otimes a^{n,y}_\bullet)_{n-1} \circ f_{n-1}^{A^n,A^n} \circ d_n(1_{[n+1]} \otimes 1_{[n+1]}) \overset{\text{(ii)}}{=} f_{n-1}^{X,Y} \circ (a^{n,x} \otimes' a^{n,y})_{n-1} \circ d_n(1_{[n+1]} \otimes 1_{[n+1]}) \\
& \overset{*}{=} f_{n-1}^{X,Y} \circ d_n \circ (a^{n,x} \otimes' a^{n,y})_n(1_{[n+1]} \otimes 1_{[n+1]}) = f_{n-1}^{X,Y} \circ d_n(a^{n,x}_{[n+1]}(1_{[n+1]}) \otimes a^{n,y}_{[n+1]}(1_{[n+1]})) \\
& = f_{n-1}^{X,Y} \circ d_n(x \otimes y) \qquad x \in X_n, y \in Y_n,
\end{aligned}$$

$$\begin{aligned}
& d_n \circ g_n^{X,Y}(x \otimes y) \overset{(81)}{=} d_n \circ (a^{p,x} \otimes' a^{n-p,y})_n(z_p) \overset{*}{=} (a^{p,x} \otimes' a^{n-p,y})_{n-1} \circ d_n(z_p) \\
& \overset{(80)}{=} (a^{p,x} \otimes' a^{n-p,y})_{n-1} \circ g_{n-1}^{A^p,A^{n-p}} \circ d_n(1_{[p+1]} \otimes 1_{[n-p+1]}) \overset{\text{(ii)}}{=} g_{n-1}^{X,Y} \circ (a^{p,x}_\bullet \otimes a^{n-p,y}_\bullet)_{n-1} \circ d_n(1_{[p+1]} \otimes 1_{[n-p+1]}) \\
& \overset{*}{=} g_{n-1}^{X,Y} \circ d_n \circ (a^{p,x}_\bullet \otimes a^{n-p,y}_\bullet)_n(1_{[p+1]} \otimes 1_{[n-p+1]}) = g_{n-1}^{X,Y} \circ d_n(a^{p,x}_{[p+1]}(1_{[p+1]}) \otimes a^{n-p,y}_{[n-p+1]}(1_{[n-p+1]})) \\
& = g_{n-1}^{X,Y} \circ d_n(x \otimes y) \qquad x \in X_p, y \in Y_{n-p},
\end{aligned}$$

where we used in * that $a^{l,x}_\bullet$, $a^{m,y}_\bullet$, $a^{l,x}_\bullet \otimes a^{m,y}_\bullet$ and $(a^{l,x} \otimes' a^{m,y})_\bullet$ are chain maps and the defining property of the simplicial morphisms $a^{p,x}$, $a^{n-p,y}$ in the last step. This establishes (i) for $k = n$.

To show that the maps $f_n^{X,Y}$, $g_n^{X,Y}$ satisfy (ii), note that for all simplicial morphisms $\alpha : X \to X'$ and $x \in X_n$, the uniqueness of $a^{n,x} : A^n \to X$ from Proposition 5.5.2 implies

$$\alpha \circ a^{n,x} = a^{n,\alpha_{[n+1]}(x)}. \tag{82}$$

With this, we obtain for all simplicial morphisms $\alpha : X \to X'$ and $\beta : Y \to Y'$

$$
\begin{aligned}
&f_n^{X',Y'} \circ (\alpha \otimes' \beta)_n(x \otimes y) \overset{*}{=} f_n^{X',Y'}(\alpha_{[n+1]}(x) \otimes \beta_{[n+1]}(y)) \overset{(81)}{=} (a_\bullet^{n,\alpha_{[n+1]}(x)} \otimes a_\bullet^{n,\beta_{[n+1]}(y)})_n(z) \\
&\overset{(82)}{=} ((\alpha \circ a^{n,x})_\bullet \otimes (\beta \circ a^{n,y})_\bullet)_n(z) \overset{**}{=} (\alpha_\bullet \otimes \beta_\bullet)_n \circ (\alpha_\bullet^{n,x} \otimes a_\bullet^{n,y})_n(z) \\
&\overset{(81)}{=} (\alpha_\bullet \otimes \beta_\bullet)_n \circ f_n^{X,Y}(x,y) \qquad x \in X_n, y \in Y_n,
\end{aligned}
$$

$$
\begin{aligned}
&g_n^{X',Y'} \circ (\alpha_\bullet \otimes \beta_\bullet)_n(x \otimes y) \overset{*}{=} g_n^{X',Y'}(\alpha_{[p+1]}(x) \otimes \beta_{[n-p+1]}(y)) \overset{(81)}{=} (a^{p,\alpha_{[p+1]}(x)} \otimes' a^{n-p,\beta_{[n-p+1]}(y)})_n(z_p) \\
&\overset{(82)}{=} ((\alpha \circ a^{p,x}) \otimes' (\beta \circ a^{n-p,y}))_n(z_p) \overset{**}{=} (\alpha \otimes' \beta)_n \circ (a^{p,x} \otimes' a^{n-p,y})_n(z_p) \\
&\overset{(81)}{=} (\alpha \otimes' \beta)_n \circ g_n^{X,Y}(x \otimes y) \qquad x \in X_p, y \in Y_{n-p},
\end{aligned}
$$

where we used the definition of $\bullet$ in * and its functoriality in **. This establishes (ii) for $k = n$ and inductively defines natural transformations $f : \bullet\otimes' \to \otimes(\bullet \times \bullet)$ and $g : \otimes(\bullet \times \bullet) \to \bullet\otimes'$.

2. We show that $f_\bullet^{X,Y}$ and $g_\bullet^{X,Y}$ are unique up to chain homotopies natural in X and Y. For this, suppose that $f' : \bullet\otimes' \to \otimes(\bullet \times \bullet)$ and $g' : \otimes(\bullet \times \bullet) \to \bullet\otimes'$ are natural transformations with $f_0'^{X,Y} = g_0'^{X,Y} = \mathrm{id} : X_0 \otimes Y_0 \to Y_0 \otimes X_0$. We inductively construct chain homotopies $h_\bullet^{X,Y} : f_\bullet'^{X,Y} \Rightarrow f_\bullet^{X,Y}$ and $k_\bullet^{X,Y} : g_\bullet'^{X,Y} \Rightarrow g_\bullet^{X,Y}$ for all simplicial objects $X, Y : \Delta^{+op} \to R$-Mod by setting $h_0^{X,Y} = 0 : X_0 \otimes Y_0 \to (X_\bullet \otimes Y_\bullet)_1$ and $k_0^{X,Y} = 0 : X_0 \otimes Y_0 \to X_1 \otimes Y_1$.

Suppose we have R-linear maps $h_l^{X,Y} : (X \otimes' Y)_l \to (X_\bullet \otimes Y_\bullet)_{l+1}$, $k_l^{X,Y} : (X_\bullet \otimes Y_\bullet)_l \to (X \otimes' Y)_{l+1}$ for all $l < n$ and simplicial objects $X, Y : \Delta^{+op} \to R$-Mod such that

(i) $d_{l+1} \circ h_l^{X,Y} + h_{l-1}^{X,Y} \circ d_l = f_l^{X,Y} - f_l'^{X,Y}$ and $d_{l+1} \circ k_l^{X,Y} + k_{l-1}^{X,Y} \circ d_l = g_l^{X,Y} - g_l'^{X,Y}$,
(ii) $h_l^{X',Y'} \circ (\alpha \otimes' \beta)_l = (\alpha_\bullet \otimes \beta_\bullet)_{l+1} \circ h_l^{X,Y}$ and $k_l^{X',Y'} \circ (\alpha_\bullet \otimes \beta_\bullet)_l = (\alpha \otimes' \beta)_{l+1} \circ k_l^{X,Y}$

for all $l < n$ and simplicial morphisms $\alpha : X \to X'$ and $\beta : Y \to Y'$.

To define R-linear maps $h_n^{X,Y} : (X \otimes' Y)_n \to (X_\bullet \otimes Y_\bullet)_{n+1}$ and $k_n^{X,Y} : (X_\bullet \otimes Y_\bullet)_n \to (X \otimes' Y)_{n+1}$ that satisfy (i) and (ii) for $l = n$, we consider again the chain complexes $A_\bullet^n$. As we have

$$
\begin{aligned}
d_n \circ (f_n^{A^n,A^n} - f_n'^{A^n,A^n} - h_{n-1}^{A^n,A^n} \circ d_n) &\overset{(i)}{=} h_{n-2}^{A^n,A^n} \circ d_{n-1} \circ d_n = 0 \\
d_n \circ (g_n^{A^p,A^{n-p}} - g_n'^{A^p,A^{n-p}} - k_{n-1}^{A^p,A^{n-p}} \circ d_n) &\overset{(i)}{=} k_{n-2}^{A^p,A^{n-p}} \circ d_{n-1} \circ d_n = 0
\end{aligned}
$$

and $A_\bullet^n \otimes A_\bullet^n$, $(A^p \otimes' A^{n-p})_\bullet$ are acyclic, there are $w \in (A_\bullet^n \otimes A_\bullet^n)_{n+1}$, $w_p \in (A^p \otimes' A^{n-p})_{n+1}$ with

$$
\begin{aligned}
d_{n+1}(w) &= (f_n^{A^n,A^n} - f_n'^{A^n,A^n} - h_{n-1}^{A^n,A^n} \circ d_n)(1_{[n+1]} \otimes 1_{[n+1]}) \qquad (83) \\
d_{n+1}(w_p) &= (g_n^{A^p,A^{n-p}} - g_n'^{A^p,A^{n-p}} - k_{n-1}^{A^p,A^{n-p}} \circ d_n)(1_{[p+1]} \otimes 1_{[n-p+1]}).
\end{aligned}
$$

For all simplicial objects $X, Y : \Delta^{+op} \to R$-Mod we define

$$
\begin{aligned}
h_n^{X,Y}(x \otimes y) &:= (a_\bullet^{n,x} \otimes a_\bullet^{n,y})_{n+1}(w) \qquad && x \in X_n, y \in Y_n \qquad (84) \\
k_n^{X,Y}(x \otimes y) &:= (a^{p,x} \otimes' a^{n-p,y})_{n+1}(w_p) && x \in X_p, y \in Y_{n-p},
\end{aligned}
$$

where $a^{n,x} : A^n \to X$ is the unique simplicial morphism with $a_{[n+1]}^{n,x}(1_{[n+1]}) = x$ from Proposition

5.5.2. This implies

$$
\begin{aligned}
&d_{n+1}\circ h_n^{X,Y}(x\otimes y) \overset{(84)}{=} d_{n+1}\circ (a_\bullet^{n,x}\otimes a_\bullet^{n,y})_{n+1}(w) \overset{*}{=} (a_\bullet^{n,x}\otimes a_\bullet^{n,y})_n\circ d_{n+1}(w)\\
&\overset{(83)}{=} (a_\bullet^{n,x}\otimes a_\bullet^{n,y})_n\circ (f_n^{A^n,A^n} - f_n'^{A^n,A^n} - h_{n-1}^{A^n,A^n}\circ d_n)(1_{[n+1]}\otimes 1_{[n+1]})\\
&\overset{**}{=} (f_n^{X,Y} - f_n'^{X,Y} - h_{n-1}^{X,Y}\circ d_n)\circ (a^{n,x}\otimes' a^{n,y})_n(1_{[n+1]}\otimes 1_{[n+1]})\\
&= (f_n^{X,Y} - f_n'^{X,Y} - h_{n-1}^{X,Y}\circ d_n)(x\otimes y) \qquad x\in X_n, y\in Y_n,
\end{aligned}
$$

$$
\begin{aligned}
&d_{n+1}\circ k_n^{X,Y}(x\otimes y) \overset{(84)}{=} d_{n+1}\circ (a^{p,x}\otimes' a^{n-p,y})_{n+1}(w_p) \overset{*}{=} (a^{p,x}\otimes' a^{n-p,y})_n\circ d_{n+1}(w_p)\\
&\overset{(83)}{=} (a^{p,x}\otimes' a^{n-p,y})_n\circ (g_n^{A^p,A^{n-p}} - g_n'^{A^p,A^{n-p}} - k_{n-1}^{A^p,A^{n-p}}\circ d_n)(1_{[p+1]}\otimes 1_{[n-p+1]})\\
&\overset{**}{=} (g_n^{X,Y} - g_n'^{X,Y} - k_{n-1}^{X,Y}\circ d_n)\circ (a_\bullet^{p,x}\otimes a_\bullet^{n-p,y})_n(1_{[p+1]}\otimes 1_{[n-p+1]})\\
&= (g_n^{X,Y} - g_n'^{X,Y} - k_{n-1}^{X,Y}\circ d_n)(x\otimes y) \qquad x\in X_p, y\in Y_{n-p},
\end{aligned}
$$

where we used in * and ** that $a_\bullet^{l,x}\otimes a_\bullet^{m,y}$, $(a^{l,x}\otimes' a^{m,y})_\bullet$ are chain maps, in ** the identities (ii) for f, f', h_{n-1}, k_{n-1} and in the last step the definitions of the standard chain complex functor and of $a^{l,x}$ and $a^{m,y}$. This proves (i) for $l=n$.

To show that (ii) holds for $l=n$, we compute for simplicial morphisms $\alpha: X\to X'$, $\beta: Y\to Y'$

$$
\begin{aligned}
&h_n^{X',Y'}\circ(\alpha\otimes'\beta)_n(x\otimes y) \overset{*}{=} h_n^{X',Y'}(\alpha_{[n+1]}(x)\otimes\beta_{[n+1]}(y)) \overset{(84)}{=} (a_\bullet^{n,\alpha_{[n+1]}(x)}\otimes a_\bullet^{n,\beta_{[n+1]}(y)})_{n+1}(w)\\
&\overset{(82)}{=} ((\alpha\circ a^{n,x})_\bullet\otimes(\beta\circ a^{n,y})_\bullet)_{n+1}(w) \overset{**}{=} (\alpha_\bullet\otimes\beta_\bullet)_{n+1}\circ(a_\bullet^{n,x}\otimes a_\bullet^{n,y})_{n+1}(w)\\
&\overset{(84)}{=} (\alpha_\bullet\otimes\beta_\bullet)_{n+1}\circ h_n^{X,Y}(x\otimes y) \qquad x\in X_n, y\in Y_n
\end{aligned}
$$

$$
\begin{aligned}
&k_n^{X',Y'}\circ(\alpha_\bullet\otimes\beta_\bullet)_n(x\otimes y) \overset{*}{=} k_n^{X',Y'}(\alpha_{[p+1]}(x)\otimes\beta_{[n-p+1]}(y)) \overset{(84)}{=} (a^{p,\alpha_{[p+1]}(x)}\otimes' a^{n-p,\beta_{[n+1]}(y)})_{n+1}(w_p)\\
&\overset{(82)}{=} ((\alpha\circ a^{p,x})\otimes'(\beta\circ a^{n-p,y}))_{n+1}(w_p) \overset{**}{=} (\alpha\otimes'\beta)_{n+1}\circ(a^{p,x}\otimes' a^{n-p,y})_{n+1}(w_p)\\
&\overset{(84)}{=} (\alpha\otimes'\beta)_{n+1}\circ k_n^{X,Y}(x\otimes y) \qquad x\in X_p, y\in Y_{n-p},
\end{aligned}
$$

where we used in * the definition of the standard chain complex functor and in ** its functoriality. This shows that (i) and (ii) hold for $n=l$, and inductively we obtain chain homotopies $h_\bullet^{X,Y}: f_\bullet'^{X,Y}\Rightarrow f_\bullet^{X,Y}$ and $k_\bullet^{X,Y}: g_\bullet'^{X,Y}\Rightarrow g_\bullet^{X,Y}$ that are natural in X, Y.

3. We show that for natural transformations $F: \bullet\otimes'\to\bullet\otimes'$ and $G: \otimes(\bullet\times\bullet)\to\otimes(\bullet\times\bullet)$ with $F_0^{X,Y}=G_0^{X,Y}=\mathrm{id}: X_0\otimes Y_0\to X_0\otimes Y_0$ for all simplicial objects $X,Y:\Delta^{+op}\to R$-Mod, the component morphisms $F_\bullet^{X,Y}$ and $G_\bullet^{X,Y}$ are all chain homotopic to the identity chain maps with chain homotopies that are natural in X and Y. The third claim then follows by applying this statement to $F=g\circ f$ and $G=f\circ g$.

For this we inductively construct chain homotopies

$$
h_\bullet^{X,Y}: \mathrm{id}_{(X\otimes' Y)_\bullet}\to F_\bullet^{X,Y} \qquad\qquad k_\bullet^{X,Y}: \mathrm{id}_{X_\bullet\otimes Y_\bullet}\to G_\bullet^{X,Y}
$$

by setting $h_0^{X,Y}=0: X_0\otimes Y_0\to X_1\otimes Y_1$ and $k_0^{X,Y}=0: X_0\otimes Y_0\to (X_\bullet\otimes Y_\bullet)_1$.

Suppose we constructed $h_l^{X,Y}: (X\otimes' Y)_l\to (X\otimes' Y)_{l+1}$ and $k_l^{X,Y}: (X_\bullet\otimes Y_\bullet)_l\to (X_\bullet\otimes Y_\bullet)_{l+1}$ for all simplicial modules $X,Y:\Delta^{+op}\to R$-Mod and $l<n$ such that

(i) $d_{l+1}\circ h_l^{X,Y}+h_{l-1}^{X,Y}\circ d_l = F_l^{X,Y}-\mathrm{id}_{X_l\otimes Y_l}$ and $d_{l+1}\circ k_l^{X,Y}+k_{l-1}^{X,Y}\circ d_l = G_l^{X,Y}-\mathrm{id}_{(X_\bullet\otimes Y_\bullet)_l}$,

(ii) $h_l^{X',Y'} \circ (\alpha \otimes' \beta)_l = (\alpha \otimes' \beta)_{l+1} \circ h_l^{X,Y}$ and $k_l^{X',Y'} \circ (\alpha_\bullet \otimes \beta_\bullet)_l = (\alpha_\bullet \otimes \beta_\bullet)_{l+1} \circ k_l^{X,Y}$

for all $l < n$ and simplicial morphisms $\alpha : X \to X'$ and $\beta : Y \to Y'$.

