# Hopf Algebras and Representation Theory of Hopf Algebras

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In preparing this lecture, I used the references in the bibliography and the following textbooks and lecture notes:


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# Contents

# 1 Algebras and modules over algebras

## 1.1 Algebras

In this subsection, we recall some basic concepts, definitions and constructions and discuss important examples of algebras. In the following we will always take *algebra* to mean *associative unital algebra*, and all algebra homomorphisms are assumed to be *unital* as well. For the tensor product of vector spaces $V$ and $W$ over $\mathbb{F}$, we use the notation $V\otimes W = V\otimes_{\mathbb{F}} W$.

**Definition 1.1.1:**

1. An **algebra** $A$ over a field $\mathbb{F}$ is a vector space $A$ over $\mathbb{F}$ together with a multiplication map $\cdot : A\times A \to A$ such that $(A, +, \cdot)$ is a unital ring and $a\cdot(\lambda a') = (\lambda a)\cdot a' = \lambda(a\cdot a')$ for all $a, a' \in A$ and $\lambda \in \mathbb{F}$. An algebra $A$ is called **commutative** if $a\cdot a' = a'\cdot a$ for all $a, a' \in A$.

2. An **algebra homomorphism** from an algebra $A$ to an algebra $B$ over $\mathbb{F}$ is an $\mathbb{F}$-linear map $\phi : A \to B$ that is also a unital ring homomorphism, i. e. satisfies $\phi(1_A) = 1_B$ and

$$\phi(a + a') = \phi(a) + \phi(a'), \quad \phi(\lambda a) = \lambda\phi(a), \quad \phi(a \cdot a') = \phi(a) \cdot \phi(a') \qquad \forall a, a' \in A, \lambda \in \mathbb{F}.$$

As an algebra can be viewed as a unital ring with a compatible vector space structure, the concepts of a unital subring, of a left right or two-sided ideal and of a quotient by an ideal have direct analogues for algebras. In particular, a left, right or two-sided ideal in an *algebra* $A$ is simply a left, right or two-sided ideal in the *ring* $A$. That such an ideal is also a linear subspace of $A$ follows because $\lambda a = (\lambda 1) \cdot a = a \cdot (\lambda 1) \in I$ for all $a \in I$ and $\lambda \in \mathbb{F}$. Consequently, the quotient $A/I$ by a two sided ideal $I \subset A$ is not only a ring, but also inherits a vector space structure and hence the structure of an algebra.

**Definition 1.1.2:** Let $\mathbb{F}$ be a field and $A$ an algebra over $\mathbb{F}$.

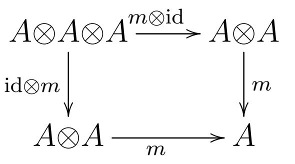
1. A **subalgebra** of $A$ is a subset $B \subset A$ that is an algebra with the restriction of the addition, scalar multiplication and multiplication, i. e. a subset $B \subset A$ with $1_A \in B$, $b + b' \in B$, $\lambda b \in B$, and $b \cdot b' \in B$ for all $b, b' \in B$ and and $\lambda \in \mathbb{F}$.

2. The **quotient algebra** of $A$ by a two-sided ideal $I \subset A$ is the quotient vector space $A/I$ with the multiplication map $\cdot : A/I \times A/I \to A/I$, $(a + I, a' + I) \mapsto aa' + I$.

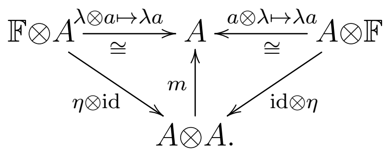
Before proceeding with examples, we give an equivalent definition of an algebra that is formulated purely in terms of vector spaces and linear maps. For this, note that we can view the unit $1 \in A$ as a linear map $\eta : \mathbb{F} \to A$, $\lambda \mapsto \lambda 1$. Similarly, we can interpret the multiplication as an $\mathbb{F}$-linear map $m : A\otimes A \to A$ instead of a map $\cdot : A \times A \to A$ that is compatible with scalar multiplication and satisfies the distributive laws. This follows because the distributive laws and the compatibility condition on scalar multiplication and algebra multiplication are equivalent to the statement that the map $\cdot$ is $\mathbb{F}$-bilinear. By the universal property of the tensor product, it therefore induces a unique linear map $m : A\otimes A \to A$ with $m(a\otimes b) = a \cdot b$. The remaining conditions are the associativity of the multiplication map $m$ and the condition that 1 is a unit, which can be stated as follows.

**Definition 1.1.3:**

1. An **algebra** $(A, m, \eta)$ over a field $\mathbb{F}$ is a vector space $A$ over $\mathbb{F}$ together with linear maps $m : A \otimes A \to A$ and $\eta : \mathbb{F} \to A$, the **multiplication** and the **unit**, such that the following two diagrams commute



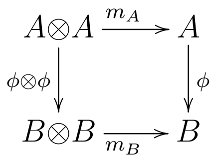

**associativity**

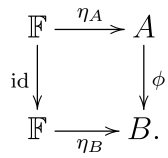



**unitality**

   An algebra $A$ is called **commutative** if $m^{op} := m \circ \tau = m$, where $\tau : A \otimes A \to A \otimes A$, $a \otimes b \mapsto b \otimes a$ is called the **flip map**.

2. An **algebra homomorphism** from an $\mathbb{F}$-algebra $A$ to an $\mathbb{F}$-algebra $B$ is a linear map $\phi : A \to B$ such that the following two diagrams commute




**Remark 1.1.4:**

1. Definition 1.1.3 can be generalised by replacing the field $\mathbb{F}$ with a commutative unital ring $R$, $\mathbb{F}$-vector spaces by $R$-modules, tensor products of vector spaces by tensor products of $R$-modules and $\mathbb{F}$-linear maps by homomorphisms of $R$-modules. This leads to the more general notion of an **algebra over a commutative unital ring**.

2. Note that the multiplication and unit do not play the same role in Definition 1.1.3. The multiplication map $m$ for a vector space $A$ is a *structure* - there may be many associative multiplication maps, and an algebra is specified by *choosing* one of them. The existence of a unit map $\eta$ that satisfies the conditions in Definition 1.1.3 is a *property* of the pair $(A, m)$. As two-sided units in monoids are unique, there there is at most one unit for $m$.

**Example 1.1.5:**

1. Every field $\mathbb{F}$ is an algebra over itself. If $\mathbb{F} \subset \mathbb{G}$ is a field extension, then $\mathbb{G}$ is an algebra over $\mathbb{F}$.
2. For every field $\mathbb{F}$, the $(n \times n)$-matrices with entries in $\mathbb{F}$ form an algebra $\mathrm{Mat}(n \times n, \mathbb{F})$ with the matrix addition, scalar multiplication and matrix multiplication. The diagonal matrices, the upper triangular matrices and the lower triangular matrices form subalgebras of $\mathrm{Mat}(n \times n, \mathbb{F})$.

3. For any $\mathbb{F}$-vector space $V$, the linear endomorphisms of $V$ form an algebra $\mathrm{End}_{\mathbb{F}}(V)$ with the pointwise addition and scalar multiplication and composition.

4. For any algebra $A$, the vector space $A$ with the opposite multiplication $m_{op} : A \otimes A \to A$, $a \otimes b \mapsto b \cdot a$ is an algebra. It is called the **opposite algebra** and denoted $A^{op}$.

5. For two $\mathbb{F}$-algebras $A$ and $B$, the vector space $A\otimes B$ has a canonical algebra structure with multiplication and unit

$$\begin{aligned} m_{A\otimes B} &: (A\otimes B)\otimes(A\otimes B) \to A\otimes B & (a\otimes b)\otimes(a'\otimes b') &\mapsto (aa')\otimes(bb') \\ \eta_{A\otimes B} &: \mathbb{F} \to A\otimes B, & \lambda &\mapsto \lambda(1\otimes 1). \end{aligned}$$

This algebra is called the **tensor product** of the algebras $A$ and $B$ and denoted $A\otimes B$.

6. The maps $f : \mathbb{N}_0 \to \mathbb{F}$ form an associative algebra over $\mathbb{F}$ with

$$(f+g)(n) = f(n) + g(n) \qquad (\lambda f)(n) = \lambda\, f(n) \qquad (f\cdot g)(n) = \Sigma_{k=0}^{n} f(n-k)g(k).$$

This is called the algebra of **formal power series** with coefficients in $\mathbb{F}$ and denoted $\mathbb{F}[[x]]$. The name is due to the following. If we describe a power series $\Sigma_{n\in\mathbb{N}_0} a_n x^n$ by its coefficient function $f : \mathbb{N}_0 \to \mathbb{F}$, $n \mapsto a_n$, then the formulas above give the familiar addition, scalar multiplication and multiplication law for power series

$$\begin{aligned} &\Sigma_{n\in\mathbb{N}_0} a_n x^n + \Sigma_{n\in\mathbb{N}_0} b_n x^n = \Sigma_{n\in\mathbb{N}_0} (a_n + b_n) x^n \\ &\lambda \Sigma_{n\in\mathbb{N}_0} a_n x^n = \Sigma_{n\in\mathbb{N}_0} \lambda a_n x^n \\ &(\Sigma_{n\in\mathbb{N}_0} a_n x^n) \cdot (\Sigma_{n\in\mathbb{N}_0} b_n x^n) = \Sigma_{n\in\mathbb{N}_0} (\Sigma_{k=0}^{n} a_{n-k} b_k) x^n. \end{aligned}$$

7. The polynomials with coefficients in $\mathbb{F}$ form a subalgebra

$$\mathbb{F}[x] = \{f : \mathbb{N}_0 \to \mathbb{F} \,|\, f(n) = 0 \text{ for almost all } n \in \mathbb{N}_0\} \subset \mathbb{F}[[x]].$$

8. For any set $M$ and any $\mathbb{F}$-algebra $A$, the maps $f : M \to A$ form an algebra over $\mathbb{F}$ with the pointwise addition, scalar multiplication and multiplication.

An important example of an algebra that will be used extensively in the following is the tensor algebra of a vector space $V$ over $\mathbb{F}$. As a vector space, it is the direct sum $T(V) = \oplus_{n=0}^{\infty} V^{\otimes n}$, where $V^{\otimes 0} := \mathbb{F}$ and $V^{\otimes n} := V\otimes ...\otimes V$ is the $n$-fold tensor product of $V$ with itself for $n \in \mathbb{N}$. Its algebra structure is given by the concatenation, and the unit is the element $1 = 1_{\mathbb{F}} \in \mathbb{F}$. The symmetric and the exterior algebra of $V$ are two further examples of algebras associated with a vector space $V$. They are obtained by taking quotients of $T(V)$ by two-sided ideals.

**Example 1.1.6:** Let $V$ be a vector space over $\mathbb{F}$.

1. The **tensor algebra** of $V$ is the vector space $T(V) = \oplus_{n=0}^{\infty} V^{\otimes n}$ with the multiplication

$$(v_1\otimes ...\otimes v_m) \cdot (w_1\otimes ...\otimes w_n) = v_1\otimes ...\otimes v_m\otimes w_1\otimes ...\otimes w_n$$

for all $v_1, ..., v_m, w_1, ..., w_n \in V$ and $n, m \in \mathbb{N}_0$, where $v_1\otimes ...\otimes v_n := 1_{\mathbb{F}}$ for $n = 0$. It is an algebra over $\mathbb{F}$ with unit $1_{\mathbb{F}} \in V^0$. The injective $\mathbb{F}$-linear map $\iota_V : V \to T(V)$, $v \mapsto v$ is called the **inclusion map**.

2. If $B$ is a basis of $V$, then $B_\otimes = \{b_1\otimes ...\otimes b_n \,|\, n \in \mathbb{N}_0, b_i \in B\}$ is a basis of $T(V)$.

3. The tensor algebra is $\mathbb{N}$-**graded**: it is given as the direct sum $T(V) = \oplus_{n=0}^{\infty} V^{\otimes n}$ of the linear subspaces $V^{\otimes n}$, and one has $V^{\otimes m} \cdot V^{\otimes n} \subset V^{\otimes (n+m)}$ for all $m, n \in \mathbb{N}_0$.

4. The tensor algebra has the following **universal property**:

   For every $\mathbb{F}$-linear map $\phi : V \to A$ into an $\mathbb{F}$-algebra $A$, there is a unique algebra homomorphism $\tilde{\phi} : T(V) \to A$ such that the following diagram commutes

$$\begin{array}{ccc} V & \xrightarrow{\ \phi\ } & A \\ {\scriptstyle \iota_V}\big\downarrow & \nearrow_{\exists!\tilde{\phi}} & \\ T(V). & & \end{array}$$

5. The **symmetric algebra** of $V$ is the quotient algebra of $T(V)$

$$S(V) = T(V)/(v \otimes w - w \otimes v)$$

   by the two-sided ideal $(v \otimes w - w \otimes v)$ generated by the elements $v \otimes w - w \otimes v$ for $v, w \in V$.

6. The **exterior algebra** of $V$ or **alternating algebra** of $V$ is the quotient algebra of $T(V)$

$$\Lambda V = T(V)/(v \otimes v)$$

   by the two-sided ideal $(v \otimes v)$ generated by the elements $v \otimes v$ for $v \in V$.

Tensor algebras plays a similar role for algebras as free groups for groups and free modules for modules. In particular, it allows one to describe an algebra in terms of *generators* and *relations*. If $A$ is an algebra over $\mathbb{F}$, then by the universal property of the tensor algebra, the linear map $\mathrm{id}_A : A \to A$ induces an algebra homomorphism $\phi : T(A) \to A$. As its kernel is a two-sided ideal in $T(A)$, the quotient $T(A)/\ker(\phi)$ has a canonical algebra structure, and the algebra homomorphism $\phi$ induces an algebra isomorphism $\phi' : T(A)/\ker(\phi) \xrightarrow{\sim} A$. Hence, one can view any algebra as a quotient of its tensor algebra.

If $B$ is a basis of $A$, then the set $B_\otimes$ from Example 1.1.6 is a basis of $T(A)$, and we can express any element in $T(A)$ as a linear combination of elements of $B_\otimes$, or, equivalently, linear combinations of products of elements in $B$, where $1_\mathbb{F} = V^{\otimes 0}$ is viewed as the empty product. Hence, we can characterise $A$ uniquely by specifying a basis of $A$, specifying a subset $U \subset \ker(\phi)$ that generates the two-sided ideal $\ker(\phi)$, and expressing the elements of $U$ as linear combinations of products of basis elements. This is called a *presentation* of $A$.

**Definition 1.1.7:** Let $A$ be an algebra and $\phi : T(A) \to A$ the algebra homomorphism with $\phi \circ \iota_A = \mathrm{id}_A$ induced by the universal property of the tensor algebra. A **presentation** of $A$ is a pair $(B, U)$ of a basis $B \subset A$ and a subset $U \subset \ker(\phi)$ that generates the two-sided ideal $\ker(\phi) \subset T(A)$. The elements of $B$ are called **generators** and the elements of $U$ **relations**. One often lists the relations $u \in U$ as equations $u = 0$ for $u \in U$.

For simplicity, one usually presents an algebra $A$ with as few generators and relations as possible. In particular, one requires that the generators are linearly independent and that no proper subset $U' \subsetneq U$ generates $\ker(\phi)$. However, even if these additional conditions are imposed, an algebra may have many different presentations that are not related in an obvious way. In general, it is very difficult to decide if two algebras presented in terms of generators and relations are isomorphic, and there are no algorithms that solve this problem in the general case.

In some textbooks, presentations of algebras are defined in terms of the **free algebra** generated by a set $B$ and relations in this free algebra. This is equivalent to our definition, since the tensor algebra of a vector space $V$ with basis $B$ is canonically isomorphic to the *free algebra* generated by the set $B$ - both are characterised by the same universal property. For a detailed discussion, see [Ka, Chapter I.2 and Chapter II.5].

Another algebra that will be important in the following is the *universal enveloping algebra* of a Lie algebra, which is obtained as a quotient of its tensor algebra. This example is important because its universal property relates Lie algebra homomorphisms and modules over Lie algebras to algebra homomorphisms and modules over algebras. It is also the starting point for the construction of many Hopf algebra structures that arise from Lie algebras.

**Definition 1.1.8:** Let $\mathbb{F}$ be a field.

1. A **Lie algebra** over $\mathbb{F}$ is an $\mathbb{F}$-vector space $\mathfrak{g}$ together with an antisymmetric $\mathbb{F}$-linear map $[\,,\,]: \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}$, $x\otimes y \mapsto [x, y]$, the **Lie bracket**, that satisfies the **Jacobi identity**
$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0 \qquad \forall x, y, z \in \mathfrak{g}.$$

2. A **Lie algebra homomorphism** from $(\mathfrak{g}, [\,,\,]_{\mathfrak{g}})$ to $(\mathfrak{h}, [\,,\,]_{\mathfrak{h}})$ is an $\mathbb{F}$-linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ with
$$[\,,\,]_{\mathfrak{h}} \circ (\phi\otimes\phi) = \phi \circ [\,,\,]_{\mathfrak{g}}.$$

Every associative (not necessarily unital) algebra $A$ has a canonical Lie algebra structure with the **commutator** $[\,,\,] : A\otimes A \to A$, $a\otimes b \mapsto [a, b] = a \cdot b - b \cdot a$ as the Lie bracket, whose Jacobi identity follows from the associativity of $A$. If we talk about the Lie algebra structure of an associative algebra or Lie algebra homomorphisms into an associative algebra $A$, we assume that $A$ is equipped with this Lie bracket.

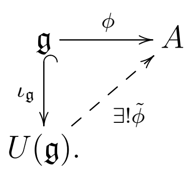
**Example 1.1.9:** Let $\mathfrak{g}$ be a Lie algebra.

1. The **universal enveloping algebra** of $\mathfrak{g}$ is the quotient algebra $U(\mathfrak{g}) = T(\mathfrak{g})/I$, where $I = (x\otimes y - y\otimes x - [x, y])$ is the two-sided ideal generated by the elements $x\otimes y - y\otimes x - [x, y]$ for $x, y \in \mathfrak{g}$.

2. If $\mathfrak{g}$ is finite-dimensional with a basis $B = \{x_1, ..., x_n\}$ then the Lie bracket of the basis elements takes the form $[x_i, x_j] = \Sigma_{k=1}^n f_{ij}{}^k x_k$ with coefficients $f_{ij}{}^k \in \mathbb{F}$, the **structure constants** of $\mathfrak{g}$. In this case $U(\mathfrak{g})$ is presented with generators $x_1, ..., x_n$ and relations $x_i\otimes x_j - x_j\otimes x_i = \Sigma_{k=1}^n f_{ij}{}^k x_k$.

3. The universal enveloping algebra has the following **universal property**:

   The **inclusion maps** $\iota_{\mathfrak{g}} : \mathfrak{g} \to U(\mathfrak{g})$, $x \mapsto x + I$ are Lie algebra homomorphisms. For any Lie algebra homomorphism $\phi : \mathfrak{g} \to A$ into an algebra $A$, there is a unique algebra homomorphism $\tilde{\phi} : U(\mathfrak{g}) \to A$ such that the following diagram commutes



4. If $B=(b_i)_{i\in I}$ is an ordered basis of $\mathfrak{g}$, then the Lie bracket of $\mathfrak{g}$ is given by
$$[b_i,b_j]=\Sigma_{k\in I}f_{ij}{}^k b_k$$
with $f_{ij}{}^k=-f_{ji}{}^k\in\mathbb{F}$, $f_{ij}^k=0$ for almost all $k\in I$, and the universal enveloping algebra $U(\mathfrak{g})$ is presented with generators $b_i$, $i\in I$, and relations
$$b_i\otimes b_j-b_j\otimes b_i=[b_i,b_j]=\Sigma_{k\in I}f_{ij}{}^k b_k.$$
The set $B'=\{\iota_{\mathfrak{g}}(b_{i_1})\cdots\iota_{\mathfrak{g}}(b_{i_n})\,|\,n\in\mathbb{N}_0, i\in I, i_1\leq i_2\leq ...\leq i_n\}$ is a basis of $U(\mathfrak{g})$, the **Poincaré-Birkhoff-Witt basis** of $U(\mathfrak{g})$.

5. The universal enveloping algebra is a **filtered algebra**:
It is the union $U(\mathfrak{g})=\cup_{n=0}^{\infty}U^n(\mathfrak{g})$ of subspaces $U^n(\mathfrak{g})=\oplus_{k=0}^n V^{\otimes k}/I$, which satisfy $U^0(\mathfrak{g})\subset U^1(\mathfrak{g})\subset ...$ and $U^m(\mathfrak{g})\cdot U^n(\mathfrak{g})\subset U^{n+m}(\mathfrak{g})$ for all $n,m\in\mathbb{N}_0$.

The universal property of the universal enveloping algebra is a direct consequence of the universal property of the tensor algebra and the fact that for any Lie algebra morphism $\phi:\mathfrak{g}\to A$, the induced algebra homomorphism $\phi':T(V)\to A$ satisfies $\phi'(x\otimes y-y\otimes x-[x,y])=0$ for all $x,y\in\mathfrak{g}$. The proof of the Poincaré-Birkhoff-Witt Theorem, which states that the Poincaré-Birkhoff-Witt basis is a basis of $U(\mathfrak{g})$, and the proof that $U(\mathfrak{g})$ is filtered are more cumbersome and proceed by induction. These proofs and more details on universal enveloping algebras can be found in [Di] and [Se, Chapter II].

Another class of important examples are *group algebras*. They will become the simplest examples of Hopf algebras. The group algebra of a group $G$ is simply its group ring $R[G]$, in the case where the ring $R=\mathbb{F}$ is a field. In this case, the group ring becomes an algebra over $\mathbb{F}$ with the pointwise multiplication by $\mathbb{F}$ as scalar multiplication.

**Example 1.1.10:** (The **group algebra** $\mathbb{F}[G]$)
Let $G$ be a group and $\mathbb{F}$ a field. The free $\mathbb{F}$-vector space generated by $G$
$$\langle G\rangle_{\mathbb{F}}=\{f:G\to\mathbb{F}\,|\,f(g)=0\text{ for almost all }g\in G\}$$
with the pointwise addition and scalar multiplication and the **convolution product**:
$$(f_1+f_2)(g)=f_1(g)+f_2(g)\qquad(\lambda f)(g)=\lambda f(g)\qquad(f_1\cdot f_2)(g)=\Sigma_{h\in G}f_1(gh^{-1})\cdot f_2(h)$$
is an associative unital $\mathbb{F}$-algebra, called the group **group algebra** of $G$ and denoted $\mathbb{F}[G]$. The maps $\delta_g:G\to\mathbb{F}$ with $\delta_g(g)=1$ and $\delta_g(h)=0$ for $g\neq h$ form a basis of $\mathbb{F}[G]$.

**Remark 1.1.11:**

1. In terms of the maps $\delta_g:G\to\mathbb{F}$ the multiplication of $\mathbb{F}[G]$ takes the form $\delta_g\cdot\delta_h=\delta_{gh}$ for all $g,h\in G$. In the following we therefore write $g$ for $\delta_g$ and denote elements of $\mathbb{F}[G]$ by $f=\Sigma_{g\in G}\lambda_g\,g$ with $\lambda_g\in\mathbb{F}$ for all $g\in G$. The algebra structure of $\mathbb{F}[G]$ is then given by
$$(\Sigma_{g\in G}\lambda_g\,g)+(\Sigma_{h\in G}\mu_h\,h)=\Sigma_{g\in G}(\lambda_g+\mu_g)\,g$$
$$\lambda\,(\Sigma_{g\in G}\lambda_g\,g)=\Sigma_{g\in G}(\lambda\lambda_g)\,g$$
$$(\Sigma_{g\in G}\lambda_g\,g)\cdot(\Sigma_{h\in G}\mu_h\,h)=\Sigma_{g\in G}\left(\Sigma_{h\in G}\lambda_{gh^{-1}}\mu_h\right)\,g.$$

2. A group homomorphism $\rho:G\to H$ induces an algebra homomorphism $\phi_\rho:\mathbb{F}[G]\to\mathbb{F}[H]$, $\Sigma_{g\in G}\lambda_g\,g\mapsto\Sigma_{g\in G}\lambda_g\,\rho(g)$, but not every algebra homomorphism $\phi:\mathbb{F}[G]\to\mathbb{F}[H]$ arises from a group homomorphism. Similarly, for every subgroup $U\subset G$, the linear subspace $\mathrm{span}_{\mathbb{F}}(U)\cong\mathbb{F}[U]\subset\mathbb{F}[G]$ is a subalgebra, but not all subalgebras of $\mathbb{F}[G]$ arise this way.

## 1.2 Modules over algebras

In this section, we discuss basic properties and examples of modules over algebras. As an algebra $A$ over $\mathbb{F}$ is a unital ring with a compatible vector space structure over $\mathbb{F}$, a module over an algebra is simply defined as a module over the underlying ring. In particular, this ensures that all known constructions for modules over rings such as submodules, quotients, direct sums, products and tensor products can be carried out for algebras as well. Some basic results and constructions for modules over rings are summarised in Appendix A.1.

The only difference to modules over general rings is that modules over an algebra $A$ are vector spaces over $\mathbb{F}$ and all module homomorphisms between them are $\mathbb{F}$-linear maps. Hence, we can also view submodules, quotients, direct sums and tensor products of modules over an algebra $A$ as submodules, quotients, direct sums and tensor products of vector spaces that carry additional structure, namely a representation of $A$.

**Definition 1.2.1:** Let $\mathbb{F}$ be a field, $A$ an algebra over $\mathbb{F}$ and $G$ a group.

1. A **left module** over $A$ or a **representation** of $A$ is an abelian group $(V,+)$ together with a map $\rhd : A \times V \to V$, $(a,v) \mapsto a \rhd v$ that satisfies for all $a, b \in A$, $v, v' \in V$

   $$a\rhd(v+v') = a\rhd v+a\rhd v', \quad (a+b)\rhd v = a\rhd v+b\rhd v, \quad (a\cdot b)\rhd v = a\rhd(b\rhd v), \quad 1\rhd v = v.$$

2. A **homomorphism of representations**, an **$A$-linear map** or a **homomorphism of $A$-left modules** from $(V, \rhd_V)$ to $(W, \rhd_W)$ is a group homomorphism $\phi : (V,+) \to (W,+)$ with $\phi(a \rhd_V v) = a \rhd_W \phi(v)$ for all $a \in A$ and $v \in V$.

3. A **representation** of $G$ over $\mathbb{F}$ is an $\mathbb{F}[G]$-left module. A **homomorphism of group representations** is a homomorphism of $\mathbb{F}[G]$-left modules.

Note that the left modules over $A$ form a category $A$-Mod. The objects of $A$-Mod are left modules over $A$, and the morphisms of $A$-Mod are homomorphisms of $A$-left modules. In fact, the category $A$-Mod has additional structure, namely that of $\mathbb{F}$-*linear category.*

There are analogous concepts of right modules over $A$ and of $(A, A)$-bimodules, see Remark 1.1.2 in Appendix A.1. The former are equivalent to left modules over the algebra $A^{op}$ and the latter to left modules over the algebra $A\otimes A^{op}$. In the following we use the term *module* over $A$ as a synonym of *left module* over $A$.

It important to note that there are several equivalent definitions of algebra and group representations in the literature, which are summarised in the following remark and then used interchangeably, without further comments.

**Remark 1.2.2:**

1. A representation of a $\mathbb{F}$-algebra $A$ can be defined equivalently as a pair $(V, \rho)$ of an $\mathbb{F}$-vector space $V$ and an algebra homomorphism $\rho : A \to \mathrm{End}_{\mathbb{F}}(V)$.

   This holds because every $A$-module $(V, \rhd)$ has a canonical $\mathbb{F}$-vector space structure with the scalar multiplication $\lambda v := (\lambda 1) \rhd v$, and the map $\rho : A \to \mathrm{End}_{\mathbb{F}}(V)$ with $\rho(a)v := a \rhd v$ is an algebra homomorphism. Conversely, each algebra homomorphism $\rho : A \to \mathrm{End}_{\mathbb{F}}(V)$ determines an $A$-left module structure on $V$ given by $a \rhd v := \rho(a)v$.

2. A representation of a group $G$ can be defined equivalently as a pair $(V, \rho)$ of a vector space $V$ and a group homomorphism $\rho : G \to \mathrm{Aut}_{\mathbb{F}}(V)$.

   This follows because a representation of $G$ is equivalent to a pair $(V, \rho')$ of an $\mathbb{F}$-vector space $V$ and an algebra homomorphism $\rho' : \mathbb{F}[G] \to \mathrm{End}_{\mathbb{F}}(V)$ by 1. As every algebra homomorphism $\rho' : \mathbb{F}[G] \to \mathrm{End}_{\mathbb{F}}(V)$ induces a group homomorphism $\rho : G \to \mathrm{End}_{\mathbb{F}}(V)$, $g \mapsto \rho'(\delta_g)$ and vice versa, this corresponds to the choice of an $\mathbb{F}$-vector space $V$ and a group homomorphism $\rho : G \to \mathrm{End}_{\mathbb{F}}(V)$. As $\rho(g^{-1}) \circ \rho(g) = \rho(g) \circ \rho(g)^{-1} = \mathrm{id}_V$ for all $g \in G$, one has $\rho(g) \in \mathrm{Aut}_{\mathbb{F}}(V)$ for all $g \in G$.

3. Equivalently, we can view a representation of an $\mathbb{F}$-algebra $A$ as a pair $(V, \rhd)$ of an $\mathbb{F}$-vector space $V$ and an $\mathbb{F}$-linear map $\rhd : A \otimes V \to V$ such that the following diagrams commute

$$\begin{array}{ccc} A\otimes A\otimes V & \xrightarrow{\mathrm{id}\otimes\rhd} & A\otimes V \\ {\scriptstyle m\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle\rhd} \\ A\otimes V & \xrightarrow[\rhd]{} & V \end{array} \qquad \begin{array}{ccc} V & \xleftarrow[\cong]{\lambda\otimes v\mapsto\lambda v} & \mathbb{F}\otimes V \\ {\scriptstyle\rhd}\uparrow & \swarrow{\scriptstyle\eta\otimes\mathrm{id}} & \\ A\otimes V. & & \end{array}$$

   A homomorphism of representations can then be defined as a $\mathbb{F}$-linear map $\phi : V \to W$ such that the following diagram commutes

$$\begin{array}{ccc} A\otimes V & \xrightarrow{\rhd_V} & V \\ {\scriptstyle \mathrm{id}\otimes\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ A\otimes W & \xrightarrow{\rhd_W} & W. \end{array}$$

   This follows from the $\mathbb{F}$-bilinearity of the map $\rhd : A \times V \to V$. By the universal property of the tensor product, it induces a linear map $\rhd : A \otimes V \to V$. The commutativity of the diagrams is then a direct consequence of the definitions.

**Example 1.2.3:**

1. Any group $G$ can be represented on any $\mathbb{F}$-vector space $V$ by the **trivial representation** $\rho : G \to \mathrm{Aut}_{\mathbb{F}}(V)$, $g \mapsto \mathrm{id}_V$.

2. Any $\mathbb{F}$-vector space $V$ carries representations of $\mathrm{Aut}_{\mathbb{F}}(V)$ and $\mathrm{End}_{\mathbb{F}}(V)$.

3. A representation of the group $\mathbb{Z}/2\mathbb{Z}$ on an $\mathbb{F}$-vector space $V$ corresponds to the choice of an involution on $V$, i. e. an $\mathbb{F}$-linear map $I : V \to V$ with $I \circ I = \mathrm{id}_V$. If $\mathrm{char}(\mathbb{F}) \neq 2$, this amounts to a decomposition $V = V_+ \oplus V_-$, where $V_\pm = \ker(I \mp \mathrm{id}_V)$.

4. A representations of the group $\mathbb{Z}$ on an $\mathbb{F}$-vector space $V$ corresponds to the choice of an automorphism $\phi \in \mathrm{Aut}_{\mathbb{F}}(V)$. This holds because a group homomorphism $\rho : \mathbb{Z} \to \mathrm{Aut}_{\mathbb{F}}(V)$ is determined uniquely by the automorphism $\rho(1) = \phi$, and every automorphism $\phi \in \mathrm{Aut}_{\mathbb{F}}(V)$ determines a representation of $\mathbb{Z}$ given by $\rho(z) = \phi^z$.

5. For any $\mathbb{F}$-vector space $V$ there is a representation of $S_n$ on $V^{\otimes n}$, which is given by $\rho : S_n \to \mathrm{Aut}_{\mathbb{F}}(V^{\otimes n})$ with $\rho(\sigma)(v_1 \otimes ... \otimes v_n) = v_{\sigma(1)} \otimes .... \otimes v_{\sigma(n)}$ for all $v_1, .., v_n \in V$.

6. A representation of the polynomial algebra $\mathbb{F}[x]$ on an $\mathbb{F}$-vector space $V$ amounts to the choice of an endomorphism of $V$.

This follows because any algebra homomorphism $\phi : \mathbb{F}[x] \to \mathrm{End}_\mathbb{F}(V)$ is determined uniquely by $\rho(x) \in \mathrm{End}_\mathbb{F}(V)$ and any element $\phi \in \mathrm{End}_\mathbb{F}(V)$ determines an algebra homomorphism $\rho : \mathbb{F}[x] \to \mathrm{End}_\mathbb{F}(V)$, $\Sigma_{n\in\mathbb{N}_0} a_n x^n \mapsto \Sigma_{n\in\mathbb{N}_0} a_n \phi^n$.

7. Let $V, W$ be vector spaces over $\mathbb{F}$. The representations of the tensor algebra $T(V)$ on $W$ correspond bijectively to $\mathbb{F}$-linear maps $\phi : V \to \mathrm{End}_\mathbb{F}(W)$.

   This follows because the restriction of a representation $\rho : T(V) \to \mathrm{End}_\mathbb{F}(W)$ to $\iota_V(V) \subset T(V)$ defines an $\mathbb{F}$-linear map from $V$ to $\mathrm{End}_\mathbb{F}(W)$. Conversely, every $\mathbb{F}$-linear map $\phi : V \to \mathrm{End}_\mathbb{F}(W)$ induces an algebra homomorphism $\rho : T(V) \to \mathrm{End}_\mathbb{F}(W)$ with $\rho \circ \iota = \phi$ by the universal property of the tensor algebra.

8. Let $\mathfrak{g}$ be a Lie algebra and $V$ a vector space over $\mathbb{F}$. Representations of the universal enveloping algebra $U(\mathfrak{g})$ on $V$ correspond bijectively to representations of $\mathfrak{g}$ on $V$, i. e. Lie algebra homomorphisms $\phi : \mathfrak{g} \to \mathrm{End}_\mathbb{F}(V)$.

   This follows because any algebra homomorphism $\rho : U(\mathfrak{g}) \to \mathrm{End}_\mathbb{F}(V)$ satisfies

   $$[\rho \circ \iota_\mathfrak{g}(x), \rho \circ \iota_\mathfrak{g}(y)] = \rho(\iota_\mathfrak{g}(x) \cdot \iota_\mathfrak{g}(y) - \iota_\mathfrak{g}(y) \cdot \iota_\mathfrak{g}(x)) = \rho \circ \iota_\mathfrak{g}([x, y]) \quad \forall x, y \in \mathfrak{g}.$$

   Hence $\rho \circ \iota_\mathfrak{g} : \mathfrak{g} \to \mathrm{End}_\mathbb{F}(V)$ is a Lie algebra morphism. Conversely, for any Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathrm{End}_\mathbb{F}(V)$ there is a unique algebra homomorphism $\rho : U(\mathfrak{g}) \to \mathrm{End}_\mathbb{F}(V)$ with $\rho \circ \iota_\mathfrak{g} = \phi$ by the universal property of $U(\mathfrak{g})$.

9. Any algebra is a left module over itself with the module structure given by left multiplication $\rhd : A \otimes A \to A$, $a \otimes b \mapsto a \cdot b$ and a left module over $A^{op}$ with respect to right multiplication $\rhd : A^{op} \otimes A \to A$, $a \otimes b \mapsto b \cdot a$. Combining the two yields an $A \otimes A^{op}$-left module structure on $A$ with $\rhd : (A \otimes A^{op}) \otimes A \to A$, $(a \otimes b) \otimes c \mapsto a \cdot c \cdot b$.

10. If $\phi : A \to B$ is an algebra homomorphism, then every $B$-module $V$ becomes an $A$-module with the module structure given by $a \rhd v := \phi(a) \rhd v$ for all $v \in V$. This is called the **pullback** of the $B$-module structure on $V$ by $\phi$.

11. In particular, for any subalgebra $U \subset A$, the inclusion map $\iota : U \to A$, $u \mapsto u$ is an injective algebra homomorphism and induces a $U$-left module structure on any $A$-left module $V$. This is called the **restriction** of the $A$-module structure to $U$.

12. If $(V, \rhd_V)$ and $(W, \rhd_W)$ are modules over $\mathbb{F}$-algebras $A$ and $B$, then

    $$\rhd : (A \otimes B) \otimes (V \otimes W) \to V \otimes W, \qquad (a \otimes b) \rhd (v \otimes w) = (a \rhd_V v) \otimes (b \rhd_W w)$$

    defines an $A \otimes B$ module structure on $V \otimes W$.

We will now focus on representations of groups or, equivalently, group algebras. In contrast to representations of algebras, group representations are naturally compatible with the tensor product of vector spaces. More specifically, if $V$ and $W$ are vector spaces over $\mathbb{F}$ that carry representations of a group $G$, then their tensor product $V \otimes W$ also carries a canonical representation of $G$. Moreover, we have a trivial representation of $G$ on $\mathbb{F}$, and for each representation of $G$ on $V$, a representation on the dual vector space $V^* = \mathrm{Hom}_\mathbb{F}(V, \mathbb{F})$. This follows by a direct computation verifying the conditions in Definition 1.2.1.

**Proposition 1.2.4:** Let $G$ be a group, $\mathbb{F}$ a field and $\rho_V : G \to \mathrm{Aut}_\mathbb{F}(V)$, $\rho_W : G \to \mathrm{Aut}_\mathbb{F}(W)$ representations of $G$ on $\mathbb{F}$-vector spaces $V$, $W$. Then the following are representations of $G$:

- the **trivial representation of group** $\rho_{\mathbb{F}} : G \to \mathrm{Aut}_{\mathbb{F}}(\mathbb{F})$, $g \mapsto \mathrm{id}_V$.
- the **tensor product representation**: $\rho_{V\otimes W} : G \to \mathrm{Aut}_{\mathbb{F}}(V\otimes_{\mathbb{F}} W)$, $g \mapsto \rho_V(g)\otimes\rho_W(g)$.
- the **dual representation**: $\rho_{V^*} : G \to \mathrm{Aut}_{\mathbb{F}}(V^*)$, $g \mapsto \rho_{V^*}(g)$ with $\rho_{V^*}(g)\alpha = \alpha \circ \rho(g^{-1})$.

Proposition 1.2.4 defines group representations on multiple tensor products of representation spaces of a group $G$, of the underlying field $\mathbb{F}$ and of dual vector spaces of representation spaces. However, when working with tensor product of vector spaces, one is not only interested in these tensor products themselves but also in certain canonical isomorphisms or homomorphisms between them. These are

- the **associativity isomorphism**

$$a_{U,V,W} : (U\otimes V)\otimes W \xrightarrow{\sim} U\otimes(V\otimes W), \quad (u\otimes v)\otimes w \mapsto u\otimes(v\otimes w)$$

- the **right and left unit isomorphisms**

$$r_V : V\otimes\mathbb{F} \xrightarrow{\sim} V, \quad v\otimes\lambda \mapsto \lambda v \qquad\qquad l_V : \mathbb{F}\otimes V \xrightarrow{\sim} V, \quad \lambda\otimes v \mapsto \lambda v$$

- the **flip isomorphism** $c_{U,V} : U\otimes V \xrightarrow{\sim} V\otimes U, \quad u\otimes v \mapsto v\otimes u$
- the **left and right evaluation maps**

$$\mathrm{ev}^L_V : V\otimes V^* \to \mathbb{F}, \quad v\otimes\alpha \mapsto \alpha(v) \qquad\qquad \mathrm{ev}^R_V : V^*\otimes V \to \mathbb{F}, \quad \alpha\otimes v \mapsto \alpha(v)$$

If $V$ is finite-dimensional with an ordered basis $B = (b_1, ..., b_n)$ and dual basis $B^* = (\beta^1, .., \beta^n)$,

- the **left and right coevaluation maps**

$$\mathrm{coev}^L_V : \mathbb{F} \to V^*\otimes V, \quad \lambda \mapsto \lambda\Sigma_{i=1}^n \beta^i\otimes b_i \qquad\qquad \mathrm{coev}^R_V : \mathbb{F} \to V\otimes V^*, \quad \lambda \mapsto \lambda\Sigma_{i=1}^n b_i\otimes\beta^i.$$

Note that the left and right coevaluation maps do not depend on the choice of the basis of $V$, although a basis and its dual are used in their definition. This follows because any other ordered basis $C = (c_1, ..., c_n)$ of $V$ with dual basis $C^* = (\gamma^1, ..., \gamma^n)$ is related to $B$ by an automorphism $\phi \in \mathrm{Aut}_{\mathbb{F}}(V)$ with $c_i = \phi(b_i) = \Sigma_{j=1}^n \phi_{ji} b_j$ and $\beta^i = \phi^*(\gamma^i) = \Sigma_{j=1}^n \phi_{ij}\gamma^j$. This implies that the coevaluation maps for the two bases are equal.

Given these canonical linear maps associated with tensor products of vector spaces, it is natural to ask if these linear maps become homomorphisms of representations, when the tensor products are equipped with the tensor product of group representations, the field $\mathbb{F}$ with the trivial representation and dual vector spaces with dual representations. That this is indeed the case follows by a simple computation verifying the conditions from Definition 1.2.1, 2. and 3., which is left as an exercise.

**Proposition 1.2.5:** Let $G$ be a group and $\mathbb{F}$ a field. Then:

1. The **associativity isomorphism** $a_{U,V,W}$ is an isomorphism of representations from $\rho_{(U\otimes V)\otimes W}$ to $\rho_{U\otimes(V\otimes W)}$ for all representations $\rho_U$, $\rho_V$, $\rho_W$ of $G$ on $U, V, W$.
2. The **right and left unit isomorphisms** $r_V$ and $l_V$ are isomorphisms of representations from $\rho_{\mathbb{F}\otimes V}$ and to $\rho_{V\otimes\mathbb{F}}$ to $\rho_V$ for all representations $\rho_V$ of $G$ on $V$

3. The **flip isomorphism** $c_{U,V}$ is an isomorphism of representations from $\rho_{U\otimes V}$ to $\rho_{V\otimes U}$ for all representations $\rho_U$, $\rho_V$ of $G$ on $U, V$.

4. The **left and right evaluation maps** $\mathrm{ev}^L_V$ and $\mathrm{ev}^R_V$ are are homomorphisms of representations from $\rho_{V\otimes V^*}$ and $\rho_{V^*\otimes V}$ to $\rho_{\mathbb{F}}$ for all representations $\rho_V$ of $G$ on $V$.

5. If $\dim_{\mathbb{F}}(V) < \infty$, the **left and right coevaluation maps** $\mathrm{coev}^L_V$ and $\mathrm{coev}^R_V$ are homomorphisms of representations from $\rho_{\mathbb{F}}$ to $\rho_{V^*\otimes V}$ and $\rho_{V\otimes V^*}$ for all representations $\rho_V$ of $G$ on $V$.

Clearly, Propositions 1.2.4 and 1.2.5 have no counterparts for representations of algebras. For an $\mathbb{F}$-algebra $A$, the map $\rho_{\mathbb{F}} : A \to \mathrm{End}_{\mathbb{F}}(\mathbb{F})$, $a \mapsto \mathrm{id}_{\mathbb{F}}$ is in general *not* an algebra homomorphism, since this would imply $\rho_{\mathbb{F}}(\mu a) = \mathrm{id}_{\mathbb{F}} = \mu\rho(a) = \mu \mathrm{id}_{\mathbb{F}}$ for all $\mu \in \mathbb{F}$ and $a \in A$. Neither does one obtain an algebra homomorphism $\rho_{\mathbb{F}} : A \to \mathrm{End}_F(F)$ by setting $\rho(a)\lambda = 0$ for all $\lambda \in \mathbb{F}$ and $a \in A$, since this contradicts the condition $\rho_{\mathbb{F}}(1) = \mathrm{id}_{\mathbb{F}}$.

Similarly, if $\rho_V : A \to \mathrm{End}_{\mathbb{F}}(V)$ and $\rho_W : A \to \mathrm{End}_{\mathbb{F}}(W)$ are algebra homomorphisms, then the map $\rho_{V\otimes W} : A \to \mathrm{End}_{\mathbb{F}}(V\otimes W)$, $a \mapsto \rho_V(a)\otimes\rho_W(a)$ is in general not an algebra homomorphism, since this would imply $\rho_{V\otimes W}(\mu a) = \mu\rho_{V\otimes W}(a) = \rho_V(\mu a)\otimes\rho_W(\mu a) = \mu^2\rho_{V\otimes W}(a)$ for all $\mu \in \mathbb{F}$ and $a \in A$. Finally, it is not possible to define a representation on the dual vector space $V^*$ in analogy to the one for a group since the elements of an algebra do not necessarily have multiplicative inverses. The linear map $\rho' : A \to \mathrm{End}_{\mathbb{F}}(V^*)$, $\rho'(a)\alpha = \alpha \circ \rho(a)$ is a representation of $A^{op}$, not of $A$, since $\rho'(a \cdot b) = \rho'(b) \circ \rho'(a)$ for all $a, b \in A$.

Hence, to obtain counterparts of Propositions 1.2.4 and 1.2.5 for representations of algebras over $\mathbb{F}$, we need to consider algebras that are equipped with *additional structures* that define a trivial representation on the underlying field, representations on tensor product and representations on dual vector spaces. The requirement that the linear maps from Proposition 1.2.5 are homomorphisms of representations then induce compatibility conditions between these structures. This leads us to the concepts of *bialgebras* and *Hopf algebras*.

# 2 Bialgebras and Hopf algebras

## 2.1 Bialgebras

We now investigate which additional structures are required on an algebra $A$ to ensure that its representations satisfy counterparts of Propositions 1.2.4 and 1.2.5 for group representations. We start by considering only the trivial representation and representations on tensor products in Proposition 1.2.4 and the the associativity and the left and right unit isomorphisms in Proposition 1.2.5.

To obtain a representation of $A$ on $\mathbb{F}$, we require an algebra homomorphism $\epsilon : A \to \mathbb{F}$ and set $\rho_{\mathbb{F}} : A \to \mathrm{End}_{\mathbb{F}}(\mathbb{F})$, $\rho_{\mathbb{F}}(a)\lambda = \epsilon(a)\lambda$ for all $a \in A$, $\lambda \in \mathbb{F}$. To obtain representations on tensor products of representation spaces, we require an algebra homomorphism $\Delta : A \to A\otimes A$ and set $\rho_{V\otimes W} = (\rho_V \otimes \rho_W) \circ \Delta : A \to \mathrm{End}_{\mathbb{F}}(V\otimes W)$.

The requirements that the associativity isomorphism $a_{U,V,W} : (U\otimes V)\otimes W \to U\otimes(V\otimes W)$ and the left and right unit isomorphisms $r_V : V\otimes\mathbb{F} \to V$ and $l_V : \mathbb{F}\otimes V \to V$ from Proposition 1.2.5 are homomorphisms of representations impose additional conditions on the algebra homomorphisms $\epsilon$ and $\Delta$. If we define representations on tensor products as above, then the representations of $A$ on $U\otimes(V\otimes W)$ and on $U\otimes(V\otimes W)$ are given by

$$\rho_{(U\otimes V)\otimes W} = ((\rho_U\otimes\rho_V)\otimes\rho_W) \circ (\Delta\otimes\mathrm{id}_A) \circ \Delta \qquad \rho_{U\otimes(V\otimes W)} = ((\rho_U\otimes\rho_V)\otimes\rho_W) \circ (\mathrm{id}_A\otimes\Delta) \circ \Delta$$

Hence the associativity isomorphism is an isomorphism of representations for all $\rho_U, \rho_V, \rho_W$ if

$$(\Delta\otimes\mathrm{id}_A) \circ \Delta = (\mathrm{id}_A\otimes\Delta) \circ \Delta. \tag{1}$$

Similarly, we find that the representations of $A$ on $V\otimes\mathbb{F}$ and $\mathbb{F}\otimes V$ are given by

$$\rho_{V\otimes\mathbb{F}} = (\rho_V\otimes\rho_{\mathbb{F}}) \circ \Delta \qquad \rho_{\mathbb{F}\otimes V} = (\rho_{\mathbb{F}}\otimes\rho_V) \circ \Delta$$

with $\rho_{\mathbb{F}}$ as above. The linear maps $r_V : V\otimes\mathbb{F} \to V$, $v\otimes\lambda \mapsto \lambda v$ and $l_V : \mathbb{F}\otimes V \to V$, $\lambda\otimes V \mapsto \lambda v$ are isomorphisms of representations for all representations $\rho_V$ of $A$ on $V$ if

$$r_V \circ (\epsilon\otimes\mathrm{id}_A) \circ \Delta = l_V \circ (\mathrm{id}_A\otimes\epsilon) \circ \Delta = \mathrm{id}_A. \tag{2}$$

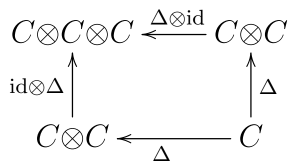
That conditions (1) and (2) are not only sufficient but necessary follows by considering tensor products of the representation of $A$ on itself by left multiplication. In this case one has

$$((\rho_A\otimes\rho_A) \circ \rho_A)(a)1_A = (\Delta\otimes\mathrm{id}) \circ \Delta(a) \qquad (\rho_A\otimes(\rho_A\otimes\rho_A))(a)1_A = (\mathrm{id}\otimes\Delta) \circ \Delta(a)$$
$$(\rho_A\otimes\rho_{\mathbb{F}})(a)(1_A\otimes 1_{\mathbb{F}}) = (\mathrm{id}\otimes\epsilon)(a) \qquad (\rho_{\mathbb{F}}\otimes\rho_A)(a)(1_{\mathbb{F}}\otimes 1_A) = (\epsilon\otimes\mathrm{id})(a)$$

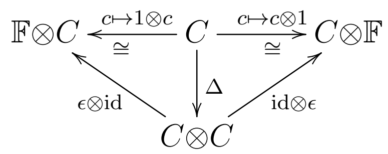
A vector space $A$ over $\mathbb{F}$ together with linear maps $\epsilon : A \to \mathbb{F}$ and $\Delta : A \to A\otimes A$ subject to (1) and (2) is called a *coalgebra*. If we also require that the linear maps $\epsilon : A \to \mathbb{F}$ and $\Delta : A \to A\otimes A$ are *algebra homomorphisms*, we obtain the concept of a *bialgebra*.

**Definition 2.1.1:**

1. A **coalgebra** over a field $\mathbb{F}$ is a triple $(C, \Delta, \epsilon)$ of an $\mathbb{F}$-vector space $C$ and linear maps $\Delta : C \to C\otimes C$, $\epsilon : C \to \mathbb{F}$, the **comultiplication** and the **counit**, such that the following diagrams commute



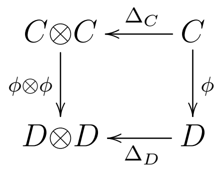

**coassociativity**

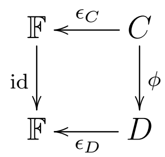



**counitality**

A coalgebra $(C, \Delta, \epsilon)$ is called **cocommutative** if $\Delta^{op} := \tau \circ \Delta = \Delta$, where $\tau : C\otimes C \to C\otimes C$, $c\otimes c' \mapsto c'\otimes c$ is called the **flip map**.

2. A **homomorphism of coalgebras** or **coalgebra map** from $(C, \Delta_C, \epsilon_C)$ to $(D, \Delta_D, \epsilon_D)$ is a linear map $\phi : C \to D$ for which the following diagrams commute




Note that the comultiplication $\Delta$ is a *structure* on $C$, whereas the counit is a *property*. One can show that for each pair $(C, \Delta)$ there is at most one linear map $\epsilon : C \to \mathbb{F}$ that satisfies the counitality condition (Exercise).

Note also that the commuting diagrams in Definition 2.1.1 are obtained from the corresponding diagrams for algebras and algebra homomorphisms in Definition 1.1.3 by reversing all arrows labelled by $m$ or $\eta$ and labelling them with $\Delta$ and $\epsilon$ instead. In this sense, the concepts of an algebra and a coalgebra are dual to each other, which motivates the name *coalgebra*.

**Remark 2.1.2:** For a coalgebra $(C, \Delta, \epsilon)$ we use the symbolic notation $\Delta(c) = \Sigma_{(c)}\, c_{(1)}\otimes c_{(2)}$, where $c_{(1)}$ and $c_{(2)}$ are interpreted as elements of $C$ and $\Sigma_{(c)}$ as a finite sum over elements of $C\otimes C$. This notation is called **Sweedler notation**.

It is symbolic since the properties of the tensor product imply that the elements $c_{(1)}$ and $c_{(2)}$ are not defined uniquely. However, this ambiguity causes no problems as long as all maps composed with $\Delta$ are $\mathbb{F}$-linear. The coassociativity of $\Delta$ then implies for all $c \in C$

$$(\Delta\otimes\mathrm{id}) \circ \Delta(c) = \Sigma_{(c)} c_{(1)(1)}\otimes c_{(1)(2)}\otimes c_{(2)} = \Sigma_{(c)} c_{(1)}\otimes c_{(2)(1)}\otimes c_{(2)(2)} = (\mathrm{id}\otimes\Delta) \circ \Delta(c).$$

This allows us to renumber the factors in the tensor product as

$$\Sigma_{(c)} c_{(1)}\otimes c_{(2)}\otimes c_{(3)} = \Sigma_{(c)} c_{(1)(1)}\otimes c_{(1)(2)}\otimes c_{(2)} = \Sigma_{(c)} c_{(1)}\otimes c_{(2)(1)}\otimes c_{(2)(2)}$$

and similarly for higher composites of $\Delta$.

**Example 2.1.3:**

1. For any coalgebra $(C, \Delta, \epsilon)$, the opposite comultiplication $\Delta^{op} = \tau \circ \Delta : C \to C\otimes C$ defines another coalgebra structure on $C$ with counit $\epsilon$. The coalgebra $(C, \Delta^{op}, \epsilon)$ is called the **opposite coalgebra** and denoted $C^{cop}$.

2. For any pair of coalgebras $(C, \Delta_C, \epsilon_C)$ and $(D, \Delta_D, \epsilon_D)$ the vector space $C \otimes D$ has a canonical coalgebra structure given by

$$\begin{aligned} &\Delta_{C\otimes D} = \tau_{23} \circ (\Delta_C \otimes \Delta_D) : C\otimes D \to (C\otimes D)\otimes(C\otimes D) && c\otimes d \mapsto \Sigma_{(c),(d)}\, c_{(1)}\otimes d_{(1)}\otimes c_{(2)}\otimes d_{(2)} \\ &\epsilon_{C\otimes D} = \epsilon_C \otimes \epsilon_D : C\otimes D \to \mathbb{F} && c\otimes d \mapsto \epsilon_C(c)\epsilon_D(d), \end{aligned}$$

with $\tau_{23} : C\otimes C\otimes D\otimes D \to C\otimes D\otimes C\otimes D$, $c\otimes c'\otimes d\otimes d' \mapsto c\otimes d\otimes c'\otimes d'$. This coalgebra structure on $C\otimes D$ is called the **tensor product** of the coalgebras $C$, $D$.

3. If $(A, m, \cdot)$ is a finite-dimensional algebra over $\mathbb{F}$, then the dual vector space $A^*$ has a coalgebra structure $(A^*, m^*, \eta^*)$, where $m^* : A^* \to (A\otimes A)^* = A^*\otimes A^*$ and $\eta^* : A^* \to \mathbb{F}$ are the duals of the multiplication and unit map given by

$$m^*(\alpha)(a\otimes b) = \alpha(ab) \qquad \eta^*(\alpha) = \alpha(1) \qquad \forall \alpha \in A^*, a, b \in A.$$

If $A$ is infinite-dimensional, then the dual of the multiplication is a linear map $m^* : A^* \to (A\otimes A)^*$. However, in this case we can have $A^*\otimes A^* \subsetneq (A\otimes A)^*$, and then $M^*$ does not define a coalgebra structure on $A^*$. However, we obtain a coalgebra structure on the **finite dual** $A^\circ = \{\alpha \in A^* \,|\, m^*(\alpha) \in A^*\otimes A^*\}$ (Exercise).

4. The dual statement of 3. holds also in the infinite-dimensional case. If $(C, \Delta, \epsilon)$ is a coalgebra over $\mathbb{F}$, then the $(C^*, \Delta^*|_{C^*\otimes C^*}, \epsilon^*)$ is an algebra over $\mathbb{F}$.

5. We consider the algebra $\mathrm{Mat}(n\times n, \mathbb{F})$ with the basis given by the elementary matrices $E_{ij}$ that have the entry 1 in the $i$th row and $j$th column and zero elsewhere. The dual basis of $\mathrm{Mat}(n \times n, \mathbb{F})^*$ is given by the **matrix elements** $M_{ij} : \mathrm{Mat}(n \times n, \mathbb{F}) \to \mathbb{F}$, $M \mapsto m_{ij}$. The comultiplication and counit of $\mathrm{Mat}(n \times n, \mathbb{F})^*$ are given by

$$\Delta(M_{ij}) = \Sigma_{k=1}^n M_{ik}\otimes M_{kj} \qquad \epsilon(M_{ij}) = \delta_{ij}.$$

As we can view a coalgebra as the dual of an algebra, we can also introduce subcoalgebras and left, right and two-sided coideals by dualising the concepts of subalgebras, left, right and two-sided ideals. In particular, we can take the quotient of a coalgebra by a two-sided coideal and obtain another coalgebra.

**Definition 2.1.4:** Let $(C, \Delta, \epsilon)$ be a coalgebra.

1. A **subcoalgebra** of $C$ is a linear subspace $I \subset C$ with $\Delta(I) \subset I\otimes I$.

2. A **left coideal** in $C$ is a linear subspace $I \subset C$ with $\Delta(I) \subset C\otimes I$, a **right coideal** is a linear subspace $I \subset C$ with $\Delta(I) \subset I\otimes C$ and a **coideal** is a linear subspace $I \subset C$ with $\Delta(I) \subset I\otimes C + C\otimes I$ and $\epsilon(I) = 0$.

**Proposition 2.1.5:** If $C$ is a coalgebra and $I \subset C$ a coideal, then the quotient space $C/I$ inherits a canonical coalgebra structure with the following **universal property**:

The canonical surjection $\pi : C \to C/I$ is a coalgebra map. For any coalgebra map $\phi : C \to D$ with $\ker(\phi) \subset I$ there is a unique coalgebra map $\tilde{\phi} : C/I \to D$ with $\tilde{\phi} \circ \pi = \phi$.

**Proof:**
As $I \subset C$ is a coideal, the map $\Delta' : C/I \to C/I \otimes C/I$, $c + I \mapsto (\pi\otimes\pi)\Delta(c)$ is well defined and satisfies $\Delta' \circ \pi = (\pi\otimes\pi) \circ \Delta$. Its coassociativity follows directly from the coassociativity of $\Delta$ and the surjectivity of $\pi$

$$\begin{aligned}(\Delta'\otimes\mathrm{id}) \circ \Delta' \circ \pi &= (\Delta'\otimes\mathrm{id}) \circ (\pi\otimes\pi) \circ \Delta = (\pi\otimes\pi\otimes\pi) \circ (\Delta\otimes\mathrm{id}) \circ \Delta \\ &= (\pi\otimes\pi\otimes\pi) \circ (\mathrm{id}\otimes\Delta) \circ \Delta = (\mathrm{id}\otimes\Delta') \circ (\pi\otimes\pi) \circ \Delta = (\mathrm{id}\otimes\Delta') \circ \Delta' \circ \pi.\end{aligned}$$

As $I$ is a coideal, we have $I \subset \ker(\epsilon)$ and obtain a linear map $\epsilon' : C/I \to \mathbb{F}$ with $\epsilon' \circ \pi = \epsilon$. The counitality of $\epsilon'$ then follows directly from the counitality of $\epsilon$ and the surjectivity of $\pi$

$$\begin{aligned}(\epsilon'\otimes\mathrm{id}) \circ \Delta' \circ \pi &= (\epsilon'\otimes\mathrm{id}) \circ (\pi\otimes\pi) \circ \Delta = (\mathrm{id}\otimes\pi) \circ (\epsilon\otimes\mathrm{id}) \circ \Delta = 1_{\mathbb{F}}\otimes\pi \\ (\mathrm{id}\otimes\epsilon') \circ \Delta' \circ \pi &= (\mathrm{id}\otimes\epsilon') \circ (\pi\otimes\pi) \circ \Delta = (\mathrm{id}\otimes\pi) \circ (\mathrm{id}\otimes\epsilon) \circ \Delta = 1_{\mathbb{F}}\otimes\pi.\end{aligned}$$

□

In a similar manner, we can dualise the concept of a *module* over an *algebra* to obtain the notion of a *comodule* over a *coalgebra*. One also can define subcomodules, quotients of comodules by subcomodules and related structures. All of them are obtained by taking the corresponding diagrams for modules over algebras and reversing all arrows labelled by $m$, $\eta$ and $\rhd$.

**Definition 2.1.6:** Let $(C, \Delta, \epsilon)$ be a coalgebra over $\mathbb{F}$.

1. A **left comodule** over $C$ is a pair $(V, \delta)$ of a vector space $V$ over $\mathbb{F}$ and a linear map $\delta : V \to C\otimes V$ such that the following diagrams commute

$$\begin{array}{ccc} C\otimes C\otimes V & \xleftarrow{\mathrm{id}\otimes\delta} & C\otimes V \\ {\scriptstyle \Delta\otimes\mathrm{id}}\uparrow & & \uparrow{\scriptstyle\delta} \\ C\otimes V & \xleftarrow{\delta} & V \end{array} \qquad \begin{array}{ccc} V & \xrightarrow[\cong]{v\mapsto 1\otimes v} & \mathbb{F}\otimes V \\ {\scriptstyle\delta}\downarrow & \nearrow{\scriptstyle \epsilon\otimes\mathrm{id}} & \\ C\otimes V. & & \end{array}$$

2. A **homomorphism of left comodules** or an $C$-**colinear map** from $(V, \delta_V)$ to $(W, \delta_W)$ is an $\mathbb{F}$-linear map $\phi : V \to W$ for which the following diagram commutes

$$\begin{array}{ccc} C\otimes V & \xleftarrow{\delta_V} & V \\ {\scriptstyle\mathrm{id}\otimes\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ C\otimes W & \xleftarrow{\delta_W} & W. \end{array}$$

Analogously, one defines right comodules over $C$ as left modules over $C^{cop}$ and $(C, C)$-bicomodules as left comodules over $C\otimes C^{cop}$. One often uses a variant of Sweedler notation and denotes the map $\delta : V \to V\otimes C$ for a *right* comodule $V$ by $\delta(v) = \Sigma_{(v)} v_{(0)}\otimes v_{(1)}$, where $v_{(0)}$ is understood as an element of $V$, $v_{(1)}$ as an element of $C$ and $\Sigma_{(v)}$ as a finite sum over elements of $C\otimes V$. By definition of a *right* comodule, one then has

$$(\delta\otimes\mathrm{id}_C)\circ\delta(v) = \Sigma_{(v)} v_{(0)(0)}\otimes v_{(0)(1)}\otimes v_{(1)} =: \Sigma_{(v)} v_{(0)}\otimes v_{(1)}\otimes v_{(2)} := \Sigma_{(v)} v_{(0)}\otimes v_{(1)(1)}\otimes v_{(1)(2)} = (\mathrm{id}\otimes\Delta)\circ\delta(v).$$

**Example 2.1.7:**

1. Every coalgebra $(C, \Delta, \epsilon)$ is a left comodule over itself with the comultiplication $\delta = \Delta$ and a right comodule over itself with the opposite comultiplication $\delta = \Delta^{op}$. This gives $C$ the structure of a $(C, C)$-bicomodule.

2. If $V$ is a left comodule over a coalgebra $(C, \Delta, \epsilon)$ with $\delta : V \to C \otimes V$, $v \mapsto \Sigma_{(v)} v_{(1)} \otimes v_{(0)}$, then it is a right module over $(C^*, \Delta^*, \epsilon^*)$ with $\rhd : V \otimes C^* \to V$, $v \otimes \alpha \mapsto \Sigma_{(v)} \alpha(v_{(1)})\, v_{(0)}$. However, not every right module over $(C^*, \Delta^*, \epsilon^*)$ arises from a comodule over $C$, if $C$ is infinite-dimensional. The modules that arise in this way are called **rational modules**.

3. If $(C, \Delta, \epsilon)$ is a coalgebra and $I \subset C$ a linear subspace, then the comultiplication of $C$ induces a left (right) module structure on the quotient $C/I$ if and only if $I$ is a left (right) coideal in $C$.

If we require that a coalgebra over $\mathbb{F}$ also has an algebra structure and that its comultiplication and counit are algebra homomorphisms, we obtain the notion of a bialgebra. Note that the condition that comultiplication and the counit are algebra homomorphisms is equivalent to imposing that the multiplication and unit are coalgebra homomorphisms. Hence, the coalgebra structure and the algebra structure enter the definition of a bialgebra on an equal footing.

**Definition 2.1.8:**

1. A **bialgebra** over a field $\mathbb{F}$ is a pentuple $(B, m, \eta, \Delta, \epsilon)$ such that $(B, m, \eta)$ is an algebra over $\mathbb{F}$, $(B, \Delta, \epsilon)$ is a coalgebra over $\mathbb{F}$ and $\Delta : B \to B \otimes B$ and $\epsilon : B \to \mathbb{F}$ are algebra homomorphisms.

2. A **bialgebra homomorphism** from a bialgebra $(B, m, \eta, \Delta, \epsilon)$ to a bialgebra $(B', m', \eta', \Delta', \epsilon')$ is a linear map $\phi : B \to B'$ that is a homomorphism of algebras and a homomorphism of coalgebras:

$$m' \circ (\phi \otimes \phi) = \phi \circ m \qquad \phi \circ \eta = \eta' \qquad \Delta' \circ \phi = (\phi \otimes \phi) \circ \Delta \qquad \epsilon' \circ \phi = \epsilon.$$

**Example 2.1.9:**

1. For any bialgebra $(B, m, \eta, \Delta, \epsilon)$, reversing the multiplication or the comultiplication yields another bialgebra structure on $B$. The three new bialgebras obtained in this way are given by $(B, m^{op}, \eta, \Delta, \epsilon)$, $(B, m, \eta, \Delta^{op}, \epsilon)$ and $(B, m^{op}, \eta, \Delta^{op}, \epsilon)$ and denoted $B^{op}$, $B^{cop}$ and $B^{op,cop}$, respectively.

2. For any two bialgebras $B, C$ over $\mathbb{F}$, the vector space $B \otimes C$ becomes a bialgebra when equipped with the tensor product algebra and coalgebra structures. This is called the **tensor product bialgebra** and denoted $B \otimes C$.

3. For any finite-dimensional bialgebra $(B, m, \eta, \Delta, \epsilon)$, the dual vector space has a canonical bialgebra structure given by $(B^*, \Delta^*, \epsilon^*, m^*, \eta^*)$. If $B$ is infinite-dimensional, the **finite dual** $B^\circ = \{b \in B \,|\, m^*(b) \in B^* \otimes B^*\}$ is a bialgebra with the restriction of the maps $m^* : B^* \to B^* \otimes B^*$, $\eta^* : B^* \to \mathbb{F}$, $\Delta^* : B^* \otimes B^* \to B^*$ and $\epsilon^* : \mathbb{F} \to B^*$ (Exercise).

## 2.2 Hopf algebras

We will now investigate which additional structure is needed on a bialgebra $H$ to obtain a representation on the dual vector space $V^*$ for every representation $\rho_V : H \to \mathrm{End}_{\mathbb{F}}(V)$ such that the evaluation maps and for finite-dimensional vector spaces the coevaluation maps from Proposition 1.2.5 become homomorphisms of representations. For this, we suppose that $V$ is a finite-dimensional vector space over $\mathbb{F}$ with an ordered basis $B = (b_1, ..., b_n)$ and denote by $B^* = (\beta^1, ..., \beta^n)$ the dual basis of $V^*$. Then the evaluation and coevaluation maps are given by

$$\begin{aligned} &\mathrm{ev}_V^R : V^*\otimes V \to \mathbb{F}, \quad \alpha\otimes v \mapsto \alpha(v) && \mathrm{ev}_V^L : V\otimes V^* \to \mathbb{F}, \quad v\otimes\alpha \mapsto \alpha(v) \\ &\mathrm{coev}_V^L : \mathbb{F} \to V^*\otimes V, \quad \lambda \mapsto \lambda\Sigma_{i=1}^n \beta^i\otimes b_i && \mathrm{coev}_V^R : \mathbb{F} \to V\otimes V^*, \quad \lambda \mapsto \lambda\Sigma_{i=1}^n b_i\otimes\beta^i. \end{aligned}$$

To obtain a representation of $H$ on $V^*$ for each representation $\rho_V : H \to \mathrm{End}_{\mathbb{F}}(V)$ we require an algebra homomorphism $S : H \to H^{op}$ and set $\rho_{V^*}(h)\alpha = \alpha \circ \rho_V \circ S(h)$ for all $h \in H$ and $\alpha \in V^*$. By definition of the representations on tensor products and of the right evaluation and coevaluation we then have for all $v \in V$, $\alpha \in V^*$ and $h \in H$

$$\begin{aligned} (\mathrm{ev}_V^R \circ \rho_{V^*\otimes V}(h))(\alpha\otimes v) &= (\mathrm{ev}_V^R \circ (\rho_{V^*}\otimes\rho_V) \circ \Delta(h))(\alpha\otimes v) = \Sigma_{(h)}\, \mathrm{ev}_V^R(\rho_{V^*}(h_{(1)})\alpha\otimes\rho_V(h_{(2)})v) \\ &= \Sigma_{(h)}\, \mathrm{ev}_V^R(\alpha \circ \rho_V(S(h_{(1)}))\otimes\rho_V(h_{(2)})v) = \Sigma_{(h)}\, (\alpha \circ \rho_V(S(h_{(1)})))(\rho_V(h_{(2)})v) \\ &= \Sigma_{(h)}\, (\alpha \circ \rho_V(S(h_{(1)})) \circ \rho_V(h_{(2)}))v = \Sigma_{(h)}\, (\alpha \circ \rho_V(S(h_{(1)}) \cdot h_{(2)}))(v) \end{aligned}$$

$$\begin{aligned} (\rho_{V\otimes V^*}(h) \circ \mathrm{coev}_V^R)(\lambda) &= \lambda\, \Sigma_{i=1}^n(\rho_{V\otimes V^*}(h))(b_i\otimes\beta^i) = \lambda\Sigma_{i=1}^n\Sigma_{(h)}\, \rho_V(h_{(1)})b_i\otimes\rho_{V^*}(h_{(2)})\beta^i \\ &= \lambda\Sigma_{i=1}^n\Sigma_{(h)}\, \rho_V(h_{(1)})b_i\otimes(\beta^i \circ \rho_V(S(h_{(2)})) = \lambda\Sigma_{i=1}^n\Sigma_{(h)}\, \rho_V(h_{(1)} \cdot S(h_{(2)}))b_i\otimes\beta^i, \end{aligned}$$

where we used the identity $\Sigma_{i=1}^n\phi(b_i)\otimes\beta^i = \Sigma_{i=1}^n b_i\otimes\phi^*(\beta^i)$ for $\phi \in \mathrm{End}_{\mathbb{F}}(V)$ and Sweedler notation for the comultiplication of $H$. As we have by definition of the trivial representation

$$(\rho_{\mathbb{F}}(h)\circ\mathrm{ev}_V^R)(\alpha\otimes v) = \rho_{\mathbb{F}}(h)\alpha(v) = \epsilon(h)\alpha(v) \quad (\mathrm{coev}_V^R\circ\rho_{\mathbb{F}}(h))(\lambda) = \epsilon(h)\mathrm{coev}_V^R(\lambda) = \lambda\epsilon(h)\, \Sigma_{i=1}^n b_i\otimes\beta^i,$$

we find that the right evaluation and coevaluation are homomorphisms of representations iff

$$m \circ (S\otimes\mathrm{id})\circ\Delta(h) = \Sigma_{(h)}S(h_{(1)}) \cdot h_{(2)} = \eta\circ\epsilon(h) = \Sigma_{(h)}h_{(1)} \cdot S(h_{(2)}) = m \circ (\mathrm{id}\otimes S)\circ\Delta(h) \quad (3)$$

for all $h \in H$. An analogous computation for the left evaluation and coevaluation yields

$$\mathrm{ev}_V^L \circ \rho_{V\otimes V^*}(h)(v\otimes\alpha) = \alpha \circ \rho_V(\Sigma_{(h)}S(h_{(2)}) \cdot h_{(1)})(v)$$

$$\rho_{V\otimes V^*}(h) \circ \mathrm{coev}_V^L(\lambda) = \lambda\Sigma_{i=1}^n\Sigma_{(h)}\, \beta^i\otimes\rho_V(S(h_{(2)}) \cdot h_{(1)})b_i$$

and hence the left evaluation and coevaluation are homomorphisms of representations iff

$$m^{op}\circ(S\otimes\mathrm{id})\circ\Delta(h) = \Sigma_{(h)}h_{(2)} \cdot S(h_{(1)}) = \eta\circ\epsilon(h) = \Sigma_{(h)}S(h_{(2)})h_{(1)} = m^{op}\circ(\mathrm{id}\otimes S)\circ\Delta(h) \quad (4)$$

for all $h \in H$. We will see in the following that imposing both (3) and (4) is too restrictive since it eliminates too many examples. Moreover, if $V$ is finite-dimensional we can obtain the left evaluation and coevaluation for a vector space $V$ from the right evaluation and coevaluation for $V^*$ by composing the latter with the canonical isomorphism $\mathrm{can} : V \xrightarrow{\sim} V^{**}$, $v \mapsto f_v$ with $f(\alpha) = \alpha(v_f)$ for all $\alpha \in V^*$. Hence, at least in the finite-dimensional case the left and right (co)evaluations are not independent and for this reason, we consider only the conditions (3) for the right evaluation. We will also see that these conditions imply that $S : H \to H^{op}$ is an algebra homomorphism and hence we do not impose this in our definition. This yields the concept of a Hopf algebra.

**Definition 2.2.1:** A bialgebra $(H, m, \eta, \Delta, \epsilon)$ is called a **Hopf algebra** if there is a linear map $S : H \to H$, called the **antipode**, with

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = m \circ (\mathrm{id} \otimes S) \circ \Delta = \eta \circ \epsilon.$$

**Remark 2.2.2:**

1. In Sweedler notation, the axioms for a Hopf algebra read:

$$\begin{array}{ll} \Sigma_{(a)} a_{(1)(1)} \otimes a_{(1)(2)} \otimes a_{(2)} = \Sigma_{(a)} a_{(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)} & \Sigma_{(a)} \epsilon(a_{(1)}) a_{(2)} = \Sigma_{(a)} a_{(1)} \epsilon(a_{(2)}) = a \\ \Sigma_{(ab)} (ab)_{(1)} \otimes (ab)_{(2)} = \Sigma_{(a)} \Sigma_{(b)} a_{(1)} b_{(1)} \otimes a_{(2)} b_{(2)} & \epsilon(ab) = \epsilon(a) \epsilon(b) \\ \Sigma_{(a)} S(a_{(1)}) a_{(2)} = \Sigma_{(a)} a_{(1)} S(a_{(2)}) = \epsilon(a)\, 1 & \forall a, b \in H. \end{array}$$

2. As indicated by the wording of Definition 2.2.1, the existence of the antipode is a *property* of a given bialgebra $(H, m, \eta, \Delta, \epsilon)$. An antipode either exists or it does not, but we will prove that there is at most one antipode for a given bialgebra structure on $H$.

3. Although the antipode replaces the inverse for representations of a finite group, it does not need to be an involution. In fact, it is not even guaranteed that the antipode of a Hopf algebra is invertible.

Before considering concrete examples, we give an alternative interpretation to the defining condition on the antipode in Definition 2.2.1. This condition is well-motivated from the representation theoretical viewpoint since it was obtained from the condition that the right evaluation and coevaluation maps should be homomorphisms of representations. However, it looks rather odd from a purely algebraic viewpoint, and it should be possible to gain a conceptual understanding without considering representations. This can be achieved by considering the *convolution product* of a bialgebra $H$, which is defined more generally for a pair $(A, C)$ of an algebra $A$ and a coalgebra $C$.

**Lemma 2.2.3:** Let $(A, m, \eta)$ be an algebra and $(C, \Delta, \epsilon)$ a coalgebra over $\mathbb{F}$.

1. The map $* : \mathrm{Hom}_{\mathbb{F}}(C, A) \otimes \mathrm{Hom}_{\mathbb{F}}(C, A) \to \mathrm{Hom}_{\mathbb{F}}(C, A)$, $f \otimes g \mapsto f * g = m \circ (f \otimes g) \circ \Delta$ defines an algebra structure on $\mathrm{Hom}_{\mathbb{F}}(C, A)$ with unit $\eta \circ \epsilon : C \to A$.

   The vector space $\mathrm{Hom}_{\mathbb{F}}(C, A)$ with this algebra structure is called the **convolution algebra** of $C$ and $A$, and $*$ is called the **convolution product** on $\mathrm{Hom}_{\mathbb{F}}(C, A)$.

2. $f \in \mathrm{Hom}_{\mathbb{F}}(C, A)$ is called **convolution invertible** if there is a $g \in \mathrm{Hom}_{\mathbb{F}}(C, A)$ with $g * f = f * g = \eta \circ \epsilon$. The convolution inverse of an element $f \in \mathrm{Hom}_{\mathbb{F}}(C, A)$ is unique, and the convolution invertible elements in $\mathrm{Hom}_{\mathbb{F}}(C, A)$ form a group with unit $\eta \circ \epsilon$.

**Proof:**
That the map $*$ is $\mathbb{F}$-linear follows from the $\mathbb{F}$-linearity of $\Delta : C \to C \otimes C$, $m : A \otimes A \to A$ and the properties of the tensor product. The associativity of $*$ follows from the associativity of $m$ and the coassociativity of $\Delta$

$$\begin{aligned} (f * g) * h &= m \circ ((f * g) \otimes h) \circ \Delta = m \circ (m \otimes \mathrm{id}) \circ (f \otimes g \otimes h) \circ (\Delta \otimes \mathrm{id}) \circ \Delta \\ &= m \circ (\mathrm{id} \otimes m) \circ (f \otimes g \otimes h) \circ (\mathrm{id} \otimes \Delta) \circ \Delta = m \circ (f \otimes (g * h)) \circ \Delta = f * (g * h). \end{aligned}$$

That $\eta \circ \epsilon : C \to A$ is a unit for $*$ follows because $\eta$ is the unit of $A$ and $\epsilon$ the counit of $C$

$$(\eta \circ \epsilon) * f = m \circ ((\eta \circ \epsilon) \otimes f) \circ \Delta = m \circ (\mathrm{id} \otimes f) \circ (\eta \otimes \mathrm{id}) \circ (\epsilon \otimes \mathrm{id}) \circ \Delta = m \circ (1_A \otimes f) = f$$
$$f * (\eta \circ \epsilon) = m \circ (f \otimes (\eta \circ \epsilon)) \circ \Delta = m \circ (f \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \eta) \circ (\mathrm{id} \otimes \epsilon) \circ \Delta = m \circ (f \otimes 1_A) = f.$$

This shows that the vector space $\mathrm{Hom}_{\mathbb{F}}(C, A)$ with the convolution product is an associative algebra over $\mathbb{F}$. The uniqueness of two-sided inverses and the statement that the elements with a two-sided inverse form a group holds for any monoid and hence for any associative algebra. $\square$

If $(B, m, \eta, \Delta, \epsilon)$ is a bialgebra, we can choose $(C, \Delta, \epsilon) = (B, \Delta, \epsilon)$ and $(A, m, \eta) = (B, m, \eta)$ and consider the convolution product on $\mathrm{End}_{\mathbb{F}}(B)$. Then it is natural to ask if the element $\mathrm{id}_B \in \mathrm{End}_{\mathbb{F}}(B)$ is convolution invertible. It turns out that this is the case if and only if $B$ is a Hopf algebra, and in this case, the convolution inverse is the antipode of $B$. This gives a more conceptual interpretation to the defining condition on the antipode in Definition 2.2.1. Moreover, the convolution product allows us to derive the basic properties of the antipode which are obtained from the following proposition.

**Proposition 2.2.4:** Let $(B, m, \eta, \Delta, \epsilon)$ and $(B', m', \eta', \Delta', \epsilon')$ bialgebras over $\mathbb{F}$.

1. The identity map $\mathrm{id}_B : B \to B$ is convolution invertible if and only if $(B, m, \eta, \Delta, \epsilon)$ is a Hopf algebra, and in this case the convolution inverse of $\mathrm{id}_B$ is the antipode $S : B \to B$.
2. If $f \in \mathrm{Hom}_{\mathbb{F}}(B, B)$ is a convolution invertible algebra homomorphism, then its convolution inverse $f^{-1} : B \to B$ is an anti-algebra homomorphism: $m^{op} \circ (f^{-1} \otimes f^{-1}) = f^{-1} \circ m$.
3. If $f \in \mathrm{Hom}_{\mathbb{F}}(B, B)$ is a convolution invertible coalgebra homomorphism, then its convolution inverse $f^{-1} : B \to B$ is an anti-coalgebra homomorphism: $\Delta \circ f^{-1} = (f^{-1} \otimes f^{-1}) \circ \Delta^{op}$.
4. If $\phi : B \to B'$ is an algebra homomorphism, then $L_\phi : \mathrm{Hom}_{\mathbb{F}}(B, B) \to \mathrm{Hom}_{\mathbb{F}}(B, B')$, $f \mapsto \phi \circ f$ is an algebra homomorphism with respect to the convolution products.
5. If $\psi : B \to B'$ is a coalgebra homomorphism, then $R_\phi : \mathrm{Hom}_{\mathbb{F}}(B', B') \to \mathrm{Hom}_{\mathbb{F}}(B', B)$, $g \mapsto g \circ \phi$ is an algebra homomorphism with respect to the convolution products.

**Proof:**
By definition, the identity map $\mathrm{id}_B$ is convolution invertible if and only if there is a linear map $f : B \to B$ with $f * \mathrm{id}_B = m \circ (f \otimes \mathrm{id}_B) \circ \Delta = \eta \circ \epsilon = m \circ (\mathrm{id}_B \otimes f) \circ \Delta = \mathrm{id}_B * f$. This is precisely the condition on the antipode in a Hopf algebra.

For 2. we note that $f^{-1} \circ \eta = m \circ (f^{-1} \circ \eta \otimes f \circ \eta) = m \circ (f^{-1} \otimes f) \circ \Delta \circ \eta = \eta' \circ \epsilon \circ \eta = \eta'$ for any algebra homomorphism $f$. We then consider the convolution algebra $\mathrm{Hom}_{\mathbb{F}}(B \otimes B, B)$, where $B \otimes B$ is equipped with the tensor product coalgebra structure from Example 2.1.3, 2. and show that both, $f^{-1} \circ m_B : B \otimes B \to B$ and $m^{op} \circ (f^{-1} \otimes f^{-1}) : B \otimes B \to B$ are convolution inverses of $m \circ (f \otimes f) = f \circ m : B \otimes B \to B$. The uniqueness of the convolution inverse then implies $m^{op} \circ (f^{-1} \otimes f^{-1}) = f^{-1} \circ m$, and this shows that $f$ is an anti-algebra homomorphism. To show that both, $f^{-1} \circ m_B$ and $m^{op} \circ (f^{-1} \otimes f^{-1})$, are convolution inverses of $f \circ m$, we compute

$$\begin{aligned}(f^{-1} \circ m) * (f \circ m)(b \otimes c) &= m \circ (f^{-1} \circ m \otimes f \circ m) \circ \Delta_{B \otimes B} = m \circ (f^{-1} \otimes f) \circ \Delta_B \circ m \\ &= (f^{-1} * f) \circ m = \eta \circ \epsilon \circ m = \eta \circ (\epsilon \otimes \epsilon) = \eta \circ \epsilon_{B \otimes B}.\end{aligned}$$

To evaluate the other term, we note that in Sweedler notation we have $f{*}g(b) = \Sigma_{(b)} f(b_{(1)})g(b_{(2)})$ and $\Delta_{B\otimes B}(b\otimes c) = \Sigma_{(b)(c)} b_{(1)}\otimes c_{(1)}\otimes b_{(2)}\otimes c_{(2)}$. With this, we obtain

$$\begin{aligned}
&(m^{op}\circ(f^{-1}\otimes f^{-1})) * (f\circ m)(b\otimes c)\\
&= \Sigma_{(b)(c)}(m^{op}\circ(f^{-1}\otimes f^{-1}))(b_{(1)}\otimes c_{(1)})\cdot(f\circ m)(b_{(2)}\otimes c_{(2)}) = \Sigma_{(b)(c)} f^{-1}(c_{(1)})\cdot f^{-1}(b_{(1)})f(b_{(2)}c_{(2)})\\
&= \Sigma_{(b)(c)} f^{-1}(c_{(1)})\cdot f^{-1}(b_{(1)})f(b_{(2)})f(c_{(2)}) = \Sigma_{(b)(c)} f^{-1}(c_{(1)})\cdot(f^{-1}*f)(b)f(c_{(2)})\\
&= \epsilon(b)\Sigma_{(c)} f^{-1}(c_{(1)})\cdot f(c_{(2)}) = \epsilon(b)(f^{-1}*f)(c) = \epsilon(b)\epsilon(c)1_B = \eta\circ\epsilon_{B\otimes B}(b\otimes c)
\end{aligned}$$

and a similar computation proves $(f\circ m)*(f^{-1}\circ m) = (f*m)\circ(m^{op}\circ(f^{-1}\circ f^{-1}) = \eta\circ\epsilon_{B\otimes B}$. The proof for 3. is analogous. One considers the convolution algebra $\mathrm{Hom}_{\mathbb{F}}(B, B\otimes B)$, where $B\otimes B$ is the tensor product of $B$ with itself from Example 1.1.5, 5. and proves that both $\Delta\circ f^{-1}$ and $(f^{-1}\otimes f^{-1})\circ\Delta^{op}$ are convolution inverses of $\Delta\circ f = (f\otimes f)\circ\Delta : B\to B\otimes B$ (Exercise).

For 4. and 5. we compute for an algebra homomorphism $\phi : B\to B'$, a coalgebra homomorphism $\psi : B\to B'$ and linear maps $f, g\in\mathrm{Hom}_{\mathbb{F}}(B,B)$, $h,k\in\mathrm{Hom}_{\mathbb{F}}(B',B')$

$$\begin{aligned}
&\phi\circ(f*g) = \phi\circ m\circ(f\otimes g)\circ\Delta = m'\circ(\phi\otimes\phi)\circ(f\otimes g)\circ\Delta = (\phi\circ f)*'(\phi\circ g)\\
&(h*k)\circ\psi = m'\circ(h\otimes k)\circ\Delta'\circ\psi = m'\circ(h\otimes k)\circ(\psi\otimes\psi)\circ\Delta = (h\circ\psi)*'(k\circ\psi).
\end{aligned}$$

As for any algebra homomorphism $\phi : B\to B'$ one has $L_\phi(\eta\circ\epsilon) = \phi\circ\eta\circ\epsilon = \eta'\circ\epsilon$ and for any coalgebra homomorphism $\psi : B\to B'$, one has $R_\psi(\eta'\circ\epsilon') = \eta'\circ\epsilon'\circ\psi = \eta'\circ\epsilon$, this shows that $L_\phi : \mathrm{Hom}_{\mathbb{F}}(B,B)\to\mathrm{Hom}_{\mathbb{F}}(B,B')$ and $R_\phi : \mathrm{Hom}_{\mathbb{F}}(B',B')\to\mathrm{Hom}_F(B,B')$ are algebra homomorphisms. □

Proposition 2.2.4 allows us to draw conclusions about the basic properties of an antipode in a Hopf algebra $H$. The first is its uniqueness, which follows directly from the uniqueness of the convolution inverse established in Lemma 2.2.3 and the fact that the antipode is the convolution inverse of the identity map. Proposition 2.2.4, 2. and 3. imply that the antipode defines an algebra homomorphism $S : H\to H^{op,cop}$ since the identity map $\mathrm{id}_H : H\to H$ is an algebra and a coalgebra homomorphism. Finally, Proposition 2.2.4, 4. and 5. show that the antipode of a Hopf algebra is automatically compatible with bialgebra homomorphisms since any bialgebra homomorphism $\phi : H\to H'$ induces algebra homomorphisms $L_\phi : \mathrm{End}_{\mathbb{F}}(H)\to\mathrm{Hom}_{\mathbb{F}}(H,H')$, $f\mapsto\phi\circ f$ and $R_\phi : \mathrm{End}_{\mathbb{F}}(H')\to\mathrm{Hom}_{\mathbb{F}}(H,H')$, $g\mapsto g\circ\phi$ by Proposition 2.2.4, 4. and 5. which map convolution inverses to convolution inverses.

**Corollary 2.2.5:** (**Properties of the antipode**)

1. If a bialgebra $(B, m, \eta, \Delta, \epsilon)$ is a Hopf algebra, then its antipode $S : B\to B$ is unique.
2. The antipode of a Hopf algebra is an anti-algebra and anti-coalgebra homomorphism
$$m^{op}\circ(S\otimes S) = S\circ m \qquad S\circ\eta = \eta \qquad \Delta\circ S = (S\otimes S)\circ\Delta^{op} \qquad \epsilon\circ S = \epsilon.$$
3. If $(B, m, \eta, \Delta, \epsilon, S)$ and $(B', m', \eta', \Delta', \epsilon', S')$ are Hopf algebras, then any homomorphism of bialgebras $\phi : B\to B'$ satisfies $S'\circ\phi = \phi\circ S$.

**Proof:**
The first two claims are obvious. The last claim follows directly from Proposition 2.2.4, 4. since for any bialgebra homomorphism $\phi : B\to B'$, one has

$$\begin{aligned}
&(\phi\circ S)*\phi = L_\phi(S*\mathrm{id}_B) = L_\phi(\eta\circ\epsilon) = \eta'\circ\epsilon = L_\phi(\eta\circ\epsilon) = L_\phi(\mathrm{id}_B * S) = \phi*(\phi\circ S)\\
&(S'\circ\phi)*\phi = R_\phi(S'*\mathrm{id}_{B'}) = R_\phi(\eta'\circ\epsilon') = \eta'\circ\epsilon = R_\phi(\eta'\circ\epsilon') = R_\phi(\mathrm{id}_{B'}*S') = \phi*(S'\circ\phi).
\end{aligned}$$

This shows that both, $S' \circ \phi$ and $\phi \circ S$ are convolution inverses of $\phi : B \to B'$ in $\mathrm{Hom}_{\mathbb{F}}(B, B')$ and the uniqueness of the convolution inverse then implies $S' \circ \phi = \phi \circ S$. □

Before considering examples, it remains to clarify the dependence of our definitions on the choices involved in the process. Our definition of a Hopf algebra took as the defining condition for the antipode equation (3), which ensured that the *right* evaluation and coevaluation maps are homomorphisms of representations. It is natural to ask how the corresponding condition (4) for the *left* evaluation and coevaluation is reflected in the properties of the antipode. It turns out that imposing both conditions is equivalent to the requirement that the antipode is an involution. More generally, if the antipode is invertible, the antipode satisfies condition (3) and the inverse of the antipode satisfies condition (4).

**Lemma 2.2.6:** (**Properties of the antipode**)
Let $(H, m, \eta, \Delta, \epsilon, S)$ be a Hopf algebra.

1. If $S$ is invertible, then $m^{op} \circ (S^{-1} \otimes \mathrm{id}) \circ \Delta = m^{op} \circ (\mathrm{id} \otimes S^{-1}) \circ \Delta = \eta \circ \epsilon$.
2. $S^2 = \mathrm{id}_H$ if and only if $m^{op} \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \epsilon = m^{op} \circ (\mathrm{id} \otimes S) \circ \Delta$.
3. If $H$ is commutative or cocommutative, then $S^2 = \mathrm{id}_H$.

**Proof:**
1. If $S^{-1} : H \to H$ is the inverse of the antipode $S : H \to H$, one has

$$S \circ m^{op} \circ (S^{-1} \otimes \mathrm{id}) \circ \Delta = m \circ (S \otimes S) \circ (S^{-1} \otimes \mathrm{id}) \circ \Delta = m \circ (\mathrm{id} \otimes S) \circ \Delta = \eta \circ \epsilon$$
$$S \circ m^{op} \circ (\mathrm{id} \otimes S^{-1}) \circ \Delta = m \circ (S \otimes S) \circ (\mathrm{id} \otimes S^{-1}) \circ \Delta = m \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \epsilon.$$

As $S \circ \eta \circ \epsilon = \eta \circ \epsilon = S^{-1} \circ \eta \circ \epsilon$, applying $S^{-1}$ to both sides of these equations proves 1.

2. If $S^2 = \mathrm{id}_H$, then $S = S^{-1}$ and from 1. one obtains $m^{op} \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \epsilon = m^{op} \circ (\mathrm{id} \otimes S) \circ \Delta$. To prove the other implication, one computes with the convolution product in $\mathrm{End}_{\mathbb{F}}(H)$

$$S * S^2 = m \circ (S \otimes S^2) \circ \Delta = S \circ m^{op} \circ (\mathrm{id} \otimes S) \circ \Delta = m \circ (\mathrm{id} \otimes S) \circ \Delta^{op} \circ S \tag{5}$$
$$S^2 * S = m \circ (S^2 \otimes S) \circ \Delta = S \circ m^{op} \circ (S \otimes \mathrm{id}) \circ \Delta = m \circ (S \otimes \mathrm{id}) \circ \Delta^{op} \circ S.$$

If $m^{op} \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \epsilon = m^{op} \circ (\mathrm{id} \otimes S) \circ \Delta$, this implies $S * S^2 = S^2 * S = S \circ \eta \circ \epsilon = \eta \circ \epsilon$. This means that $S^2$ and $\mathrm{id}_H$ are both convolution inverses of $S$, and from the uniqueness of the convolution inverse one has $S^2 = \mathrm{id}_H$. Claim 3. also follows directly from (5) since $m^{op} = m$ or $\Delta^{op} = \Delta$ imply $S * S^2 = S^2 * S = \mathrm{id}_H$ in (5) and hence $S^2 = \mathrm{id}_H$. □

With these results on the properties of the antipode, we can now consider our first examples, which are rather trivial but structurally important, because they care used in many constructions. More interesting and advanced examples will be considered in the next subsection.

**Example 2.2.7:**

1. For any Hopf algebra $(H, m, \eta, \Delta, \epsilon, S)$, reversing the multiplication *and* the comultiplication yields another Hopf algebra structure on $H$, namely the Hopf algebra $H^{op,cop} = (H, m^{op}, \eta, \Delta^{op}, \epsilon, S)$. If $S$ is *invertible*, then reversing the multiplication *or* the comultiplication and taking the inverse of the antipode yields new Hopf algebra structures $H^{op} = (H, m^{op}, \eta, \Delta, \epsilon, S^{-1})$ and $H^{cop} = (H, m, \eta, \Delta^{op}, \epsilon, S^{-1})$ on $H$.

2. For any two Hopf algebras $H, K$ over $\mathbb{F}$, the tensor product bialgebra $H \otimes K$ is a Hopf algebra with antipode $S = S_H \otimes S_K$. This is called the **tensor product Hopf algebra** and denoted $H \otimes K$.

3. For any *finite-dimensional* Hopf algebra $(H, m, \eta, \Delta, \epsilon, S)$, the dual bialgebra $(H^*, \Delta^*, \epsilon^*, m^*, \eta^*)$ is a Hopf algebra with antipode $S^*$. For any Hopf algebra $(H, m, \eta, \Delta, \epsilon, S)$, the finite dual $(H^\circ, \Delta^*|_{H^\circ \otimes H^\circ}, \epsilon^*, m^*|_{H^\circ}, \eta^*|_{H^\circ})$ from Example 2.1.9, 3. is a Hopf algebra with antipode $S^*|_{H^\circ}$ (Exercise).

**Example 2.2.8:** Let $G$ be a group and $\mathbb{F}$ a field.
The group algebra $\mathbb{F}[G]$ is a cocommutative Hopf algebra with the algebra structure from Example 1.1.10, comultiplication $\Delta : \mathbb{F}[G] \to \mathbb{F}[G] \otimes \mathbb{F}[G]$, $g \mapsto g \otimes g$, counit $\epsilon : \mathbb{F}[G] \to \mathbb{F}$, $g \mapsto 1$ and antipode $S : \mathbb{F}[G] \to \mathbb{F}[G]$, $g \mapsto g^{-1}$.

**Proof:**
As the elements of $\mathbb{F}[G]$ are finite linear combinations $\Sigma_{g \in G} \lambda_g g$ with $\lambda_g \in \mathbb{F}$, it is sufficient to verify that the axioms hold for the basis elements. This follows by a direct computation

$$
\begin{aligned}
&(\Delta \otimes \mathrm{id}) \circ \Delta(g) = \Delta(g) \otimes g = g \otimes g \otimes g = g \otimes \Delta(g) = (\mathrm{id} \otimes \Delta) \circ \Delta(g) \\
&(\epsilon \otimes \mathrm{id}) \circ \Delta(g) = \epsilon(g) \otimes g = 1 \otimes g \qquad (\mathrm{id} \otimes \epsilon) \circ \Delta(g) = g \otimes \epsilon(g) = g \otimes 1 \\
&\Delta(g \cdot h) = (gh) \otimes (gh) = (g \otimes g) \cdot (h \otimes h) = \Delta(g) \cdot \Delta(h) \\
&\epsilon(g \cdot h) = 1 = 1 \cdot 1 = \epsilon(g) \cdot \epsilon(h) \\
&m \circ (S \otimes \mathrm{id}) \circ \Delta(g) = m(g^{-1} \otimes g) = g^{-1} g = 1 = \eta(\epsilon(g)) = g g^{-1} = m(g \otimes g^{-1}) = m \circ (\mathrm{id} \otimes S) \circ \Delta(g).
\end{aligned}
$$

□

**Example 2.2.9:** Let $G$ be a finite group and $\mathbb{F}$ a field.
Then the dual vector space $\mathbb{F}[G]^*$ is isomorphic to the vector space $\mathrm{Fun}_{\mathbb{F}}(G)$ of functions $f : G \to \mathbb{F}$ with the pointwise addition and scalar multiplication. The functions $\delta_g : G \to \mathbb{F}$ with $\delta_g(g) = 1$ and $\delta_g(h) = 0$ for $g \neq h$ form a basis of $\mathrm{Fun}_{\mathbb{F}}(G)$. In terms of this basis, the Hopf algebra structure $(\mathrm{Fun}_{\mathbb{F}}(G), \Delta^*, \epsilon^*, m^*, \eta^*, S^*)$ that is dual to $(\mathbb{F}[G], m, \eta, \Delta, \epsilon, S)$ is given by

$$
\begin{aligned}
&\Delta^*(\delta_g \otimes \delta_h) = \delta_g \cdot \delta_h = \delta_g(h)\, \delta_h \qquad \epsilon^*(\lambda) = \lambda \Sigma_{g \in G} \delta_g \\
&m^*(\delta_g) = \Sigma_{h \in G} \delta_h \otimes \delta_{h^{-1} g} \qquad \eta^*(\delta_g) = \delta_g(e) \qquad S'(\delta_g) = S^*(\delta_g) = \delta_{g^{-1}}
\end{aligned}
$$

for all $g, h \in G$. This Hopf algebra is commutative, and its algebra structure is given by the pointwise multiplication of functions $f : G \to \mathbb{F}$.

**Proof:**
This follows by a direct computation from the definition of the dual Hopf algebra structure. We have for all $g, h, u, v \in G$

$$
\begin{aligned}
&\Delta^*(\delta_g \otimes \delta_h)(u) = (\delta_g \otimes \delta_h)(\Delta(u)) = (\delta_g \otimes \delta_h)(u \otimes u) = \delta_g(u) \delta_h(u) = \delta_g(h)\, \delta_h(u) \\
&\epsilon^*(\lambda)(u) = \lambda \epsilon(u) = \lambda = \Sigma_{g \in G} \lambda \delta_g(u) \\
&m^*(\delta_g)(u \otimes v) = \delta_g(u \cdot v) = \Sigma_{h \in G} \delta_h(u) \delta_g(hv) = \Sigma_{h \in G} \delta_h(u) \delta_{h^{-1} g}(v) \\
&\eta^*(\delta_g) = \delta_g(e) \\
&S^*(\delta_g)(u) = \delta_g(S(u)) = \delta_g(u^{-1}) = \delta_{g^{-1}}(u).
\end{aligned}
$$

In particular, this implies for all $u \in G$

$$
\begin{aligned}
(f_1 \cdot f_2)(u) &= \Delta^*(f_1 \otimes f_2)(u) = \Sigma_{g,h \in G} f_1(g) f_2(h) \Delta^*(\delta_g \otimes \delta_h)(u) = \Sigma_{g,h \in G} f_1(g) f_2(h) \delta_g(u) \delta_h(u) \\
&= f_1(u) f_2(u).
\end{aligned}
$$

□

## 2.3 Examples

In this section, we consider more interesting and advanced examples of Hopf algebras, which show that the concept goes far beyond group algebras and other familiar constructions. In particular, we construct parameter dependent examples that are non-commutative and non-cocommutative and can be viewed as deformations of other, more basic Hopf algebras with a simpler structure. We start with two most basic examples, namely the tensor algebra of a vector space and the universal enveloping algebra of a Lie algebra.

**Example 2.3.1:** The tensor algebra $T(V)$ of a vector space $V$ over $\mathbb{F}$ is a cocommutative Hopf algebra over $\mathbb{F}$ with the algebra structure from Example 1.1.6 and the comultiplication, counit and antipode given by

$$\begin{aligned}\Delta(v_1\otimes\ldots\otimes v_n) &= \Sigma_{p=0}^n \Sigma_{\sigma\in \mathrm{Sh}(p,n-p)} (v_{\sigma(1)}\otimes\ldots\otimes v_{\sigma(p)})\otimes(v_{\sigma(p+1)}\otimes\ldots\otimes v_{\sigma(n)})\\ \epsilon(v_1\otimes\ldots\otimes v_n) &= 0 \qquad S(v_1\otimes...\otimes v_n) = (-1)^n v_n\otimes...\otimes v_1,\end{aligned}$$

where $\mathrm{Sh}(p,q)$ is the set of $(p,q)$-**shuffle permutations**

$$\sigma\in S_{p+q} \quad \text{with} \quad \sigma(1)<\sigma(2)<...<\sigma(p) \text{ and } \sigma(p+1)<\sigma(p+2)<...<\sigma(p+q),$$

and we set $v_{\sigma(1)}\otimes\ldots\otimes v_{\sigma(p)}=1$ for $p=0$ and $v_{\sigma(p+1)}\otimes\ldots\otimes v_{\sigma(n)}=1$ for $p=n$.

**Proof:**
By the universal property of the tensor algebra, the linear maps

$$\Delta': V\to T(V)\otimes T(V),\;\; v\mapsto 1\otimes v+v\otimes 1 \qquad \epsilon': V\to\mathbb{F},\;\; v\mapsto 0 \qquad S: V\to T(V)^{op},\;\; v\mapsto -v$$

induce algebra homomorphisms $\Delta: T(V)\to T(V)\otimes T(V)$, $\epsilon: T(V)\to\mathbb{F}$, $S: T(V)\to T(V)^{op}$ with $\Delta\circ\iota_V=\Delta'$, $\epsilon\circ\iota_V=\epsilon'$ and $S\circ\iota_V=S'$. To show that $\Delta$ and $\epsilon$ are coassociative and counital and $S$ is an antipode it is sufficient to prove that

$$\begin{aligned}&(\Delta\otimes\mathrm{id})\circ\Delta\circ\iota_V = (\mathrm{id}\otimes\Delta)\circ\Delta\circ\iota_V\\ &l_{T(V)}\circ(\epsilon\otimes\mathrm{id})\circ\Delta\circ\iota_V = \iota_V = r_{T(V)}\circ(\mathrm{id}\otimes\epsilon)\circ\Delta\circ\iota_V\\ &m\circ(S\otimes\mathrm{id})\circ\Delta\circ\iota_V = \eta\circ\epsilon\circ\iota_V = m\circ(\mathrm{id}\otimes S)\circ\Delta\circ\iota_V.\end{aligned}$$

The claim then follows from the universal property of the tensor algebra. These identities follow by a direct computation from the expressions above

$$\begin{aligned}&(\Delta\otimes\mathrm{id})\circ\Delta\circ\iota_V(v) = (\Delta\otimes\mathrm{id})(1\otimes v+v\otimes 1) = \Delta(1)\otimes v+\Delta(v)\otimes 1 = 1\otimes 1\otimes v+1\otimes v\otimes 1+v\otimes 1\otimes 1\\ &(\mathrm{id}\otimes\Delta)\circ\Delta\circ\iota_V(v) = (\mathrm{id}\otimes\Delta)(1\otimes v+v\otimes 1) = 1\otimes\Delta(v)+\Delta(1)\otimes v = 1\otimes 1\otimes v+1\otimes v\otimes 1+v\otimes 1\otimes 1\\ &l_{T(V)}\circ(\epsilon\otimes\mathrm{id})\circ\Delta\circ\iota_V(v) = l_{T(V)}(\epsilon(v)\otimes 1+\epsilon(1)\otimes v) = v = \iota_V(v)\\ &r_{T(V)}\circ(\mathrm{id}\otimes\epsilon)\circ\Delta\circ\iota_V(v) = r_{T(V)}(1\otimes\epsilon(v)+v\otimes\epsilon(1)) = v = \iota_V(v)\\ &m\circ(S\otimes\mathrm{id})\circ\Delta\circ\iota_V(v) = m(S(1)\otimes v+S(v)\otimes 1) = 1\cdot v-v\cdot 1 = 0 = \eta(\epsilon(v)) = \eta\circ\epsilon\circ\iota_V(v)\\ &m\circ(\mathrm{id}\otimes S)\circ\Delta\circ\iota_V(v) = m(1\otimes S(v)+v\otimes S(1)) = -1\cdot v+v\cdot 1 = 0 = \eta(\epsilon(v)) = \eta\circ\epsilon\circ\iota_V(v).\end{aligned}$$

This proves that the algebra homomorphisms $\Delta$ and $\epsilon$ are coassociative and counital, that $S$ is an antipode that and $(T(V),m,\eta,\Delta,\epsilon,S)$ is a Hopf algebra, The formulas for the comultiplication, counit and antipode follow by induction over $n$. If they hold for all products of vectors in $V$ of length $\leq n$, then we have by definition of $\Delta$, $\epsilon$ and $S$

$$\begin{aligned}
&\epsilon(v_1 \otimes ... \otimes v_{n+1}) = \epsilon(v_1 \otimes ... \otimes v_n) \cdot \epsilon(v_{n+1}) = 0 \\
&S(v_1 \otimes ... \otimes v_{n+1}) = S(v_{n+1}) \otimes S(v_1 \otimes ... \otimes v_n) = (-1)^{n+1} v_{n+1} \otimes v_n \otimes ... \otimes v_1 \\
&\Delta(v_1 \otimes ... \otimes v_{n+1}) = \Delta(v_1 \otimes ... \otimes v_n) \cdot \Delta(v_{n+1}) \\
&= \big(\Sigma_{p=0}^n \Sigma_{\sigma \in \mathrm{Sh}(p,n-p)} (v_{\sigma(1)} \otimes ... \otimes v_{\sigma(p)}) \otimes (v_{\sigma(p+1)} \otimes ... \otimes v_{\sigma(n)})\big) \cdot (v_{n+1} \otimes 1 + 1 \otimes v_{n+1}) \\
&= \Sigma_{p=0}^n \Sigma_{\sigma \in \mathrm{Sh}(p,n-p)} (v_{\sigma(1)} \otimes ... \otimes v_{\sigma(p)} \otimes v_{n+1}) \otimes (v_{\sigma(p+1)} \otimes ... \otimes v_{\sigma(n)}) \\
&+ \Sigma_{p=0}^n \Sigma_{\sigma \in \mathrm{Sh}(p,n-p)} (v_{\sigma(1)} \otimes ... \otimes v_{\sigma(p)}) \otimes (v_{\sigma(p+1)} \otimes ... \otimes v_{\sigma(n)} \otimes v_{n+1}) \\
&= \Sigma_{p=0}^{n+1} \Sigma_{\sigma \in \mathrm{Sh}(p,n+1-p)} (v_{\sigma(1)} \otimes ... \otimes v_{\sigma(p)}) \otimes (v_{\sigma(p+1)} \otimes ... \otimes v_{\sigma(n+1)}).
\end{aligned}$$

In the last step, we used that from every shuffle permutation $\sigma \in \mathrm{Sh}(p, n-p)$, we obtain a shuffle permutation $\sigma' \in \mathrm{Sh}(p+1, n-p)$ by setting $\sigma'(i) = \sigma(i)$ for $1 \leq i \leq p$, $\sigma'(p+1) = n+1$ and $\sigma'(i) = \sigma(i-1)$ for $p+2 \leq i \leq n+1$. We also obtain a shuffle permutation $\sigma'' \in \mathrm{Sh}(p, n+1-p)$ by setting $\sigma''(i) = \sigma(i)$ for $1 \leq i \leq n$, $\sigma''(n+1) = n+1$. Conversely, for every shuffle permutation $\pi \in \mathrm{Sh}(p, n+1-p)$, one has either $\pi(p) = n+1$ or $\pi(n+1) = n+1$. In the first case, one has $p > 0$ and $\pi = \sigma'$ for a shuffle permutation $\sigma \in \mathrm{Sh}(p-1, n+1-p)$ and in the second $\pi = \sigma''$ for a shuffle permutation $\sigma \in \mathrm{Sh}(p, n-p)$. □

**Example 2.3.2:** The universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is a cocommutative Hopf algebra with the algebra structure from Example 1.1.9 and the comultiplication, counit and antipode given by $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\epsilon(x) = 0$ and $S(x) = -x$ for all $x \in \mathfrak{g}$.

**Proof:**
The linear maps

$$\Delta' : \mathfrak{g} \to U(\mathfrak{g}) \otimes U(\mathfrak{g}),\ x \mapsto x \otimes 1 + 1 \otimes x \qquad \epsilon' : \mathfrak{g} \to \mathbb{F},\ x \mapsto 0 \qquad S' : \mathfrak{g} \to U(\mathfrak{g})^{op},\ x \mapsto -x$$

are Lie algebra homomorphisms, since one has for all $x, y \in \mathfrak{g}$

$$\begin{aligned}
&[\Delta'(x), \Delta'(y)] = \Delta'(x) \cdot \Delta'(y) - \Delta'(y) \cdot \Delta'(x) \\
&= (x \otimes 1 + 1 \otimes x) \cdot (y \otimes 1 + 1 \otimes y) - (y \otimes 1 + 1 \otimes y) \cdot (x \otimes 1 + 1 \otimes x) \\
&= (xy) \otimes 1 + x \otimes y + 1 \otimes (xy) + y \otimes x - ((yx) \otimes 1 + y \otimes x + x \otimes y - 1 \otimes (yx)) \\
&= (xy - yx) \otimes 1 + 1 \otimes (xy - yx) = [x, y] \otimes 1 + 1 \otimes [x, y] = \Delta'([x, y]) \\
&[\epsilon'(x), \epsilon'(y)]) = \epsilon'(x)\epsilon'(y) - \epsilon'(y)\epsilon'(x) = 0 = \epsilon'([x, y]) \\
&[S'(x), S'(y)] = S'(y)S'(x) - S'(x)S'(y) = y \cdot x - x \cdot y = -[x, y] = S'([x, y]).
\end{aligned}$$

By the universal property of $U(\mathfrak{g})$, they induce algebra homomorphisms $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ $\epsilon : U(\mathfrak{g}) \to \mathbb{F}$ and $S : U(\mathfrak{g}) \to U(\mathfrak{g})^{op}$ with $\Delta \circ \iota_{\mathfrak{g}} = \Delta'$, $\epsilon \circ \iota_{\mathfrak{g}} = \epsilon'$ and $S \circ \iota_{\mathfrak{g}} = S'$. To prove the coassociativity and counitality of $\Delta$ and $\epsilon$ and that $S$ is an antipode, it is sufficient to show that $(\Delta \otimes \mathrm{id}) \circ \Delta \circ \iota_{\mathfrak{g}} = (\mathrm{id} \otimes \Delta) \circ \Delta \circ \iota_{\mathfrak{g}}$, $l_{U(\mathfrak{g})} \circ (\epsilon \otimes \mathrm{id}) \circ \Delta \circ \iota_{\mathfrak{g}} = \iota_{\mathfrak{g}} = r_{U(\mathfrak{g})} \circ (\mathrm{id} \otimes \epsilon) \circ \Delta \circ \iota_{\mathfrak{g}}$ and $m \circ (S \otimes \mathrm{id}) \circ \Delta \circ \iota_{\mathfrak{g}} = \eta \circ \epsilon \circ \iota_{\mathfrak{g}} = m \circ (\mathrm{id} \otimes S) \circ \Delta \circ \iota_{\mathfrak{g}}$ . The claim then follows from the universal property of $U(\mathfrak{g})$. These identities follow by a direct computation that yields the same formulas as in the proof of Example 2.3.1. That this bialgebra is cocommutative follows from the fact that $\Delta \circ \iota_{\mathfrak{g}}(x) = 1 \otimes x + x \otimes 1 = \Delta^{op} \circ \iota_{\mathfrak{g}}(x)$ for all $x \in \mathfrak{g}$. With the universal property of $U(\mathfrak{g})$, this implies $\Delta = \Delta^{op} : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$. □

All examples of Hopf algebras treated so far are cocommutative, and hence their (finite) duals are commutative. To construct more interesting examples that are neither commutative nor cocommutative, we consider certain polynomials in $\mathbb{Z}[q]$, the so-called *q-factorials* and *q-binomials*. Their name is due to the fact that they exhibit relations that resemble the relations between factorials of natural numbers and binomial coefficients. That the variable in the polynomial ring $\mathbb{Z}[q]$ is called $q$ instead of $x$ has historical reasons and no deeper meaning.

**Definition 2.3.3:** Let $R$ be an integral domain, $\mathbb{Z}[q]$ the ring of polynomials with coefficients in $\mathbb{Z}$ and $\mathbb{Z}(q)$ the associated fraction field of rational functions. We define:

- the $q$-**natural** $(n)_q = 1 + q + ... + q^{n-1} = \frac{q^n-1}{q-1}$ for all $n \in \mathbb{N}$,
- the $q$-**factorial** $(0)!_q = 1$ and $(n)!_q = (n)_q(n-1)_q \cdots (1)_q = \frac{(q^n-1)(q^{n-1}-1)\cdots(q-1)}{(q-1)^n}$ for $n \in \mathbb{N}$,
- the $q$-**binomial** or **Gauß polynomial** $\binom{n}{k}_q = \frac{(n)!_q}{(n-k)!_q\,(k)!_q}$ for $k, n \in \mathbb{N}_0$ with $0 \leq k \leq n$.

**Lemma 2.3.4:**

1. For all $k, n \in \mathbb{N}$ with $0 \leq k \leq n$, the $q$-naturals, the $q$-factorials and the $q$-binomials are polynomials in $q$ with integer coefficients.
2. For all $k, n \in \mathbb{N}$ with $0 \leq k \leq n$ the $q$-binomials satisfy the identity
$$\binom{n}{k}_q = \binom{n}{n-k}_q$$
3. For all $k, n \in \mathbb{N}$ with $0 \leq k < n$ the $q$-binomials satisfy the $q$-**Pascal identity**
$$\binom{n+1}{k+1}_q = \binom{n}{k}_q + q^{k+1}\binom{n}{k+1}_q = \binom{n}{k+1}_q + q^{n-k}\binom{n}{k}_q$$
4. If $A$ is an algebra over $\mathbb{Z}(q)$ and $x, y \in A$ with $xy = q\,yx$ one has the $q$-**binomial formula**
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k}_q y^k x^{n-k},$$

**Proof:**
That the elements $(n)_q$ and $(n)!_q$ are polynomials in $q$ follows directly from their definition. That this also holds for the $q$-binomials follows by induction from 3. and from the fact that they are equal to 1 for $k = 0$ or $k = n$. The second claim follows directly from the definition of the $q$-binomial, and the third follows by a direct computation

$$\binom{n}{k}_q + q^{k+1}\binom{n}{k+1}_q = \frac{(q^n-1)\cdots(q^{k+1}-1)}{(q^{n-k}-1)\cdots(q-1)} + q^{k+1}\frac{(q^n-1)\cdots(q^{k+2}-1)}{(q^{n-k-1}-1)\cdots(q-1)}$$
$$= \frac{(q^n-1)\cdots(q^{k+2}-1)}{(q^{n-k}-1)\cdots(q-1)} \cdot \left(q^{k+1} - 1 + q^{k+1}(q^{n-k}-1)\right) = \frac{(q^{n+1}-1)\cdots(q^{k+2}-1)}{(q^{n-k}-1)\cdots(q-1)} = \binom{n+1}{k+1}_q.$$

4. To prove the last claim, we use the identity

$$xy^k - q^k y^k x = \Sigma_{l=0}^{k-1} q^l y^l (xy - qyx) y^{k-l-1}, \tag{6}$$

which follows by induction over $k$ and compute with 3. for general $x, y \in A$

$$(x+y)^{n+1} - \sum_{k=0}^{n+1} \binom{n+1}{k}_q y^k x^{n+1-k} = (x+y) \cdot \sum_{k=0}^{n} \binom{n}{k}_q y^k x^{n-k} - \sum_{k=0}^{n+1} \binom{n+1}{k}_q y^k x^{n+1-k}$$

$$= \sum_{k=0}^{n-1} \binom{n}{k}_q y^{k+1} x^{n-k} + \sum_{k=1}^{n} \binom{n}{k}_q x y^k x^{n-k} - \sum_{k=1}^{n} \binom{n}{k-1}_q y^k x^{n+1-k} - \sum_{k=1}^{n} \binom{n}{k}_q q^k y^k x^{n+1-k}$$

$$= \sum_{k=1}^{n} \binom{n}{k}_q (x y^k x^{n-k} - q^k y^k x^{n+1-k}) = \sum_{k=1}^{n} \sum_{l=0}^{k-1} \binom{n}{k}_q q^l y^l (xy - qyx) y^{k-l-1} x^{n-k} \in (xy - qyx).$$

If $xy = q\, yx$, then the ideal $(xy - qyx)$ is trivial and the claim follows. □

To evaluate the $q$-naturals, $q$-factorials and $q$-binomials, we recall that for every integral domain $R$, we have a unital ring homomorphism $\mathbb{Z} \to R$, $z \mapsto z = z1$, which induces a unital ring homomorphism $\mathbb{Z}[q] \to R[q]$, $\Sigma_{n \in \mathbb{N}_0} a_n q^n \mapsto \Sigma_{n \in \mathbb{N}_0} a_n 1\, q^n$. By composing it with the evaluation homomorphism $\mathrm{ev}_r : R[q] \to R$, $\Sigma_{n \in \mathbb{N}_0} a_n q^n \mapsto \Sigma_{n \in \mathbb{N}_0} a_n r^n$ for $r \in R$, we obtain a unital ring homomorphism $\mathrm{ev}'_r : \mathbb{Z}[q] \to R$, $\Sigma_{n \in \mathbb{N}_0} a_n q^n \mapsto \Sigma_{n \in \mathbb{N}_0} a_n 1\, r^n$ that allows us to evaluate every polynomial in $\mathbb{Z}[q]$ at $r \in R$.

**Definition 2.3.5:** The **evaluation** of the $q$-naturals, $q$-binomials and $q$-factorials in $r \in R$ is

$$(n)'_r = \mathrm{ev'_r}(n)_q \qquad (n)!'_r = \mathrm{ev'_r}(n)!_q \qquad \binom{n}{k}'_r = \mathrm{ev}'_r \binom{n}{k}_q .$$

Clearly, there are two cases in which the evaluation of $q$-naturals, $q$-factorials and $q$-binomials is of special interest. The first is $r = 1$, where we have $(n)'_1 = n1$, and the evaluations of $q$-factorials and $q$-binomials in $r = 1$ coincide with the usual factorials and binomials in $R$. This justifies the names $q$-naturals, $q$-binomials and $q$-factorials. The second is the case, where $r \in R$ is a primitive $n$th root of unity, i. e. $r^n = 1$ and $r^k \neq 1$ for $1 \leq k < n$. In this case, one has $(n)'_r = 0$ and $(k)'_r \neq 0$ for all $k < n$ since the roots of the polynomial $(m)_q$ are precisely the non-trivial $m$th roots of unity. This implies that the evaluations of all $q$-factorials $(m)!_q$ with $m \geq n$ vanish, since they contain a factor $(n)_q$. The same holds for the evaluations of all $q$-binomials with entries $0 < k < n$, since $(n)'_r = 0$ and $(k)'_r \neq 0$ for all $k < n$ implies

$$\binom{n}{k}'_r = \mathrm{ev}'_r \frac{(n)!_q}{(n-k)!_q (k)!_q} = \mathrm{ev}'_r \frac{(n)_q \cdots (n-k+1)_q}{(k)_q (k-1)_q \cdots (1)_q} = 0.$$

We will now use the $q$-naturals, $q$-factorials and $q$-binomials to construct an example of a Hopf algebra that is neither commutative nor cocommutative. The natural way to proceed is to present its algebra structure in terms of generators and relations. Clearly, the minimum number of linearly independent generators that can give rise to a non-commutative and non-cocommutative bialgebra is two. The simplest relations that can be imposed on such an algebra without making it commutative or trivial are quadratic relations in the two generators, i. e. relations of the form $x^2 - q$, $y^2 - q$ or $xy - qyx$ for some $q \in \mathbb{F}$ and generators $x, y$. While the first two yield a rather trivial algebra structure, the last one is the more promising and indeed gives rise to an infinite-dimensional non-commutative and non-cocommutative bialgebra. If we impose additional relations of the form $x^n = 0$ and $y^n = 1$ to make the bialgebra finite-dimensional, we have to take for $q$ a primitive $n$th root of unity and obtain *Taft's example*.

**Example 2.3.6:** **(Taft's example)**
Let $\mathbb{F}$ be a field, $n \in \mathbb{N}$ and $q \in \mathbb{F}$ a primitive $n$th root of unity. Let $A$ be the algebra over $\mathbb{F}$ with generators $x, y$ and relations

$$xy - q\,yx = 0, \qquad x^n = 0, \qquad y^n - 1 = 0.$$

Then $A$ is a Hopf algebra with the comultiplication, counit and antipode given by

$$\Delta(x) = 1\otimes x + x\otimes y, \quad \Delta(y) = y\otimes y, \quad \epsilon(x) = 0, \quad \epsilon(y) = 1, \quad S(x) = -xy^{n-1}, \quad S(y) = y^{n-1}.$$

**Proof:**
Let $V$ be the free $\mathbb{F}$-vector space generated by $B = \{x, y\}$. Then the algebra $A$ is given as $A = T(V)/I$, where $I = (x \cdot y - q\, y \cdot x, x^n, y^n - 1)$ is the two-sided ideal in $T(V)$ generated by the relations and $\cdot$ denotes the multiplication of the tensor algebra $T(V)$.

The linear maps $\Delta' : V \to T(V)\otimes T(V)$ and $\epsilon' : V \to \mathbb{F}$ that are determined by their values on the basis $B$ induce algebra homomorphisms $\Delta'' : T(V) \to T(V)\otimes T(V)$, $\epsilon'' : T(V) \to \mathbb{F}$ with $\Delta'' \circ \iota_V = \Delta'$ and $\epsilon'' \circ \iota_V = \epsilon'$ by the universal property of the tensor algebra. These algebra homomorphisms satisfy the coassociativity and the counitality condition, since we have

$$\begin{aligned}
&(\Delta'\otimes \mathrm{id}) \circ \Delta'(x) = \Delta'(1)\otimes x + \Delta'(x)\otimes y = 1\otimes 1\otimes x + 1\otimes x\otimes y + x\otimes y\otimes y\\
&(\mathrm{id}\otimes\Delta') \circ \Delta'(x) = 1\otimes\Delta'(x) + x\otimes\Delta'(y) = 1\otimes 1\otimes x + 1\otimes x\otimes y + x\otimes y\otimes y\\
&(\Delta'\otimes \mathrm{id}) \circ \Delta'(y) = \Delta'(y)\otimes y = y\otimes y\otimes y = y\otimes\Delta'(y) = (\mathrm{id}\otimes\Delta') \circ \Delta'(y)\\
&(\epsilon'\otimes \mathrm{id}) \circ \Delta'(x) = \epsilon'(1)\otimes x + \epsilon'(x)\otimes y = 1\otimes x \qquad (\mathrm{id}\otimes\epsilon') \circ \Delta'(x) = 1\otimes\epsilon'(x) + x\otimes\epsilon'(y) = x\otimes 1\\
&(\epsilon'\otimes \mathrm{id}) \circ \Delta'(y) = \epsilon'(y)\otimes y = 1\otimes y \qquad (\mathrm{id}\otimes\epsilon') \circ \Delta'(y) = y\otimes\epsilon'(y) = y\otimes 1.
\end{aligned}$$

This shows that $\Delta''$ and $\epsilon''$ define a bialgebra structure on the tensor algebra $T(V)$. To show that this induces a bialgebra structure on $A = T(V)/I$, it is sufficient to show that $I$ is a coideal in $T(V)$, i. e. $\Delta''(I) \subset T(V)\otimes I + I\otimes T(V)$ and $\epsilon''(I) = 0$. Proposition 2.1.5 then implies that this induces a bialgebra structure on $A$. As $I$ is the two-sided ideal generated by the relations and $\Delta''$ and $\epsilon''$ are algebra homomorphisms, it is sufficient to show that $\Delta''(r) \in I\otimes T(V) + T(V)\otimes I$ and $\epsilon''(r) = 0$ for each relation $r$. The latter follows directly from the definition of $\epsilon''$

$$\begin{aligned}
\epsilon''(xy - qyx) &= \epsilon''(x)\epsilon''(y) - q\epsilon''(y)\epsilon''(x) = 0 \cdot 1 - q \cdot 1 \cdot 0 = 0\\
\epsilon''(y^n - 1) &= \epsilon''(y)^n - 1 = 1^n - 1 = 0 = 0^n = \epsilon''(x)^n = \epsilon''(x^n).
\end{aligned}$$

For the former, we compute with the multiplication $\cdot$ of the tensor algebra

$$\begin{aligned}
\Delta''(xy - qyx) &= \Delta''(x) \cdot \Delta''(y) - q\Delta''(y) \cdot \Delta''(x)\\
&= (1\otimes x + x\otimes y) \cdot (y\otimes y) - q(y\otimes y) \cdot (1\otimes x + x\otimes y)\\
&= y\otimes(xy) + (xy)\otimes y^2 - q\, y\otimes(yx) - q(yx)\otimes y^2\\
&= y\otimes(xy - q\,yx) + (xy - qyx)\otimes y^2 \in T(V)\otimes I + I\otimes T(V).\\
\Delta''(y^n - 1) &= \Delta''(y)^n - 1\otimes 1 = (y\otimes y)^n - (1\otimes 1) = y^n\otimes y^n - 1\otimes 1\\
&= y^n\otimes(y^n - 1) + (y^n - 1)\otimes 1 \in T(V)\otimes I + I\otimes T(V).
\end{aligned}$$

To prove this for $\Delta''(x^n) = (1\otimes x + x\otimes y)^n$, note that the proof of Lemma 2.3.4, 4. implies

$$(1\otimes x + x\otimes y)^n - \sum_{k=0}^{n} \binom{n}{k}'_q (x\otimes y)^k \cdot (1\otimes x)^{n-k} \in J$$

where the binomial coefficient is evaluated in $q \in \mathbb{F}$ and $J$ is the two-sided idea generated by the element $(1\otimes x) \cdot (x\otimes y) - q\,(x\otimes y)(1\otimes x) = x\otimes(xy) - q\,x\otimes(yx) = x\otimes(xy - qyx) \in T(V)\otimes I$. As $T(V)\otimes I$ is a two-sided ideal in $T(V)\otimes T(V)$, we have $J \subset T(V)\otimes I$, and as $q$ is a primitive $n$th root of unity, the evaluations of the binomial coefficients for $0 < k < n$ vanish. This yields

$$\Delta''(x^n) = (1\otimes x + x\otimes y)^n = 1\otimes x^n + x^n\otimes y^n + T(V).\otimes I \in T(V)\otimes I + I\otimes T(V).$$

Hence, we have shown that $\Delta'$ and $\epsilon'$ induce algebra homomorphisms $\Delta : A \to A\otimes A$ and $\epsilon : A \to \mathbb{F}$ with $(\pi\otimes\pi)\circ\Delta' = \Delta\circ\pi$ and $\epsilon' = \epsilon\circ\pi$, where $\pi : T(V) \to T(V)/I$ is the canonical surjection, and that this defines a bialgebra structure on $A = T(V)/I$.

To show that $A$ is a Hopf algebra, we consider the linear map $S' : V \to T(V)^{op}$ defined by its values on the basis. By the universal property of the tensor algebra, it induces an algebra homomorphism $S'' : T(V) \to T(V)^{op}$ with $S''\circ\iota_V = S'$. By composing $S''$ with the canonical surjection, we obtain an algebra homomorphism $\pi\circ S'' : T(V) \to A^{op}$, and we have to show that $\pi\circ S''(r) = 0$ for all relations $r$ of $I$. With the definition of the antipode and (6), we compute

$$\begin{aligned}
\pi\circ S''(xy - q\,yx) &= \pi(S''(y))\pi(S''(x)) - q\,\pi(S''(x))\pi(S''(y)) = -y^{n-1}xy^{n-1} + q\,xy^{n-1}\cdot y^{n-1}\\
&\overset{(6)}{=} -y^{n-1}\,xy^{n-1} + q^n y^{n-1}xy^{n-1} = (q^n - 1)y^{n-1}xy^{n-1} = 0,\\
\pi\circ S''(x^n) &= \pi(S''(x)^n) = (-xy^{n-1})^n = (-1)^n(xy^{n-1})\cdots(xy^{n-1}) = (-1)^n q^{(n-1)(1+\ldots+n)}y^n x^n = 0,\\
\pi\circ S''(y^n - 1) &= \pi(S''(y)^n) - 1 = y^{n(n-1)} - 1 = 1^{n-1} - 1 = 0,
\end{aligned}$$

and this shows that the map $\pi\circ S''$ induces an algebra homomorphism $S : A \to A^{op}$. Because

$$\begin{aligned}
m\circ(S\otimes\mathrm{id})\circ\Delta(x) &= m(S(1)\otimes x + S(x)\otimes y) = 1\cdot x - (xy^{n-1})\cdot y = x(1 - y^n) = 0 = \epsilon(x),\\
m\circ(\mathrm{id}\otimes S)\circ\Delta(x) &= m(1\otimes S(x) + x\otimes S(y)) = -1\cdot(xy^{n-1}) + x\cdot y^{n-1} = 0 = \epsilon(x),\\
m\circ(S\otimes\mathrm{id})\circ\Delta(y) &= m(y^{n-1}\otimes y) = y^{n-1}\cdot y = y^n = 1 = \epsilon(y)1,\\
m\circ(\mathrm{id}\otimes S)\circ\Delta(y) &= m(y\otimes S(y)) = y\cdot y^{n-1} = y^n = 1 = \epsilon(y)1,
\end{aligned}$$

it follows from the universal property of the tensor algebra and the quotient algebra that $S$ is an antipode for $A$ and $A$ is a Hopf algebra. □

**Remark 2.3.7:**

1. Taft's example for $q = -1$, $n = 2$ is also known as **Sweedler's example**.

2. The elements $x^i y^j \in A$ with $0 \leq i, j \leq n-1$ generate the vector space $A$. This follows because every mixed monomial in $x$ and $y$ can be transformed into one of them by applying the relations. However, they are not linearly independent (why?).

3. It follows from the proof of Example 2.3.6 that the algebra with generators $x, y$ and the relation $xy - qyx$ is also a bialgebra for any $q \in \mathbb{F}$, because the ideal $I' = (xy - qyx)$ is a coideal in $T(V)$. This infinite-dimensional bialgebra is sometimes called the **quantum plane**. It is not cocommutative, commutative if and only if $q = 1$, and the elements $x^i y^j$ for $i, j \in \mathbb{N}_0$ form a basis of this bialgebra. .

4. The antipode in Taft's example satisfies $S^2(y) = y$ and $S^2(x) = yxy^{-1}$. This shows that $S^2$ and hence $S$ are invertible, but we do not have $S^2 = \mathrm{id}$.

Our next example of a non-cocommutative and non-commutative bialgebra and Hopf algebra are the so-called $q$-deformed matrix algebras $\mathrm{M}_q(2, \mathbb{F})$ and $\mathrm{SL}_q(2, \mathbb{F})$. They are again presented in terms of generators and relations and their coalgebra structure will be interpreted later as a generalisation and deformation of the coalgebra $\mathrm{Mat}(2\times 2,\mathbb{F})^*$ from Example 2.1.3, 5. We first describe their bialgebra and Hopf algebra structure and then relate them to $\mathrm{Mat}(2\times 2,\mathbb{F})^*$.

**Example 2.3.8:** Let $\mathbb{F}$ be a field and $q\in\mathbb{F}\setminus\{0\}$.

1. The **matrix algebra** $\mathrm{M}_q(2,\mathbb{F})$ is the algebra over $\mathbb{F}$ with generators $a,b,c,d$ and relations

$$ba = q\,ab,\quad db = q\,bd,\quad ca = q\,ac,\quad dc = q\,cd,\quad bc = cb,\quad da-ad=(q-q^{-1})\,bc. \tag{7}$$

   It has a bialgebra structure with comultiplication and counit given by

$$\begin{array}{llll}\Delta(a)=a\otimes a+b\otimes c, & \Delta(b)=a\otimes b+b\otimes d, & \Delta(c)=c\otimes a+d\otimes c, & \Delta(d)=c\otimes b+d\otimes d\\ \epsilon(a)=1 & \epsilon(b)=0 & \epsilon(c)=0 & \epsilon(d)=1.\end{array} \tag{8}$$

2. The $q$**-determinant** $\det_q = ad-q^{-1}bc$ is central in $\mathrm{M}_q(2,\mathbb{F})$ with

$$\Delta(\det_q)=\det_q\otimes\det_q \qquad \epsilon(\det_q)=1.$$

3. The bialgebra structure of $\mathrm{M}_q(2,\mathbb{F})$ induces a Hopf algebra structure on the algebra $\mathrm{SL}_q(2,\mathbb{F})=\mathrm{M}_q(2,\mathbb{F})/(\det_q-1)$ with the antipode given by

$$S(a)=d,\quad S(b)=-qb,\quad S(c)=-q^{-1}c,\quad S(d)=a.$$

**Proof:**
1. The proof is similar to the one of Example 2.3.6. The algebra $\mathrm{M}_q(2,\mathbb{F})$ is given as the quotient $\mathrm{M}_q(2,\mathbb{F})=T(V)/I$, where $V$ is the free vector space with basis $\{a,b,c,d\}$ and $I\subset T(V)$ the two-sided ideal generated by the six relations in (7). By the universal property of the tensor algebra, the maps $\Delta': V\to T(V)\otimes T(V)$ and $\epsilon': V\to\mathbb{F}$ specified by (8) induce algebra homomorphisms $\Delta'': T(V)\to T(V)\otimes T(V)$ and $\epsilon'': T(V)\to\mathbb{F}$. To show that $\Delta''$ and $\epsilon''$ are coassociative and counital, it is again sufficient to show that $(\Delta''\otimes\mathrm{id})\circ\Delta''(x)=(\mathrm{id}\otimes\Delta'')\circ\Delta''(x)$ and $l_{T(V)}\circ(\epsilon''\otimes\mathrm{id})\circ\Delta''(x)=r_{T(V)}\circ(\mathrm{id}\otimes\epsilon'')\circ\Delta''(x)$ for $x\in\{a,b,c,d\}$. This follows by a direct computation from (8), which yields

$$\begin{aligned}
(\Delta''\otimes\mathrm{id})\circ\Delta''(a)&=\Delta''(a)\otimes a+\Delta''(b)\otimes c=a\otimes a\otimes a+b\otimes c\otimes a+a\otimes b\otimes c+b\otimes d\otimes c\\
(\mathrm{id}\otimes\Delta'')\circ\Delta''(a)&=a\otimes\Delta''(a)+b\otimes\Delta''(c)=a\otimes a\otimes a+a\otimes b\otimes c+b\otimes c\otimes a+b\otimes d\otimes c\\
(\Delta''\otimes\mathrm{id})\circ\Delta''(b)&=\Delta''(a)\otimes b+\Delta''(b)\otimes d=a\otimes a\otimes b+b\otimes c\otimes b+a\otimes b\otimes d+b\otimes d\otimes d\\
(\mathrm{id}\otimes\Delta'')\circ\Delta''(b)&=a\otimes\Delta''(b)+b\otimes\Delta(d)=a\otimes a\otimes b+a\otimes b\otimes d+b\otimes c\otimes b+b\otimes d\otimes d\\
(\Delta''\otimes\mathrm{id})\circ\Delta''(c)&=\Delta''(c)\otimes a+\Delta''(d)\otimes c=c\otimes a\otimes a+d\otimes c\otimes a+c\otimes b\otimes c+d\otimes d\otimes c\\
(\mathrm{id}\otimes\Delta'')\circ\Delta''(c)&=c\otimes\Delta''(a)+d\otimes\Delta''(c)=c\otimes a\otimes a+c\otimes b\otimes c+d\otimes c\otimes a+d\otimes d\otimes c\\
(\Delta''\otimes\mathrm{id})\circ\Delta''(d)&=\Delta''(c)\otimes b+\Delta''(d)\otimes d=c\otimes a\otimes b+d\otimes c\otimes b+c\otimes b\otimes d+d\otimes d\otimes d\\
(\mathrm{id}\otimes\Delta'')\circ\Delta''(d)&=c\otimes\Delta''(b)+d\otimes\Delta''(c)=c\otimes a\otimes b+c\otimes b\otimes d+d\otimes c\otimes b+d\otimes d\otimes d.
\end{aligned}$$

To show that this induces a bialgebra structure on $\mathrm{M}_q(2,\mathbb{F})$ it is sufficient to prove that $I$ is a two-sided coideal in $T(V)$, i. e. that we have $\Delta''(r)\in I\otimes T(V)+T(V)\otimes I$ and $\epsilon''(r)=0$ for each relation $r$. For the latter, note that $\epsilon''(xy)=0$ if $x,y\in\{a,b,c,d\}$ with $\{x,y\}\cap\{b,c\}\neq\emptyset$. This proves that $\epsilon''(r)=0$ for the first five relations. For the last relation, we have

$$\epsilon''(da-ad)=\epsilon''(d)\epsilon''(a)-\epsilon''(a)\epsilon''(d)=1-1=0=(q-q^{-1})\epsilon''(b)\epsilon''(c)=(q-q^{-1})\epsilon''(bc).$$

The identities $\Delta''(r) \in I\otimes T(V)+T(V)\otimes I$ follow from a direct computation, which we perform for the first relation, since the other computations are similar

$$\Delta''(ba-qab) = (a\otimes b+b\otimes d)\cdot(a\otimes a+b\otimes c)-q(a\otimes a+b\otimes c)\cdot(a\otimes b+b\otimes d)$$
$$= a^2\otimes ba+ab\otimes bc+ba\otimes da+b^2\otimes dc-q(a^2\otimes ab+ab\otimes ad+ba\otimes cb+b^2\otimes cd)$$
$$= a^2\otimes(ba-q\,ab)+b^2\otimes(dc-q\,cd)+(ba-q\,ab)\otimes da+q\,ab\otimes(da-ad+(q^{-1}-q)bc)+ab\otimes(cb-bc).$$

2. That the element $\det_q$ is central in $\mathrm{M}_q(2,\mathbb{F})$ follows from the relations in $\mathrm{M}_q(2,\mathbb{F})$:

$$\begin{aligned}
a\cdot\det_q &= a\cdot(ad-q^{-1}bc) = a^2d-q^{-1}abc = ada+(q^{-1}-q)abc-q^{-1}abc\\
&= ada-q\,abc = ada-bac = ada-q^{-1}bca = (ad-q^{-1}bc)a = \det_q\cdot a,\\
b\cdot\det_q &= b\cdot(ad-q^{-1}bc) = bad-q^{-1}b^2c = qabd-q^{-1}bcb = (ad-q^{-1}bc)b = \det_q\cdot b,\\
c\cdot\det_q &= c\cdot(ad-q^{-1}bc) = cad-q^{-1}cbc = qacd-q^{-1}bc^2 = (ad-q^{-1}bc)c = \det_q\cdot c,\\
d\cdot\det_q &= d\cdot(ad-q^{-1}bc) = dad-q^{-1}dbc = ad^2+(q-q^{-1})bcd-q^{-1}dbc\\
&= ad^2+(q-q^{-1})bcd-bdc = ad^2+(q-q^{-1}-q)bcd = (ad-q^{-1}bc)d = \det_q\cdot d.
\end{aligned}$$

For the coproduct and the counit of the $q$-determinant, we compute

$$\begin{aligned}
&\epsilon(\det_q) = \epsilon(a)\epsilon(d)-q^{-1}\epsilon(b)\epsilon(c) = 1\\
&\Delta(\det_q) = \Delta(a)\cdot\Delta(d)-q^{-1}\Delta(b)\Delta(c)\\
&= (a\otimes a+b\otimes c)\cdot(c\otimes b+d\otimes d)-q^{-1}\,(a\otimes b+b\otimes d)\cdot(c\otimes a+d\otimes c)\\
&= ac\otimes ab+ad\otimes ad+bc\otimes cb+bd\otimes cd-q^{-1}(ac\otimes ba+ad\otimes bc+bc\otimes da+bd\otimes dc)\\
&= ad\otimes(ad-q^{-1}bc)+bc\otimes bc-q^{-1}bc\otimes da = ad\otimes(ad-q^{-1}bc)-q^{-1}bc\otimes(ad-q^{-1}bc)\\
&= \det_q\otimes\det_q.
\end{aligned}$$

3. As we have $\Delta(\det_q-1) = \det_q\otimes\det_q-1\otimes 1 = \det_q\otimes(\det_q-1)+(\det_q-1)\otimes 1$, $\epsilon(\det_q-1)=0$, the two-sided ideal $(\det_q-1)$ in $\mathrm{M}_q(2,\mathbb{F})$ is a coideal in $\mathrm{M}_q(2,\mathbb{F})$. This implies that the quotient $\mathrm{M}_q(2,\mathbb{F})/(\det_q-1)$ inherits a bialgebra structure from $\mathrm{M}_q(2,\mathbb{F})$. To show that this bialgebra is a Hopf algebra, we compute with the expressions for $S$ in $\mathrm{SL}_q(2,\mathbb{F})$

$$\begin{aligned}
&m\circ(S\otimes\mathrm{id})\circ\Delta(a) = S(a)\cdot a+S(b)\cdot c = da-qbc = ad-q^{-1}bc = \det_q = 1 = \epsilon(a)\\
&m\circ(\mathrm{id}\otimes S)\circ\Delta(a) = a\cdot S(a)+b\cdot S(c) = ad-q^{-1}bc = \det_q = 1 = \epsilon(a)\\
&m\circ(S\otimes\mathrm{id})\circ\Delta(b) = S(a)\cdot b+S(b)\cdot d = db-qbd = 0 = \epsilon(b)\\
&m\circ(\mathrm{id}\otimes S)\circ\Delta(b) = a\cdot S(b)+b\cdot S(d) = -qab+ba = 0 = \epsilon(b)\\
&m\circ(S\otimes\mathrm{id})\circ\Delta(c) = S(c)\cdot a+S(d)\cdot c = -q^{-1}ca+ac = 0 = \epsilon(c)\\
&m\circ(\mathrm{id}\otimes S)\circ\Delta(c) = c\cdot S(a)+d\cdot S(c) = cd-q^{-1}dc = 0 = \epsilon(c)\\
&m\circ(S\otimes\mathrm{id})\circ\Delta(d) = S(c)\cdot b+S(d)\cdot d = -q^{-1}cb+ad = \det_q = 1 = \epsilon(d)\\
&m\circ(\mathrm{id}\otimes S)\circ\Delta(d) = c\cdot S(b)+d\cdot S(d) = -qcb+da = ad-q^{-1}bc = \det_q = 1 = \epsilon(d).
\end{aligned}$$

This shows that $S$ is an antipode for the bialgebra $\mathrm{SL}_q(2,\mathbb{F})$ and $\mathrm{SL}_q(2,\mathbb{F})$ is a Hopf algebra. □

To understand the names $\mathrm{M}_q(2,\mathbb{F})$ and $\mathrm{SL}_q(2,\mathbb{F})$ for these algebras, we note that for $q=1$ the relations of the matrix algebra $M_q(2,\mathbb{F})$ in (7) imply that $M_1(2,\mathbb{F})$ is a commutative algebra with four generators and the coalgebra structure given by (8). If we interpret the generators $a,b,c,d$ as linear maps $a,b,c,d\in\mathrm{Mat}(2\times 2,\mathbb{F})^*$ given by

$$a:\begin{pmatrix}a' & b'\\ c' & d'\end{pmatrix}\mapsto a',\quad b:\begin{pmatrix}a' & b'\\ c' & d'\end{pmatrix}\mapsto b',\quad c:\begin{pmatrix}a' & b'\\ c' & d'\end{pmatrix}\mapsto c',\quad d:\begin{pmatrix}a' & b'\\ c' & d'\end{pmatrix}\mapsto d'$$

then $M_1(2,\mathbb{F})$ is isomorphic to the algebra of functions $f: \mathrm{Mat}(2\times 2,\mathbb{F}) \to \mathbb{F}$ that are polynomials in the entries $a', b', c', d'$, with the pointwise addition, scalar multiplication and multiplication. Moreover, the coalgebra structure defined by (8) coincides with the one from Example 2.1.3, 5. Hence, we extended the coalgebra structure on $\mathrm{Mat}(2\times 2,\mathbb{F})$ to the commutative algebra $\mathrm{M}_1(2,\mathbb{F})$ and obtained a bialgebra structure on $\mathrm{M}_1(2,\mathbb{F})$. We can therefore interpret $q$ as a deformation parameter that changes the algebra structure of the commutative bialgebra $\mathrm{M}_1(2,\mathbb{F})$ to a non-commutative one given by (7). This justifies the name $M_q(2,\mathbb{F})$.