To define R-linear maps $h_n^{X,Y} : (X \otimes' Y)_n \to (X \otimes' Y)_{n+1}$ and $k_n^{X,Y} : (X_\bullet \otimes Y_\bullet)_n \to (X_\bullet \otimes Y_\bullet)_{n+1}$ that satisfy (i) and (ii) we use again the acyclic complexes $(A^n \otimes' A^n)_\bullet$ and $A^p_\bullet \otimes A^{n-p}_\bullet$. Because these complexes are acyclic and we have

$$d_n \circ (F_n^{A^n,A^n} - \mathrm{id}_{A^n_n \otimes A^n_n} - h_{n-1}^{A^n,A^n} \circ d_n) = h_{n-2}^{A^n,A^n} \circ d_{n-1} \circ d_n = 0$$
$$d_n \circ (G_n^{A^p,A^{n-p}} - \mathrm{id}_n^{(A^p_\bullet \otimes A^{n-p}_\bullet)_n} - k_{n-1}^{A^p,A^{n-p}} \circ d_n) = k_{n-2}^{A^p,A^{n-p}} \circ d_{n-1} \circ d_n = 0,$$

there are $v \in (A^n \otimes' A^n)_{n+1}$ and $v_p \in (A^p_\bullet \otimes A^{n-p}_\bullet)_{n+1}$ with

$$(F_n^{A^n,A^n} - \mathrm{id}_{A^n_n \otimes A^n_n} - h_{n-1}^{A^n,A^n} \circ d_n)(1_{[n+1]} \otimes 1_{[n+1]}) = d_{n+1}(v) \tag{85}$$
$$(G_n^{A^p,A^{n-p}} - \mathrm{id}_n^{(A^p_\bullet \otimes A^{n-p}_\bullet)_n} - k_{n-1}^{A^p,A^{n-p}} \circ d_n) = d_{n+1}(v_p).$$

For all simplicial objects $X, Y : \Delta^{+op} \to R$-Mod we define

$$h_n^{X,Y}(x \otimes y) := (a^{n,x} \otimes' a^{n,y})_{n+1}(v) \qquad x \in X_n, y \in Y_n \tag{86}$$
$$k_n^{X,Y}(x \otimes y) := (a^{p,x}_\bullet \otimes a^{n-p,y}_\bullet)_{n+1}(v_p) \qquad x \in X_p, y \in Y_{n-p},$$

where $a^{n,x} : A^n \to X$ is the unique simplicial morphism with $a^{n,x}_{[n+1]}(1_{[n+1]}) = x$ from Proposition 5.5.2. With this definition, we obtain

$$\begin{aligned}
&d_{n+1} \circ h_n^{X,Y}(x,y) \overset{(86)}{=} d_{n+1} \circ (a^{n,x} \otimes' a^{n,y})_{n+1}(v) \overset{*}{=} (a^{n,x} \otimes' a^{n,y})_n \circ d_{n+1}(v)\\
&\overset{(85)}{=} (a^{n,x} \otimes' a^{n,y})_n \circ (F_n^{A^n,A^n} - \mathrm{id}_{A^n_n \otimes A^n_n} - h_{n-1}^{A^n,A^n} \circ d_n)(1_{[n+1]} \otimes 1_{[n+1]})\\
&\overset{**}{=} (F_n^{X,Y} - \mathrm{id}_{X_n \otimes Y_n} - h_{n-1}^{X,Y} \circ d_n) \circ (a^{n,x} \otimes' a^{n,y})_n(1_{[n+1]} \otimes 1_{[n+1]})\\
&= (F_n^{X,Y} - \mathrm{id}_{X_n \otimes Y_n} - h_{n-1}^{X,Y} \circ d_n)(x \otimes y) \qquad x \in X_n, y \in Y_n
\end{aligned}$$

$$\begin{aligned}
&d_{n+1} \circ k_n^{X,Y}(x,y) \overset{(86)}{=} d_{n+1} \circ (a^{p,x}_\bullet \otimes a^{n-p,y}_\bullet)_{n+1}(v_p) \overset{*}{=} (a^{p,x}_\bullet \otimes a^{n-p,y}_\bullet)_n \circ d_{n+1}(v_p)\\
&\overset{(85)}{=} (a^{p,x}_\bullet \otimes a^{n-p,y}_\bullet)_n \circ (G_n^{A^p,A^{n-p}} - \mathrm{id}_{(A^p_\bullet \otimes A^{n-p}_\bullet)_n} - k_{n-1}^{A^p,A^{n-p}} \circ d_n)(1_{[p+1]} \otimes 1_{[n-p+1]})\\
&\overset{**}{=} (G_n^{X,Y} - \mathrm{id}_{(X_\bullet \otimes Y_\bullet)_n} - k_{n-1}^{X,Y} \circ d_n) \circ (a^{p,x}_\bullet \otimes a^{n-p,y}_\bullet)_n(1_{[p+1]} \otimes 1_{[n-p+1]})\\
&= (G_n^{X,Y} - \mathrm{id}_{(X_\bullet \otimes Y_\bullet)_n} - k_{n-1}^{X,Y} \circ d_n)(x \otimes y) \qquad x \in X_p, y \in Y_{n-p},
\end{aligned}$$

where we used in *, ** that $(a^{l,x} \otimes' a^{m,y})_\bullet$ and $a^{l,x}_\bullet \otimes a^{m,y}_\bullet$ are chain maps, in ** the naturality of F, G, $h_{n-1}^{X,Y}$ and $k_{n-1}^{X,Y}$ and then the definition of the standard chain complex functor and of a^x, a^y. This shows that $h_n^{X,Y}$ and $k_n^{X,Y}$ satisfy (i).

To show that $h_n^{X,Y}$ and $k_n^{X,Y}$ satisfy (ii) for all simplicial morphisms $\alpha : X \to X'$ and $\beta : Y \to Y'$ we compute

$$\begin{aligned}
&h_n^{X',Y'} \circ (\alpha \otimes' \beta)_n(x,y) \overset{*}{=} h_n^{X',Y'}(\alpha_{[n+1]}(x) \otimes \beta_{[n+1]}(y)) \overset{(86)}{=} (a^{n,\alpha_{[n+1]}(x)} \otimes' a^{n,\beta_{[n+1]}(y)})_{n+1}(v)\\
&\overset{(82)}{=} ((\alpha \circ a^{n,x}) \otimes' (\beta \circ a^{n,y}))_{n+1}(v) \overset{**}{=} (\alpha \otimes' \beta)_{n+1} \circ (a^{n,x} \otimes' a^{n,y})_{n+1}(v)\\
&\overset{(86)}{=} (\alpha \otimes' \beta)_{n+1} \circ h_n^{X,Y}(x \otimes y) \qquad x \in X_n, y \in Y_n
\end{aligned}$$

$$\begin{aligned}
&k_n^{X',Y'} \circ (\alpha_\bullet \otimes \beta_\bullet)_n(x,y) \overset{*}{=} k_n^{X',Y'}(\alpha_{[p+1]}(x) \otimes \beta_{[n-p+1]}(y)) \overset{(86)}{=} (a_\bullet^{p,\alpha_{[p+1]}(x)} \otimes a_\bullet^{n-p,\beta_{[n-p+1]}(y)})_{n+1}(v_p)\\
&\overset{(82)}{=} ((\alpha \circ a^{p,x})_\bullet \otimes (\beta \circ a^{n-p,y})_\bullet)_{n+1}(v_p) \overset{**}{=} (\alpha_\bullet \otimes \beta_\bullet)_{n+1} \circ (a^{p,x}_\bullet \otimes a^{n-p,y}_\bullet)_{n+1}(v_p)\\
&\overset{(86)}{=} (\alpha_\bullet \otimes \beta_\bullet)_{n+1} \circ k_n^{X,Y}(x \otimes y) \qquad x \in X_p, y \in Y_{n-p}
\end{aligned}$$

where we used in * the definition of the standard chain complex functor and in ** its functoriality. By induction the maps $h_n^{X,Y}$ and $k_n^{X,Y}$ then define chain homotopies $h_\bullet^{X,Y} : \mathrm{id}_{(X\otimes' Y)_\bullet} \Rightarrow F_\bullet^{X,Y}$ and $k_\bullet^{X,Y} : \mathrm{id}_{X_\bullet \otimes Y_\bullet} \Rightarrow G_\bullet^{X,Y}$ that are natural in X and Y. $\square$

The Eilenberg-Zilber Theorem does not give a concrete formula for the natural transformations f and g, although such a formula could be derived from their inductive construction. However, the existence of these natural transformations and the statement that the maps $f_\bullet^{X,Y} : (X\otimes Y)_\bullet \to X_\bullet \otimes Y_\bullet$ and $g_\bullet^{X,Y} : X_\bullet \otimes Y_\bullet \to (X\otimes Y)_\bullet$ form chain homotopy equivalence for all simplicial objects X, Y in R-Mod are sufficient to draw conclusions about the homologies.

Corollary 5.5.5: Let $X, Y : \Delta^{+op} \to R$-Mod simplicial objects in R-Mod. Then for all $n \in \mathbb{N}_0$

$$H_n((X\otimes' Y)_\bullet) \cong H_n(X_\bullet \otimes Y_\bullet)$$

Example 5.5.6:

1. By Example 5.1.7 the Hochschild complex of a k-algebra A with values in an (A, A)-bimodule M is the standard chain complex of a simplicial object $F^{A,M} : \Delta^{+op} \to k$-Mod with $F^{A,M}([n+1]) = M\otimes_k A^{\otimes n}$.

   Given two k-algebras A and B, an (A, A)-bimodule M and a (B, B)-bimodule N, we have a canonical $(A\otimes B, A\otimes B)$-bimodule structure on $M\otimes N$ with

   $$(a\otimes b) \rhd (m\otimes n) = (a \rhd m)\otimes(b \rhd n) \qquad (m\otimes n) \lhd (a\otimes b) = (m \lhd a)\otimes(n \lhd b),$$

   and the k-linear maps

   $$\begin{aligned} \phi_j :&(M\otimes_k A^{\otimes j})\otimes_k(N\otimes_k B^{\otimes j}) \to (M\otimes_k N)\otimes_k(A\otimes_k B)^{\otimes j}, \\ &(m\otimes a_1\otimes ...\otimes a_j)\otimes(n\otimes b_1\otimes ...\otimes b_j) \mapsto (m\otimes n)\otimes(a_1\otimes b_1)\otimes ...\otimes(a_j\otimes b_j) \end{aligned}$$

   define a natural isomorphism $\phi : F^{A,M}\otimes' F^{B,N} \to F^{A\otimes_k B, M\otimes_k N}$.

   Corollary 5.5.5 states that the Hochschild homology of $A\otimes_k B$ with values in $M\otimes_k N$ is the homology of the tensor product of the Hochschild complex of A with values in M and the Hochschild complex of B with values in N:

   $$H_n(A\otimes_k B, M\otimes_k N) = H_n(F_\bullet^{A\otimes_k B, M\otimes_k N}) \cong H_n(F_\bullet^{A,M}\otimes F_\bullet^{B,N})$$

   By specialising this to group algebras $A = k[G]$ and $B = k[H]$ and $k[G]$ and an $k[H]$-right modules M, N with the trivial $k[G]$- and $k[H]$-left module structure, one obtains an analogous statment for group homologies.

2. By Example 5.1.8 the singular chain complex for a topological space X with values in k is the standard chain complex $C_\bullet^X$ of the simplicial object $C^X : \Delta^{+op} \to k$-Mod with

   $$\begin{aligned} &C^X([n+1]) = \langle \mathrm{Hom}_{\mathrm{Top}}(\Delta^n, X)\rangle_k, \\ &C^X(\delta_n^i) : C_n^X \to C_{n-1}^X, \sigma \mapsto \sigma \circ f_i^n, \qquad C_n^X(\sigma_n^i) : C_n^X \to C_{n+1}^X, \sigma \mapsto \sigma \circ s_i^n. \end{aligned}$$

   By the universal property of the product topology we have

   $$\mathrm{Hom}_{\mathrm{Top}}(\Delta^n, X\times Y) \cong \mathrm{Hom}_{\mathrm{Top}}(\Delta^n, X) \times \mathrm{Hom}_{\mathrm{Top}}(\Delta^n, Y).$$

With the universal properties of the free R-modules and the tensor product we then obtain a natural isomorphism $\eta : C^{X\times Y} \to C^X \otimes' C^Y$. Corollary 5.5.5 states that the singular homologies of the product space $X \times Y$ are the homologies of the tensor product of the singular chain complex of X and the singular chain complex of Y:

$$H_n(X \times Y, k) = H_n(C_\bullet^{X\times Y}) \cong H_n((C^X \otimes' C^Y)_\bullet) \cong H_n(C_\bullet^X \otimes C_\bullet^Y).$$

The Künneth formula for chain complexes from Theorem 4.6.3 then implies Theorem 4.6.6: for all commutative rings k and topological spaces X, Y there is a short exact sequence

$$0 \to \bigoplus_{j=0}^{n} H_j(X,k) \otimes_k H_{n-j}(Y,k) \to H_n(X\times Y, k) \to \bigoplus_{j=0}^{n-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_j(X,k), H_{n-j-1}(Y,k)) \to 0.$$

Although the abstract formulation in the Eilenberg-Zilber Theorem is sufficient to construct isomorphisms between the homologies, some applications require concrete expressions for the natural transformations $f : \bullet\otimes' \to \otimes(\bullet \times \bullet)$ and $g : \otimes(\bullet \times \bullet) \to \bullet\otimes'$ in Theorem 5.5.4. In particular, these are needed to investigate their interaction with the associativity constraints in the monoidal categories $\mathrm{Fun}(\Delta^{+op}, R\text{-Mod})$ and $\mathrm{Ch}_{R\text{-Mod}}$. The component morphisms of the natural transformations f and g in Theorem 5.5.4 are unique only up to natural chain homotopies. Up to this, they are given by the *Alexander-Whitney* and *Eilenberg-Zilber maps*.

Definition 5.5.7: Let $X, Y : \Delta^{+op} \to R$-Mod be simplicial objects and $n \in \mathbb{N}_0$.

1. The nth **Alexander-Whitney map** for X, Y is the R-linear map

$$f_n^{X,Y} : X_n \otimes Y_n \to \oplus_{p=0}^n X_p \otimes Y_{n-p}, \quad x \otimes y \mapsto \sum_{p=0}^{n} X(\phi_n^p)x \otimes Y(\psi_n^p)y,$$

with $\phi_n^p : [p+1] \to [n+1]$, $i \mapsto i$ and $\psi_n^p : [n-p+1] \to [n+1]$, $i \mapsto i+p$.

2. The nth **Eilenberg-Zilber map** for X, Y is the R-linear map

$$\begin{aligned} g_n^{X,Y} &: \oplus_{p=0}^n X_p \otimes Y_{n-p} \to X_n \otimes Y_n \\ &X_p \otimes Y_{n-p} \ni x \otimes y \mapsto \sum_{\pi \in \mathrm{Sh}(p,n-p)} \mathrm{sgn}(\pi) X(\sigma_{p+1}^{\pi(p)} \circ \ldots \circ \sigma_n^{\pi(n-1)})x \otimes Y(\sigma_{n-p+1}^{\pi(0)} \circ \ldots \circ \sigma_n^{\pi(p-1)})y, \end{aligned}$$

where $\mathrm{Sh}(p, n-p)$ is the set of $(p, n-p)$-**shuffle permutations** of $[n] = \{0, ..., n-1\}$:

$$\mathrm{Sh}(p, n-p) = \{\pi \in \mathrm{Perm}_{[n]} \mid \pi(0) < \pi(1) < \ldots < \pi(p-1), \pi(p) < \pi(p+1) < \ldots < \pi(n-1)\}.$$

The $(p, n-p)$-shuffles partition the set $[n]$ into the subsets $\{0, .., p-1\}$ and $\{p, ..., n-1\}$ and mix these subsets in any way that preserves the order in each subset. This is precisely what happens when one shuffles a deck of cards: one partitions it into two stacks and then merges them, in such a way that the order of cards is preserved within each stack. This explains the name *shuffle permutations*.

Note also that the $(p, n-p)$ shuffles in the Eilenberg-Zilber map have a geometric interpretation. They describe the subdivision of a product $\Delta^p \times \Delta^{n-p}$ of two standard simplexes into n-simplexes. Every affine linear map $f : \Delta^n \to \Delta^p \times \Delta^{n-p}$ that sends vertices to vertices and respects the order of the vertices in Δ^p and in Δ^{n-p} is given by a permutation $\pi \in S_n$ with $\pi(0) < ... < \pi(p-1)$ and $\pi(p) < ... < \pi(n-1)$. Such affine linear maps generalise the prism maps introduced in Proposition 3.3.6, which correspond to the product $\Delta^1 \times \Delta^{n-1}$ and to $(1, n-1)$-shuffles.

Theorem 5.5.8: The Alexander-Whitney and the Eilenberg-Zilber maps define natural transformations $f : \bullet\otimes' \to \otimes(\bullet\times\bullet)$ and $g : \otimes(\bullet\times\bullet) \to \bullet\otimes'$ with the properties in the Eilenberg-Zilber Theorem 5.5.4.

Proof. 1. The Alexander-Whitney and Eilenberg Zilber map satisfy $f_0^{X,Y} = g_0^{X,Y} = \mathrm{id}_{X_0\otimes Y_0}$. Their naturality follows, because they are defined in terms of simplicial objects and morphisms in the simplex category. For simplicial morphisms $\alpha : X \to X'$ and $\beta : Y \to Y'$, we have

$$\begin{aligned}
&f_n^{X',Y'} \circ (\alpha\otimes'\beta)_{[n+1]} = \Sigma_{p=0}^n \,(X'(\phi_n^p) \circ \alpha_{[n+1]})\otimes(Y'(\psi_n^p) \circ \beta_{[n+1]}) \\
&\overset{\text{nat}}{=} \Sigma_{p=0}^n(\alpha_{[p+1]} \circ X(\phi_n^p))\otimes(\beta_{[n-p+1]} \circ Y(\psi_n^p)) = (\alpha_\bullet\otimes\beta_\bullet)_n \circ f_n^{X,Y}
\end{aligned}$$

$$\begin{aligned}
&g_n^{X',Y'} \circ (\alpha_\bullet\otimes\beta_\bullet)_n \circ \iota_p = g_n^{X',Y'} \circ (\alpha_{[p+1]}\otimes\beta_{[n-p+1]}) \circ \iota_p \\
&= \Sigma_{\pi\in\mathrm{Sh}(p,n-p)}\mathrm{sgn}(\pi)(X'(\sigma_{p+1}^{\pi(p)} \circ ... \circ \sigma_n^{\pi(n-1)}) \circ \alpha_{[p+1]})\otimes(Y'(\sigma_{n-p+1}^{\pi(0)} \circ ... \circ \sigma_n^{\pi(p-1)}) \circ \beta_{[n-p+1]}) \circ \iota_p \\
&\overset{\text{nat}}{=} \Sigma_{\pi\in\mathrm{Sh}(p,n-p)}\mathrm{sgn}(\pi)(\alpha_{[n+1]} \circ X(\sigma_{p+1}^{\pi(p)} \circ ... \circ \sigma_n^{\pi(n-1)}))\otimes(\beta_{[n+1]} \circ Y(\sigma_{n-p+1}^{\pi(0)} \circ ... \circ \sigma_n^{\pi(p-1)})) \circ \iota_p \\
&= (\alpha\otimes'\beta)_{[n+1]} \circ g_n^{X,Y} \circ \iota_p,
\end{aligned}$$

where $\iota_p : X_p\otimes Y_{n-p} \to (X_\bullet\otimes Y_\bullet)_n$ is the inclusion for the direct sum.

2. It remains to show that the Alexander-Whitney and Eilenberg-Zilber maps are chain maps. By construction of $f : \bullet\otimes' \to \otimes(\bullet \times \bullet)$ and $g : \otimes(\bullet \times \bullet) \to \bullet\otimes'$ in the proof of Theorem 5.5.4 and by the naturality of the Alexander-Whitney and Eilenberg-Zilber map from step 1, it is sufficient to prove that for all $n \in \mathbb{N}_0$ and $p \in \{0, ..., n\}$

$$\begin{aligned}
d_n \circ f_n^{A^n,A^n}(1_{[n+1]}\otimes 1_{[n+1]}) &= f_{n-1}^{A^n,A^n} \circ d_n(1_{[n+1]}\otimes 1_{[n+1]}) \\
d_n \circ g_n^{A^p,A^{n-p}}(1_{[p+1]}\otimes 1_{[n-p+1]}) &= g_{n-1}^{A^p,A^{n-p}} \circ d_n(1_{[p+1]}\otimes 1_{[n-p+1]}),
\end{aligned}$$

where $A^n_\bullet$ is the acyclic chain complex from (77) with the universal property from Proposition 5.5.2. To prove the claim for f, we use the following identities for the maps $\phi_n^p : [p+1] \to [n+1]$, $i \mapsto i$ and $\psi_n^p : [n-p+1] \to [n+1]$, $i \mapsto i+p+1$ from Definition 5.5.7, which follow by a direct computation

$$\delta_n^i \circ \phi_{n-1}^p = \begin{cases} \phi_n^{p+1} \circ \delta_{p+1}^i & 0 \leq i \leq p \\ \phi_n^p & p < i \leq n \end{cases} \qquad \delta_n^i \circ \psi_{n-1}^p = \begin{cases} \psi_n^{p+1} & 0 \leq i \leq p \\ \psi_n^p \circ \delta_{n-p}^{i-p} & p < i \leq n. \end{cases} \tag{87}$$

With these identities, the definition of the chain complex $A^n_\bullet$ in (77) and the expression for the

Alexander-Whitney map in Definition 5.5.7 we compute

$$\begin{aligned}
&d_n \circ f_n^{A_n,A_n}(1_{[n+1]}\otimes 1_{[n+1]}) \overset{\text{Def 5.5.7}}{=} \Sigma_{p=0}^n d_n(\phi_n^p\otimes\psi_n^p) = \Sigma_{p=0}^n d_p(\phi_n^p)\otimes\psi_n^p + (-1)^p\phi_n^p\otimes d_{n-p}(\psi_n^p)\\
&\overset{(77)}{=} \Sigma_{p=1}^n\Sigma_{i=0}^p(-1)^i(\phi_n^p\circ\delta_p^i)\otimes\psi_n^p + \Sigma_{p=0}^{n-1}\Sigma_{i=0}^{n-p}(-1)^{p+i}\phi_n^p\otimes(\psi_n^p\circ\delta_{n-p}^i)\\
&= \Sigma_{p=1}^n\Sigma_{i=0}^p(-1)^i(\phi_n^p\circ\delta_p^i)\otimes\psi_n^p + \Sigma_{p=0}^{n-1}\Sigma_{i=p}^n(-1)^i\phi_n^p\otimes(\psi_n^p\circ\delta_{n-p}^{i-p})\\
&\overset{*}{=} \Sigma_{p=1}^n\Sigma_{i=0}^{p-1}(-1)^i(\phi_n^p\circ\delta_p^i)\otimes\psi_n^p + \Sigma_{p=0}^{n-1}\Sigma_{i=p+1}^n(-1)^i\phi_n^p\otimes(\psi_n^p\circ\delta_{n-p}^{i-p})\\
&= \Sigma_{p=0}^{n-1}\Sigma_{i=0}^p(-1)^i(\phi_n^{p+1}\circ\delta_{p+1}^i)\otimes\psi_n^{p+1} + \Sigma_{p=0}^{n-1}\Sigma_{i=p+1}^n(-1)^i\phi_n^p\otimes(\psi_n^p\circ\delta_{n-p}^{i-p})\\
&\overset{(87)}{=} \Sigma_{p=0}^{n-1}\Sigma_{i=0}^n(-1)^i(\delta_n^i\circ\phi_{n-1}^p)\otimes(\delta_n^i\circ\psi_{n-1}^p) \overset{(77)}{=} \Sigma_{p=0}^{n-1}\Sigma_{i=0}^n(-1)^i(A^n(\phi_{n-1}^p)\delta_n^i)\otimes(A^n(\psi_{n-1}^p)\delta_n^i)\\
&\overset{\text{Def 5.5.7}}{=} \Sigma_{i=0}^n(-1)^i f_{n-1}^{A^n,A^n}\circ(\delta_n^i\otimes\delta_n^i) \overset{(77)}{=} f_{n-1}^{A^n,A^n}\circ d_n(1_{[n+1]}\otimes 1_{[n+1]}),
\end{aligned}$$

where we used in * the identity $(\phi_n^p\circ\delta_p^p)\otimes\psi_n^p = \phi_n^{p-1}\otimes(\psi_n^{p-1}\circ\delta_{n-p}^0)$ that makes the terms for $i=p$ in the first summand and for $i=p$ in the second summand cancel. This shows that the Alexander-Whitney maps define a chain map.

To prove the corresponding identity for the Eilenberg-Zilber map, we compute with the chain complex $A_\bullet^n$ from (77) and the expression for the Eilenberg-Zilber map in Definition 5.5.7

$$\begin{aligned}
&d_n\circ g_n^{A^p,A^{n-p}}(1_{[p+1]}\otimes 1_{[n-p+1]}) &(88)\\
&\overset{\text{Def 5.5.7}}{=} \Sigma_{\pi\in\mathrm{Sh}(p,n-p)}\mathrm{sgn}(\pi)d_n(\sigma_{p+1}^{\pi(p)}\circ\ldots\circ\sigma_n^{\pi(n-1)})\otimes d_n(\sigma_{n-p+1}^{\pi(0)}\circ\ldots\circ\sigma_n^{\pi(p-1)})\\
&\overset{(77)}{=} \Sigma_{i=0}^n\Sigma_{\pi\in\mathrm{Sh}(p,n-p)}(-1)^i\mathrm{sgn}(\pi)(\sigma_{p+1}^{\pi(p)}\circ\ldots\circ\sigma_n^{\pi(n-1)}\circ\delta_n^i)\otimes(\sigma_{n-p+1}^{\pi(0)}\circ\ldots\circ\sigma_n^{\pi(p-1)}\circ\delta_n^i).
\end{aligned}$$

An analogous computation yields

$$\begin{aligned}
&g_{n-1}^{A^p,A^{n-p}}\circ d_n(1_{[p+1]}\otimes 1_{[n-p+1]}) &(89)\\
&\overset{(77)}{=} \Sigma_{k=0}^p(-1)^k g_{n-1}^{A^p,A^{n-p}}(\delta_p^k\otimes 1_{[n-p+1]}) + \Sigma_{k=0}^{n-p}(-1)^{k+p}g_{n-1}^{A^p,A^{n-p}}(1_{[p+1]}\otimes\delta_{n-p}^k)\\
&\overset{\text{Def 5.5.7}}{=} \Sigma_{k=0}^p\Sigma_{\tau\in\mathrm{Sh}(p-1,n-p)}(-1)^k\mathrm{sgn}(\tau)(\delta_p^k\circ\sigma_p^{\tau(p-1)}\circ\ldots\circ\sigma_{n-1}^{\tau(n-2)})\otimes(\sigma_{n-p+1}^{\tau(0)}\circ\ldots\circ\sigma_{n-1}^{\tau(p-2)})\\
&\qquad + \Sigma_{k=0}^{n-p}\Sigma_{\tau\in\mathrm{Sh}(p,n-p-1)}(-1)^{k+p}\mathrm{sgn}(\tau)(\sigma_{p+1}^{\tau(p)}\circ\ldots\circ\sigma_{n-1}^{\tau(n-2)})\otimes(\delta_{n-p}^k\circ\sigma_{n-p}^{\tau(0)}\circ\ldots\circ\sigma_{n-1}^{\tau(p-1)})\\
&= \Sigma_{k=0}^p\Sigma_{\tau\in\mathrm{Sh}(p-1,n-p)}(-1)^k\mathrm{sgn}(\tau)(\delta_p^k\circ\sigma_p^{\tau(p-1)}\circ\ldots\circ\sigma_{n-1}^{\tau(n-2)})\otimes(\sigma_{n-p+1}^{\tau(0)}\circ\ldots\circ\sigma_{n-1}^{\tau(p-2)})\\
&+ \Sigma_{k=p}^n\Sigma_{\tau\in\mathrm{Sh}(p,n-p-1)}(-1)^k\mathrm{sgn}(\tau)(\sigma_{p+1}^{\tau(p)}\circ\ldots\circ\sigma_{n-1}^{\tau(n-2)})\otimes(\delta_{n-p}^{k-p}\circ\sigma_{n-p}^{\tau(0)}\circ\ldots\circ\sigma_{n-1}^{\tau(p-1)}).
\end{aligned}$$

To establish the identity for the Eilenberg-Zilber map, we have to transform the last line in (88) into the last line of (89). This is achieved with formula (59), which allows one to move the face map δ_n^i past the degeneracies in (88).

For this, note that by (59) the canonical factorisation (58) of the morphism $\sigma_{n-k+1}^{j_1}\circ\ldots\circ\sigma_n^{j_k}\circ\delta_n^i$ in Δ^+ involves only degeneracies if $\{i,i-1\}\cap\{j_1,...,j_k\}\neq\emptyset$. Otherwise, it involves exactly one face map and k degeneracies. We thus distinguish the cases in which degeneracies with upper indices i and $i-1$ occur both in the first factor, both in the second factor or in different factors in the tensor products in (88). The case $i=0$ has to be treated separately using the observation that in this case one either has $\pi(0)=0$ or $\pi(p)=0$ for each shuffle permutation $\pi\in\mathrm{Sh}(p,n-p)$ in (88). This leads to the following five cases:

(i) If $i = 0 = \pi(0)$, we obtain

$$\begin{aligned}
&\big(\sigma_{p+1}^{\pi(p)} \circ ... \circ \sigma_n^{\pi(n-1)} \circ \delta_n^0\big) \otimes \big(\sigma_{n-p+1}^{\pi(0)} \circ ... \circ \sigma_n^{\pi(p-1)} \circ \delta_n^0\big)\\
&\overset{(59)}{=} \big(\delta_p^0 \circ \sigma_p^{\pi(p)-1} \circ ... \circ \sigma_{n-1}^{\pi(n-1)-1}\big) \otimes \big(\sigma_{n-p}^{\pi(1)-1} \circ ... \circ \sigma_{n-1}^{\pi(p-1)-1}\big)\\
&= \big(\delta_p^0 \circ \sigma_p^{\tau(p-1)} \circ ... \circ \sigma_{n-1}^{\tau(n-2)}\big) \otimes \big(\sigma_{n-p+1}^{\tau(0)} \circ ... \circ \sigma_{n-1}^{\tau(p-2)}\big),
\end{aligned}$$

where $\tau : \{0, ..., n-2\} \to \{0, ..., n-2\}$ is the $(p-1, n-p)$ shuffle with $\tau(j) = \pi(j+1)-1$ for $j \in \{0, ..., n-2\}$ and $\mathrm{sgn}(\tau) = \mathrm{sgn}(\pi)$.

(ii) If $i = 0 = \pi(p)$, we obtain analogously

$$\begin{aligned}
&\big(\sigma_{p+1}^{\pi(p)} \circ ... \circ \sigma_n^{\pi(n-1)} \circ \delta_n^0\big) \otimes \big(\sigma_{n-p+1}^{\pi(0)} \circ ... \circ \sigma_n^{\pi(p-1)} \circ \delta_n^0\big)\\
&\overset{(59)}{=} \big(\sigma_{p-1}^{\pi(p-1)-1} \circ ... \circ \sigma_{n-1}^{\pi(n-1)-1}\big) \otimes \big(\delta_{n-p}^0 \circ \sigma_{n-p}^{\pi(0)-1} \circ ... \circ \sigma_{n-1}^{\pi(p-1)-1}\big)\\
&= \big(\sigma_{p-1}^{\tau(p-1)} \circ ... \circ \sigma_{n-1}^{\tau(n-2)}\big) \otimes \big(\delta_{n-p}^0 \circ \sigma_{n-p}^{\tau(0)} \circ ... \circ \sigma_{n-1}^{\tau(p-2)}\big),
\end{aligned}$$

where $\tau : \{0, ..., n-2\} \to \{0, ..., n-2\}$ is the $(p, n-p-1)$ shuffle with $\tau(j) = \pi(j) - 1$ for $j < p$, $\tau(j) = \pi(j+1) - 1$ for $j \geq p$ and $\mathrm{sgn}(\tau) = \mathrm{sgn}(\pi)(-1)^p$.

(iii) If $i \neq 0$ and the degeneracies with upper indices i and $i-1$ occur in different factors of the tensor products in (88), then there are $k \in \{0, ..., p-1\}$ and $l \in \{p, ..., n-1\}$ with $\{\pi(k), \pi(l)\} = \{i-1, i\}$. Then $\rho := (i, i-1) \circ \pi$ is also a $(p, n-p)$ shuffle permutation with $\mathrm{sgn}(\rho) = -\mathrm{sgn}(\pi)$ and

$$\begin{aligned}
&\big(\sigma_{p+1}^{\pi(p)} \circ ... \circ \sigma_n^{\pi(n-1)} \circ \delta_n^i\big) \otimes \big(\sigma_{n-p+1}^{\pi(0)} \circ ... \circ \sigma_n^{\pi(p-1)} \circ \delta_n^i\big)\\
&= \big(\sigma_{p+1}^{\rho(p)} \circ ... \circ \sigma_n^{\rho(n-1)} \circ \delta_n^i\big) \otimes \big(\sigma_{n-p+1}^{\rho(0)} \circ ... \circ \sigma_n^{\rho(p-1)} \circ \delta_n^i\big)\\
&\overset{(59)}{=} \big(\sigma_{p+1}^{\pi(p)} \circ ... \circ \sigma_{l-1}^{\pi(l-1)} \circ \sigma_l^{\pi(l+1)-1} \circ ... \circ \sigma_{n-1}^{\pi(n-1)-1}\big) \otimes \big(\sigma_{n-p+1}^{\pi(0)} \circ ... \circ \sigma_{n-p+k-1}^{\pi(k-1)} \circ \sigma_{n-p+k}^{\pi(k+1)-1} \circ ... \circ \sigma_{n-1}^{\pi(p-1)-1}\big).
\end{aligned}$$

If follows that the terms for π and ρ in (88) cancel.