Note also that for $q=1$, we have $\det_1 = ad - bc$ and hence can interpret $\det_1 \in \mathrm{M}_1(2,\mathbb{F})$ as the determinant $\det: \mathrm{Mat}(2\times 2,\mathbb{F}) \to \mathbb{F}$. The algebra $\mathrm{SL}_1(2,\mathbb{F}) = \mathrm{M}_1(2,\mathbb{F})/(\det_1 - 1)$ is obtained from $\mathrm{M}_1(2,\mathbb{F})$ by identifying those polynomial functions that agree on the subset $\mathrm{SL}(2,\mathbb{F}) = \{M \in \mathrm{Mat}(2\times 2,\mathbb{F}) \,|\, \det(M) = 1\}$. Hence, we can interpret $\mathrm{SL}_1(2,\mathbb{F})$ as the bialgebra of functions $f: \mathrm{SL}(2,\mathbb{F}) \to \mathbb{F}$ that are polynomials in the matrix entries, with the pointwise addition, scalar multiplication and multiplication. The antipode of $\mathrm{SL}_1(2,\mathbb{F})$ is given by $S(a) = d$, $S(c) = -b$, $S(b) = -c$ and $S(d) = a$, and we can interpret it as a map that sends the matrix elements of a matrix in $\mathrm{SL}(2,\mathbb{F})$ to the matrix elements of the inverse matrix. The algebra $\mathrm{SL}_q(2,\mathbb{F})$ for general $q$ can then be viewed a deformation of this algebra, in which the multiplication becomes non-commutative, and the matrix elements of the inverse matrix are replaced by their image under the antipode.

Our last important example of a $q$-deformation are the so-called *$q$-deformed universal enveloping algebras*. The simplest non-trivial one is the $q$-deformed universal enveloping algebra $U_q(\mathfrak{sl}(2))$, which is related to the Lie algebra $\mathfrak{sl}(2)$ of traceless $(2\times 2)$-matrices with the Lie bracket given by the matrix commutator. We first give its bialgebra structure in the simplest presentation and then discuss its relation to the Lie algebra $\mathfrak{sl}(2)$ and its universal enveloping algebra $U(\mathfrak{sl}(2))$.

**Example 2.3.9:** Let $\mathbb{F}$ be a field and $q \in \mathbb{F}\setminus\{0,1,-1\}$.
The **$q$-deformed universal enveloping algebra** $U_q(\mathfrak{sl}_2)$ is the algebra over $\mathbb{F}$ with generators $E, F, K, K^{-1}$ and relations

$$K^{\pm 1}K^{\mp 1} = 1, \quad KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2}F, \quad [E,F] = \frac{K-K^{-1}}{q-q^{-1}}. \tag{9}$$

A Hopf algebra structure on $U_q(\mathfrak{sl}_2)$ is given by

$$\begin{array}{lll} \Delta(K^{\pm 1}) = K^{\pm 1}\otimes K^{\pm 1}, & \Delta(E) = 1\otimes E + E\otimes K, & \Delta(F) = F\otimes 1 + K^{-1}\otimes F \\ \epsilon(K^{\pm 1}) = 1, & \epsilon(E) = 0, & \epsilon(F) = 0 \\ S(K^{\pm 1}) = K^{\mp 1} & S(E) = -EK^{-1} & S(F) = -KF. \end{array} \tag{10}$$

**Proof:**
The proof of the claims in Example 2.3.9 is analogous to the one for the previous two examples and is left as an exercise. □

**Remark 2.3.10:** One can show that the set $B = \{E^i F^j K^k \,|\, i,j \in \mathbb{N}_0, k \in \mathbb{N}\}$ is a basis of $U_q(\mathfrak{sl}_2)$ and that the Hopf algebra $\mathrm{SL}_q(2,\mathbb{F})$ from Example 2.3.8 is the finite dual of $U_q(\mathfrak{sl}_2)$ and vice versa. The duality is given by the unique bilinear map $\langle\,,\,\rangle : \mathrm{SL}_q(2)\otimes U_q(\mathfrak{sl}_2) \to \mathbb{F}$ with

$$\langle a, K^{\pm 1}\rangle = q^{\mp 1} \qquad \langle d, K^{\pm 1}\rangle = q^{\pm 1} \qquad \langle b, E\rangle = 1 \qquad \langle c, F\rangle = 1$$

and $\langle x, U\rangle = 0$ for all other combinations of $x \in \{a,b,c,d\}$ and $U \in \{K^{\pm 1}, E, F\}$. The proofs of these statements, which are are lengthy and technical, are given in [Ka].

We will now relate the bialgebra $U_q(\mathfrak{sl}_2)$ to the universal enveloping algebra of the Lie algebra $\mathfrak{sl}_2$ of traceless $(2\times 2)$-matrices. However, the presentation of $U_q(\mathfrak{sl}_2)$ in Example 2.3.9 is not suitable for this task since it is ill-defined for $q=1$. It turns out that this is not a problem with its Hopf algebra structure but with its presentation in terms of generators and relations. We show that there is a bialgebra $U'_q(\mathfrak{sl}_2)$ defined for all $q\in\mathbb{F}\setminus\{0\}$ which is isomorphic to $U_q(\mathfrak{sl}_2)$ for $q\neq\pm 1$ and closely related to the universal enveloping algebra $U(\mathfrak{sl}_2)$ for $q=1$. The price one has to pay is a higher number of generators and relations.

**Proposition 2.3.11:** Let $q\in\mathbb{F}\setminus\{0\}$.

For $q\neq\pm1$ the algebra $U_q(\mathfrak{sl}_2)$ is isomorphic to the algebra $U'_q(\mathfrak{sl}_2)$ over $\mathbb{F}$ with generators $e,f,k,k^{-1},l$ and relations

$$\begin{aligned} &kk^{-1}=k^{-1}k=1, && kek^{-1}=q^2e, && kfk^{-1}=q^{-2}f, && [e,f]=l,\\ &(q-q^{-1})l=k-k^{-1}, && [l,e]=q(ek+k^{-1}e), && [l,f]=-q^{-1}(fk+k^{-1}f). && \end{aligned} \tag{11}$$

For $q=1$, the element $k$ is central in $U'_q(\mathfrak{sl}_2)$ with $k^2=1$ and $U'_1(\mathfrak{sl}_2)/(k-1)$ is isomorphic to $U(\mathfrak{sl}_2)$ as a bialgebra.

**Proof:**
1. Let $V$ be the free vector space generated by $E,F,K^{\pm1}$ and $V'$ be the free vector space generated by $e,f,k^{\pm1},l$. Let $I\subset T(V)$ and $I'\subset T(V')$ be the two-sided ideals generated by the relations (9) and (11), respectively. To show that $U_q(\mathfrak{sl}_2)$ and $U'_q(\mathfrak{sl}_2)$ are isomorphic, we consider for $q\neq\pm1$ the linear maps

$$\begin{aligned} &\phi: V\to T(V') && \text{with} && \phi(E)=e,\ \phi(F)=f,\ \phi(K^{\pm1})=k^{\pm1}\\ &\psi: V'\to T(V) && \text{with} && \psi(e)=E,\ \psi(f)=F,\ \psi(k^{\pm1})=K^{\pm1},\ \psi(l)=[E,F]. \end{aligned}$$

By the universal property of the tensor algebra, there are unique algebra homomorphisms $\phi': T(V)\to T(V')$ and $\psi': T(V')\to T(V)$ with $\phi'\circ\iota_V=\phi$ and $\psi'\circ\iota_{V'}=\psi$. To prove that the latter descend to algebra homomorphisms between $U_q(\mathfrak{sl}_2)$ and $U'_q(\mathfrak{sl}_2)$, we have to show that $\phi'(r)\in I'$ and $\psi'(r')\in I$ for each relation $r$ of $U_q(\mathfrak{sl}_2)$ and $r'$ of $U'_q(\mathfrak{sl}_2)$. For the first four relations of $U_q(\mathfrak{sl}_2)$ and the first five relations of $U'_q(\mathfrak{sl}_2)$, this is obvious. For the 5th relation of $U_q(\mathfrak{sl}_2)$ and the 6th relation of $U'_q(\mathfrak{sl}_2)$, we have

$$\begin{aligned} &\phi'([E,F]-(q-q^{-1})^{-1}(K-K^{-1}))=[e,f]-(q-q^{-1})^{-1}(k-k^{-1})=l-l=0 \bmod I'\\ &\psi'((q-q)^{-1}l-k+k^{-1})=(q-q^{-1})[E,F]-K+K^{-1}=0 \bmod I, \end{aligned}$$

and for the last two relations in $I'$, we obtain

$$\begin{aligned} &\psi'([l,e]-q(ek+k^{-1}e))=[[E,F],E]-q(EK+K^{-1}E)\\ &=(q-q^{-1})^{-1}[K-K^{-1},E]-q(EK+K^{-1}E)\ \text{mod}\ I\\ &=(q^2-1)(q-q^{-1})^{-1}(EK+K^{-1}E)-q(EK+K^{-1}E)\ \text{mod}\ I=0\ \text{mod}\ I \end{aligned}$$

$$\begin{aligned} &\psi'([l,f]+q^{-1}(fk+k^{-1}f))=[[E,F],F]+q^{-1}(FK+K^{-1}F)\\ &=(q-q^{-1})^{-1}[K-K^{-1},F]+q^{-1}(FK+K^{-1}F)\ \text{mod}\ I\\ &=(q^{-2}-1)(q-q^{-1})^{-1}(FK+K^{-1}F)+q^{-1}(FK+K^{-1}F)\ \text{mod}\ I=0\ \text{mod}\ I, \end{aligned}$$

where we use the shorthand notation $a=b \bmod I$ for $a-b\in I$. This shows that $\phi'$ and $\psi'$ induce algebra homomorphisms $\phi: U_q(\mathfrak{sl}_2)\to U'_q(\mathfrak{sl}_2)$ and $\psi: U'_q(\mathfrak{sl}(2))\to U_q(\mathfrak{sl}_2)$. That the latter are isomorphisms follows with the universal properties of the tensor algebra and the quotient algebra from the identities

$$\psi\circ\phi(E)=E,\qquad \psi\circ\phi(F)=F,\qquad \psi\circ\phi(K^{\pm 1})=K^{\pm 1},$$
$$\phi\circ\psi(e)=e,\qquad \phi\circ\psi(f)=f,\qquad \phi\circ\psi(k^{\pm 1})=k^{\pm 1},\qquad \phi\circ\psi(l)=[e,f]=l.$$

In particular, we find that the Hopf algebra structure of $U'_q(\mathfrak{sl}_2)$ is given by

$$\begin{array}{llll}\Delta(k^{\pm 1})=k^{\pm 1}\otimes k^{\pm 1}, & \Delta(e)=1\otimes e+e\otimes k, & \Delta(f)=f\otimes 1+k^{-1}\otimes f, & \Delta(l)=l\otimes k+k^{-1}\otimes l\\ \epsilon(k^{\pm 1})=1, & \epsilon(e)=0, & \epsilon(f)=0, & \epsilon(l)=0,\\ S(k^{\pm 1})=k^{\mp 1}, & S(e)=-ek^{-1}, & S(f)=-kf, & S(l)=-l.\end{array}$$

2. The algebra $U'_q(\mathfrak{sl}_2)$ is defined for $q=1$. In this case its relations given in (11) reduce to

$$k^2=1,\qquad [k,e]=0,\qquad [k,f]=0,\qquad [e,f]=l,\qquad [l,e]=2ek,\qquad [l,f]=-2fk,$$

and its Hopf algebra structure is given by $\epsilon(k)=1$, $\epsilon(e)=\epsilon(f)=\epsilon(l)=0$, and

$$\begin{array}{llll}\Delta(k)=k\otimes k, & \Delta(e)=1\otimes e+e\otimes k, & \Delta(f)=f\otimes 1+k\otimes f, & \Delta(l)=l\otimes k+k\otimes l\\ S(k)=k\otimes k, & S(e)=-e, & S(f)=-f, & S(l)=-l.\end{array}$$

As $k$ is central in $U'_1(\mathfrak{sl}(2))$ with $k^2=1$ and $\Delta(k)=k\otimes k$, the quotient $U'_1(\mathfrak{sl}(2))/(k-1)$ inherits a bialgebra structure from $U'_1(\mathfrak{sl}_2)$. Its algebra structure is given by

$$[e,f]=l,\quad [l,e]=2e,\quad [l,f]=-2f, \tag{12}$$

and its Hopf algebra structure by

$$\epsilon(X)=0,\quad \Delta(X)=X\otimes 1+1\otimes X,\quad S(X)=-X\qquad \forall X\in\{e,f,l\}. \tag{13}$$

If we choose as a basis of $\mathfrak{sl}_2=\{M\in \mathrm{Mat}(2\times 2,\mathbb{F})\,|\,\mathrm{tr}(M)=0\}$ the matrices

$$l=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\qquad e=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\qquad f=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} \tag{14}$$

then the Lie bracket of $\mathfrak{sl}_2$ is given by (12), and the bialgebra structure of the universal enveloping algebra $U(\mathfrak{sl}_2)$ from Example 2.3.2 by (12) and (13). This shows that the Hopf algebras $U'_1(\mathfrak{sl}_2)/(k-1)$ and $U(\mathfrak{sl}_2)$ are isomorphic. □

Proposition 2.3.11 motivates the name $q$-deformed universal enveloping algebra and the notation $U_q(\mathfrak{sl}_2)$, since it relates $U_q(\mathfrak{sl}_2)$ for $q=1$ to the universal enveloping algebra of the Lie algebra $\mathfrak{sl}_2$ of traceless $2\times 2$-matrices. Besides $q=1$, there are other values of $q$, for which the $q$-deformed universal enveloping algebra $U_q(\mathfrak{sl}_2)$ has a particularly interesting structure, namely the case where $q$ is a root of unity. In this case, one can take a quotient of $U_q(\mathfrak{sl}_2)$ by a two-sided ideal to obtain a finite-dimensional Hopf algebra. This finite-dimensional Hopf algebra is often called the **$q$-deformed universal enveloping algebra $U_q(\mathfrak{sl}_2)$ at a root of unity**, but the name is slightly misleading since it is a quotient of $U_q(\mathfrak{sl}_2)$. The proof of the following proposition is left as an exercise.

**Proposition 2.3.12:** Let $\mathbb{F}$ be a field, $q \in \mathbb{F} \setminus \{1, -1\}$ a primitive $d$th root of unity and $r := d$ if $d$ is odd and $r := d/2$ if $d$ is even.

1. The elements $K^{\pm r}$, $E^r$, $F^r$ are central in $U_q(\mathfrak{sl}_2)$.
2. $U_q^r(\mathfrak{sl}_2) = U_q(\mathfrak{sl}_2)/(F^r, E^r, K^r - 1)$ inherits a Hopf algebra structure from $U_q(\mathfrak{sl}_2)$.
3. $U_q^r(\mathfrak{sl}_2)$ is finite-dimensional and spanned by $\{E^i F^j K^k \,|\, i, j, k = 0, 1, ..., r-1\}$.

Clearly, the $q$-deformed universal enveloping algebra $U_q(\mathfrak{sl}_2)$, its counterpart $U'_q(\mathfrak{sl}_2)$ and its quotient $U_q^r(\mathfrak{sl}_2)$ at a root of unity have a complicated mathematical structure, and it is not obvious at all how to generalise this construction to other Lie algebras in a systematic way. Nevertheless, they are part of a general construction that is possible for all complex, simple Lie algebras and can be generalised to affine Kac-Moody algebras. These are the so-called *Drinfeld-Jimbo deformations* of of universal enveloping algebras. For complex simple Lie-algebras of type $A, D, E$, they take a particularly simple form.

**Remark 2.3.13:** (**Drinfeld-Jimbo deformations**)

Let $\mathfrak{g}$ be a complex, simple Lie algebra and $B = \{H_i, E_i, F_i \,|\, i = 1, ..., r\}$, the **Chevalley basis** of $\mathfrak{g}$, in which the Lie bracket takes the form

$$\begin{array}{llll} [H_i, H_j] = 0, & [H_i, E_j] = a_{ij} E_j, & [H_i, F_j] = -a_{ij} F_j, & [E_i, E_j] = \delta_{ij} H_i, \\ (\mathrm{ad}_{E_i})^{1-a_{ij}} E_j = 0, & & (\mathrm{ad}_{F_i})^{1-a_{ij}} F_j = 0, & \end{array}$$

where $\mathrm{ad}_X$ is the linear map $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$, $Y \mapsto [X, Y]$ and $A = (a_{ij}) \in \mathrm{Mat}(r \times r, \mathbb{Z})$ the **Cartan matrix** of $\mathfrak{g}$.

If $\mathfrak{g}$ is a complex simple Lie algebra of type $A, D$ or $E$, its Cartan matrix is positive definite and symmetric with $a_{ii} = 2$ for $i \in \{1, ..., r\}$ and $a_{ij} \in \{0, -1\}$ for $i \neq j$. In this case, the $q$-deformed universal enveloping algebra $U_q(\mathfrak{g})$ is has generators $\{K_i, E_i, F_i \,|\, i = 1, ..., r\}$ and relations

$$\begin{array}{llll} K_i^{\pm 1} K_i^{\pm 1} = 1, \quad [K_i, K_j] = 0, & & & \\ [E_i, F_j] = \delta_{ij} (q - q^{-1})^{-1} (K_i - K_i^{-1}) & K_i E_j K_i^{-1} = q^{a_{ij}} E_j, & K_i F_j K_i^{-1} = q^{-a_{ij}} F_j, & \\ [E_i, E_j] = 0 & [F_i, F_j] = 0 & & \text{if } a_{ij} = 0, \\ E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = 0 & F_i^2 F_j - (q + q^{-1}) F_i F_j F_i + F_j F_i^2 = 0 & & \text{if } a_{ij} = -1. \end{array}$$

Its Hopf algebra structure is given by

$$\begin{array}{lll} \Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1} & \Delta(E_i) = 1 \otimes E_i + E_i \otimes K_i & \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i \\ \epsilon(K_i^{\pm 1}) = 1 & \epsilon(E_i) = 0 & \epsilon(F_i) = 0 \\ S(K_i^{\pm 1}) = K_i^{\mp 1} & S(E_i) = -E_i K_i^{-1} & S(F_i) = -K_i F_i. \end{array}$$

There is also a presentation of $U_q(\mathfrak{g})$ similar to the one in Proposition 2.3.11 that is well-defined at $q = 1$ and relates the Hopf algebra $U_q(\mathfrak{g})$ to the universal enveloping algebra $U(\mathfrak{g})$. If $q$ is a root of unity, then there is a finite-dimensional quotient $U_q^r(\mathfrak{g})$, which inherits a Hopf algebra structure from $U_q(\mathfrak{g})$ and generalises the Hopf algebra $U_q^r(\mathfrak{sl}_2)$ from Proposition 2.3.12.

## 2.4 Grouplike and primitive elements

In this section, we investigate elements of Hopf algebras that behave in a similar way to the elements $g \in G$ in the group algebra $\mathbb{F}[G]$ and the elements $v \in V$ in the tensor algebra

$T(V)$ or the elements $x \in \mathfrak{g}$ in the universal enveloping algebra $U(\mathfrak{g})$. Such elements play an important role in the classification of Hopf algebras, especially in the cocommutative and finite-dimensional case. Although these classification results are rather involved and cannot be proven here, the investigation of these elements is helpful to develop an intuition for Hopf algebras.

**Definition 2.4.1:** Let $H$ be a Hopf algebra.

1. An element $g \in H \setminus \{0\}$ is called **grouplike** if $\Delta(g) = g \otimes g$. The set of grouplike elements of $H$ is denoted $\mathrm{Gr}(H)$.
2. An element $h \in H$ is called **primitive** if $\Delta(h) = 1 \otimes h + h \otimes 1$. The set of primitive elements of $H$ is denoted $\mathrm{Pr}(H)$.

**Example 2.4.2:**

1. The elements $g \in G$ are grouplike elements of the group algebra $\mathbb{F}[G]$. The element $y$ in Taft's example and the elements $K^{\pm 1}$ in $U_q(\mathfrak{sl}_2)$ are grouplike.
2. The elements $v \in V$ are primitive elements of the tensor algebra $T(V)$ and the elements $x \in \mathfrak{g}$ are primitive elements of the universal enveloping algebra $U(\mathfrak{g})$.
3. Let $H^*$ be the (finite) dual of a Hopf algebra $H$. An element $\alpha \in H^* \setminus \{0\}$ is grouplike if and only if for all $h, k \in H$ $\alpha(h \cdot k) = \Delta(\alpha)(h \otimes k) = (\alpha \otimes \alpha)(h \otimes k) = \alpha(h)\alpha(k)$. As this implies $\alpha(1) = \epsilon(\alpha) = 1$, the grouplike elements of $H^*$ are the algebra homomorphisms $\alpha : H \to \mathbb{F}$. An algebra homomorphism $\alpha : H \to \mathbb{F}$ is also called a **character** of $H$. An element $\beta \in H^*$ is primitive if and only if for all $h, k \in H$ it satisfies the condition $\beta(h \cdot k) = \Delta(\beta)(h \otimes k) = (1 \otimes \beta + \beta \otimes 1)(h \otimes k) = \epsilon(h)\beta(k) + \epsilon(k)\beta(h)$.

The reason for the name *grouplike element* is not only that grouplike elements mimic the behaviour of elements $g \in G$ in the group algebra $\mathbb{F}[G]$, but one can show that they form indeed a group. Similarly, *primitive elements* of a Hopf algebra could in principle be called *Lie algebra-like* since they form a Lie algebra with the commutator. Moreover, one can show that both, grouplike and primitive elements generate Hopf subalgebras of $H$.

**Proposition 2.4.3:** Let $H$ be a Hopf algebra.

1. Every grouplike element $g \in H$ satisfies $\epsilon(g) = 1$ and $S(g) = g^{-1}$.
2. The set $\mathrm{Gr}(H) \subset H$ is a group and $\mathrm{span}_{\mathbb{F}}\mathrm{Gr}(H) \subset H$ is a Hopf subalgebra.
3. Every primitive element $h \in H$ satisfies $\epsilon(h) = 0$ and $S(h) = -h$.
4. The set $\mathrm{Pr}(H) \subset H$ is a Lie subalgebra of the Lie algebra $H$ with the commutator, and the subalgebra of $H$ generated by $\mathrm{Pr}(H)$ is a Hopf subalgebra.
5. If $g \in H$ is grouplike and $h \in H$ primitive, then $ghg^{-1}$ is primitive.

**Proof:**
If $g \in H$ is grouplike, then $\Delta(g) = g \otimes g$ and $g \neq 0$. The counitality condition implies

$$1 \otimes g = (\epsilon \otimes \mathrm{id}) \circ \Delta(g) = (\epsilon \otimes \mathrm{id})(g \otimes g) = \epsilon(g) \otimes g = 1 \otimes \epsilon(g) g.$$

As $g \neq 0$ it follows that $\epsilon(g) = 1$. Similarly, the condition on the antipode implies

$$\eta \circ \epsilon(g) = 1 = m \circ (S \otimes \mathrm{id}) \circ \Delta(g) = S(g) \cdot g = m \circ (\mathrm{id} \otimes S) \circ \Delta(g) = g \cdot S(g).$$

This shows that $S(g)$ is an inverse of $g$. As $\Delta$ is an algebra homomorphism, we have $\Delta(1) = 1 \otimes 1$ and $1 \in \mathrm{Gr}(H)$. If $g, h \in \mathrm{Gr}(h)$, then we have $\Delta(gh) = \Delta(g) \cdot \Delta(h) = (g \otimes g)(h \otimes h) = gh \otimes gh$ and hence $gh \in \mathrm{Gr}(H)$. Similarly, $\Delta(S(g)) = (S \otimes S) \circ \Delta^{op}(g) = S(g) \otimes S(g)$. As $S(g) = g^{-1} \neq 0$, we have $g^{-1} \in \mathrm{Gr}(H)$ for all $g \in \mathrm{Gr}(H)$. This shows that $\mathrm{Gr}(H)$ is a group. By definition of a grouplike element, one has $\Delta(g) = g \otimes g$ and hence $\mathrm{span}_{\mathbb{F}} \mathrm{Gr}(H) \subset H$ is a Hopf subalgebra.

If $h \in H$ is primitive, then the counitality condition implies

$$1 \otimes h = (\epsilon \otimes \mathrm{id}) \circ \Delta(h) = (\epsilon \otimes \mathrm{id}) \circ (1 \otimes h + h \otimes 1) = \epsilon(1) \otimes h + \epsilon(h) \otimes 1 = 1 \otimes h + \epsilon(h) \otimes 1,$$

and it follows that $\epsilon(h) = 0$. Similarly, the antipode condition implies

$$m \circ (S \otimes \mathrm{id}) \circ \Delta(h) = (m \circ S)(1 \otimes h + h \otimes 1) = S(1) \cdot h + S(h) \cdot 1 = h + S(h) = \eta \circ \epsilon(h) = 0$$

and hence $S(h) = -h$. It also follows from the linearity of the comultiplication that $\mathrm{Pr}(H) \subset H$ is a linear subspace. If $h, k \in \mathrm{Pr}(H)$, then their commutator $[h, k] = h \cdot k - k \cdot h$ satisfies

$$\begin{aligned}\Delta([h,k]) &= [\Delta(h), \Delta(k)] = (1 \otimes h + h \otimes 1) \cdot (1 \otimes k + k \otimes 1) - (1 \otimes k + k \otimes 1) \cdot (1 \otimes h + h \otimes 1) \\ &= 1 \otimes hk + k \otimes h + h \otimes k + hk \otimes 1 - (1 \otimes kh + h \otimes k + k \otimes h + kh \otimes 1) = 1 \otimes [h,k] + [h,k] \otimes 1.\end{aligned}$$

This shows that $[h, k] \in \mathrm{Pr}(H)$ and hence $\mathrm{Pr}(H) \subset H$ is a Lie subalgebra of the Lie algebra $H$ with the commutator. As $\Delta(h) = 1 \otimes h + h \otimes 1$ and $S(h) = -h$ for every primitive element $h \in H$ and the maps $\Delta : H \to H \otimes H$ and $S : H \to H^{op}$ are algebra homomorphisms, it follows that the subalgebra of $H$ generated by the primitive elements is a Hopf subalgebra of $H$. Finally, if $g \in H$ is grouplike and $h \in H$ primitive, then

$$\Delta(ghg^{-1}) = \Delta(g)\Delta(h)\Delta(g^{-1}) = (g \otimes g)(1 \otimes h + h \otimes 1)(g^{-1} \otimes g^{-1}) = 1 \otimes ghg^{-1} + ghg^{-1} \otimes 1$$

and hence $ghg^{-1}$ is primitive. □

Proposition 2.4.3 suggests that every Hopf algebra $H$ contains a Hopf subalgebra $K$ that is a semidirect product $K = \mathbb{F}[\mathrm{Gr}(H)] \ltimes A$ of the group algebra of $\mathrm{Gr}(H)$ and the Hopf subalgebra $A \subset H$ generated by the primitive elements, i. e. $K \cong \mathbb{F}[G] \otimes A$ as a vector space with the multiplication law $(a \otimes g) \cdot (b \otimes h) = a(gbg^{-1}) \otimes gh$. for all $a, b \in A$ and $g, h \in \mathrm{Gr}(H)$. To show that this is indeed the case, we need to prove that different grouplike elements of $H$ are linearly independent, i. e. that $\mathrm{span}_{\mathbb{F}} \mathrm{Gr}(H) \cong \mathbb{F}[\mathrm{Gr}(H)]$ and that $\mathrm{Gr}(H) \cap A = \{1_H\}$, i. e. that the only grouplike element in the Hopf subalgebra generated by primitive elements is the unit of $H$.

**Proposition 2.4.4:** Let $H$ be a Hopf algebra over $\mathbb{F}$.

1. The set $\mathrm{Gr}(H)$ of grouplike elements is linearly independent.
2. If $H$ is generated as an algebra by primitive elements, then $\mathrm{Gr}(H) = \{1\}$.

**Proof:**
1. We show by induction over $n$ that $\Sigma_{i=1}^n \lambda_i g_i = 0$ with $\lambda_i \in \mathbb{F}$ and $g_i \in \mathrm{Gr}(H)$ pairwise distinct implies $\lambda_1 = ... = \lambda_n = 0$. For $n = 1$, this follows from the fact that $g \neq 0$ for all $g \in \mathrm{Gr}(H)$.

Suppose the claim holds for all linear combinations with at most $n$ nontrivial coefficients, and let $\Sigma_{i=1}^{n+1}\lambda_i g_i = 0$ with pairwise distinct $g_i \in \mathrm{Gr}(H)$.

Let $\iota : H \to H^{**}$, $h \mapsto h'$ be the canonical injection defined by $h'(\alpha) = \alpha(h)$ for all $\alpha \in H^*$. Then the elements $g_i' \in H^{**}$ are characters of the algebra $H^*$, i. e. they satisfy $g_i'(\alpha \cdot \beta) = g_i'(\alpha) \cdot g_i'(\beta)$ for all $\alpha, \beta \in H^*$ and $g_i'(1) = 1_{\mathbb{F}}$. As $g_{n+1} \notin \{g_1, ..., g_n\}$ there is an element $\alpha \in H^*$ with $g_{n+1}'(\alpha) = \alpha(g_{n+1}) = 1$ and $g_i'(\alpha) = \alpha(g_i) \neq 1$ for all $i \in \{1, ..., n\}$. This implies for all $\beta \in H^*$

$$0 = \Sigma_{i=1}^{n+1}\lambda_i g_i'(\beta) - \Sigma_{i=1}^{n+1}\lambda_i g_i'(\alpha \cdot \beta) = \Sigma_{i=1}^{n+1}\lambda_i(1 - g_i'(\alpha))g_i'(\beta) = \beta(\Sigma_{i=1}^{n}\lambda_i(1 - \alpha(g_i))g_i)$$

and hence $\Sigma_{i=1}^{n}\lambda_i(1 - \alpha(g_i))g_i = 0$. With the induction hypothesis and $\alpha(g_i) \neq 1$ one obtains $\lambda_1 = ... = \lambda_n = 0$, and this implies $\lambda_{n+1} = 0$ since $g_{n+1} \neq 0$.

2. Let $H_0 = \mathbb{F}1_H$, $X \subset H$ a set of primitive generators and $H_n$ the linear subspace of $H$ spanned by all elements of the form $x_{i_1}^{m_1} \cdots x_{i_k}^{m_k}$ with $x_{i_j} \in X$ and $m_1 + ... + m_k \leq n$. Then we have $H = \cup_{n=0}^{\infty} H_n$, $H_n \subset H_m$ for all $m \geq n$, and $H_n \cdot H_m \subset H_{n+m}$. Moreover, it follows by induction that for any primitive element $h \in H$, one has

$$\Delta(h^n) = \sum_{k=0}^{n} \binom{n}{k} h^k \otimes h^{n-k}. \tag{15}$$

This implies

$$\Delta(x_{i_1}^{m_1} \cdots x_{i_k}^{m_k}) = \Delta(x_{i_1})^{m_1} \cdots \Delta(x_{i_k})^{m_k} = \sum_{l_1=0}^{m_1} \cdots \sum_{l_k=0}^{m_k} \binom{m_1}{l_1} \cdots \binom{m_k}{l_k} x_{i_1}^{l_1} \cdots x_{i_k}^{l_k} \otimes x_{i_1}^{m_1-l_1} \cdots x_{i_k}^{m_k-l_k}$$

and hence $\Delta(H_n) \subset \Sigma_{k=0}^{n} H_k \otimes H_{n-k}$. If $g \in H$ is grouplike with $m = \min\{n \in \mathbb{N}_0 \,|\, g \in H_n\} \geq 1$, then there is an $\alpha \in H^*$ with $\alpha(g) = 1$ and $\alpha(1_H) = \{0\}$, and this implies

$$1_{\mathbb{F}} \otimes g = (\alpha \otimes \mathrm{id})(g \otimes g) = (\alpha \otimes \mathrm{id}) \circ \Delta(g) \in (\alpha \otimes \mathrm{id})\,(\Sigma_{k=0}^{m} H_k \otimes H_{m-k}) \subset 1_{\mathbb{F}} \otimes H_{m-1}$$

where we used in the last step that $\alpha(H_0) = \alpha(\mathbb{F}1_H) = \{0\}$. As $g \neq 0$, it follows that $g \in H_{m-1}$, which contradicts the minimality of $m$. Hence $\mathrm{Gr}(H) \subset H_0$, and the only grouplike element in $H_0$ is $1_H$. □

**Corollary 2.4.5:** Let $G$ be a group and $\mathbb{F}$ a field. Then $\mathrm{Pr}(\mathbb{F}[G]) = \{0\}$ and $\mathrm{Gr}(\mathbb{F}[G]) = G$.

**Proof:**
If $x = \Sigma_{g \in G}\lambda_g\, g$ is primitive, then $\Delta(x) = \Sigma_{g \in G}\lambda_g\, g \otimes g = \Sigma_{g \in G}\lambda_g(1 \otimes g + g \otimes 1)$. As the set $\{g \otimes h \,|\, g, h \in G\}$ is a basis of $\mathbb{F}[G] \otimes \mathbb{F}[G]$, this implies $\lambda_g = 0$ for all $g \in G$ and $x = 0$. Clearly, every element $g \in G$ is grouplike. If there was a grouplike element $y \in \mathbb{F}[G] \setminus G$, then the set $G \cup \{y\} \supsetneq G$ would be linearly independent by Proposition 2.4.4, a contradiction to the fact that $G \subset \mathbb{F}[G]$ is a basis of $\mathbb{F}[G]$. □

This corollary confirms the expectation that the only grouplike elements in a group algebra $\mathbb{F}[G]$ are the group elements $g \in G$ and that the group algebra contains no non-trivial primitive elements. Similarly, Proposition 2.4.4 implies that the only grouplike element in the tensor algebra $T(V)$ and in a universal enveloping algebra $U(\mathfrak{g})$ is the unit element, since both Hopf algebras are generated by primitive elements. In analogy to the statement about the grouplike elements in a group algebra $\mathbb{F}[G]$, one would expect that the primitive elements in a universal enveloping algebra $U(\mathfrak{g})$ are precisely the elements of the Lie algebra $\mathfrak{g} \subset U(\mathfrak{g})$. However, the following proposition shows that this is only true for Lie algebras over fields of characteristic zero.

**Proposition 2.4.6:** Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{F}$ and $U(\mathfrak{g})$ its universal enveloping algebra.

1. If $\mathrm{char}(\mathbb{F}) = 0$ then $\mathrm{Pr}(U(\mathfrak{g})) = \mathfrak{g}$.
2. If $\mathrm{char}(\mathbb{F}) = p$ then $\mathrm{Pr}(U(\mathfrak{g})) = \mathrm{span}_{\mathbb{F}}\{x^{p^l} \,|\, x \in \mathfrak{g}, l \in \mathbb{N}_0\}$.

**Proof:**
Every element $x \in \mathfrak{g} \subset U(\mathfrak{g})$ is primitive, and hence $\mathfrak{g} \subset \mathrm{Pr}(U(\mathfrak{g}))$. If $B = (x_1, ..., x_n)$ is an ordered basis of $\mathfrak{g}$, then the Poincaré-Birkhoff-Witt basis $B = \{x_1^{m_1} \cdots x_n^{m_n} \,|\, m_1, ..., m_n \in \mathbb{N}_0\}$ is a basis of $U(\mathfrak{g})$, and hence every element $x \in U(\mathfrak{g})$ can be expressed as a linear combination

$$x = \sum_{m_1=0}^{K} \cdots \sum_{m_n=0}^{K} \lambda_{m_1...m_n} x_1^{m_1} \cdots x_n^{m_n}$$

with $\lambda_{m_1...m_n} \in \mathbb{F}$ for some $K \in \mathbb{N}$. Equation (15) from the proof of Proposition 2.4.4 implies

$$\Delta(x) = \sum_{m_1=0}^{K} \cdots \sum_{m_n=0}^{K} \sum_{k_1=0}^{m_1} \cdots \sum_{k_n=0}^{m_n} \lambda_{m_1...m_n} \binom{m_1}{k_1} \cdots \binom{m_n}{k_n} x_1^{k_1} \cdots x_n^{k_n} \otimes x_1^{m_1-k_1} \cdots x_n^{m_n-k_n}.$$

As the set $\{x_1^{k_1} \cdots x_n^{k_n} \,|\, k_1, ..., k_n \in \{0, ..., K\}\}$ is linearly independent by the Poincaré-Birkhoff-Witt Theorem, this shows that $x$ cannot be primitive, unless it is of the form $x = \Sigma_{i=1}^{n} \mu_i x_i^{m_i}$ for some $\mu_i \in \mathbb{F}$. In this case, one has

$$\Delta(x) = \sum_{i=1}^{n} \sum_{k=0}^{m_i} \mu_i \binom{m_i}{k} x_i^k \otimes x_i^{m_i-k}.$$

If $\mathrm{char}(\mathbb{F}) = 0$, all binomial coefficients in this formula are non-zero, and this shows that $x$ can only be primitive if $m_i \in \{0, 1\}$ for all $i \in \{1, ..., n\}$ with $\mu_i \neq 0$, which implies $x \in \mathfrak{g}$. If $\mathrm{char}(\mathbb{F}) = p$, then all binomial coefficients for $i \in \{1, ..., n\}$ with $\mu_i \neq 0$ and $k = 1 < m_i$ must vanish in order for $x$ to be primitive. This is the case is and only if $m_i = p^{l_i}$ for some $l_i \in \mathbb{N}$ and all $i \in \{1, ..., n\}$ with $\mu_i \neq 0$. Conversely, if $m_i = p^{l_i}$ with $l_i \in \mathbb{N}_0$, then all binomial coefficients for $k \notin \{0, m_i\}$ vanish, since they are divisible by $p$, and this shows that $x$ is a linear combination of elements $y^{p^l}$ with $y \in \mathfrak{g}$ and $l \in \mathbb{N}_0$. □

As the restrictions of the comultiplication of a Hopf algebra $H$ to the Hopf subalgebras $\mathrm{span}_{\mathbb{F}}\mathrm{Gr}(H)$ and to the Hopf subalgebra generated by the set $\mathrm{Pr}(H)$ are cocommutative, one cannot hope in general that every Hopf algebra can be decomposed into Hopf subalgebras spanned by grouplike or generated by primitive elements, since this would imply that $H$ is cocommutative. However, one can show that this is indeed possible for every *cocommutative Hopf algebra* over an *algebraically closed field of characteristic zero*. This is known as the *Cartier-Kostant-Milnor-Moore Theorem*. Parts of the proof are given in [Mo, Chapter 5].

**Theorem 2.4.7:** (**Cartier-Kostant-Milnor-Moore Theorem**)
If $H$ is a cocommutative Hopf algebra over an algebraically closed field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) = 0$, then $H$ is isomorphic as an algebra to the semidirect product $H \cong \mathbb{F}[G] \ltimes U(\mathfrak{g})$, where $G = \mathrm{Gr}(H)$ and $\mathfrak{g} = \mathrm{Pr}(H)$.

## 2.5 *Construction of $q$-deformed universal enveloping algebras

In this section, we show how $q$-deformed universal enveloping algebras can be constructed in a systematic way from Lie algebras with additional structure, the so-called *Lie bialgebras*. This construction requires formal *power series with coefficients in an algebra* and *tensor products of formal power series*. The idea is then to construct the multiplication and comultiplication of the $q$-deformed universal enveloping algebra inductively, order for order, from the corresponding structures of the Lie bialgebra, which characterise the lowest terms in these formal power series. We start by introducing the required notions of *formal power series* and their *tensor products*.

Let $V$ be a vector space over $\mathbb{F}$. We consider the $\mathbb{N}_0$-fold product $\Pi_{n\in\mathbb{N}_0}V$ of the vector space $V$ with itself, i. e. the vector space of sequences with values in $V$ with the pointwise addition and scalar multiplication. For a sequence with values in $V$, we write $v = \Sigma_{n=0}^\infty \hbar^n v_n$ instead of $v = (v_n)_{n\in\mathbb{N}_0}$. Then one has $\Sigma_{n=0}^\infty \hbar^n v_n = \Sigma_{n=0}^\infty \hbar^n w_n$ if and only if $v_n = w_n$ for all $n \in \mathbb{N}_0$, and the vector addition and scalar multiplication take the form

$$(\Sigma_{n=0}^\infty \hbar^n v_n) + (\Sigma_{n=0}^\infty \hbar^n w_n) = \Sigma_{n=0}^\infty \hbar^n (v_n + w_n) \qquad \lambda\,(\Sigma_{n=0}^\infty \hbar^n v_n) = \Sigma_{n=0}^\infty \hbar^n (\lambda v_n).$$

In analogy to the algebra $\mathbb{F}[[\hbar]]$ of formal power series from Example 1.1.5, 6, we can interpret the product space $\Pi_{n\in\mathbb{N}_0}V$ as the vector space of formal power series with coefficients in $V$.

**Definition 2.5.1:** Let $V$ be a vector space over $\mathbb{F}$.

1. The vector space of **formal power series** with values in $V$ is the product vector space $V[[\hbar]] = \Pi_{n\in\mathbb{N}_0}V$. The injective $\mathbb{F}$-linear maps $\iota_n : V \to V[[\hbar]]$, $v \mapsto \hbar^n v$ are called the **canonical inclusions**.

2. The linear subspace $V_n[[\hbar]] = \iota_n(V) = \hbar^n V$ is called the **subspace of order $\hbar^n$** . We write $x = y + O(\hbar^n)$ if $x - y \in \cup_{k\geq n} V_k[[\hbar]]$.

The goal is now to define Hopf algebra structures on vector spaces of formal power series in analogy to Hopf algebras over a field. As the multiplication and comultiplication of a Hopf algebra $H$ over $\mathbb{F}$ are $\mathbb{F}$-linear maps $m : H\otimes H \to H$ and $\Delta : H \to H\otimes H$, this requires an appropriate concept of linear maps and of tensor products for vector spaces of formal power series. This notion of linearity must be stronger than simple $\mathbb{F}$-linearity, because it must take into account the formal power series structure, i. e. the different powers of $\hbar$. Similarly, the appropriate notion of a tensor product must take into account both, the tensor product of vector spaces over $\mathbb{F}$ and the multiplication of formal power series in $\mathbb{F}[[\hbar]]$ and combine them in a non-trivial and coherent way. The sensible way to proceed is to define a suitable notion of linearity and bilinearity and then to require that the tensor product is characterised by via a universal property that generalises the one for the tensor product of vector spaces over $\mathbb{F}$.

**Definition 2.5.2:** Let $U, V, W$ be vector spaces over $\mathbb{F}$

1. A map $\phi : V[[\hbar]] \to W[[\hbar]]$ is called **$\hbar$-linear**, if it is $\mathbb{F}$-linear and satisfies

$$\phi(\Sigma_{n=0}^\infty \hbar^n v_n) = \Sigma_{n=0}^\infty \hbar^n \phi(v_n) \quad \text{for all} \quad \Sigma_{n=0}^\infty \hbar^n v_n \in V[[\hbar]]$$

2. A map $\alpha : U[[\hbar]] \times V[[\hbar]] \to W[[\hbar]]$ is called **$\hbar$-bilinear** if it is $\mathbb{F}$-bilinear and

$$\alpha\,(\Sigma_{n=0}^\infty \hbar^n u_n, \Sigma_{n=0}^\infty \hbar^n v_n) = \Sigma_{n=0}^\infty \hbar^n\, \Sigma_{k=0}^n \alpha(u_k, v_{n-k}).$$

   for all $u = \Sigma_{n=0}^\infty \hbar^n u_n \in U[[h]]$ and $v = \Sigma_{n=0}^\infty \hbar^n v_n \in V[[\hbar]]$.

It follows directly from the definition that $\hbar$-linear maps $\phi : V[[\hbar]] \to W[[\hbar]]$ are in bijection with $\mathbb{F}$-linear maps $\phi' : V \to W[[\hbar]]$. More precisely, an $\mathbb{F}$-linear map $\psi' = \Sigma_{n=0}^{\infty}\hbar^n\psi_n : V \to W[[\hbar]]$ corresponds to a family $(\psi_n)_{n\in\mathbb{N}_0}$ of $\mathbb{F}$-linear maps $\psi_n : V \to W$. The associated $\hbar$-linear map $\psi : V[[\hbar]] \to W[[\hbar]]$ is the unique $\mathbb{F}$-linear map with $p_n^W \circ \psi = \psi_n \circ p_n^V$ induced by the universal property of products of vector spaces, where $p_n^V : V[[\hbar]] \to V$, $\Sigma_{m\in\mathbb{N}_0}\hbar^m v_m \mapsto v_n$ and $p_n^W : W[[\hbar]] \to W$, $\Sigma_{m\in\mathbb{N}_0}\hbar^m w_m \mapsto w_n$ are the canonical projections.

Similarly, $\hbar$-bilinear maps $\alpha : U[[\hbar]] \times V[[\hbar]] \to W[[\hbar]]$ are in bijection with $\mathbb{F}$-bilinear maps $\alpha' : U \times V \to W[[\hbar]]$, and an $\mathbb{F}$-bilinear map $\alpha' = \Sigma_{n=0}^{\infty}\hbar^n\alpha_n : U \times V \to W[[\hbar]]$ can be viewed as a family $(\alpha_n)_{n\in\mathbb{N}_0}$ of $\mathbb{F}$-bilinear maps $\alpha_n : U \times V \to W$.

By the universal property of the tensor product of vector spaces, $\mathbb{F}$-bilinear maps $\alpha_n : U \times V \to W$ are in bijection with $\mathbb{F}$-linear maps $\tilde{\alpha}_n : U\otimes V \to W$. By the universal property of products of vector spaces, every family $(\alpha_n)_{n\in\mathbb{N}_0}$ of $\mathbb{F}$-bilinear maps $\alpha_n : U \times V \to W$ induces a unique $\mathbb{F}$-linear map $\tilde{\alpha} : (U\otimes V)[[\hbar]] \to W[[\hbar]]$ with $p_n^W \circ \tilde{\alpha} = \tilde{\alpha}_n \circ p_n^{U\otimes V}$. This suggests an interpretation of the vector space $(U\otimes V)[[\hbar]]$ as a tensor product of $U[[\hbar]]$ and $V[[\hbar]]$, whose universal property relates $\hbar$-bilinear maps $\alpha : U[[\hbar]] \times V[[\hbar]] \to W[[\hbar]]$ to $\hbar$-linear maps $\tilde{\alpha} : (U\otimes V)[[\hbar]] \to W[[\hbar]]$. We have proven the following proposition.

**Proposition 2.5.3:** Let $U, V, W$ be vector spaces over $\mathbb{F}$. Define the **$\hbar$-tensor product** as $U[[\hbar]]\otimes_\hbar V[[\hbar]] := (U\otimes V)[[\hbar]]$ and consider the canonical surjection

$$\pi_{U\otimes V} : U[[\hbar]] \times V[[\hbar]] \to (U\otimes V)[[\hbar]], \quad (\Sigma_{n=0}^{\infty}\hbar^n u_n, \Sigma_{n=0}^{\infty}\hbar^n v_n) \mapsto \Sigma_{n=0}^{\infty}\hbar^n\, \Sigma_{k=0}^{n} u_k\otimes v_{n-k}.$$

Then the pair $(U[[\hbar]]\otimes_\hbar V[[\hbar]], \pi_{U\otimes V})$ has the following **universal property**:

The map $\pi_{U\otimes V}$ is $\hbar$-bilinear, and for every $\hbar$-bilinear map $\phi : U[[\hbar]] \times V[[\hbar]] \to W[[\hbar]]$ there is a unique $\hbar$-linear map $\tilde{\alpha} : U[[\hbar]]\otimes_\hbar V[[\hbar]] \to W[[\hbar]]$ with $\tilde{\alpha} \circ \pi_{U\otimes V} = \alpha$.

**Remark 2.5.4:** It follows directly from the definition that the $\hbar$-tensor product has properties analogous to the usual tensor product of vector spaces or, more generally, modules over commutative rings. In particular, there are canonical $\hbar$-linear isomorphisms

- $(U[[\hbar]] \otimes_\hbar V[[\hbar]]) \otimes_\hbar W[[\hbar]] \cong U[[\hbar]] \otimes_\hbar (V[[\hbar]]\otimes_\hbar W[[\hbar]])$
- $U[[\hbar]]\otimes_\hbar \mathbb{F}[[\hbar]] \cong U[[\hbar]] \cong \mathbb{F}[[\hbar]] \otimes_\hbar U[[\hbar]]$
- $U[[\hbar]]\otimes_\hbar(V[[\hbar]] \oplus W[[\hbar]]) \cong U[[\hbar]]\otimes_\hbar V[[\hbar]] \oplus U[[\hbar]]\otimes_\hbar W[[\hbar]]$
- $(U[[\hbar]] \oplus V[[\hbar]])\otimes_\hbar W[[\hbar]]) \cong U[[\hbar]]\otimes_\hbar W[[\hbar]] \oplus V[[\hbar]]\otimes_\hbar W[[\hbar]]$
- $U[[\hbar]]\otimes_\hbar V[[\hbar]] \cong V[[\hbar]]\otimes_\hbar U[[\hbar]]$.

For every pair of $\hbar$-linear maps $\phi : U[[\hbar]] \to V[[\hbar]]$ and $\psi : W[[\hbar]] \to X[[\hbar]]$ there is a unique $\hbar$-linear map $\phi\otimes\psi : U[[\hbar]]\otimes_\hbar W[[\hbar]] \to V[[\hbar]]\otimes_\hbar X[[\hbar]]$ with $(\phi\otimes\psi) \circ \pi_{U\otimes V} = \pi_{V\otimes W} \circ (\phi \times \psi)$. This is called the **$\hbar$-tensor product** of the maps $\phi$ and $\psi$.

**Remark 2.5.5:** For any vector space $V$ over $\mathbb{F}$, the vector space $V[[\hbar]]$ is a module over the algebra $\mathbb{F}[[\hbar]]$ of formal power series from Example 1.1.5, 6. with

$$\rhd : \mathbb{F}[[\hbar]] \times V[[\hbar]] \to V[[\hbar]], \quad (\Sigma_{n=0}^{\infty}\hbar^n\lambda_n) \rhd (\Sigma_{n=0}^{\infty}\hbar^n v_n) = \Sigma_{n=0}^{\infty}\hbar^n\Sigma_{k=0}^{n}\lambda_k v_{n-k}.$$

However, the notion of an $\mathbb{F}[[\hbar]]$-linear map, i. e. a module homomorphism with respect to these module structures, is weaker than the one of an $\hbar$-linear map $\phi : V[[\hbar]] \to W[[\hbar]]$. Any $\hbar$-linear map $\phi : V[[\hbar]] \to W[[\hbar]]$ is $\mathbb{F}[[\hbar]]$-linear, but there are $\mathbb{F}[[\hbar]]$-linear maps from $V[[\hbar]]$ to $W[[\hbar]]$ that are *not* $\hbar$-linear. Similarly, the tensor product $U[[\hbar]]\otimes_\hbar V[[\hbar]] = (U\otimes V)[[\hbar]]$ *does in general not coincide* with the tensor product $U[[\hbar]]\otimes_{\mathbb{F}[[\hbar]]} V[[\hbar]]$ over the algebra $\mathbb{F}[[\hbar]]$ from Proposition 1.1.15.

One can view the $\hbar$-tensor product $U[[\hbar]]\otimes_\hbar V[[\hbar]]$ as the closure of $U[[\hbar]]\otimes_{\mathbb{F}[[\hbar]]} V[[\hbar]]$ with respect to a certain topology, the so-called **$\hbar$-adic topology**. However, it is also well-motivated from a purely algebraic perspective since it is the simplest way of combining the formal power series structure of $U[[\hbar]]$ and $V[[\hbar]]$ and the tensor product of the vector spaces $U$ and $V$.

With the tensor product over $\hbar$, we can now define algebra, coalgebra, bialgebra and Hopf algebra structures on the vector spaces $V[[\hbar]]$ in direct analogy with Definition 1.1.3, Definition 2.1.1, Definition 2.1.8 and Definition 2.2.1. The only difference is that $\mathbb{F}$-linear maps are replaced by $\hbar$-linear maps and tensor products over $\mathbb{F}$ are replaced by $\hbar$-tensor products.

**Definition 2.5.6:** Let $\mathbb{F}$ be a field.

1. An **algebra over** $\mathbb{F}[[\hbar]]$ is a triple of a vector space $A[[\hbar]]$ and $\hbar$-linear maps $m : (A\otimes A)[[\hbar]] \to A[[\hbar]]$ and $\eta : \mathbb{F}[[\hbar]] \to A[[\hbar]]$ that satisfy the associativity and unit axioms in Definition 1.1.3.

2. A **coalgebra over** $\mathbb{F}[[\hbar]]$ is a triple of a vector space $C[[\hbar]]$ and $\hbar$-linear maps $\Delta : C[[\hbar]] \to (C\otimes C)[[\hbar]]$, $\epsilon : C[[\hbar]] \to \mathbb{F}[[\hbar]]$ that satisfy the coassociativity and counit axioms in Definition 2.1.1.

3. A **bialgebra over** $\mathbb{F}[[\hbar]]$ is a pentuple of a vector space $B[[\hbar]]$ and $\hbar$-linear maps $m : (B\otimes B)[[\hbar]] \to B[[\hbar]]$, $\eta : \mathbb{F}[[\hbar]] \to B[[\hbar]]$, $\Delta : B[[\hbar]] \to (B\otimes B)[[\hbar]]$, $\epsilon : B[[\hbar]] \to \mathbb{F}[[\hbar]]$ that satisfy the conditions in Definition 2.1.8.

4. A bialgebra $B[[\hbar]]$ over $\mathbb{F}[[\hbar]]$ is called a **Hopf algebra over** $\mathbb{F}[[\hbar]]$ if there is an $\hbar$-linear map $S : B[[\hbar]] \to B[[\hbar]$ that satisfies the conditions in Definition 2.2.1.

A bialgebra or a Hopf algebra over $\mathbb{F}[[\hbar]]$ is sometimes called a **topological bialgebra** or **topological Hopf algebra**.

By the universal properties of tensor products and products of vector spaces, any algebra, coalgebra, bialgebra or Hopf algebra structure on a vector space $V$ over $\mathbb{F}$ induces an algebra, coalgebra, bialgebra or Hopf algebra structure on $V[[\hbar]]$, but not all algebra, coalgebra, bialgebra or Hopf algebra structures on $V[[\hbar]]$ arise in this way.

To construct more general Hopf algebra structures over $\mathbb{F}[[\hbar]]$ one presents their algebra structures in terms of generators and relations. As in the case of algebras over $\mathbb{F}$, any algebra over $\mathbb{F}[[\hbar]]$ can be presented in terms of generators and relations, i. e. as a quotient of the tensor algebra $T(V[[\hbar]]) \cong T(V)[[\hbar]]$ by a two-sided ideal in $T(V)[[\hbar]]$. The goal is to start with a finite-dimensional Lie algebra $(\mathfrak{g}, [\,,\,]_\mathfrak{g})$ over $\mathbb{F}$ and to construct a Hopf algebra over $\mathbb{F}[[\hbar]]$ that is presented as a quotient of the algebra $T(\mathfrak{g})[[\hbar]]$ with elements of an ordered basis of

$\mathfrak{g}$ as generators and with a set $R \subset T(\mathfrak{g})[[\hbar]]$ of relations. To obtain a bialgebra structure on $T(\mathfrak{g})[[\hbar]]/(R)$, we then have have to find linear maps

$$\Delta = \Sigma_{n=0}^{\infty}\hbar^n\Delta_n : \mathfrak{g} \to T(\mathfrak{g})[[\hbar]]\otimes T(\mathfrak{g})[[\hbar]] \qquad\qquad \epsilon = \Sigma_{n=0}^{\infty}\hbar^n\epsilon_n : \mathfrak{g} \to \mathbb{F}[[\hbar]]$$

that are given by families of $\mathbb{F}$-linear maps $\Delta_n : \mathfrak{g} \to T(\mathfrak{g})\otimes T(\mathfrak{g})$ and $\epsilon_n : \mathfrak{g} \to \mathbb{F}$ and that satisfy the coassociativity and counitality conditions on each generator. This defines a bialgebra structure on $T(\mathfrak{g})[[\hbar]]$, and if the two-sided ideal $(R) \subset T(\mathfrak{g})[[\hbar]]$ is a coideal for $(T(\mathfrak{g})[[\hbar]], \Delta, \epsilon)$, this bialgebra structure induces a bialgebra structure on $T(\mathfrak{g})[[\hbar]]/(R)$. The resting bialgebra is a Hopf algebra if and only if there is an $\mathbb{F}$-linear map $S = \Sigma_{n=0}^{\infty}\hbar^n S_n : \mathfrak{g} \to T(\mathfrak{g})[[\hbar]]$ with $S(r) \in (R)$ for each relation $r \in R$ that satisfies the antipode condition on each generator. Hence, to obtain a Hopf algebra as a quotient of $T(\mathfrak{g})[[\hbar]]$, one needs to:

- Specify relations $r = \Sigma_{n=0}^{\infty}\hbar^n r_n \in T(\mathfrak{g})[[\hbar]]$ with $r_n \in T(\mathfrak{g})$.
- Specify linear maps $\Delta = \Sigma_{n=0}^{\infty}\hbar^n\Delta_n : \mathfrak{g} \to T(\mathfrak{g})[[\hbar]]\otimes T(\mathfrak{g})[[\hbar]]$, $\epsilon = \Sigma_{n=0}^{\infty}\hbar^n\epsilon_n : \mathfrak{g} \to \mathbb{F}[[\hbar]]$ such that $\epsilon(r) = 0$ and $\Delta(r) \in (R)\otimes T(\mathfrak{g})[[\hbar]] + T(\mathfrak{g})[[\hbar]]\otimes(R)$ for every relation $r \in R$ and the coassociativity and counitality condition are satisfied for each generator.
- Show that there is a linear map $S = \Sigma_{n=0}^{\infty}\hbar^n S_n : \mathfrak{g} \to T(\mathfrak{g})[[\hbar]]$ such that $S(r) \in (R)$ for each relation $r$ and the antipode condition is satisfied on each generator.

If this Hopf algebra is to be interpreted as a deformation of the universal enveloping algebra $U(\mathfrak{g})$ with a deformation parameter $\hbar$, then its *structures*, the relations and the comultiplication, should take the same form as for $U(\mathfrak{g})$ in lowest order in $\hbar$. Moreover, one imposes a homogeneity condition on $\Delta$ that relates the number of generators occurring in $\Delta_n(x)$ to power of $\hbar$. Hence, we require that the following additional conditions are satisfied:

- $\Delta_0(x) = 1\otimes x + x\otimes 1$ for all $x \in \mathfrak{g}$,
- there are exactly $\frac{1}{2}\dim\mathfrak{g}(\dim\mathfrak{g} - 1)$ relations, which are of the form

$$r_{ij} = x_i x_j - x_j x_i - [x_i, x_j]_{\mathfrak{g}} + O(\hbar) \in T(\mathfrak{g})[[\hbar]] \quad \text{for} \quad 1 \le i < j \le \dim(\mathfrak{g}),$$

- $\Delta_n(x) \in \oplus_{k=0}^{n+1}\, \mathfrak{g}^k\otimes\mathfrak{g}^{n+1-k}$ for all $x \in \mathfrak{g}$.

The structure that characterises the comultiplication $\Delta$ data in lowest order of $\hbar$ is then a linear map $\delta : \mathfrak{g} \to \mathfrak{g}\otimes\mathfrak{g}$ defined by

$$\Delta(x) - \Delta^{op}(x) = \hbar(\Delta_1(x) - \Delta_1^{op}(x)) + O(\hbar^2) = \hbar\delta(x) + O(\hbar^2) \qquad \forall x \in \mathfrak{g},$$

which relates the linear map $\Delta_1$ to its opposite $\Delta_1^{op}$ and describes the deviation of $\Delta$ from the cocommutative map $\Delta_0$ in lowest order in $\hbar$.

The idea is to classify and construct deformations of the universal enveloping algebra $U(\mathfrak{g})$ by requirements on $\delta$ that arise from the condition that the comultiplication $\Delta$ is a coassociative algebra homomorphism. In order $\hbar^0$, these conditions take the form $\Delta_0([x, y]_{\mathfrak{g}}) = [\Delta_0(x), \Delta_0(y)]$ and $(\Delta_0\otimes\mathrm{id}) \circ \Delta_0 = (\mathrm{id}\otimes\Delta_0) \circ \Delta_0$. These conditions are satisfied because $\Delta_0$ coincides with the comultiplication of $U(\mathfrak{g})$. To see that the coassociativity condition is also satisfied in order $\hbar$, we express the linear map $\Delta_1 : \mathfrak{g} \to \mathfrak{g}\otimes\mathfrak{g}$ in terms of a basis $B = \{x_1, ..., x_{\dim\mathfrak{g}}\}$ if $\mathfrak{g}$ as $\Delta_1(x_l) = \Sigma_{i,j=1}^{k} C_l^{ij} x_i\otimes x_j$ with constants $C_l^{ij} \in \mathbb{F}$. As the condition $\Delta(1) = \Delta_0(1) = 1\otimes 1$

implies $\Delta_n(1) = 0$ for all $n \in \mathbb{N}$, this yields for all $l \in \{1, ..., \dim \mathfrak{g}\}$

$$
\begin{aligned}
&(\Delta\otimes\mathrm{id}) \circ \Delta(x_l) - (\mathrm{id}\otimes\Delta) \circ \Delta(x_l) \\
&= \hbar\left((\Delta_1\otimes\mathrm{id}) \circ \Delta_0(x_l) + (\Delta_0\otimes\mathrm{id}) \circ \Delta_1(x_l) - (\mathrm{id}\otimes\Delta_1) \circ \Delta_0(x_l) - (\mathrm{id}\otimes\Delta_0) \circ \Delta_1(x_l)\right) + O(\hbar^2) \\
&= \hbar\Sigma_{i,j=1}^k C_l^{ij}(x_i\otimes x_j\otimes 1 - 1\otimes x_i\otimes x_j + 1\otimes x_i\otimes x_j + x_i\otimes 1\otimes x_j - x_i\otimes 1\otimes x_j - x_i\otimes x_j\otimes 1) + O(\hbar^2) \\
&= O(\hbar^2).
\end{aligned}
$$

Hence, the first condition on $\delta$ from the coassociativity on $\Delta$ arises in order $\hbar^2$. To obtain a condition that involves only $\delta$ and not $\Delta_1$ or $\Delta_1^{op}$, we take the sum over the cyclic permutations of factors in tensor products $\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}$ and combine the coassociativity conditions for $\Delta$ and $\Delta^{op}$. This yields

$$
\begin{aligned}
0 &= \Sigma_{\mathrm{cyc}}(\Delta\otimes\mathrm{id}) \circ \Delta - (\mathrm{id}\otimes\Delta) \circ \Delta + (\Delta^{op}\otimes\mathrm{id}) \circ \Delta^{op} - (\mathrm{id}\otimes\Delta^{op}) \circ \Delta^{op} \\
&= \Sigma_{\mathrm{cyc}}(\Delta\otimes\mathrm{id}) \circ \Delta - \tau_{231} \circ (\mathrm{id}\otimes\Delta^{op}) \circ \Delta^{op} - \tau_{231} \circ (\mathrm{id}\otimes\Delta) \circ \Delta + (\Delta^{op}\otimes\mathrm{id}) \circ \Delta^{op} \\
&= \Sigma_{\mathrm{cyc}}(\Delta\otimes\mathrm{id}) \circ \Delta - (\Delta^{op}\otimes\mathrm{id}) \circ \Delta - (\Delta\otimes\mathrm{id}) \circ \Delta^{op} + (\Delta^{op}\otimes\mathrm{id}) \circ \Delta^{op} \\
&= \Sigma_{\mathrm{cyc}}((\Delta - \Delta^{op})\otimes\mathrm{id}) \circ (\Delta - \Delta^{op}) = \hbar^2\Sigma_{\mathrm{cyc}}(\delta\otimes\mathrm{id}) \circ \delta + O(\hbar^3),
\end{aligned}
$$

where $\tau_{ijk} \in S_3$ is the permutation with $\tau_{ijk}(i) = 1$, $\tau_{ijk}(j) = 2$ and $\tau_{ijk}(k) = 3$. This equation is satisfied in order $\hbar^2$ if and only if $\delta$ satisfies the *coJacobi identity* $\Sigma_{\mathrm{cyc}}(\delta\otimes\mathrm{id}) \circ \delta(x) = 0$ for all $x \in \mathfrak{g}$. To determine the conditions on $\delta$ that arise from the requirement that $\Delta$ is an algebra homomorphism, we note that the relations imply $x \cdot y - y \cdot x = [x, y]_\mathfrak{g} + O(\hbar^2)$ for all $x, y \in \mathfrak{g}$. The condition that $\Delta$ is an algebra homomorphism and the definition of $\delta$ then imply

$$
\begin{aligned}
&\Delta([x, y]) - \Delta^{op}([x, y]) = [\Delta(x), \Delta(y)] - [\Delta^{op}(x), \Delta^{op}(y)] \\
&= \Delta(x)\Delta(y) - \Delta(y)\Delta(x) - \Delta^{op}(x)\Delta^{op}(y) + \Delta^{op}(y)\Delta^{op}(x) \\
&= [\Delta(x), \Delta(y) - \Delta^{op}(y)] - [\Delta(y), \Delta(x) - \Delta^{op}(x)] \\
&= \hbar(\mathrm{id}\otimes\mathrm{ad}_x + \mathrm{ad}_x\otimes\mathrm{id})\delta(y) - \hbar(\mathrm{id}\otimes\mathrm{ad}_y + \mathrm{ad}_y\otimes\mathrm{id})\delta(x) + O(\hbar^2) \\
&= \hbar\delta([x, y]) + O(\hbar^2) = \hbar\delta([x, y]_\mathfrak{g}) + O(\hbar^2),
\end{aligned}
$$

where $[\,,\,]$ stands for the commutator in $T(\mathfrak{g})[[\hbar]]$, the expression $[\,,\,]_\mathfrak{g}$ for the Lie bracket in $\mathfrak{g}$ and $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$, $y \mapsto [x, y]_\mathfrak{g}$ for the adjoint action of $x \in \mathfrak{g}$ on $\mathfrak{g}$. This equation is satisfied in order $\hbar$ if and only if $\delta$ satisfies $(\mathrm{id}\otimes\mathrm{ad}_x + \mathrm{ad}_x\otimes\mathrm{id})\delta(y) - \hbar(\mathrm{id}\otimes\mathrm{ad}_y + \mathrm{ad}_y\otimes\mathrm{id})\delta(x) = \delta([x, y]_\mathfrak{g})$. By combining this condition with the coJacobi identity, we obtain the following definition.

**Definition 2.5.7:** A **Lie bialgebra** over $\mathbb{F}$ is a Lie algebra $(\mathfrak{g}, [\,,\,])$ over $\mathbb{F}$ together with an antisymmetric $\mathbb{F}$-linear map $\delta : \mathfrak{g} \to \mathfrak{g}\otimes\mathfrak{g}$, the **cocommutator**, that satisfies for all $x, y \in \mathfrak{g}$

1. the **co-Jacobi identity**: $\Sigma_{\mathrm{cyc}}(\delta\otimes\mathrm{id}) \circ \delta(x) = 0$,
2. the **cocycle condition**: $\delta([x, y]) = (\mathrm{ad}_x\otimes\mathrm{id} + \mathrm{id}\otimes\mathrm{ad}_x)\delta(y) - (\mathrm{ad}_y\otimes\mathrm{id} + \mathrm{id}\otimes\mathrm{ad}_y)\delta(x)$,

where $\mathrm{ad}_x$ is the adjoint action $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$, $y \mapsto [x, y]$ for all $x \in \mathfrak{g}$.

We can view a Lie bialgebra as the infinitesimal concept associated with a bialgebra. The Lie bracket characterises the multiplication and relates it to the opposite multiplication in lowest order in $\hbar$, and the Jacobi identity is the infinitesimal version of the associativity condition for the multiplication. The cocommutator characterises the comultiplication and relates it to the opposite comultiplication in lowest order of $\hbar$, and the coJacobi identity is the infinitesimal version of coassociativity. The cocycle condition is a compatibility condition between Lie bracket and cocommutator that can be viewed as the infinitesimal version of the condition that the comultiplication is an algebra homomorphism. The duality between a bialgebra $H$ and its (finite) dual $H^*$ also has an infinitesimal counterpart that associates to each Lie bialgebra $(\mathfrak{g}, [\,,\,], \delta)$ a dual Lie bialgebra $(\mathfrak{g}^*, \delta^*, [\,,\,]^*)$.

**Lemma 2.5.8:** If $(\mathfrak{g}, [\,,\,], \delta)$ is a finite-dimensional Lie bialgebra over $\mathbb{F}$, then $(\mathfrak{g}^*, \delta^*, [\,,\,]^*)$ is a Lie bialgebra over $\mathbb{F}$. It is called the **dual Lie bialgebra** of $(\mathfrak{g}, [\,,\,], \delta)$.

**Proof:**
As $\mathfrak{g}$ is finite-dimensional, we have $(\mathfrak{g}\otimes\mathfrak{g})^* \cong \mathfrak{g}^*\otimes\mathfrak{g}^*$. The $\mathbb{F}$-linear map $\delta^*: \mathfrak{g}^*\otimes\mathfrak{g}^* \to \mathfrak{g}^*$ is antisymmetric, and the coJacobi identity for $\delta$ implies that $\delta^*$ satisfies the Jacobi identity

$$0 = \left(\Sigma_{\text{cyc}}\delta^* \circ (\delta^*\otimes\text{id})\right)(\alpha\otimes\beta\otimes\gamma) = \delta^*(\delta^*(\alpha\otimes\beta), \gamma) + \delta^*(\delta^*(\gamma\otimes\alpha), \beta) + \delta^*(\delta^*(\beta\otimes\gamma), \alpha)$$

for all $\alpha, \beta, \gamma \in \mathfrak{g}^*$. Hence $(\mathfrak{g}^*, \delta^*)$ is a Lie algebra. The dual of the Lie bracket on $\mathfrak{g}$ is an antisymmetric $\mathbb{F}$-linear map $[\,,\,]^*: \mathfrak{g}^* \to \mathfrak{g}^*\otimes\mathfrak{g}^*$ that satisfies the coJacobi identity since $[\,,\,]$ satisfies the Jacobi identity. Moreover, we have for all $x, y, z \in \mathfrak{g}$ and $\alpha, \beta, \gamma \in \mathfrak{g}^*$

$$\begin{aligned}
&\left([\,,\,]^* \circ \delta^*\right)(\alpha\otimes\beta)(x\otimes y) = \left((\delta \circ [\,,\,])^*(\alpha\otimes\beta)\right)(x\otimes y) = (\alpha\otimes\beta)(\delta([x, y]))\\
&= (\alpha\otimes\beta)((\text{ad}_x\otimes\text{id} + \text{id}\otimes\text{ad}_x)\delta(y) - (\text{ad}_y\otimes\text{id} + \text{id}\otimes\text{ad}_y)\delta(x))\\
&= \left((\delta^*(\alpha\otimes -)\otimes\text{id} + \text{id}\otimes\delta^*(\alpha\otimes -)) \circ [\,,\,]^*(\beta) - (\delta^*(\beta\otimes -)\otimes\text{id} + \text{id}\otimes\delta^*(\beta\otimes -)) \circ [\,,\,]^*(\alpha)\right)(x\otimes y),
\end{aligned}$$

where $\delta^*(\alpha\otimes -): \mathfrak{g}^* \to \mathfrak{g}^*$, $\beta \mapsto \delta^*(\alpha\otimes\beta)$ is the adjoint action of $\mathfrak{g}^*$ on itself. This shows that $[\,,\,]^*$ satisfies the cocycle condition. □

**Example 2.5.9:**

1. Any finite-dimensional Lie algebra $(\mathfrak{g}, [\,,\,])$ becomes a Lie bialgebra when equipped with the trivial cocommutator $\delta \equiv 0$. In this case, the dual Lie bialgebra has a trivial Lie bracket and the cocommutator is given by $[\,,\,]^*$.
2. Let $B = \{E, F, G\}$ the basis of the Lie algebra $\mathfrak{sl}_2$, in which the Lie bracket is given by

   $$[H, E] = 2E \qquad [H, F] = -2F \qquad [E, F] = H.$$

   Then a Lie bialgebra structure on $\mathfrak{sl}_2$ is given by

   $$\delta(E) = E\otimes H - H\otimes E \qquad \delta(F) = F\otimes H - H\otimes F \qquad \delta(H) = 0,$$

   and another Lie bialgebra structure is given by

   $$\delta(E) = 0, \qquad \delta(H) = H\otimes E - E\otimes H, \qquad \delta(F) = F\otimes E - E\otimes F.$$

3. If $\mathfrak{g}$ is a complex simple Lie algebra and $B = \{E_i, F_i, H_i \mid i = 1, ..., r\}$ is the Chevalley basis of $\mathfrak{g}$ from Remark 2.3.13, then a Lie bialgebra structure on $\mathfrak{g}$ is given by

   $$\delta(H_i) = 0, \qquad \delta(E_i) = E_i\otimes H_i - H_i\otimes E_i, \qquad \delta(F_i) = F_i\otimes H_i - H_i\otimes F_i.$$

   This is called the **standard Lie bialgebra structure** on $\mathfrak{g}$ and coincides with the first cocommutator for $\mathfrak{g} = \text{sl}_2$.

The advantage of working with Lie bialgebra structures is that the *nonlinear* relations and the *nonlinear* expressions for the coproduct in a bialgebra are replaced by *linear* structures from the context of Lie algebras. In particular, the dependence of the structures on the choice of generators becomes more transparent, since this corresponds to the choice of a basis of $\mathfrak{g}$. It is also much simpler to *classify* Lie bialgebra structures. As every Lie bialgebra consists of a Lie algebra and a dual Lie algebra that satisfy the cocycle condition, the classification of Lie bialgebras of a given dimension $d$ can be achieved by classifying all $d$-dimensional Lie algebras and investigating which pairs of $d$-dimensional Lie algebras satisfy the cocycle condition.

Due to these simplifications, it is natural to attempt to construct $q$-deformed universal enveloping algebras from Lie bialgebra structures and to require that the latter determine the $q$-deformed universal enveloping algebra in lowest order in $\hbar$. From a physics viewpoint, this can be interpreted as a quantisation, which motivates the following definition.

**Definition 2.5.10:** Let $(\mathfrak{g}, [\,,\,]_\mathfrak{g}, \delta)$ be a Lie bialgebra over $\mathbb{F}$. A **quantisation** of $\mathfrak{g}$ is a Hopf algebra over $\mathbb{F}[[\hbar]]$ that is generated by a basis of $\mathfrak{g}$ with relations of the form

$$x \cdot y - y \cdot x = [x, y]_\mathfrak{g} + O(\hbar) \quad \forall x, y \in \mathfrak{g},$$

and whose comultiplication satisfies the conditions

$$\Delta(x) = 1 \otimes x + x \otimes 1 + O(\hbar) \qquad \Delta(x) - \Delta^{op}(x) = \hbar\delta(x) + O(\hbar^2) \qquad \forall x \in \mathfrak{g}.$$

It has been shown by Drinfeld that every finite-dimensional Lie bialgebra $(\mathfrak{g}, [\,,\,], \delta)$ has a quantisation that is unique up to bialgebra isomorphisms. In principle, it is possible to construct the multiplication relations and the coproduct of this quantisation order for order in $\hbar$ by imposing coassociativity and the condition that $\Delta$ is an algebra homomorphism. As the comultiplication is given by $\Delta(x) = \Sigma_{n=0}^{\infty} \hbar^n \Delta_n(x)$ with $\Delta_n(x) \in \mathfrak{g}^{\otimes n}$ for all $x \in \mathfrak{g}$, one obtains

$$(\Delta \otimes \text{id}) \circ \Delta = \sum_{n=0}^{\infty} \hbar^n \sum_{k=0}^{n} (\Delta_k \otimes \text{id}) \circ \Delta_{n-k} \qquad (\text{id} \otimes \Delta) \circ \Delta = \sum_{n=0}^{\infty} \hbar^n \sum_{k=0}^{n} (\text{id} \otimes \Delta_k) \circ \Delta_{n-k}$$

and the coassociativity condition leads to a recursion relation

$$\sum_{k=0}^{n} ((\Delta_k \otimes \text{id}) \circ \Delta_{n-k} - (\text{id} \otimes \Delta_k) \circ \Delta_{n-k})(x) = 0 \qquad \forall x \in \mathfrak{g}.$$

Similarly, as the relations are given by $[x, y] = \Sigma_{n=0}^{\infty} \hbar^n r_n(x \otimes y)$, one obtains for all $x, y \in \mathfrak{g}$

$$\Delta([x, y]) = \sum_{n=0}^{\infty} \hbar^n \sum_{k=0}^{n} \Delta_k \circ r_k(x \otimes y) \qquad [\Delta(x), \Delta(y)] = \Sigma_{n=0}^{\infty} \hbar^n \sum_{k=0}^{n} [\Delta_k(x), \Delta_{n-k}(y)].$$

The condition that $\Delta$ is an algebra homomorphism leads to the recursion relations

$$\sum_{k=0}^{n} \Delta_k \circ r_{n-k}(x \otimes y) - [\Delta_k(x), \Delta_{n-k}(y)] = 0 \qquad \forall x, y \in \mathfrak{g}$$

which characterise the relation $r_n$ in terms of the coproduct and the relations $r_k$ with $k < n$.

Although it is possible in principle, to construct the quantisation of the Lie bialgebra in this way, this is too cumbersome and complicated in practice. Instead of determining the coalgebra structure and the relations order for order in $\hbar$, one uses the following shortcuts or rules of thumb:

- If $\delta(x) = 0$ for some $x \in \mathfrak{g}$, one sets $\Delta(x) = 1\otimes x + x\otimes 1$.
- If $\delta(x) = 0$ and $\delta(y) = y\otimes x - x\otimes y$ for $x, y \in \mathfrak{g}$ then one sets $\Delta(y) = f_1(x)\otimes y + y\otimes f_1(x)$ with formal power series $f_i(x) = \Sigma_{n=0}^{\infty}\hbar^n a_n^i x^n$, where $a_n^i \in \mathbb{F}$.
- If $\delta(x) = 0$ and $[x, y]_{\mathfrak{g}} = \alpha y$ for $x, y \in \mathfrak{g}$ and $\alpha \in \mathbb{F}$, one sets $[x, y] = \alpha y$.

**Example 2.5.11:** We construct a quantisation for the Lie algebra $\mathfrak{sl}_2$ over $\mathbb{F}$ with the standard Lie bialgebra structure. In this case, there is a basis $B = \{E, F, H\}$ of $\mathfrak{g}$ in which the Lie bracket and the cocommutator are given by

$$[H, E]_{\mathfrak{sl}_2} = 2E \qquad [H, F]_{\mathfrak{sl}_2} = -2F \qquad [E, F]_{\mathfrak{sl}_2} = H$$
$$\delta(E) = E\otimes H - H\otimes E \qquad \delta(F) = F\otimes H - H\otimes F \qquad \delta(H) = 0.$$

**Step 1:**
Due to the form of the cocommutator, it is reasonable to assume that $H$ is primitive and to set

$$\Delta(H) = H\otimes 1 + 1\otimes H \qquad \Delta(E) = E\otimes e_1(H) + e_2(H)\otimes E \qquad \Delta(F) = F\otimes f_1(H) + f_2(H)\otimes F$$

with formal power series $e_1, e_2, f_1, f_2$ satisfying $e_i(H) = 1+\hbar H+O(\hbar^2)$, $f_i(H) = 1+\hbar H+O(\hbar^2)$.

**Step 2:**
We determine the conditions on $e_i$ and $f_i$ that arise from the coassociativity condition on $\Delta$. For this, we compute

$$\begin{aligned}
(\Delta\otimes\mathrm{id}) \circ \Delta(E) &= \Delta(E)\otimes e_1(H) - \Delta(e_2(H))\otimes E \\
&= E\otimes e_1(H)\otimes e_1(H) - e_2(H)\otimes E\otimes e_1(H) - \Delta(e_2(H))\otimes E \\
(\mathrm{id}\otimes\Delta) \circ \Delta(E) &= E\otimes\Delta(e_1(H)) - e_2(H)\otimes\Delta(E) \\
&= E\otimes\Delta(e_1(H)) - e_2(H)\otimes E\otimes e_1(H) - e_2(H)\otimes e_2(H)\otimes E,
\end{aligned}$$

and a similar computation for $F$ shows that the coassociativity of $\Delta$ implies that $e_i(H)$ and $f_i(H)$ must be grouplike. As $H$ is primitive, its coproduct is given by

$$\Delta(H^n) = \sum_{k=0}^{n} \binom{n}{k} H^k\otimes H^{n-k}.$$

It follows that an element $X = \Sigma_{n=0}^{\infty}\hbar^n a_n H^n$ with $a_n \in \mathbb{F}$ is grouplike if and only if

$$\Delta(X) = \sum_{n=0}^{\infty} \hbar^n \sum_{k=0}^{n} a_n \binom{n}{k} H^k\otimes H^{n-k} = \sum_{n=0}^{\infty} \hbar^n \sum_{k=0}^{n} a_k a_{n-k} H^k\otimes H^{n-k} = X\otimes X,$$

which is the case if and only if its coefficients satisfy

$$a_n \binom{n}{k} = a_k a_{n-k} \; \forall k, n \in \mathbb{N}_0 \qquad \Leftrightarrow \qquad a_n = \frac{\lambda^n}{n!} \text{ for some } \lambda \in \mathbb{F}.$$

This shows that the only grouplike elements that are homogeneous power series in $H$ are of the form $e^{\alpha\hbar H}$ with $\alpha \in \mathbb{F}$. Hence we set $e_i(H) = e^{\lambda_i\hbar H}$, $f_i(H) = e^{\mu_i\hbar H}$ with $\mu_i, \lambda_i \in \mathbb{F} \setminus \{0\}$ and

$$\Delta(E) = E\otimes e^{\hbar\lambda_1 H} + e^{\hbar\lambda_2 H}\otimes E \qquad \Delta(F) = F\otimes e^{\hbar\mu_1 H} + e^{\hbar\mu_2 H}\otimes F.$$

As $H$ is primitive, the coassociaitvity condition is satisfied for $H$, and we obtain a coalgebra structure with the counit given by $\epsilon(E)=\epsilon(F)=\epsilon(H)=0$. The condition $\Delta^{op}-\Delta=\hbar\delta+O(\hbar^2)$ leads to restrictions on the parameters $\lambda_i$ and $\mu_i$ since we have

$$\begin{aligned}&\Delta(E)-\Delta^{op}(E)=E\otimes e^{\hbar\lambda_1 H}+e^{\hbar\lambda_2 H}\otimes E-E\otimes e^{\hbar\lambda_2 H}-e^{\hbar\lambda_1 H}\otimes E\\&=\hbar(\lambda_1-\lambda_2)E\otimes H-\hbar(\lambda_1-\lambda_2)H\otimes E+O(\hbar^2)\\&=\hbar\delta(E)+O(\hbar^2)=\hbar(E\otimes H-H\otimes E)+O(\hbar^2)\end{aligned}$$

$$\begin{aligned}&\Delta(F)-\Delta^{op}(F)=F\otimes e^{\hbar\mu_1 H}+e^{\hbar\mu_2 H}\otimes F-F\otimes e^{\hbar\mu_2 H}-e^{\hbar\mu_1 H}\otimes F\\&=\hbar(\mu_1-\mu_2)F\otimes H-\hbar(\mu_1-\mu_2)H\otimes F+O(\hbar^2)\\&=\hbar\delta(F)+O(\hbar^2)=\hbar(F\otimes H-H\otimes F)+O(\hbar^2),\end{aligned}$$

which implies $\lambda_1-\lambda_2=1$ and $\mu_1-\mu_2=1$. By a rescaling $E\to e^{-\hbar\lambda_2 H}E$ and $F\to Fe^{-\hbar\mu_1 H}$, we can then achieve that $\lambda_2=\mu_1=0$, $\lambda_1=-\mu_2=1$, and the comultiplication takes the form

$$\Delta(H)=1\otimes H+H\otimes 1\qquad \Delta(E)=1\otimes E+E\otimes e^{\hbar H}\qquad \Delta(F)=F\otimes 1+e^{-\hbar H}\otimes F. \tag{16}$$

From the formulas above, we find that this comultiplication map is coassociative and counital in all orders of $\hbar$ with counit $\epsilon(H)=\epsilon(F)=\epsilon(E)=0$.

**Step 3:**
It remains to determine the algebra structure, i. e. the relations, in all orders of $\hbar$. These are obtained by the requirement that $\Delta$ is an algebra homomorphism. A direct computation yields

$$\begin{aligned}&\Delta([H,E])=[\Delta(H),\Delta(E)]\\&=(1\otimes H+H\otimes 1)\cdot(E\otimes e^{\hbar H}+1\otimes E)-(E\otimes e^{\hbar H}+1\otimes E)\cdot(1\otimes H+H\otimes 1)\\&=[H,E]\otimes e^{\hbar H}+1\otimes[H,E]\end{aligned}$$

$$\begin{aligned}&\Delta([H,F])=[\Delta(H),\Delta(F)]\\&=(1\otimes H+H\otimes 1)\cdot(F\otimes 1+e^{-\hbar H}\otimes F)-(F\otimes 1+e^{-\hbar H}\otimes F)\cdot(1\otimes H+H\otimes 1)\\&=[H,F]\otimes 1+e^{-\hbar H}\otimes[H,F]\end{aligned}$$

$$\begin{aligned}&\Delta([E,F])=[\Delta(E),\Delta(F)]\\&=[E,F]\otimes e^{\hbar H}+e^{-\hbar H}\otimes[E,F]+Ee^{-\hbar H}\otimes e^{\hbar H}F-e^{-\hbar H}E\otimes Fe^{\hbar H}.\end{aligned}$$

Together with the condition that in lowest order, the commutator must agree with the Lie bracket, the first two equations suggests to set $[H,E]=2E$ and $[H,F]=-2F$. From the formulas for $\Delta(E)$ and $\Delta(F)$ it is then apparent that the ideal generated by these relations is a coideal. To evaluate the last expression further, we prove by induction the identity

$$[H^k,E]=\sum_{s=1}^{k}\binom{k}{s}2^s EH^{k-s}.$$

If it holds for $k$, then one obtains for $k+1$

$$\begin{aligned}[H^{k+1},E]&=H[H^k,E]+[H,E]H^k=[H^k,E]H+[H,[H^k,E]]+EH^k=[H^k,E](H+2)+EH^k\\&=EH^k+\sum_{s=1}^{k}\binom{k}{s}2^sE(2H^{k-s}+H^{k+1-s})=E+\sum_{s=1}^{k}\left(\binom{k}{s-1}+\binom{k}{s}\right)2^sEH^{k+1-s}\\&=\sum_{s=1}^{k+1}\binom{k+1}{s}2^sEH^{k+1-s}\end{aligned}$$

where we used the addition formula for the binomial coefficients. This implies

$$[e^{\hbar H},E]=\sum_{k=1}^{\infty}\frac{\hbar^k}{k!}[H^k,E]=\sum_{k=1}^{\infty}\sum_{s=1}^{k}\frac{\hbar^k}{k!}\binom{k}{s}2^sEH^{k-s}=\sum_{k=1}^{\infty}\sum_{s=1}^{k}\frac{\hbar^k}{s!(k-s)!}2^sEH^{k-s}$$
$$=\sum_{l=0}^{\infty}\sum_{s=1}^{\infty}\frac{2^s\hbar^s}{s!}\frac{\hbar^l}{l!}EH^l=(e^{2\hbar}-1)Ee^{\hbar H}.$$

An analogous computation yields $[e^{\hbar\lambda H},F]=(e^{-2\hbar}-1)Fe^{\hbar H}$, and these commutation relations are equivalent to the relations

$$e^{\pm\hbar H}E=e^{\pm 2\hbar}Ee^{\pm 2\hbar H}\qquad e^{\pm\hbar H}F=e^{\mp 2\hbar}Fe^{\pm 2\hbar H}. \tag{17}$$

Inserting this into the expression for $\Delta([E,F])$, we obtain

$$\Delta([E,F])=[E,F]\otimes e^{\hbar H}+e^{-\hbar H}\otimes[E,F].$$

As we must have $[E,F]=H+O(\hbar)$ it is reasonable to assume that $[E,F]$ is of the form

$$[E,F]=\sum_{n=0}^{\infty}a_n\hbar^{n-1}H^n$$

with $a_n\in\mathbb{F}$, $a_1=1$ and $a_0=0$. This yields

$$\Delta([E,F])=\sum_{n=0}^{\infty}\hbar^{n-1}\sum_{k=0}^{n}\frac{a_n n!}{k!(n-k)!}H^k\otimes H^{n-k}$$
$$[E,F]\otimes e^{\hbar H}=\sum_{n=0}^{\infty}\hbar^{n-1}\sum_{k=0}^{n}\frac{a_k}{(n-k)!}H^k\otimes H^{n-k}$$
$$e^{-\hbar H}\otimes[E,F]=\sum_{n=0}^{\infty}\hbar^{n-1}\sum_{k=0}^{n}\frac{(-1)^k a_{n-k}}{k!}H^k\otimes H^{n-k}$$
$$\Delta([E,F])-[E,F]\otimes e^{\hbar H}-e^{-\hbar H}\otimes[E,F]=\sum_{n=0}^{\infty}\hbar^{n-1}\sum_{k=0}^{n}\left(\frac{a_n n!-a_k k!-(-1)^k a_{n-k}(n-k)!}{(n-k)!k!}\right)H^k\otimes H^{n-k}.$$

The last expression vanishes if and only if the coefficients $a_n$ satisfy the recursion relation

$$a_n n!=a_k k!+(-1)^k a_{n-k}(n-k)!\ \ \forall n\in\mathbb{N}_0, k\in\{0,1,...,n\}.$$

With the conditions on $a_0$ and $a_1$, this is equivalent to $a_{2n}=0$ and $a_{2n+1}=1/(2n+1)!$ for all $n\in\mathbb{N}_0$. Hence, we obtain the multiplication and comultiplication relations

$$\begin{aligned}&[E,F]=\frac{e^{\hbar H}-e^{-\hbar H}}{\hbar} && [H,E]=2E, && [H,F]=2F,\\ &\Delta(H)=1\otimes H+H\otimes 1, && \Delta(E)=1\otimes E+E\otimes e^{\hbar H}, && \Delta(F)=F\otimes 1+e^{-\hbar H}F.\end{aligned} \tag{18}$$

A direct computation shows that this defines a Hopf algebra structure with counit and antipode

$$\epsilon(H)=\epsilon(E)=\epsilon(F)=0,\qquad S(H)=-H,\quad S(E)=-Ee^{-\hbar H},\quad S(F)=-e^{\hbar H}F.$$

**Step 4:**
By rescaling $E$ or $F$ with a formal power series in $\hbar$, we can achieve that the commutator of $E$ and $F$ is given by

$$[E,F]=\frac{e^{\hbar H}-e^{-\hbar H}}{e^{\hbar}-e^{-\hbar}},$$

while all other relations and the expressions for the comultiplication, the counit and antipode remain unchanged. If we replace the last two multiplication relations by (17) and the expression for the coproduct, counit and antipode of $H$ by the expression for the coproduct, counit and antipode of $e^{\pm\hbar H}$, we then obtain the Hopf algebra structure

$$
\begin{aligned}
&[E,F]=\frac{e^{\hbar H}-e^{-\hbar H}}{e^{\hbar}-e^{-\hbar}} && e^{\hbar H}Ee^{-\hbar H}=e^{2\hbar}E, && e^{\hbar H}Fe^{-\hbar}=e^{-2\hbar H}F,\\
&\Delta(e^{\pm\hbar H})=e^{\pm\hbar H}\otimes e^{\pm\hbar H}, && \Delta(E)=1\otimes E+E\otimes e^{\hbar H}, && \Delta(F)=F\otimes 1+e^{-\hbar H}F\\
&\epsilon(e^{\pm\hbar H})=1, && \epsilon(E)=0, && \epsilon(F)=0,\\
&S(e^{\pm\hbar H})=e^{\mp\hbar H}, && S(E)=-Ee^{-\hbar H}, && S(F)=-e^{\hbar H}F.
\end{aligned}
$$

If we set $q=e^{\hbar}$ and $K^{\pm 1}=e^{\pm\hbar H}$, this coincides with the Hopf algebra structure from Example 2.3.9. This shows that the Hopf algebra $U_q(\mathfrak{sl}_2)$ can be realised as a Hopf-subalgebra of the Hopf algebra over $\mathbb{F}[[\hbar]]$ with generators $E, F, H$ and with relations and coalgebra structure (18).

**Exercise 1:** Let $\mathbb{F}$ be a field.

(a) Show that a formal power series $x=\Sigma_{n=0}^{\infty}\hbar^n x_n\in\mathbb{F}[[x]]$ has a multiplicative inverse in $\mathbb{F}[[\hbar]]$ if and only if $x_0\neq 0$.

(b) Show that the quotient field of the ring $\mathbb{F}[[\hbar]]$ can be identified with the set of power series $\Sigma_{n=k}^{\infty}\hbar^n x_n$ for $k\in\mathbb{Z}$ and with the product given by

$$\left(\sum_{n=k}^{\infty}\hbar^n x_n\right)\cdot\left(\sum_{n=l}^{\infty}\hbar^n y_n\right)=\sum_{n=k+l}^{\infty}\hbar^n\left(\sum_{j=k}^{n=l}x_j y_{n-j}\right).$$

**Exercise 2:** Consider the Lie algebra $\mathfrak{b}_+$ with basis $\{x,y\}$ and Lie bracket $[x,y]=y$.

(a) Construct its $q$-deformed universal enveloping algebra $U_q(\mathfrak{b}_+)$.

(b) Construct the associated $q$-deformed universal enveloping algebra $U^r(\mathfrak{b}_+)$ at a primitive $r$th root of unity $q$.

(c) Compare the result with Taft's example.

# 3 Modules over Hopf algebras

## 3.1 (Co)module (co)algebras

In this section we consider (co)module (co)algebras over bialgebras and Hopf algebras and their (co)invariants. These structures can be viewed as a generalisation of the algebra of functions on a space with a group action and are motivated among others by their applications in *non-commutative geometry*. The basic idea is to describe a space $X$ with an action of a symmetry group $G$ in terms of the algebra $\mathrm{Fun}(X, \mathbb{F})$ of functions $f : X \to \mathbb{F}$ into a field $\mathbb{F}$ with the pointwise addition, multiplication and scalar multiplication. This algebra of functions includes coordinate functions on $X$, but also functions that are invariant under the group action and describe its *orbits*.

More precisely, if $G$ is a group, $X$ a set and $\rhd : G \times X \to X$ a **group action** of $G$ on $X$, i. e. a map that satisfies $(gh) \rhd x = g \rhd (h \rhd x)$ and $1 \rhd x = x$ for all $g, h \in G$ and $x \in X$, then the **orbit** of an element $x \in X$ is the set $G \rhd x = \{g \rhd x \,|\, g \in G\}$, and the **orbit space** the set $O = \{G \rhd x \,| x \in X\}$ of orbits. The algebra $\mathrm{Fun}(X, \mathbb{F})$ becomes a right module over the group algebra $\mathbb{F}[G]$ with the induced right action

$$\lhd : \mathrm{Fun}(X, \mathbb{F}) \otimes \mathbb{F}[G] \to \mathrm{Fun}(X, \mathbb{F}), \quad f \mapsto f \lhd g \quad \text{where} \quad (f \lhd g)(x) = f(g \rhd x)\ \forall x \in X.$$

Functions $\tilde{f} : O \to \mathbb{F}$ are in bijection with functions $f : X \to \mathbb{F}$ that are invariant under the action of $G$, i. e. satisfy $f \lhd g = f$ for all $g \in G$. Every function $\tilde{f} : O \to \mathbb{F}$ gives rise to an invariant function $f : X \to \mathbb{F}$ defined by $f(x) = \tilde{f}(G \rhd x)$ and vice versa

$$\mathrm{Fun}(O, \mathbb{F}) \cong \mathrm{Fun}(X, \mathbb{F})^G = \{f : X \to \mathbb{F} \,|\, f(g \rhd x) = f(x)\, \forall g \in G, x \in X\}.$$

The set $\mathrm{Fun}(X, \mathbb{F})^G$ is a subalgebra of $\mathrm{Fun}(X, \mathbb{F})$ since it is a linear subspace, the constant function $1_X : X \to \mathbb{F}$, $x \mapsto 1$ is contained in $\mathrm{Fun}(X, \mathbb{F})$ and for all $f_1, f_2 \in \mathrm{Fun}(X, \mathbb{F})^G$ one has $(f_1 \cdot f_2)(g \rhd x) = f_1(g \rhd x) \cdot f_2(g \rhd x) = f_1(x) \cdot f_2(x) = (f_1 \cdot f_2)(x)$ for all $x \in X$ and hence $f_1 \cdot f_2 \in \mathrm{Fun}(X, \mathbb{F})^G$. We can view this subalgebra as the algebra of functions on the orbit space.

For instance, if we consider the space $X = \mathbb{R}^n$, the group $G = \mathrm{O}(n, \mathbb{R})$ and the group action $\rhd : \mathrm{O}(n, \mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}^n$, $(M, x) \mapsto M \cdot x$, then the orbits are the set $S_0^{n-1} = \{0\}$ and spheres $S_r^{n-1} = \{x \in \mathbb{R}^n \,|\, \Sigma_{i=1}^n x_i^2 = r^2\}$ for $r > 0$. The invariant functions are functions whose value at a point $x \in \mathbb{R}^n$ depends only on the euclidean distance of $x$ to the origin. If we choose $X = G$ for some group $G$ and the conjugation action $\rhd : G \times G \to G$, $(g, h) \mapsto ghg^{-1}$, then the orbits are conjugacy classes in $G$ and the invariant functions are **class functions**, functions $f : G \to \mathbb{F}$ with $f(ghg^{-1}) = f(h)$ for all $g, h \in G$.

In *non-commutative geometry* these structures are generalised by replacing the group $G$ by a Hopf algebra $H$ and the algebra of functions $\mathrm{Fun}(X, \mathbb{F})$ by a (usually non-commutative) algebra $A$ that is a module over the Hopf algebra $H$. One also obtains a generalised notion of invariant elements, and the requirement that these invariant elements form a subalgebra of $A$ leads to a compatibility condition between the algebra structure and the $H$-module structure of $A$. This is encoded in notion of a *module algebra* over a Hopf algebra. As both, the Hopf algebra $H$ and the algebra $A$ can be dualised, one also obtains three dual concepts, namely a *module coalgebra*, a *comodule algebra* and a *comodule coalgebra*. These concepts can also be defined in more generality for bialgebras.

**Definition 3.1.1:** Let $B$ be a bialgebra over $\mathbb{F}$. A **(left) module algebra** over $B$ is an algebra $(A, m_A, \eta_A)$ together with a $B$-left module structure $\rhd : B\otimes A \to A$ such that the following diagrams commute

$$
\begin{array}{ccccc}
B\otimes A\otimes A & \xrightarrow{\mathrm{id}\otimes m_A} & B\otimes A & \xrightarrow{\rhd} & A \\
\downarrow{\scriptstyle \Delta\otimes\mathrm{id}\otimes\mathrm{id}} & & & & \uparrow{\scriptstyle m_A} \\
B\otimes B\otimes A\otimes A & \xrightarrow[\mathrm{id}\otimes\tau\otimes\mathrm{id}]{} & B\otimes A\otimes B\otimes A & \xrightarrow[\rhd\otimes\rhd]{} & A\otimes A
\end{array}
\qquad
\begin{array}{ccccc}
B & \xrightarrow[\cong]{h\mapsto h\otimes 1} & B\otimes\mathbb{F} & \xrightarrow{\mathrm{id}\otimes\eta_A} & B\otimes A \\
\downarrow{\scriptstyle \epsilon} & & & \swarrow{\scriptstyle \rhd} & \\
\mathbb{F} & \xrightarrow[\eta_A]{} & A. & &
\end{array}
\qquad (19)
$$

In formulas: for all $b \in B$ and $a, a' \in A$, one has

$$b \rhd (a \cdot a') = \Sigma_{(b)}(b_{(1)} \rhd a) \cdot (b_{(2)} \rhd a') \qquad b \rhd 1_A = \epsilon(b)\, 1_A.$$

A **right module algebra** over $B$ is a left module algebra over $B^{op}$, and a **bimodule algebra** over $B$ is a left module algebra over $B\otimes B^{op}$.

**Example 3.1.2:**

1. If $G$ is a group and $\rhd : G \times X \to X$ a group action of $G$ on a set $X$, then the algebra of functions $\mathrm{Fun}(X, \mathbb{F})$ is a right module algebra over the group algebra $\mathbb{F}[G]$ with $\lhd : \mathrm{Fun}(X,\mathbb{F})\otimes\mathbb{F}[G] \to \mathrm{Fun}(X,\mathbb{F})$, $f\otimes g \mapsto f \lhd g$ with $(f \lhd g)(x) = f(g \rhd x)$.

2. Let $B$ be a bialgebra over $\mathbb{F}$. Every algebra $A$ over $\mathbb{F}$ becomes a $B$-module algebra when equipped with the trivial $B$-module structure $\rhd : B\otimes A \to A$, $b \rhd a = \epsilon(b)\, a$.

3. If $A$ is a module algebra over a Hopf algebra $H$ with $\rhd : H\otimes A \to A$, then $A$ is a module algebra over $H^{op,cop}$ with $\rhd' : H\otimes A \to A$, $h \rhd' a = S(h) \rhd a$.

4. Every Hopf algebra $H$ is a module algebra over itself with the **adjoint action**

$$\rhd_{ad} : H\otimes H \to H,\ h\otimes k \mapsto h \rhd_{ad} k \quad \text{with} \quad h \rhd_{ad} k = \Sigma_{(h)}\, h_{(1)} \cdot k \cdot S(h_{(2)}).$$

   If $H = \mathbb{F}[G]$ is a group algebra, this corresponds to the conjugation action of $G$ on itself $g \rhd_{ad} h = ghg^{-1}$ for all $g, h \in G$. (Exercise)

5. For every bialgebra $(B, m, \eta, \Delta, \epsilon)$, the dual algebra $(B^*, \Delta^*, \epsilon^*)$ is a module algebra over $B$ with the **right dual action**

$$\rhd^*_R : B\otimes B^* \to B^*,\ b\otimes\alpha \mapsto b \rhd^*_R \alpha \quad \text{with} \quad b \rhd^*_R \alpha = \Sigma_{(\alpha)}\alpha_{(2)}(b)\otimes\alpha_{(1)}.$$

   and a module algebra over $B^{op}$ with the **left dual action**

$$\rhd^*_L : B^{op}\otimes B^* \to B^*,\ b\otimes\alpha \mapsto b \rhd^*_L \alpha \quad \text{with} \quad b \rhd^*_L \alpha = \Sigma_{(\alpha)}\alpha_{(1)}(b)\, \alpha_{(2)}$$

   This gives $B^*$ the structure of a bimodule algebra over $B$. (Exercise).

6. If $G$ is a group and $\mathbb{F}[G]$ its group algebra, then the dual algebra is $\mathbb{F}[G]^* = \mathrm{Fun}(G, \mathbb{F})$ with the pointwise addition, multiplication and scalar multiplication. In this case the right and left dual action correspond to the group actions

$$\begin{aligned} &\rhd^*_R : \mathbb{F}[G] \times \mathrm{Fun}(G,\mathbb{F}) \to \mathrm{Fun}(G,\mathbb{F}), \quad (g \rhd^*_R f)(h) = f(h \cdot g) \\ &\rhd^*_L : \mathbb{F}[G]^{op} \times \mathrm{Fun}(G,\mathbb{F}) \to \mathrm{Fun}(G,\mathbb{F}), \quad (g \rhd^*_L f)(h) = f(g \cdot h). \end{aligned}$$

A more interesting example that illustrates the interpretation of comodule algebras as deformed or quantised algebras of functions is the *quantum plane*.

**Example 3.1.3:** Let $\mathbb{F}$ be a field and $q \in \mathbb{F}\setminus\{0,1,-1\}$. The **quantum plane** $\mathbb{F}_q[x,y]$ over $\mathbb{F}$ is the algebra over $\mathbb{F}$ with generators $x, y$ and the relation $xy = qyx$. The quantum plane is a module algebra over $U_q(\mathfrak{sl}_2)$ with $\rhd : U_q(\mathfrak{sl}_2)\otimes\mathbb{F}_q[x,y] \to \mathbb{F}_q[x,y]$, $U\otimes w \mapsto U \rhd w$ given by

$$\begin{aligned} &E \rhd (y^m x^n) = q^{1-n}(n)_{q^2} y^{m+1}x^{n-1} && F \rhd (y^m x^n) = q^{1-m}(m)_{q^2} y^{m-1}x^{n+1} \\ &K \rhd (y^m x^n) = q^{m-n} y^m x^n && K^{-1} \rhd (y^m x^n) = q^{n-m} y^m x^n. \end{aligned} \tag{20}$$

**Proof:**
Let $V$ be the free vector space with basis $E, F, K, K^{-1}$. Then (20) defines a linear map $\rho' : V \to \mathrm{End}_{\mathbb{F}}(\mathbb{F}_q[x,y])$, $X \mapsto \rho'(X)$ with $\rho(X)w = X \rhd w$. By the universal property of the tensor algebra $T(V)$, this induces an algebra homomorphism $\rho : T(V) \to \mathrm{End}_{\mathbb{F}}(\mathbb{F}_q[x,y])$ with $\rho\circ\iota_V = \rho'$. To obtain an algebra homomorphism $\rho : U_q(\mathfrak{sl}_2) \to \mathrm{End}_{\mathbb{F}}(\mathbb{F}_q[x,y])$, we need to show that $r \in \ker(\rho)$ for all relations $r$ in (9). As the set $\{y^m x^n \,|\, m,n \in \mathbb{N}_0\}$ is a basis of $\mathbb{F}_q[x,y]$, it is sufficient to show that $\rho(r)(y^m x^n) = 0$ for all relations $r$ in (9) and $m,n \in \mathbb{N}_0$. This follows by a direct computation from (20) and the relations in (9) and shows that $\rho : T(V) \to \mathrm{End}_{\mathbb{F}}(\mathbb{F}_q[x,y])$ induces an algebra homomorphism $\rho : U_q(\mathfrak{sl}_2) \to \mathrm{End}_{\mathbb{F}}(\mathbb{F}_q[x,y])$. Hence, $\mathbb{F}_q[x,y]$ is a module over $U_q(\mathfrak{sl}_2)$. To show that $\mathbb{F}_q[x,y]$ is a module *algebra* over $U_q(\mathfrak{sl}_2)$, we have to show that $m\circ(\Delta(X) \rhd (y^m x^n \otimes y^r x^s) = X \rhd m(y^m x^n \otimes y^r x^s)$ for all $m,n,r,s \in \mathbb{N}_0$ and $X \in \{E, F, K^{\pm1}\}$. This follows again by a direct but lengthy computation from the relation $xy = qyx$ and the expressions for the coproduct in (10). □

The geometrical meaning and the name *quantum plane* becomes more transparent if one considers the case $q = 1$. For this, we consider instead of $U_q(\mathfrak{sl}_2)$ the $q$-deformed universal enveloping algebra $U'_q(\mathfrak{sl}_2)$ with generators $e, f, l$ from Example 2.3.11 and set

$$\begin{aligned} &e \rhd (y^m x^n) = q^{1-n}(n)_{q^2} y^{m+1}x^{n-1} \qquad f \rhd (y^m x^n) = q^{1-m}(m)_{q^2} y^{m-1}x^{n+1} \\ &l \rhd (y^m x^n) = [e,f] \rhd (y^m x^n) = (q^{n-m}(m)_{q^2} - q^{m-n}(n)_{q^2})\, y^m x^n, \end{aligned} \tag{21}$$

For $q = 1$, we obtain the commutative quantum plane $\mathbb{F}_1[x,y] = \mathbb{F}[x,y]$ with a left module structure over the universal enveloping algebra $U(\mathfrak{sl}_2)$ given by

$$e \rhd (y^m x^n) = n y^{m+1}x^{n-1} \qquad f \rhd (y^m x^n) = m y^{m-1}x^{n+1} \qquad l \rhd (y^m x^n) = (m-n)\, y^m x^n. \tag{22}$$

This can be interpreted as a representation $\rho : \mathfrak{sl}_2 \to \mathrm{End}_{\mathbb{F}}\mathbb{F}[x,y]$ of the Lie algebra $\mathfrak{sl}_2$ on the polynomial algebra $\mathbb{F}[x,y]$ by formal differential operators

$$\rho(e) = y\frac{\partial}{\partial x} \qquad \rho(f) = x\frac{\partial}{\partial y} \qquad \rho(l) = y\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial y}\right) - x\frac{\partial}{\partial y}\left(y\frac{\partial}{\partial x}\right) = y\frac{\partial}{\partial y} - x\frac{\partial}{\partial x}.$$

On the other hand, if we interpret the elements $x$ and $y$ as linear maps $x : \mathbb{F}^2 \to \mathbb{F}$, $(x', y') \to x'$ and $y : \mathbb{F}^2 \to \mathbb{F}$, $(x', y') \mapsto y'$, we can consider the representation $\rho : \mathfrak{sl}_2 \to \mathrm{End}_{\mathbb{F}}(\mathbb{F}[x,y])$ that is induced by the representation of the Lie algebra $\mathfrak{sl}_2^{op}$ with the opposite commutator in terms of the matrices $e, f, l$ from (14). As we have

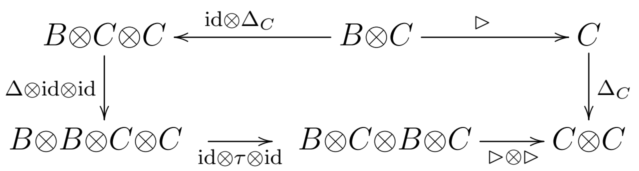
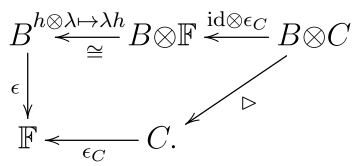
$$\begin{aligned} &x\left(\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\cdot\begin{pmatrix}x'\\ y'\end{pmatrix}\right) = x\begin{pmatrix}y'\\ 0\end{pmatrix} = y' && y\left(\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\cdot\begin{pmatrix}x'\\ y'\end{pmatrix}\right) = x\begin{pmatrix}y'\\ 0\end{pmatrix} = 0 \\ &x\left(\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\cdot\begin{pmatrix}x'\\ y'\end{pmatrix}\right) = x\begin{pmatrix}0\\ x'\end{pmatrix} = 0 && y\left(\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\cdot\begin{pmatrix}x'\\ y'\end{pmatrix}\right) = x\begin{pmatrix}0\\ x'\end{pmatrix} = x' \\ &x\left(\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}x'\\ y'\end{pmatrix}\right) = x\begin{pmatrix}-x'\\ y'\end{pmatrix} = -x' && y\left(\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}x'\\ y'\end{pmatrix}\right) = y\begin{pmatrix}-x'\\ y'\end{pmatrix} = y', \end{aligned}$$

this representation $\rho : \mathfrak{sl}_2 \to \mathrm{End}_{\mathbb{F}}(\mathbb{F}[x,y])$ induces the representation in (22). Hence we can view the $U_q(\mathfrak{sl}_2)$-module algebra structure on the quantum plane as a deformation or quantisation of the $\mathfrak{sl}_2$-module structure on the algebra of polynomial functions on the plane that is induced by the standard $\mathfrak{sl}_2$-representation on $\mathbb{F}^2$.

By passing from from a finite-dimensional algebra $A$ to the dual coalgebra $A^*$ or from a finite-dimensional bialgebra $B$ acting on $A$ to its dual $B^*$, we obtain concepts analogous to the module algebra in Definition 3.1.1. They are obtained, respectively, by reversing the arrows labelled by the multiplication and unit of $A$ in (19) and labelling them with a comultiplication and counit instead or by reversing the arrows labelled by the comultiplication and counit of $B$ and by the action $\rhd$ in (19) and labelling them with the multiplication and unit and a coaction $\delta$ instead. This leads to the concepts of a *module coalgebra*, a *comodule algebra* and a *comodule coalgebra* over a bialgebra.

**Definition 3.1.4:** Let $B$ be a bialgebra over $\mathbb{F}$.

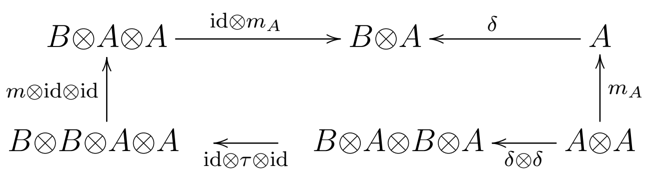
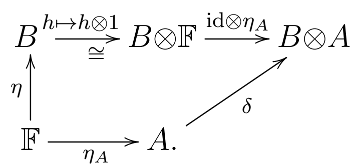
1. A $B$-**left module coalgebra** is a coalgebra $(C, \Delta_C, \epsilon_C)$ together with a $B$-left module structure $\rhd : B \otimes C \to C$ such that the following two diagrams commute

In formulas: for all $b \in B$ and $c \in C$, one has

$$\Sigma_{(b\rhd c)}(b \rhd c)_{(1)} \otimes (b \rhd c)_{(2)} = \Sigma_{(b),(c)}(b_{(1)} \rhd c_{(1)}) \otimes (b_{(2)} \rhd c_{(2)}) \qquad \epsilon_C(b \rhd c) = \epsilon(b)\epsilon_C(c).$$

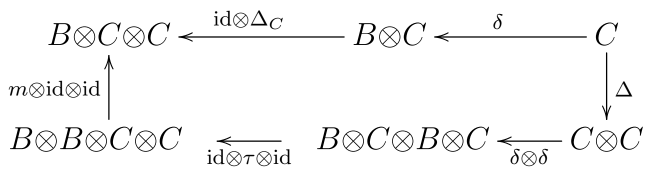
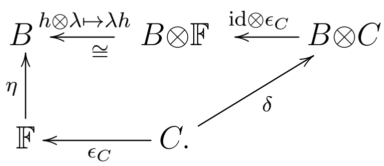
2. A $B$-**left comodule algebra** is an algebra $(A, m_A, \eta_A)$ together with an $B$-left comodule structure $\delta : A \to B \otimes A$, $a \mapsto \Sigma_{(a)} a_{(1)} \otimes a_{(0)}$ such that the following two diagrams commute

In formulas: for all $b \in B$ and $a, a' \in A$, one has

$$\Sigma_{(aa')}(aa')_{(1)} \otimes (aa')_{(0)} = \Sigma_{(a)(a')} a_{(1)} a'_{(1)} \otimes a_{(0)} a'_{(0)} \qquad \Sigma_{(1_A)} 1_{A(1)} \otimes 1_{A(0)} = 1_B \otimes 1_A$$

3. A $B$-**left comodule coalgebra** is a coalgebra $(C, \Delta_C, \epsilon_C)$ with an $B$-left comodule structure $\delta : C \to B \otimes C$, $c \mapsto \Sigma_{(c)} c_{(1)} \otimes c_{(0)}$ such that the following two diagrams commute

In formulas: for all $c \in C$, one has

$$\Sigma_{(c)} c_{(1)} \otimes c_{(0)(1)} \otimes c_{(0)(2)} = \Sigma_{(c)} c_{(1)(1)} c_{(2)(1)} \otimes c_{(1)(0)} \otimes c_{(2)(0)} \qquad \Sigma_{(c)} c_{(1)} \epsilon_C(c_{(0)}) = \epsilon_C(c) 1_B.$$

A **right module coalgebra** over $B$ is a left module coalgebra over $B^{op}$ and a **module bi-coalgebra** over $B$ is a left module coalgebra over $B\otimes B^{op}$. A **right comodule (co)algebra** over $B$ is a left comodule (co)algebra over $B^{cop}$ and a **comodule bi(co)algebra** over $B$ is a left comodule (co)algebra over $B\otimes B^{cop}$.

**Remark 3.1.5:**

1. If $V$ is a comodule (co)algebra over a bialgebra $B$ with $\delta : V \to B\otimes V$, then $V$ is a $B^{*op}$ module (co)algebra with $\rhd : B^*\otimes V \to V$, $\beta \rhd v = (\beta\otimes\text{id})\circ\delta(v)$. If $B$ is finite-dimensional, this defines a bijection between $B$-comodule (co)algebra and $B^{*op}$-module (co)algebra structures on $V$.

2. If $(A, m_A, \eta_A)$ is a finite-dimensional module algebra over a finite-dimensional bialgebra $B$, then $(A^*, m_A^*, \eta_A^*)$ is a comodule coalgebra over $B^*$ and vice versa. (Exercise)

3. In the infinite-dimensional case, the second statement in 1. and statement 2. do not hold. There are modules over $B^{*op}$ that do not arise from comodules over $B$, the dual of an infinite-dimensional algebra $A$ does not have a canonical coalgebra structure and that in that case one may have $B^*\otimes A^* \subsetneq (B\otimes A)^*$. This is the reason for the introduction of the dual concepts in Definition 3.1.4.

**Example 3.1.6:**

1. If $G$ is a finite group acting on a finite set $X$ via $\rhd : G \times X \to X$, then:
   (a) The algebra $\text{Fun}(X, \mathbb{F})$ is a comodule algebra over $\text{Fun}(G, \mathbb{F})$ with
   $$\delta : \text{Fun}(X, \mathbb{F}) \to \text{Fun}(G, \mathbb{F})\otimes\text{Fun}(X, \mathbb{F}), \quad f \mapsto \Sigma_{g\in G}\delta_g\otimes(f \lhd g).$$
   (b) The free vector space $\langle X\rangle_{\mathbb{F}}$ generated by $X$ is a module coalgebra over $\mathbb{F}[G]$ with module structure $\rhd : \mathbb{F}[G]\otimes\langle X\rangle_{\mathbb{F}} \to \langle X\rangle_{\mathbb{F}}$, $g\otimes x \mapsto g \rhd x$ and coalgebra structure $\Delta : \langle X\rangle_{\mathbb{F}} \to \langle X\rangle_{\mathbb{F}}\otimes\langle X\rangle_{\mathbb{F}}$, $x \mapsto x\otimes x$.
   (c) The free vector space $\langle X\rangle_{\mathbb{F}}$ is a comodule coalgebra over $\text{Fun}(G, \mathbb{F})$ with comodule structure $\delta : \langle X\rangle_{\mathbb{F}} \to \text{Fun}(G, \mathbb{F})\otimes\langle X\rangle_{\mathbb{F}}$, $x \mapsto \Sigma_{g\in G}\, \delta_g\otimes g \rhd x$ and coalgebra structure $\Delta : \langle X\rangle_{\mathbb{F}} \to \langle X\rangle_{\mathbb{F}}\otimes\langle X\rangle_{\mathbb{F}}$, $x \mapsto x\otimes x$.

2. Every bialgebra $B$ is a module coalgebra over itself with the **left regular action**
   $$\rhd_L : B\otimes B \to B, \quad b \rhd c = b \cdot c,$$
   a module coalgebra over $B^{op}$ with the **right regular action**
   $$\lhd_R : B^{op}\otimes B \to B, \quad c \lhd b = c \cdot b,$$
   and a comodule algebra over itself with its multiplication and comultiplication.

3. A comodule algebra over a group algebra $\mathbb{F}[G]$ is the same as a $G$-**graded algebra**, i. e. an algebra $A$ that is given as a direct sum $A = \oplus_{g\in G} A_g$ with linear subspaces $A_g \subset A$ such that $1_A \in A_e$ and $A_g \cdot A_h \subseteq A_{gh}$ (Exercise).

Given a module algebra $A$ over a bialgebra $B$, one can associate to this pair an algebra structure on the vector space $A\otimes B$, the so-called *cross product* or *smash product*. This generalises the construction of a semidirect product of groups and plays an important role in many constructions with Hopf algebras.

**Proposition 3.1.7:** Let $B$ be a bialgebra and $(A, \rhd)$ a module algebra over $B$. Then

$$(a\otimes b) \cdot (a'\otimes b') = \Sigma_{(b)}\, a(b_{(1)} \rhd a')\otimes b_{(2)}b'$$

defines an algebra structure on $A\otimes H$ with unit $1_A\otimes 1_H$, such that $\iota_A : A \to A\otimes H$, $a \mapsto a\otimes 1$ and $\iota_H : H \to A\otimes H$, $h \mapsto 1\otimes h$ are injective algebra homomorphisms. The vector space $A\otimes H$ with these algebra structure is called the **smash product** or **cross product** of $A$ and $H$ and denoted $A\#H$.

**Proof:**
That $\iota_A$ and $\iota_H$ are algebra homomorphisms is obvious. The remaining claims follow by a direct computation. To check associativity, we compute

$$\begin{aligned}
&\big((a\otimes h) \cdot (b\otimes k)\big) \cdot (c\otimes l)\\
&= \Sigma_{(h)}(a(h_{(1)} \rhd b)\otimes h_{(2)}k) \cdot (c\otimes l) = \Sigma_{(h)(h_{(2)}k)} a(h_{(1)} \rhd b)((h_{(2)}k)_{(1)} \rhd c)\otimes (h_{(2)}k)_{(2)}l\\
&= \Sigma_{(h)(k)} a(h_{(1)} \rhd b)((h_{(2)(1)}k_{(1)}) \rhd c)\otimes h_{(2)(2)}k_{(2)}l = \Sigma_{(h)(k)} a(h_{(1)} \rhd b)((h_{(2)}k_{(1)}) \rhd c)\otimes h_{(3)}k_{(2)}l
\end{aligned}$$

$$\begin{aligned}
&(a\otimes h) \cdot \big((b\otimes k) \cdot (c\otimes l)\big)\\
&= \Sigma_{(k)}(a\otimes h) \cdot (b(k_{(1)} \rhd c)\otimes k_{(2)}l) = \Sigma_{(h)(k)} a(h_{(1)} \rhd (b(k_{(1)} \rhd c)))\otimes h_{(2)}k_{(2)}l\\
&= \Sigma_{(h)(k)} a(h_{(1)(1)} \rhd b)(h_{(1)(2)} \rhd (k_{(1)} \rhd c))\otimes h_{(2)}k_{(2)}l = \Sigma_{(h)(k)} a(h_{(1)} \rhd b)((h_{(2)}k_{(1)}) \rhd c)\otimes h_{(3)}k_{(2)}l,
\end{aligned}$$

and to show that $1_A\otimes 1_H$ is a unit, we compute for all $a \in A$ and $h \in H$

$$\begin{aligned}
(a\otimes h) \cdot (1_A\otimes 1_H) &= \Sigma_{(h)} a(h_{(1)} \rhd 1_A)\otimes h_{(2)}1_H = \Sigma_{(h)}\epsilon(h_{(1)})a\otimes h_{(2)} = a\otimes h\\
(1_A\otimes 1_H) \cdot (a\otimes h) &= \Sigma_{(1_H)} 1_A(1_{H(1)} \rhd a)\otimes 1_{H(2)}h = 1_A a\otimes 1_H h = a\otimes h.
\end{aligned}$$

□

**Example 3.1.8:**

1. Let $G, H$ be groups and $\rho : G \to \mathrm{Aut}(H)$ a group homomorphism. Then $\mathbb{F}[H]$ is a module algebra over $\mathbb{F}[G]$ with $\rhd : \mathbb{F}[G]\otimes\mathbb{F}[H] \to \mathbb{F}[H]$, $g\otimes h \mapsto \rho(g)h$. The associated cross product $H\#G$ is given by the multiplication law

$$(h\otimes g) \cdot (h'\otimes g') = h(\rho(g)h')\otimes gg'$$

for all $g, g' \in G$ and $h, h' \in H$. This coincides with the multiplication law of the semidirect product $H \rtimes G$ and hence one has $\mathbb{F}[H]\#\mathbb{F}[G] \cong \mathbb{F}[H \rtimes G]$.

2. Let $B$ be a bialgebra over $\mathbb{F}$ and consider the dual algebra $A = B^*$ with the right dual action $\rhd^*_R : B\otimes B^* \to B^*$, $b \rhd^*_R \alpha = \Sigma_{(\alpha)}\alpha_{(2)}(b)\, \alpha_{(1)}$. Then the multiplication of the cross product $B^*\#B$ is given by

$$(\alpha\otimes b)(\alpha'\otimes b') = \Sigma_{(h)(\alpha')}\alpha'_{(2)}(b_{(1)})\ \alpha\alpha'_{(1)}\otimes b_{(2)}b'$$

This cross product is also called the **Heisenberg double** of $B$ and denoted $\mathcal{H}(B)$.

## 3.2 (Co)invariants and (co)integrals

The concepts of (co)module (co)algebras over Hopf algebras from the last subsection allow one to generalise group actions $G$ on a set $X$ and the associated algebra of functions $\mathrm{Fun}(X,\mathbb{F})$ to Hopf algebra (co)actions on (co)algebras. In this section, we introduce the notions of (co)invariants for (co)modules over Hopf algebras that allows us to generalise the notion of a function that is invariant under a group action and the notion of functions on the orbit space.

Invariants can be defined for modules over a bialgebra by linearising the notion of an invariant for a group representation. If $\rhd : G\times V \to V$ is a representation of a group $G$ on a vector space $V$, then an element $v\in V$ is called *invariant* if the group $G$ acts trivially on $v$, i. e. $g\rhd v = v$ for all $g\in G$. When passing from a group $G$ to a Hopf algebra, then the invariance condition needs to be replaced by an appropriate condition that is linear in $H$. Given the available structures, the natural choice is to set $h\rhd v = \epsilon(h)\, v$ for all $h\in H$ and $v\in V$. The dual notion of a *coinvariant* for a comodule over $B$ is obtained by requiring that for a finite-dimensional bialgebra $B$, coinvariants for a left comodule $(V,\delta)$ over $B$ coincide with the invariants of the associated $B^{*op}$-module structure on $V$ that is given by $\beta\rhd v = (\beta\otimes\mathrm{id})\circ\delta(v)$.

**Definition 3.2.1:** Let $B$ be a bialgebra over $\mathbb{F}$.

1. Let $(M,\rhd)$ be a module over $B$. An element $m\in M$ is called an **invariant** of $M$ if $b\rhd m = \epsilon(b)\, m$ for all $b\in B$. The submodule of invariants of $M$ is denoted $M^B$.
2. Let $(M,\delta)$ be a comodule over $H$. An element $m\in M$ is called a **coinvariant** of $M$ if $\delta(m) = 1_B\otimes m$. The subcomodule of invariants in $M$ is denoted $M^{coB}$.

**Example 3.2.2:**

1. Let $G$ be a group acting on a set $X$ by $\rhd : G\times X\to X$. Let $\lhd : \mathrm{Fun}(X,\mathbb{F})\otimes\mathbb{F}[G]\to \mathrm{Fun}(X,\mathbb{F})$ be the associated right action of $G$ on $\mathrm{Fun}(X,\mathbb{F})$ with $(f\lhd g)(x) = f(g\rhd x)$ for all $g\in G$ and $x\in X$. Then this defines a $\mathbb{F}[G]$-module structure on $\mathrm{Fun}(X,\mathbb{F})$ with
$$\mathrm{Fun}(X,\mathbb{F})^{\mathbb{F}[G]} = \{f: X\to\mathbb{F}\,|\, f(g\rhd x) = f(x)\,\forall g\in G\}.$$
2. Let $H$ be a Hopf algebra acting on itself via the adjoint action $\rhd_{ad} : H\otimes H\to H$with $h\rhd_{ad} k = \Sigma_{(h)}\, h_{(1)}\cdot k\cdot S(h_{(2)})$. Then the submodule of invariants is the centre of $H$
$$H^{\rhd_{ad}} = Z(H) = \{k\in H\,|\, h\cdot k = k\cdot h\,\forall h\in H\}.$$
This can be seen as follows. If $k\in Z(H)$, then $h\rhd_{ad} k = \Sigma_{(h)} h_{(1)}\cdot k\cdot S(h_{(2)}) = \Sigma_{(h)}(h_{(1)}S(h_{(2)}))\cdot k = \epsilon(h)\, k$ for all $h\in H$. Conversely, if $k\in H^{\rhd_{ad}}$ then for all $h\in H$
$$\begin{aligned} h\cdot k &= (\Sigma_{(h)} h_{(1)}\epsilon(h_{(2)}))\cdot k = \Sigma_{(h)} h_{(1)} k (S(h_{(2)(1)}) h_{(2)(2)}) = \Sigma_{(h)} (h_{(1)(1)} k S(h_{(1)(2)})) h_{(2)} \\ &= \Sigma_{(h)} (h_{(1)}\rhd_{ad} k) h_{(2)} = \Sigma_{(h)}\epsilon(h_{(1)})\, k h_{(2)} = k\cdot\big(\Sigma_{(h)}\epsilon(h_{(1)}) h_{(2)}\big) = k\cdot h. \end{aligned}$$

In both examples, the module over the Hopf algebra $H$ has the structure of a module algebra, and the invariants of the module form a subalgebra. This is a general pattern that holds for any module algebra over a bialgebra and allows us to view the invariants of a module algebra $A$ over a Hopf algebra $H$ as a generalisation of functions on the orbit space of a group action.

**Proposition 3.2.3:** Let $B$ be a bialgebra over $\mathbb{F}$

1. If $A$ is a module algebra over $B$, then $A^B \subset A$ is a subalgebra.
2. If $A$ is a comodule algebra over $B$, then $A^{coB} \subset A$ is a subalgebra.

**Proof:**
Let $A$ be a module algebra over $B$. Clearly, the submodule $A^B \subset A$ is a linear subspace of $A$. Moreover, one has $1_A \in A^B$ since $b \rhd 1_A = \epsilon(h)\, 1$ for all $b \in B$ by definition of the module algebra, and for all $a, a' \in A^B$ and $b \in B$

$$\begin{aligned} b \rhd (aa') &= \Sigma_{(b)}(b_{(1)} \rhd a) \cdot (b_{(2)} \rhd a') = \Sigma_{(b)}(\epsilon(b_{(1)})a) \cdot (\epsilon(b_{(2)})a') = \Sigma_{(b)}\epsilon(b_{(1)})\epsilon(h_{(b)})\, aa' \\ &= \epsilon(\Sigma_{(b)}\epsilon(b_{(1)})b_{(2)})\, aa' = \epsilon(b)\, aa', \end{aligned}$$

where we used the definition of a module algebra and of the invariants, then the linearity of the multiplication in $A$ and of $\epsilon$ and finally the counitality of $\epsilon$. This shows that $aa' \in A^B$ and that $A^B$ is a subalgebra of $A$. Similarly, if $A$ is a comodule algebra over $H$, then $A^{coB} \subset A$ is a linear subspace and $\delta(1_A) = 1_B \otimes 1_A$ by definition of a comodule algebra. The definition of a comodule algebra also implies for all $a, a' \in A^{coB}$

$$\delta(aa') = \delta(a) \cdot \delta(a') = (1_B \otimes a) \cdot (1_B \otimes a') = 1_B \otimes aa',$$

and hence $A^{coB} \subset A$ is a subalgebra of $A$. $\square$

Another nice example of invariants (although not in the context of module algebras) is given by the following lemma, which shows that *module homomorphisms* between modules $M, N$ over a Hopf algebra can also be viewed as the invariants of a certain module, namely the module $\mathrm{Hom}_{\mathbb{F}}(M, N)$ of *linear maps* $f : M \to N$ with $H$-module structure induced by the ones of $M$ and $N$. In this case, an antipode is necessary to ensure that the pre-and post composition with the action of $H$ can be combined into a module structure on $\mathrm{Hom}_{\mathbb{F}}(M, N)$.

**Lemma 3.2.4:** Let $H$ be a Hopf algebra and $M, N$ modules over $H$. Then

$$\rhd :\ H \otimes \mathrm{Hom}_{\mathbb{F}}(M, N) \to \mathrm{Hom}_{\mathbb{F}}(M, N), \qquad (h \rhd f)(m) = \Sigma_{(h)}\, h_{(1)} \rhd_N f(S(h_{(2)}) \rhd_N m)$$

is a $H$-left module structure on $\mathrm{Hom}_{\mathbb{F}}(M, N)$ whose invariants are the module maps $f : M \to N$.

**Proof:**
With the formula for the module structure we compute $(1 \rhd f)(m) = 1 \rhd_N f(S(1) \rhd_M m) = f(m)$ for all $m \in M$, and for all $h, k \in H$ we obtain

$$\begin{aligned} &(h \rhd (k \rhd f))(m) = \Sigma_{(h)} h_{(1)} \rhd_N ((k \rhd f)(S(h_{(2)}) \rhd m)) \\ &= \Sigma_{(h)(k)} h_{(1)} \rhd_N (k_{(1)} \rhd_N f)(S(k_{(2)}) \rhd_M (S(h_{(2)}) \rhd_M m)) \\ &= \Sigma_{(h)(k)} (h_{(1)} k_{(1)}) \rhd_N f((S(k_{(2)}) S(h_{(2)})) \rhd_M m) = \Sigma_{(h)(k)} (h_{(1)} k_{(1)}) \rhd_N f(S(h_{(2)} k_{(2)}) \rhd_M m) \\ &= \Sigma_{(hk)} (hk)_{(1)} \rhd_N f(S((hk)_{(2)}) \rhd_M m) = ((hk) \rhd f)(m). \end{aligned}$$

This shows that $\mathrm{Hom}_{\mathbb{F}}(M, N)$ is a module over $H$. For each module homomorphism $f : M \to N$ we have for all $m \in M$ and $h \in H$

$$(h \rhd f)(m) = \Sigma_{(h)} h_{(1)} \rhd_N f(S(h_{(2)}) \rhd_M m) = \Sigma_{(h)}\, (h_{(1)} S(h_{(2)})) \rhd_N f(m) = \epsilon(h) f(m)$$

and hence $f \in \mathrm{Hom}_{\mathbb{F}}(M,N)^H$. Conversely, if $f \in \mathrm{Hom}_{\mathbb{F}}(M,N)^H$, then for all $m \in M$ and $h \in H$

$$\begin{aligned}
&h \rhd_N f(m) = \Sigma_{(h)}\, \epsilon(h_{(2)})\, h_{(1)} \rhd_N f(m) = \Sigma_{(h)} h_{(1)} \rhd_N f((S(h_{(2)})h_{(3)}) \rhd_M m) \\
&= \Sigma_{(h)}(h_{(1)} \rhd f)(h_{(2)} \rhd_M m) = \Sigma_{(h)}\epsilon(h_{(1)})\, f(h_{(2)} \rhd_M h) = \Sigma_{(h)} f((\Sigma_{(h)}\epsilon(h_{(1)})h_{(2)}) \rhd_M m) \\
&= f(h \rhd_M m),
\end{aligned}$$

and hence $f$ is a module homomorphism from $M$ to $N$. □

We now consider the invariants for the left and right regular action of a Hopf algebra $H$ on itself. These invariants play a special role and are called *integrals*. The reason for this is that they are the bialgebra or Hopf algebra counterparts of left or right invariant integrals on a Lie group. For the same reason, an integral in a bialgebra $H$ that is both left and right invariant is sometimes called a *Haar integral* in $H$. Just as Haar integrals over compact Lie groups can be used to construct invariant functions on Lie groups, integrals in a bialgebra that satisfy certain normalisation conditions define projectors on invariant submodules over $H$.

**Definition 3.2.5:** Let $H$ be a Hopf algebra over $\mathbb{F}$.

1. A **left (right) integral** in $H$ is an invariant for the left (right) regular action of $H$ on itself, i. e. an element $\ell \in H$ with $h \cdot \ell = \epsilon(h)\,\ell$ (with $\ell \cdot h = \epsilon(h)\,\ell$) for all $h \in H$. The linear subspaces of left and right integrals in $H$ are denoted $I_L(H)$ and $I_R(H)$. If $I_L(H) = I_R(H)$, then the Hopf algebra $H$ is called **unimodular**.

2. A **left (right) cointegral** for $H$ is an element $\lambda \in H^*$ with $(\mathrm{id}\otimes\lambda)(\Delta(h)) = 1_H \otimes \lambda(h)$ (with $(\lambda\otimes\mathrm{id})(\Delta(h)) = \lambda(h)\otimes 1_H$) for all $h \in H$. The linear subspaces of left and right cointegrals for $H$ are denoted $C_L(H)$ and $C_R(H)$.

3. A left or right integral $\ell \in H$ is called **normalised** if $\epsilon(\ell) = 1$. A (normalised) element $\ell \in I_L(H) \cap I_R(H)$ is sometimes called a **(normalised) Haar integral**.

**Remark 3.2.6:**

1. A right (co)integral for a Hopf algebra $H$ can also be defined as a left (co)integral for the Hopf algebra $H^{op,cop}$ and vice versa.
2. If $H$ is finite-dimensional, then a left (right) cointegral for $H$ is the same as a left (right) integral for the dual Hopf algebra $H^*$, since one has $(\mathrm{id}\otimes\lambda)(\Delta(h)) = 1_H\otimes\lambda(h)$ for all $h \in H$ if and only if $(\alpha \cdot \lambda)(h) = (\alpha\otimes\lambda)(\Delta(h)) = \alpha(1)\otimes\lambda(h) = \epsilon(\alpha)\lambda(h)$ for all $h \in H$ and $\alpha \in H^*$, which is the case if and only if $\alpha \cdot \lambda = \epsilon(\alpha)\,\lambda$ for all $\alpha \in H^*$.

**Example 3.2.7:**

1. If $G$ is a finite group, then $\mathbb{F}[G]$ and $\mathrm{Fun}(G,\mathbb{F})$ are unimodular with

$$I_L(\mathbb{F}[G]) = I_R(\mathbb{F}[G]) = \mathrm{span}_{\mathbb{F}}\{\Sigma_{g\in G}\, g\} \qquad C_L(\mathbb{F}[G]) = C_R(\mathbb{F}[G]) = \mathrm{span}_{\mathbb{F}}\{\delta_e\}\,.$$

   The cointegral $\delta_e$ is normalised. The integral $\Sigma_{g\in G} g$ can be normalised iff $\mathrm{char}(\mathbb{F}) \not| |G|$.

2. The Taft algebra from Example 2.3.6. is not unimodular. One has

$$I_L(H) = \mathrm{span}_{\mathbb{F}}\{\Sigma_{j=0}^{n-1} y^j x^{n-1}\} \qquad I_R(H) = \mathrm{span}_{\mathbb{F}}\{\Sigma_{j=0}^{n-1} q^j y^j x^{n-1}\}.$$

3. The $q$-deformed universal enveloping algebra $U_q(\mathfrak{sl}_2)$ from Example 2.3.9 has no non-trivial left or right integrals: $I_L(U_q(\mathfrak{sl}_2)) = I_R(U_q(\mathfrak{sl}_2)) = \{0\}$. The $q$-deformed universal enveloping algebra $U_q^r(\mathfrak{sl}_2)$ at a root of unity from Proposition 2.3.12 is unimodular with

$$I_L(U^r(\mathfrak{sl}_2)) = I_R(U^r(\mathfrak{sl}_2)) = \mathrm{span}_{\mathbb{F}}\{\Sigma_{j=0}^{r-1} K^j E^{r-1} F^{r-1}\}$$

The importance and usefulness of integrals comes from the fact that given a normalised left or right integral for a Hopf algebra $H$, we can construct a projector on the invariants of any left or right module over $H$. This allows one to determine the invariants of any module over $H$ explicitly and systematically.

**Lemma 3.2.8:** Let $H$ be a Hopf algebra and $\ell \in H$ a normalised left integral. Then for any $H$-module $(M, \rhd)$, the linear map $P : M \to M$, $m \mapsto \ell \rhd m$ is a projector on $M^H$.

**Proof:**
This follows directly from the properties of the normalised left integral $\ell \in H$. One has

$$h \rhd P(m) = h \rhd (\ell \rhd m) = (h\ell) \rhd m = (\epsilon(h)\ell) \rhd m = \epsilon(h)\,(\ell \rhd m) = \epsilon(h)\,P(m)$$
$$(P \circ P)(m) = \ell \rhd (\ell \rhd m) = (\ell^2) \rhd m = (\epsilon(\ell)\,\ell) \rhd m = \epsilon(\ell)\,\ell \rhd m = \ell \rhd m = P(m)$$

for all $m \in M$ and $h \in H$, and this shows that $P$ is a projector with $\mathrm{im}(P) \subset M^H$. Conversely, if $m \in M^H$, one has $P(m) = \ell \rhd m = \epsilon(\ell)\,m = m$ and hence $m \in P(M)$. □

Given the usefulness and importance of integrals, we will now determine under which conditions a Hopf algebra $H$ admits non-trivial or normalised left and right integrals and under which conditions left and right integrals coincide. The essential concept that allows one to address this question is the notion of a *Hopf module*.

We will see in the following that in the finite-dimensional case a Hopf module over $H$ is nothing but a right module over a cross product $H^{*op}\# H^{cop}$, where $H^{cop}$ acts on $H^{*op}$ via the right dual action. However, as we wish to consider infinite-dimensional modules and Hopf algebras as well, it is advantageous to dualise this notion and to transform the action of $H^{*op}$ into a coaction of $H$, as in Remark 3.1.5. This leads to a formulation that is free of dual Hopf algebras and hence avoids problems with duals of tensor products and the need to pass to finite duals. As we wish to use Sweedler notation for comodules (see Definition 2.1.6), which is simpler and more intuitive for *right* comodules, it is advantageous to work with *right* Hopf modules.

**Definition 3.2.9:** Let $H$ be a Hopf algebra over $\mathbb{F}$.

1. A **right Hopf module** over $H$ is a vector space $V$ over $\mathbb{F}$ with a right $H$-module structure and a right $H$-comodule structure

$$\lhd : V \otimes H \to V, \quad h \otimes v \mapsto v \lhd h \qquad \delta : V \to V \otimes H, \quad v \mapsto \Sigma_{(v)} v_{(0)} \otimes v_{(1)},$$

such that $\delta$ is a morphism of $H$-right modules with respect to the $H$-right module structure on $V$ and the right regular action of $H$ on itself:

$$\delta(v \lhd h) = \delta(v) \lhd h = \Sigma_{(v)(h)}\,(v_{(0)} \lhd h_{(1)}) \otimes (v_{(1)} \cdot h_{(2)})$$

A **left Hopf module** over $H$ is a right Hopf module over $H^{op,cop}$.

2. A **homomorphism of right Hopf modules** from $(V, \rhd_V, \delta_V)$ to $(W, \rhd_W, \delta_W)$ is a linear map $\phi : V \to W$ that is a homomorphism of right $H$-modules and right $H$-comodules:

$$\phi(v \lhd_V h) = \phi(v) \lhd_W h \qquad (\phi \otimes \mathrm{id}_H)(\delta_V(v)) = \delta_W(\phi(v)) \qquad \forall v \in V, w \in W.$$

**Remark 3.2.10:** Right and left Hopf modules are closely related to modules over cross products. One can show that every right Hopf module $M$ over $H$ is a right module over the cross product $H^{*op} \# H^{cop}$ from Proposition 3.1.7 with the multiplication law

$$(\alpha \otimes h) \cdot (\beta \otimes k) = \Sigma_{(\beta)(h)} \beta_{(2)}(h_{(2)})\, \beta_{(1)} \alpha \otimes h_{(1)} k. \tag{23}$$

The $H^{*op} \# H^{cop}$ right module structure on the right Hopf module $M$ is given by

$$\lhd : M \otimes (H^{*op} \# H^{cop}) \to M, \quad m \lhd (\alpha \otimes h) = \Sigma_{(m)} \alpha(m_{(1)})\, m_{(0)} \lhd h$$

If $H$ and $M$ are finite-dimensional, this induces a bijection between $H$-right Hopf module structures and $H^{*op} \# H^{cop}$-right module structures on $M$.

**Example 3.2.11:**

1. Every Hopf algebra $H$ is a right Hopf module over itself with the right regular action $\lhd_R : H \otimes H \to H$, $k \lhd h = k \cdot h$ and the comultiplication $\delta = \Delta : H \to H \otimes H$.
2. For every right module $(M, \lhd)$ over $H$, the vector space $M \otimes H$ is a Hopf module with

$$\begin{aligned} &\lhd : (M \otimes H) \otimes H \to M \otimes H, && (m \otimes k) \lhd h = \Sigma_{(h)} (m \lhd h_{(1)}) \otimes (k \cdot h_{(2)}) \\ &\delta = (\mathrm{id}_M \otimes \Delta) : M \otimes H \to (M \otimes H) \otimes H, && \delta(m \otimes k) = \Sigma_{(k)} m \otimes k_{(1)} \otimes k_{(2)}. \end{aligned}$$

3. In particular, for every vector space $V$ over $\mathbb{F}$, the vector space $V \otimes H$ is a Hopf module over $H$ with the **trivial Hopf module structure**

$$\begin{aligned} &\lhd = \mathrm{id} \otimes m : (V \otimes H) \otimes H \to V \otimes H, && (v \otimes k) \lhd h = v \otimes kh \\ &\delta = (\mathrm{id} \otimes \Delta) : V \otimes H \to (V \otimes H) \otimes H, && \delta(v \otimes k) = \Sigma_{(k)} v \otimes k_{(1)} \otimes k_{(2)}. \end{aligned}$$

The distinguishing property of Hopf modules is that they *factorise* into the submodule of coinvariants and the underlying Hopf algebra $H$. Every Hopf module is isomorphic as a Hopf module to the tensor product of its coinvariants with $H$. In the finite-dimensional case, this will allow us later to determine the dimension of the linear subspace of coinvariants from the dimension of $M$ and the dimension of $H$.