(iv) If $i \neq 0$ with $0 \leq \pi^{-1}(i-1) < \pi^{-1}(i) =: k \leq p-1$, then there is a unique $l \in \{p, ..., n-1\}$ with $\pi(l) < i-1 < i < \pi(l+1)$, namely $l = i + p - k$. We then obtain

$$\begin{aligned}
&\big(\sigma_{p+1}^{\pi(p)} \circ ... \circ \sigma_n^{\pi(n-1)} \circ \delta_n^i\big) \otimes \big(\sigma_{n-p+1}^{\pi(0)} \circ ... \circ \sigma_n^{\pi(p-1)} \circ \delta_n^i\big)\\
&\overset{(59)}{=} \big(\delta_p^{i-l+p} \circ \sigma_p^{\pi(p)} \circ ... \circ \sigma_{l-1}^{\pi(l)} \circ \sigma_{l+1}^{\pi(l+1)-1} \circ ... \circ \sigma_{n-1}^{\pi(n-1)-1}\big)\\
&\otimes \big(\sigma_{n-p+1}^{\pi(0)} \circ ... \circ \sigma_{n-p+k}^{\pi(k-1)} \circ \sigma_{n-p+k+1}^{\pi(k+1)-1} \circ ... \circ \sigma_{n-1}^{\pi(p-1)-1}\big)\\
&= \big(\delta_p^k \circ \sigma_p^{\tau(p-1)} \circ ... \circ \sigma_{n-1}^{\tau(n-2)}\big) \otimes \big(\sigma_{n-p+1}^{\tau(0)} \circ ... \circ \sigma_{n-1}^{\tau(p-2)}\big)
\end{aligned}$$

where $\tau : \{0, ..., n-2\} \to \{0, ..., n-2\}$ is the $(p-1, n-p)$ shuffle permutation with

$$\tau(j) = \begin{cases} \pi(j) & 0 \leq j < k \\ \pi(j+1) - 1 & k \leq j < p \text{ or } l \leq j \leq n-2 \\ \pi(j+1) & p \leq j < l. \end{cases}$$

Note that this implies $\mathrm{sgn}(\tau) = (-1)^{i-k}\mathrm{sgn}(\pi)$.

(v) If $i \neq 0$ with $p \leq \pi^{-1}(i-1) < \pi^{-1}(i) =: k \leq n-1$, then there is a unique $l \in \{0, ..., p-1\}$ with $\pi(l) < i-1 < i < \pi(l+1)$, namely $l = i-k+p-1$, and we have

$$\begin{aligned}
&\big(\sigma_{p+1}^{\pi(p)} \circ ... \circ \sigma_n^{\pi(n-1)} \circ \delta_n^i\big) \otimes \big(\sigma_{n-p+1}^{\pi(0)} \circ ... \circ \sigma_n^{\pi(p-1)} \circ \delta_n^i\big) \\
\overset{(59)}{=} &\big(\sigma_{p+1}^{\pi(p)} \circ ... \circ \sigma_{k-2}^{\pi(k-1)} \circ \sigma_{k+1}^{\pi(k+1)-1} \circ ... \circ \sigma_{n-1}^{\pi(n-1)-1}\big) \\
\otimes &\big(\delta_{n-p}^{i-l-1} \circ \sigma_{n-p}^{\pi(0)} \circ ... \circ \sigma_{n-p+l}^{\pi(l)} \circ \sigma_{n-p+l+1}^{\pi(l+1)-1} \circ ... \sigma_{n-1}^{\pi(p-1)-1}\big) \\
= &\big(\sigma_{p+1}^{\tau(p)} \circ ... \circ \sigma_{n-1}^{\tau(n-2)}\big) \otimes \big(\delta_{n-p}^{k-p} \circ \sigma_{n-p}^{\tau(0)} \circ ... \circ \sigma_{n-1}^{\tau(p-1)}\big),
\end{aligned}$$

where $\tau : \{0, ..., n-2\} \to \{0, ..., n-2\}$ is the $(p, n-p-1)$ shuffle with

$$\tau(j) = \begin{cases} \pi(j) & 0 \leq j \leq l \text{or} p \leq j < k \\ \pi(j)-1 & l < j \leq p-1 \\ \pi(j-1)-1 & k \leq j \leq n-1, \end{cases}$$

and $\text{sgn}(\tau) = (-1)^{k-i}\text{sgn}(\pi)$.

Splitting the summation in the last line of (88) into these five cases, inserting in each case the expression for the tensor product and for $\text{sgn}(\pi)$ as a function of $\text{sgn}(\tau)$ yields the last line in (89). This proves that the Eilenberg-Zilber maps are chain maps. □

We now show that the Alexander-Whitney and the Eilenberg-Zilber maps equip the standard chain complex functor with the structure of a lax and op-lax monoidal functor. In addition to the natural transformations $g : \otimes(\bullet \times \bullet) \to \bullet\otimes'$ and $f : \bullet\otimes' \to \otimes(\bullet \times \bullet)$ this requires chain maps $\kappa_\bullet : E_\bullet \to E'_\bullet$ and $\rho_\bullet : E'_\bullet \to E_\bullet$, where E' and $E_\bullet$ are the tensor units in the categories of simplicial objects and in the category of chain complexes, respectively. The former is the simplicial object $E' : \Delta^{+op} \to R$-Mod that sends every object to R and every morphism to id_R. The latter is the chain complex that has entry R in degree 0 and 0 elsewhere. Thus we have

$$E_\bullet = 0 \to R \to 0 \qquad\qquad E'_\bullet = \ldots \to R \xrightarrow{0} R \xrightarrow{\text{id}} R \xrightarrow{0} R \xrightarrow{\text{id}} R \xrightarrow{0} R \to 0.$$

Obvious candidates for the chain maps $\kappa_\bullet$ and $\rho_\bullet$ are

$$\begin{aligned}
&\kappa_\bullet : E_\bullet \to E'_\bullet, \qquad && \kappa_0 = \text{id}_R,\ \kappa_n = 0 : 0 \to R \text{ for } n \in \mathbb{N} \\
&\rho_\bullet : E'_\bullet \to E_\bullet, && \rho_0 = \text{id}_R,\ \rho_n = 0 : R \to 0 \text{ for } n \in \mathbb{N}.
\end{aligned} \tag{90}$$

By combining them with the Alexander-Whitney and Eilenberg-Zilber maps we then obtain a lax and op-lax monoidal structure for the standard chain complex functor.

Theorem 5.5.9: The standard chain complex functor $\bullet : \text{Fun}(\Delta^{+op}, R\text{-Mod}) \to \text{Ch}_{R\text{-Mod}\geq 0}$ is

- lax monoidal with the natural transformation $g : \otimes(\bullet \times \bullet) \to \bullet\otimes'$ defined by the Eilenberg-Zilber maps and the chain map $\kappa_\bullet : E_\bullet \to E'_\bullet$ from (90),
- op-lax monoidal with the natural transformation $f : \bullet\otimes' \to \otimes(\bullet \times \bullet)$ defined by the Alexander-Whitney maps and the chain map $\rho_\bullet : E'_\bullet \to E_\bullet$ from (90).

The component morphisms of the structure morphisms f, g and the structure morphisms $\kappa_\bullet, \rho_\bullet$ form chain homotopy equivalences.

Proof. By Theorem 5.5.8 the Eilenberg-Zilber maps define a natural transformation $g : \otimes(\bullet \times \bullet) \to \bullet\otimes'$ and the Alexander-Whitney maps define a natural transformation $f : \bullet\otimes' \to \otimes(\bullet \times \bullet)$ whose component morphisms are chain homotopy equivalences.

A direct computation shows that $\kappa_\bullet$ and $\rho_\bullet$ in (90) are indeed chain maps and that they satisfy $\rho_\bullet \circ \kappa_\bullet = \mathrm{id}_\bullet : E_\bullet \to E_\bullet$. The R-linear maps $h_n = \mathrm{id}_R : R \to R$ define a chain homotopy from $\kappa_\bullet \circ \rho_\bullet : E'_\bullet \to E'_\bullet$ to $\mathrm{id}_\bullet : E'_\bullet \to E'_\bullet$, as we have

$$d_{n+1} \circ h_n + h_{n-1} \circ d_n = \left\{ \begin{array}{ll} 0 & n = 0 \\ \mathrm{id}_R & n > 0 \text{ even} \\ \mathrm{id}_R & n \text{ odd} \end{array} \right\} = \mathrm{id}_n - \kappa_n \circ \rho_n.$$

To show that g and $\kappa_\bullet$ define a lax and f and $\rho_\bullet$ an op-lax monoidal structure for the standard chain complex functor, it remains to verify the associativity and unitality condition from Definition 5.3.4. This amounts to the identies

$$\begin{aligned}
& g_\bullet^{X,Y\otimes'Z} \circ (\mathrm{id}_{X_\bullet} \otimes g_\bullet^{Y,Z}) = g_\bullet^{X\otimes'Y,Z} \circ (g_\bullet^{X,Y} \otimes \mathrm{id}_{Z_\bullet}) : X_\bullet \otimes Y_\bullet \otimes Z_\bullet \to (X\otimes'Y\otimes'Z)_\bullet \qquad (91)\\
& (\mathrm{id}_{X_\bullet} \otimes f_\bullet^{Y,Z}) \circ f_\bullet^{X,Y\otimes'Z} = (f_\bullet^{X,Y} \otimes \mathrm{id}_{Z_\bullet}) \circ f_\bullet^{X\otimes'Y,Z} : (X\otimes'Y\otimes'Z)_\bullet \to X_\bullet \otimes Y_\bullet \otimes Z_\bullet \\
& (L_X)_\bullet \circ g_\bullet^{E',X} \circ (\kappa_\bullet \otimes \mathrm{id}_{X_\bullet}) \circ l_{X_\bullet}^{-1} = \mathrm{id}_{X_\bullet} = (R_X)_\bullet \circ g_\bullet^{X,E'} \circ (\mathrm{id}_{X_\bullet} \otimes \kappa_\bullet) \circ r_{X_\bullet}^{-1} \\
& l_{X_\bullet} \circ (\rho_\bullet \otimes \mathrm{id}_{X_\bullet}) \circ f_\bullet^{E',X} \circ (L_X^{-1})_\bullet = \mathrm{id}_{X_\bullet} = r_{X_\bullet} \circ (\mathrm{id}_{X_\bullet} \otimes \rho_\bullet) \circ f_\bullet^{X,E'} \circ (R_X^{-1})_\bullet,
\end{aligned}$$

where $l_{X_\bullet} : E_\bullet \otimes X_\bullet \to X_\bullet$, $(r\otimes x) \mapsto r \rhd x$, $r_{X_\bullet} : X_\bullet \otimes E_\bullet \to X_\bullet$, $(x\otimes r) \mapsto r \rhd x$ are the left and right unit constraints in the category of chain complexes and chain maps in R-Mod and $L_X : E'\otimes'X \to X$, $R_X : X\otimes'E' \to E$ are the left and right unit constraints in the category of simplicial objects and morphisms in R-Mod.

The last two identities in (91) follow directly from the expressions for the Eilenberg-Zilber and Alexander-Whitney map in Definition 5.5.7 and the expressions for $\kappa_\bullet$ and $\rho_\bullet$ in (90). Due to the naturality of the Alexander-Whitney and Eilenberg-Zilber map, it is sufficient to prove the first two identities in (91) for the chain complexes $(A^n\otimes'A^n\otimes'A^n)_\bullet$ and $A^p_\bullet \otimes A^q_\bullet \otimes A^r_\bullet$, where $A^n_\bullet$ is the chain complex from (77).

For the Alexander-Whitney map we compute

$$\begin{aligned}
& (\mathrm{id}_{A^n_\bullet} \otimes f_\bullet^{A^n,A^n})_n \circ f_n^{A^n,A^n\otimes'A^n}(1_{[n+1]} \otimes 1_{[n+1]} \otimes 1_{[n+1]}) \\
& = \sum_{p=0}^{n} \phi_n^p \otimes (A^n\otimes'A^n)(\psi_n^p)(1_{[n+1]} \otimes 1_{[n+1]}) = \sum_{p=0}^{n} \sum_{q=0}^{n-p} \phi_n^p \otimes (\psi_n^p \circ \phi_{n-p}^q) \otimes (\psi_n^p \circ \psi_{n-p}^q) \\
& = \sum_{p=0}^{n} \sum_{q=0}^{n-p} (\phi_n^p \circ \phi_{n-p}^q) \otimes (\phi_n^p \circ \psi_{n-p}^q) \otimes \psi_n^p = \sum_{p=0}^{n} (A^n\otimes'A^n)(\phi_n^p)(1_{[n+1]} \otimes 1_{[n+1]}) \otimes \psi_n^p \\
& = (f_\bullet^{A^n,A^n} \otimes \mathrm{id}_{A^n_\bullet})_n \circ f_n^{A^n\otimes'A^n,A^n}(1_{[n+1]} \otimes 1_{[n+1]} \otimes 1_{[n+1]}).
\end{aligned}$$

For the Eilenberg-Zilber map Definition 5.5.7 yields

$$\begin{aligned}
& g_{q+r}^{A^p,A^q\otimes'A^r} \circ (\mathrm{id}_{A^\bullet_p} \otimes g_\bullet^{A^q\otimes'A^r})_{q+r}(1_{[p+1]} \otimes 1_{[q+1]} \otimes 1_{[r+1]}) \\
& = \sum_{\substack{\pi \in \mathrm{Sh}(p,q+r) \\ \tau \in \mathrm{Sh}(q,r)}} \mathrm{sgn}(\tau \circ \pi)\, (\sigma_{p+1}^{\tau(p)} \circ \ldots \circ \sigma_{p+q+r}^{\tau(p+q+r-1)}) \otimes (\sigma_{q+1}^{\pi(q)} \circ \ldots \circ \sigma_{q+r}^{\pi(q+r-1)} \circ \sigma_{q+r+1}^{\tau(0)} \circ \ldots \circ \sigma_{p+q+r}^{\tau(p-1)}) \\
& \qquad \otimes (\sigma_{r+1}^{\pi(0)} \circ \ldots \circ \sigma_{q+r}^{\pi(q-1)} \circ \sigma_{q+r+1}^{\tau(0)} \circ \ldots \circ \sigma_{p+q+r}^{\tau(p-1)}).
\end{aligned}$$

Using the second identity in (59), we can permute the factors $\sigma_{\ldots}^{\pi(\ldots)}$ and $\sigma_{\ldots}^{\tau(\ldots)}$ in the second and third factor of the tensor product to ensure that all factors $\sigma_{\ldots}^{j}$ in them are ordered with j

strictly increasing from the left to the right. After the reordering (i) the factors σ^j are ordered with j strictly increasing from left to right in all three factors of the tensor product, and (ii) each index $k \in \{0, ..., p+q+r-1\}$ occurs as an upper index $\sigma^k_{...}$ in exactly two factors.

All possible combinations of indices that satisfy (i) and (ii) occur when π, τ range over all $(p, q+r)$ and (q, r) shuffles. We have thus rewritten the expression in a form that is symmetric in the three factors of the tensor product. The term $\mathrm{sgn}(\tau \circ \pi)$ can also be rewritten in a way that is symmetric in all three factors. By applying the same argument to the right-hand-side side of the associativity condition in (91), we find that the two expressions agree. This shows that the Eilenberg-Zilber maps are associative. □

Remark 5.5.10: The Alexander-Whitney and Eilenberg-Zilber maps also induce natural transformations $f : N\otimes' \to \otimes(N \times N)$ and $g : \otimes(N \times N) \to N\otimes'$ for the normalised chain complex functor $N : \mathrm{Fun}(\Delta^{+op}, R\text{-Mod}) \to \mathrm{Ch}_{R\text{-Mod}\geq 0}$ from Proposition 5.2.2.

For all simplicial objects $X, Y : \Delta^{+op} \to R$-Mod their component morphisms form chain homotopy equivalences with $f^{X,Y}_\bullet \circ g^{X,Y}_\bullet = \mathrm{id}_\bullet : N(X)_\bullet \otimes N(Y)_\bullet \to N(X)_\bullet \otimes N(Y)_\bullet$.

Together with the chain maps induced by (90) they equip the normalised chain complex functor $N : \mathrm{Fun}(\Delta^{+op}, R\text{-Mod}) \to \mathrm{Ch}_{R\text{-Mod}\geq 0}$ with a lax and op-lax monoidal structure.

6 Exercises

6.1 Exercises for Chapter 1

Exercise 1: Let $\mathbb{F}$ be a field and $\mathbb{F}[X]$ the $\mathbb{F}$-algebra of polynomials with coefficients in $\mathbb{F}$.

(a) Show that $\mathbb{F}[X]$-modules are in bijection with pairs (M, ϕ) of an $\mathbb{F}$-vector space M and an $\mathbb{F}$-linear map $\phi : M \to M$.

(b) Characterise morphisms of $\mathbb{F}[X]$-modules in terms of vector spaces over $\mathbb{F}$ and $\mathbb{F}$-linear maps.

(c) Let M be a finite-dimensional vector space over $\mathbb{F}$ and $\phi : M \to M$ an $\mathbb{F}$-linear map. As $\mathbb{F}[X]$ is a principal ideal domain, there is a polynomial $p \in \mathbb{F}[X]$ that generates the annihilator of the associated $\mathbb{F}[X]$-module M: $\langle p\rangle_{\mathbb{F}[X]} = \mathrm{Ann}(M)$. Characterise this polynomial with concepts from linear algebra.

(d) Let $p \in \mathbb{F}[X] \setminus \{0\}$ be a polynomial and $\langle p\rangle_{\mathbb{F}[X]} \subset \mathbb{F}[X]$ the left ideal in $\mathbb{F}[X]$ generated by p. Show that the quotient module $M = \mathbb{F}[X]/\langle p\rangle_{\mathbb{F}[X]}$ is a cyclic $\mathbb{F}[X]$-module and a finite-dimensional vector space over $\mathbb{F}$. Determine its dimension $\dim_{\mathbb{F}}(M)$ and its annihilator $\mathrm{Ann}(M)$.

(e) Consider the $\mathbb{F}[X]$-module $M = \mathbb{F}[X]/\langle (x-\lambda)^n\rangle_{\mathbb{F}[X]}$ for some $\lambda \in \mathbb{F}$ and $n \in \mathbb{N}$. Show that M has a vector space basis for which the transformation matrix of $\phi = x \rhd - : M \to M$, $m \mapsto x \rhd m$ is a Jordan block

$$\begin{pmatrix} \lambda & 1 & 0 & \ldots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \lambda & 1 \\ 0 & \ldots & \ldots & 0 & \lambda \end{pmatrix} \qquad \text{mit } \lambda \in \mathbb{F}.$$

(f) Conclude that for every finite dimensional complex vector space M and every $\mathbb{C}$-linear map $\phi : M \to M$ the $\mathbb{C}[X]$-module (M, ϕ) is isomorphic to a direct sum

$$\mathbb{C}[X]/\langle (x-\lambda_1)^{n_1}\rangle_{\mathbb{C}[X]} \oplus ... \oplus \mathbb{C}[X]/\langle (x-\lambda_k)^{n_k}\rangle_{\mathbb{C}[X]} \qquad \lambda_1, ..., \lambda_k \in \mathbb{C}, \quad n_1, ..., n_k \in \mathbb{N}.$$

Hint: In (a), consider the maps $\rhd|_{\mathbb{F}\times M} : \mathbb{F}\times M \to M$, $(\lambda, m) \mapsto \lambda \rhd m$ and $\phi = x \rhd - : M \to M$, $m \mapsto x \rhd m$.

Exercise 2: Consider the abelian group $G = \langle\, x, y \mid ax + by\,\rangle_{\mathbb{Z}}$ for fixed $a, b \in \mathbb{Z}$. Determine $n \in \mathbb{N}_0$ and $q_1, ..., q_r \in \mathbb{N}$ such that $G \cong \mathbb{Z}^n \times \mathbb{Z}/q_1\mathbb{Z} \times ... \times \mathbb{Z}/q_r\mathbb{Z}$.

Exercise 3: Prove that the tensor product of $\mathbb{Z}$-modules satisfies

$$\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/\gcd(m, n)\mathbb{Z} \quad \forall m, n \in \mathbb{N}.$$

Exercise 4: Let V be a vector space over $\mathbb{F}$ with a countable basis $B = \{b_n \mid n \in \mathbb{N}\}$ and $R = \mathrm{End}_{\mathbb{F}}(V)$ the endomorphism ring of V.

(a) Consider the $\mathbb{F}$-linear maps $\phi, \psi : V \to V$ with

$$\phi(b_{2n}) = b_n,\ \phi(b_{2n-1}) = 0 \qquad \psi(b_{2n}) = 0,\ \psi(b_{2n-1}) = b_n \qquad \forall n \in \mathbb{N}$$

and show that there are $\mathbb{F}$-linear maps $\alpha, \beta : V \to V$ with $\mathrm{id}_V = \alpha \circ \phi + \beta \circ \psi$.

(b) Conclude that the ring R as a left module over itself is the direct sum $R = (R \rhd \phi) \oplus (R \rhd \psi)$, where $R \rhd \chi := \{r \rhd \chi \mid r \in R\}$ for all $\chi \in \mathrm{End}_{\mathbb{F}}(V)$.

(c) Conclude that $R^k \cong R^l$ for all $k, l \in \mathbb{N}$.

Exercise 5: Let A, B, C be sets.

- A **relation** between A and B is a subset $R \subset A \times B$.
- A relation $R \subset A \times B$ is called a **map** from A to B, if for every $a \in A$ there is a unique $b \in B$ with $(a, b) \in R$.
- The composite of two relations $R \subset A \times B$ and $S \subset B \times C$ is the relation

$$S \circ R = \{(a, c) \in A \times C \mid \exists b \in B :\ (a, b) \in R, (b, c) \in S\} \subset A \times C.$$

(a) Show that sets and relations form a category **Rel** with $\mathrm{Hom}_{\mathrm{Rel}}(A, B) = \mathcal{P}(A \times B)$.

(b) Determine the isomorphisms in **Rel**.

(c) Show that the disjoint union of sets defines both, a product and a coproduct in Rel

Exercise 6: Let $\mathcal{C}$ be a small category and $\mathcal{D}$ a category in which products (coproducts) exist for all (finite) families $(D_i)_{i \in I}$ of objects in $\mathcal{D}$. Show that then products (coproducts) exist for any (finite) family $(F_i)_{i \in I}$ of objects in the functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Exercise 7: The **abelisation** of a group G is the factor group $G/[G, G]$ with respect to the normal subgroup $[G, G] = \{[g, g'] \mid g, g' \in G\} \subset G$, where $[g, g'] = gg'g^{-1}g'^{-1}$.

(a) Show that the abelisation has the following universal property:
For every group homomorphism $f : G \to A$ into an abelian group A, there is a unique group homomorphism $f' : G/[G, G] \to A$ with $f' \circ \pi = f$, where $\pi : G \to G/[G, G]$, $g \mapsto g[G, G]$ is the canonical surjection.

(b) The abelisation defines a functor $\mathrm{Ab} : \mathrm{Grp} \to \mathrm{Ab}$.

(c) The functor $\mathrm{Ab} : \mathrm{Grp} \to \mathrm{Ab}$ is left adjoint to the inclusion functor $I : \mathrm{Ab} \to \mathrm{Grp}$.

Exercise 8: Two rings A, B are called **Morita equivalent** if there is an (A, B)-bimodule P and a (B, A)-bimodule Q such that $P \otimes_B Q \cong A$ and $Q \otimes_A P \cong B$, respectively, as (A, A)- and (B, B)-bimodules.

(a) Show that there is a category[3] URg$'$ whose objects are unital rings, whose morphisms from A to B are isomorphism classes of (B, A)-bimodules and with the composition $[N] \circ [M] = [N \otimes_B M]$ for $[M] : A \to B$ and $[N] : B \to C$. Determine its isomorphisms.

(b) Show that Morita equivalence is an equivalence relation on the class of unital rings.

(c) Show that if A, B are Morita equivalent, then the categories A-Mod-A and B-Mod-B of (A, A)- and (B, B)-bimodules are equivalent.

(d) Show that for any unital ring A and $n \in \mathbb{N}$ the ring $\mathrm{Mat}(n \times n, A)$ of $n \times n$-matrices with entries in A is Morita equivalent to A.

Remark: The converse of (c) is true as well, if one supposes that the equivalence is additive, but this is more difficult to show.

Exercise 9: Let R be a unital ring and M an R-module.

(a) Show that taking the direct sum with M defines a functor $M \oplus - : R\text{-Mod} \to R\text{-Mod}$.

(b) Determine the R-modules M for which it has a left or right adjoint.

Exercise 10: Let R be a ring, $(M_i)_{i \in I}$ a family of R-modules and N an R-module.

(a) Show that the abelian groups $\mathrm{Hom}_R(\amalg_{i \in I} M_i, N)$ and $\Pi_{i \in I} \mathrm{Hom}_R(M_i, N)$ are isomorphic.

(b) Show that the abelian groups $\mathrm{Hom}_R(N, \Pi_{i \in I} M_i)$ and $\Pi_{i \in I} \mathrm{Hom}_R(N, M_i)$ are isomorphic.

(c) Find a family $(M_i)_{i \in I}$ of R-modules and an R-module N, such that $\mathrm{Hom}_R(N, \amalg_{i \in I} M_i)$ and $\amalg_{i \in I} \mathrm{Hom}_R(N, M_i)$ are not isomorphic.

(d) Find a family $(M_i)_{i \in I}$ of R-modules and an R-module N, such that $\mathrm{Hom}_R(\Pi_{i \in I} M_i, N)$ and $\Pi_{i \in I} \mathrm{Hom}_R(M_i, N)$ are not isomorphic.

6.2 Exercises for Chapter 2

Exercise 11: Let k be a commutative ring and $X = \amalg_{i \in I} X_i$ a topological sum. Prove that

$$H_n(X, k) = \amalg_{i \in I} H_n(X_i, k) \quad \forall\, n \in \mathbb{N}_0.$$

Exercise 12: Let k be a commutative ring and $\emptyset \neq X \subset \mathbb{R}^m$ star-shaped with respect to $p \in X$. Show that $H_n(X, k) = 0$ for all $n \in \mathbb{N}$ and $H_0(X, k) \cong k$.

Proceed as follows. Consider the k-linear map $P_n : C_n(X, k) \to C_{n+1}(X, k)$ with

$$P_n(\sigma)(\Sigma_{i=0}^{n+1} \lambda_i e_i) = \begin{cases} \lambda_{n+1}\, p + (1 - \lambda_{n+1})\, \sigma(\Sigma_{i=0}^{n} \frac{\lambda_i}{1-\lambda_{n+1}}\, e_i) & \lambda_{n+1} \neq 1 \\ p & \lambda_{n+1} = 1 \end{cases}$$

for all $n \in \mathbb{N}_0$ and singular n-simplexes $\sigma : \Delta^n \to X$.

[3] In this exercise we do not suppose that the category is *locally small*, i. e. that mophisms between given objects form sets.

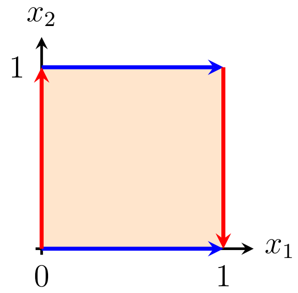
(a) Show that for all singular n-simplexes $\sigma : \Delta^n \to X$

$$P_n(\sigma) \circ f_{n+1}^{n+1} = \sigma \qquad P_n(\sigma) \circ f_i^{n+1} = P_{n-1}(\sigma \circ f_i^n) \quad \forall i \in \{0, ..., n\},$$

where $f_i^n : \Delta^{n-1} \to \Delta^n$ denote the face maps.

(b) Use (a) to compute $d_{n+1}(P_n(\sigma))$ and to show that $Z_n(X, k) = B_n(X, k)$ for all $n \in \mathbb{N}$. Treat the case $n = 0$ separately.

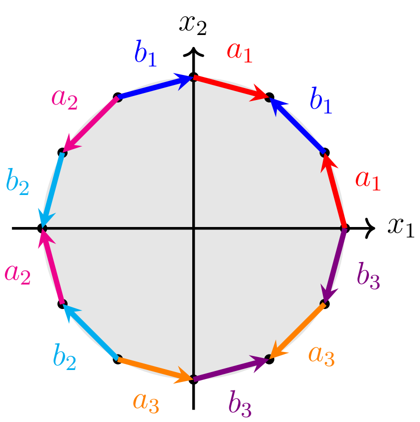
Exercise 13: The **Klein bottle** is the quotient space $K = [0, 1] \times [0, 1]/\sim$ with respect to the equivalence relation $(0, y) \sim (1, 1 - y)$ and $(x, 0) \sim (x, 1)$ for all $x, y \in [0, 1]$.



(a) Choose a semisimplicial complex structure Δ on K and compute its simplicial homologies $H_n(\Delta, k)$ for $n \in \mathbb{N}_0$ and (i) $k = \mathbb{Z}$, (ii) $k = \mathbb{F}$ a field, (iii) $k = \mathbb{Z}/6\mathbb{Z}$.

(b) Compute the cohomologies $H^n(\Delta, M)$ of the semisimplicial complex Δ from (a) with values in the $\mathbb{Z}$-module M for (i) $M = \mathbb{Z}$, (ii) $M = \mathbb{F}$ a field and (iii) $M = \mathbb{Z}[x]$.

Exercise 14: An **oriented surface** of **genus** $g \geq 1$ is the quotient space $\Sigma_g = P_{4g}/\sim$ of a regular $4g$-gon $P_{4g} \subset \mathbb{R}^2$ with respect to the equivalence relation $\sim$, which identifies its sides pairwise as follows:



Give a semisimplicial complex structure on Σ_g and compute the homologies $H_n(\Delta, k)$ for a commutative ring k.

Exercise 15: Let $\mathbb{F}$ be a field and $\Delta = (X, \{\sigma_\alpha\}_{\alpha \in I})$ a finite semisimplicial complex. The **Euler characteristic** of Δ is defined as

$$\chi(\Delta) = \Sigma_{k=0}^{\infty} (-1)^k \dim_{\mathbb{F}} C_k(\Delta, \mathbb{F}).$$

(a) Compute the Euler characteristic of the semisimplicial complex Δ from Exercise 14.

(b) Convince yourself that the family of affine linear maps

$$M = \{f^{n+1}_{j_{n+1}} \circ f^n_{j_n} \circ ... \circ f^{n+1-k}_{j_{n+1-k}} : \Delta^{n-k} \to \Delta^{n+1} \mid k \in \{0, ..., n\},\ j_i \in \{0, ..., i\}\}$$

equips the boundary $\partial\Delta^{n+1} \subset \mathbb{R}^{n+1}$ of the standard $(n+1)$-simplex $\Delta^{n+1} \subset \mathbb{R}^{n+1}$ with the structure of a simplicial complex. Compute its Euler-characteristic.

Exercise 16: Let A be an algebra over a commutative ring k and M an (A, A)-bimodule with structure maps $\rhd : A \times M \to M$ and $\lhd : M \times A \to M$. Compute the Hochschild homologies $H_0(A, M)$ and $H_1(A, M)$.

Exercise 17: We consider a commutative ring k as an algebra over itself and a k-module M, interpreted as a (k, k)-bimodule with $k \rhd m = m \lhd k$. Compute all Hochschild homologies $H_n(k, M)$ and Hochschild cohomologies $H^n(k, M)$ for $n \in \mathbb{N}_0$.

Exercise 18: Let k be a commutative ring, A a *commutative* algebra over k and M an A-module, interpreted as an (A, A)-bimodule with $a \rhd m = m \lhd a$. The A-module $\Omega_{A/k}$ of **Kähler differentials** is the A-module with one generator da for each element $a \in A$ and relations $d(a + b) - da - db$ and $d(ab) - a \rhd db - b \rhd da$ for all $a, b \in A$:

$$\Omega_{A/k} = \frac{\langle\, \{da \mid a \in A\}\, \rangle_A}{\langle\, \{d(a + b) - da - db, d(ab) - a \rhd db - b \rhd da \mid a, b \in A\}\, \rangle}.$$

(a) Prove that the Kähler differentials have the following **universal property**:
The map $d : A \to \Omega_{A/k}$, $a \mapsto da$ is a derivation, and for any derivation $f : A \to M$ there is a unique A-linear map $\phi : \Omega_{A/k} \to M$ with $f = \phi \circ d$.

(b) Express the first Hochschild homology and Hochschild cohomology with values in M in terms of Kähler differentials.

Exercise 19: An algebra A over a commutative ring k is called **separable** if the multiplication map $m : A \otimes_k A \to A$ has a right-inverse $\sigma : A \to A \otimes_k A$ that is a homomorphism of (A, A)-bimodules with respect to $b \rhd a \lhd c = bac$ and $b \rhd (a \otimes a') \lhd c = (ba) \otimes (a'c)$.

(a) Show that a k-algebra A is separable if and only if there is an element $\ell \in A \otimes_k A$ mit $m(\ell) = 1$ and $(a \otimes 1) \cdot \ell = \ell \cdot (1 \otimes a)$ for all $a \in A$, a **separability idempotent**. Show that any separability idempotent is an idempotent in $A \otimes_k A^{op}$.

(b) Show that for a separable k-algebra A and every (A, A)-bimodule M one has $H^n(A, M) = H_n(A, M) = 0$ for all $n \in \mathbb{N}$.

(c) Show that the following are separable algebras: (i) any matrix algebra $\mathrm{Mat}(n, k)$, (ii) any group algebra $k[G]$ of a finite group G over a field k with char $\nmid |G|$, (iii) any finite-dimensional semisimple algebra over an algebraically closed field k.

Hint: In (b) consider for $f \in Z^n(A, M)$ the map

$$f' : A^{\otimes(n-1)} \to M, \quad a_1 \otimes ... \otimes a_{n-1} \mapsto \Sigma_{i=1}^k x_i \rhd f(y_i \otimes a_1 \otimes .. \otimes a_n),$$

where $\ell = \Sigma_{i=1}^k x_i \otimes y_i$ is a separability idempotent for A.

Exercise 20: Let G be a group, $(M, \rhd)$ a $\mathbb{Z}[G]$-module and $M \rtimes G$ the associated semidirect product, the set $M \times G$ with group multiplication $(m, g) \cdot (m', g') = (m + g \rhd m', gg')$.

One says two actions $\rhd, \rhd' : H \times X \to X$ of a group H on a set X are related by conjugation, if there is a $k \in H$ with $h \rhd' x = (k^{-1}hk) \rhd x$ for all $h \in H$, $x \in X$.

Prove the following:

(a) The map $\rhd_f : (M \rtimes G) \times M \to M$, $(m, g) \rhd_f m' = m + g \rhd m' + f(g)$ for a map $f : G \to M$ is a group action of $M \rtimes G$ on M if and only if f is a 1-cocycle.

(b) The group actions $\rhd_f$, $\rhd_{f'}$ for two 1-cocycles $f, f' : G \to M$ are related by conjugation with an element $(m, 1) \in M \rtimes G$ if and only if $f - f'$ is a 1-coboundary.

Exercise 21: Let G be a group and M an abelian group equipped with the trivial $\mathbb{Z}[G]$-module structure.

(a) Characterise the group cohomologies $H^1(G, M)$ and the group homologies $H_1(G, \mathbb{Z})$ in terms of the abelisation of G.

(b) Compute $H^1(G, M)$ for a finitely generated abelian group G.

(c) Show that $H^1(G, \mathbb{Z}) = 0$ for every finite group G.

(d) Compute $H^1(G, M)$ for $G = S_n$, $n \geq 2$, and $M = \mathbb{Z}/2\mathbb{Z}$.

Hint: In (b) use the classification theorem for finitely generated abelian groups, which states that every finitely generated abelian group is isomorphic to a group of the form

$$G \cong \mathbb{Z}^n \times \mathbb{Z}/q_1\mathbb{Z} \times \ldots \times \mathbb{Z}/q_k\mathbb{Z} \quad \text{with} \quad n \in \mathbb{N}_0, q_i \in \mathbb{N}.$$

Exercise 22: Compute $H^2(\mathbb{Z}/3\mathbb{Z}, M)$ for the following abelian groups M with the trivial $\mathbb{Z}[\mathbb{Z}/3\mathbb{Z}]$-module structure (i) $M = \mathbb{Z}$, (ii) $M = \mathbb{Z}/3\mathbb{Z}$ and (iii) $M = \mathbb{Z}/2\mathbb{Z}$.