**Theorem 3.2.12:** (**Fundamental theorem of Hopf modules**)
Let $H$ be a Hopf algebra over $\mathbb{F}$ and $(M, \lhd, \delta)$ a Hopf module over $H$. Then

$$\phi = \lhd : M^{coH} \otimes H \to M, \quad m \otimes h \mapsto m \lhd h$$

is an isomorphism of Hopf modules if $M^{coH} \otimes H$ is equipped with the trivial Hopf module structure from Example 3.2.11, 3.

**Proof:**
1. We denote by $\lhd : M\otimes H \to M$ the $H$-right module structure and by $\delta : M \to M\otimes H$ the $H$-right comodule structure on $M$. Then we have in Sweedler notation $\delta(m) = \Sigma_{(m)} m_{(0)}\otimes m_{(1)}$ and the Hopf module conditions on $\delta$ read

$$(\delta\otimes\mathrm{id}) \circ \delta(m) = \Sigma_{(m)} m_{(0)}\otimes m_{(1)}\otimes m_{(2)} = (\mathrm{id}\otimes\Delta) \circ \delta(m)$$
$$\delta(m \lhd h) = \delta(m) \lhd h = \Sigma_{(m)(h)} m_{(0)} \lhd h_{(1)}\otimes m_{(1)} h_{(2)}.$$

The trivial Hopf module structure on $M^{coH}\otimes H$ is given by

$$(m\otimes k) \lhd' h = m\otimes kh \qquad \delta'(m\otimes k) = \Sigma_{(k)} m\otimes k_{(1)}\otimes k_{(2)}.$$

2. We show that the map $\phi : M^{coH}\otimes H \to M$ is a homomorphism of Hopf modules. For all $m \in M^{coH}$ and $h, k \in H$, we have

$$\phi(m\otimes k) \lhd h = (m \lhd k) \lhd h = m \lhd (kh) = \phi(m\otimes kh) = \phi((m\otimes k) \lhd' h)$$
$$(\delta \circ \phi)(m\otimes k) = \delta(m \lhd k) = \delta(m) \lhd k = (m\otimes 1) \lhd k = \Sigma_{(k)}(m \lhd k_{(1)})\otimes k_{(2)}$$
$$= (\phi\otimes\mathrm{id}) \left(\Sigma_{(k)} m\otimes k_{(1)}\otimes k_{(2)}\right) = ((\phi\otimes\mathrm{id}) \circ \delta')(m\otimes k).$$

3. To show that $\phi$ is an isomorphism of Hopf modules, we consider the linear map

$$\psi : M \to M, \quad m \mapsto \lhd \circ (\mathrm{id}\otimes S) \circ \delta(m) = \Sigma_{(m)} m_{(0)} \lhd S(m_{(1)})$$

and prove that $\psi$ is a projector on $M^{coH}$ that is invariant under the action of $H$ on $M$. We first show that $\psi$ takes values in $M^{coH}$ and is invariant under the action of $H$ on $M$. For all $m \in M$ and $h \in H$ we have

$$\begin{aligned}
\delta(\psi(m)) &= \delta\left(\Sigma_{(m)} m_{(0)} \lhd S(m_{(1)})\right) = \Sigma_{(m)} \delta(m_{(0)}) \lhd S(m_{(1)})\\
&= \Sigma_{(m)}(m_{(0)}\otimes m_{(1)}) \lhd S(m_{(2)}) = \Sigma_{(m)} m_{(0)} \lhd S(m_{(2)})_{(1)}\otimes m_{(1)} S(m_{(2)})_{(2)}\\
&= \Sigma_{(m)} m_{(0)} \lhd S(m_{(2)(2)})\otimes m_{(1)} S(m_{(2)(1)}) = \Sigma_{(m)} m_{(0)} \lhd S(m_{(3)})\otimes m_{(1)} S(m_{(2)})\\
&= \epsilon(m_{(1)})\Sigma_{(m)} m_{(0)} \lhd S(m_{(2)})\otimes 1 = \Sigma_{(m)} m_{(0)} \lhd S(m_{(1)})\otimes 1 = \psi(m)\otimes 1
\end{aligned}$$

$$\begin{aligned}
\psi(m \lhd h) &= \lhd \circ (\mathrm{id}\otimes S) \circ \delta(m \lhd h) = \lhd \circ (\mathrm{id}\otimes S)(\delta(m) \lhd h)\\
&= \lhd \circ (\mathrm{id}\otimes S)(\Sigma_{(m)(h)} m_{(0)} \lhd h_{(1)}\otimes m_{(1)} h_{(2)})\\
&= \Sigma_{(m)(h)}\ (m_{(0)} \lhd h_{(1)}) \lhd S(m_{(1)} h_{(2)}) = \Sigma_{(m)(h)}(m_{(0)} \lhd h_{(1)}) \lhd (S(h_{(2)}) S(m_{(1)}))\\
&= \Sigma_{(m)(h)}(m_{(0)} \lhd (h_{(1)} S(h_{(2)}))) \lhd S(m_{(1)})) = \epsilon(h)\Sigma_{(m)}(m_{(0)} \lhd 1) \lhd S(m_{(1)})\\
&= \epsilon(h)\, \psi(m).
\end{aligned}$$

This shows that $\psi(m) \in M^{coH}$ and $\psi(m \lhd h) = \epsilon(h)\psi(m)$ for all $m \in M$ and $h \in H$. For all coinvariants $n \in M^{coH}$ we then obtain

$$\psi(n) = \lhd \circ (\mathrm{id}\otimes S) \circ \delta(n) = \lhd \circ (\mathrm{id}\otimes S)(n\otimes 1) = n \lhd S(1) = n \lhd 1 = n.$$

This implies $(\psi \circ \psi)(m) = \psi(m)$ for all $m \in M$ and shows that $\psi$ is a projector on $M^{coH}$.

4. We prove that the linear map

$$\chi = (\psi\otimes\mathrm{id})\circ\delta : M \to M^{coH}\otimes H, \quad m \mapsto \Sigma_{(m)} \psi(m_{(0)})\otimes m_{(1)} = \Sigma_{(m)} m_{(0)}\lhd S(m_{(1)})\otimes m_{(2)}$$

is inverse to $\phi$. Using the fact that $\psi$ is a projector on $M^{coH}$ that is invariant under the action of $H$ on $M$, we obtain for all $m \in M$, $n \in M^{coH}$ and $h \in H$

$$\phi \circ \chi(m) = \phi(\Sigma_{(m)} m_{(0)} \lhd S(m_{(1)}) \otimes m_{(2)}) = \Sigma_{(m)}(m_{(0)} \lhd S(m_{(1)})) \lhd m_{(2)}$$
$$= \Sigma_{(m)} m_{(0)} \lhd (S(m_{(1)}) m_{(2)}) = \Sigma_{(m)} m_{(0)} \lhd (\epsilon(m_{(1)})1) = \Sigma_{(m)}(\epsilon(m_{(1)}) m_{(0)}) \lhd 1 = m$$

$$\chi \circ \phi(n \otimes h) = \chi(n \lhd h) = (\psi \otimes \mathrm{id})(\delta(n \lhd h)) = (\psi \otimes \mathrm{id})(\delta(n) \lhd h) = (\psi \otimes \mathrm{id})((n \otimes 1) \lhd h)$$
$$= \Sigma_{(h)} \psi(n \lhd h_{(1)}) \otimes h_{(2)} = \Sigma_{(h)} \epsilon(h_{(1)}) \psi(n) \otimes h_{(2)} = \psi(n) \otimes (\Sigma_{(h)} \epsilon(h_{(1)}) h_{(2)}) = n \otimes h.$$

This proves that $\chi = \phi^{-1}$ and that $\phi : M^{coH} \otimes H \to M$ is an isomorphism of Hopf modules. □

**Corollary 3.2.13:** Let $H$ be a finite-dimensional Hopf algebra over $\mathbb{F}$ and $I \subset H$ a right (left) ideal and right (left) coideal in $H$. Then $I = H$ or $I = \{0\}$.

**Proof:**
If $I \subset H$ is a right ideal and a right coideal in $H$, it is a Hopf submodule of the right Hopf module $H$ from Example 3.2.11, 2. with $k \lhd h = k \cdot h$ and $\delta = \Delta : H \to H \otimes H$. The fundamental theorem of Hopf modules implies $I \cong I^{coH} \otimes H$ and hence $\dim_{\mathbb{F}}(H) \geq \dim_{\mathbb{F}}(I) = \dim_{\mathbb{F}}(I^{coH}) \cdot \dim_{\mathbb{F}} H$. Hence one has $\dim_{\mathbb{F}}(I^{coH}) = 1$ and $I = H$ or $\dim_{\mathbb{F}}(I^{coH}) = 0$ and $I = \{0\}$. The claims for left ideals and coideals follow by replacing $H$ with $H^{op,cop}$. □

We will now apply the fundamental theorem of Hopf modules to determine the dimension of the linear subspaces of left and right integrals in a finite-dimensional Hopf algebra $H$. For this we require a finite-dimensional right Hopf module whose coinvariants are the left or right integrals of the Hopf algebra $H$ or its dual $H^*$. As all Hopf algebras involved are finite-dimensional, this is equivalent to a module over the cross product $H^{*op} \# H^{cop}$ from Remark 3.2.10 whose invariants with respect to $H^*$ are the left or right integrals of $H^*$. Such a module over $H^{*op} \# H^{cop}$ is provided by the following lemma.

**Lemma 3.2.14:** Let $H$ be a finite-dimensional Hopf algebra. Then $H^*$ is a right module over the cross product $H^{*op} \# H^{cop}$ from Remark 3.2.10 with

$$\lhd : H^* \otimes (H^{*op} \# H^{cop}) \to H^*, \quad \gamma \lhd (\alpha \otimes h) = \Sigma_{(\alpha)(\gamma)} (\alpha_{(2)} \gamma_{(2)})(S(h))\, \alpha_{(1)} \gamma_{(1)}.$$

This defines a Hopf module structure on $H^*$ with $(H^*)^{coH} = I_L(H^*)$.

**Proof:**
That $(H^*, \lhd)$ is a module over the cross product $H^{*op} \# H^{cop}$ from Remark 3.2.10 follows by a direct but lengthy computation. To show that $H^*$ is a right module over $H^{*op} \# H$, we compute for $\alpha, \beta, \gamma \in H^*$ and $h, k \in H$

$$\gamma \lhd (1_{H^*} \otimes 1_H) = \Sigma_{(\gamma)} \gamma_{(2)}(1_H)\, \gamma_{(1)} = \Sigma_{(\gamma)} \epsilon(\gamma_{(2)})\, \gamma_{(1)} = \gamma$$
$$(\gamma \lhd (\alpha \otimes h)) \lhd (\beta \otimes k) = \Sigma_{(\gamma)(\alpha)}\, (\alpha_{(2)} \gamma_{(2)})(S(h))\, (\alpha_{(1)} \gamma_{(1)}) \lhd (\beta \otimes k)$$
$$= \Sigma_{(\alpha)(\beta)(\gamma)}\, (\alpha_{(2)} \gamma_{(2)})(S(h))\, (\beta_{(2)} (\alpha_{(1)} \gamma_{(1)})_{(2)})(S(k))\, \beta_{(1)} (\alpha_{(1)} \gamma_{(1)})_{(1)}$$
$$= \Sigma_{(\alpha)(\beta)(\gamma)}\, (\alpha_{(3)} \gamma_{(3)})(S(h))\, (\beta_{(2)} \alpha_{(2)} \gamma_{(2)})(S(k))\, \beta_{(1)} \alpha_{(1)} \gamma_{(1)}$$

where we used the definition of $\lhd$ in the 2nd line and to pass from the 2nd to the 3rd line and then the coassociativity in $H^*$ and the fact that the comultiplication is an algebra homomorphism to pass to the 4th line. Similarly, we compute

$$\begin{aligned}
&\gamma \lhd ((\alpha\otimes h)\cdot(\beta\otimes k)) = \Sigma_{(\beta)(h)}\, \beta_{(2)}(h_{(2)})\, \gamma \lhd (\beta_{(1)}\alpha\otimes h_{(1)}k)\\
&= \Sigma_{(\alpha)(\beta)(\gamma)(h)}\, \beta_{(2)}(h_{(2)})\, (\beta_{(1)(2)}\alpha_{(2)}\gamma_{(2)})(S(h_{(1)}k))\, \beta_{(1)(1)}\alpha_{(1)}\gamma_{(1)}\\
&= \Sigma_{(\alpha)(\beta)(\gamma)(h)}\, \beta_{(3)}(h_{(2)})\, (\beta_{(2)}\alpha_{(2)}\gamma_{(2)})(S(k)S(h_{(1)}))\, \beta_{(1)}\alpha_{(1)}\gamma_{(1)}\\
&= \Sigma_{(\alpha)(\beta)(\gamma)(h)}\, \beta_{(3)}(h_{(2)})\, (\beta_{(2)(1)}\alpha_{(2)(1)}\gamma_{(2)(1)})(S(k))(\beta_{(2)(2)}\alpha_{(2)(2)}\gamma_{(2)(2)})S(h_{(1)}))\, \beta_{(1)}\alpha_{(1)}\gamma_{(1)}\\
&= \Sigma_{(\alpha)(\beta)(\gamma)(h)}\, \beta_{(4)}(h_{(2)})\, (\beta_{(2)}\alpha_{(2)}\gamma_{(2)})(S(k))(\beta_{(3)}\alpha_{(3)}\gamma_{(3)})S(h_{(1)}))\, \beta_{(1)}\alpha_{(1)}\gamma_{(1)}\\
&= \Sigma_{(\alpha)(\beta)(\gamma)(h)}\, \beta_{(4)}(h_{(2)})\, (\beta_{(2)}\alpha_{(2)}\gamma_{(2)})(S(k))\beta_{(3)}(S(h_{(1)})_{(1)})\, (\alpha_{(3)}\gamma_{(3)})S(h_{(1)})_{(2)})\, \beta_{(1)}\alpha_{(1)}\gamma_{(1)}\\
&= \Sigma_{(\alpha)(\beta)(\gamma)(h)}\, \beta_{(4)}(h_{(2)})\, (\beta_{(2)}\alpha_{(2)}\gamma_{(2)})(S(k))\beta_{(3)}(S(h_{(1)(2)}))\, (\alpha_{(3)}\gamma_{(3)})(S(h_{(1)(1)}))\, \beta_{(1)}\alpha_{(1)}\gamma_{(1)}\\
&= \Sigma_{(\alpha)(\beta)(\gamma)(h)}\, \beta_{(3)}(S(h_{(1)(2)})h_{(2)})\, (\beta_{(2)}\alpha_{(2)}\gamma_{(2)})(S(k))(\alpha_{(3)}\gamma_{(3)})(S(h_{(1)(1)}))\, \beta_{(1)}\alpha_{(1)}\gamma_{(1)}\\
&= \Sigma_{(\alpha)(\beta)(\gamma)(h)}\, \epsilon(\beta_{(3)})\epsilon(h_{(2)})(\beta_{(2)}\alpha_{(2)}\gamma_{(2)})(S(k))(\alpha_{(3)}\gamma_{(3)})(S(h_{(1)}))\, \beta_{(1)}\alpha_{(1)}\gamma_{(1)}\\
&= \Sigma_{(\alpha)(\beta)(\gamma)}\, (\beta_{(2)}\alpha_{(2)}\gamma_{(2)})(S(k))(\alpha_{(3)}\gamma_{(3)})(S(h))\, \beta_{(1)}\alpha_{(1)}\gamma_{(1)} = \gamma \lhd ((\alpha\otimes h)\cdot(\beta\otimes k)),
\end{aligned}$$

where we used the multiplication law (23) of $H^{*op}\#H^{cop}$ in the 1st line, the definition of $\lhd$ to pass to the 2nd line, the coassociativity to pass from the 2nd to the 3rd, from the 4th to the 5th and from the 9th to the 10th line, the fact that $S: H \to H$ is an anti-algebra and anti-coalgebra homomorphism to pass from the 2nd to the 3rd and from the 6th to the 7th line, the duality between the product in $H$ and the coproduct in $H^*$ to pass form the 3rd to the 4th, the 5th to the 6th and from the 7th to the 8th line, the antipode condition in $H$ to pass from the 8th to the 9th and the counitality conditions in $H$ and $H^*$ to pass from the 9th to the 10th line.

As $H$ and $H^*$ are finite-dimensional, this defines a right $H$-Hopf module structure on $H^*$ by Remark 3.2.10 with $(\mathrm{id}\otimes\alpha)\circ\delta(\beta) = (\beta \lhd (\alpha\otimes 1_H))\otimes 1_{\mathbb{F}}$ and $\beta \lhd h = \beta \lhd (1_{H^*}\otimes h)$. An element $\beta \in H^*$ is a coinvariant for this Hopf module structure if and only if $\delta(\beta) = \beta\otimes 1_H$ which is equivalent to $(\beta \lhd (\alpha\otimes 1_H))\otimes 1_{\mathbb{F}} = (\mathrm{id}\otimes\alpha)\circ\delta(\beta) = \beta\otimes\alpha(1_H) = \epsilon(\alpha)\beta\otimes 1_{\mathbb{F}}$ for all $\alpha \in H^*$. As we have $\beta \lhd (\alpha\otimes 1_H) = \alpha\beta$, this is equivalent to $\beta \in I_L(H^*)$. □

By combining this Lemma with the fundamental theorem of Hopf modules, we can determine the dimension of the space of left integrals for $H^*$ and and $H \cong H^{**}$ for any finite-dimensional Hopf algebra $H$. As the module structure from Lemma 3.2.14 involves the antipode of $H$, this also allows us to draw conclusions about the dimension of the kernel of the antipode and to conclude that for any finite-dimensional Hopf algebra $H$ the antipode is invertible. Moreover, we obtain an isomorphism of $H$-right modules between $H$ and $H^*$, the so-called *Frobenius map*.

**Theorem 3.2.15:** Let $H$ be a finite-dimensional Hopf algebra. Then:

1. $\dim_{\mathbb{F}} I_L(H) = \dim_{\mathbb{F}} I_R(H) = 1$.
2. The antipode of $H$ is bijective with $S^{\pm 1}(I_L(H)) = I_R(H)$ and $S^{\pm 1}(I_R(H)) = I_L(H)$.
3. For any $\lambda \in I_L(H^*) \setminus \{0\}$, the **Frobenius map**

$$\phi_\lambda : H \to H^*, \quad h \mapsto S(h) \rhd^*_R \lambda = \Sigma_{(\lambda)}\lambda_{(2)}(S(h))\, \lambda_{(1)}$$

is an isomorphism of right $H$-modules with respect to the right regular action of $H$ on itself and the right action $\lhd : H^*\otimes H \to H^*$, $\alpha\otimes h \mapsto S(h) \rhd^*_R \alpha$.

**Proof:**
Equip $H^*$ with the $H$-Hopf module structure from Lemma 3.2.14. Then $(H^*)^{coH} \cong I_L(H^*)$ and

$(H^{*coH})\otimes H \cong H^*$ by the fundamental theorem of Hopf modules. As $H$ is finite-dimensional, this implies $\dim_{\mathbb{F}}(H)\cdot\dim_{\mathbb{F}}(I_L(H^*)) = \dim_{\mathbb{F}}(H^*) = \dim_{\mathbb{F}}(H)$ and hence $\dim_{\mathbb{F}}(I_L(H^*)) = 1$. By exchanging $H \cong H^{**}$ and $H^*$, we also obtain $\dim_{\mathbb{F}}(I_L(H)) = 1$. By the fundamental theorem of Hopf modules the linear map

$$\lhd : I_L(H^*)\otimes H \to H^*, \quad \lambda\otimes h \mapsto \lambda \lhd h = S(h) \rhd_R^* \lambda = \Sigma_{(\lambda)}\lambda_{(2)}(S(h))\,\lambda_{(1)}$$

is an isomorphism of Hopf modules. As $\dim_{\mathbb{F}}(I_L(H^*)) = 1$, this implies that the Frobenius map $\phi_\lambda$ is a linear isomorphism for all $\lambda \in H^* \setminus \{0\}$. Moreover, one has

$$\begin{aligned}\phi_\lambda(k \lhd h) &= \phi_\lambda(kh) = \Sigma_{(\lambda)}\lambda_{(2)}(S(kh))\,\lambda_{(1)} = \Sigma_{(\lambda)}\lambda_{(2)}(S(h)S(k))\,\lambda_{(1)}\\ &= \Sigma_{(\lambda)}\lambda_{(2)}(S(h))\,\lambda_{(3)}(S(k))\lambda_{(1)} = S(h) \rhd_R^* \phi_\lambda(\Sigma_{(\lambda)}\lambda_{(2)}(S(k))\,\lambda_{(1)}) = \phi_\lambda(k) \lhd h\end{aligned}$$

and hence $\phi_\lambda$ is an isomorphism of right $H$-modules. If $h \in \ker(S)$, then $\phi_\lambda(h) = S(h) \rhd_R^* \lambda = 0$ and hence $\ker(S) \subset \ker(\phi_\lambda) = \{0\}$. This shows that the antipode is injective. Because $H$ is finite-dimensional with $\dim_{\mathbb{F}} H = \dim_{\mathbb{F}} H^*$, it follows that $S$ is bijective. As $S^{\pm 1} : H \to H^{op,cop}$ is an algebra and coalgebra homomorphism, one has for all $\ell \in I_L(H)$, $\ell' \in I_R(H)$ and $h \in H$

$$\begin{aligned} S^{\pm1}(\ell)\cdot h &= S^{\pm1}(S^{\mp1}(h)\cdot \ell) = \epsilon(S^{\mp1}(h))S^{\pm1}(\ell) = \epsilon(h)S^{\pm1}(\ell) && \Rightarrow \quad S^{\pm1}(\ell) \in I_R(H)\\ h\cdot S^{\pm1}(\ell') &= S^{\pm1}(\ell'\cdot S^{\mp1}(h)) = \epsilon(S^{\mp1}(h))\,S^{\pm1}(\ell') = \epsilon(h)S^{\pm1}(\ell') && \Rightarrow \quad S^{\pm1}(\ell') \in I_L(H).\end{aligned}$$

This shows that $S^{\pm1}(I_L(H)) = I_R(H)$ and $S^{\pm1}(I_R(H)) = I_L(H)$. □

Theorem 3.2.15 clarifies the existence and uniqueness of left and right integrals for finite-dimensional Hopf algebras $H$. It shows that every finite-dimensional Hopf algebra $H$ has non-trivial left and right integrals and that the vector spaces of left and right integrals are one-dimensional. One can show that the finite-dimensionality of $H$ is not only a sufficient but also a necessary condition for the existence of non-trivial left and right integrals: if a Hopf algebra $H$ has a left or right integral integral $\ell \neq 0$, then it follows that $H$ is finite-dimensional. We will not prove this statement here. A proof is given in [R, Prop. 10.2.1].

In particular, Theorem 3.2.15 implies that a finite-dimensional Hopf algebra $H$ is unimodular if and only if its antipode maps each left or right integral to a scalar multiple of itself. An alternative criterion for the unimodularity of $H$ is obtained by realising that the linear sub-space space of left integrals is invariant under right multiplication with $H$. This implies that multiplying a left integral $\ell$ on the right by an element $h \in H$ yields a scalar multiple $\alpha_h\,\ell$ of the left integral $\ell$. The linear map that assigns to each element $h \in H$ the scalar $\alpha_h \in \mathbb{F}$ has interesting properties and gives a criterion for the unimodularity of $H$.

**Proposition 3.2.16:** Let $H$ be a finite-dimensional Hopf algebra over $\mathbb{F}$. Then:

1. There is a unique element $\alpha \in H^*$ with $\ell\cdot h = \alpha(h)\ell$ for all $h \in H$ and $\ell \in I_L(H)$.
2. One has $h\cdot \ell' = \alpha^{-1}(h)$ for all $h \in H$ and $\ell' \in I_R(H)$.
3. The element $\alpha \in H^*$ is grouplike and is called the **modular element** of $H$.
4. The Hopf algebra $H$ is unimodular if and only if $\alpha = \epsilon$.

**Proof:**
As $\dim_{\mathbb{F}} I_L(H) = 1$ by Theorem 3.2.15, we have $I_L(H) = \mathrm{span}_{\mathbb{F}}\{\ell\}$ for all non-trivial left integrals $\ell \in I_L(H)\setminus\{0\}$. For every non-trivial left integral $\ell \in I_L(H)\setminus\{0\}$ and $h \in H$, one

has $\ell h \in I_L(H)$ since $k \cdot (\ell \cdot h) = (k \cdot \ell) \cdot h = \epsilon(k)\, \ell \cdot h$ for all $k \in H$. As $\dim_{\mathbb{F}} I_L(H) = 1$ and the multiplication is linear, this implies $\ell h = \alpha(h)\ell$ for some element $\alpha \in H^*$. We have

$$\alpha(hk)\,\ell = \ell hk = (\ell h)k = (\alpha(h)\ell)k = \alpha(h)\,\ell k = \alpha(h)\alpha(k)\ell \qquad \ell = \ell 1_H = \ell = \alpha(1_H)\,\ell.$$

for all $h, k \in H$, and this shows that $\alpha : H \to \mathbb{F}$ is an algebra homomorphism and hence a grouplike element of $H^*$ by Example 2.4.2. If $\ell' \in I_L(h)$ is another left integral, then $\ell' = \mu\ell$ for some $\mu \in \mathbb{F}$, and we have $\ell' h = \mu\,\ell h = \mu\alpha(h)\,\ell = \alpha(h)\,\ell'$ for all $h \in H$. This shows that the identity $\ell \cdot h = \alpha(h)\,\ell$ holds for all $\ell \in I_L(H)$. As we have $I_R(H) = S^{\pm}(I_L(H))$ and $I_L(H) = S^{\pm}(I_R(H))$ by Theorem 3.2.15 and $S(\alpha) = \alpha \circ S = \alpha^{-1}$ for any $\alpha \in \mathrm{Gr}(H^*)$, we obtain

$$h \cdot \ell' = S^{-1}(S(h)) \cdot S^{-1}(S(\ell)) = S^{-1}(S(\ell) \cdot S(h)) = \alpha(S(h))\, S^{-1}(S(\ell)) = \alpha^{-1}(h)\,\ell'$$

for all right integrals $\ell' \in I_R(H)$. Finally, the Hopf algebra $H$ is unimodular if and only if $I_L(H) = I_R(H)$. This is the case if and only if $\ell h = \alpha(h)\ell = \epsilon(h)\ell$ for all $h \in H$ and left integrals $\ell \in I_L(H)$, which is equivalent to $\alpha = \epsilon$ since $\dim_{\mathbb{F}} I_L(H) = 1$. □

## 3.3 Integrals and Frobenius algebras

Theorem 3.2.15 not only allows us to draw conclusions about the left and right integrals of a finite-dimensional Hopf algebra $H$ and to obtain criteria for its unimodularity but also relates finite-dimensional Hopf algebras $H$ to another type of algebras, namely *Frobenius algebras*. Frobenius algebras play an important role in modern mathematical physics, in particular in *conformal field theories* and *topological quantum field theories*. A Frobenius algebra can be viewed as an algebra $A$ over $\mathbb{F}$ with a non-degenerate bilinear form $\kappa : A \times A \to \mathbb{F}$ that satisfies a compatibility condition with the algebra multiplication. Using tensor products over $\mathbb{F}$, we can also interpret $\kappa$ as a linear map $\kappa : A \otimes A \to \mathbb{F}$, and obtain the following definition.

**Definition 3.3.1:** A **Frobenius algebra** over $\mathbb{F}$ is an algebra $A$ over $\mathbb{F}$ together with a linear map $\kappa : A \otimes A \to \mathbb{F}$, $a \otimes b \mapsto \kappa(a \otimes b)$, the **Frobenius form** such that

1. $\kappa((h \cdot k) \otimes l) = \kappa(h \otimes (k \cdot l))$ for all $h, k, l \in H$.
2. $\kappa$ is **non-degenerate**: $\kappa(a \otimes b) = 0$ for all $a \in A$ implies $b = 0$

Note that the condition on the Frobenius form implies $\kappa(a \otimes b) = \kappa(ab \otimes 1) = \kappa(1 \otimes ab)$ for all $a, b \in A$, and hence the Frobenius form $\kappa$ is determined uniquely by its values on elements $1 \otimes a$ or $a \otimes 1$. Hence, every Frobenius form $\kappa : A \otimes A \to \mathbb{F}$ arises from a linear form $\lambda : A \to \mathbb{F}$ with $\lambda(a) = \kappa(a \otimes 1)$ and $\kappa(a \otimes b) = \lambda(a \cdot b)$ for all $a, b \in A$. It also follows from the definition that the change of the Frobenius form under a flip of its arguments is given by an automorphism of $A$, the so-called *Nakayama automorphism*. Symmetric Frobenius algebras are then characterised by the condition that the Nakayama automorphism is the identity map.

**Proposition 3.3.2:** Let $(A, \kappa)$ be a finite-dimensional Frobenius algebra over $\mathbb{F}$. Then there is a unique linear map $\rho : A \to A$ with $\kappa(a \otimes b) = \kappa(\rho(b) \otimes a)$ for all $a, b \in A$. The map $\rho$ is an algebra automorphism and is called the **Nakayama automorphism**. The Frobenius algebra $(A, \kappa)$ is called **symmetric** if $\rho = \mathrm{id}_A$.

**Proof:**
The non-degeneracy of the Frobenius form implies that the linear map

$$\chi_\kappa : A \to A^*, \quad a \mapsto \kappa_a \quad \text{with} \quad \kappa_a(b) = \kappa(b\otimes a)\ \forall b \in B$$

is injective. As $A$ is finite-dimensional and $\dim_{\mathbb{F}} A = \dim_{\mathbb{F}} A^*$ it follows that $\chi_\kappa : A \to A^*$ is a linear isomorphism. The bijectivity of $\chi_\kappa$ then implies that for each element $a \in A$ there is a unique element $\rho'(a) \in A$ with $\chi_\kappa(\rho'(a)) = {}_a\kappa : A \to \mathbb{F}$, $b \mapsto \kappa(a\otimes b)$. Hence, we obtain a unique linear map $\rho' : A \to A$, $a \mapsto \rho(a)$ with $\kappa(a\otimes b) = \kappa(b\otimes\rho'(a))$ for all $a, b \in A$. If $b \in \ker(\rho')$ then $\kappa(a\otimes b) = \kappa(b\otimes\rho'(a)) = \kappa(\rho'(a)\otimes\rho'(b)) = 0$ for all $a \in A$ and the non-degeneracy of the Frobenius form implies $b = 0$. This shows that $\rho'$ is a linear isomorphism. Its inverse $\rho = \rho'^{-1} : A \to A$ satisfies $\kappa(\rho(b)\otimes a) = \kappa(a\otimes\rho'(\rho(b))) = \kappa(a\otimes b)$ for all $a, b \in A$. With the condition on the Frobenius form, we then obtain for all $a, b, c \in A$

$$\kappa(\rho(1)\otimes c) = \kappa(c\otimes 1) = \kappa(1\otimes c)$$
$$\kappa(\rho(ab)\otimes c) = \kappa(c\otimes ab) = \kappa(ca\otimes b) = \kappa(\rho(b)\otimes ca) = \kappa(\rho(b)c\otimes a) = \kappa(\rho(a)\otimes\rho(b)c) = \kappa(\rho(a)\rho(b)\otimes c).$$

Due to the non-degeneracy of $\kappa$ this implies $\rho(1) = 1$ and $\rho(ab) = \rho(a)\cdot\rho(b)$ for all $a, b \in A$. This shows that $\rho$ is an algebra automorphism with $\kappa(\rho(b)\otimes a) = \kappa(a\otimes b)$ for all $a, b \in A$. □

**Example 3.3.3:**

1. Let $A \in \mathrm{Mat}(n\times n, \mathbb{F})$ a matrix of rank $n$. Then $\kappa : \mathrm{Mat}(n\times n, \mathbb{F}) \otimes \mathrm{Mat}(n\times n, \mathbb{F}) \to \mathbb{F}$, $\kappa(M\otimes N) = \mathrm{Tr}(M\cdot N\cdot A)$ is a Frobenius form on $\mathrm{Mat}(n\times n, \mathbb{F})$. The Nakayama automorphism is given by $\rho : \mathrm{Mat}(n\times n, \mathbb{F}) \to \mathrm{Mat}(n\times n, \mathbb{F})$, $N \mapsto ANA^{-1}$ and the Frobenius algebra is symmetric if and only if $A = \lambda 1_n$ for some $\lambda \in \mathbb{F}$.

2. Let $G$ be a finite group. Then $\kappa : \mathbb{F}[G]\otimes\mathbb{F}[G] \to \mathbb{F}$, $\kappa(g\otimes h) = \delta_e(g\cdot h)$ is a Frobenius form on $\mathbb{F}[G]$ and $\kappa' : \mathrm{Fun}(G, \mathbb{F})\otimes\mathrm{Fun}(G, \mathbb{F}) \to \mathbb{F}$, $\kappa'(f\otimes h) = \Sigma_{g\in G} f(g)h(g)$ is a Frobenius form on $\mathrm{Fun}(G, \mathbb{F})$. Both Frobenius algebras are symmetric.

3. If $(A, \kappa)$ is a Frobenius algebra and $a \in A$ is invertible, then

$$\kappa_a : A\otimes A \to \mathbb{F}, \quad b\otimes c \mapsto \kappa(b\otimes ca) \qquad \kappa'_a : A\otimes A \to \mathbb{F}, \quad b\otimes c \mapsto \kappa(ab\otimes c)$$

are Frobenius forms on $A$ as well. One says they are obtained by **twisting** with $a$.

4. The tensor product $A\otimes B$ of two Frobenius algebras $(A, \kappa_A)$ and $(B, \kappa_B)$ over $\mathbb{F}$ has a natural Frobenius algebra structure with the Frobenius form

$$\kappa : (A\otimes B)\otimes(A\otimes B) \to \mathbb{F}, \quad (a\otimes b)\otimes(a'\otimes b') \mapsto \kappa_A(a\otimes a')\kappa_B(b\otimes b').$$

5. If $(A, \kappa)$ is a commutative Frobenius algebra, then $A$ is symmetric. However, the previous examples show that not every symmetric Frobenius algebra is commutative.

We will now show that for every finite-dimensional Hopf algebra $H$ non-trivial left and right integrals $\lambda \in H^*$ define Frobenius forms on $H$. While every element $\lambda \in H^*$ gives rise to a linear map $\kappa_\lambda : H\otimes H \to \mathbb{F}$, $h\otimes k \mapsto \lambda(h\cdot k)$ that satisfies the first condition in Definition 3.3.1, the non-degeneracy condition is more subtle. It is equivalent to the condition that the linear map $\psi_\kappa : H \to H^*$, $h \mapsto \kappa_h$ with $\kappa_h(k) = \kappa(k\otimes h)$ for all $k \in H$ from the proof of Proposition 3.3.2 is a linear isomorphism. From the definition of $\kappa$ we find that $\kappa_h = h \rhd^*_R \lambda$ and by composing $\phi_\kappa$ with the antipode of $H$, we obtain the Frobenius map from Theorem 3.2.15. This shows that the non-degeneracy condition is satisfied for all non-trivial integrals $\lambda \in I_L(H^*)$.

**Theorem 3.3.4:** Let $H$ be a finite-dimensional Hopf algebra and $0 \neq \lambda \in I_L(H^*)$ a left integral. Then $H$ is a Frobenius algebra with the linear map

$$\kappa : H \otimes H \to \mathbb{F}, \quad \kappa(h \otimes k) = \lambda(h \cdot k).$$

**Proof:**
The linearity of $\kappa$ follows directly from the linearity of the evaluation and the multiplication, and the Frobenius condition follows from the associativity of the product in $H$

$$\kappa((h \cdot k) \otimes l) = \lambda((h \cdot k) \cdot l) = \lambda(h \cdot (k \cdot l)) = \kappa(h \otimes (k \cdot l)) \qquad \forall h, k, l \in H.$$

That $\kappa$ is non-degenerate follows because by Theorem 3.2.15 the antipode $S : H \to H$ and the Frobenius map $\phi_\lambda : H \to H^*$, $h \mapsto \phi_\lambda(h) = \Sigma_{(\lambda)} \lambda_{(2)}(S(h))\, \lambda_{(1)}$ are linear isomorphisms and

$$\kappa(h \otimes k) = \lambda(h \cdot k) = \Sigma_{(\lambda)} \lambda_{(1)}(h) \lambda_{(2)}(k) = \phi_\lambda(S^{-1}(k))(h).$$

Hence $\kappa(h \otimes k) = 0$ for all $h \in H$ implies $\phi_\lambda(S^{-1}(k)) = 0$, and this implies $k = 0$. □

It remains to determine the Nakayama automorphism for the Frobenius algebra in Theorem 3.3.4. In particular, this will allow us to derive conditions under which the Frobenius algebra defined by an integral in a finite-dimensional Hopf algebra is symmetric. This is not only relevant in the context of Frobenius algebras but will also lead to conclusions about the properties of the antipode. To determine the Nakayama automorphism, we use the concept of *dual bases*, which is useful whenever one considers a finite-dimensional algebra $A$ with a linear map $\lambda : A \to \mathbb{F}$.

**Lemma 3.3.5:** Let $A$ be an algebra of dimension $n \in \mathbb{N}$, $\lambda \in A^*$ and $r_1, ..., r_n \in A$ and $l_1, ..., l_n \in A$. Then the following two conditions are equivalent

(i) $a = \Sigma_{i=1}^n \lambda(l_i \cdot a)\, r_i$ for all $a \in A$.
(ii) $a = \Sigma_{i=1}^n \lambda(a \cdot r_i)\, l_i$ for all $a \in A$.

If these conditions are satisfied, then $\{r_1, ..., r_n\}$ and $\{l_1, ..., l_n\}$ are called **dual bases** for $\lambda$.

**Proof:**
This follows by a direct computation using matrices. If $r_1, .., r_n$ and $l_1, ..., l_n$ satisfy condition (i), then $\{r_1, ..., r_n\}$ is a basis for $A$, and we have for all $j \in \{1, ..., n\}$

$$\begin{aligned} r_j &= \Sigma_{i=1}^n \lambda(l_i \cdot r_j) r_i \quad \Rightarrow \quad \lambda(l_i \cdot r_j) = \delta_{ij} \quad \forall i, j \in \{1, ..., n\} \\ l_j &= \Sigma_{i=1}^n \lambda(l_i \cdot l_j) r_i \quad \Rightarrow \quad \delta_{jk} = \lambda(l_j r_k) = \Sigma_{i=1}^n \lambda(l_i \cdot l_j) \lambda(r_i r_k) \quad \forall j, k \in \{1, ..., n\}. \end{aligned} \tag{24}$$

This implies that the matrix $M \in \mathrm{Mat}(n \times n, \mathbb{F})$ with entries $M_{ij} = \lambda(r_j \cdot r_i)$ is invertible, and its inverse has entries $M^{-1}_{ij} = \lambda(l_i \cdot l_j)$. With this result, we obtain (ii):

$$\begin{aligned} \Sigma_{i=1}^n \lambda(a \cdot r_i) l_i &\overset{\text{(i)}}{=} \Sigma_{i,j=1}^n \lambda(l_j \cdot a)\, \lambda(r_j \cdot r_i) l_i \overset{(24)}{=} \Sigma_{i,j,k=1}^n \lambda(l_j \cdot a)\, \lambda(r_j \cdot r_i) \lambda(l_k \cdot l_i) r_k \\ &= \Sigma_{i,j,k=1}^n \lambda(l_j \cdot a) M_{ij} M^{-1}_{ki} = \Sigma_{j,k=1}^n \lambda(l_j \cdot a) \delta_{jk}\, r_k = \Sigma_{j=1}^n \lambda(l_j \cdot a)\, r_j \overset{\text{(i)}}{=} a. \end{aligned}$$

The proof of (ii)⇒(i) is analogous. □

We now consider a finite-dimensional Hopf algebra $H$, fix a non-trivial right cointegral $\lambda \in C_R(H) = I_R(H^*)$ and show that every left integral $\ell \in I_L(H)$ with $\lambda(\ell) = 1$ defines two pairs of dual bases for $\lambda$. Note that such a left integral $\ell \in I_L(H)$ exists if and only if $\lambda|_{I_L(H)} \neq \{0\}$ and that this condition does not depend on the choice of $\lambda$, since $\dim_{\mathbb{F}} I_R(H^*) = 1$. These pairs of dual bases will allow us to determine the Nakayama automorphism for the associated Frobenius algebra $(H, \kappa)$ with $\kappa(h \otimes k) = \lambda(h \cdot k)$.

**Proposition 3.3.6:** Let $H$ be a finite-dimensional Hopf algebra, $\alpha \in \mathrm{Gr}(H^*)$ and $a \in \mathrm{Gr}(H)$ the modular elements for $H$ and $H^*$ from Proposition 3.2.16 and $\lambda \in I_R(H^*) \setminus \{0\}$. Then:

1. There is a left integral $\ell \in I_L(H)$ with $\lambda(\ell) = 1$.
2. The element $(\mathrm{id} \otimes S)(\Delta(\ell)) = \Sigma_{(\ell)}\, \ell_{(1)} \otimes S(\ell_{(2)})$ defines dual bases for $\lambda$:

$$h = \Sigma_{(\ell)}\, \lambda(h\ell_{(1)})\, S(\ell_{(2)}) = \Sigma_{(\ell)}\, \lambda(S(\ell_{(2)})h)\, \ell_{(1)} \qquad \forall h \in H. \tag{25}$$

3. The element $(a^{-1} \otimes 1) \cdot (S^{-1} \otimes \mathrm{id}) \circ \Delta(\ell) = \Sigma_{(\ell)}\, a^{-1} S^{-1}(\ell_{(1)}) \otimes \ell_{(2)}$ defines dual bases for $\lambda$:

$$h = \Sigma_{(\ell)} \lambda(h\ell_{(2)})\, a^{-1} S^{-1}(\ell_{(1)}) = \Sigma_{(\ell)} \lambda(a^{-1} S^{-1}(\ell_{(1)}) h)\, \ell_{(2)} \qquad \forall h \in H. \tag{26}$$

4. The Nakayama automorphism for $(H, \kappa)$ with $\kappa(h \otimes k) = \lambda(h \cdot k)$ is given by

$$\rho(h) = \Sigma_{(h)}\, \alpha(h_{(1)})\, S^2(h_{(2)}) = a^{-1} \cdot (\Sigma_{(h)}\, \alpha(h_{(2)})\, S^{-2}(h_{(1)})) \cdot a. \tag{27}$$

**Proof:**
1. Let $\lambda \in I_R(H^*) \setminus \{0\}$ be a right integral and $\ell \in I_L(H) \setminus \{0\}$ a left integral. Then by Theorem 3.2.15 the Frobenius map $\phi_\lambda : H \to H^*$, $\phi_\lambda(h) = \Sigma_{(\lambda)} \lambda_{(2)}(S(h))\, \lambda_{(1)}$ is a linear isomorphism and one has $\phi_\lambda(S^{-1}(\ell)) \neq 0$. Hence there is a $k \in H$ with

$$0 \neq \phi_\lambda(S^{-1}(\ell))k = \Sigma_{(\lambda)} \lambda_{(2)}(\ell)\, \lambda_{(1)}(k) = \lambda(k \cdot \ell) = \epsilon(k)\lambda(\ell),$$

and this implies $\lambda(\ell) \neq 0$. By multiplying $\ell$ with an element $\mu \in \mathbb{F} \setminus \{0\}$, we obtain $\lambda(\ell) = 1$.

2. Let now $\ell \in I_L(H)$, $\lambda \in I_R(H^*)$ be left and right integrals with $\lambda(\ell) = 1$ and $\alpha \in \mathrm{Gr}(H^*)$ the modular element for $H$ from Proposition 3.2.16. Then we obtain for all $h \in H$

$$\begin{aligned} &\Sigma_{(\ell)}\, h\ell_{(1)} \otimes S(\ell_{(2)}) = \Sigma_{(\ell)}\, \ell_{(1)} \otimes S(\ell_{(2)})h \\ &\Sigma_{(\ell)}\, \ell_{(1)}h \otimes S(\ell_{(2)}) = \Sigma_{(\ell)(h)}\, \alpha(h_{(1)})\, \ell_{(1)} \otimes S^2(h_{(2)})S(\ell_{(2)}). \end{aligned} \tag{28}$$

This follows by a direct computation using the fact that $\ell \in I_L(H)$ is a left integral, which implies $\ell \cdot h = \alpha(h)\, \ell$ for all $h \in H$, and the counitality and antipode condition:

$$\begin{aligned} &\Sigma_{(\ell)}\, h\ell_{(1)} \otimes S(\ell_{(2)}) = \Sigma_{(\ell)(h)} h_{(1)}\ell_{(1)} \otimes S(\epsilon(h_{(2)})\ell_{(2)}) = \Sigma_{(\ell)(h)}\, h_{(1)}\ell_{(1)} \otimes S(S^{-1}(h_{(3)})h_{(2)}\ell_{(2)}) \\ &= \Sigma_{(\ell)(h)}\, h_{(1)}\ell_{(1)} \otimes S(h_{(2)}\ell_{(2)})h_{(3)} = \Sigma_{(h)} (\mathrm{id} \otimes S)(\Delta(h_{(1)}\ell)) \cdot (1 \otimes h_{(2)}) \\ &= \Sigma_{(h)} \epsilon(h_{(1)})(\mathrm{id} \otimes S)(\Delta(\ell)) \cdot (1 \otimes h_{(2)}) = (\mathrm{id} \otimes S)(\Delta(\ell)) \cdot (1 \otimes h) = \Sigma_{(\ell)} \ell_{(1)} \otimes S(\ell_{(2)})h \end{aligned}$$

$$\begin{aligned} &\Sigma_{(\ell)}\, \ell_{(1)}h \otimes S(\ell_{(2)}) = \Sigma_{(\ell)(h)} \ell_{(1)}h_{(1)} \otimes S(\ell_{(2)}\epsilon(h_{(2)})) = \Sigma_{(\ell)(h)}\, \ell_{(1)}h_{(1)} \otimes S(\ell_{(2)}h_{(2)}S(h_{(3)})) \\ &= \Sigma_{(\ell)(h)}\, \ell_{(1)}h_{(1)} \otimes S^2(h_{(3)})S(\ell_{(2)}h_{(2)}) = \Sigma_{(h)}\, (1 \otimes S^2(h_{(2)})) \cdot (\mathrm{id} \otimes S)(\Delta(\ell h_{(1)})) \\ &= \Sigma_{(h)} \alpha(h_{(1)})(1 \otimes S^2(h_{(2)})) \cdot (\mathrm{id} \otimes S)(\Delta(\ell)) = \Sigma_{(\ell)(h)} \alpha(h_{(1)})\ell_{(1)} \otimes S^2(h_{(2)})S(\ell_{(2)}) \end{aligned}$$

As $\lambda \in I_R(H^*)$ is a right integral, we have $(\lambda \otimes \mathrm{id})(\Delta(h)) = \Sigma_{(h)}\, \lambda(h_{(1)}) \otimes h_{(2)} = \lambda(h) \otimes 1$ for all $h \in H$. Combining this with the condition $\lambda(\ell) = 1$ and the first identity in (28) we obtain

$$\Sigma_{(\ell)}\, \lambda(h\ell_{(1)})\, S(\ell_{(2)}) = \Sigma_{(\ell)}\, \lambda(\ell_{(1)})\, (S(\ell_{(2)})h) = \lambda(\ell)\, S(1) \cdot h = h.$$

This proves the first identity in (25). Moreover, if $(1\otimes S)(\Delta(\ell)) = \Sigma_{(\ell)}\, \ell_{(1)}\otimes S(\ell_{(2)}) = \Sigma_{i=1}^n r_i\otimes l_i$ with $r_i, l_i \in H$, then $\{r_1, ..., r_n\}$ and $\{l_1, ..., l_n\}$ are dual bases for $\lambda$. The second identity in (25) then follows from Lemma 3.3.5.

3. In analogy to 2. we derive the formulas

$$\begin{aligned}&\Sigma_{(\ell)}\, S^{-1}(\ell_{(1)})h\otimes\ell_{(2)} = \Sigma_{(\ell)}\, S^{-1}(\ell_{(1)})\otimes h\ell_{(2)}\\ &\Sigma_{(\ell)}\, \alpha(h_{(2)})S^{-2}(h_{(1)})\ell_{(1)}\otimes\ell_{(2)} = \Sigma_{(\ell)(h)}\, S^{-1}(\ell_{(1)})\otimes\ell_{(2)}h,\end{aligned} \tag{29}$$

which follow again by a direct computation from the fact that $\ell \in I_L(H)$:

$$\begin{aligned}&\Sigma_{(\ell)}\, S^{-1}(\ell_{(1)})\otimes h\ell_{(2)} = \Sigma_{(\ell)(h)}S^{-1}(\epsilon(h_{(1)})\ell_{(1)})\otimes h_{(2)}\ell_{(2)}\\ &= \Sigma_{(\ell)(h)}S^{-1}(S(h_{(1)})h_{(2)}\ell_{(1)})\otimes h_{(3)}\ell_{(2)} = \Sigma_{(\ell)(h)}S^{-1}(h_{(2)}\ell_{(1)})h_{(1)}\otimes h_{(3)}\ell_{(2)}\\ &= \Sigma_{(h)}(S^{-1}\otimes\mathrm{id})(\Delta(h_{(2)}\cdot\ell))\cdot(1\otimes h_{(1)}) = \Sigma_{(h)}\epsilon(h_{(2)})(S^{-1}\otimes\mathrm{id})(\Delta(\ell))\cdot(h_{(1)}\otimes 1)\\ &= (S^{-1}\otimes\mathrm{id})(\Delta(\ell))\cdot(h\otimes 1) = \Sigma_{(\ell)}\, S^{-1}(\ell_{(1)})h\otimes\ell_{(2)}\end{aligned}$$

$$\begin{aligned}&\Sigma_{(\ell)}\, S^{-1}(\ell_{(1)})\otimes\ell_{(2)}h = \Sigma_{(\ell)(h)}S^{-1}(\ell_{(1)}\epsilon(h_{(1)}))\otimes\ell_{(2)}h_{(2)}\\ &= \Sigma_{(\ell)(h)}\, S^{-1}(\ell_{(1)}h_{(2)}S^{-1}(h_{(1)}))\otimes\ell_{(2)}h_{(3)} = \Sigma_{(\ell)(h)}\, S^{-2}(h_{(1)})S^{-1}(\ell_{(1)}h_{(2)})\otimes\ell_{(2)}h_{(3)}\\ &= \Sigma_{(h)}\, (S^{-2}(h_{(1)})\otimes 1)\cdot(S^{-1}\otimes\mathrm{id})(\Delta(\ell\cdot h_{(2)}))\\ &= \Sigma_{(h)}\, \alpha(h_{(2)})\,(S^{-2}(h_{(1)})\otimes 1)\cdot(S^{-1}\otimes\mathrm{id})(\Delta(\ell)) = \Sigma_{(\ell)}\alpha(h_{(2)})\, S^{-2}(h_{(1)})S^{-1}(\ell_{(1)})\otimes\ell_{(2)}.\end{aligned}$$

Let $a \in \mathrm{Gr}(H)$ be the modular element for $H^*$. As $\lambda$ is a right integral, we have $\beta\cdot\lambda = \beta(a^{-1})\lambda$ for all $\beta \in H^*$ by Proposition 3.2.16. This implies $\beta(a^{-1})\lambda(h) = (\beta\cdot\lambda)(h) = (\beta\otimes\lambda)(\Delta(h))$ and $(\mathrm{id}\otimes\lambda)(h) = \Sigma_{(h)}\, h_{(1)}\otimes\lambda(h_{(2)}) = a^{-1}\otimes h$ for all $h \in H$. Multiplying with $a$ then yields

$$\Sigma_{(h)}\, h_{(1)}a\otimes\lambda(h_{(2)}) = (a^{-1}\otimes h)\cdot(a\otimes 1) = 1\otimes h.$$

By combining this formula with the first formula in (29) we obtain for all $h \in H$

$$\Sigma_{(\ell)}\lambda(h\ell_{(2)})a^{-1}S^{-1}(\ell_{(1)}) = \Sigma_{(\ell)}\lambda(\ell_{(2)})S^{-1}(\ell_{(1)}a)h = \big(\Sigma_{(\ell)}\lambda(\ell_{(2)})S^{-1}(\ell_{(1)}a)\big)\cdot h = \lambda(\ell)\, S^{-1}(1)h = h.$$

This proves the first identity in (26) and shows that if $\Sigma_{(\ell)}\, aS^{-1}(\ell_{(1)})\otimes\ell_{(2)} = \Sigma_{i=1}^n l_i\otimes r_i$, then $\{r_1, ..., r_n\}$ and $\{l_1, ..., l_n\}$ are dual bases for $\lambda$. The second identity in (26) then follows from Lemma 3.3.5.

4. To determine the Nakayama automorphism for the Frobenius form $\kappa$ with $\kappa(h\otimes k) = \lambda(h\cdot k)$, we compute with the formulas from 2. and 3.

$$\begin{aligned}\lambda(k\cdot h) &\overset{(25)}{=} \lambda\big(\,\big(\Sigma_{(\ell)}\,\lambda(S(\ell_{(2)})k)\,\ell_{(1)}\big)\cdot h\,\big) = \Sigma_{(\ell)}\,\lambda(S(\ell_{(2)})k)\,\lambda(\ell_{(1)}h)\\ &\overset{(28)}{=} \Sigma_{(\ell)(h)}\,\alpha(h_{(1)})\,\lambda(S^2(h_{(2)})S(\ell_{(2)})k)\,\lambda(\ell_{(1)}) \overset{(28)}{=} \Sigma_{(\ell)(h)}\,\alpha(h_{(1)})\,\lambda(S^2(h_{(2)})S(\ell_{(2)}))\,\lambda(k\ell_{(1)})\\ &= \lambda\big(\,\big(\Sigma_{(h)}\alpha(h_{(1)})\,S^2(h_{(2)})\big)\cdot\big(\Sigma_{(\ell)}\lambda(k\ell_{(1)})S(\ell_{(2)})\big)\,\big) \overset{(25)}{=} \lambda\big(\,\big(\Sigma_{(h)}\alpha(h_{(1)})\,S^2(h_{(2)})\big)\cdot k\,\big)\end{aligned}$$

$$\begin{aligned}\lambda(k\cdot h) &\overset{(26)}{=} \lambda\big(\,\big(\,\Sigma_{(\ell)}\lambda(a^{-1}S^{-1}(\ell_{(1)})k)\,\ell_{(2)}\,\big)\cdot h\,\big) = \Sigma_{(\ell)}\,\lambda(a^{-1}S^{-1}(\ell_{(1)})k)\,\lambda(\ell_{(2)}h)\\ &\overset{(29)}{=} \Sigma_{(\ell)(h)}\,\alpha(h_{(2)})\,\lambda(a^{-1}S^{-2}(h_{(1)})S^{-1}(\ell_{(1)})k)\,\lambda(\ell_{(2)})\\ &\overset{(29)}{=} \Sigma_{(\ell)(h)}\alpha(h_{(2)})\,\lambda(a^{-1}S^{-2}(h_{(1)})S^{-1}(\ell_{(1)}))\,\lambda(k\ell_{(2)})\\ &= \lambda\big(\,a^{-1}\cdot\big(\Sigma_{(h)}\alpha(h_{(2)})\,S^{-2}(h_{(1)})\,\big)\cdot a\cdot\big(\Sigma_{(\ell)}\lambda(k\ell_{(2)})\,a^{-1}S^{-1}(\ell_{(1)})\big)\big)\\ &\overset{(26)}{=} \lambda\big(\,a^{-1}\cdot\big(\Sigma_{(h)}\alpha(h_{(2)})\,S^{-2}(h_{(1)})\,\big)\cdot a\cdot k\,\big)\end{aligned}$$

As the Nakayama automorphism is defined by $\kappa(k\otimes h) = \lambda(k\cdot h) = \lambda(\rho(h)\cdot k) = \kappa(\rho(h)\otimes k)$, this yields formula (27). □

By making use of the two pairs of dual bases in Proposition 3.3.6, one can also derive an explicit formula for the fourth power of the antipode in a finite-dimensional Hopf algebra $H$. This formula is known as *Radford's formula* and characterises the fourth power of the antipode in terms of the modular elements of $H$ and $H^*$. It is obtained directly from the two formulas for the Nakayama automorphism in Proposition 3.3.6 and the fact that the modular elements are grouplike.

**Theorem 3.3.7: (Radford's formula)**
Let $H$ be a finite-dimensional Hopf algebra over $\mathbb{F}$ and $a \in \mathrm{Gr}(H)$ and $\alpha \in \mathrm{Gr}(H^*)$ the modular elements from Proposition 3.2.16. Then for all $h \in H$ one has

$$S^4(h) = a^{-1} \cdot (\alpha \rhd_R^* h \lhd_L^* \alpha^{-1}) \cdot a = \alpha \rhd_R^* (a^{-1}ha) \lhd_L^* \alpha^{-1}$$

where $\alpha \rhd_R^* h = \Sigma_{(h)} \alpha(h_{(2)})\, h_{(1)}$ and $h \lhd_L^* \alpha^{-1} = \Sigma_{(h)} \alpha^{-1}(h_{(1)})\, h_{(2)}$ denote the left and right dual action of $H^*$ on $H^{**} \cong H$.

**Proof:**
By Proposition 3.3.6 the Nakayama automorphism is given by

$$\rho(h) = \Sigma_{(h)}\, \alpha(h_{(1)})\, S^2(h_{(2)}) = a^{-1} \cdot (\Sigma_{(h)}\, \alpha(h_{(2)})\, S^{-2}(h_{(1)})) \cdot a. \tag{30}$$

We now consider the element $\Sigma_{(h)}\; \alpha^{-1}(h_{(1)})S^2(\rho(h_{(2)}))$. Inserting the first expression for the Nakayama automorphism into this formula yields

$$\begin{aligned}\Sigma_{(h)}\; \alpha^{-1}(h_{(1)})S^2(\rho(h_{(2)})) &= \Sigma_{(h)}\; \alpha^{-1}(h_{(1)})\alpha(h_{(2)})\; S^4(h_{(3)}) = \Sigma_{(h)}\; (\alpha^{-1} \cdot \alpha)(h_{(1)})\; S^4(h_{(2)})\\ &= \Sigma_{(h)}\; \epsilon(h_{(1)})S^4(h_{(2)}) = S^4(h)\end{aligned}$$

and inserting the second expression for the Nakayama automorphism, we obtain

$$\begin{aligned}\Sigma_{(h)}\; \alpha^{-1}(h_{(1)})S^2(\rho(h_{(2)})) &= \Sigma_{(h)}\; \alpha^{-1}(h_{(1)})\alpha(h_{(3)})\, S^2(a^{-1} \cdot S^{-2}(h_{(2)}) \cdot a)\\ &= \Sigma_{(h)}\; \alpha^{-1}(h_{(1)})\alpha(h_{(3)})\, a^{-1} \cdot h_{(2)} \cdot a = a^{-1} \cdot \big(\Sigma_{(h)}\alpha^{-1}(h_{(1)})\alpha(h_{(3)})h_{(2)}\big) \cdot a\\ &= a^{-1} \cdot (\alpha \rhd_R^* h \lhd_L^* \alpha^{-1}) \cdot a.\end{aligned}$$

This proves the first equation in the theorem. The second follows by a direct computation from the fact that $\alpha^{\pm 1} \in \mathrm{Gr}(H^*)$ are grouplike elements and hence algebra homomorphisms $\alpha : H \to \mathbb{F}$ by Example 2.4.2. This yields for all $h \in H$

$$\begin{aligned}\alpha \rhd_R^* (a^{-1}ha) \lhd_L^* \alpha^{-1} &= \Sigma_{(h)}\, \alpha(a^{-1}h_{(3)}a)\alpha^{-1}(a^{-1}h_{(1)}a)\, a^{-1}h_{(2)}a\\ &= \Sigma_{(h)}\, \alpha(a^{-1})\alpha(h_{(3)})\alpha(a)\alpha^{-1}(a^{-1})\alpha^{-1}(h_{(1)})\alpha^{-1}(a)\, a^{-1}h_{(2)}a\\ &= \Sigma_{(h)}\alpha(aa^{-1}h_{(3)})\alpha^{-1}(aa^{-1}h_{(1)})\, a^{-1}h_{(2)}a = \Sigma_{(h)}\, \alpha(h_{(3)})\alpha^{-1}(h_{(1)})\, a^{-1}h_{(2)}a\\ &= a^{-1}(\alpha \rhd_R^* h \lhd_L^* \alpha^{-1})a.\end{aligned}$$

□

An important conclusion that follows from Radford's formula is that the antipode of every finite-dimensional Hopf algebra has finite order, i. e. there is an $n \in \mathbb{N}$ with $S^n = \mathrm{id}_H$. This is a direct consequence of Radford's formula and the fact that a finite-dimensional Hopf algebra can only contain finitely many grouplike elements since grouplike elements are linearly independent.

**Corollary 3.3.8:** Let $H$ be a finite-dimensional Hopf algebra over $\mathbb{F}$. Then:

1. The order of the antipode is finite.
2. If $H$ is unimodular, then $S^4(h) = a^{-1}ha$ where $a \in \mathrm{Gr}(H)$ is the modular element of $H^*$ and $S^{4k} = \mathrm{id}_H$ for some $k \in \mathbb{N}$ with $1 \leq k < \dim_{\mathbb{F}} H$.
3. If $H$ and $H^*$ are unimodular, then $S^4 = \mathrm{id}_H$.

**Proof:**
As $H$ and $H^*$ are finite-dimensional, all grouplike elements $g \in \mathrm{Gr}(H)$ and $\beta \in \mathrm{Gr}(H^*)$ must be of finite order $\leq \dim_{\mathbb{F}} H$. This follows because the sets of grouplike elements in $H$ and in $H^*$ are linearly independent by Proposition 2.4.4 and all powers of grouplike elements are grouplike. In particular, this holds for the modular elements $a \in \mathrm{Gr}(H)$ and $\alpha \in \mathrm{Gr}(H^*)$. If $a^n = 1_H$ and $\alpha^m = 1_{H^*}$ for some $n, m \in \{1, ..., \dim_{\mathbb{F}} H\}$, then by Radford's formula

$$S^{4nm}(h) = a^{-mn} \cdot (\alpha^{nm} \rhd^*_R h \lhd^*_R \alpha^{-mn}) \cdot a^{mn} = h.$$

If $H$ is unimodular, then its modular element satisfies $\alpha = \epsilon$ by Proposition 3.2.16, and Radford's formula reduces to $S^4(h) = a^{-1}ha$. It follows that there is a $k \in \{1, ..., \dim_{\mathbb{F}} H\}$ with $a^k = 1_H$, and this implies $S^{4k} = \mathrm{id}_H$. If $n = \min\{k \in \mathbb{N} \,|\, a^k = 1_H\} = \dim_{\mathbb{F}} H$, then $H = \mathbb{F}[\mathrm{Gr}(H)]$ and $S^2 = \mathrm{id}_H$. This shows that there is a $k \in \{1, ..., \dim_{\mathbb{F}} H - 1\}$ with $S^{4k} = \mathrm{id}_H$. If both, $H$ and $H^*$ are unimodular, then by Proposition 3.2.16 we have $a = 1_H$, $\alpha = 1_{H^*}$ and Radford's formula reduces to $S^4(h) = h$. □

## 3.4 Integrals and semisimplicity

In this section, we will show that for a finite-dimensional Hopf algebra $H$, the behaviour of the square $S^2$ of the antipode is related to the question if $H$ is semisimple. This is a surprising result that is again obtained from the Frobenius algebra structures of a finite-dimensional Hopf algebras $H$. We start by recalling the basic results and definitions about semisimplicity. The concept of semisimplicity is motivated by the wish to decompose every module over an algebra $A$ as a direct sum of basic building blocks, the so-called *simple modules*, which are characterised by the condition that they do not contain any non-trivial submodules.

**Definition 3.4.1:** Let $A$ be an algebra over $\mathbb{F}$.

1. A module $M$ over $A$ is called **simple** if $M \neq \{0\}$ and $M$ has no non-trivial submodules, i. e. the only submodules of $M$ are $M$ and $\{0\}$.
2. A module over $A$ is called **semisimple**, if it is the direct sum $M = \oplus_{i \in I} M_i$ of simple submodules $M_i$.
3. The algebra $A$ is called **simple** or **semisimple** if it is simple or semisimple as a left module over itself with the left multiplication.

Note that the trivial module $\{0\}$ is not simple by definition, but it is semisimple, since it is given by the direct sum over an empty index set. The following proposition gives an alternative criterion for the semisimplicity of a module that is very useful in practice.

**Proposition 3.4.2:** Let $A$ be an algebra over $\mathbb{F}$ and $M$ a module over $A$. Then the following are equivalent:

(i) $M$ is semisimple.

(ii) Every submodule $U \subset M$ is a (not necessarily direct) sum of simple submodules.

(iii) Every submodule $U \subset M$ has a **complement**, i. e. there is a submodule $V \subset M$ with $M = U \oplus V$.

**Proof:**
(i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii): Suppose $U \subset M$ is a submodule and $\{N_i\}_{i\in I}$ a maximal set of simple submodules of $M$ with $U \cap \Sigma_{i\in I} N_i = \{0\}$. We show that $M = U \oplus V$ with $V = \Sigma_{i\in I} N_i$. Because $M$ is a sum of simple submodules, for every element $m \in M$ there are simple submodules $L_1, ..., L_k$ of $M$ and elements $l_i \in L_i$ with $m = l_1 + ... + l_k$. As $L_1, ..., L_k$ are simple and $(U \oplus V) \cap L_i \subset L_i$ is a submodule, we have $(U \oplus V) \cap L_i = \{0\}$ or $(U \oplus V) \cap L_i = L_i$. If $(U \oplus V) \cap L_i = \{0\}$, then $U \cap L_i = \{0\}$, in contradiction to the maximality of $\{N_i\}_{i\in I}$. Hence $(U \oplus V) \cap L_i = L_i$ for all $i \in \{1, ..., k\}$. This implies $L_i \subset U \oplus V$ for all $i \in \{1, ..., k\}$ and $m \in U \oplus V$ for all $m \in M$.

(iii) $\Rightarrow$ (i): Let $\{U_i\}_{i\in I}$ be a maximal set of simple submodules $U_i \subset M$ with $\Sigma_{i\in I} U_i = \oplus_{i\in I} U_i$. Suppose $\oplus_{i\in I} U_i \neq M$. Then by (iii) there is a submodule $\{0\} \neq V$ with $M = \oplus_{i\in I} U_i \oplus V$. If $V$ is simple, we have a contradiction to the maximality of $\{U_i\}_{i\in I}$. Otherwise, there is a maximal submodule[1] $\{0\} \subsetneq W \subsetneq V$ and by (iii) a submodule $X \subset M$ with $M = W \oplus X$. Then $V = W \oplus (X \cap V)$ with $X \cap V \neq \{0\}$. If $X \cap V$ is simple, then we have a contradiction to the maximality of $\{U_i\}_{i\in I}$. Otherwise there is a submodule $\{0\} \subsetneq Y \subsetneq X \cap V$, which implies $W \subsetneq W \oplus Y \subsetneq V$, which contradicts the maximality of $W$. It follows that $M \oplus_{i\in I} U_i$. □

In particular, Proposition 3.4.2 implies that semisimplicity is a property that is inherited by submodules and quotients of modules with respect to submodules.

**Corollary 3.4.3:** Every submodule and every quotient of a semisimple module is semisimple.

**Proof:**
1. Let $U \subset M$ be a submodule and $\pi : M \to M/U$ the canonical surjection. Then for every submodule $N \subset M$ and every submodule $\{0\} \subsetneq L \subsetneq \pi(N)$ the submodule $\pi^{-1}(L) \subset M$ satisfies $\{0\} \subsetneq \pi^{-1}(L) \subsetneq N$. If $N \subset M$ is simple, this implies $\pi(N) = \{0\}$ or $\pi(N)$ simple. If $M = \oplus_{i\in I} N_i$ with simple modules $N_i \subset M$, one obtains $M/U = \pi(M) = \oplus_{i\in I} \pi(N_i)$ with $\pi(N_i) = \{0\}$ or $\pi(N_i)$ simple and hence $M/U$ is semisimple. This shows that quotients of semisimple modules are simple. To show that submodules of semisimple modules are simple, we note that Proposition 3.4.2 implies that every submodule $U \subset M$ has a complement $V \subset M$ with $M = U \oplus V$. Then we have $U \cong M/V$ and hence $U$ is simple. □

The fact that every module over an algebra $A$ can be described as a quotient of a free module over $A$, that is, as a quotient of a direct sum $\oplus_{i\in I} A$ for some index set $I$, then allows one to relate the semisimplicity of $A$ to the semisimplicity of modules over $A$.

[1]The existence of this maximal submodule follows with Zorn's Lemma, in analogy to the proof of the existence of maximal ideals in a ring $R$.

**Proposition 3.4.4:** Let $A$ be an algebra over $\mathbb{F}$. Then the following are equivalent:

(i) $A$ is semisimple.

(ii) Every module over $A$ is semisimple.

**Proof:**
(ii) $\Rightarrow$ (i) is obvious, since (i) is a special case of (ii). Let now $A$ be semisimple and $M$ a module over $A$. Then we consider the free $A$-module $\langle M\rangle_A = \oplus_{m\in M} A$ generated by $M$ and the module homomorphism $\phi : \langle M\rangle_A \to M$ defined by $\phi(m) = m$ for all $m \in M$. Then $\langle M\rangle_A$ is semisimple as the direct sum of semisimple $A$-modules, $\ker(\phi) \subset \langle M\rangle_A$ is a submodule and $M \cong \langle M\rangle_A/\ker(\phi)$ is semisimple as a quotient of a semisimple module by Corollary 3.4.3. □

After assembling the basic facts about semisimplicity of algebras, we now focus on Hopf algebras. While the definition of semisimplicity is the same - a Hopf algebra is called semisimple if it is semisimple as an algebra - the additional structures of a Hopf algebra allow us to give alternative characterisations of semisimplicity. They also imply that in many respects semisimple Hopf algebras behave similarly to group algebras of finite groups. Alternatively, we can view certain statements about representations of finite groups as a special case of more general statements about Hopf algebras. The first example is *Maschke's Theorem* for Hopf algebras, which relates the semisimplicity of a finite-dimensional Hopf algebra $H$ to the existence of *normalised left and right integrals.* This result that generalises *Maschke's Theorem for finite groups.*

**Theorem 3.4.5:** (**Maschke's Theorem for Hopf algebras**)
Let $H$ be a finite-dimensional Hopf algebra over $\mathbb{F}$. The the following are equivalent:

(i) $H$ is semisimple.

(ii) There is a left integral $\ell \in H$ with $\epsilon(\ell) \neq 0$.

(iii) There is a right integral $\ell \in H$ with $\epsilon(\ell) \neq 0$.

**Proof:**
We prove the claim for left integrals. The claim for right integrals then follows because $S(I_L(H)) = I_R(H)$ and $\epsilon \circ S = \epsilon$.

(i)$\Rightarrow$ (ii): The linear map $\epsilon : H \to \mathbb{F}$ is a module homomorphism with respect to the left regular action of $H$ on itself and the trivial $H$-module structure on $\mathbb{F}$, since we have $\epsilon(h \rhd_L k) = \epsilon(hk) = \epsilon(h)\epsilon(k) = h \rhd \epsilon(k)$ for all $h, k \in H$. Hence $\ker(\epsilon) \subset H$ is a submodule, i. e. a left ideal in $H$, and by Proposition 3.4.2 there is a left ideal $I \subset H$ with $H = \ker(\epsilon) \oplus I$. As we have $(h - \epsilon(h)\, 1) \cdot k \in \ker(\epsilon)$ and $h \cdot i \in I$ for all $h, k \in H$ and $i \in I$, we obtain

$$\underbrace{h \cdot i}_{\in I} = \underbrace{(h - \epsilon(h)\, 1) i}_{\in \ker(\epsilon)} + \underbrace{\epsilon(h) i}_{\in I} = \underbrace{\epsilon(h) i}_{\in I},$$

and this implies $I \subset I_L(H)$. As $\epsilon(1) = 1$ implies $\dim_{\mathbb{F}}(\ker(\epsilon)) < \dim_{\mathbb{F}} H$ and $I \cap \ker(\epsilon) = \{0\}$, we have $1 \leq \dim_{\mathbb{F}} I$, and there is a left integral $\ell \in I \subset I_L(H)$ with $\epsilon(\ell) \neq 0$.

(ii) $\Rightarrow$ (i): Suppose there is an integral $\ell \in I_L(H)$ with $\epsilon(\ell) \neq 0$. Then by multiplying with an element $\lambda \in \mathbb{F}$, one can achieve $\epsilon(\ell) = 1$. Let $M$ be a module over $H$, $U \subset M$ a submodule and choose a linear map $P : M \to U$ with $P|_U = \mathrm{id}_U$. If we equip the vector space $\mathrm{Hom}_{\mathbb{F}}(M, U)$ with the $H$-module structure from Lemma 3.2.4, then the linear map

$$\pi = \ell \rhd P : M \to U, \quad m \mapsto (\ell \rhd P)(m) = \Sigma_\ell\, \ell_{(1)} \rhd P(S(\ell_{(2)}) \rhd m)$$

is an invariant of the module $\mathrm{Hom}_{\mathbb{F}}(M, U)$ by Lemma 3.2.8 and hence a module map by Lemma 3.2.4. It follows that $\ker(\pi) \subset M$ is a submodule of $M$. Moreover, one has for all $u \in U$

$$\pi(u) = (\ell \rhd P)(u) = \Sigma_{(\ell)}\, \ell_{(1)} \rhd P(S(\ell_{(2)}) \rhd u) = \ell_{(1)} \rhd (S(\ell_{(2)}) \rhd u) = \epsilon(\ell)\, u = u.$$

This implies $\ker(\pi) \cap U = \{0\}$ and since every element $m \in M$ can be written as $m = m - \pi(m) + \pi(m)$ with $m - \pi(m) \in \ker(\pi)$ and $\pi(m) \in U$ it follows that $M = U \oplus \ker(\pi)$. Hence, every submodule of $M$ has a complement and $H$ is semisimple. □

**Corollary 3.4.6:** (**Maschke's Theorem for finite groups**)
Let $G$ be a finite group. Then the group algebra $\mathbb{F}[G]$ is semisimple if and only if $\mathrm{char}(\mathbb{F}) \not| |G|$.

**Proof:**
By Example 3.2.7, we have $I_L(\mathbb{F}[G]) = \mathrm{span}_{\mathbb{F}} \{\Sigma_{g\in G}\, g\}$. As $\epsilon\,(\Sigma_{g\in G}\, g) = \Sigma_{g\in G}\, \epsilon(g) = |G|$, it follows that $\epsilon(I_L(\mathbb{F}[G])) \neq \{0\}$ if and only if $\mathrm{char}(\mathbb{F}) \not| \ |G|$, and by Theorem 3.4.5, this is equivalent to the semisimplicity of $\mathbb{F}[G]$. □

**Corollary 3.4.7:** Every finite-dimensional semisimple Hopf algebra is unimodular.

**Proof:**
As $H$ is finite-dimensional semisimple, there is a left integral $\ell \in I_L(H)$ with $\epsilon(\ell) = 1$ by Theorem 3.4.5. Then the modular element $\alpha \in \mathrm{Gr}(H^*)$ with $\ell \cdot h = \alpha(h)\,\ell$ for all $h \in H$ from Proposition 3.2.16 satisfies

$$\alpha(h)\,\ell = \alpha(h)\,\epsilon(\ell)\ell = \alpha(h)\,\ell^2 = (\ell h)\ell = \ell(h\ell) = \epsilon(h)\,\ell^2 = \epsilon(h)\epsilon(\ell)\,\ell = \epsilon(h)\,\ell.$$

As $\ell \neq 0$, this implies $\alpha(h) = \epsilon(h)$ for all $h \in H$ and by Proposition 3.2.16 $H$ is unimodular. □

We will now show that the semisimplicity of a finite-dimensional Hopf algebra $H$ is related to the square of its antipode. The key result that links these two concepts are the results on dual bases in Proposition 3.3.6. The fact that every element of $H$ can be expressed in terms of dual bases for a right integral $\lambda \in I_R(H^*)$ allows one to express the traces of a linear maps $\phi : H \to H$ in terms of integrals and to relate the existence of normalised left or right integrals to the square of the antipode of $H$.

**Proposition 3.4.8:** Let $H$ be a finite-dimensional Hopf algebra over $\mathbb{F}$, $\ell \in I_L(H)$ a left integral and $\lambda \in I_R(H^*)$ a right integral with $\lambda(\ell) = 1$, as in Proposition 3.3.6. Then:

1. For all linear maps $\phi : H \to H$ one has $\mathrm{Tr}(\phi) = \Sigma_{(\ell)}\, \lambda(S(\ell_{(2)})\phi(\ell_{(1)}))$,
2. The square of the antipode satisfies $\mathrm{Tr}(S^2) = \epsilon(\ell)\,\lambda(1)$.

**Proof:**
We choose an ordered basis $(x_1, ..., x_n)$ of $H$ and the dual basis $(\alpha^1, ..., \alpha^n)$ of $H^*$. Then we have $\mathrm{Tr}(\phi) = \Sigma_{i=1}^n \alpha^i(\phi(x_i))$ for all $\phi \in \mathrm{End}_{\mathbb{F}}(H)$. As $\phi(x_i) = \Sigma_{(\ell)}\ \lambda(S(\ell_{(2)})\phi(x_i))\,\ell_{(1)}$ for all $i = 1, ..., n$ by formula (25) in Proposition 3.3.6, we obtain

$$\begin{aligned}\mathrm{Tr}(\phi) &= \Sigma_{i=1}^n\ \alpha^i\big(\Sigma_{(\ell)}\ \lambda(S(\ell_{(2)})\phi(x_i))\ \ell_{(1)}\big) = \Sigma_{i=1}^n \Sigma_{(\ell)} \alpha^i(\ell_{(1)})\ \lambda(S(\ell_{(2)})\phi(x_i)) \\ &= \Sigma_{(\ell)}\ \lambda\big(S(\ell_{(2)}) \cdot \phi(\Sigma_{i=1}^n \alpha^i(\ell_{(1)})x_i)\,\big) = \Sigma_{(\ell)} \lambda\big(S(\ell_{(2)})\phi(\ell_{(1)})\big)\end{aligned}$$

Inserting $\phi = S^2$ into this equation and using the condition on the antipode yields

$$\mathrm{Tr}(S^2) = \Sigma_{(\ell)}\ \lambda(S(\ell_{(2)}) \cdot S^2(\ell_{(1)})) = \lambda\big(\Sigma_{(\ell)}\, S(\, S(\ell_{(1)})\ell_{(2)})\,\big) = \lambda(\epsilon(\ell)S(1)) = \epsilon(\ell)\lambda(1). \qquad \square$$

**Corollary 3.4.9:** Let $H$ be a finite-dimensional Hopf algebra over $\mathbb{F}$.

1. $H$ and $H^*$ are semisimple if and only if $\mathrm{Tr}(S^2) \neq 0$.
2. If $S^2 = \mathrm{id}_H$ and $\mathrm{char}(\mathbb{F}) \not| \dim_{\mathbb{F}} H$, then $H$ and $H^*$ are semisimple.

**Proof:**
1. By Proposition 3.3.6 there are integrals $\ell \in I_L(H)$ and $\lambda \in I_R(H^*)$ with $\lambda(\ell) = 1$, and by Proposition 3.4.8 we then have $\mathrm{Tr}(S^2) = \epsilon(\ell)\lambda(1)$. Hence, we have $\mathrm{Tr}(S^2) \neq 0$ if and only if $\epsilon(\ell), \lambda(1) \neq 0$. As $\dim_{\mathbb{F}} I_L(H) = \dim_{\mathbb{F}} I_R(H^*) = 1$, this is equivalent to the existence of a left integral $\ell \in I_L(H)$ and a right integral $\lambda \in I_R(H^*)$ with $\epsilon(\ell) \neq 0$ and $\lambda(1) \neq 0$. By Theorem 3.4.5 the first condition is equivalent to the semisimplicity of $H$ and the second to the semisimplicity of $H^*$.

2. If $S^2 = \mathrm{id}_H$, then by Proposition 3.4.8 $\mathrm{Tr}(S^2) = \mathrm{Tr}(\mathrm{id}_H) = \dim_{\mathbb{F}}(H)$. If $\mathrm{char}(\mathbb{F}) \not| \dim_{\mathbb{F}}(H)$, this implies $\mathrm{Tr}(S^2) \neq 0$, and with 1. it follows that $H$ and $H^*$ are semisimple. □

Corollary 3.4.9 relates the the semisimplicity of a finite-dimensional Hopf algebra $H$ and its dual to the square of the antipode. In fact, for fields of characteristic zero there is a stronger result, theorem of Larson and Radford. A proof is given in the original article [LR]

**Theorem 3.4.10: (Larson-Radford Theorem)**
Let $H$ be a finite-dimensional Hopf algebra over a field $\mathbb{F}$ of characteristic zero. Then the following are equivalent:

(i) $H$ is semisimple.
(ii) $H^*$ is semisimple.
(iii) $S^2 = \mathrm{id}$.

In particular, the theorem by Larson and Radford allows us to derive sufficient conditions under which the Frobenius algebra from Proposition 3.3.4 is symmetric. By combining Theorem 3.4.10 with formula (27) for the Nakayama automorphism, one finds that the semisimplicity of $H$ is sufficient in characteristic zero. This implies in particular that the coproduct of an integral in a finite-dimensional semisimple Hopf algebra is always symmetric.

**Corollary 3.4.11:** Let $H$ be a finite-dimensional semisimple Hopf algebra over a field $\mathbb{F}$ of characteristic zero. Then:

1. The Frobenius algebra from Proposition 3.3.4 is symmetric.
2. For all integrals $\ell \in I_L(H) = I_R(H)$ one has $\Delta(\ell) = \Delta^{op}(\ell)$.

**Proof:**
1. Let $H$ be a finite-dimensional semisimple Hopf algebra over $\mathbb{F}$. The by Theorem 3.4.10 one has $S^2 = \mathrm{id}_H$, the dual $H^*$ is semisimple as well and $H$ and $H^*$ are unimodular by Corollary 3.4.7. By Proposition 3.2.16 this implies that the modular element $\alpha \in \mathrm{Gr}(H^*)$ is given by $\alpha = \epsilon$. Formula (27) for the Nakayama automorphism then yields for all $h \in H$

$$\rho(h) = \Sigma_{(h)}\, \alpha(h_{(1)})\, S^2(h_{(2)}) = \Sigma_{(h)}\, \epsilon(h_{(1)})\, h_{(2)} = h,$$

and this shows that the Frobenius algebra from Proposition 3.3.4 is symmetric.

2. Clearly, the identity holds if $\ell = 0$. For every integral $\ell \in I_L(H) = I_R(H)\setminus\{0\}$, the associated Frobenius form $\kappa$ on $H^*$ is symmetric by 1. This implies for all $\alpha, \beta \in H^*$

$$(\alpha\otimes\beta)(\Delta(\ell)) = (\alpha \cdot \beta)(\ell) = \kappa(\alpha\otimes\beta) = \kappa(\beta\otimes\alpha) = (\beta \cdot \alpha)(\ell) = (\alpha\otimes\beta)(\Delta^{op}(\ell))$$

and hence $\Delta(\ell) = \Delta^{op}(\ell)$. □

**Example 3.4.12:** Let $\mathbb{F}$ be a field of characteristic zero.

1. The $q$-deformed universal enveloping algebra $U_q^r(\mathfrak{sl}_2)$ at a primitive $n$th root of unity from Proposition 2.3.12 is unimodular, but it is not semisimple. This follows because the square of its antipode is given by

$$S^2(K) = K, \qquad S^2(E) = KEK^{-1} = q^2E \qquad S^2(F) = -KFK^{-1} = q^{-2}F$$

   Hence, the antipode of $U_q^r(\mathfrak{sl}_2)$ has order $2r > 2$ with $r = n/2$ for $n$ even and $r = n$ for $n$ odd and $U_q^r(\mathfrak{sl}_2)$ cannot be semisimple by Theorem 3.4.10.

2. The Taft algebra From Example 2.3.6 is not semisimple, since one has

$$S^2(x) = S(-xy^{-1}) = -S(xy^{-1}) = -S(y)^{-1}S(x) = -y(-xy^{-1}) = yxy^{-1} = q^{-1}x \neq x.$$

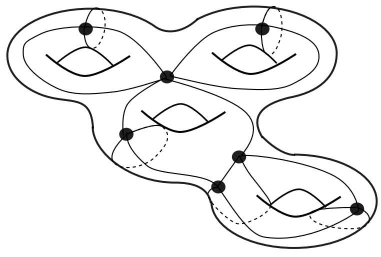
## 3.5 Application: Kitaev models

Kitaev models were first introduced in 2003 by A. Kitaev [Ki] to obtain a realistic model for a quantum computer that would be protected against errors by topological effects and could in principle be realised in the framework of solid state physics. While the original model was based on group algebra $\mathbb{C}[\mathbb{Z}/p\mathbb{Z}]$, it was then generalised to models based on the group algebra $\mathbb{C}[G]$ for a finite group $G$ in [BMD] and to finite-dimensional semisimple Hopf algebras in [BMCA]. These models became very prominent and are a topic of current research in condensed matter physics. They are also interesting from the mathematical perspective since they are related to topological quantum field theories and define manifold invariants.

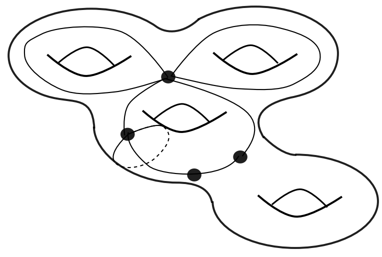
The ingredients of the Kitaev model are

- a finite-dimensional semisimple Hopf algebra $H$ over field $\mathbb{F}$ of characteristic zero ,
- an oriented surface $\Sigma$, i. e. a connected, compact oriented topological manifold $\Sigma$ of dimension two,
- a finite directed graph $\Gamma$ embedded into $\Sigma$ such that $\Sigma \setminus \Gamma$ is a disjoint union of discs.

Recall that compact oriented surfaces are classified up to homeomorphisms by their genus, i. e. the number of handles. The last condition ensures that the graph $\Gamma$ is sufficiently refined to resolve the topology of the surface $\Sigma$, i. e. that we can realise a set of generators of the fundamental group $\pi_1(\Sigma)$ as paths in the embedded graph $\Gamma$:



surface with an admissible graph



surface with a graph that is not admissible.

We denote by $E$ and $V$, respectively, the sets of edges and vertices of $\Gamma$ and use the same letters for their cardinalities. We also require the notion of a *face*. A face in $\Gamma$ is represented by a closed path in $\Gamma$ that starts and ends at a vertex $v \in V$, turns maximally right at each vertex and traverses each edge at most once in each direction. More precisely, a **face** of $\Gamma$ is an equivalence class of such a path under *cyclic permutations*, i. e. up to the choice of the starting vertex of the path. The set of faces of $\Gamma$ and its cardinality are denoted $F$.

We assume for simplicity that $\Gamma$ is a graph without loops, i. e. the starting and target vertex of each edge are different vertices, and that all paths representing faces traverse each edge of $\Gamma$ at most once. By placing a marking at a vertex $v \in V$ between two incident edges at $v$, we then obtain an ordering of the edges at $v$ by counting them counterclockwise from the marking. Similarly, if we place a marking at one of the vertices in a face, we obtain an ordering of the edges in the face by counting them counterclockwise from the marking. A vertex together with a marking is called a **marked vertex** and a face with a marking a **marked face**. In the following, we assume that the markings are chosen in such a way that each face and each vertex carries exactly one marking, as in Figure 1. This defines a partition of the set $V \times F$ into pairs $(v, f)$ that share a marking. Such a pair is called a **site**.

With these preliminary definitions, we can now introduce the Kitaev model associated with the triple $(\Sigma, \Gamma, H)$ of an oriented surface $\Sigma$, a marked directed graph $\Gamma$ embedded into $\Sigma$ that satisfies the conditions above and a finite-dimensional semisimple Hopf algebra $H$ over field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) = 0$. Throughout this section, we assume that the data $(\Sigma, \Gamma, H)$ is fixed and satisfies the conditions above.

**Definition 3.5.1:** The **Kitaev model** for $(\Sigma, \Gamma, H)$ consists of the following data:

1. The **extended Hilbert space**: the vector space $H^{\otimes E}$.
2. The **edge operators:**

   The edge operators for a triple $(e, h, \alpha)$ of an edge $e \in E$ and elements $h \in H$, $\alpha \in H^*$ are the linear maps $L^h_{e\pm}, T^\alpha_{e\pm} : H^{\otimes E} \to H^{\otimes E}$ given by

$$\begin{aligned} L^h_{e+} &: ...\otimes k^e \otimes ... \mapsto ...\otimes (h \rhd_L k^e) \otimes ... = ...\otimes (hk^e) \otimes ... \\ T^\alpha_{e+} &: ...\otimes k^e \otimes ... \mapsto ...\otimes (\alpha \rhd^*_R k^e) \otimes ... = ...\otimes (\Sigma_{(k^e)} \alpha(k^e_{(2)}) k^e_{(1)}) \otimes ... \\ L^h_{e-} = S \circ L^h_{e+} \circ S &: ...\otimes k^e \otimes ... \mapsto ...\otimes (k^e \lhd_R S(h)) \otimes ... = ...\otimes (k^e S(h)) \otimes ... \\ T^\alpha_{e-} = S \circ T^\alpha_{e+} \circ S &: ...\otimes k^e \otimes ... \mapsto ...\otimes (k^e \lhd^*_L S(\alpha)) \otimes ... = ...\otimes (\Sigma_{(k^e)} \alpha(S(k^e_{(1)})) k^e_{(2)}) \otimes ... \end{aligned} \tag{31}$$

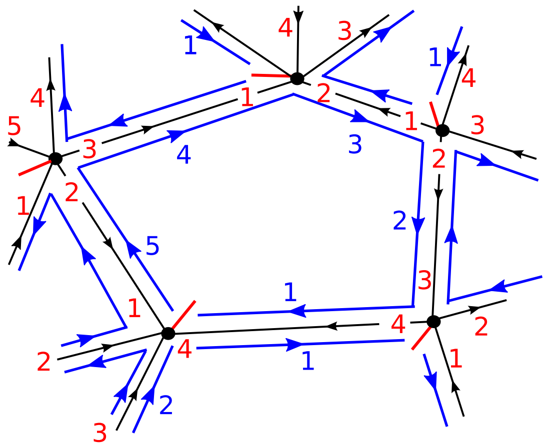


Figure 1: Embedded graph Γ with markings and the induced ordering of edges at the vertices and faces of Γ.

3. The **vertex and face operators:**
   - The **vertex operator** for a pair $(v, h)$ of a vertex $v \in V$ and $h \in H$ is the linear map
   $$A_v^h = \Sigma_{(h)}\, L^{h_{(1)}}_{e_1,\epsilon_1} \circ L^{h_{(2)}}_{e_2,\epsilon_2} \circ ... \circ L^{h_{(n)}}_{e_n,\epsilon_n} : H^{\otimes E} \to H^{\otimes E}$$
   where $e_1, ..., e_n$ are the incident edges at $v$, numbered counterclockwise from the marking at $v$, $\epsilon_i = +$ if $e_i$ is incoming at $v$ and and $\epsilon_i = -$ if $e_i$ is outgoing from $v$.
   - The **face operator** for a pair $(f, \alpha)$ of a face $f \in F$ and $\alpha \in H^*$ is the linear map
   $$B_f^\alpha = \Sigma_{(\alpha)}\, T^{\alpha_{(1)}}_{e_1,\epsilon_1} \circ T^{\alpha_{(2)}}_{e_2,\epsilon_2} \circ ... \circ T^{\alpha_{(n)}}_{e_n,\epsilon_n} : H^{\otimes E} \to H^{\otimes E},$$
   where $e_1, ..., e_n$ are the edges in $f$, numbered counterclockwise from the marking, $\epsilon_i = +$ if $e_i$ is traversed in its orientation and $\epsilon_i = -$ if $e_i$ is traversed against its orientation.

4. The **protected space** or **ground state**:

   The protected space of a Kitaev model is the linear subspace
   $$H^{\otimes E}_{inv} = \{x \in H^{\otimes E} \,|\, A_v^h x = \epsilon(h)\, x,\ B_f^\alpha(x) = \epsilon(\alpha)\, x \quad \forall h \in H, \alpha \in H^*, v \in V, f \in F\}.$$

These Definitions can be generalised to graphs Γ with loops or with faces that traverse certain edges more than once. In this case, one simply replaces the word *edge* in the definition by *edge ends*. However, to keep notation simple, we will not consider this in the following.

The main reason why Kitaev models are interesting from the mathematics perspective is that their protected space $H^{\otimes E}_{inv}$ does not depend on the choice of the graph Γ or its embedding into Σ but only on the homeomorphism class of the surface Σ. It is a topological invariant of Σ, which was shown to be related to certain topological quantum field theories [BK].

**Theorem 3.5.2** ([BMCA]): The protected space of a Kitaev model is a **topological invariant**: Its dimension depends only on the homeomorphism class of the surface Σ an not on the embedded graph Γ.

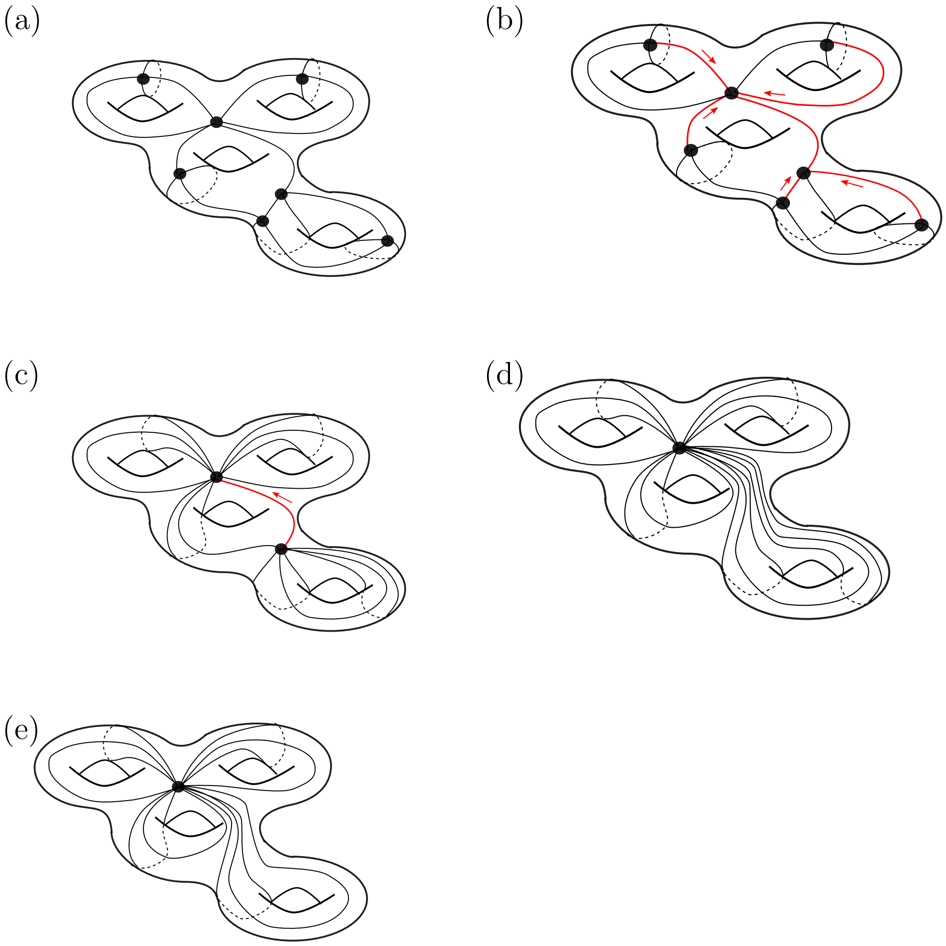



Figure 2: Contracting the edges of a maximal tree in a graph: (b)→(c), (c)→(d) and removing loops: (d)→(e).

**Sketch of Proof:** The proof is performed by selecting a maximal tree $T \subset \Gamma$ as in Figure 2 (b). This is a subgraph $T \subset \Gamma$ with no non-trivial closed paths or, equivalently, with trivial fundamental group that contains each vertex of $\Gamma$. One then contracts all edges in the tree towards a chosen vertex as in Figure 2 (b),(c),(d). One can show that these edge contractions induce isomorphisms between the protected spaces of the associated Kitaev models. By contracting all edges in the tree $T$ one obtains a graph $\Gamma'$ as in Figure 2 (d) whose ground state is isomorphic to the one of $\Gamma$ and which contains only a single vertex. By removing loops of $\Gamma'$ that can be removed without violating the condition that $\Sigma \setminus \Gamma$ is a disjoint union of discs as in Figure 2 (d),(e) one obtains another graph $\Gamma''$ whose protected space is isomorphic to the one of $\Gamma$, with a single vertex, a single face and $2g$ edges, where $g$ is the genus of $\Sigma$. After performing some further graph transformations which again induce isomorphisms between the protected spaces of the associated Kitaev models, one obtains a standard graph $\Gamma'''$ which depends only on the genus of the surface $\Sigma$ and such that the protected space of the associated Kitaev model is isomorphic to the one for $\Gamma$. This shows that the protected space of the Kitaev model for $(\Sigma, \Gamma, H)$ depends only on $H$ and the genus of $\Sigma$. □

We will now investigate the mathematical structure of the Kitaev model and and show that it is an application of the concepts introduced in this chapter. We first show that the edge operators span an algebra that is related to cross product $H\#H^*$ from Proposition 3.1.7 with the multiplication law

$$(h\otimes\alpha)\cdot(k\otimes\beta)=\Sigma_{(\alpha)(k)}\ \alpha_{(1)}(k_{(2)})\ hk_{(1)}\otimes\alpha_{(2)}\beta$$

This allows us to identify the algebra generated by the edge operators $L^h_{e\pm}$ and $T^\alpha_{e\pm}$ for edges $e\in E$ with an $E$-fold tensor product of the cross product $H\#H^*$ with itself and to show that it is isomorphic to $\mathrm{End}_{\mathbb{F}}(H^{\otimes E})$.

**Proposition 3.5.3** ([BMCA])**:** For all edges $e\in E$ the linear map

$$\rho_e: H\#H^*\to\mathrm{End}_{\mathbb{F}}(H^{\otimes E}),\quad h\otimes\alpha\mapsto L^h_{e+}\circ T^\alpha_{e+}$$

defines a representation of $H\#H^*$ on $H^{\otimes E}$. This induces an algebra isomorphism

$$\rho:(H\#H^*)^{\otimes E}\to\mathrm{End}_{\mathbb{F}}(H^{\otimes E}).$$

**Proof:**
That the maps $\rho_e$ define a representation of $H\#H^*$ on $H^{\otimes E}$ follows by a direct computation. By definition of the edge operators, we have $\rho_e(1_H\otimes 1_{H^*})=L^1_{e+}T^\epsilon_{e+}=\mathrm{id}_{H^{\otimes E}}$ and

$$\begin{aligned}&\rho_e(h\otimes\alpha)\rho_e(k\otimes\beta)(...\otimes l^e\otimes...)=L^h_{e+}T^\alpha_{e+}L^k_{e+}T^\beta_{e+}(...\otimes l^e\otimes...)=\Sigma_{(l^e)}\beta(l^e{}_{(2)})L^h_vT^\alpha_f(...\otimes kl^e{}_{(1)}\otimes...)\\&=\Sigma_{(k)(l^e)}\beta(l^e{}_{(2)})\alpha((kl^e{}_{(1)})_{(2)})\ ...\otimes h(kl^e{}_{(1)})_{(1)}\otimes...=\Sigma_{(k)(l^e)}\beta(l^e{}_{(3)})\alpha(k_{(2)}l^e{}_{(2)})\ ...\otimes hk_{(1)}l^e{}_{(1)}\otimes...\\&=\Sigma_{(k)(l^e)(\alpha)}\alpha_{(1)}(k_{(2)})\ (\alpha_{(2)}\beta)(l^e{}_{(2)})\ ...\otimes hk_{(1)}l^e{}_{(1)}\otimes...=\Sigma_{(k)(\alpha)}\alpha_{(1)}(k_{(2)})\ L^{hk_{(1)}}_{e+}T^{\alpha_{(2)}\beta}_{e+}(...\otimes l^e\otimes...)\\&=\Sigma_{(k)(\alpha)}\alpha_{(1)}(k_{(2)})\ \rho(hk_{(1)}\otimes\alpha_{(2)}\beta)(....\otimes l^e\otimes...)=\rho_e((h\otimes\alpha)\cdot(k\otimes\beta))(...\otimes l^e\otimes...)\end{aligned}$$

for all $h,k,l^e\in H$ and $\alpha,\beta\in H^*$. This shows that $\rho_e$ is an algebra homomorphism. To show that this induces an algebra isomorphism $\rho:(H\#H^*)^{\otimes E}\to\mathrm{End}_{\mathbb{F}}(H^{\otimes E})$, we consider the algebra homomorphism $\rho': H\#H^*\to\mathrm{End}_{\mathbb{F}}(H)$ with $\rho'(h\otimes\alpha)k=\Sigma_{(k)}\ \alpha(k_{(2)})\ hk_{(1)}$, which is related to $\rho_e$ by the condition

$$\rho_e(\alpha\otimes h)(...\otimes l^e\otimes...)=L^h_{e+}T^\alpha_{e+}(...\otimes l^e\otimes...)=\Sigma_{(l^e)}\ \alpha(l^e_{(1)})\ ...\otimes hl^e_{(1)}\otimes...=...\otimes\rho'(\alpha\otimes h)(l^e)\otimes....$$

As the edge operators for different edges commute by definition, this induces an algebra homomorphism $\rho=\otimes_{e\in E}\rho':(H\#H^*)^{\otimes E}\to\mathrm{End}_{\mathbb{F}}(H^{\otimes E})$. To show that $\rho$ is an algebra *isomorphism*, it is sufficient to show that the algebra homomorphism $\rho'$ is surjective. As $H$ is finite-dimensional, this implies that $\rho'$ is an isomorphism and it follows that $\rho=\otimes_{e\in E}\rho':(H\#H^*)^{\otimes E}\to\mathrm{End}_{\mathbb{F}}(H^{\otimes E})$ is an isomorphism as well.

As $H$ is finite-dimensional semisimple, by Theorem 3.4.10 the dual Hopf algebra $H^*$ is finite-dimensional semisimple as well, and by Theorem 3.4.5 and Corollary 3.4.7 there are normalised Haar integrals $\ell\in I_L(H)=I_R(H)$ and $\lambda\in I_L(H^*)=I_R(H^*)$. We consider an ordered basis $(x_1,...,x_n)$ of $H$ and the dual basis $(\alpha^1,...,\alpha^n)$ of $H^*$. With this, we compute for $i,j,k\in\{1,..,n\}$

$$\begin{aligned}&\rho'((x_i\otimes\lambda)\cdot(\ell\otimes\alpha^j))x_k=\Sigma_{(\lambda)(\ell)}\lambda_{(1)}(\ell_{(2)})\ \rho(\lambda_{(2)}\alpha^j\otimes x_i\ell_{(1)})x_k\\&=\Sigma_{(\lambda)(\ell)(x_k)}\lambda_{(1)}(\ell_{(2)})\ (\lambda_{(2)}\alpha^i)(x_{k(2)})\ x_i\ell_{(1)}x_{k(1)}=\Sigma_{(\ell)(\lambda)(x_k)}\lambda_{(1)}(\ell_{(2)})\lambda_{(2)}(x_{k(2)})\alpha^j(x_{k(3)})\ x_i\ell_{(1)}x_{k(1)}\\&=\Sigma_{(\ell)(x_k)}\lambda(\ell_{(2)}x_{k(2)})\alpha^j(x_{k(3)})\ x_i\ell_{(1)}x_{k(1)}=\Sigma_{(x_k)}\lambda(\ell x_{k(1)})\ \alpha^j(x_{k(2)})x_i=\lambda(\ell)\ \alpha^j(x_k)x_i=\lambda(\ell)\delta^j_kx_i,\end{aligned}$$

where we used the multiplication law of $H\#H^*$ in the first line, the definition of $\rho'$ to pass to the second line, the duality between the multiplication in $H^*$ and the comultiplication in $H$ in the second line and to pass to the third line, then the identity $(\lambda\otimes\mathrm{id})\circ\Delta(k)=1\otimes\lambda(k)$ for all $k \in H$ and then the fact that $\ell \in H$ is an integral. As $\lambda(\ell) \neq 0$ by Proposition 3.3.6 and the linear maps $\phi_{ij} \in \mathrm{End}_{\mathbb{F}}(H)$ with $\phi_{ij}(x_k) = \delta^j_k x_i$ form a basis of $\mathrm{End}_{\mathbb{F}}(H)$, the claim follows. □

We have shown that the algebra generated by the edge operators in the Kitaev model is just the $E$-fold tensor product of the cross product $H\#H^*$ with itself. Proposition 3.5.3 also implies that *every linear map* $\phi : H^{\otimes E} \to H^{\otimes E}$ can be realised as a a linear combination of finite composites of edge operators $L^{e+}_h$ and $T^{\alpha}_{e-}$. Hence, the edge operators in the Kitaev model can be viewed as a particularly nice generating set of the algebra $\mathrm{End}_{\mathbb{F}}(H^{\otimes E})$ that is adapted to the embedded graph $\Gamma$.

We now consider the vertex and face operators in the Kitaev model. From the definition of the model it is apparent that they play the role of a symmetry algebra that acts on the extended Hilbert space and defines the protected space by the condition that their action on an element of $H^{\otimes E}_{inv}$ is trivial, i. e. given by the counits of $H$ and $H^*$. As vertices and faces of $\Gamma$ are grouped into pairs $(v, f)$ that share a common marking, it is natural to combine the vertex and face operators $A^h_v$ and $B^\alpha_f$ for such a pair. It is then natural to expect that each such pair defines a Hopf algebra structure on $H^*\otimes H$, that the extended Hilbert space $H^{\otimes E}$ is a module over this Hopf algebra and that the protected space is its submodule of invariants. However, it turns out that the relevant Hopf algebra is not simply the tensor product of $H^*$ and $H$ but another Hopf algebra structure on the vector space $H^*\otimes H$, the so-called *Drinfeld double* $D(H)$.

**Lemma 3.5.4:** Let $H$ be a finite-dimensional Hopf algebra over $\mathbb{F}$ and $H^*$ its dual. Then the following defines a Hopf algebra structure on $H^*\otimes H$

$$
\begin{aligned}
&(\alpha\otimes h)\cdot(\beta\otimes k) = \Sigma_{(h),(\beta)}\, \beta_{(3)}(h_{(1)})\, \beta_{(1)}(S^{-1}(h_{(3)}))\, \alpha\beta_{(2)}\otimes h_{(2)}k \qquad && 1 = 1_{H^*}\otimes 1_H\\
&\Delta(\alpha\otimes h) = \Sigma_{(h),(\alpha)}\, \alpha_{(2)}\otimes h_{(1)}\otimes \alpha_{(1)}\otimes h_{(2)} && \epsilon(\alpha\otimes h) = \epsilon_{H^*}(\alpha)\epsilon_H(h)\\
&S(\alpha\otimes h) = (1\otimes S(h))\cdot(S(\alpha)\otimes 1) = \Sigma_{(h)(\alpha)} \alpha_{(1)}(h_{(3)})\alpha_{(3)}(S^{-1}(h_{(1)}))\, S(\alpha_{(2)})\otimes S(h_{(2)}).
\end{aligned}
$$

It is called the **Drinfeld double** or **quantum double** of $H$ and denoted $D(H)$.

**Proof:**
The coassociativity and counitality follow directly from the coassociativity and counitality for $H$ and $H^*$. That $\Delta$ and $\epsilon$ are algebra homomorphisms follows, because this holds in $H$ and $H^*$ and from the identity $\alpha\otimes h = (\alpha\otimes 1)\cdot(1\otimes h)$ for all $h \in H$ and $\alpha \in H^*$. The same holds for the antipode condition, since we have

$$
\begin{aligned}
&\Sigma_{(h)(\alpha)}\, (1\otimes S(h_{(1)}))\cdot(S(\alpha_{(1)})\otimes 1)\cdot(\alpha_{(2)}\otimes 1)\cdot(1\otimes h_{(2)}) = (1\otimes S(h_{(1)}))\cdot(S(\alpha_{(1)})\alpha_{(2)}\otimes 1)\cdot(1\otimes h_{(2)})\\
&= \epsilon(\alpha)\Sigma_{(h)}\, 1\otimes S(h_{(1)})h_{(2)} = \epsilon(\alpha)\epsilon(h)\, 1_{H^*}\otimes 1_H
\end{aligned}
$$

and similarly for the other equation in the antipode condition. It follows directly from the formulas that $1_{H^*}\otimes 1_H$ is a unit for the multiplication

$$
\begin{aligned}
(\alpha\otimes h)\cdot(1_{H^*}\otimes 1_H) &= \Sigma_{(h)}\, 1_{H^*}(h_{(1)})\, 1_{H^*}(S^{-1}(h_{(3)}))\, \alpha 1_{H^*}\otimes h_{(2)}1_H\\
&= \Sigma_{(h)}\, \epsilon(h_{(1)})\, \epsilon(S^{-1}(h_{(3)}))\, \alpha\epsilon\otimes h_{(2)} = \alpha\otimes h\\
(1_{H^*}\otimes 1_H)\cdot(\beta\otimes k) &= \Sigma_{(\beta)}\, \beta_{(3)}(1_H)\, \beta_{(1)}(S^{-1}(1_H))\, 1_{H^*}\beta_{(2)}\otimes 1_H k\\
&= \Sigma_{(\beta)}\, \epsilon(\beta_{(3)})\epsilon(\beta_{(1)})\beta_{(2)}\otimes k = \beta\otimes k.
\end{aligned}
$$

It remains to prove the associativity of the multiplication, which follows by a direct computation:

$$\big((\alpha\otimes h)\cdot(\beta\otimes k)\big)\cdot(\gamma\otimes l)=\Sigma_{(h),(\beta)}\,\beta_{(3)}(h_{(1)})\,\beta_{(1)}(S^{-1}(h_{(3)}))\;(\alpha\beta_{(2)}\otimes h_{(2)}k)\cdot(\gamma\otimes l)$$
$$=\Sigma_{(h),(k),(\beta),(\gamma)}\;\beta_{(3)}(h_{(1)})\,\beta_{(1)}(S^{-1}(h_{(3)}))\gamma_{(3)}((h_{(2)}k)_{(1)})\gamma_{(1)}(S^{-1}((h_{(2)}k)_{(3)}))\;\alpha\beta_{(2)}\gamma_{(2)}\otimes(h_{(2)}k)_{(2)}l$$
$$=\Sigma_{(h),(k),(\beta),(\gamma)}\;\beta_{(3)}(h_{(1)})\,\beta_{(1)}(S^{-1}(h_{(5)}))\gamma_{(3)}(h_{(2)}k_{(1)})\gamma_{(1)}(S^{-1}(h_{(4)}k_{(3)}))\;\alpha\beta_{(2)}\gamma_{(2)}\otimes h_{(3)}k_{(2)}l$$

$$(\alpha\otimes h)\cdot\big((\beta\otimes k)\cdot(\gamma\otimes l)\big)=\Sigma_{(k),(\gamma)}\,\gamma_{(3)}(k_{(1)})\,\gamma_{(1)}(S^{-1}(k_{(3)}))\;(\alpha\otimes h)\cdot(\beta\gamma_{(2)}\otimes k_{(2)}l)$$
$$=\Sigma_{(k),(h),(\beta),(\gamma)}\,\gamma_{(3)}(k_{(1)})\,\gamma_{(1)}(S^{-1}(k_{(3)}))(\beta\gamma_{(2)})_{(3)}(h_{(1)})(\beta\gamma_{(2)})_{(1)}(S(h_{(3)}))\;\alpha(\beta\gamma_{(2)})_{(2)}\otimes h_{(2)}k_{(2)}l$$
$$=\Sigma_{(k),(h),(\beta),(\gamma)}\,\gamma_{(5)}(k_{(1)})\,\gamma_{(1)}(S^{-1}(k_{(3)}))(\beta_{(3)}\gamma_{(4)})(h_{(1)})(\beta_{(1)}\gamma_{(2)})(S^{-1}(h_{(3)}))\;\alpha\beta_{(2)}\gamma_{(3)}\otimes h_{(2)}k_{(2)}l$$
$$=\Sigma_{(k),(h),(\beta),(\gamma)}\,\beta_{(3)}(h_{(1)})\gamma_{(3)}(h_{(2)}k_{(1)})\,\gamma_{(1)}(S^{-1}(h_{(4)}k_{(3)}))\beta_{(1)}(S^{-1}(h_{(5)}))\;\alpha\beta_{(2)}\gamma_{(2)}\otimes h_{(3)}k_{(2)}l.\qquad\square$$

We will later derive Lemma 3.5.4 in a more conceptual way and gain more insight into the meaning of the Drinfeld double. We will now show that the Hopf algebra symmetry associated with a pair $(v,f)$ of a vertex $v$ and a face $f$ that share a marking is a representation of the Drinfeld double. By combining these symmetries for all such pairs, we find that $H^{\otimes E}$ is a module algebra over the $V$-fold product of the Drinfeld double $D(H)$ and that its submodule of invariants is the protected space.

**Proposition 3.5.5** ([BMCA]):

1. If a vertex $v$ and face $f$ share a marking, then the associated vertex and face operator define a representation of $D(H)$ on $H^{\otimes E}$
$$\rho_{(v,f)}:D(H)\to\mathrm{End}_{\mathbb{F}}(H^{\otimes E}),\quad \alpha\otimes h\mapsto B^{\alpha}_f\circ A^h_v.$$
2. This induces a representation $\rho: D(H)^{\otimes V}\to\mathrm{End}_{\mathbb{F}}(H^{\otimes E})$, whose submodule of invariants is the protected space: $(H^{\otimes E})^{D(H)^{\otimes V}}=H^{\otimes E}_{inv}$.
3. Let $\lambda\in H^*$ and $\ell\in H$ be normalised Haar integrals. Then the vertex and face operators $A^{\ell}_v$ and $B^{\lambda}_f$ do not depend on the choice of markings at $v$ and $f$. All vertex and face operators $A^{\ell}_v$ and $B^{\lambda}_f$ commute, and a projector on $H^{\otimes E}_{inv}$ is given by
$$P=\rho(\,(\lambda\otimes\ell)^{\otimes E})=\prod_{(v,f)}B^{\lambda}_f A^{\ell}_v: H^{\otimes E}\to H^{\otimes E}$$

**Proof:**
1. Suppose that $e_1,...,e_n$ are the edges at $v$, numbered counterclockwise from the marking as in Figure 1, and that all edges $e_1,...,e_n$ are incoming. Then the vertex operator for $v$ is

$$A^h_v=\Sigma_{(h)}\,L^{h_{(1)}}_{e_1+}\circ L^{h_{(2)}}_{e_2+}\circ...\circ L^{h_{(n)}}_{e_n+}=L^{h_{(1)}}_{e_1+}\circ X(h_{(2)})\circ L^{h_{(3)}}_{e_n+}$$

with a linear map $X:H\to\mathrm{End}_{\mathbb{F}}(H^{\otimes E})$, $h\mapsto X(h)$ such that $X(h)$ commutes with $L^k_{e_1+}$ and $L^k_{e_n+}$ for all $h,k\in H$. As the face $f$ shares a marking with $v$ and turns maximally right at each vertex, the edge $e_n$ is the first edge in $f$ and traversed with its orientation, and the edge $e_1$ is the last edge in $f$ and traversed against its orientation, as shown in Figure 1. Hence, the associated face operator is of the form

$$B^{\alpha}_f=\Sigma_{(\alpha)}\,T^{\alpha_{(1)}}_{e_n+}\circ Y(\alpha_{(2)})\circ T^{\alpha_{(3)}}_{e_1-}$$

with a linear map $Y:H^*\to\mathrm{End}_{\mathbb{F}}(H^{\otimes E})$, $\alpha\mapsto Y(\alpha)$. For all $\alpha,\beta\in H^*$ and $h\in H$, the element $Y(\alpha)$ commutes with $T^{\beta}_{e_1-}$, $T^{\beta}_{e_n+}$, $L^h_{e_1+}$ and $L^h_{e_n+}$, since every edge is traversed at most once by

$f$ and the edge operators for different edges commute. If $f$ does not traverse any edges incident at $v$ except $e_1, e_n$ then this also implies that $Y(\alpha)$ commutes with $X(h)$ for all $h \in H$.

If $f$ traverses any edge $e_i$ with $i \in \{2, .., n-1\}$ in the direction of its orientation, it also traverses the edge $e_{i+1}$ against its orientation and $i < n-1$, since $f$ turns maximally right at each vertex and traverses each edge only once. Conversely, if $f$ traverses the edge $e_{i+1}$ against its orientation, then it also traverses $e_i$ with its orientation. This is pictured in Figure 1. The contribution for the edges $e_i$ and $e_{i+1}$ to the face operator is then of the form $\Sigma_{(\alpha)} T^{\alpha_{(1)}}_{e_{i+1}-} \circ T^{\alpha_{(2)}}_{e_i+}$ with $i \in \{2, ..., n-2\}$. This commutes with $L^h_{e_k+}$ for all $k \notin \{i, i+1\}$. To show that it commutes with $X(h)$ for all $h \in H$, we use the definition of the edge operators and the identity $S^2 = \mathrm{id}_H$, which follows with the semisimplicity of $H$ from Theorem 3.4.10. This yields for all $e \in E$

$$\begin{aligned}
&T^\alpha_{e+} \circ L^h_{e_+} = \Sigma_{(\alpha)(h)}\, \alpha_{(1)}(h_{(2)})\, L^{h_{(1)}}_{e+} \circ T^{\alpha_{(2)}}_{e+} && T^\alpha_{e-} \circ L^h_{e_+} = \Sigma_{(\alpha)(h)}\, \alpha_{(2)}(S(h_{(1)}))\, L^{h_{(2)}}_{e+} \circ T^{\alpha_{(1)}}_{e-}\\
&L^h_{e_+} \circ T^\alpha_{e+} = \Sigma_{(\alpha)(h)}\, \alpha_{(1)}(S(h_{(2)}))\, T^{\alpha_{(2)}}_{e+} \circ L^{h_{(1)}}_{e+} && L^h_{e_+} \circ T^\alpha_{e-} = \Sigma_{(\alpha)(h)}\, \alpha_{(2)}(h_{(1)})\, L^{h_{(2)}}_{e+} T^{\alpha_{(1)}}_{e-}.
\end{aligned}$$

It follows that the contributions of the edges $e_{i+1}$ and $e_i$ to the vertex and face operator commute

$$\begin{aligned}
&\big(\Sigma_{(\alpha)}\, T^{\alpha_{(1)}}_{e_{i+1}-} T^{\alpha_{(2)}}_{e_i+}\big)\, \big(\Sigma_{(h)}\, L^{h_{(1)}}_{e_i+} L^{h_{(2)}}_{e_{i+1}+}\big)\\
&= \Sigma_{(\alpha)(h)}\, \alpha_{(2)(1)}(h_{(1)(2)})\alpha_{(1)(2)}(S(h_{(2)(1)}))\, L^{h_{(1)(1)}}_{e_i+} L^{h_{(2)(2)}}_{e_{i+1}+} T^{\alpha_{(1)(1)}}_{e_{i+1}-} T^{\alpha_{(2)(2)}}_{e_i+}\\
&= \Sigma_{(\alpha)(h)}\, \alpha_{(3)}(h_{(2)})\alpha_{(2)}(S(h_{(3)}))\, L^{h_{(1)}}_{e_i+} L^{h_{(4)}}_{e_{i+1}+} T^{\alpha_{(1)}}_{e_{i+1}-} T^{\alpha_{(4)}}_{e_i+}\\
&= \Sigma_{(\alpha)(h)}\, \alpha_{(2)}(S(h_{(3)})h_{(2)})\, L^{h_{(1)}}_{e_i+} L^{h_{(4)}}_{e_{i+1}+} T^{\alpha_{(1)}}_{e_{i+1}-} T^{\alpha_{(3)}}_{e_i+}\\
&= \Sigma_{(\alpha)(h)}\, \epsilon(\alpha_{(2)})\epsilon(h_{(2)})\, L^{h_{(1)}}_{e_i+} L^{h_{(3)}}_{e_{i+1}+} T^{\alpha_{(1)}}_{e_{i+1}-} T^{\alpha_{(3)}}_{e_i+}\\
&= (\Sigma_{(h)}\, L^{h_{(1)}}_{e_i+} L^{h_{(2)}}_{e_{i+1}+})(\Sigma_{(\alpha)}\, T^{\alpha_{(1)}}_{e_{i+1}-} T^{\alpha_{(2)}}_{e_i+}),
\end{aligned}$$

and this implies that the elements $Y(\alpha)$ and $X(h)$ commute for all $h \in H$ and $\alpha \in H^*$. With these results, we compute

$$\begin{aligned}
&\rho_{(v,f)}(\alpha \otimes h)\rho_{(v,f)}(\beta \otimes k)\\
&= \Sigma_{\substack{(h)(k)\\(\alpha)(\beta)}}\, T^{\alpha_{(1)}}_{e_n+} Y(\alpha_{(2)}) T^{\alpha_{(3)}}_{e_1-} L^{h_{(1)}}_{e_1+} X(h_{(2)}) L^{h_{(3)}}_{e_n+} T^{\beta_{(1)}}_{e_n+} Y(\beta_{(2)}) T^{\beta_{(3)}}_{e_1-} L^{k_{(1)}}_{e_1+} X(k_{(2)}) L^{k_{(3)}}_{e_n+}\\
&= \Sigma_{\substack{(h)(h)\\(\alpha)(\beta)}}\, \beta_{(1)(1)}(S(h_{(3)(2)}))\beta_{(3)(2)}(h_{(1)(1)})\\
&\qquad T^{\alpha_{(1)}}_{e_n+} T^{\beta_{(1)(2)}}_{e_n+} Y(\alpha_{(2)}) Y(\beta_{(2)}) T^{\alpha_{(3)}}_{e_1-} T^{\beta_{(3)(1)}}_{e_1-} L^{h_{(1)(1)}}_{e_1+} L^{k_{(1)}}_{e_1+} X(h_{(2)}) X(k_{(2)}) L^{h_{(3)(1)}}_{e_n+} L^{k_{(3)}}_{e_n+}\\
&= \Sigma_{\substack{(h)(k)\\(\alpha)(\beta)}}\, \beta_{(1)}(S(h_{(5)}))\beta_5(h_{(1)})\ T^{\alpha_{(1)}\beta_{(2)}}_{e_n+} Y(\alpha_{(2)}\beta_{(3)}) T^{\alpha_{(3)}\beta_{(4)}}_{e_1-} L^{h_{(2)}k_{(1)}}_{e_1+} X(h_{(3)}k_{(2)}) L^{h_{(4)}k_{(3)}}_{e_n+}\\
&= \Sigma_{(\beta)(h)}\, \beta_{(1)}(S(h_{(3)}))\beta_3(h_{(1)})\ \rho_{(v,f)}(\alpha\beta_{(2)} \otimes h_{(2)}k)\\
&= \rho_{(v,f)}(\Sigma_{(\beta)(h)}\, \beta_{(1)}(S(h_{(3)}))\beta_3(h_{(1)})\alpha\beta_{(2)} \otimes h_{(2)}k)
\end{aligned}$$

As $S^2 = \mathrm{id}_H$, this coincides with the formula for the multiplication of $D(H)$ in Lemma 3.5.4. As we also have $\rho(1_{H^*} \otimes 1_H) = B^{1_{H^*}}_f A^{1_H}_v = \mathrm{id}_{H^{\otimes E}}$, this proves that $\rho_{(v,f)}$ is a representation of $D(H)$. The proof for the case where some of the edges at $v$ are outgoing is analogous.

2. To prove that the representations $\rho_{(v,f)}$ induce a representation $\rho : D(H)^{\otimes V} \to \mathrm{End}_{\mathbb{F}}(H^{\otimes E})$, it is sufficient to show that the vertex operators $A^h_v$ and $A^k_w$ commute for $v \neq w$, the face operators $B^\alpha_f$ and $B^\beta_g$ commute for $f \neq g$ and that a vertex operator $A^h_v$ commutes with a face operator $B^\alpha_f$ if $v$ and $f$ do not share a marking.

If $v \neq w$ are vertices of $\Gamma$ such that there is no edge that is incident at both vertices, then the vertex operators $A^h_v$ and $A^k_w$ commute. If $e \in E$ is an edge that is incident at both $v$ and $w$, then it is incoming at one vertex and outgoing at the other. This implies that one of the vertex operators contains an edge operator $L^x_{e+}$ and the other the edge operator $L^y_{e-}$ for some $x, y \in H$. It follows directly from the expression for these edge operators in Definition 3.5.1 that these edge operators commute and hence $A^h_v$ and $A^k_w$ commute. Similarly, if $e \in E$ is an edge that is traversed by both faces $f, g$ then it is traversed with its orientation by one of the two faces and against its orientation by the other. Hence, one of the face operators contains the edge operator $T^\gamma_{e+}$ and the other an edge operator $T^\delta_{e-}$ for some $\gamma, \delta \in H^*$. Again it follows directly from the expressions for these edge operators in Definition 3.5.1 that these edge operators commute and hence the two face operators commute as well.

If $v$ and $f$ do not share a marking, then $f$ cannot start or end at $v$ or traverse the marking at $v$, since each face contains exactly one marking. By the same reasoning as in 1. one finds that if $e_1, ..., e_n$ are the edges incoming at $v$, ordered counterclockwise from the cilium, then $f$ either traverses none of the edges $e_1, ..., e_n$ or it traverses two consecutive edges $e_i$ and $e_{i+1}$. The same reasoning as in 1. then shows that $A^h_v$ and $B^\alpha_f$ commute for all $h \in H$ and $\alpha \in H^*$. This proves that $\rho = \otimes_{v \in V} \rho_{(v,f)} : D(H)^{\otimes V} \to \mathrm{End}_{\mathbb{F}}(H^{\otimes E})$ is a representation. By definition of $\rho$, its submodule of invariants is the ground state $H^{\otimes E}_{inv}$.

3. As $H$ is finite-dimensional semisimple, by Theorem 3.4.10, the dual Hopf algebra $H^*$ is finite-dimensional semisimple as well, and by Theorem 3.4.5 and Corollary 3.4.7 there are normalised Haar integrals $\ell \in I_L(H) = I_R(H)$ and $\lambda \in I_L(H^*) = I_R(H^*)$. By Corollary 3.4.11, one has $\Delta^{op}(\ell) = \Delta(\ell)$ and $\Delta^{op}(\lambda) = \Delta(\lambda)$. This implies that $\Delta^{(n-1)}(\ell) = (\Delta \otimes \mathrm{id}^{\otimes n-2}) \circ ... \circ (\Delta \otimes \mathrm{id}) \circ \Delta(\ell)$ and $\Delta^{(n-1)}(\lambda)$ are invariant under cyclic permutations of the $n$ factors in the tensor products for all $n \in \mathbb{N}$ (see Exercise 41). From the multiplication law of the Drinfeld double $D(H)$ in Lemma 3.5.4 it then follows that $\lambda \otimes \ell$ is a normalised Haar integral for $D(H)$ since

$$
\begin{aligned}
&(\alpha \otimes h) \cdot (\lambda \otimes \ell) = \Sigma_{(h),(\lambda)}\, \lambda_{(3)}(h_{(1)})\, \lambda_{(1)}(S(h_{(3)}))\, \alpha\lambda_{(2)} \otimes h_{(2)}\ell \\
&= \Sigma_{(h),(\lambda)}\, \epsilon(h_{(2)}) \lambda_{(3)}(h_{(1)})\, \lambda_{(1)}(S(h_{(3)}))\, \alpha\lambda_{(2)} \otimes \ell = \Sigma_{(h),(\lambda)}\, \lambda_{(3)}(h_{(1)})\, \lambda_{(1)}(S(h_{(2)}))\, \alpha\lambda_{(2)} \otimes \ell \\
&= \Sigma_{(h),(\lambda)}\, (\lambda_{(3)} S(\lambda_{(1)}))(h)\, \alpha\lambda_{(2)} \otimes \ell = \Sigma_{(h),(\lambda)}\, (\lambda_{(1)} S(\lambda_{(2)}))(h)\, \alpha\lambda_{(3)} \otimes \ell = \Sigma_{(h),(\lambda)}\, \epsilon(\lambda_{(1)}) \epsilon(h)\, \alpha\lambda_{(2)} \otimes \ell \\
&= \epsilon(h)\, \alpha\lambda \otimes \ell = \epsilon(h)\epsilon(\alpha)\, \lambda \otimes \ell
\end{aligned}
$$

where we used the identity $S^2 = \mathrm{id}_H$, then the fact that $\ell \in H$ is a Haar integral to pass to the second line, then the duality between multiplication in $H^*$ and comultiplication in $H$ to pass to the third line, then the cyclic invariance of $\Delta^{(2)}(\ell)$, the antipode condition for $H^*$ and the fact that $\lambda$ is a Haar integral for $H^*$. This shows that $\lambda \otimes \ell$ is a left integral for $D(H)$, and by a similar computation, it follows that it is a right integral. As we have $\epsilon(\lambda \otimes \ell) = \epsilon_{H^*}(\lambda)\epsilon(\ell) = 1$, it is normalised, and hence $(\lambda \otimes \ell)^{\otimes V}$ is a normalised Haar integral for $D(H)^{\otimes V}$. Lemma 3.2.8 then implies that $P$ is a projector on $H^{\otimes E}_{inv}$.

As the different choices of the markings at a vertex $v$ correspond to a cyclic permutation of the edge numbering at $v$ and hence to a cyclic permutation of the tensor factors of $\Delta^{(n-1)}(\ell)$ in $A^\ell_v$, it follows from the cyclic invariance of $\Delta^{(n-1)}(\ell)$ that $A^\ell_v$ does not depend on the choice of the marking at $v$. Similarly, different choices of a marking of a face $f$ correspond to cyclic permutations of the numbering of edges traversed by $f$ and hence to cyclic permutations of the tensor factors of $\Delta^{(m-1)}(\lambda)$ in $B^\lambda_f$. It then follows from the cyclic invariance of $\Delta^{(m-1)}(\lambda)$ that $B^\lambda_f$ does not depend on the choice of the marking at $f$.

It remains to show that the vertex and face operators $A^\ell_v$ and $B^\lambda_f$ commute if the face $f$ and the vertex $\ell$ share a marking. This follows by a direct computation from the fact that they define a representation of the Drinfeld double:

$$A_v^\ell \circ B_f^\lambda = \rho_{(v,f)}(1\otimes\ell) \circ \rho_{(v,f)}(\lambda\otimes 1) = \rho_{(v,f)}(\Sigma_{(\ell)(\lambda)}\, \lambda_{(3)}(\ell_{(1)})\, \lambda_{(1)}(S(\ell_{(3)}))\lambda_{(2)}\otimes\ell_{(2)}\,)$$
$$= \rho_{(v,f)}(\Sigma_{(\ell)(\lambda)}\, \lambda_{(1)}(\ell_{(2)})\, \lambda_{(2)}(S(\ell_{(1)}))\lambda_{(3)}\otimes\ell_{(3)}\,) = \rho_{(v,f)}(\Sigma_{(\ell)(\lambda)}\, \lambda_{(1)}(\ell_{(2)}S(\ell_{(1)}))\lambda_{(2)}\otimes\ell_{(3)}\,)$$
$$= \rho_{(v,f)}(\Sigma_{(\ell)(\lambda)}\, \epsilon(\lambda_{(1)})\epsilon(\ell_{(1)})\lambda_{(2)}\otimes\ell_{(2)}\,) = \rho_{(v,f)}(\lambda\otimes\ell) = \rho_{(v,f)}(\lambda\otimes 1) \circ \rho_{(v,f)}(1\otimes\ell) = B_f^\lambda \circ A_v^\ell,$$

where we used first the fact that $\rho_{(v,f)}$ is an algebra homomorphism and the multiplication law of $D(H)$, then the cyclic invariance of $\Delta^{(2)}(\ell)$ and $\Delta^{(2)}(\lambda)$ to pass to the second line, then the duality between multiplication in $H$ and comultiplication in $H^*$, the antipode condition to pass to the third line, the counit condition and again the fact that $\rho_{(v,f)}$ is an algebra homomorphism and the multiplication law of $D(H)$. □

We will now investigate how the Hopf algebra $D(H)^{\otimes V}$ formed by the vertex and face operators acts on the edge operators, or, equivalently, on linear maps $\phi : H^{\otimes E} \to H^{\otimes E}$.

**Proposition 3.5.6:**

1. If a vertex $v$ and a face $f$ share a marking, then $(H\#H^*)^{\otimes E} \cong \mathrm{End}_{\mathbb{F}}(H^{\otimes E})$ is a right module algebra over $D(H)$ with

$$\lhd_{(v,f)} : \mathrm{End}_{\mathbb{F}}(H^{\otimes E})\otimes D(H) \to \mathrm{End}_{\mathbb{F}}(H^{\otimes E})$$
$$Y \lhd_{(v,f)} (\alpha\otimes h) = \Sigma_{(h),(\alpha)} A_v^{S(h_{(2)})} \circ B_f^{S(\alpha_{(1)})} \circ Y \circ B_f^{\alpha_{(2)}} \circ A_v^{h_{(1)}}.$$

2. This induces a $D(H)^{\otimes V}$-right module structure $\lhd : \mathrm{End}_{\mathbb{F}}(H^{\otimes E})\otimes D(H)^{\otimes V} \to \mathrm{End}_{\mathbb{F}}(H^{\otimes E})$ on $\mathrm{End}_{\mathbb{F}}(H^{\otimes E})$ that gives $\mathrm{End}_{\mathbb{F}}(H^{\otimes E})$ the structure of a $D(H)^{\otimes V}$-right module algebra.

3. The protected space $H_{inv}^{\otimes E}$ is a module over the subalgebra of invariants $\mathrm{End}_{\mathbb{F}}(H^{\otimes E})^{D(H)^{\otimes V}}$.

**Proof:**
The first claim follows because the linear map $\rho_{(v,f)} : D(H) \to \mathrm{End}_{\mathbb{F}}(H^{\otimes E})$ from Proposition 3.5.5 is an algebra homomorphism. One can show that for any finite-dimensional semisimple Hopf algebra $K$, any algebra $A$ and any algebra homomorphism $\rho : K \to A$ the linear map

$$\lhd : A\otimes K \to A, \qquad k\otimes a \mapsto \Sigma_{(k)}\, \rho(S(k_{(2)})) \cdot a \cdot k_{(1)}$$

defines a $K$-right module algebra structure on $A$ (Exercise). As the vertex and face operators for vertices and faces that do not share a marking commute, this defines a $D(H)^{\otimes V}$-module algebra structure on $\mathrm{End}_{\mathbb{F}}(H^{\otimes E})$. To prove the last claim, it is sufficient to show that $Y(H_{inv}^{\otimes E}) \subset H_{inv}^{\otimes E}$ for $Y \in \mathrm{End}_{\mathbb{F}}(H^{\otimes E})^{D(H)^{\otimes V}}$. The $\mathrm{End}_{\mathbb{F}}(H^{\otimes E})^{D(H)^{\otimes V}}$-module structure on $H_{inv}^{\otimes E}$ is then given by restricting the $\mathrm{End}_{\mathbb{F}}(H^{\otimes E})$ module structure on $H^{\otimes E}$ from Definition 3.5.1. For this, we show first that $Y \circ B_f^\lambda \circ A_v^\ell = B_f^\lambda \circ A_v^\ell \circ Y$ for all $Y \in \mathrm{End}_{\mathbb{F}}(H^{\otimes E})^{D(H)^{\otimes V}}$, vertices $v$ and faces $f$ that share a marking. This follows by a direct computation

$$Y \circ B_f^\lambda \circ A_v^\ell = \Sigma_{(\ell),(\lambda)}(B_f^{\lambda_{(1)}} A_v^{\ell_{(3)}} A_v^{S(\ell_{(2)})} B_f^{S(\lambda_{(2)})} Y B_f^{\lambda_{(3)}} A_v^{\ell_{(1)}})$$
$$= \Sigma_{(\ell),(\lambda)}(B_f^{\lambda_{(1)}} A_v^{\ell_{(2)}} (Y \lhd_{v,f} (\lambda_{(2)}\otimes\ell_{(1)})) = \Sigma_{(\ell),(\lambda)}\epsilon(\lambda_{(2)})\epsilon(\ell_{(1)})\, B_f^{\lambda_{(1)}} A_v^{\ell_{(2)}} Y = B_f^\lambda \circ A_v^\ell \circ Y,$$

where we used that $\ell \in H$, $\lambda \in H^*$ are Haar integrals, the antipode condition, that $A_v^\ell$ and $B_f^\lambda$ commute and the definition of an invariant. By Proposition 3.5.5, this implies $Y \circ P = P \circ Y$ for all $Y \in \mathrm{End}_{\mathbb{F}}(H^{\otimes E})^{D(H)^{\otimes V}}$ and hence $Y(x) \in H_{inv}^{\otimes E}$ for all $x \in H_{inv}^{\otimes E}$. □

Hence, we have clarified the mathematical stucture of Kitaev models and shown that they form a simple application of the concepts introduced in this section:

- The edge operators associated with an edge $e \in E$ form an algebra that is isomorphic to the cross product $H\#H^*$ and to the algebra $\mathrm{End}_{\mathbb{F}}(H)$ .
- The vertex and face operator for a given marking define a representation of the Hopf algebra $D(H)$ on the extended Hilbert space $H^{\otimes E}$, and by combining the operators for all markings, one obtains a representation of $D(H)^{\otimes V}$.
- The submodule of invariants of this $D(H)^{\otimes V}$-module is the protected space. The normalised Haar integrals for $H$ and $H^*$ define a projector on the protected space.
- Similarly, the vertex and face operator for a given marking define a $D(H)$-module algebra structure on the algebra of edge operators for the different edges $e \in E$. By combining the vertex and face operators for different markings, one obtains a $D(H)^{\otimes V}$ module algebra structure on the algebra of edge operators.
- The subalgebra of invariants of this $D(H)$-module algebra structure acts on the protected space. The normalised Haar integrals of $H$ and $H^*$ define a projector on the subalgebra of invariants.

## 3.6 Representations of $U_q(\mathfrak{sl}_2)$

In this section, we consider the representations of an infinite-dimensional Hopf algebra, namely the $q$-deformed universal enveloping algebra $U_q(\mathfrak{sl}_2)$ for the case where $q \in \mathbb{C}$ is not a root of unity. This is one of the most important examples which gives rise to many applications in knot theory. As we will see in the following, in the case where $q$ is not a root of unity, the finite-dimensional simple complex representations of $U_q(\mathfrak{sl}_2)$ can be classified in a very similar manner to the finite-dimensional representations of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ and its universal enveloping algebra $U(\mathfrak{sl}_2)$. In both cases, the key concepts are the so-called *highest weight vectors* and *highest weight modules*.

**Definition 3.6.1:** Let $q \in \mathbb{C}$ and $(V, \rhd)$ be a module over $U_q(\mathfrak{sl}_2)$.

1. A **weight** of $V$ is an eigenvalue of the linear map $\phi_K : V \to V$, $v \mapsto K \rhd v$. The associated eigenspace is denoted $V^\lambda = \ker(\phi_K - \lambda \mathrm{id}_V) = \{v \in V \mid K \rhd v = \lambda v\}$.
2. An eigenvector $v \in V^\lambda \setminus \{0\}$ is called a **highest weight vector** of weight $\lambda$ if $E \rhd v = 0$.
3. The module $V$ is called a **highest weight module** of highest weight $\lambda$ if it is generated by a highest weight vector $v$ of weight $\lambda$.

As a highest weight vector $v \in V^\lambda$ is annihilated by the generator $E \in U_q(\mathfrak{sl}_2)$ and mapped to a multiple of itself by the generators $K^{\pm 1} \in U_q(\mathfrak{sl}_2)$, the only way of obtaining a basis of a highest weight module $V$ from $v$ is to act on it with the generators $F$. The relations of $U_q(\mathfrak{sl}_2)$ in (9) ensure that this yields eigenvectors of $\phi_K$ for other eigenvalues and hence a set of linearly independent vectors in $V$. By normalising these vectors with the $q$-binomials and $q$-factorials from Definition 2.3.3 and their evaluations from Definition 2.3.5, we obtain the following lemma.

**Lemma 3.6.2:** Suppose that $q \in \mathbb{C}$ is not a root of unity and $(V, \rhd)$ a module over $U_q(\mathfrak{sl}_2)$.

1. Then one has $E \rhd V^\lambda \subset V^{q^2\lambda}$ and $F \rhd V^\lambda \subset V^{q^{-2}\lambda}$ for all weights $\lambda$.
2. For an eigenvector $v \in V^\lambda \setminus \{0\}$ we define $v_0 := v$ and $v_p := q^{p(p-1)/2}/(p)!_{q^2}\, F^p \rhd v$ for all $p \in \mathbb{N}$. Then one has for all $p \in \mathbb{N}_0$

$$K \rhd v_p = \lambda q^{-2p}\, v_p, \qquad E \rhd v_{p+1} = \frac{q^{-p}\lambda - q^p\lambda^{-1}}{q - q^{-1}}\, v_p, \qquad F \rhd v_p = q^{-p}(p+1)_{q^2}\;\, v_{p+1}.$$

**Proof:**
From the relations of $U_q(\mathfrak{sl}_2)$ in (9), we obtain for every $v \in V^\lambda$

$$\begin{aligned} K \rhd (E \rhd v) &= (KE) \rhd v = (q^2 EK) \rhd v = q^2 E \rhd (K \rhd v) = q^2\lambda\, E \rhd v \\ K \rhd (F \rhd v) &= (KF) \rhd v = (q^{-2} FK) \rhd v = q^{-2} F \rhd (K \rhd v) = q^{-2}\lambda\, F \rhd v. \end{aligned}$$

This shows that $E \rhd V^\lambda \subset V^{q^2\lambda}$ and $F \rhd V^\lambda \subset V^{q^{-2}\lambda}$. To prove 2. note that $v_p$ is well-defined for all $p \in \mathbb{N}_0$ since $q \in \mathbb{C}$ is not a root of unity and hence $(p)!_{q^2} \neq 0$. The first and the last identity in 2. then follow directly from the relations of $U_q(\mathfrak{sl}_2)$ and the definition of $v_p$

$$\begin{aligned} F \rhd v_p &= \frac{q^{p(p-1)/2}}{(p)!_{q^2}} F \rhd (F^p \rhd v_0) = (p+1)_{q^2} q^{-p}\, \frac{q^{p(p+1)/2}}{(p+1)!_{q^2}} F^{p+1} \rhd v_0 = (p+1)_{q^2} q^{-p}\, v_{p+1} \\ K \rhd v_p &= \frac{q^{p(p-1)/2}}{(p)!_{q^2}} (KF^p) \rhd v_0 = q^{-2p}\, \frac{q^{p(p-1)/2}}{(p)!_{q^2}} (F^p K) \rhd v_0 = \lambda q^{-2p}\, \frac{q^{p(p-1)/2}}{(p)!_{q^2}}\, F^p \rhd v_0 = \lambda q^{-2p}\, v_p. \end{aligned}$$

The remaining identity follows by induction over $p$. If it holds for $p$, then one has

$$\begin{aligned} E \rhd v_{p+1} &= \frac{q^p}{(p+1)_{q^2}}\, E \rhd (F \rhd v_p) = \frac{q^p}{(p+1)_{q^2}} \left( F \rhd (E \rhd v_p) + \frac{K - K^{-1}}{q - q^{-1}} \rhd v_p \right) \\ &= \frac{q^p}{(p+1)_{q^2}} \left( \frac{q^{1-p}\lambda - q^{p-1}\lambda^{-1}}{q - q^{-1}} F \rhd v_{p-1} + \frac{\lambda q^{-2p} - \lambda^{-1} q^{2p}}{q - q^{-1}}\, v_p \right) \\ &= \frac{q^p}{(p+1)_{q^2}}\, \frac{(q^{1-p}\lambda - q^{p-1}\lambda^{-1}) q^{1-p} (p)_{q^2} + \lambda q^{-2p} - \lambda^{-1} q^{2p}}{q - q^{-1}}\, v_p \\ &= \frac{q^p}{(p+1)_{q^2}}\, \frac{(q^{1-p}\lambda - q^{p-1}\lambda^{-1}) q^{1-p} (q^{2p} - 1) + (\lambda q^{-2p} - \lambda^{-1} q^{2p})(q^2 - 1)}{(q - q^{-1})(q^2 - 1)}\, v_p \\ &= \frac{q^p}{(p+1)_{q^2}}\, \frac{\lambda(q^2 - q^{2-2p} + q^{2-2p} - q^{-2p}) + \lambda^{-1}(1 - q^{2p} - q^{2p+2} + q^{2p})}{(q - q^{-1})(q^2 - 1)}\, v_p \\ &= q^p\, \frac{\lambda q^2 (1 - q^{-2p-2}) + \lambda^{-1}(1 - q^{2p+2})}{(q - q^{-1})(q^{2p+2} - 1)}\, v_p = \frac{\lambda q^{-p} - \lambda^{-1} q^p}{(q - q^{-1})}\, v_p. \end{aligned}$$

□

The expressions in Lemma 3.6.2 for the action of the generators $K^{\pm 1}, E, F$ on an eigenvector $v \in V^\lambda$ and the associated vectors $v_p$ closely resemble the expressions that one would obtain if for the Lie algebra $\mathfrak{sl}_2$ or, equivalently, the universal enveloping algebra $U(\mathfrak{sl}_2)$. The only difference is that binomials and factorials are replaced by $q$-binomials and $q$-factorials. In particular, it follows that every *finite-dimensional* complex representation of $U_q(\mathfrak{sl}_2)$ must have a highest weight vector. This is a direct consequence of the fact that there can be only finitely many linearly independent eigenvectors of $\phi_K$.