Exercise 23: The integer **Heisenberg group** is the subgroup

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{Z} \right\} \subset \mathrm{SL}(3, \mathbb{Z})$$

(a) Show that H is a central extension of $\mathbb{Z} \times \mathbb{Z}$ by $\mathbb{Z}$.

(b) Compute the associated $\mathbb{Z} \times \mathbb{Z}$-action on $\mathbb{Z}$ and the associated element in $H^2(\mathbb{Z} \times \mathbb{Z}, \mathbb{Z})$. Show that $H^2(\mathbb{Z} \times \mathbb{Z}, \mathbb{Z}) \neq 0$.

Exercise 24: Determine all isomorphism classes of (not necessarily central) extensions of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}/3\mathbb{Z}$.

6.3 Exercises for Chapter 3

Exercise 25: Let $\mathcal{C}$ be an additive category.

Show that an object X in $\mathcal{C}$ is a (co)product of a finite family $(C_k)_{k\in I}$ of objects in $\mathcal{C}$ if and only if there are families of morphisms $(i_k : C_k \to X)_{k\in I}$ and $(p_k : X \to C_k)_{k\in I}$ with $p_k \circ i_k = 1_{C_k}$ for all $k \in I$, $p_j \circ i_k = 0 : C_k \to C_j$ for $j \neq k$ and $\Sigma_{k\in I} i_k \circ p_k = 1_X$.

Exercise 26: Let $\mathcal{C}$, $\mathcal{D}$ be additive categories and $F : \mathcal{C} \to \mathcal{D}$ a functor. Show that the following statements are equivalent:

(i) F is additive.

(ii) F preserves finite products: $F(\Pi_{i\in I} C_i) \cong \Pi_{i\in I} F(C_i)$ for all finite families $(C_i)_{i\in I}$ of objects C_i in $\mathcal{C}$.

(iii) F preserves finite coproducts: $F(\amalg_{i\in I} C_i) \cong \amalg_{i\in I} F(C_i)$ for all finite families $(C_i)_{i\in I}$ of objects C_i in $\mathcal{C}$.

Hint: Use Exercise 25.

Exercise 27: Let $\mathcal{C}$ be a category with a zero object, $f : C \to D$ a morphism in $\mathcal{C}$, $g : D \to E$ a monomorphism and $h : B \to C$ an epimorphism in $\mathcal{C}$. Prove the following:

(a) A morphism $\pi : D \to A$ is a cokernel of f if and only if π is a cokernel of $f \circ h$.

(b) A morphism $\iota : B \to X$ is a kernel of f if and only if ι is a kernel of $g \circ f$.

Exercise 28: Let $\mathcal{C}$ be a category with a zero object. Prove the following:

(a) Kernels in $\mathcal{C}$ are unique up to unique isomorphism: if $\iota : \ker(f) \to X$, $\iota' : \ker(f)' \to X$ are kernels of a morphism $f : X \to Y$ in $\mathcal{C}$, then there is a unique morphism $\phi : \ker(f) \to \ker(f)'$ with $\iota' \circ \phi = \iota$, and ϕ is an isomorphism.

(b) $\mathcal{C}^{op}$ has a zero object and $\iota : W \to X$ ($\pi : Y \to Z$) is a kernel (cokernel) of $f : X \to Y$ in $\mathcal{C}$ if and only if $\iota : X \to W$ ($\pi : Z \to Y$) is a cokernel (kernel) of $f : Y \to X$ in $\mathcal{C}^{op}$.

Exercise 29: Let $\mathcal{A}$ be an abelian category. Prove the following: If a morphism $f : X \to Z$ in $\mathcal{A}$ is given by $f = \iota \circ \pi$ with a monomorphism $\iota : Y \to Z$ and an epimorphism $\pi : X \to Y$, then $\iota : Y \to Z$ is an image of f and $\pi : X \to Y$ a coimage of f.

Exercise 30: Let $\mathcal{A}$ be an abelian category. Prove that every morphism $f : X \to Y$ that is both a monomorphism and an epimorphism is an isomorphism.

Exercise 31: Show that a category $\mathcal{A}$ is abelian if and only if $\mathcal{A}^{op}$ is abelian and kernels and cokernels in $\mathcal{A}$ then correspond to cokernels and kernels in $\mathcal{A}^{op}$.

Exercise 32: Show that the full subcategory $\mathcal{C}$ of Ab $= \mathbb{Z}$-Mod with finitely generated free $\mathbb{Z}$-modules as objects is additive, but not abelian.

Exercise 33: Show that the full subcategory of $\mathrm{Vect}_{\mathbb{F}}$ with even-dimensional vector spaces as objects is additive, but contains morphisms without kernels or cokernels.

Exercise 34: Let $\mathcal{C}$ be a small category and $\mathcal{A}$ an abelian category. Prove the following:

(a) The category $\mathrm{Fun}(\mathcal{C}, \mathcal{A})$ is abelian.

(b) For every object C in $\mathcal{C}$, the functor $\mathrm{ev}_C : \mathrm{Fun}(\mathcal{C}, \mathcal{A}) \to \mathcal{A}$ that sends a functor $F : \mathcal{C} \to \mathcal{A}$ to the object $F(C)$ and a natural transformation $\eta : F \to G$ to the component morphism $\eta_C : F(C) \to G(C)$ is exact.

Exercise 35: Prove that for any abelian category $\mathcal{A}$, the Cartesian product $\mathcal{A} \times \mathcal{A}$ is abelian and the functor $\Pi : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is exact.

Exercise 36: Let $\mathcal{A}$ be an abelian category. Show that the category $\mathrm{Ch}_{\mathcal{A}}$ of chain complexes and chain maps in $\mathcal{A}$ is abelian as well.

Exercise 37: Let k be a commutative ring, G a group, and denote by $\langle M \rangle_k$ the free k-module generated by M. Consider the chain complexes

- $X_\bullet$ with $X_n = \langle G^{\times(n+1)} \rangle_k$, the $k[G]$-module structure $g \rhd (g_0, ..., g_n) = (gg_0, ..., gg_n)$ on X_n and boundary operator $d_n = \Sigma_{i=0}^n (-1)^i d_n^i : X_n \to X_{n-1}$ for $n \in \mathbb{N}_0$ with
$$d_n^i(g_0, ..., g_n) = (g_0, ..., \widehat{g_i}, ..., g_n).$$
- $X'_\bullet$ with $X'_n = \langle G^{\times n} \rangle_{k[G]}$, $k[G]$-linear boundary operator $d'_n = \Sigma_{i=0}^{n+1} (-1)^i d'^i_n : X'_n \to X'_{n-1}$ for $n \in \mathbb{N}_0$ with
$$d'^i_n(g_1, ..., g_n) = \begin{cases} g_1 \rhd (g_2, ..., g_n) & i = 0 \\ (..., g_i g_{i+1}, ...) & 1 \leq i \leq n-1 \\ (g_1, ..., g_{n-1}) & i = n. \end{cases}$$

Show that the $k[G]$-linear maps $f_n : X'_n \to X_n$, $(g_1, ..., g_n) \mapsto (1, g_1, g_1 g_2, \ldots, g_1 g_2 \cdots g_{n-1} g_n)$ define an invertible chain map $f_\bullet : X'_\bullet \to X_\bullet$.

Exercise 38: Let $\mathcal{A}$ be an abelian category, $X_\bullet, X'_\bullet$ chain complexes in $\mathcal{A}$ and $p \in \mathbb{Z}$. We define a chain complex $T_p(X_\bullet)$
$$T_p(X_\bullet)_n = X_{n+p}, \qquad T_p(d_\bullet)_n = (-1)^p d_{n+p} \qquad \forall n \in \mathbb{Z}.$$
and for every chain map $f_\bullet : X_\bullet \to X'_\bullet$ a chain map
$$T_p(f_\bullet) : T_p(X_\bullet) \to T_p(X'_\bullet), \qquad T_p(f_\bullet)_n = f_{n+p}.$$
Show that this defines a functor $T_p : \mathrm{Ch}_{\mathcal{A}} \to \mathrm{Ch}_{\mathcal{A}}$ that satisfies
$$H_n(T_p(X_\bullet)) = H_{n+p}(X_\bullet).$$
This functor is called the **translation functor**.

Exercise 39: Let $(X_\bullet, d_\bullet)$ be a chain complex in R-Mod for a ring R. Show that this defines chain complexes $Z_\bullet(X_\bullet) = (Z_n(X_\bullet))_{n\in\mathbb{Z}}$, $B_\bullet(X_\bullet) = (B_n(X_\bullet))_{n\in\mathbb{Z}}$ and $H_\bullet(X_\bullet) = (H_n(X_\bullet))_{n\in\mathbb{Z}}$ and exact sequences in $\mathrm{Ch}_{R\text{-Mod}}$

$$0_\bullet \to Z_\bullet(X_\bullet) \xrightarrow{f_\bullet} X_\bullet \xrightarrow{g_\bullet} B_\bullet^{(-1)}(X_\bullet) \to 0_\bullet$$
$$0_\bullet \to H_\bullet(X_\bullet) \xrightarrow{h_\bullet} X_\bullet/B_\bullet(X_\bullet) \xrightarrow{k_\bullet} Z_\bullet^{(-1)}(X_\bullet) \xrightarrow{l_\bullet} H_\bullet^{(-1)}(X_\bullet) \to 0_\bullet,$$

where $0_\bullet = (0)_{n\in\mathbb{Z}}$ and $X_\bullet^{-1} = (X_{n-1})_{n\in\mathbb{Z}}$ for a chain complex $X_\bullet = (X_n)_{n\in\mathbb{Z}}$ denotes the shifted chain complex. Determine the boundary operators of $Z_\bullet(X_\bullet)$, $B_\bullet(X_\bullet)$ and $H_\bullet(X_\bullet)$ and the chain maps $f_\bullet, g_\bullet, h_\bullet, k_\bullet, l_\bullet$.

Exercise 40: Let R be a ring. Compute and interpret the homologies of the following chain complexes in R-Mod.

(a) $C_\bullet = 0 \to R \xrightarrow{d_1: s\mapsto r\cdot s} R \to 0$
(b) $C_\bullet = 0 \to R \xrightarrow{d_2: r\mapsto(-b\cdot r, a\cdot r)} R^2 \xrightarrow{d_1:(r,r')\mapsto ar+br'} R \to 0$ for a commutative ring R and $a, b \in R$.

Exercise 41: Prove the **5-lemma**:

Suppose that the following diagram in R-Mod commutes and all rows are exact

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r, "h"] \arrow[d, "\gamma"] & D \arrow[r, "k"] \arrow[d, "\delta"] & E \arrow[d, "\epsilon"] \\
A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & D' \arrow[r, "k'"] & E'.
\end{tikzcd}$$

Then:

(i) If β, δ are monomorphisms and α is an epimorphism, then γ is a monomorphism.
(ii) If β, δ are epimorphisms and ϵ is a monomorphism, then γ is an epimorphism.
(iii) If $\alpha, \beta, \delta, \epsilon$ are isomorphisms, then γ is an isomorphism as well.

Exercise 42: Prove the **9-lemma**:

Let R be a ring. Suppose the following diagram in R-Mod commutes, has exact rows and all vertical composites of morphisms vanish

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r, "\iota_A"] \arrow[d, "\phi"] & A' \arrow[r, "\pi_A"] \arrow[d, "\phi'"] & A'' \arrow[r] \arrow[d, "\phi''"] & 0 \\
0 \arrow[r] & B \arrow[r, "\iota_B"] \arrow[d, "\psi"] & B' \arrow[r, "\pi_B"] \arrow[d, "\psi'"] & B'' \arrow[r] \arrow[d, "\psi''"] & 0 \\
0 \arrow[r] & C \arrow[r, "\iota_C"] \arrow[d] & C' \arrow[r, "\pi_C"] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0. &
\end{tikzcd}$$

If two of the columns are short exact sequences then so is the third one.

Exercise 43: A chain complex $(X_\bullet, d_\bullet)$ in an abelian category $\mathcal{A}$ is called **split** if there is a family $(s_n)_{n\in\mathbb{N}_0}$ of morphisms $s_n : X_n \to X_{n+1}$ with $d_n \circ s_{n-1} \circ d_n = d_n$ for all $n \in \mathbb{Z}$ and **split exact** if it is split and exact. Prove the following for $\mathcal{A} = R$-Mod:

(a) For any family $(s_n)_{n\in\mathbb{Z}}$ of morphisms $s_n : X_n \to X_{n+1}$ the morphisms

$$f_n = s_{n-1} \circ d_n + d_{n+1} \circ s_n : X_n \to X_n$$

define a chain map $f_\bullet : X_\bullet \to X_\bullet$ with $H_n(f_\bullet) = 0 : H_n(X_\bullet) \to H_n(X_\bullet)$.

(b) A chain complex $X_\bullet$ is split exact if and only if $1_{X_\bullet} : X_\bullet \to X_\bullet$ is chain homotopic to $0_\bullet : X_\bullet \to X_\bullet$.

(c) A chain complex $(X_\bullet, d_\bullet)$ in $\mathcal{A}$ is split if and only if there are families $(C_n)_{n\in\mathbb{Z}}$ and $(D_n)_{n\in\mathbb{Z}}$ of objects C_n, D_n with $X_n = C_n \amalg \ker(d_n)$ and $\ker(d_n) = D_n \amalg \mathrm{im}(d_{n+1})$, and in this case its homologies are given by $H_n(X_\bullet) = D_n$.

Exercise 44: Let R be a ring.

(a) Show that the chain complex $X_\bullet = \ldots \mathbb{Z}/4\mathbb{Z} \xrightarrow{z\mapsto 2z} \mathbb{Z}/4\mathbb{Z} \xrightarrow{z\mapsto 2z} \mathbb{Z}/4\mathbb{Z} \xrightarrow{z\mapsto 2z} \mathbb{Z}/4\mathbb{Z} \xrightarrow{z\mapsto 2z} \ldots$ is exact but not split exact.

(b) Show that every exact, bounded below chain complex $X_\bullet$ in R-Mod with projective R-modules X_n is split exact.

Exercise 45: Let X, Y be topological spaces and $C_\bullet(X, k)$ and $C_\bullet(Y, k)$ the associated singular chain complexes. The **prism maps** are the affine linear maps

$$T^j_n : \Delta^{n+1} \to [0,1] \times \Delta^n, \quad T^j_n(e_k) = \begin{cases} (0, e_k) & 0 \leq k \leq j \leq n \\ (1, e_{k-1}) & 0 \leq j < k \leq n+1. \end{cases}$$

(a) Prove that the prism maps satisfy the relations

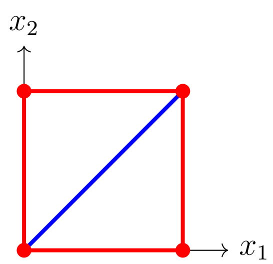
$$T^j_n \circ f^{n+1}_i = (\mathrm{id}_{[0,1]} \times f^n_i) \circ T^{j-1}_{n-1} \quad \forall j > i \quad T^j_n \circ f^{n+1}_i = (\mathrm{id}_{[0,1]} \times f^n_{i-1}) \circ T^j_{n-1} \quad \forall j < i-1$$
$$T^i_n \circ f^{n+1}_i = T^{i-1}_n \circ f^{n+1}_i \quad \forall i \in \{1, ..., n\} \quad T^0_n \circ f^{n+1}_0 = i_1, \quad T^n_n \circ f^{n+1}_{n+1} = i_0, \tag{92}$$

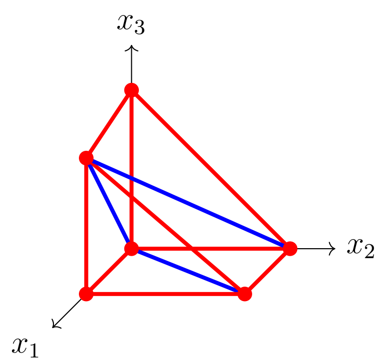
where $i_t : \Delta^n \to [0,1] \times \Delta^n$, $x \mapsto (t, x)$ and $f^{n+1}_j : \Delta^n \to \Delta^{n+1}$ denote the face maps.

(b) Let $f, g : X \to Y$ continuous maps and $h : [0,1] \times X \to Y$ a homotopy from f to g. Prove that the k-linear maps

$$C_n(h, k) : C_n(X, k) \to C_{n+1}(Y, k), \qquad \sigma \mapsto \Sigma^n_{j=0}(-1)^j \, h \circ (\mathrm{id}_{[0,1]} \times \sigma) \circ T^j_n.$$

define a chain homotopy $C_\bullet(h, k) : C_\bullet(f, k) \Rightarrow C_\bullet(g, k)$.





The prism maps T_n^j for $n = 2, 3$.

Exercise 46: Let A be an algebra over a commutative ring k and $(M, \rhd, \lhd)$ an (A, A)-bimodule and $c \in Z(A)$ an element in the centre of A. Prove the following:

(a) The maps $c \rhd - : M \to M$, $m \mapsto c \rhd m$ and $- \lhd c : M \to M$, $m \mapsto m \lhd c$ are homomorphisms of (A, A)-bimodules and induce chain maps $c\rhd_\bullet, \lhd c_\bullet : C_\bullet(A, M) \to C_\bullet(A, M)$ on the Hochschild complex.

(b) The chain maps $c\rhd_\bullet$ and $\lhd c_\bullet$ are chain homotopic.

Hint: In (b), consider for $0 \leq i \leq n$ the morphisms

$$h_n^i : M \otimes_k A^{\otimes n} \to M \otimes_k A^{\otimes(n+1)}, \quad m \otimes a_1 \otimes ... \otimes a_n \mapsto m \otimes a_1 \otimes ... \otimes a_i \otimes c \otimes a_{i+1} \otimes ... \otimes a_n$$

and combine them into a chain homotopy.