**Corollary 3.6.3:** Suppose that $q \in \mathbb{C}$ is not a root of unity. Then every finite-dimensional complex module over $U_q(\mathfrak{sl}_2)$ has a highest weight vector, and every simple finite-dimensional complex module over $U_q(\mathfrak{sl}_2)$ is a highest weight module.

**Proof:**
If $V$ is a finite-dimensional $U_q(\mathfrak{sl}_2)$-module over $\mathbb{C}$, then the linear map $\phi_K : V \to V$, $v \mapsto K \rhd v$ has at least one eigenvalue $\mu \in \mathbb{C}$ and because $\phi_{K^{-1}} : V \to V$, $v \mapsto K^{-1} \rhd v$ is inverse to $\phi_K$ one has $\mu \neq 0$. If $v \in V^\mu \setminus \{0\}$ is an associated eigenvector, then by Lemma 3.6.2, one has $E^n \rhd v \in V^{q^{2n}\mu}$ for all $n \in \mathbb{N}_0$ and hence $E^n \rhd v$ is either an eigenvector for $\phi_K$ for the eigenvalue $q^{2n}\mu$ or $E^n \rhd v = 0$. As $q$ is not a root of unity and $\mu \neq 0$, all values $\mu q^{2n}$ are different and their eigenvectors are linearly independent. As $V$ is finite-dimensional, the number of linearly independent eigenvectors must be finite, and it follows that there is an eigenvector $v'$ of of $\phi_K$ with $E \rhd v' = 0$. This is a highest weight vector. If $V$ is simple, then the submodule of $V$ generated by $v'$ must be $V$, since it contains the vector $v' \in V \setminus \{0\}$, and hence $V$ is a highest weight module. □

With Lemma 3.6.2 one can show that the structure of a finite-dimensional highest weight module is determined uniquely by its weight, and that the weight is determined, up to a sign, by the dimension of the module. It also follows that a highest weight vector is unique up to multiplication by a scalar and that every finite-dimensional highest weight module is simple. This is again a direct analogue of the situation for finite-dimensional modules over the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$, and the proof proceeds along the same lines as the proof for $\mathfrak{sl}(2, \mathbb{C})$. The only difference is that the eigenvalues of the map $\phi_K : V \to V$, $v \mapsto K \rhd v$ and the eigenvectors are modified by powers, factorials and binomials of $q$.

**Theorem 3.6.4:** Suppose that $q \in \mathbb{C}$ is not a root of unity and $(V, \rhd)$ an $(n+1)$-dimensional highest weight module over $U_q(\mathfrak{sl}_2)$ that is generated by a highest weight vector $v$ of weight $\lambda$.

1. With $v_p \in V$ as in Lemma 3.6.2 one has $v_p = 0$ for $p > n$, and $\{v_0, v_1, ..., v_n\}$ is a basis of $V$.
2. The weight $\lambda$ is given by $\lambda = \epsilon q^n$ with $\epsilon \in \{\pm 1\}$ and $\phi_K : V \to V$, $v \mapsto K \rhd v$ is diagonalisable with eigenvalues $\epsilon q^n, \epsilon q^{n-2}, ..., \epsilon q^{-n+2}, \epsilon q^{-n}$.
3. Every highest weight vector of $V$ is of weight $\lambda$ and a scalar multiple of $v$.
4. The module $V$ is simple.

**Proof:**
1. If $v \in V$ is a highest weight vector of weight $\lambda$ and $V$ a highest weight module, then the relations from Lemma 3.6.2 imply $V = \text{span}_{\mathbb{F}}\{v_p \,|\, p \in \mathbb{N}_0\}$. As $K \rhd v_p = \lambda q^{-2p} v_p$ for all $p \in \mathbb{N}_0$ by Lemma 3.6.2, one has either $v_p = 0$ or $v_p$ is an eigenvector of $K$ for the eigenvalue $q^{-2p}\lambda$. As $q$ is not a root of unity and $\lambda \neq 0$, all eigenvalues $q^{-2p}\lambda$ are different. As eigenvectors for different eigenvalues are linearly independent and $V$ is finite-dimensional, there must be a $n \in \mathbb{N}_0$ with $v_n \neq 0$ and $v_{n+1} = 0$. As $v_k = 0$ implies $v_p = 0$ for all $p \geq k$, it follows that $v_p = 0$ for all $p \geq n+1$ and $v_p \neq 0$ for all $0 \leq p \leq n$. This shows that $\{v_0, ..., v_n\}$ is a basis of $V$.

2. By Lemma 3.6.2 one has

$$0 = E \rhd 0 = E \rhd v_{n+1} = \frac{\lambda q^{-n} - \lambda^{-1} q^n}{q - q^{-1}} v_n \quad \overset{v_n \neq 0}{\Rightarrow} \quad \lambda^2 = q^{2n} \quad \Rightarrow \quad \lambda = \epsilon q^n \text{ with } \epsilon \in \{\pm 1\}.$$

This implies that $\phi_K : V \to V$, $v \mapsto K \rhd v$ is diagonalisable with $n+1$ distinct eigenvalues $\lambda = \epsilon q^n$, $\lambda q^{-2} = \epsilon q^{n-2}$,..., $\lambda q^{-2n+2} = \epsilon q^{-n+1}$, $\lambda q^{-2n} = \epsilon q^{-n}$.

3. If $w \in V$ is another highest weight vector of weight $\lambda'$, then $w$ is an eigenvector of $\phi_K$. As $\phi_K$ is diagonalisable with $n+1$ pairwise distinct eigenvalues and $\dim_{\mathbb{F}}(V) = n+1$, this implies

$w = \mu v_i$ with $\mu \in \mathbb{F} \setminus \{0\}$ and $i \in \{0, ..., n\}$. By Lemma 3.6.2, the condition $E \rhd w = 0$ then implies $i = 0$ and $w = \mu v$.

4. Suppose that $\{0\} \subsetneq U \subset V$ is a submodule. Then $U$ has a highest weight vector $u \in U \setminus \{0\}$ by Corollary 3.6.3 and $u$ is also a highest weight vector for $V$. By 4. this implies $u = \mu v$ with $\mu \in \mathbb{F} \setminus \{0\}$ and $U = V$. □

By combining this Theorem with Corollary 3.6.3, we obtain a complete classification of all finite-dimensional complex simple $U_q(\mathfrak{sl}_2)$-modules. As every finite-dimensional complex simple $U_q(\mathfrak{sl}_2)$-module is a highest weight module by Corollary 3.6.3, it only remains to show that highest weight modules of the same weight are isomorphic as $U_q(\mathfrak{sl}_2)$-modules. This is a direct consequence of the fact that any highest weight module is spanned by its highest weight vector $v \in V^\lambda$ and the vectors $F^p \rhd v$ for $p \in \mathbb{N}$.

**Corollary 3.6.5:** Suppose that $q \in \mathbb{C}$ is not a root of unity. Then:

1. Finite-dimensional complex highest weight modules over $U_q(\mathfrak{sl}_2)$ of the same weight are isomorphic as $U_q(\mathfrak{sl}_2)$-modules.
2. Every $n$-dimensional simple complex $U_q(\mathfrak{sl}_2)$-module is a highest weight module of weight $\lambda = \pm q^{n-1}$.

**Proof:**
By Corollary 3.6.3 every finite-dimensional simple module $V$ over $U_q(\mathfrak{sl}_2)$ has a highest weight vector $v \in V$ of weight $\lambda$. As $V$ is simple and $v \neq 0$, the submodule generated by $v$ must be $V$ and hence $V$ is a highest weight module. If $V$ and $W$ are finite-dimensional highest weight modules of weight $\lambda$, then there are highest weight vectors $v \in V$ and $w \in W$ of weight $\lambda$. By Theorem 3.6.4 one has $V = \text{span}_{\mathbb{F}}\{v_0, ..., v_n\}$ and $W = \text{span}_{\mathbb{F}}\{w_0, ..., w_n\}$ with $\lambda = \epsilon q^{2n}$ and $v_p$ and $w_p$ defines as in Lemma 3.6.2. By Lemma 3.6.2, the linear map $\psi : V \to W$ with $\psi(v_p) = w_p$ for all $p \in \{0, 1, ..., n\}$ is a linear isomorphism, and by Lemma 3.6.2 it is an isomorphism of representations. By Corollary 3.6.3 every simple complex $n$-dimensional $U_q(\mathfrak{sl}_2)$-module is a highest weight module, and by Theorem 3.6.4 is weight is given by $\lambda = \pm q^{n-1}$. □

To conclude our discussion of finite-dimensional complex simple $U_q(\mathfrak{sl}_2)$ modules, we note that there is a universal module $V(\lambda)$ from which every finite-dimensional complex $U_q(\mathfrak{sl}_2)$-module of weight $\lambda$ can be obtained by taking the quotient with respect to a suitable submodule. As there are simple complex $U_q(\mathfrak{sl}_2)$-modules of any dimension, it is clear that any such universal module must be infinite-dimensional. Moreover, by the same considerations as in the proof of Lemma 3.6.2, we find that it must be generated by a highest weight vector $v$ of weight $\lambda$, from which all other vectors in a basis of $V(\lambda)$ can be obtained by acting on it with the generators $F^p \in U_q(\mathfrak{sl}_2)$. This leads to the concept of a *Verma module*.

**Theorem 3.6.6:** Suppose that $q \in \mathbb{C}$ is not a root of unity and $\lambda \neq 0$.

1. Let $V$ be the complex vector space with basis $\{v_p\}_{p \in \mathbb{N}_0}$. Then

$$K^{\pm 1} \rhd v_p = \lambda^{\pm 1} q^{\mp 2p}\, v_p, \quad E \rhd v_p = \frac{q^{1-p}\lambda - q^{p-1}\lambda^{-1}}{q - q^{-1}}\, v_{p-1}, \quad F \rhd v_p = q^{-p}(p+1)_{q^2}\, v_{p+1}$$

with $v_{-1} := 0$ defines a $U_q(\mathfrak{sl}_2)$-module structure on $V$. This module is denoted $V(\lambda)$ and called the **Verma module** of highest weight $\lambda$.

2. The vector $v_0 \in V$ generates $V(\lambda)$ and is a highest weight vector of weight $\lambda$.

3. The Verma module $V(\lambda)$ has the following **universal property**: Any highest weight $U_q(\mathfrak{sl}_2)$-module of highest weight $\lambda$ is a quotient module of $V(\lambda)$.

**Proof:**
To show that this defines a $U_q(\mathfrak{sl}_2)$-module structure on $V$ it is sufficient to verify the multiplication relations of $U_q(\mathfrak{sl}_2)$ from (9)

$$(K^{\mp 1}K^{\pm 1}) \rhd v_p = K^{\pm 1} \rhd (K^{\pm 1} \rhd v_p) = \lambda^{\pm 1} q^{\mp 2p}\, K^{\mp 1} \rhd v_p = \lambda^{\pm 1} q^{\mp 2p}\, \lambda^{\mp 1} q^{\pm 2p}\, v_p = v_p = 1 \rhd v_p$$

$$\begin{aligned}(KE) \rhd v_p &= K \rhd (E \rhd v_p) = \frac{q^{1-p}\lambda - q^{p-1}\lambda^{-1}}{q-q^{-1}}\, K \rhd v_{p-1} = \lambda q^{2-2p} \frac{q^{1-p}\lambda - q^{p-1}\lambda^{-1}}{q-q^{-1}}\, v_{p-1} \\ &= q^2 \lambda q^{-2p} \frac{q^{1-p}\lambda - q^{p-1}\lambda^{-1}}{q-q^{-1}}\, v_{p-1} = q^2 E \rhd (\lambda q^{-2p} v_p) = (q^2 EK) \rhd v_p\end{aligned}$$

$$\begin{aligned}(KF) \rhd v_p &= K \rhd (F \rhd v_p) = q^{-p}(p+1)_{q^2}\, K \rhd v_{p+1} = \lambda q^{-2p-2} q^{-p}(p+1)_{q^2}\, v_{p+1} \\ &= q^{-2} F \rhd (\lambda q^{-2p} v_p) = q^{-2} F \rhd (K \rhd v_p) = (q^{-2} FK) \rhd v_p\end{aligned}$$

$$\begin{aligned}([E,F]) \rhd v_p &= E \rhd (F \rhd v_p) - F \rhd (E \rhd v_p) \\ &= q^{-p}(p+1)_{q^2} E \rhd v_{p+1} - \frac{q^{1-p}\lambda - q^{p-1}\lambda^{-1}}{q-q^{-1}}\, F \rhd v_{p-1} \\ &= \frac{q^{-p}(q^{2p+2}-1)(\lambda q^{-p} - \lambda^{-1} q^p) - q^{1-p}(q^{2p}-1)(q^{1-p}\lambda - q^{p-1}\lambda^{-1})}{(q^2-1)(q-q^{-1})}\, v_p \\ &= \frac{\lambda(q^2 - q^{-2p} - q^2 + q^{2-2p}) + \lambda^{-1}(1 - q^{2p+2} - 1 + q^{2p})}{(q^2-1)(q-q^{-1})}\, v_p = \frac{\lambda q^{-2p} - q^{2p}\lambda^{-1}}{(q-q^{-1})}\, v_p \\ &= (q-q^{-1})(K - K^{-1}) \rhd v_p\end{aligned}$$

for all $p \in \mathbb{N}_0$. This shows that $V(\lambda)$ is a $U_q(\mathfrak{sl}_2)$-module. As we have $E \rhd v_0 = 0$ and $K \rhd v_0 = \lambda v_0$ by definition of $V(\lambda)$, the vector $v_0$ is a highest weight vector of weight $\lambda$. As we also have by definition $V = \mathrm{span}_{\mathbb{F}}\{v_p \,|\, p \in \mathbb{N}_0\}$ and the relation $F \rhd v_p = q^{-p}(p+1)_{q^2} v_{p+1}$ implies $v_p = \mu_p F^p \rhd v_0$ for some $\mu_p \neq 0$, the vector $v_0$ generates $V(\lambda)$. To verify the universal property of the Verma module $V(\lambda)$, suppose that $W$ is a highest weight module of highest weight $\lambda$ and $w \in W$ a highest weight vector. Then the linear map

$$\psi : V(\lambda) \to W, \qquad v_p \mapsto \frac{q^{p(p-1)/2}}{(p)!_{q^2}} F^p \rhd w$$

is a surjective homomorphism of representations. This implies $W \cong V(\lambda)/\ker(\psi)$ as a $U_q(\mathfrak{sl}_2)$-module. □

Finally, we note that every finite-dimensional $U_q(\mathfrak{sl}_2)$ module is semisimple and hence can be decomposed as a direct sum of the highest weight modules from Theorem 3.6.4. The proof proceeds by constructing a complement of a submodule $U \subset V$ of a finite-dimensional $U_q(\mathfrak{sl}_2)$-module $V$, where the cases $\dim_F V - \dim_{\mathbb{F}} U = 1$ and $\dim_{\mathbb{F}} V - \dim_{\mathbb{F}} U > 1$ are treated separately and one performs an induction over the dimension of $U$. As it is lengthy and technical and mainly based on techniques from linear algebra, we omit this proof and refer to [Ka, Theorem VII.2.2].

**Theorem 3.6.7:** Suppose that $q \in \mathbb{C}$ is not a root of unity. Then every finite-dimensional $U_q(\mathfrak{sl}_2)$-module is semisimple.

Although the classification of finite-dimensional complex simple $U_q(\mathfrak{sl}_2)$-modules is achieved by the same methods and structurally very similar to the classification of finite-dimensional complex simple modules over $\mathfrak{sl}_2$ and $U(\mathfrak{sl}_2)$, there is an important difference between modules over $U_q(\mathfrak{sl}_2)$ and over $U(\mathfrak{sl}_2)$: representations of $U_q(\mathfrak{sl}_2)$ and $U(\mathfrak{sl}_2)$ on tensor products of $U_q(\mathfrak{sl}_2)$-modules and $U(\mathfrak{sl}_2)$-modules are constructed with the comultiplication of $U_q(\mathfrak{sl}_2)$ and $U(\mathfrak{sl}_2)$. While the comultiplication of $U(\mathfrak{sl}_2)$ is cocommutative, this is not the case for $U_q(\mathfrak{sl}_2)$. Consequently, if $V$ and $W$ are modules over $U(\mathfrak{sl}_2)$, then the flip map $\tau : V \otimes W \to W \otimes V$, $v \otimes w \mapsto w \otimes v$ is a module isomorphism between the $U(\mathfrak{sl}_2)$-modules $V \otimes W$ and $W \otimes V$, while this does not hold in general for modules over $U_q(\mathfrak{sl}_2)$. The $U_q(\mathfrak{sl}_2)$-module structure of the tensor product of $U_q(\mathfrak{sl}_2)$-modules depends on the order of the factors in the tensor product.

To conclude the discussion of the finite-dimensional $U_q(\mathfrak{sl}_2)$-modules, we comment on the case where $q \in \mathbb{C}$ is a root of unity. In this case, the relevant Hopf algebra is the finite-dimensional Hopf algebra $U_q^r(\mathfrak{sl}_2)$ from Proposition 2.3.12 which is obtained as a quotient of $U_q(\mathfrak{sl}_2)$. It was shown in Example 3.4.12 that this Hopf algebra is not semisimple. Hence, it exhibits finite-dimensional representations that are not simple and cannot be decomposed as a direct sum of simple representations.

It is nevertheless instructive to consider its finite-dimensional simple representations. One find that in addition to a finite set of highest weight modules from Theorem 3.6.4 for certain values of $\lambda$, there is another set of finite-dimensional simple modules $V'_{\lambda,r}$, given as vector spaces by $V'_{\lambda,r} = \text{span}\{v_0, ..., v_{r-1}\}$ and with the $U_q(\mathfrak{sl}_2)$-module structure

$$K \rhd v_p = \lambda q^{-2p} v_p \qquad E \rhd v_p = q^{1-p}(p)_{q^2} \left( \frac{q^{1-p}\lambda - q^{p-1}\lambda^{-1}}{q - q^{-1}} \right) v_{p-1} \qquad F \rhd v_p = v_{p+1}$$

where we set $v_{-1} = v_r = 0$. The classification of finite-dimensional simple complex $U_q^r(\mathfrak{sl}_2)$-modules is then given as follows.

**Remark 3.6.8:** Suppose that $q \in \mathbb{C}$ is a primitive $n$th root of unity and set $r = n$ if $n$ is odd and $r = n/2$ if $p$ is even. Then every finite-dimensional nonzero simple module over the Hopf algebra $U_q^r(\mathfrak{sl}_2)$ from Proposition 2.3.12 is of the form

- $r = n$ **odd:** A highest weight module from Theorem 3.6.4 with $\lambda = q^m$ and $0 \leq m < r - 1$ or the module $V'_{q^{-1},r}$.
- $n$ **even,** $r = n/2$ **even:** A highest weight module from Theorem 3.6.4 with $\lambda = \pm q^m$ with $0 \leq m < r - 1$ even.
- $n$ **even,** $r = n/2$ **odd:** A highest weight module from Theorem 3.6.4 with $\lambda = q^m$ and $0 \leq m < r - 1$ even, or with $\lambda = -q^m$ and $0 < m < r - 1$ odd, or the module $V'_{-q^{-1},r}$.

# 4 Monoidal categories and monoidal functors

## 4.1 Monoidal categories

Bialgebras were introduced as algebras with additional structure that ensured that their representation theory behaves like the one of a group. More precisely, we required a trivial representation on the underlying field $\mathbb{F}$ and a canonical representation on the tensor product of two representation spaces. In addition to these requirements, we imposed that the canonical linear isomorphisms $a_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$, $l_V : \mathbb{F} \otimes V \to V$ and $r_V : V \otimes \mathbb{F} \to V$ are isomorphisms of representations.

In this section, we take a more abstract viewpoint and formulate the concept of a tensor product for general categories. This requires a formulation that relies only on objects, morphisms, functors and natural transformations. For this, recall that the tensor product of vector spaces induces tensor products of linear maps. For each pair $(f, g)$ of linear maps $f : V \to V'$ and $g : W \to W'$ there is a linear map $f \otimes g = (\mathrm{id}_{V'} \otimes g) \circ (f \otimes \mathrm{id}_W) = (f \otimes \mathrm{id}_{W'}) \circ (\mathrm{id}_V \otimes g) : V \otimes W \to V' \otimes W'$. This states that the tensor product is a functor $\otimes : \mathrm{Vect}_\mathbb{F} \times \mathrm{Vect}_\mathbb{F} \to \mathrm{Vect}_\mathbb{F}$, since the composition of morphisms in the category $\mathrm{Vect}_\mathbb{F} \times \mathrm{Vect}_\mathbb{F}$ is given by the composition in $\mathrm{Vect}_\mathbb{F}$ and the relation $f \times g = (f \times \mathrm{id}_{W'}) \circ (\mathrm{id}_V \times g) = (\mathrm{id}_{V'} \times g) \circ (f \times \mathrm{id}_W)$.

The associativity isomorphisms $a_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ relate the value of the functors $\otimes(\otimes \times \mathrm{id}_{\mathrm{Vect}_\mathbb{F}})$ and $\otimes(\mathrm{id}_{\mathrm{Vect}_\mathbb{F}} \times \otimes)$ on the triple $(U, V, W)$ of objects in $\mathrm{Vect}_\mathbb{F}$. Similarly, if $\mathbb{F} \times \mathrm{id}_{\mathrm{Vect}_\mathbb{F}} : \mathrm{Vect}_\mathbb{F} \to \mathrm{Vect}_\mathbb{F} \times \mathrm{Vect}_\mathbb{F}$ denotes the functor that assigns to a vector space $V$ the pair $(\mathbb{F}, V)$ and to a linear map $f : V \to W$ the pair $(\mathrm{id}_\mathbb{F}, f)$, then the left and right unit isomorphisms $l_V : \mathbb{F} \otimes V \to V$ and $r_V : V \otimes \mathbb{F} \to V$ relate the values of the functors $\otimes(\mathbb{F} \times \mathrm{id}_{\mathrm{Vect}_\mathbb{F}})$ and $\otimes(\mathrm{id}_{\mathrm{Vect}_\mathbb{F}} \times \mathbb{F})$ on an object $V$ to the values of the functor $\mathrm{id}_{\mathrm{Vect}_\mathbb{F}}$. Moreover, the associativity and unit isomorphisms are compatible with linear maps in the sense that

$$a_{U',V',W'} \circ ((f \otimes g) \otimes h) = (f \otimes (g \otimes h)) \circ a_{U,V,W}, \quad l_{U'} \circ (\mathrm{id}_\mathbb{F} \otimes f) = f \circ l_U, \quad r_{U'} \circ (f \otimes \mathrm{id}_\mathbb{F}) = f \circ r_U$$

for all linear maps $f : U \to U'$, $g : V \to V'$ and $h : W \to W'$. We can therefore interpret them as component morphisms of natural isomorphisms $a : \otimes(\otimes \times \mathrm{id}_{\mathrm{Vect}_\mathbb{F}}) \to \otimes(\mathrm{id}_{\mathrm{Vect}_\mathbb{F}} \times \otimes)$, $l : \otimes(\mathbb{F} \times \mathrm{id}_{\mathrm{Vect}_\mathbb{F}}) \to \mathrm{id}_{\mathrm{Vect}_\mathbb{F}}$ and $r : \otimes(\mathrm{id}_{\mathrm{Vect}_\mathbb{F}} \times \mathbb{F}) \to \mathrm{id}_{\mathrm{Vect}_\mathbb{F}}$. If we also take into account the compatibility conditions between multiple composites of the associativity isomorphisms and between associativity and unit morphisms, we obtain the following definition that generalises the tensor product of vector spaces to tensor products in a general category.

**Definition 4.1.1:**

A **monoidal category** or **tensor category** is a sextuple $(\mathcal{C}, \otimes, e, a, l, r)$ consisting of

- a category $\mathcal{C}$,
- a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, the **tensor product**,
- an object $e$ in $\mathcal{C}$, the **tensor unit**,
- a natural isomorphism $a : \otimes(\otimes \times \mathrm{id}_\mathcal{C}) \to \otimes(\mathrm{id}_\mathcal{C} \times \otimes)$, the **associator**,
- natural isomorphisms $r : \otimes(\mathrm{id}_\mathcal{C} \times e) \to \mathrm{id}_\mathcal{C}$ and $l : \otimes(e \times \mathrm{id}_\mathcal{C}) \to \mathrm{id}_\mathcal{C}$, the **unit constraints**,

subject to the following two conditions:

1. **pentagon axiom**: for all objects $U, V, W, X$ of $\mathcal{C}$ the following diagram commutes

$$\begin{array}{ccccc} ((U\otimes V)\otimes W)\otimes X & \xrightarrow{a_{U\otimes V,W,X}} & (U\otimes V)\otimes(W\otimes X) & \xrightarrow{a_{U,V,W\otimes X}} & U\otimes(V\otimes(W\otimes X)) \\ \downarrow{\scriptstyle a_{U,V,W}\otimes 1_X} & & & \nearrow{\scriptstyle 1_U\otimes a_{V,W,X}} & \\ (U\otimes(V\otimes W))\otimes X & \xrightarrow[a_{U,V\otimes W,X}]{} & U\otimes((V\otimes W)\otimes X). & & \end{array}$$

2. **triangle axiom**: for all objects $V, W$ of $\mathcal{C}$ the following diagram commutes

$$\begin{array}{ccc} (V\otimes e)\otimes W & \xrightarrow{a_{V,e,W}} & V\otimes(e\otimes W) \\ & {\scriptstyle r_V\otimes 1_W}\searrow \quad \swarrow{\scriptstyle 1_V\otimes l_W} & \\ & V\otimes W. & \end{array}$$

A monoidal category is called **strict** if the natural isomorphisms $a$, $r$ and $l$ are the identity natural transformations.

**Remark 4.1.2:**

1. The tensor unit and the unit constraints are determined by $\otimes$ uniquely up to unique isomorphism: If $e'$ is an object in $\mathcal{C}$ with natural isomorphisms $r' : \mathrm{id}_\mathcal{C} \times e' \to \mathrm{id}_\mathcal{C}$ and $l' : e' \times \mathrm{id}_\mathcal{C} \to \mathrm{id}_\mathcal{C}$, then there is a unique isomorphism $\phi : e \to e'$ with $r'_X \circ (1_X \times \phi) = r_X$ and $l'_X \circ (\phi \times 1_X) = l_X$. (Exercise).

2. One can show that if $C, D$ are objects of a monoidal category $(\mathcal{C}, \otimes, e, a, l, r)$ and $f, g : C \to D$ morphisms in $\mathcal{C}$ that are obtained by composing identity morphisms, component morphisms of the associator $a$ and component morphisms of the left and right unit constraints $l, r$ with the composition of morphisms and the tensor product, then $f$ and $g$ are equal. This is MacLane's famous **coherence theorem**. A proof of this statement can be found in [McL, Chapter VI.2] and [Ka, Chapter XI.5].

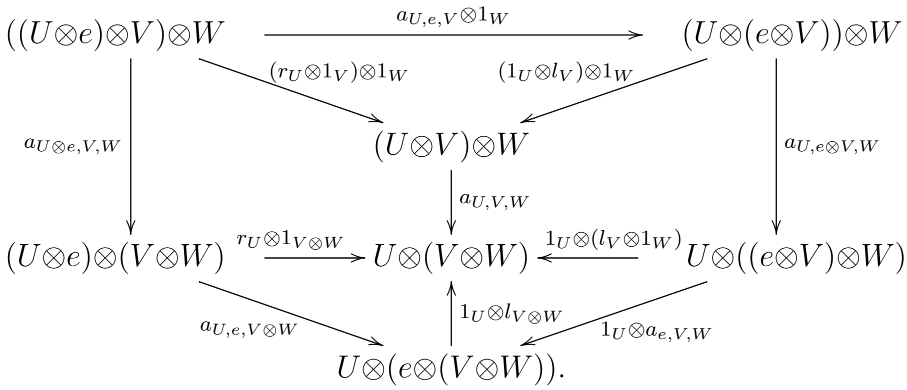
Examples of Mac Lane's coherence theorem are the following lemma and corollary about the properties of the tensor unit and the associated left and right unit constraints. They motivate the name *monoidal category*.

**Lemma 4.1.3:** Let $(\mathcal{C}, \otimes, e, a, l, r)$ be a monoidal category. Then the diagrams

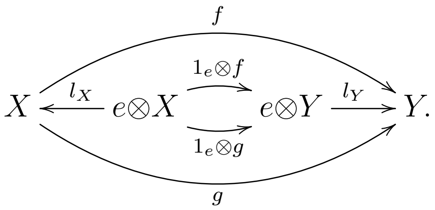
$$\begin{array}{ccc} (e\otimes V)\otimes W & \xrightarrow{a_{e,V,W}} & e\otimes(V\otimes W) \\ & {\scriptstyle l_V\otimes 1_W}\searrow \quad \swarrow{\scriptstyle l_{V\otimes W}} & \\ & V\otimes W & \end{array} \qquad \begin{array}{ccc} (V\otimes W)\otimes e & \xrightarrow{a_{V,W,e}} & V\otimes(W\otimes e) \\ & {\scriptstyle r_{V\otimes W}}\searrow \quad \swarrow{\scriptstyle 1_V\otimes r_W} & \\ & V\otimes W & \end{array}$$

commute for all objects $V, W$, and one has $l_{e\otimes V} = 1_e\otimes l_V$, $r_{V\otimes e} = r_V\otimes 1_e$ and $l_e = r_e$.

**Proof:**
1. We consider for objects $U, V, W$ of $\mathcal{C}$ the diagram

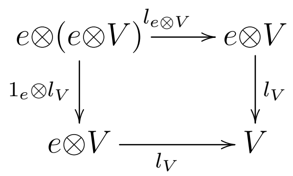


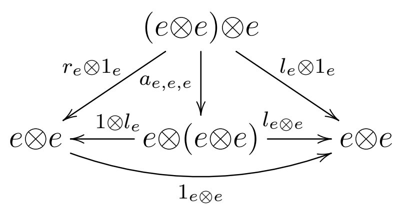

The outer pentagon in this diagram commutes by the pentagon axiom, the upper triangle and the lower left triangle commute by the triangle axiom and the two quadrilaterals by the naturality of $a : \otimes(\otimes \times \mathrm{id}_{\mathcal{C}}) \to \otimes(\mathrm{id}_{\mathcal{C}} \times \otimes)$. As all arrows in this diagram are isomorphisms, it follows that the lower right triangle commutes as well. To show that this implies the commutativity of the first triangle in the lemma, we choose $U = e$ and use the naturality of $l : \otimes(e \times \mathrm{id}_{\mathcal{C}}) \to \mathrm{id}_{\mathcal{C}}$, which implies $f = g$ for all morphisms $f, g : X \to Y$ with $1_e \otimes f = 1_e \otimes g$:




This shows that the first triangle commutes, and the proof for the second triangle is similar.

2. To prove the last three identities in the lemma, we consider the commutative diagrams

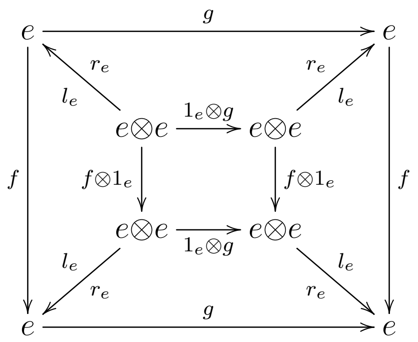



The first diagram commutes by the naturality of $l : \otimes(e \times \mathrm{id}_{\mathcal{C}}) \to \mathrm{id}_{\mathcal{C}}$. Because $l_V : e \otimes V \to V$ is an isomorphism, it follows that $l_{e \otimes V} = 1_e \otimes l_V$. The proof of the identity $r_{V \otimes e} = r_V \otimes 1_e$ is analogous. In the second diagram, the lower triangle is the identity $l_{e \otimes e} = 1_e \otimes l_e$, which follows from the first diagram with $V = e$, the left triangle commutes by the triangle axiom and the right triangle commutes by 1. Hence, the outer triangle commutes as well and $l_e \otimes 1_e = r_e \otimes 1_e$. By the same argument as in 1. this implies $r_e = l_e$. $\square$

**Corollary 4.1.4:** The endomorphisms of the tensor unit in a monoidal category form a commutative monoid.

**Proof:**
In any category $\mathcal{C}$ and for any object $C$ the set $\mathrm{Hom}_{\mathcal{C}}(C,C)$ is a monoid with the composition of morphisms. To show that $\mathrm{Hom}_{\mathcal{C}}(e,e)$ is commutative, we consider the diagram




where we used the identity $l_e = r_e$ from Lemma 4.1.3. The inner rectangle commutes because $\otimes : \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ is a functor. The inner quadrilaterals commute by the naturality of $r$ and $l$, and hence the outer square commutes as well. □

Many categories from algebra or topology have a monoidal structure. This includes in particular any category with finite products or coproducts. Note that a given category can have several non-equivalent monoidal structures. Specifying the functor $\otimes : \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ and the associativity constraint $a$ amount to a choice of *structure*, while the tensor unit and unit constraints are essentially determined by the functor $\otimes$.

**Example 4.1.5:**

1. The category $\mathrm{Vect}_{\mathbb{F}}$ of vector spaces over $\mathbb{F}$ is a monoidal category with:
   - the functor $\otimes : \mathrm{Vect}_{\mathbb{F}}\times\mathrm{Vect}_{\mathbb{F}}\to\mathrm{Vect}_{\mathbb{F}}$ that assigns to a pair $(V,W)$ of vector spaces over $\mathbb{F}$ the vector space $V\otimes W$ and to a pair $(f,g)$ of linear maps $f: V\to V'$, $g: W\to W'$ the linear map $f\otimes g : V\otimes W\to V'\otimes W'$, $v\otimes w\mapsto f(v)\otimes g(w)$,
   - the tensor unit $e=\mathbb{F}$,
   - the associator with component isomorphisms
     $a_{U,V,W} : (U\otimes V)\otimes W \xrightarrow{\sim} U\otimes(V\otimes W)$, $(u\otimes v)\otimes w\mapsto u\otimes(v\otimes w)$
   - the unit constraints with component morphisms
     $r_V : V\otimes\mathbb{F}\xrightarrow{\sim} V$, $v\otimes\lambda\mapsto\lambda v$ and $l_V:\mathbb{F}\otimes V\xrightarrow{\sim} V$, $\lambda\otimes v\mapsto\lambda v$.

2. More generally, for any commutative ring $R$, the category $R$-Mod of modules over $R$ is a monoidal category with:
   - the functor $\otimes_R : R\text{-Mod}\times R\text{-Mod}\to R\text{-Mod}$ that assigns to a pair $(M,N)$ of $R$-modules the module $M\otimes_R N$ and to a pair $(f,g)$ of $R$-linear maps $f: M\to M'$, $g: N\to N'$ the $R$-linear map $f\otimes_R g : M\otimes_R N\to M'\otimes_R N'$, $m\otimes n\mapsto f(m)\otimes g(n)$,
   - the tensor unit $e=R$,
   - the associator with component morphisms
     $a_{M,N,P} : (M\otimes_R N)\otimes_R P\xrightarrow{\sim} M\otimes_R(N\otimes_R P)$, $(m\otimes n)\otimes p\mapsto m\otimes(n\otimes p)$
   - the unit constraints with component morphisms
     $r_M : M\otimes_R R\xrightarrow{\sim} M$, $m\otimes r\mapsto r\rhd m$ and $l_M : R\otimes_R M\xrightarrow{\sim} M$, $r\otimes m\mapsto r\rhd m$.

3. For any small category $\mathcal{C}$, the category $\mathrm{End}(\mathcal{C})$ of endofunctors $F : \mathcal{C} \to \mathcal{C}$ and natural transformations between them is a *strict* monoidal category with:

   - the functor $\otimes : \mathrm{End}(\mathcal{C}) \times \mathrm{End}(\mathcal{C}) \to \mathrm{End}(\mathcal{C})$ that assigns to a pair $(F, G)$ of functors $F, G : \mathcal{C} \to \mathcal{C}$ the functor $FG : \mathcal{C} \to \mathcal{C}$ and to a pair $(\mu, \eta)$ of natural transformations $\mu : F \to F'$, $\eta : G \to G'$ the natural transformation $\mu\eta : FG \to F'G'$ with component morphisms $(\mu\eta)_C = \mu_{G'(C)} \circ F(\eta_C) = F'(\eta_C) \circ \mu_{G(C)} : FG(C) \to F'G'(C)$,
   - the identity functor as the tensor unit: $e = \mathrm{id}_{\mathcal{C}}$.

4. The categories Set and Top are monoidal categories with:

   - the functor $\otimes$ : Set $\times$ Set $\to$ Set that assigns to a pair of sets $(X, Y)$ their cartesian product $X \times Y$ and to a pair $(f, g)$ of maps $f : X \to X'$, $g : y \to Y'$ the product $f \times g : X \times Y \to X' \times Y'$, $(x, y) \mapsto (f(x), g(y))$ and the functor $\otimes$ : Top $\times$ Top $\to$ Top that assigns to a pair $(X, Y)$ of topological spaces the product space $X \times Y$ and to a pair of continuous maps $f : X \to X'$, $g : Y \to Y'$ the product map $f \times g$,
   - the one-point set $\{p\}$ and the one-point space $\{p\}$ as the tensor unit,
   - the associators with component morphisms
     $a_{X,Y,Z} : (X \times Y) \times Z \to X \times (Y \times Z)$, $((x, y), z) \mapsto (x, (y, z))$,
   - the unit constraints with component morphisms
     $r_X : X \times \{p\} \to X$, $(x, p) \mapsto x$ and $l_X : \{p\} \times X \to X$, $(p, x) \mapsto x$.

5. More generally, any category $\mathcal{C}$ with finite (co)products is a tensor category with:

   - the functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ that assigns to a pair of objects their (co)product and to a pair of morphisms the corresponding morphism between (co)products that is induced by their universal property,
   - the empty (co)product, i. e. the initial or final object in $\mathcal{C}$ as the tensor unit,
   - the associators induced by the universal properties of the (co)products,
   - the unit constraints induced by the universal properties of the (co)products.

   This includes:

   - the category Set with the disjoint union of sets and the empty set,
   - the category Top with the sum of topological spaces and the empty space,
   - the category $\mathrm{Mfld}_n$ of topological or smooth $n$-dimensional manifolds with the disjoint union and the empty manifold,
   - the category $R$-Mod for any ring $R$ with direct sums and the null module,
   - the category $\mathrm{Top}^1$ of pointed topological spaces with wedge sums and the one-point space or with products of pointed spaces and the one-point space,
   - the category Grp with the direct product of groups and the trivial group or with the free product of groups and the trivial group.

6. For any commutative ring $R$, the category $\mathrm{Ch}_{R\text{-Mod}}$ of chain complexes in $R$-Mod is a strict monoidal category with the tensor product of chain complexes given by

$$(C_\bullet \otimes D_\bullet)_n = \oplus_{k=0}^n C_k \otimes_R D_{n-k}, \quad d^{C\otimes D}(c\otimes d) = d^C(c)\otimes d + (-1)^{\deg(c)} c \otimes d^D(d)$$

   and the induced tensor product of chain maps.

**Example 4.1.6:** The **simplex category** $\Delta$ has as objects **ordinal numbers** $[0] = \emptyset$ and $[n] = \{0, 1, ..., n-1\}$ for $n \in \mathbb{N}$ and as morphisms $f : [n] \to [m]$ monotone maps, i. e. maps $f : \{0, ..., n-1\} \to \{0, ..., m-1\}$ with $f(i) \le f(j)$ for all $0 \le i \le j < n$. It is a strict tensor category with:

- the functor $\otimes : \Delta \times \Delta \to \Delta$ that assigns to a pair $([m], [n])$ of ordinals the ordinal $[m+n]$ and to a pair $(f, g)$ of monotone maps $f : [m] \to [m']$ and $g : [n] \to [n']$ the map $f \otimes g : [m+n] \to [m'+n']$ with $(f \otimes g)(i) = f(i)$ for $0 \le i \le m-1$ and $(f \otimes g)(i) = m' + g(i-m)$ for $m \le i \le n+m-1$,
- the ordinal $[0] = \emptyset$ as the tensor unit.

Our algebraic main example of a monoidal category is the representation category of a bialgebra. The condition that the representations of a bialgebra should form a tensor category was our original motivation for the bialgebra axioms. We can now formulate this result in the language of monoidal categories.

**Theorem 4.1.7:** Let $(A, m, \eta)$ be an algebra over $\mathbb{F}$, $\Delta : A \to A \otimes A$, $\epsilon : A \to \mathbb{F}$ algebra homomorphisms and $(V, \rhd_V)$, $(W, \rhd_W)$ modules over $A$.

1. Then $\mathbb{F}$ and $V \otimes W$ become $A$-modules with

$$\rhd_{\mathbb{F}} : A \otimes \mathbb{F} \to \mathbb{F}, \quad a \otimes \lambda \mapsto \epsilon(a)\lambda$$
$$\rhd_{V \otimes W} : A \otimes (V \otimes W) \to V \otimes W, \; a \otimes (v \otimes w) \mapsto \Sigma_{(a)} (a_{(1)} \rhd_V v) \otimes (a_{(2)} \rhd_W w).$$

2. The induced functor $\otimes : A\text{-Mod} \times A\text{-Mod} \to A\text{-Mod}$ equips $A$-Mod with the structure of a monoidal category if and only if $(A, m, \eta, \Delta, \epsilon)$ is a bialgebra.

**Proof:**
It was shown in Section 2.1 that any algebra homomorphism $\Delta : A \to A \otimes A$ defines an $A$-module structure on $V \otimes W$. It also follows directly that for any pair $(f, g)$ of $A$-module homomorphisms $f : V \to V'$ and $g : W \to W'$ the linear map $f \otimes g : V \otimes W \to V' \otimes W'$ is a module isomorphism if $V \otimes W$ and $V' \otimes W'$ are equipped with this $A$-module structure. Hence, for any algebra homomorphism $\Delta : A \to A \otimes A$, we obtain a functor $\otimes : A\text{-Mod} \times A\text{-Mod} \to A\text{-Mod}$.

It was also shown in Section 2.1 that any algebra homomorphism $\epsilon : A \to \mathbb{F}$ defines an $A$-module structure on $\mathbb{F}$ and that the linear isomorphisms $a_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$, $(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w)$, $l_V : \mathbb{F} \otimes V \to V$, $\lambda \otimes v \to \lambda v$ and $r_V : V \otimes \mathbb{F} \to V$, $v \otimes \lambda \mapsto \lambda v$ are module isomorphisms if $(A, m, \eta, \Delta, \epsilon)$ is a bialgebra. As they are natural and satisfy the pentagon and the triangle axioms, this shows that for any bialgebra $(A, m, \eta, \Delta, \epsilon)$ the category $A$-Mod becomes a monoidal category when equipped with these structures.

Conversely, if $A$-Mod is a monoidal category when equipped with this tensor functor and the trivial module structure on $\mathbb{F}$, then the linear maps $a_{A,A,A} : (A \otimes A) \otimes A \to A \otimes (A \otimes A)$, $l_A : \mathbb{F} \otimes A \to A$ and $r_A : A \otimes \mathbb{F} \to A$ are isomorphisms of representations when $A$ is equipped with its canonical $A$-module structure by left multiplication. This implies

$$(\Delta \otimes \mathrm{id}) \circ \Delta(a) = a \rhd ((1_A \otimes 1_A) \otimes 1_A) = a \rhd (1_A \otimes (1_A \otimes 1_A)) = (\mathrm{id} \otimes \Delta) \circ \Delta(a)$$
$$l_A \circ (\epsilon \otimes \mathrm{id}) \circ \Delta(a) = l_A(a \rhd (1_{\mathbb{F}} \otimes 1_A)) = a \rhd 1_A = a$$
$$r_A \circ (\mathrm{id} \otimes \epsilon) \circ \Delta(a) = r_A(a \rhd (1_A \otimes 1_{\mathbb{F}})) = a \rhd 1_A = a$$

for all $a \in A$, and hence $(A, m, \eta, \Delta, \epsilon)$ is a bialgebra. $\square$

Two important examples of monoidal categories with a geometrical interpretation arise from a construction involving groups. For this, note that a group can be viewed as a category with a single object, with the group elements as morphisms and the composition of morphisms given by the group multiplication. Given a family $(G_n)_{n\in\mathbb{N}_0}$ of groups, one can construct a category whose objects are number $n \in \mathbb{N}_0$, with $\mathrm{Hom}(n,n) = G_n$ and all other morphism sets empty. If there is a family of group homomorphisms relating the groups $G_m$ and $G_n$ to the group $G_{m+n}$ and subject to some consistency conditions, then this category becomes monoidal.

**Example 4.1.8:** Suppose $(G_n)_{n\in\mathbb{N}_0}$ is a family of groups with $G_0 = \{e\}$ and $(\rho_{m,n})_{m,n\in\mathbb{N}_0}$ a family of group homomorphisms $\rho_{m,n} : G_m \times G_n \to G_{m+n}$ such that $\rho_{0,m}$ and $\rho_{m,0}$ are given by $\rho_{0,m} : \{e\} \times G_m \to G_m$, $(e,g) \mapsto g$ and $\rho_{m,0} : G_m \times \{e\} \to G_m$, $(g,e) \mapsto g$ and

$$\rho_{m+n,p} \circ (\rho_{m,n} \times \mathrm{id}_{G_p}) = \rho_{m,n+p} \circ (\mathrm{id}_{G_M} \times \rho_{n,p}) \quad \forall m,n,p \in \mathbb{N}_0. \tag{32}$$

Then one obtains a strict tensor category $\mathcal{C}$ as follows:

- The objects of $\mathcal{C}$ are nonnegative integers $n \in \mathbb{N}_0$.
- The set of morphisms $\mathrm{Hom}_{\mathcal{C}}(n,m)$ is given by

$$\mathrm{Hom}_{\mathcal{C}}(m,n) = \begin{cases} \emptyset & n \neq m \\ G_n & n = m, \end{cases}$$

- The tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ given by $m \otimes n = m+n$ for all $n,m \in \mathbb{N}_0$ and $f \otimes g = \rho_{m,n}(f,g)$ for all morphisms $f \in G_m$, $g \in G_n$ and the tensor unit $e = 0$.

In particular, the construction from Example 4.1.8 can be applied to the braid groups and permutation groups, which both form families of groups with the required families of group homomorphisms.

**Definition 4.1.9:** For $n \in \mathbb{N}$ the **braid group** $B_n$ on $n$ strands is the group presented by generators $\sigma_1, .., \sigma_{n-1}$ and relations

$$\begin{aligned} &\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} && \forall i \in \{1,...,n-2\} \\ &\sigma_i\sigma_j = \sigma_j\sigma_i && \forall i,j \in \{1,...,n-1\} \text{ with } |i-j|>1. \end{aligned}$$

The braid group $B_n$ is closely related to the permutation group $S_n$. This is easy to see if one presents the latter in terms of generators and relations. The **permutation group** $S_n$ is presented with generators $\pi_1, ..., \pi_{n-1}$ and relations

$$\begin{aligned} &\pi_i\pi_{i+1}\pi_i = \pi_{i+1}\pi_i\pi_{i+1} && \forall i \in \{1,...,n-2\} \\ &\pi_i\pi_j = \pi_j\pi_i && \forall i,j \in \{1,...,n-1\} \text{ with } |i-j|>1 \\ &\pi_i^2 = 1 && \forall i \in \{1,...,n-1\}, \end{aligned}$$

where $\pi_i$ is the elementary transposition with $\pi_i(i) = i+1$, $\pi_i(i+1) = i$ and $\pi_i(j) = j$ for $j \notin \{i, i+1\}$. As the permutation group $S_n$ is presented by the same relations as the braid group $B_n$ and the additional relations $\pi_i^2 = 1$, there is a unique group homomorphism $\Pi_n : B_n \to S_n$ with $\Pi_n(\sigma_i) = \pi_i$.

To apply the construction from Example 4.1.8 to the families $(B_n)_{n\in\mathbb{N}_0}$ and $(S_n)_{n\in\mathbb{N}_0}$, note that the relations of the braid and permutation groups imply that the maps

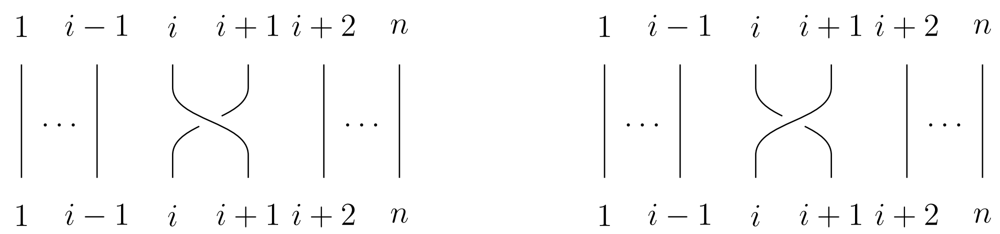
$$\rho_{m,n} : B_m \times B_n \to B_{n+m},\ (\sigma_i, \sigma_j) \mapsto \sigma_i \circ \sigma_{m+j} \qquad \rho'_{m,n} : S_m \times S_n \to S_{m+n},\ (\pi_i, \pi_j) \mapsto \pi_i \circ \pi_{m+j}$$

are group homomorphisms that satisfy the condition (32).

**Definition 4.1.10:**

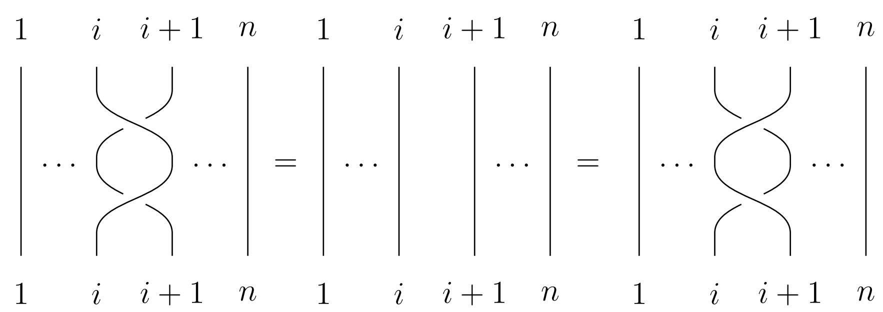
1. The **braid category** $\mathcal{B}$ is the strict monoidal category from Example 4.1.8 associated with the family $(B_n)_{n\in\mathbb{N}_0}$ of braid groups and the group homomorphisms $\rho_{m,n} : B_m \times B_n \to B_{n+m}$ above.
2. The **permutation category** $\mathcal{S}$ is the strict monoidal category from Example 4.1.8 associated with the family $(S_n)_{n\in\mathbb{N}_0}$ of permutation groups and the group homomorphisms $\rho'_{m,n} : S_m \times S_n \to S_{n+m}$ above.

The name *braid group* stems from the fact that elements $\sigma \in B_n$ can be visualised by braid diagrams, which involve two parallel horizontal lines with $n$ marked points, labelled from the left to the right by $1, ..., n$. The diagram for an element $\sigma \in B_n$ is obtained by drawing $n$ smooth curves that are nowhere horizontal and connect the point $i$ on the upper line to the point $\Pi_n(\sigma)(i)$ on the lower line. The intersection points of these curves are then changed to overcrossings and undercrossings, in such a way that at each intersection point exactly one curve crosses over another. The diagrams for the generators $\sigma_i$ and their inverses $\sigma_i^{-1}$ are



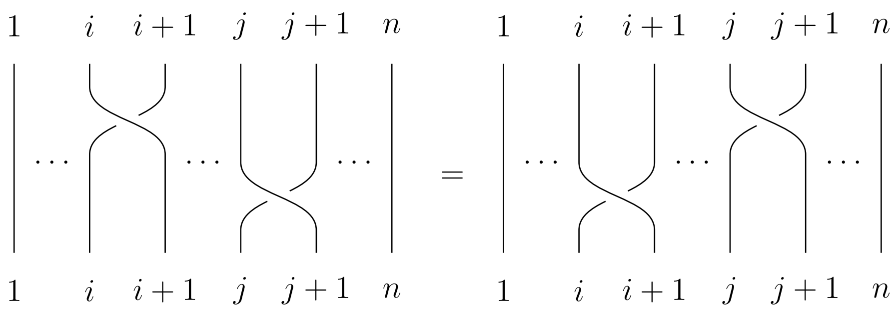

The generator $\sigma_i \in B_n$. The generator $\sigma_i^{-1} \in B_n$.

The group multiplication in $B_n$ is given by the vertical composition of diagrams. The diagram for the group element $\tau \circ \sigma \in B_n$ is obtained by putting the diagram for $\tau$ below the one for $\sigma$ such that the points on the horizontal lines match, erasing the middle lines, connecting the strands of the two diagrams and tightening them to remove unnecessary crossings. This corresponds to applying the relations $\sigma_i^{\mp 1} \circ \sigma_i^{\pm 1} = 1$:

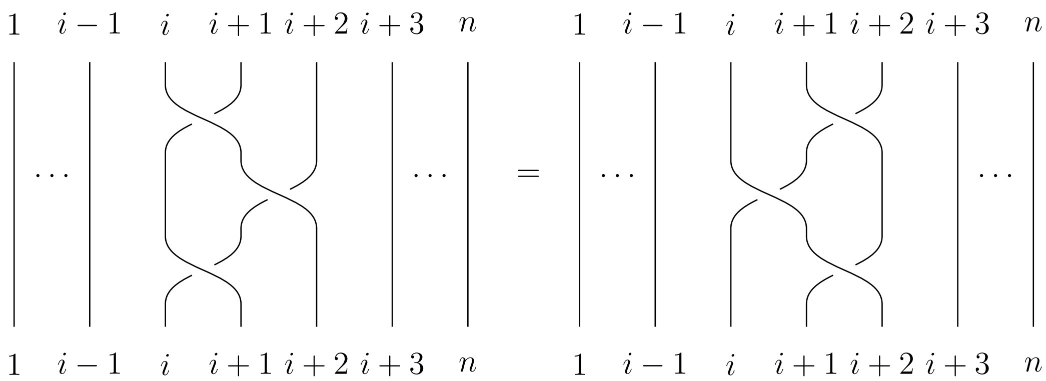



The relation $\sigma_i^{-1} \circ \sigma_i = 1 = \sigma_i \circ \sigma_i^{-1}$.

The remaining relations of the braid group in Definition 4.1.9 correspond to sliding two crossings that do not share a strand past each other and to sliding one crossing point in a triple crossing below the remaining strand.

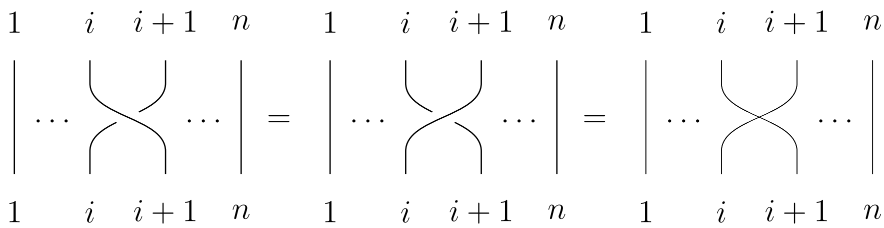



The relation $\sigma_i \circ \sigma_j = \sigma_j \circ \sigma_i$ for $|i-j|>1$.



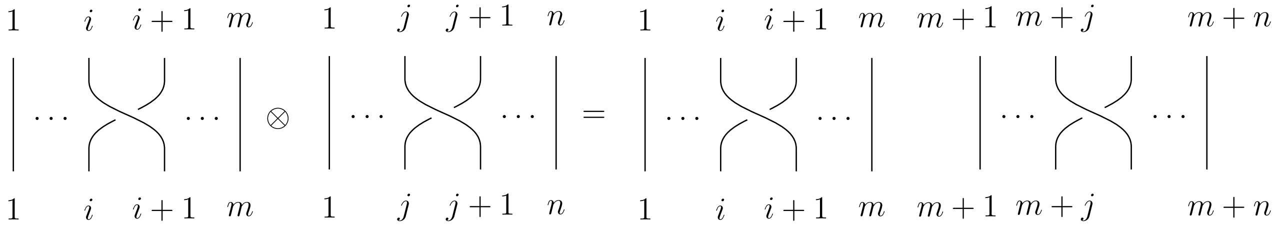

The relation $\sigma_i \circ \sigma_{i+1} \circ \sigma_i = \sigma_{i+1} \circ \sigma_i \circ \sigma_{i+1}$.

Elements of the permutation group $S_n$ are represented by the same diagrams, but with crossings instead of overcrossings and undercrossings. This corresponds to the additional relations $\pi_i^2 = 1$ or, equivalently, $\pi_i = \pi_i^{-1}$ for all $i \in \{1, ..., n-1\}$, which identify overcrossings and undercrossings. This implies that the group homomorphism $\Pi_n : B_n \to S_n$, $\sigma_i \mapsto \pi_i$ is represented graphically by changing each overcrossing or undercrossing in a braid diagram to a crossing:




The relation $\pi_i = \pi_i^{-1}$ in $S_n$.

Elements of the braid category and the permutation category are visualised by the same diagrams. The only difference is that in addition to the *vertical composition* of diagrams that corresponds to the composition of morphisms, there is also a *horizontal composition* corresponding to the tensor product. The tensor product $f \otimes g : m+n \to m+n$ of two morphisms $f : m \to m$ and $g : n \to n$ is obtained by putting the diagram for $g$ with $n$ strands to the right of the diagram for $f$ and adding $m$ to each number in the diagram for $g$.



Many applications of Hopf algebras arise from the fact that their representation categories can be related to certain monoidal categories from low dimensional topology such as the braid category and certain categories of manifolds. Such a relation must be given by a functor and requires a notion of functor that is compatible with the monoidal structures of the two categories, i. e. the tensor product, the tensor units, the associator and the unit constraints. Compatibility with the former takes the form of additional structure, namely certain (natural) isomorphisms associated with the functor, while compatibility with the latter leads to consistency conditions, i. e. requirements that certain diagrams involving this additional structure commute. Natural transformation between such functors should also compatible with the additional structure for the functors. This takes the form of consistency conditions, i. e. certain diagrams involving their component morphisms and the additional structure for the functors commute.

**Definition 4.1.11:**
Let $(\mathcal{C}, \otimes_\mathcal{C}, e_\mathcal{C}, a^\mathcal{C}, l^\mathcal{C}, r^\mathcal{C})$ and $(\mathcal{D}, \otimes_\mathcal{D}, e_\mathcal{D}, a^\mathcal{D}, l^\mathcal{D}, r^\mathcal{D})$ be monoidal categories.

1. A **monoidal functor** or **tensor functor** from $\mathcal{C}$ to $\mathcal{D}$ is a triple $(F, \phi^e, \phi^\otimes)$ of
   - a functor $F : \mathcal{C} \to \mathcal{D}$,
   - an isomorphism $\phi^e : e_\mathcal{D} \to F(e_\mathcal{C})$ in $\mathcal{D}$,
   - a natural isomorphism $\phi^\otimes : \otimes_\mathcal{D}(F \times F) \Rightarrow F\otimes_\mathcal{C}$,

   that satisfy the following axioms:

   (a) **compatibility with the associativity constraint**:
   for all objects $U, V, W$ of $\mathcal{C}$ the following diagram commutes

$$\begin{array}{ccc}
(F(U)\otimes F(V))\otimes F(W) & \xrightarrow{a^{\mathcal{D}}_{F(U),F(V),F(W)}} & F(U)\otimes(F(V)\otimes F(W)) \\
\downarrow{\scriptstyle \phi^{\otimes}_{U,V}\otimes 1_{F(W)}} & & \downarrow{\scriptstyle 1_U\otimes\phi^{\otimes}_{V,W}} \\
F(U\otimes V)\otimes F(W) & & F(U)\otimes F(V\otimes W)\\
\downarrow{\scriptstyle \phi^{\otimes}_{U\otimes V,W}} & & \downarrow{\scriptstyle\phi^\otimes_{U,V\otimes W}}\\
F((U\otimes V)\otimes W) & \xrightarrow[F(a^{\mathcal{C}}_{U,V,W})]{} & F(U\otimes(V\otimes W)).
\end{array}$$

   (b) **compatibility with the unit constraints**:
   for all objects $V$ of $\mathcal{C}$ the following diagrams commute

$$\begin{array}{ccc}
e_\mathcal{D}\otimes F(V) & \xrightarrow{\phi^e\otimes 1_{F(V)}} & F(e_\mathcal{C})\otimes F(V) \\
\downarrow{\scriptstyle l^\mathcal{D}_{F(V)}} & & \downarrow{\scriptstyle \phi^\otimes_{e_\mathcal{C},V}} \\
F(V) & \xleftarrow[F(l^\mathcal{C}_V)]{} & F(e_\mathcal{C}\otimes V)
\end{array}
\qquad
\begin{array}{ccc}
F(V)\otimes e_\mathcal{D} & \xrightarrow{1_{F(V)}\otimes\phi^e} & F(V)\otimes F(e_\mathcal{C}) \\
\downarrow{\scriptstyle r^\mathcal{D}_{F(V)}} & & \downarrow{\scriptstyle \phi^\otimes_{V,e_\mathcal{C}}} \\
F(V) & \xleftarrow[F(r^\mathcal{C}_V)]{} & F(V\otimes e_\mathcal{C}).
\end{array}$$

   A monoidal functor $(F, \phi^e, \phi^\otimes)$ is called **strict** if $\phi^e = 1_{e_\mathcal{D}}$ and $\phi^\otimes = \mathrm{id}_{F\otimes_\mathcal{C}}$ is the identity natural transformation. It is called a **monoidal equivalence** if $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories.

2. Let $(F, \phi^e, \phi^\otimes), (F', \phi'^e, \phi'^\otimes) : \mathcal{C} \to \mathcal{D}$ be monoidal functors. A **monoidal natural transformation** from $F$ to $F'$ is a natural transformation $\eta : F \to F'$ that satisfies:

(a) **compatibility with $\phi^e$ and $\phi'^e$**: the following diagram commutes

$$\begin{array}{ccc} F(e_{\mathcal{C}}) & \xrightarrow{\eta_{e_{\mathcal{C}}}} & F'(e_{\mathcal{C}}) \\ & {\scriptstyle \phi^e}\nwarrow \quad \nearrow{\scriptstyle \phi'^e} & \\ & e_{\mathcal{D}}. & \end{array}$$

(b) **compatibility with $\phi^{\otimes}$ and $\phi'^{\otimes}$**: For all objects $V, W$ of $\mathcal{C}$ the diagram

$$\begin{array}{ccc} F(V)\otimes F(W) & \xrightarrow{\eta_V\otimes\eta_W} & F'(V)\otimes F'(W) \\ {\scriptstyle \phi^{\otimes}_{V,W}}\downarrow & & \downarrow{\scriptstyle \phi'^{\otimes}_{V,W}} \\ F(V\otimes W) & \xrightarrow[\eta_{V\otimes W}]{} & F'(V\otimes W). \end{array}$$

commutes. A monoidal natural transformation $\eta : F \Rightarrow F'$ is called **monoidal isomorphism** if for all objects $V$ of $\mathcal{C}$ the morphism $\eta_V : F(V) \to F(V')$ is an isomorphism.

**Remark 4.1.12:**

1. The isomorphism $\phi^e$ and the natural isomorphism $\phi^{\otimes}$ are sometimes called the **coherence data** of the monoidal functor $(F, \phi^e, \phi^{\otimes})$.
2. One can show that for any monoidal category there is a monoidal equivalence of categories to a strict monoidal category. This is MacLane's **strictification theorem** and directly related to the coherence theorem. The proof is given in [McL, VI.2] and [Ka, XI.5].

**Example 4.1.13:**

1. The forgetful functor $F : \mathrm{Top} \to \mathrm{Set}$, that assigns to a topological space the underlying set and sends each continuous map to itself is a strict monoidal functor with respect to the product of topological spaces and the cartesian product of sets and with respect to the sum of topological spaces and the disjoint union of sets.

2. For any bialgebra $B$ over $\mathbb{F}$, the forgetful functor $F : B\text{-Mod} \to \mathrm{Vect}_{\mathbb{F}}$ that assigns to a $B$-module $M$ the underlying vector space $M$ over $\mathbb{F}$ and to a $B$-module homomorphism $f : M \to N$ the linear map $f : M \to N$ is a strict monoidal functor.

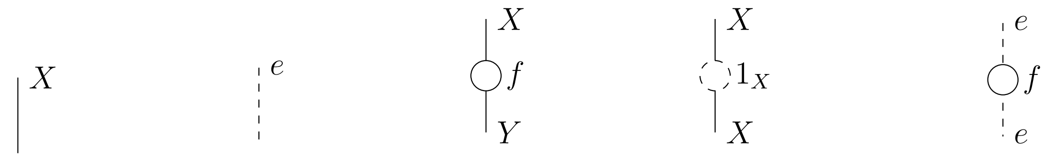
3. The functor $F : \mathrm{Set} \to \mathrm{Vect}_{\mathbb{F}}$ that assigns to a set $X$ the free vector space $F(X) = \langle X\rangle_{\mathbb{F}}$ generated by $X$ and to a map $f : X \to Y$ the unique linear map $F(f) : \langle X\rangle_{\mathbb{F}} \to \langle Y\rangle_{\mathbb{F}}$ with $F(f)|_X = f$ is a monoidal equivalence with respect to the cartesian product of sets and the tensor product of vector spaces. Its coherence data is given by $\phi^e : \mathbb{F} \to \langle\{p\}\rangle_{\mathbb{F}}$, $\lambda \mapsto \lambda p$, and $\phi^{\otimes}_{X,Y} : \langle X\rangle_{\mathbb{F}}\otimes\langle Y\rangle_{\mathbb{F}} \to \langle X \times Y\rangle_{\mathbb{F}}$, $x\otimes y \mapsto (x, y)$.

4. A bialgebra homomorphism $\phi : B \to C$ induces a monoidal functor $F_\phi : C\text{-mod} \to B\text{-Mod}$ that assigns to a $B$-module $(M, \rhd_C)$ the $B$-module $(M, \rhd_B)$ with module structure $b \rhd_B m = \phi(b) \rhd_C m$ and each a $C$-module map $f : M \to N$ to itself.

5. Let $H$ be a Hopf algebra and $g \in H$ grouplike. Then one obtains a monoidal functor $F : H\text{-Mod} \to H\text{-Mod}$ by assigning to a $H$-module $(M, \rhd)$ the module $(M, \rhd')$ with $h \rhd' m = (ghg^{-1}) \rhd m$ and to a $H$-module morphism $f : (M, \rhd) \to (N, \rhd)$ the $H$-module morphism $f : (M', \rhd) \to (N', \rhd)$. The morphisms $\mu_M : (M, \rhd) \to (M, \rhd')$, $m \mapsto g \rhd m$ define a monoidal natural isomorphism $\mu : \mathrm{id}_{H\text{-Mod}} \to F$.

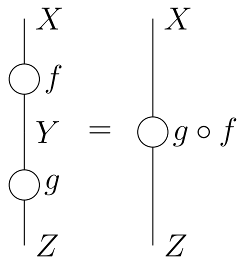
Our aim in the following sections will be to construct interesting monoidal functors between monoidal categories from low-dimensional topology and between the representation categories of bialgebras and Hopf algebras. As a preparation, we introduce a diagrammatic calculus for monoidal categories that will give us a more geometrical picture.

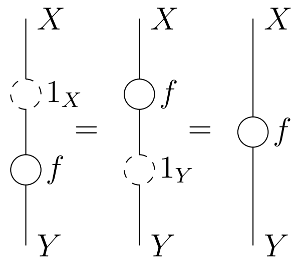
Objects in a monoidal category $\mathcal{C}$ are represented by vertical lines labelled with the name of the object. The unit object $e$ is represented by the empty line, i. e. not drawn in the diagrams. A morphism $f : X \to Y$ is represented as a vertex on a vertical line that divides the line in to an upper part labelled by $X$ and a lower part labelled by $Y$. Unit morphisms in $\mathcal{C}$ are not represented by vertices in the diagrams.



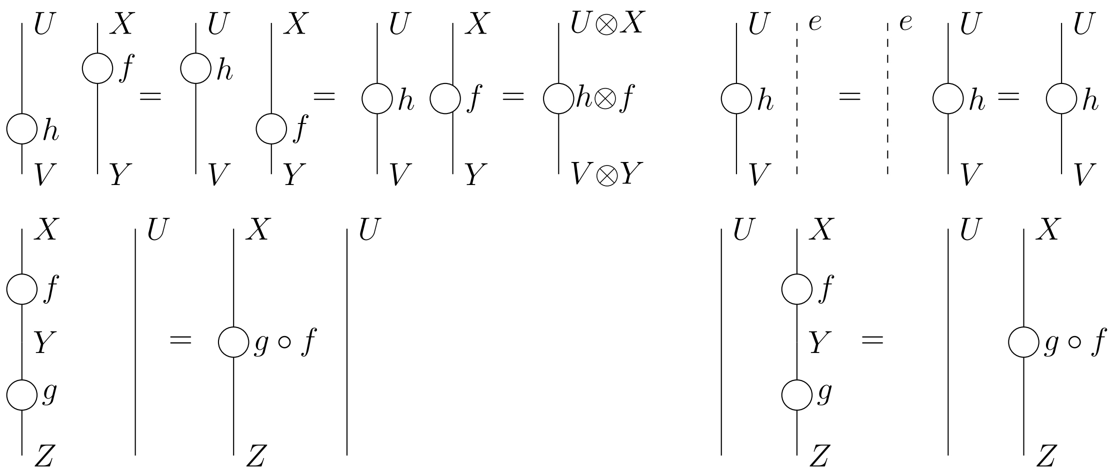

the object $X$ — the tensor unit $e$ — a morphism $f : X \to Y$ — identity morphism $1_X : X \to X$ — an endomorphism $f : e \to e$

The composition of morphisms is given by the *vertical composition* of diagrams, whenever the object at the bottom of one diagram matches the object at the top of the other. More precisely, the composite $g \circ f : X \to Z$ of two morphisms $f : X \to Y$ and $g : X \to Z$ is obtained by putting the diagram for $g$ below the one for $f$. The associativity of the composition of morphisms and the properties of the unit morphisms ensure that this is consistent for multiple composites and that it is possible to omit identity morphisms.




Tensor products of objects and morphisms are given by the *horizontal composition* of diagrams. The diagram for the tensor product $U \otimes X$ involves two parallel vertical lines, the one on the left labelled by $U$ and the one on the right labelled by $X$. The tensor product of morphisms is represented by vertices on such lines. The condition that $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a functor allows one to slide these vertices past each other and to compose them individually on each line:



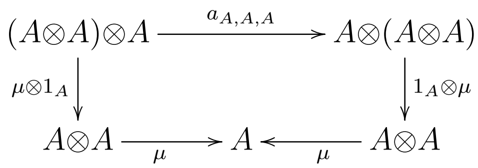
(33)

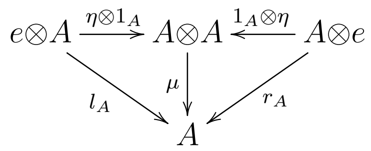
Note that just as the tensor unit and the identity morphisms, the component morphisms of the associator and of the left and right unit constraints are not represented in this diagrammatical calculus. This is consistent because of MacLane's **coherence theorem**. As any two morphisms represented by the same diagram are related by identity morphisms, component morphisms of the associator and the left and right unit constraints, the coherence theorem implies that two morphisms represented by the same diagram are related by a unique isomorphism built up of identity morphisms and these component morphisms.

In particular, this graphical representation gives rise to a graphical calculus for algebras, coalgebras modules and comodules in monoidal categories. As the definition of an algebra or coalgebra can be given purely in terms of linear maps $m : A\otimes A \to A$, $\eta : \mathbb{F} \to A$ or $\Delta : A \to A\otimes A$, $\epsilon : A \to \mathbb{F}$, subject to (co)associativity and (co)unitality axioms, these notions have a direct generalisation to monoidal categories. They are obtained by replacing linear maps with morphisms, and the field $\mathbb{F}$ by the tensor unit. Similarly, one obtains a general notion of modules and comodules in a monoidal category.