Exercise 47: Let $\mathcal{A}, \mathcal{B}$ be abelian categories and $P_n^{\mathcal{A}} : \mathrm{Ch}_{\mathcal{A}} \to \mathcal{A}$ the additive functor with

- $P_n^{\mathcal{A}}(X_\bullet) = X_n$ for every chain complex $X_\bullet = (X_n)_{n \in \mathbb{Z}}$,
- $P_n^{\mathcal{A}}(f_\bullet) = f_n : X_n \to X'_n$ for every chain map $f_\bullet = (f_n)_{n \in \mathbb{Z}} : X_\bullet \to X'_\bullet$.

Prove the following:

(a) For every additive functor $F : \mathcal{A} \to \mathcal{B}$ there is an additive functor $F' : \mathrm{Ch}_{\mathcal{A}} \to \mathrm{Ch}_{\mathcal{B}}$ with $P_n^{\mathcal{B}} F' = F P_n^{\mathcal{A}}$ for all $n \in \mathbb{Z}$.

(b) If $f_\bullet, f'_\bullet : X_\bullet \to X'_\bullet$ are chain homotopic, then $F'(f_\bullet), F'(f'_\bullet) : F(X_\bullet) \to F'(X'_\bullet)$ are chain homotopic as well. The functor $F' : \mathrm{Ch}_{\mathcal{A}} \to \mathrm{Ch}_{\mathcal{B}}$ induces a functor $F'' : K(\mathcal{A}) \to K(\mathcal{B})$.

(c) Let $\mathcal{G}(X_\bullet, X'_\bullet)$ the abelian groupoid with chain maps $f_\bullet : X_\bullet \to X'_\bullet$ as objects and chain homotopies between them as morphisms. Show that every additive functor $F : \mathcal{A} \to \mathcal{B}$ induces a functor $F' : \mathcal{G}(X_\bullet, X'_\bullet) \to \mathcal{G}(F'(X_\bullet), F'(X'_\bullet))$.

(d) If $\mathcal{A}$ is small, then this induces a functor $\Phi : \mathrm{Fun}_{add}(\mathcal{A}, \mathcal{B}) \to \mathrm{Fun}_{add}(\mathrm{Ch}_{\mathcal{A}}, \mathrm{Ch}_{\mathcal{B}})$, where $\mathrm{Fun}_{add}(\mathcal{A}, \mathcal{B})$ is the full subcategory with additive functors $F : \mathcal{A} \to \mathcal{B}$ as objects.

Exercise 48: Let G be a group.

(a) Consider the abelian groups $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$ with the trivial $\mathbb{Z}[G]$-module structure. Show that there is an exact sequence of cohomology groups

$$\ldots \xrightarrow{\partial^{k-1}} H^k(G, \mathbb{Z}) \xrightarrow{[f] \mapsto [nf]} H^k(G, \mathbb{Z}) \xrightarrow{[f] \mapsto [f + n\mathbb{Z}]} H^k(G, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\partial^k} H^{k+1}(G, \mathbb{Z}) \xrightarrow{[f] \mapsto [nf]} \ldots$$

(b) Show that $H^2(G, \mathbb{Z}) \neq 0$ for every finite group G with $\mathrm{Ab}(G) \neq \{e\}$.

Exercise 49: Let $(X_\bullet, d_\bullet)$ and $(X'_\bullet, d'_\bullet)$ be chain complexes and $f_\bullet : X_\bullet \to X'_\bullet$ a chain map in an abelian category $\mathcal{A}$.

The **mapping cone** $\mathrm{cone}(f_\bullet) = Y_\bullet$ is the chain complex with $Y_n = X_{n-1} \amalg X'_n$ and coboundary operator $d_n^Y : Y_n \to Y_{n-1}$ given by

$$\begin{aligned} \pi_{n-1} \circ d_n^Y \circ \iota_n &= -d_{n-1} & \qquad \pi'_{n-1} \circ d_n^Y \circ \iota_n &= -f_{n-1} \\ \pi_{n-1} \circ d_n^Y \circ \iota'_n &= 0 & \qquad \pi'_{n-1} \circ d_n^Y \circ \iota'_n &= d'_n, \end{aligned}$$

where $\iota_n : X_{n-1} \to Y_n$, $\iota'_n : X'_n \to Y_n$ and $\pi_n : Y_n \to X_{n-1}$, $\pi'_n : Y_n \to X'_n$ are the inclusions and projections for the coproduct.

(a) Show that $\mathrm{cone}(f_\bullet)$ is a chain complex in $\mathcal{A}$ for all chain maps $f_\bullet : X_\bullet \to Y_\bullet$.

(b) Show that $\mathrm{cone}(1_{X_\bullet})$ is homotopy equivalent to the trivial chain complex $0_\bullet$.

(c) Show that $f_\bullet : X_\bullet \to X'_\bullet$ extends to a chain map $f'_\bullet : \mathrm{cone}(1_{X_\bullet}) \to X'_\bullet$ with $f'_n \circ \iota'_n = f_n$ if and only if $f_\bullet$ is homotopic to $0_\bullet : X_\bullet \to X'_\bullet$.

(d) Show that the mapping cone induces a long exact homology sequence

$$\ldots \to H_n(X'_\bullet) \xrightarrow{H_n(\iota'_\bullet)} H_n(Y_\bullet) \xrightarrow{H_n(\pi_\bullet)} H_{n-1}(X_\bullet) \xrightarrow{\partial_n} H_{n-1}(X'_\bullet) \xrightarrow{H_{n-1}(\iota'_\bullet)} H_{n-1}(Y_\bullet) \to \ldots$$

(e) Show that $H_n(X_\bullet) = 0$ for all $n \in \mathbb{Z}$ implies $H_n(Y_\bullet) \cong H_n(X'_\bullet)$ for all $n \in \mathbb{Z}$ and $H_n(Y_\bullet) = 0$ for all $n \in \mathbb{Z}$ implies $H_n(X'_\bullet) \cong H_{n-1}(X_\bullet)$ for all $n \in \mathbb{Z}$.

Hint: Use the identities $\iota_n \circ \pi_n + \iota'_n \circ \pi'_n = 1_{Y_n}$ and $\pi_n \circ \iota_n = 1_{X_{n-1}}$, $\pi'_n \circ \iota'_n = 1_{X'_n}$ and $\pi'_n \circ \iota_n = 0$, $\pi_n \circ \iota'_n = 0$ (cf. Exercise 6, Sheet 4). In (e) use the long exact homology sequence.

6.4 Exercises for Chapter 4

Exercise 50: Show that every object in the category Set is projective and injective. Use the projectivity and injectivity criteria from Lemma 3.1.21.

Exercise 51: Let R_1, R_2 be rings and $R = R_1 \times R_2$ their product. Show that $A = R_1 \times \{0\}$ with the R-module structure $(r_1, r_2) \rhd (r'_1, 0) = (r_1, r_2) \cdot (r'_1, 0) = (r_1 r'_1, 0)$ is a projective R-module but not a free R-module.

Exercise 52: Let R be a ring and $0 \to L \xrightarrow{\iota} M \xrightarrow{\pi} N \to 0$ an exact sequence in R-Mod. Show that the following statements are equivalent:

(i) The R-linear map $\pi : M \to N$ has a **section**:
there is a R-linear map $\psi : N \to M$ with $\pi \circ \psi = \mathrm{id}_N$.

(ii) The R-linear map $\iota : L \to M$ has a **retraction**:
there is a R-linear map $\phi : M \to L$ with $\phi \circ \iota = \mathrm{id}_L$.

(iii) There is an R-linear isomorphism $\chi : M \to L \oplus N$ with $\chi \circ \iota = \iota_1$ and $\pi_2 \circ \chi = \pi$, where $\iota_1 : L \to L \oplus N$, $l \mapsto (l, 0)$ is the inclusion and $\pi_2 : L \oplus N \to N$, $(l, n) \mapsto n$ the projection

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "\iota"] \arrow[dr, "\iota_1"'] & M \arrow[r, "\pi"] \arrow[d, "\chi"] & N \arrow[r] & 0 \\ & & L \oplus N \arrow[ur, "\pi_2"'] & & \end{tikzcd}$$

If one of these conditions is satisfied, one says the exact sequence **splits**.

Exercise 53: Let R be a ring and A an R-module. Prove the following:

(a) A is projective if and only if every short exact sequence $0 \to L \xrightarrow{\iota} M \xrightarrow{\pi} A \to 0$ splits.

(b) A is injective if and only if every short exact sequence $0 \to A \xrightarrow{\iota} M \xrightarrow{\pi} N \to 0$ splits.

Exercise 54: Determine if the following abelian categories have enough projectives and injectives.

(a) The category Ab^{fin} of finite abelian groups.

(b) The category $\mathrm{Ab}^{fin\,gen}$ of finitely generated abelian groups.

Exercise 55: Let R be a ring.

(a) Show that an R-right module $M = \oplus_{i\in I} M_i$ is flat if and only if M_i is flat for all $i \in I$.

(b) Show that any projective R-right module is flat.

Hint: Use an argument similar to the one in the proof of Lemma 4.2.2.

Exercise 56: Let $\mathcal{A} = R$-Mod for some ring R. Show that a chain complex $P_\bullet = (P_n)_{n\in\mathbb{Z}}$ is a projective object in $\mathrm{Ch}_\mathcal{A}$ if and only if $P_\bullet$ is split exact and all objects P_n are projective.

Exercise 57: Show that if $\mathcal{A}$ is an abelian category with enough projectives, then $\mathrm{Ch}_\mathcal{A}$ has enough projectives as well.

Exercise 58: Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories such that $\mathcal{A}$ has enough projectives, $F, F' : \mathcal{A} \to \mathcal{B}$ right exact functors and $\eta : F \to F'$ a natural transformation. Show that η induces a family $(L_n\eta)_{n\in\mathbb{N}_0}$ of natural transformations $L_n\eta : L_nF \to L_nF'$ with $L_0\eta = \eta : F \to F'$.

Show that $L_n\eta : L_nF \to L_nF'$ is an isomorphism for any natural isomorphism $\eta : f \to F'$.

Exercise 59: Let $\mathcal{A}$ be an abelian category with enough projectives and $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor into an abelian category $\mathcal{B}$. Show that the left derived functors of F are additive.

Exercise 60: Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories such that $\mathcal{A}$ and $\mathcal{B}$ have enough projectives $F : \mathcal{A} \to \mathcal{B}$ right exact and $G : \mathcal{B} \to \mathcal{C}$ exact. Show that $L_n(GF) = G(L_nF)$ for all $n \in \mathbb{N}_0$.

Exercise 61: Let G be a group. The invariants and coinvariants of a $\mathbb{Z}[G]$-module M are the abelian subgroups $M^G, M_G \subset M$ given by

$$M^G = \{m \in M : \; g \rhd m = m \; \forall g \in G\} \qquad M_G = M/\langle \{g \rhd m - m \mid g \in G, m \in M\}\rangle_M.$$

(a) Show that the invariants and coinvariants define additive functors $(-)^G : \mathbb{Z}[G]\text{-Mod} \to \text{Ab}$ and $(-)_G : \mathbb{Z}[G]\text{-Mod} \to \text{Ab}$.

(b) Construct natural isomorphisms $\eta : (-)^G \to \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, -)$ and $\tau : (-)_G \to \mathbb{Z} \otimes_{\mathbb{Z}[G]} -$, where $\mathbb{Z}$ is equipped with the trivial $\mathbb{Z}[G]$-module structure.

(c) Show that the group homologies and cohomologies $H_n(G, M)$ and $H^n(G, M)$ are the left and right derived functors of $(-)_G : \mathbb{Z}[G]\text{-Mod} \to \text{Ab}$ and $(-)^G : \mathbb{Z}[G]\text{-Mod} \to \text{Ab}$, respectively.

Exercise 62: Let M be an $\mathbb{Z}[\mathbb{Z}]$-module. Compute the group homologies and cohomologies of $\mathbb{Z}$ with coefficients in M.

Hint: Show first that the following is a free resolution of the trivial $\mathbb{Z}[\mathbb{Z}]$-module $\mathbb{Z}$

$$0 \to \mathbb{Z}[\mathbb{Z}] \xrightarrow{\iota} \mathbb{Z}[\mathbb{Z}] \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

with $\iota : \mathbb{Z}[\mathbb{Z}] \to \mathbb{Z}[\mathbb{Z}],\ \Sigma_{z\in\mathbb{Z}}\lambda_z z \mapsto \Sigma_{z\in\mathbb{Z}}(\lambda_z - \lambda_{z-1})z$ and $\epsilon : \mathbb{Z}[\mathbb{Z}] \to \mathbb{Z},\ \Sigma_{z\in\mathbb{Z}}\lambda_z z \mapsto \Sigma_{z\in\mathbb{Z}}\lambda_z$.

Exercise 63: Let R be a principal ideal domain. Compute $\text{Ext}^n_R(M, N)$ for all finitely generated R-modules M, N.

Exercise 64: Let $\mathbb{F}$ be a field, $p = \Sigma_{k=0}^n a_k x^k \in \mathbb{F}[x]$ with $a_n = 1$ and consider the $\mathbb{F}$-algebra $A = \mathbb{F}[x]/(p)$, where $(p) \subset \mathbb{F}[x]$ is the ideal generated by p and $\pi : \mathbb{F}[x] \to A,\ q \mapsto \bar{q}$ the canonical surjection.

(a) Show that

$$\begin{aligned}
&\ldots \xrightarrow{f} A\otimes_{\mathbb{F}}A \xrightarrow{g} A\otimes_{\mathbb{F}}A \xrightarrow{f} A\otimes_{\mathbb{F}}A \xrightarrow{g} A\otimes_{\mathbb{F}}A \xrightarrow{f} A\otimes_{\mathbb{F}}A \xrightarrow{g} A\otimes_{\mathbb{F}}A \xrightarrow{f} A\otimes_{\mathbb{F}}A \xrightarrow{\mu} A \to 0\\
&f : A\otimes_{\mathbb{F}}A \to A\otimes_{\mathbb{F}}A,\ a\otimes b \mapsto (\bar{x}\otimes 1 - 1\otimes\bar{x}) \cdot (a\otimes b)\\
&g : A\otimes_{\mathbb{F}}A \to A\otimes_{\mathbb{F}}A,\ a\otimes b \mapsto q \cdot (a\otimes b) \qquad \text{with} \qquad q = \Sigma_{k=0}^n \Sigma_{j=0}^{k-1} a_k\, \bar{x}^{k-1-j} \otimes \bar{x}^j\\
&\mu : A\otimes_{\mathbb{F}}A \to A,\ a\otimes b \mapsto ab.
\end{aligned}$$

is a free resolution of the $A\otimes_{\mathbb{F}}A$-module A in $A\otimes_{\mathbb{F}}A$-Mod.

(b) Show that applying the functor $A\otimes_{A\otimes_{\mathbb{F}}A}- : A\otimes_{\mathbb{F}}A\text{-Mod} \to \mathbb{F}\text{-Mod}$ and omitting the entry A yields a chain complex isomorphic to

$$\ldots \xrightarrow{0} A \xrightarrow{a\mapsto\bar{p}'a} A \xrightarrow{0} A \xrightarrow{a\mapsto\bar{p}'a} A \xrightarrow{0} A \xrightarrow{a\mapsto\bar{p}'a} A \xrightarrow{0} A \to 0,$$

where $p' = \Sigma_{k=0}^n k a_k x^{k-1}$ is the derivative of p.

(c) Derive a formula for the Hochschild homologies $H_n(A, A)$ for $n \in \mathbb{N}_0$ in terms of p and p'.

(d) Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{Z}/k\mathbb{Z}$ with $k \in \mathbb{N}$ prime. Show that $H_{2n}(A, A) = 0$ if $p \in \mathbb{F}[x]$ is irreducible and compute the Hochschild homologies for $p = x^k$.

Exercise 65: Let $\mathbb{F}$ be a field of characteristic 0, $V = \mathbb{F}^2$ and $\{q, p\}$ a basis of V. The **Weyl algebra** W is the quotient $W = T(V)/(p\otimes q - q\otimes p - 1)$ of the tensor algebra $T(V)$ by the two-sided ideal $I = (p\otimes q - q\otimes p - 1)$ generated by the element $p\otimes q - q\otimes p - 1 \in T(V)$.

(a) Show that $q \rhd x^k y^l = x^{k+1} y^l$ and $p \rhd x^k y^l = k x^{k-1} y^l + x^k y^{l+1}$ define a W-module structure on the polynomial ring $\mathbb{F}[x, y]$.

(b) Show that the elements

$$q^i p^j := \underbrace{q\otimes....\otimes q}_{i\times} \otimes \underbrace{p\otimes....\otimes p}_{j\times} + I \qquad i, j \in \mathbb{N}_0$$

form a basis of the Weyl algebra.

(c) Show that the following is a free resolution of W in $W\otimes_{\mathbb{F}} W^{op}$-Mod:

$$0 \to W\otimes_{\mathbb{F}} W^{op} \xrightarrow{g} W\otimes_{\mathbb{F}} W^{op}\otimes_{\mathbb{F}}\mathbb{F}^2 \xrightarrow{f} W\otimes_{\mathbb{F}} W^{op} \xrightarrow{\mu} W \to 0$$

where $\mu : W\otimes_{\mathbb{F}} W^{op} \to W$, $a\otimes a' \mapsto a \cdot a'$ is the multiplication map and

$f : W\otimes_{\mathbb{F}} W^{op}\otimes_{\mathbb{F}}\mathbb{F}^2 \to W\otimes_{\mathbb{F}} W^{op}, \quad a\otimes b\otimes x \mapsto a\otimes(xb) - (ax)\otimes b,$

$g : W\otimes_{\mathbb{F}} W^{op} \to W\otimes_{\mathbb{F}} W^{op}\otimes\mathbb{F}^2, \quad a\otimes b \mapsto a\otimes(qb)\otimes p - (aq)\otimes b\otimes p - a\otimes(pb)\otimes q + (ap)\otimes b\otimes q.$

(d) Compute the Hochschild homologies $H_n(W, W)$ for all $n \in \mathbb{N}_0$.

Exercise 66: Let R be a ring, $X_\bullet$ a chain complex in R-Mod and consider the chain complex $\Delta^1_\bullet = 0 \to R \xrightarrow{(\mathrm{id}, -\mathrm{id})} R \oplus R \to 0$ in R-Mod-R, with the left and right module structures given by left and right multiplication.

(a) Compute the homologies of $\Delta^1_\bullet\otimes X_\bullet$ with the Künneth formula.
(b) Show that the chain complex $\Delta^1_\bullet\otimes X_\bullet$ is chain homotopy equivalent to $X_\bullet$.

Exercise 67: Let R be a commutative ring.

(a) Show that the tensor product of chain complexes defines a functor

$$\otimes : \mathrm{Ch}_{R\text{-Mod}} \times \mathrm{Ch}_{R\text{-Mod}} \to \mathrm{Ch}_{R\text{-Mod}}.$$

(b) Denote by $\tau : \mathrm{Ch}_{R\text{-Mod}} \times \mathrm{Ch}_{R\text{-Mod}} \to \mathrm{Ch}_{R\text{-Mod}} \times \mathrm{Ch}_{R\text{-Mod}}$ the flip functor with $\tau(X_\bullet, X'_\bullet) = (X'_\bullet, X_\bullet)$ and $\tau(f_\bullet, g_\bullet) = (g_\bullet, f_\bullet)$ for all chain complexes $X_\bullet, X'_\bullet$ and chain maps $f_\bullet$, $g_\bullet$. Show that the functors $\otimes$ and $\otimes^{op} = \otimes\tau$ are naturally isomorphic.

Exercise 68: Let G be an abelian group and $n \in \mathbb{N}$. The **Moore space** $M(G, n)$ is a path-connected topological space $X = M(G, n)$ with $H_n(X, \mathbb{Z}) \cong G$ and $H_k(X, \mathbb{Z}) \cong 0$ for all $k \in \mathbb{N} \setminus \{n\}$.

Compute the singular homologies $H_m(X, k)$ of a Moore space $X = M(G, n)$ with coefficients in $k = \mathbb{Z}/q\mathbb{Z}$ for (i) $G = \mathbb{Z}$, (ii) $G = \mathbb{Z}/r\mathbb{Z}$ with $q, r \in \mathbb{N}$.

Exercise 69: Let G be a group and M a trivial $\mathbb{Z}[G]$-module.

(a) Show that the following sequence is exact:

$$0 \to H_n(G, \mathbb{Z})\otimes_{\mathbb{Z}} M \to H_n(G, M) \to \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(G, \mathbb{Z}), M) \to 0.$$

(b) Show that $H_2(F, M) \cong H_2(F, \mathbb{Z})\otimes_{\mathbb{Z}} M$ for any free group F and trivial $\mathbb{Z}[G]$-module M.

6.5 Exercises for Chapter 5

Exercise 70: Let G be a group.

(a) Show that the family of sets $(G^{\times n})_{n\in\mathbb{N}_0}$ is a simplicial set with the maps

$$s_n^i : G^{\times n} \to G^{\times(n+1)},\ (g_1, ..., g_n) \mapsto (g_1, ..., g_i, 1, g_{i+1}, ..., g_n)$$
$$d_n^i : G^{\times n} \to G^{\times(n-1)},\ (g_1, ..., g_n) \mapsto \begin{cases} (g_2, ..., g_n) & i = 0 \\ (g_1, ..., g_i g_{i+1}, ..., g_n) & 1 \leq i \leq n-1 \\ (g_1, ..., g_{n-1}) & i = n \end{cases}$$

(b) Determine under which conditions on G this is a simplicial object in Grp.

Exercise 71: Let V be a set. A **combinatorial simplicial complex** is a subset $K \subset \mathcal{P}(V)$ consisting of finite non-empty subsets $M \subset V$ such that $M' \in K$ implies $M \in K$ for all subsets $\emptyset \neq M \subset M'$. A combinatorial simplicial complex is called **ordered** if the set V is ordered.

(a) Show that every simplicial complex $(X, \{\sigma_\alpha : \Delta^{n_\alpha} \to X\}_{\alpha\in I})$ determines a combinatorial simplicial complex given by

$$V = \{\sigma_\alpha(e_k) \mid \alpha \in I, k \in \{0, ..., n_\alpha\}\} \qquad K = \{\sigma_\alpha(\{e_0, ..., e_{n_\alpha}\}) \mid \alpha \in I\}.$$

(b) Show that every ordered combinatorial simplicial complex $K \subset \mathcal{P}(V)$ defines a simplicial set $S^K : \Delta^{+op} \to \text{Set}$ given by

$$S^K([n+1]) = \{(v_0, ..., v_n) \mid \{v_0, ..., v_n\} \in K, v_0 \leq v_1 \leq .... \leq v_n\}$$
$$S^K(\alpha)(v_0, ..., v_n) = (v_{\alpha(0)}, ..., v_{\alpha(m)}) \quad \text{for} \quad \alpha \in \text{Hom}_{\Delta^+}([m+1], [n+1])$$

and determine $S(\delta_n^i)$ and $S(\sigma_{n+1}^i)$ for $n \in \mathbb{N}$ and $0 \leq i \leq n$.

(c) Show that $S^K = S^{K'}$ for ordered combinatorial simplicial complexes K, K' implies $K = K'$.

Remark: Recall that an ordered set $(V, \leq)$ is a set V with a relation $\leq$ that satisfies (i) $v \leq v$ for all $v \in V$, (ii) $v \leq w$ and $w \leq v$ implies $v = w$, (iii) $u \leq v$ and $v \leq w$ implies $u \leq w$ and (iv) for all $v, w \in V$ one has $v \leq w$ or $w \leq v$.

Exercise 72: Let V be an ordered set with $|V| = n+1$ and $K = \mathcal{P}(V) \setminus \{\emptyset\}$. Show that the geometric realisation $\text{Geom}(S^K)$ of the associated simplicial set $S^K : \Delta^{+op} \to \text{Set}$ is homeomorphic to Δ^n.

Exercise 73: Let $S : \Delta^{+op} \to \text{Set}$ be a simplicial set. Show that the geometric realisation $\text{Geom}(S)$ has the structure of a semisimplicial complex by proceeding as follows:

(a) An element $x \in S_n = S([n+1])$ is called non-degenerate if the only monotonic surjection $\sigma : [n+1] \to [k+1]$ with $x \in S(\sigma)(S_k)$ is $\sigma = 1_{[n+1]}$.

Show that for every element $x \in S_n$, there is a unique $k \in \{0, ..., n\}$ and a unique non-degenerate $y \in S_k$ with $x = S(\sigma)y$ for some monotonic surjection $\sigma : [n+1] \to [k+1]$.

(b) Define $I = \cup_{n\in\mathbb{N}_0}\{x \in S_n \mid x \text{ non-degenerate}\}$ and consider for $x \in I \cap S_n$ the continuous maps $\sigma_x : \Delta^n \to \mathrm{Geom}(S), p \mapsto [(x,p)]$. Show that $(\mathrm{Geom}(S), \{\sigma_x\}_{x\in I})$ is a semisimplicial complex.

Exercise 74: Let Δ^+_{inj} be the subcategory of the simplex category Δ^+ with objects $[n+1]$ for $n \in \mathbb{N}_0$ and with $\mathrm{Hom}_{\Delta^+_{inj}}([n+1],[m+1]) = \{\alpha : [n+1] \to [m+1] \mid \alpha \text{ monotonic and injective}\}$. A **semisimplicial object** in $\mathcal{C}$ is a functor $K : \Delta^{+op}_{inj} \to \mathcal{C}$ and a **morphism of semisimplicial objects** is a natural transformation $\eta : K \to K'$.

For a semisimplicial object K in an abelian category $\mathcal{A}$ we define for $n \in \mathbb{N}_0$ and each morphism $\alpha : [m+1] \to [n+1]$ in Δ^+

$$LK([n+1]) = LK_n = \coprod_{\substack{0\leq p\leq n,\\ \sigma:[n+1]\twoheadrightarrow[p+1]}} K_p, \qquad\qquad LK(\alpha)\circ\iota_\sigma = \iota_{\sigma_\alpha}\circ K(\alpha_\sigma)$$

where $K_p = K([p+1])$, the coproduct runs over all monotonic surjections $\sigma : [n+1] \to [p+1]$ with $0 \leq p \leq n$, $\iota_\sigma : K_p \to LK_n$ denote the canonical inclusions, $\sigma_\alpha : [m+1] \to [q+1]$ is the unique surjection and $\alpha_\sigma : [q+1] \to [p+1]$ the unique injection in Δ^+ with $\alpha_\sigma \circ \sigma_\alpha = \sigma \circ \alpha$.

(a) Show that this defines a simplicial object $LK : \Delta^{+op} \to \mathcal{A}$.

(b) Show that this induces a functor $L : \mathrm{Fun}(\Delta^{+op}_{\mathrm{inj}}, \mathcal{A}) \to \mathrm{Fun}(\Delta^{+op}, \mathcal{A})$ from the category of semisimplicial objects in $\mathcal{A}$ to the category of simplicial objects in $\mathcal{A}$.

(c) Show that the functor $K : \mathrm{Ch}_{\mathcal{A}\geq 0} \to \mathrm{Fun}(\Delta^{+op}, \mathcal{A})$ in the Dold-Kan correspondence factorises as $K = LG$ with an appropriate functor $G : \mathrm{Ch}_{\mathcal{A}\geq 0} \to \mathrm{Fun}(\Delta^{+op}_{inj}, \mathcal{A})$.

Exercise 75: Let $\mathcal{A}$ be an abelian category and $R : \mathrm{Fun}(\Delta^{+op}, \mathcal{A}) \to \mathrm{Fun}(\Delta^{+op}_{\mathrm{inj}}, \mathcal{A})$ the functor that restricts a simplicial object $S : \Delta^{+op} \to \mathcal{A}$ to the subcategory $\Delta^{+op}_{\mathrm{inj}} \subset \Delta^{+op}$.

Denote by $K : \mathrm{Ch}_{\mathcal{A}\geq 0} \to \mathrm{Fun}(\Delta^+_{op}, \mathcal{A})$ and $N' : \mathrm{Fun}(\Delta^{+op}, \mathcal{A}) \to \mathrm{Ch}_{\mathcal{A}\geq 0}$ the functors from the proof of the Dold-Kan correspondence and by $N'' : \mathrm{Fun}(\Delta^{+op}_{inj}, \mathcal{A}) \to \mathrm{Ch}_{\mathcal{A}\geq 0}$ the corresponding functor for semisimpicial objects with $N' = N''R$.

(a) Show that the functor $L : \mathrm{Fun}(\Delta^{+op}_{\mathrm{inj}}, \mathcal{A}) \to \mathrm{Fun}(\Delta^{+op}, \mathcal{A})$ from Exercise 74 is left adjoint to the restriction functor $R : \mathrm{Fun}(\Delta^{+op}, \mathcal{A}) \to \mathrm{Fun}(\Delta^{+op}_{inj}, \mathcal{A})$.

(b) Show that the functor $G : \mathrm{Ch}_{\mathcal{A}\geq 0} \to \mathrm{Fun}(\Delta^{+op}_{inj}, \mathcal{A})$ from Exercise 74 is left adjoint to $N'' : \mathrm{Fun}(\Delta^{+op}_{inj}, \mathcal{A}) \to \mathrm{Ch}_{\mathcal{A}\geq 0}$.

(c) Conclude that the functor K is left adjoint to N'.

Exercise 76: A **left Kan extension** of a functor $K : \mathcal{C} \to \mathcal{E}$ along a functor $I : \mathcal{C} \to \mathcal{D}$ is a pair (K', η) of a functor $K' : \mathcal{D} \to \mathcal{E}$ and a natural transformation $\eta : K \to K'I$ with the following universal property: For every pair (G, γ) of a functor $G : \mathcal{D} \to \mathcal{E}$ and a natural transformation $\gamma : K \to GI$, there is a unique natural transformation $\gamma' : K' \to G$ with $\gamma = (\gamma' I) \circ \eta$.

Show that for every functor $K : \Delta^{+op}_{inj} \to \mathcal{A}$ into an abelian category $\mathcal{A}$ the functor $LK : \Delta^{+op} \to \mathcal{A}$ from Exercise 74 is a left Kan extension of K along the inclusion functor $I : \Delta^{+op}_{inj} \to \Delta^{+op}$.

Exercise 77: Show that for every group G the category $\mathbb{F}[G]$-Mod has a monoidal category structure, such that the forgetful functor $F : \mathbb{F}[G]\text{-Mod} \to \mathrm{Vect}_{\mathbb{F}}$ is a strict monoidal functor.

Exercise 78: Show that the geometric realization functor Geom : $\mathrm{Fun}(\Delta^{+op}, \mathrm{Set}) \to \mathrm{Top}$ is left adjoint to the singular functor Sing : $\mathrm{Top} \to \mathrm{Fun}(\Delta^{+op}, \mathrm{Set})$

Exercise 79: Let $(G_n)_{n\in\mathbb{N}_0}$ be a family of groups with $G_0 = \{e\}$ and $(\rho_{m,n})_{m,n\in\mathbb{N}_0}$ a family of group homomorphisms $\rho_{m,n} : G_m \times G_n \to G_{m+n}$ with $\rho_{0,m} : \{e\} \times G_m \to G_m$, $(e, g) \mapsto g$ and $\rho_{m,0} : G_m \times \{e\} \to G_m$, $(g, e) \mapsto g$ and

$$\rho_{m+n,p} \circ (\rho_{m,n} \times \mathrm{id}_{G_p}) = \rho_{m,n+p} \circ (\mathrm{id}_{G_M} \times \rho_{n,p}) \quad \forall m, n, p \in \mathbb{N}_0.$$

(a) Show that this defines a strict monoidal category $(\mathcal{C}, \otimes)$ with non-negative integers $n \in \mathbb{N}_0$ as objects, $\mathrm{Hom}_{\mathcal{C}}(n, m) = \emptyset$ if $m \neq n$ and $\mathrm{Hom}_{\mathcal{C}}(n, n) = G_n$ and the tensor product given by $m\otimes n = n + m$ for all $n, m \in \mathbb{N}_0$ and $f\otimes g = \rho_{m,n}(f, g)$ for all $f \in G_m$, $g \in G_n$.

(b) Consider the permutation groups $G_n = S_n$ and find group homomorphisms $\rho_{m,n} : S_m \times S_n \to S_{m+n}$ that satisfy the conditions.

(c) Show that in (b) any strict monoidal functor $F : \mathcal{C} \to \mathcal{D}$ into a strict monoidal category $(\mathcal{D}, \otimes)$ is determined uniquely by $F(1)$ and $F(\tau)$ for the elementary transposition $\tau \in S_2$, $(1, 2) \mapsto (2, 1)$.

Exercise 80: Show that for any monoidal category $\mathcal{C}$ the category $S\mathcal{C} := \mathrm{Fun}(\Delta^{+op}, \mathcal{C})$ of simplicial objects and morphisms in $\mathcal{C}$ is also monoidal.

Exercise 81: Let k be a commutative ring. We consider

- the functor $F : \mathrm{Grp} \to \mathrm{Alg}_k$ that assigns to a group G its group algebra $k[G]$ and to a group homomorphism $f : G \to H$ the induced group homomorphism $k[f] : k[G] \to k[H]$, $g \mapsto f(g)$,
- the functor $G : \mathrm{Alg}_k \to \mathrm{Grp}$ that assigns to an algebra A its group $A^\times$ of units and to an algebra homomorphism $f : A \to B$ the induced group homomorphism $f^\times : A^\times \to B^\times$.

(a) Show that F is left adjoint to G.
(b) Determine the monad and the comonad associated with this adjunction.

Exercise 82: We consider the inclusion functor $G : \mathrm{Ab} \to \mathrm{Grp}$ and the abelisation functor $F = \mathrm{Ab} : \mathrm{Grp} \to \mathrm{Ab}$.

(a) Show that F is left adjoint to G.
(b) Determine the associated monad and comonad.

References

[B] K. Brown, Cohomology of Groups, Springer Graduate Texts in Mathematics 87.

[D] A. Dold, Lectures on Algebraic Topology, Springer Classics in Mathematics.

[tD] T. tom Diek, Algebraic Toplogy, EMS Textbooks in Mathematics.

[F] G. Friedman, An elementary illustrated introduction to simplicial sets, arXiv:0809.4221v5 [math.AT]

[H] A. Hatcher: Algebraic Topology, Cambridge University Press.

[HS] P. J. Hilton, U. Stammbach: A Course in Homological Algebra, Springer Graduate Texts in Mathematics 4.

[JS] J. Jantzen, J. Schwermer: Algebra, Springer.

[K] C. Kassel: Quantum Groups, Springer Graduate Texts in Mathematics 155, Springer.

[LS] G. Laures, M. Szymik, Grundkurs Topologie, Spektrum Akademischer Verlag.

[Mi] B. Mitchell, Theory of Categories, Academic Press, London-New York, 1965.

[Ma] W. S. Massey: A Basic course in Algebraic Topology, Springer Graduate Texts in Mathematics 56.

[McL1] S. Mac Lane, Homology, Springer Classics in Mathematics.

[McL2] S. Mac Lane, Categories for the working mathematician, Springer Graduate Texts in Mathematics 5.

[W] C. A. Weibel, An introduction to homological algebra, Cambridge Studies in Advanced Mathematics 38.

Index