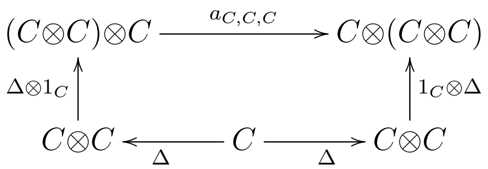
**Definition 4.1.14:** Let $(\mathcal{C}, \otimes, e, a, l, r)$ be a monoidal category.

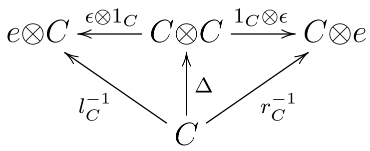
1. An **algebra** or an **algebra object** in $\mathcal{C}$ is an object $A$ in $\mathcal{C}$ together with morphisms $\mu : A\otimes A \to A$ and $\eta : e \to A$ such that the following diagrams commute



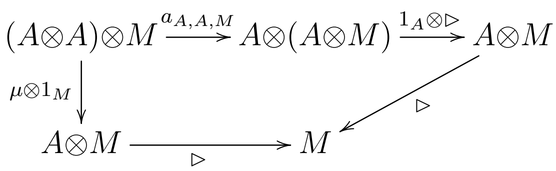


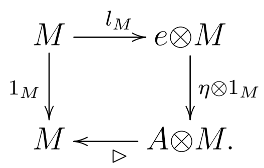

2. A **coalgebra** or a **coalgebra object** in $\mathcal{C}$ is an object $C$ in $\mathcal{C}$ together with morphisms $\Delta : C \to C\otimes C$ and $\epsilon : C \to e$ such that the following diagrams commute






3. Let $(A, \mu, \eta)$ be an algebra object in $\mathcal{C}$. An $A$-**left module** or $A$-**left module object** in $\mathcal{C}$ is an object $M$ in $\mathcal{C}$ together with a morphism $\rhd : A\otimes M \to M$ such that the following diagrams commute





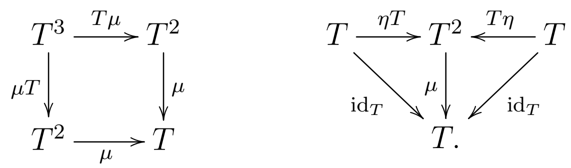

Right module and bimodule objects in $\mathcal{C}$ and left, right and bicomodule objects in $\mathcal{C}$ are defined analogously.

**Example 4.1.15:**

1. An algebra object in the category $\mathrm{Vect}_{\mathbb{F}}$ is an algebra over $\mathbb{F}$ and a coalgebra object in the category $\mathrm{Vect}_{\mathbb{F}}$ is a coalgebra over $\mathbb{F}$.

2. Let $B$ be a bialgebra and $B$-Mod the category of modules over $B$. An algebra object in $B$-Mod is a module algebra over $B$, and a coalgebra object in $B$-Mod is a module coalgebra over $B$.

3. An algebra object in the strict monoidal category $\mathrm{End}(\mathcal{C})$ for a small category $\mathcal{C}$ is also called a **monad**. It is a triple $(T, \mu, \eta)$ of a functor $T : \mathcal{C} \to \mathcal{C}$ together with natural transformations $\mu : T^2 \Rightarrow T$ and $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow T$ such that the following diagrams commute

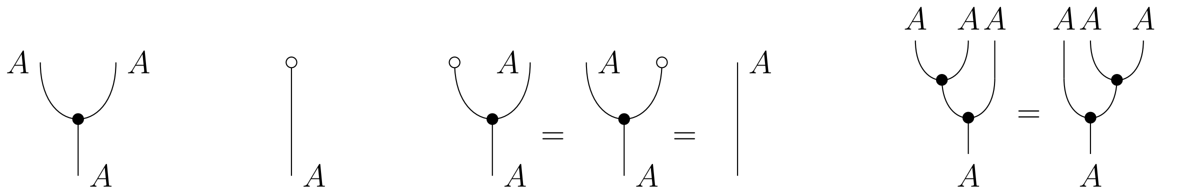


A coalgebra object in $\mathrm{End}(\mathcal{C})$ is called a **comonad**. It is a triple $(S, \Delta, \epsilon)$ of a functor $S : \mathcal{C} \to \mathcal{C}$ and natural transformations $\Delta : S \Rightarrow S^2$ and $\epsilon : S \Rightarrow \mathrm{id}_{\mathcal{C}}$ such that the diagrams dual to the monad diagrams commute.

4. Any algebra object $(A, \mu, \eta)$ in $\mathcal{C}$ is a module object over itself with $\rhd = \mu : A \otimes A \to A$.

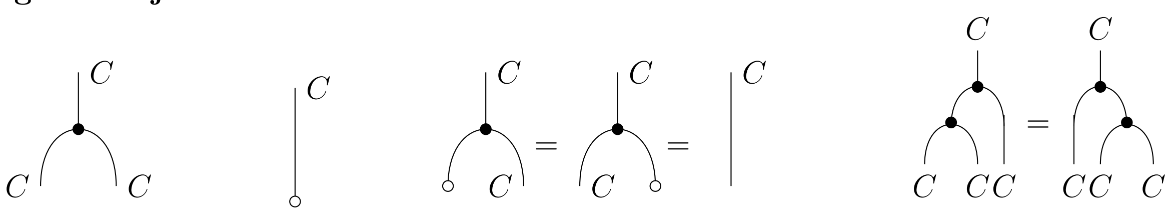
Algebra, coalgebra and module objects in a monoidal category $\mathcal{C}$ can be described graphically. They are given by the following diagrams and consistency conditions:

1. **algebra object:**



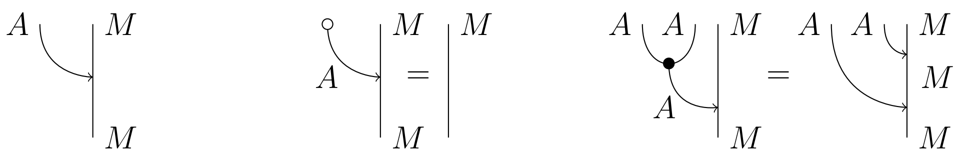
multiplication $\mu : A \otimes A \to A$ | unit $\eta : e \to A$ | unitality condition | associativity condition

2. **coalgebra object:**



comultiplication $\Delta : C \to C \otimes C$ | counit $\epsilon : C \to e$ | counitality condition | coassociativity condition

3. **module object over an algebra object:**



morphism $\rhd : A \otimes M \to M$ | compatibility with $\eta : e \to A$ | compatibility with $\mu : A \otimes A \to A$.

Note that the structures introduced so far are not sufficient to define bialgebra object in a monoidal category. This requires a generalisation of the flip map $\tau : U \otimes V \to V \otimes U$, which is needed in the condition that the comultiplication of a bialgebra is an algebra homomorphism.

## 4.2 Braided monoidal categories and the braid category

To generalise the notion of the flip map $\tau : U \otimes V \to V \otimes U$ in the category of vector spaces to a monoidal category, we require a notion of an opposite tensor product. For this, note that for any functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, there is an opposite functor $\otimes^{op} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ that assigns to a pair $(U, V)$ of objects in $\mathcal{C}$ the object $U \otimes_{op} V = V \otimes U$ and to a pair of morphisms $f : U \to U'$, $g : V \to V'$ the morphism $f \otimes_{op} g = g \otimes f : V \otimes U \to V' \otimes U'$. Moreover, if $(\mathcal{C}, \otimes, e, a, l, r)$ is a tensor category then $(\mathcal{C}, \otimes^{op}, e, a', r, l)$ becomes a tensor category as well with the associator $a'$ given by $a'_{U,V,W} = a^{-1}_{W,V,U}$. (Exercise).

In this formulation, it is apparent that the flip map should be replaced by a natural isomorphism $c : \otimes \to \otimes^{op}$ that relates the tensor product and the opposite tensor product. If this natural isomorphism is also compatible with the tensor product in the sense that flipping an object with the tensor product of two other objects is the same as performing two individual flips, then it is called a *braiding*.

**Definition 4.2.1:** Let $(\mathcal{C}, \otimes, e, a, l, r)$ be a monoidal category.

1. A **commutativity constraint** for $\mathcal{C}$ is a natural isomorphism $c : \otimes \to \otimes^{op}$. A commutativity constraint is called a **braiding** if it satisfies the **hexagon axioms**

$$\begin{array}{ccccc}
(U\otimes V)\otimes W & \xrightarrow{c_{U,V}\otimes 1_W} & (V\otimes U)\otimes W & \xrightarrow{a_{V,U,W}} & V\otimes (U\otimes W) \\
{\scriptstyle a_{U,V,W}}\downarrow & & & & \downarrow{\scriptstyle 1_V\otimes c_{U,W}} \\
U\otimes (V\otimes W) & \xrightarrow[c_{U,V\otimes W}]{} & (V\otimes W)\otimes U & \xrightarrow[a_{V,W,U}]{} & V\otimes (W\otimes U)
\end{array}$$

$$\begin{array}{ccccc}
U\otimes (V\otimes W) & \xrightarrow{1_U\otimes c_{V,W}} & U\otimes (W\otimes V) & \xrightarrow{a^{-1}_{U,W,V}} & (U\otimes W)\otimes V \\
{\scriptstyle a^{-1}_{U,V,W}}\downarrow & & & & \downarrow{\scriptstyle c_{U,W}\otimes 1_V} \\
(U\otimes V)\otimes W & \xrightarrow[c_{U\otimes V,W}]{} & W\otimes (U\otimes V) & \xrightarrow{a^{-1}_{W,U,V}} & (W\otimes U)\otimes V
\end{array} .$$

2. A braiding is called **symmetric** if for all objects $V, W$ in $\mathcal{C}$ one has $c_{W,V} = c^{-1}_{V,W}$.

3. A monoidal category together with braiding is called a **braided monoidal category** or **braided tensor category**. A monoidal category together with a symmetric braiding is called a **symmetric monoidal category** or **symmetric tensor category**.

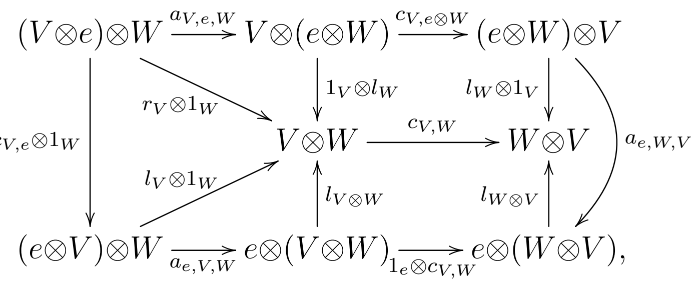
**Remark 4.2.2:**

1. If $(\mathcal{C}, \otimes, e)$ is a *strict* monoidal category, then the hexagon axioms reduce to the equations

$$c_{U\otimes V,W} = (c_{U,W} \otimes 1_V) \circ (1_U \otimes c_{V,W}) \qquad c_{U,V\otimes W} = (1_V \otimes c_{U,W}) \circ (c_{U,V} \otimes 1_W).$$

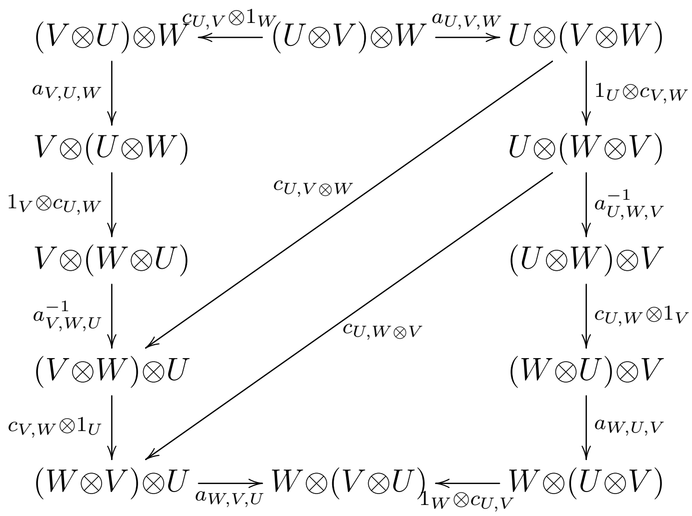
2. If $c : \otimes \to \otimes^{op}$ is a braiding for $(\mathcal{C}, \otimes, e, a, l, r)$, then $c' : \otimes \to \otimes^{op}$ with component morphisms $c'_{U,V} = c^{-1}_{V,U} : U \otimes V \to V \otimes U$ is a braiding as well (Exercise). This shows that a braiding is a choice of *structure* and not a *property*.

3. For all objects $V$ in a braided tensor category $(\mathcal{C}, \otimes, e, a, l, r, c)$ one has
$$c_{V,e} = l_V^{-1} \circ r_V = c_{V,e}^{-1}.$$
This is obtained from the diagram




in which the two rectangles commute by the naturality of $c$, the triangle on the upper left by the triangle axiom, the triangles on the lower left and on the right by Lemma 4.1.3, and the outer hexagon by the first hexagon axiom. As all arrows are labelled by isomorphisms, this implies that the middle triangle on the left commutes as well and hence $(l_V \circ c_{V,e}) \otimes 1_W = r_V \otimes 1_W$ for all objects $V, W$. Setting $W = e$ and applying the same argument as in the proof of Lemma 4.1.3 one then obtains $c_{V,e} = l_V^{-1} \circ r_V$. The proof of the second identity is analogous.

4. For all objects $U, V, W$ in in a braided tensor category $(\mathcal{C}, \otimes, e, a, l, r, c)$ the **dodecagon diagram** commutes:




This follows because the two hexagons commute by the hexagon axioms and the parallelogram commutes by the naturality of the braiding.

If $(\mathcal{C}, \otimes, e, a, l, r, c)$ is strict, the dodecagon diagram reduces to the identity
$$(c_{V,W} \otimes 1_U) \circ (1_V \otimes c_{U,W}) \circ (c_{U,V} \otimes 1_W) = (1_W \otimes c_{U,V}) \circ (c_{U,W} \otimes 1_V) \circ (1_U \otimes c_{V,W})$$

**Example 4.2.3:**

1. The category $\mathrm{Vect}_{\mathbb{F}}$ is a symmetric monoidal category with the braiding given by $c_{U,V} : U \otimes V \to V \otimes U$, $u \otimes v \mapsto v \otimes u$. More generally, for any commutative ring $R$, the category $(R\text{-Mod}, \otimes_R, R)$ is a symmetric monoidal category with $c_{U,V} : U \otimes_R V \to V \otimes_R U$, $u \otimes v \mapsto v \otimes u$.

2. If $B$ is a *cocommutative* bialgebra, then $B$-Mod is a symmetric monoidal category with the component morphisms of the braiding given by $c_{U,V} : U \otimes V \to V \otimes U$, $u \otimes v \mapsto v \otimes u$.
3. The categories Set and Top with, respectively, the cartesian product of sets and the product of topological spaces are symmetric monoidal categories with the braiding $c_{X,Y} : X \times Y \to Y \times X$, $(x, y) \mapsto (y, x)$. More generally, any monoidal category $\mathcal{C}$ whose tensor product arises from products or coproducts in $\mathcal{C}$ is a symmetric monoidal category.
4. The category $\mathrm{Ch}_{R\text{-Mod}}$ of chain complexes and chain maps is a symmetric monoidal category.

**Example 4.2.4:** Let $G$ be a group.

1. A **crossed $G$-set** is a triple $(X, \lhd, \mu)$ of a set $X$, a right action $\lhd : X \times G \to X$ and a map $\mu : X \to G$ that satisfy $\mu(x \lhd g) = g^{-1} \cdot \mu(x) \cdot g$ for all $x \in X$ and $g \in G$.
2. A **morphism of crossed $G$-sets** from $(X, \lhd_X, \mu_X)$ to $(Y, \lhd_Y, \mu_Y)$ is a map $f : X \to Y$ with $f(x \lhd_X g) = f(x) \lhd_Y g$ for all $x \in X$ and $g \in G$ and $\mu_Y \circ f = \mu_X$.
3. The **tensor product** of crossed $G$-sets $(X, \lhd_X, \mu_X)$ and $(Y, \lhd_Y, \mu_Y)$ is the crossed $G$-set $(X \times Y, \lhd, \mu)$ with $(x, y) \lhd g = (x \lhd_X g, y \lhd_Y g)$ and $\mu(x, y) = \mu_X(x) \cdot \mu_Y(y)$. The tensor product of morphisms $f : X \to Y$ and $h : U \to V$ of crossed $G$-sets is the morphism $g \times f : U \times X \to V \times Y$.

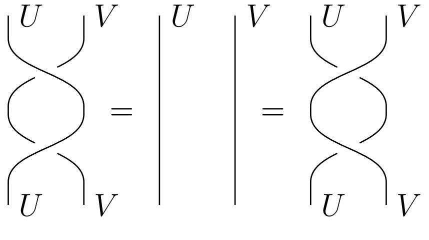
Crossed $G$-sets form a monoidal category $X(G)$, where the tensor unit is the one-point set $\{p\}$ with the trivial group action and the map $\mu : \{p\} \to G$, $p \mapsto e$. It is braided with the component morphisms of the braiding given by

$$c_{X,Y} : X \times Y \to Y \times X, \quad (x, y) \mapsto (y, x \lhd_X \mu_Y(y)).$$

As suggested by the name *braiding*, the component morphisms of the natural isomorphism $c : \otimes \to \otimes^{op}$ in a braided monoidal category are represented diagrammatically by pairs of lines that braid above or below each other:

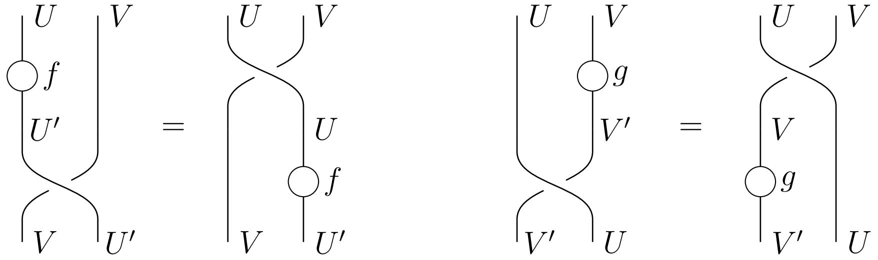

$c_{U,V} : U \otimes V \to V \otimes U$ $\quad c^{-1}_{U,V} : V \otimes U \to U \otimes V$ $\quad c_{V,U} : V \otimes U \to U \otimes V$ $\quad c^{-1}_{V,U} : U \otimes V \to V \otimes U$

The identities in Remark 4.2.2, 3. ensure that it is still consistent to omit the tensor unit from the graphical calculus, since they imply that the braiding of the tensor unit with any other object is given by the left and right unit constraints: $c_{e,V} = l_V^{-1} \circ r_V = c^{-1}_{V,e}$. The conditions $c^{-1}_{U,V} \circ c_{U,V} = 1_{U \otimes V} = c_{V,U} \circ c^{-1}_{V,U}$ read



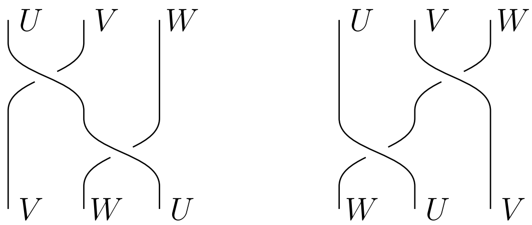

(34)

and the condition that a braiding is *symmetric* amounts to the statement that overcrossings are equal to undercrossings and can be represented by simple crossings. The naturality of the braiding implies that morphisms can slide above or below a crossing in the diagram without changing the morphism represented by this diagram



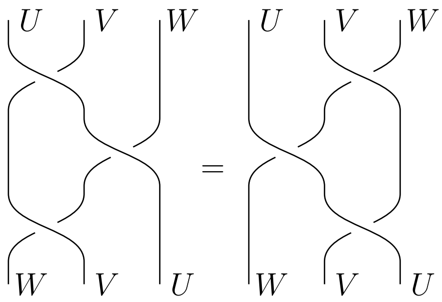
(35)

The hexagon axioms state that the two possible interpretations of the following diagrams, namely as the composite of two braidings or as a braiding of the morphism represented by one strand with the tensor product of the morphisms represented by the other two coincide:



(36)

The dodecagon identity states that the following two diagrams represent the same morphism:



(37)

That the diagrams for a braided monoidal category resemble the diagrams for the braid category $\mathcal{B}$ from Definition 4.1.10 is not coincidental. We will show that the braid category $\mathcal{B}$ is not only a strict braided monoidal category but can be viewed as the prototype of a braided monoidal category and is characterised by a universal property.

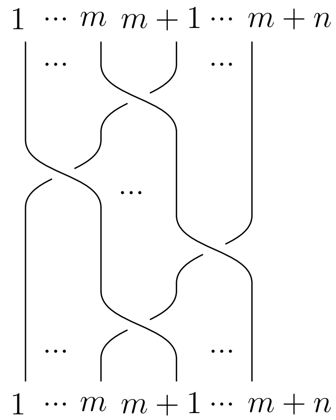
**Theorem 4.2.5:**

1. The braid category $\mathcal{B}$ is a strict braided monoidal category.
2. The permutation category $\mathcal{S}$ is a strict symmetric monoidal category.

**Proof:**
We take as the component morphisms of the braiding for the braid category $\mathcal{B}$ the morphisms

$$c_{m,n} = (\sigma_n \circ ... \circ \sigma_2 \circ \sigma_1) \circ (\sigma_{n+1} \circ ... \circ \sigma_3 \circ \sigma_2) \circ ... \circ (\sigma_{n+m-1} \circ ... \circ \sigma_{m+1} \circ \sigma_m) \quad (38)$$

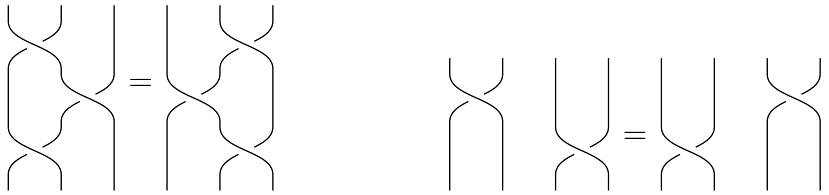
that braid the first $m$ strands over the last $n$ strands:




Then the hexagon axioms follow directly from the definition of the braiding. To prove naturality of the braiding, it is sufficient to show that

$$c_{m,n} \circ (\sigma_i \otimes \sigma_j) = (\sigma_j \otimes \sigma_i) \circ c_{m,n}$$

for all $i \in \{1, ..., m-1\}$ and $j \in \{1, ..., n-1\}$. This follows by repeatedly applying the relations



$\sigma_i \circ \sigma_{i+1} \circ \sigma_i = \sigma_{i+1} \circ \sigma_i \circ \sigma_{i+1}$
for all $i \in \{1, ..n+m-2\}$

$\sigma_i \circ \sigma_j = \sigma_j \circ \sigma_i$
for all $i \in \{1, ..n+m-2\}$, $|i-j| > 1$.

This shows that $\mathcal{B}$ is a braided monoidal category. The permutation category $\mathcal{S}$ is described by the same relations and diagrams as the braid category $\mathcal{B}$. Replacing the generators $\sigma_i$ by $\pi_i$ in (38) yields a braiding for the permutation category. As each generator $\pi_i$ of $S_n$ satisfies $\pi_i = \pi_i^{-1}$, which implies that overcrossings in the associated diagrams can be changed to undercrossings and vice versa, it follows that $\mathcal{S}$ is a symmetric monoidal category. □

To show that the braid category plays a special role among the braided monoidal categories and is characterised by a universal property, we require the concept of a functor between braided monoidal categories that is compatible not only with the monoidal structure but also with the braiding. Such a functor is called a *braided monoidal functor*, and there is also a corresponding notion of *braided natural transformation*.

**Definition 4.2.6:** Let $\mathcal{C}$, $\mathcal{D}$ be braided monoidal categories.

1. A monoidal functor $(F, \phi^e, \phi^\otimes) : \mathcal{C} \to \mathcal{D}$ is called a a **braided monoidal functor** or **braided tensor functor** from $\mathcal{C}$ to $\mathcal{D}$ if for all objects $V, W$ in $\mathcal{C}$ the following diagram commutes

$$\begin{array}{ccc} F(V)\otimes F(W) & \xrightarrow{c^{\mathcal{D}}_{F(V),F(W)}} & F(W)\otimes F(V) \\ \Big\downarrow{\scriptstyle \phi^{\otimes}_{V,W}} & & \Big\downarrow{\scriptstyle \phi^{\otimes}_{W,V}} \\ F(V\otimes W) & \xrightarrow{F(c^{\mathcal{C}}_{V,W})} & F(W\otimes V). \end{array}$$

If $\mathcal{C}$ and $\mathcal{D}$ are symmetric tensor categories, then a braided monoidal functor $F : \mathcal{C} \to \mathcal{D}$ is also called a **symmetric monoidal functor**.

2. A **braided natural transformation** is a natural transformation between braided monoidal functors. A **braided natural isomorphism** is a braided natural transformation that is a natural isomorphism.

**Example 4.2.7:**

1. The forgetful functors Top $\to$ Set, $\text{Vect}_{\mathbb{F}} \to$ Set from Example 4.1.13 and the forgetful functors $R-$ Mod $\to$ Ab for a commutative ring $R$ and $B-$Mod $\to \text{Vect}_{\mathbb{F}}$ for a *cocommutative* bialgebra $B$ over $\mathbb{F}$ are symmetric monoidal functors.

2. The functor $F : \text{Set} \to \text{Vect}_{\mathbb{F}}$ from from Example 4.1.13 that assigns to a set $X$ the free vector space $\langle X\rangle_{\mathbb{F}}$ generated by $X$ and to a map $f : X \to Y$ the unique linear map $F(f) : \langle X\rangle_{\mathbb{F}} \to \langle Y\rangle_{\mathbb{F}}$ with $F(f)|_X = f$ is a symmetric monoidal functor.

3. The family $(\Pi_n)_{n\in\mathbb{N}_0}$ of group homomorphisms $\Pi_n : B_n \to S_n$, $\sigma_i \mapsto \pi_i$ introduced after Definition 4.1.9 defines a strict braided tensor functor $F : \mathcal{B} \to \mathcal{S}$ with $F(n) = n$ for all $n \in \mathbb{N}_0$ and $F(f) = \Pi_n(f)$ for all morphisms $f \in \text{Hom}_{\mathcal{B}}(n, n) = B_n$.

With the concept of a braided tensor functor, we can now show that the braid category $\mathcal{B}$ plays a special role among the strict braided tensor categories and is characterised by a universal property that allows one to associate a unique braided tensor functor $F_V : \mathcal{B} \to \mathcal{C}$ to any object $V$ in $\mathcal{C}$. This is due to the fact that $\mathcal{B}$ is generated via the tensor product and the composition of morphisms by the object 1 and a the morphism $\sigma_1 : 2 \to 2$ and the only relations between morphisms are the ones required for the for the naturality of the tensor product, the naturality of the braiding and its compatibility with the tensor product.

**Proposition 4.2.8:** Let $(\mathcal{C}, \otimes, e, c)$ be a strict braided tensor category. Then for any object $V$ of $\mathcal{C}$ there is a unique strict braided tensor functor $F_V : \mathcal{B} \to \mathcal{C}$ with $F_V(1) = V$.

**Proof:**
If $F : \mathcal{B} \to \mathcal{C}$ is a strict tensor functor then one has $F(n) = F(1\otimes ...\otimes 1) = F(1)\otimes ...\otimes F(1)$ for all $n \in \mathbb{N}$ and $F(0) = e$. This shows that $F$ is determined uniquely on the objects by $F(1) =: V$. The condition that $F$ is a braided strict tensor functor implies that the image of the morphism $c_{1,1} = \sigma_1 : 2 \to 2$ is given by $F(c_{1,1}) = c_{V,V} : V\otimes V \to V\otimes V$ and $F(c_{1,1}^{-1}) = F(c_{1,1})^{-1} = c_{V,V}^{-1}$. To show that this determines $F$ uniquely on the morphisms, note that any morphism in $\mathcal{B}$ is a given as a composite of identity morphisms, tensor products of the morphism $c_{1,1} : 2 \to 2$ and its inverse. More specifically, we have $\sigma_i^{\pm 1} = 1_{i-1}\otimes c_{1,1}^{\pm 1}\otimes 1_{n-i-1} : n \to n$ for all $n \in \mathbb{N}$ and $i \in \{1, ..., n-1\}$. As $F$ is a strict tensor functor, this implies $F(\sigma_i^{\pm 1}) = 1_{V^{\otimes(i-1)}}\otimes c_{V,V}^{-1}\otimes 1_{V^{\otimes(n-i-1)}}$ and hence $F(c_{1,1}) = c_{V,V}$ determines $F$ uniquely on the morphisms.

Conversely, for any object $V$ in $\mathcal{C}$ setting $F_V(n) = V^{\otimes n}$ for $n \in \mathbb{N}$, $F_V(0) = e$ and defining $F_V(\sigma_i^{\pm 1}) = 1_{V^{\otimes(i-1)}}\otimes c_{V,V}^{\pm 1}\otimes 1_{V^{\otimes(n-i-1)}}$ for all $n \in \mathbb{N}$ and $\sigma_i : n \to n$ yields a functor $F_V : \mathcal{B} \to \mathcal{C}$, since the functoriality of $\otimes : \mathcal{C} \to \mathcal{C}$ implies $F_V(\sigma_i) \circ F_V(\sigma_j) = F_V(\sigma_j) \circ F_V(\sigma_i)$ for all $i, j \in \{1, ..., n-1\}$ with $|i-j| > 1$ and the dodecagon identity in $\mathcal{C}$ implies for all $i \in \{1, ..., n-2\}$

$$F_V(\sigma_i) \circ F_V(\sigma_{i+1}) \circ F_V(\sigma_i) = F_V(\sigma_{i+1}) \circ F_V(\sigma_i) \circ F_V(\sigma_{i+1}). \qquad \square$$

It is clear from the proof of this proposition that monoidal functors $\mathcal{B} \to \mathcal{C}$ into a monoidal category $\mathcal{C}$ can be defined by specifying the images of the object $1 \in \mathcal{B}$ and the morphism $\sigma_1 \in \mathrm{Hom}_{\mathcal{B}}(2,2)$ in a much more general setting. All that is required is an object $V$ in $\mathcal{C}$ and a morphism $\sigma : V \otimes V \to V \otimes V$ that satisfies the dodecagon identity, while a braiding of $V$ with other objects in $\mathcal{C}$ is not necessary.

**Definition 4.2.9:** Let $(\mathcal{C}, \otimes, e, a, l, r)$ be a monoidal category. A **Yang-Baxter operator** in $\mathcal{C}$ is an object $V$ in $\mathcal{C}$ together with an isomorphism $\sigma : V \otimes V \to V \otimes V$ such that the **dodecagon diagram** commutes

$$\begin{array}{ccccc}
(V\otimes V)\otimes V & \xleftarrow{\sigma\otimes 1_V} & (V\otimes V)\otimes V & \xrightarrow{a_{V,V,V}} & V\otimes(V\otimes V) \\
\downarrow{\scriptstyle a_{V,V,V}} & & & & \downarrow{\scriptstyle 1_V\otimes\sigma} \\
V\otimes(V\otimes V) & & & & V\otimes(V\otimes V) \\
\downarrow{\scriptstyle 1_V\otimes\sigma} & & & & \downarrow{\scriptstyle a^{-1}_{V,V,V}} \\
V\otimes(V\otimes V) & & & & (V\otimes V)\otimes V \\
\downarrow{\scriptstyle a^{-1}_{V,V,V}} & & & & \downarrow{\scriptstyle \sigma\otimes 1_V} \\
(V\otimes V)\otimes V & & & & (V\otimes V)\otimes V \\
\downarrow{\scriptstyle \sigma\otimes 1_V} & & & & \downarrow{\scriptstyle a_{V,V,V}} \\
(V\otimes V)\otimes V & \xrightarrow[a_{V,V,V}]{} & V\otimes(V\otimes V) & \xleftarrow[1_V\otimes\sigma]{} & V\otimes(V\otimes V)
\end{array}$$

**Example 4.2.10:**

1. If $\mathcal{C}$ is a braided monoidal category with braidings $c_{U,V} : U \otimes V \to V \otimes U$, then $(V, c_{V,V})$ is a Yang-Baxter operator for any object $V$ of $\mathcal{C}$.
2. If $(F, \phi^{\otimes}, \phi^e)$ is a tensor functor from a monoidal category $\mathcal{C}$ to a monoidal category $\mathcal{D}$, then for any Yang-Baxter operator $(V, \sigma)$ in $\mathcal{C}$, the pair $(F(V), \sigma')$ with $\sigma' = \phi^{\otimes\,-1}_{V,V} \circ F(\sigma) \circ \phi^{\otimes}_{V,V}$ is a Yang-Baxter operator in $\mathcal{D}$.
3. A Yang-Baxter operator in $\mathrm{Vect}_{\mathbb{F}}$ is also called a **braided vector space**. It is a pair $(V, \sigma)$ of a vector space $V$ and a linear map $\sigma : V \otimes V \to V \otimes V$ such that the dodecagon in Definition 4.2.9 commutes.
4. Let $q, \lambda \in \mathbb{F} \setminus \{0\}$ and $V$ a vector space over $\mathbb{F}$ with an ordered basis $(v_1, ..., v_n)$. Then the linear map $\sigma : V \otimes V \to V \otimes V$ with

$$\sigma(v_i \otimes v_j) = \begin{cases} \lambda\, v_j \otimes v_i & i < j \\ \lambda q\, v_i \otimes v_i & i = j \\ \lambda\, v_j \otimes v_i + \lambda(q - q^{-1}) v_i \otimes v_j & i > j \end{cases}$$

gives $V$ the structure of a braided vector space.

**Proposition 4.2.11:** Let $\mathcal{C}$ be a braided monoidal category and $(V, \sigma)$ a Yang-Baxter operator in $\mathcal{C}$. Then there is a tensor functor $F : \mathcal{B} \to \mathcal{C}$, unique up to natural isomorphisms composed of the associators and unit constraints in $\mathcal{C}$, with $F(1) = V$ and $F(c_{1,1}) = \sigma$.

**Proof:**
We prove the claim for the case where $\mathcal{C}$ is strict. The proof is analogous to the one of Proposition 4.2.8. The only difference is that the condition $F(c_{1,1}) = c_{V,V}$ in Proposition 4.2.8 that followed from the requirement that $F$ is a braided monoidal functor is replaced by the condition $F(c_{1,1}) = \sigma$. $\square$

**Corollary 4.2.12:** Let $(V, \sigma)$ be a braided vector space. Then the maps

$$\rho_n : B_n \to \mathrm{Aut}_{\mathbb{F}}(V^{\otimes n}), \quad \sigma_i \mapsto \mathrm{id}_{V^{\otimes(i-1)}} \otimes \sigma \otimes \mathrm{id}_{V^{\otimes(n-i-1)}}$$

define a family of representations of the braid groups $B_n$ on $V^{\otimes n}$.

**Proof:**
We define $\rho$ on the generators of $B_n$ by setting $\rho(\sigma_i^{\pm 1}) = \mathrm{id}_{V^{\otimes(i-1)}} \otimes \sigma^{\pm} \otimes \mathrm{id}_{V^{\otimes(n-i-1)}}$ for all $i \in \{1, ..., n-1\}$. As $(V, \sigma)$ is a Yang-Baxter operator in $\mathrm{Vect}_{\mathbb{F}}$, the functoriality of the tensor product and the dodecagon identity allow one to extend $\rho$ to a group homomorphism $\rho : B_n \to \mathrm{Aut}_{\mathbb{F}}(V^{\otimes n})$. $\square$

## 4.3 Application: topological quantum field theories

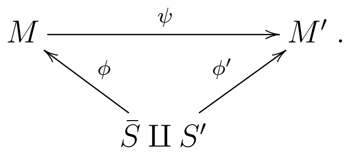
The concept of a topological quantum field theory was developed by Atiyah in [At], originally to describe quantum field theories on manifolds. The basic idea is to assign to each oriented closed $(n-1)$-dimensional manifold $S$ a vector space $Z(S)$ and to each oriented, compact $n$-manifold $M$ with boundary $\partial M = \bar{S} \amalg S'$ a linear map $Z(M) : Z(S) \to Z(S')$ in such a way that this assignment is compatible with disjoint unions of manifolds and with gluing. The latter can be implemented by requiring that the $(n-1)$-manifolds are objects in a suitable category $\mathcal{C}$ and the $n$-manifolds with boundary morphisms between them. The compatibility of the assignment with the gluing of manifolds then amounts to the statement that $Z$ is a functor. The disjoint union of manifolds should be viewed as a tensor product in the category $\mathcal{C}$, and the compatibility of $Z$ with the disjoint union of manifolds states that the category $\mathcal{C}$ should be a symmetric monoidal category and $Z$ should be a symmetric monoidal functor. We start by introducing the category $\mathcal{C}$, known as the *cobordism category*.

**Definition 4.3.1:** The **cobordism category** $\mathrm{Cob}_{n,n-1}$ for $n \in \mathbb{N}$ is the symmetric monoidal category given as follows:

- The objects of $\mathrm{Cob}_{n,n-1}$ are oriented closed smooth $(n-1)$-manifolds.
- Morphisms in $\mathrm{Cob}_{n,n-1}$ are equivalence classes of cobordisms.

  A **cobordism** from a closed oriented smooth $(n-1)$-manifold $S$ to a closed oriented smooth $(n-1)$-manifold $S'$ is a pair $(M, \phi)$ of a smooth compact oriented $n$-manifold $M$ with boundary $\partial M$ and an orientation preserving diffeomorphism $\phi : \bar{S} \amalg S' \to \partial M$, where $\bar{S}$ denotes the manifold with the reversed orientation and $\amalg$ the disjoint union of manifolds.

Two cobordisms $(M, \phi), (M', \phi') : S \to S'$ are called **equivalent** if there is an orientation preserving diffeomorphism $\psi : M \to M'$ such that the following diagram commutes



- The identity morphism $1_S$ is the equivalence class of the cobordism $([0,1] \times S, \phi_S)$ with the smooth diffeomorphism $\phi_S : \bar{S} \amalg S \to \{0,1\} \times S$ with $\phi_S(x) = (1, x)$ and $\phi_S(y) = (0, y)$ for all $x \in \bar{S}$, $y \in S$.

- The composite of morphisms $(M, \rho) : S \to S'$ and $(N, \sigma) : S' \to S''$ is the equivalence class of the cobordism $(P, \tau)$ obtained by gluing $M$ and $N$ along $S'$ to a manifold $P$, with the gluing maps given by $\rho|'_S : S' \to \partial M$ and $\sigma|_{S'} : S' \to \partial N$ and by combining the diffeomorphisms $\phi$ and $\chi$ to a diffeomorphism $\rho : \bar{S} \amalg S'' \to \partial P$. The construction of the smooth structure on $P$ involves the choice of collars around $S'$, but the the equivalence class of the resulting cobordism does not depend on this choices.

- The tensor product of cobordisms is given by the disjoint union of manifolds and the tensor unit is the empty manifold $\emptyset$, viewed as an oriented smooth $(n-1)$-manifold[2].

**Remark 4.3.2:**

1. There are other versions of topological quantum field theories based on topological or piecewise linear manifolds with boundary. For $n \leq 3$ the associated cobordism categories are equivalent. For $n \geq 4$ the smooth framework is the most common and well-developed.
2. Orientation reversal defines a functor $* : \mathrm{Cob}_{n,n-1} \to \mathrm{Cob}_{n,n-1}^{op}$ with $** = \mathrm{id}_{\mathrm{Cob}_{n,n-1}}$. This functor assigns to a smooth oriented $(n-1)$-manifold $S$ the manifold $\bar{S}$ with the opposite orientation and to the equivalence class of a cobordism $(M, \phi) : S \to S'$ the equivalence class of the cobordism $(\bar{M}, \phi) : \bar{S}' \to \bar{S}$, where $\bar{M}$ is the smooth $n$-manifold with the opposite orientation.

With the notion of the cobordism category it is simple to define a topological quantum field theory. Although one usually considers topological quantum field theories with values in the category $\mathrm{Vect}_{\mathbb{F}}^{fin}$, the notion can be generalised to any symmetric monoidal category.

**Definition 4.3.3:** Let $\mathcal{C}$ be a symmetric monoidal category.

1. An oriented $n$-dimensional **topological quantum field theory** with values in $\mathcal{C}$ is a symmetric monoidal functor $Z : \mathrm{Cob}_{n,n-1} \to \mathcal{C}$.

2. Two oriented topological $n$-dimensional quantum field theories $Z, Z' : \mathrm{Cob}_{2,1} \to \mathrm{Vect}_{\mathbb{F}}^{fin}$ are called **equivalent** if there is a monoidal natural isomorphism $\phi : Z \to Z'$.

To construct topological quantum field theories, it is reasonable to request that there is also functor $* : \mathcal{C} \to \mathcal{C}^{op}$ and a natural isomorphism $** \to \mathrm{id}_{\mathcal{C}}$ that implements orientation reversal.

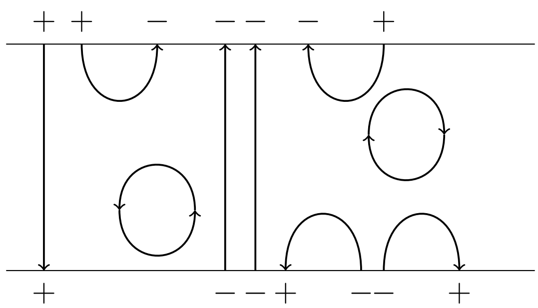
[2]Note that the empty set $\emptyset$ is by definition an $n$-dimensional smooth oriented manifold for all $n \in \mathbb{N}_0$.

If we work with the category $\mathrm{Vect}_{\mathbb{F}}^{fin}$ of finite-dimensional vector spaces over $\mathbb{F}$, the natural candidate is the functor $* : \mathrm{Vect}_{\mathbb{F}}^{fin} \to \mathrm{Vect}_{\mathbb{F}}^{fin\,op}$ that assigns to each vector space $V$ the dual vector space $V^*$ and to each linear map $f : V \to W$ the dual map $f^* : W^* \to V^*$, $\alpha \mapsto \alpha \circ f$.

It is also advantageous to describe the category $\mathrm{Cob}_{n,n-1}$ as explicitly and concretely as possible, namely to present it in terms of generators and relations. A presentation of a monoidal category is very similar to presentations of algebras or groups in terms of generators and relations, but the rigorous formulation of this is time consuming and technical. We refer to [Ka, XII.1] for details. Roughly speaking, a presentation of a monoidal category $\mathcal{C}$ consists of a set $O$ of objects, a set $G$ of generating morphisms and a set $R$ of defining relations. The morphisms of $\mathcal{C}$ are equivalence classes of free words in the morphisms in $G$ and in the identity morphisms for each object in $O$. These free words are taken with respect to two compositions, one representing the tensor product and one representing the composition of morphisms. In both cases, free words are composed by concatenation, and the equivalence relations are given by relations that implement (i) the associativity of the composition of morphisms, (ii) the properties of the identity morphisms, (iii) the properties of the tensor product, (iv) the properties of the tensor unit and (v) the relations in $R$.

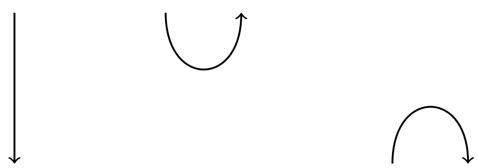
**Example 4.3.4: The cobordism category** $\mathrm{Cob}_{1,0}$

The symmetric monoidal category $\mathrm{Cob}_{1,0}$ has as objects disjoint unions of oriented points and as morphisms finite unions of oriented circles and oriented lines with endpoints at the objects such that the orientations of lines match the orientations of their endpoints. This can be depicted by a diagram in the plane as follows:

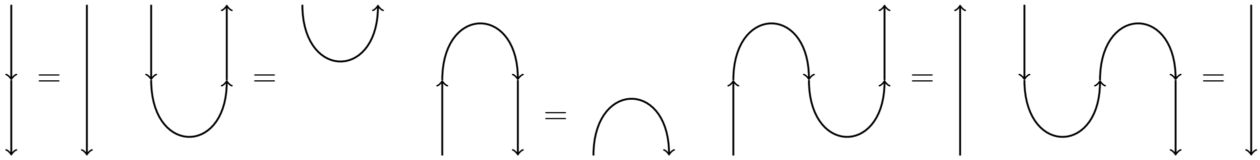



A morphism $f : (+,+,-,-,-,-,+) \to (+,-,-,+,-,+)$ in $\mathrm{Cob}_{1,0}$.

The category $\mathrm{Cob}_{1,0}$ is generated by the three morphisms



and the corresponding morphisms with reversed orientation. Its defining relations are

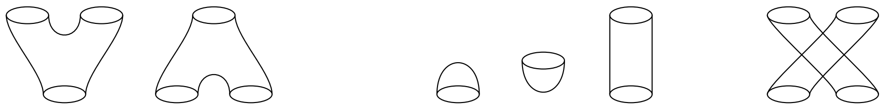


and the corresponding relations for the morphisms with the reversed orientations.

**Example 4.3.5: The cobordism category** $\mathrm{Cob}_{2,1}$

**• objects:**
The cobordism category $\mathrm{Cob}_{2,1}$ has as objects finite unions of oriented circles.

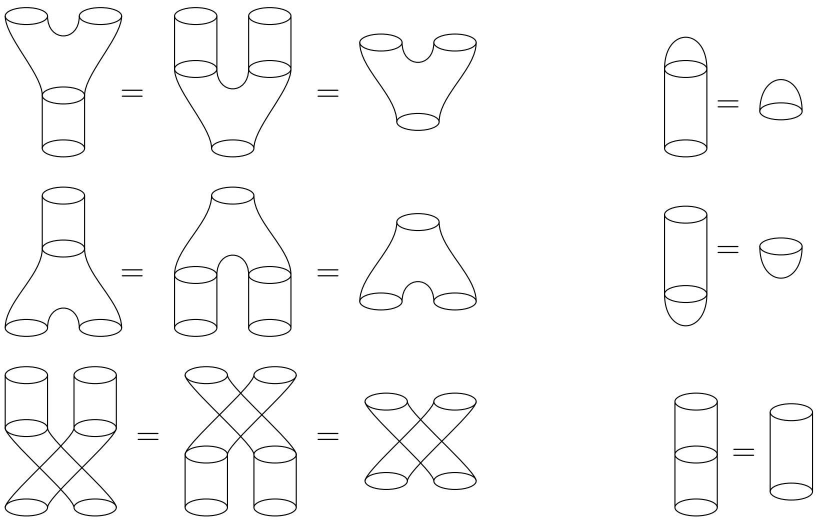
**• generating morphisms:**
The cobordism category $\mathrm{Cob}_{2,1}$ is generated by the following six morphisms. Each of these morphisms arises in two versions with opposite orientation, and the orientation of the boundary circles is understood to be induced by the orientations of the surfaces:



**trinion** **opposite trinion** **cap** **cup** **cylinder** **exchanging cylinder**

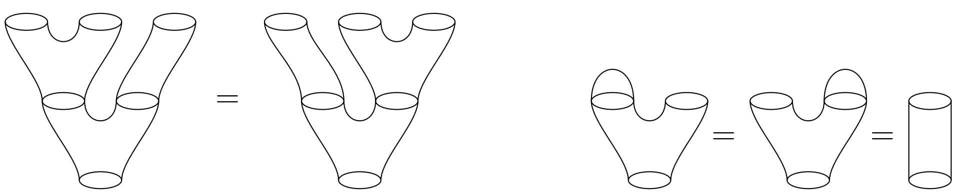
**• relations:**
The generators are subject to the following defining relations:

(a) **identity relations**

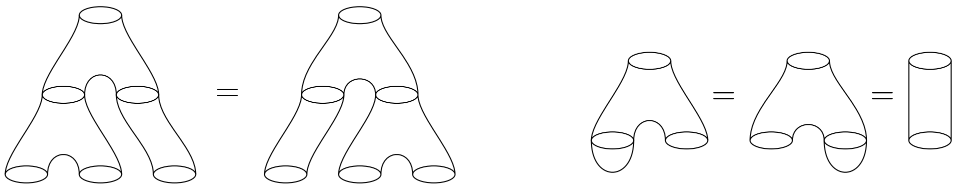


These relations state that the identity morphism on a finite union of oriented circles is the finite union of cylinders over these circles.

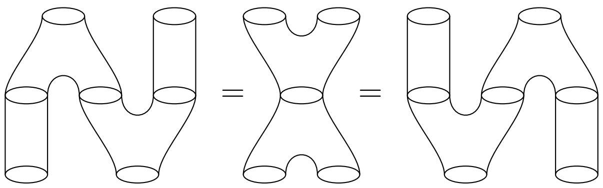
(b) **associativity and unitality**



(c) **coassociativity and counitality**

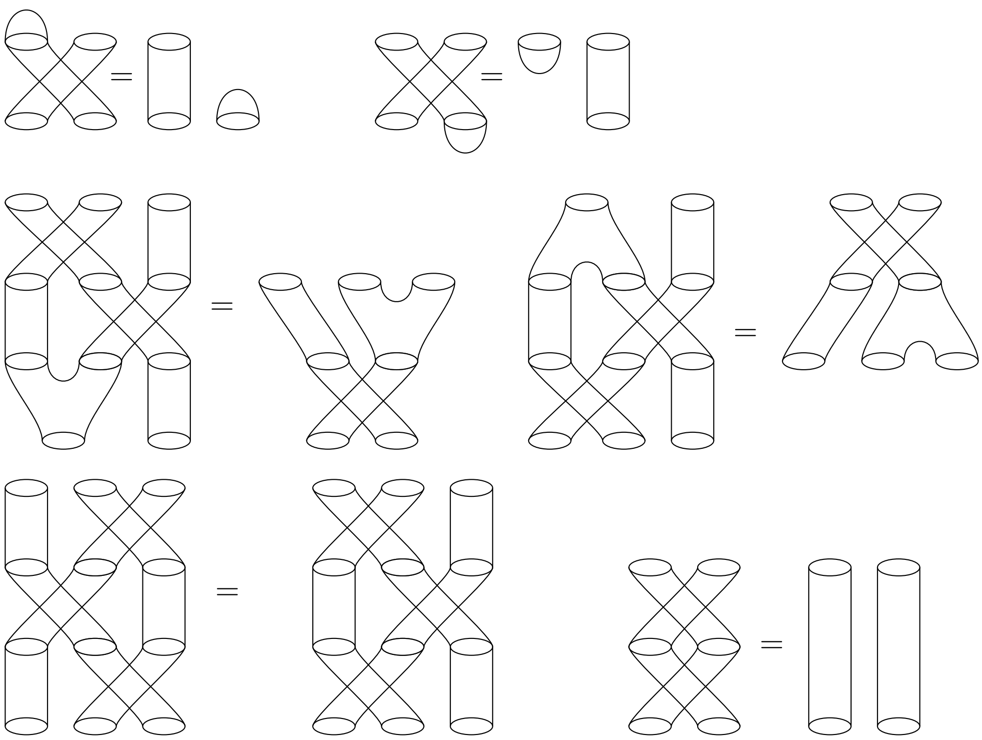


(d) **Frobenius relation**



(e) **commutativity and cocommutativity**

(f) **relations for the exchanging cylinder**



While the first four relations describe the interaction of the exchanging cylinder with cap, cup and trinions, the last two relations correspond to relations in the permutation category $\mathcal{S}$.

This presentation of the cobordism category is obtained with techniques from Morse theory, for a brief summary of the general techniques and their applications to $\mathrm{Cob}_{2,1}$ and $\mathrm{Cob}_{1,0}$, see [Kock, Chapter 1], for general background on Morse theory and Cobordisms, see [H, Chapter

6,7]. In principle, these techniques allow one to obtain a presentation of $\mathrm{Cob}_{n,n-1}$ for any $n \in \mathbb{N}$, but the description in terms of generators and relations becomes increasingly complicated with growing dimension. Given this presentation of the Cobordism category $\mathrm{Cob}_{2,1}$, one can classify all oriented 2-dimensional topological quantum field theories with values in $\mathrm{Vect}^{fin}_{\mathbb{F}}$.

**Theorem 4.3.6:** Equivalence classes of 2-dimensional oriented topological quantum field theories $Z : \mathrm{Cob}_{2,1} \to \mathrm{Vect}^{fin}_{\mathbb{F}}$ with $*Z = Z*$ are in bijection with isomorphism classes of commutative Frobenius algebras over $\mathbb{F}$.

**Proof:**
A monoidal functor $Z : \mathrm{Cob}_{2,1} \to \mathrm{Vect}^{fin}_{\mathbb{F}}$ with $*Z = Z*$ is determined uniquely (up to re-bracketing and left and right unit constraints) by its value on the positively oriented circle and on the six generating morphisms. If $Z$ assigns to the positively oriented circle a vector space $Z(O) = V$, then it assigns to the circle with the opposite orientation the dual vector space $Z(\bar{O}) = V^*$, to an $n$-fold union of circles the vector space $Z(O \amalg ... \amalg O) = V \otimes ... \otimes V$ and to the empty set the underlying field $Z(\emptyset) = \mathbb{F}$. This implies that $Z$ associates to the six generating morphisms linear maps

$m : V \otimes V \to V$ $\quad \Delta : V \to V \otimes V$ $\quad \eta : \mathbb{F} \to V$ $\quad \epsilon : V \to \mathbb{F}$ $\quad \mathrm{id}_V : V \to V$ $\quad \tau : V \otimes V \to V \otimes V$

where we took already into account the identity relations for cylinders and suppose that all circles on top of these diagrams are positively and all circles at the bottom of these diagrams are negatively oriented. In order to define a functor $Z : \mathrm{Cob}_{2,1} \to \mathrm{Vect}^{fin}_{\mathbb{F}}$ the linear maps $m$, $\eta$, $\Delta$, $\epsilon$, $\tau$ must satisfy certain relations that correspond to the defining relations of $\mathrm{Cob}_{2,1}$.

(b) **associativity and unitality:** They state that $(V, m, \eta)$ is an algebra:

$$m \circ (m \otimes \mathrm{id}) = m \circ (\mathrm{id} \otimes m) \qquad m \circ (\eta \otimes \mathrm{id}_V) = \mathrm{id}_V = m \circ (\mathrm{id}_V \otimes \eta).$$

In fact, by shrinking the cylinders in the associativity and unitality condition to lines, one obtains the diagrams for an algebra in a monoidal category from Section 4.1.

(c) **coassociativity and counitality:** They state that $(V, \Delta, \epsilon)$ is a coalgebra:

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta \qquad l_V \circ (\epsilon \otimes \mathrm{id}_V) \circ \Delta = \mathrm{id}_V = r_V \circ (\mathrm{id}_V \otimes \epsilon) \circ \Delta$$

Again, shrinking the cylinders in this relation to lines, one obtains the diagrams for a coalgebra in a monoidal category from the end of Section 4.1.

(d) **Frobenius relation:** the Frobenius relation states that the algebra and coalgebra structure on $V$ satisfy the compatibility condition:

$$(\mathrm{id}_V \otimes m) \circ (\Delta \otimes \mathrm{id}_V) = \Delta \circ m = (m \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes \Delta)$$

Together (b), (c), (d) state that $(V, m, \eta, \Delta, \epsilon)$ is a $(\Delta, \epsilon)$-Frobenius algebra and hence a Frobenius algebra by Exercise 45.

(f) **relations for the exchanging cylinder:** The last two relations state that the linear map $\tau : V \otimes V \to V \otimes V$ is an involution and defines a functor $\mathcal{S} \to \mathrm{Vect}^{fin}_{\mathbb{F}}$. The remaining ones are

$$\tau \circ (\eta \otimes \mathrm{id}_V) = \mathrm{id}_V \otimes \eta, \qquad (m \otimes \mathrm{id}) \circ (\mathrm{id}_V \otimes \tau) \circ (\tau \otimes \mathrm{id}_V) = \tau \circ (\mathrm{id}_V \otimes m),$$
$$(\mathrm{id}_V \otimes \epsilon) \circ \tau = \epsilon \otimes \mathrm{id}_V \qquad (\tau \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes \tau) \circ \Delta = (\mathrm{id}_V \otimes \Delta) \circ \tau.$$

We conclude that $\tau$ is the flip map $\tau : V \otimes V \to V \otimes V$, $v \otimes v' \mapsto v' \otimes v$.

(e) **commutativity and cocommutativity relations:** They state that the $(\Delta, \epsilon)$-Frobenius algebra $(V, m, \eta, \Delta, \epsilon)$ is commutative and cocommutative:

$$m \circ \tau = m \qquad\qquad \tau \circ \Delta = \Delta.$$

One can show (Exercise) that a $(\Delta, \epsilon)$-Frobenius algebra is commutative and cocommutative if and only if the associated Frobenius algebra from Exercise 45 is commutative. This shows that every oriented topological quantum field theory $Z : \mathrm{Cob}_{2,1} \to \mathrm{Vect}_{\mathbb{F}}^{fin}$ defines a commutative Frobenius algebra and vice versa.

Due to the conditions in Definition 4.1.11, 2. a monoidal natural isomorphism $\phi : Z \to Z'$ between two oriented topological quantum field theories $Z$ and $Z'$ is specified uniquely by the linear map $\phi_O : V = Z(O) \to Z'(O) = V'$. The naturality of $\phi$ implies that the map $\phi_O$ is an algebra and coalgebra isomorphism, which is the case if and only if $\phi$ is an algebra isomorphism that preserves the Frobenius form. Conversely, every algebra isomorphism $\phi : V = Z(O) \to Z'(O)$ defines a monoidal natural isomorphism $\phi : Z \to Z'$. □

# 5 Quasitriangular Hopf algebras

## 5.1 Quasitriangular bialgebras and Hopf algebras

In this chapter we investigate bialgebras and Hopf algebras with additional structure that relate their representation on the tensor product of two representation spaces to the one on the opposite tensor product. Clearly, the condition that the flip map $\tau : V\otimes W \to W\otimes V$ is an isomorphism of representations for all representations $V, W$ of a bialgebra $B$ would too restrictive. If this holds for all representations of $B$, then it holds also for the representation of $B$ on $B\otimes B$ by left multiplication. This implies for all $b \in B$

$$\Delta(b) = \Sigma_{(b)} b_{(1)}\otimes b_{(2)} = b \rhd (1\otimes 1) = b \rhd_{op} (1\otimes 1) = \Sigma_{(b)} b_{(2)}\otimes b_{(1)} = \Delta^{op}(b)$$

and hence $B$ must be cocommutative. Moreover, this condition is undesirable, from the representation theoretical viewpoint. It amounts to the statement that the representation category of $B$ is a *symmetric* monoidal category and hence does not carry any interesting braid group representations, i. e. braid group representations that are not induced by representations of permutation groups.

A weaker and more sensible condition is the requirement that for all $B$-modules $V, W$ there *exists* an isomorphism of representations between the representations of $B$ on $V\otimes W$ and on $W\otimes V$ and that this isomorphism is *compatible with tensor products*. This amounts to the statement that the category $B$-Mod is a braided monoidal category. We will show in the following that this is equivalent to the following conditions on the bialgebra $B$.

**Definition 5.1.1:**

1. A **quasitriangular bialgebra** is is a pair $(B, R)$ of a bialgebra $B$ and an an invertible element $R = R_{(1)}\otimes R_{(2)} \in B\otimes B$, the **universal $R$-matrix**, that satisfies

$$\Delta^{op}(b) = R \cdot \Delta(b) \cdot R^{-1} \qquad (\Delta\otimes \mathrm{id})(R) = R_{13} \cdot R_{23} \qquad (\mathrm{id}\otimes\Delta)(R) = R_{13} \cdot R_{12},$$

   where $R_{12} = R_{(1)}\otimes R_{(2)}\otimes 1$, $R_{13} = R_{(1)}\otimes 1\otimes R_{(2)}$ and $R_{23} = 1\otimes R_{(1)}\otimes R_{(2)} \in B\otimes B\otimes B$.

2. A **homomorphism** of quasitriangular bialgebras from $(B, R)$ to $(B', R')$ is a bialgebra map $\phi : B \to B'$ with $R' = (\phi\otimes\phi)(R)$.

3. A quasitriangular bialgebra is called **triangular** if its $R$-matrix satisfies $\tau \circ R = R^{-1}$, where $\tau : B\otimes B \to B\otimes B$, $b\otimes b' \mapsto b'\otimes b$ denotes the flip map.

4. A **(quasi)triangular** Hopf algebra is a (quasi)triangular bialgebra that is a Hopf algebra.

Note that the notation $R = R_{(1)}\otimes R_{(2)}$ is symbolic. It stands for a finite sum $R = \Sigma_{i=1}^n b_i\otimes b_i'$ with $b_i, b_i' \in B$. To distinguish it from the Sweedler notation for a coproduct $\Delta(b) = \Sigma_{(b)} b_{(1)}\otimes b_{(2)}$, we do not use a summation sign in this case.

**Theorem 5.1.2:** Let $B$ be a bialgebra. Then the category $B$-mod is a braided monoidal category if and only if $B$ has the structure of a quasitriangular bialgebra. It is symmetric if and only if $B$ is triangular.

**Proof:**
$\Leftarrow$: If $(B,R)$ is a quasitriangular bialgebra, then we define for a pair of $B$-modules $(V,\rhd_V)$ and $(W,\rhd_W)$ a linear map $c_{V,W}: V\otimes W\to W\otimes V$ by setting

$$c_{V,W}(v\otimes w)=\tau(R\rhd_{V\otimes W} v\otimes w)=R_{21}\rhd_{W\otimes V}\tau(v\otimes w)\qquad \forall v\in V, w\in W, \tag{39}$$

where $R_{21}=\tau(R)$ is the flipped $R$-matrix. Then $c_{V,W}$ is a linear isomorphism with inverse $c^{-1}_{V,W}(v\otimes w)=\tau(R^{-1}_{21}\rhd_{W\otimes V} w\otimes v)$, and one has

$$\begin{aligned}&b\rhd_{W\otimes V}(c_{V,W}(v\otimes w))=\tau((\Delta^{op}(b)\cdot R)\rhd_{V\otimes W} v\otimes w)=\tau((R\cdot\Delta(b))\rhd_{V\otimes W} v\otimes w)\\&=\tau(R\rhd_{V\otimes W}(b\rhd_{V\otimes W} v\otimes w))=c_{V,W}(b\rhd_{V\otimes W}(v\otimes w))\end{aligned}$$

for all $b\in B$, $v\in V$ and $w\in W$. This shows that $c_{V,W}: V\otimes W\to W\otimes V$ is a $B$-module isomorphism and hence an isomorphism in $B$-Mod. To show that this defines a natural isomorphism $c:\otimes\to\otimes^{op}$ in $B$-Mod, we consider $B$-module maps $f:V\to V'$ and $g:W\to W'$ and compute

$$\begin{aligned}(c_{V',W'}\circ(f\otimes g))(v\otimes w)&=\tau(R\rhd_{V',W'} f(v)\otimes g(w))=(\tau\circ(f\otimes g))(R\rhd_{V\otimes W} v\otimes w)\\&=((g\otimes f)\circ\tau)(R\rhd_{V\otimes W} v\otimes w)=((g\otimes f)\circ c_{V,W})(v\otimes w)\end{aligned}$$

where we used first the definition of $c_{V',W'}$, then the fact that $f$ and $g$ are $B$-module homomorphisms and then the definition of $c_{V,W}$. This proves the naturality of $c$. For the proof of the hexagon relations note that in Sweedler notation the conditions on the universal $R$-matrix read

$$\begin{aligned}(\Delta\otimes\mathrm{id})(R)=R_{13}R_{23}\quad&\Leftrightarrow\quad \Sigma_{(R_{(1)})}R_{(1)(1)}\otimes R_{(1)(2)}\otimes R_{(2)}=R'_{(1)}\otimes R_{(1)}\otimes R'_{(2)}R_{(2)}\\(\mathrm{id}\otimes\Delta)(R)=R_{13}R_{12}\quad&\Leftrightarrow\quad \Sigma_{(R_{(2)})}R_{(1)}\otimes R_{(2)(1)}\otimes R_{(2)(2)}=R'_{(1)}R_{(1)}\otimes R_{(2)}\otimes R'_{(2)}.\end{aligned}$$

With this, we obtain the two hexagon relations

$$\begin{aligned}&(1_V\otimes c_{U,W})\circ a_{V,U,W}\circ(c_{U,V}\otimes 1_W)((u\otimes v)\otimes w)=(1_V\otimes c_{U,W})(R_{(2)}\rhd_V v\otimes(R_{(1)}\rhd_U u\otimes w))\\&=R_{(2)}\rhd_V v\otimes(R'_{(2)}\rhd_W w\otimes(R'_{(1)}R_{(1)})\rhd_U u)=\Sigma_{(R_{(2)})}(R_{(2)(1)}\rhd_V v\otimes R_{(2)(2)}\rhd_W w)\otimes R_{(1)}\rhd_U u\\&=a_{V,W,U}(R_{(2)}\rhd_{V\otimes W}(v\otimes w)\otimes R_{(1)}\rhd_U u)=a_{V,W,U}\circ c_{U,V\otimes W}\circ a_{U,V,W}((u\otimes v)\otimes w)\end{aligned}$$

$$\begin{aligned}&(c_{U,W}\otimes 1_V)\circ a^{-1}_{U,W,V}\circ(1_U\otimes c_{V,W})(u\otimes(v\otimes w))=(c_{U,W}\otimes 1_V)((u\otimes R_{(2)}\rhd_W w)\otimes R_{(1)}\rhd_V v)\\&=((R'_{(2)}R_{(2)})\rhd_W w\otimes R'_{(1)}\rhd_U u)\otimes R_{(1)}\rhd_V v=\Sigma_{(R_{(1)})}(R_{(2)}\rhd_W w\otimes R_{(1)(1)}\rhd_U u)\otimes R_{(1)(2)}\rhd_V v\\&=a^{-1}_{W,U,V}(R_{(2)}\rhd_W w\otimes R_{(1)}\rhd_{U\otimes V}(u\otimes v))=a^{-1}_{W,U,V}\circ c_{U\otimes V,W}\circ a^{-1}_{U,V,W}(u\otimes(v\otimes w)).\end{aligned}$$

This shows that the $B$-module isomorphisms $c_{V,W}: V\otimes W\to W\otimes V$ define a braiding.

$\Rightarrow$: Let $B$ be a bialgebra, $c:\otimes\to\otimes^{op}$ a braiding for $B$-Mod and define $R:=\tau\circ c_{B,B}(1_B\otimes 1_B)\in B\otimes B$. Then $R$ is invertible with inverse $R^{-1}=\tau\circ c^{-1}_{B,B}(1_B\otimes 1_B)$. To prove that the conditions on the universal $R$-matrix are satisfied, note that for all $B$-modules $(V,\rhd_V)$ and $(W,\rhd_W)$ and elements $v\in V$, $w\in W$, the linear map $\phi_{v,w}: B\otimes B\to V\otimes W$, $a\otimes b\mapsto(a\rhd_V v)\otimes(b\rhd_W w)$ is a $B$-module homomorphism. The naturality of the braiding implies $c_{V,W}\circ\phi_{v,w}=\phi_{w,v}\circ c_{B,B}$ and

$$R\rhd_{V\otimes W}(v\otimes w)=\phi_{v,w}(R)=\tau(\phi_{w,v}\circ c_{B,B}(1_B\otimes 1_B))=\tau(c_{V,W}\circ\phi_{v,w}(1_B\otimes 1_B))=\tau(c_{V,W}(v\otimes w))$$

for all $v\in V$ and $w\in W$. For $v=w=1_B\in B$ we obtain

$$\Delta^{op}(b)\cdot R=\tau(b\rhd_{B\otimes B} c_{B,B}(1_B\otimes 1_B))=\tau(c_{B,B}(b\rhd_{B\otimes B} 1_B\otimes 1_B))=\tau(c_{B,B}(\Delta(b)))=R\cdot\Delta(b),$$

where we used first the definition of $R$, then that $c_{B,B}$ is a module map, the $B$-module structure on $B\otimes B$ and the identity for $R$. The remaining identities follow from the hexagon relations:

$$\begin{aligned}(\mathrm{id}_B\otimes\Delta)(R) &= R\rhd_{B\otimes(B\otimes B)}(1_B\otimes(1_B\otimes 1_B)) = \tau(c_{B,B\otimes B}(1_B\otimes(1_B\otimes 1_B)))\\ &= \tau((a^{-1}_{B,B,B}\circ(\mathrm{id}_B\otimes c_{B,B})\circ a_{B,B,B}\circ(c_{B,B}\otimes\mathrm{id}_B)\circ a^{-1}_{B,B,B})(1_B\otimes(1_B\otimes 1_B))\\ &= R_{13}R_{12}\rhd_{(B\otimes B)\otimes B}((1_B\otimes 1_B)\otimes 1_B) = R_{13}R_{12}\\ (\Delta\otimes\mathrm{id}_B)(R) &= R\rhd_{(B\otimes B)\otimes B}((1_B\otimes 1_B)\otimes 1_B) = \tau(c_{B\otimes B,B}((1_B\otimes 1_B)\otimes 1_B))\\ &= \tau(a_{B,B,B}\circ(c_{B,B}\otimes\mathrm{id}_B)\circ a^{-1}_{B,B,B}\circ(\mathrm{id}_B\otimes c_{B,B})\circ a_{B,B,B})((1_B\otimes 1_B)\otimes 1_B)\\ &= R_{13}R_{23}\rhd_{B\otimes(B\otimes B)}(1_B\otimes(1_B\otimes 1_B)) = R_{13}R_{23}.\end{aligned}$$

That the representation category $B$-Mod is symmetric if and only if $(B,R)$ is triangular follows directly from the expressions for the braiding in terms of the universal $R$-matrix since we have

$$c_{V,W}(v\otimes w)=\tau(R\rhd_{V\otimes W} v\otimes w)\qquad c^{-1}_{W,V}(v\otimes w)=\tau(R^{-1}_{21}\rhd_{V\otimes W} v\otimes w).$$

This shows that $R=R^{-1}_{21}$ if and only if $c_{V,W}=c^{-1}_{W,V}$. □

In particular, this theorem implies that quasitriangular bialgebras give rise to braid group representations. This follows directly from Corollary 4.2.12, since every object in a braided monoidal category is a Yang-Baxter operator.

**Corollary 5.1.3:** Let $(B,R)$ be a quasitriangular bialgebra, $\rho: B\to\mathrm{End}_{\mathbb{F}}(V)$ a representation of $B$ on $V$ and $\rho_{V^{\otimes n}}: B\to\mathrm{End}_{\mathbb{F}}(V^{\otimes n})$ the associated representation on $V^{\otimes n}$. Then

$$\rho: B_n\to\mathrm{End}_{\mathbb{F}}(V^{\otimes n}),\quad \sigma_i\mapsto(\mathrm{id}_{V^{\otimes(i-1)}}\otimes\tau\otimes\mathrm{id}_{V^{\otimes(n-i-1)}})\circ\rho_{V^{\otimes n}}(1^{\otimes(i-1)}\otimes R\otimes 1^{\otimes(n-i-1)})$$

is a representation of the braid group $B_n$ on $V^{\otimes n}$ for all $n\in\mathbb{N}_0$. It defines a representation of $S_n$ if and only if $(B,R)$ is triangular.

Clearly, every cocommutative bialgebra is quasitriangular with universal $R$-matrix $R=1\otimes 1$. This includes in particular group algebras of finite groups, tensor algebras of vector spaces and universal eveloping algebras of Lie algebras. However, these bialgebras do not give rise to interesting representations of the braid group. The associated representations of $B_n$ permute the factors in the tensor product $V^{\otimes n}$ via the flip map and are induced by representations of the permutation group $S_n$ via the group homomorphisms $\Pi_n: B_n\to S_n$. This gives a strong motivation to consider bialgebras that are quasitriangular but not cocommutative.

**Example 5.1.4:**

1. A *commutative* bialgebra $B$ is quasitriangular if and only if it is cocommutative, since in this case, one has $\Delta^{op}(b)=R\cdot\Delta(b)\cdot R^{-1}$ for all $b\in B$ if and only if $\Delta=\Delta^{op}$. This shows that the algebra of functions $\mathrm{Fun}_{\mathbb{F}}(G)$ on a non-abelian finite group $G$ cannot be quasitriangular, since it is commutative, but not cocommutative.

2. The group algebra $\mathbb{C}[\mathbb{Z}/n\mathbb{Z}]$ is quasitriangular with universal $R$-matrix

$$R=\tfrac{1}{n}\Sigma_{j,k=0}^{n-1}e^{2\pi i\, jk/n}\,\bar{\jmath}\otimes\bar{k}.$$

3. For $n=2$ and $\mathrm{char}(\mathbb{F})\neq 2$, the Taft algebra from Example 2.3.6 is presented with generators $x, y$ and relations $xy=-yx$, $y^2=1$, $x^2=0$, and Hopf algebra structure

$$\Delta(y)=y\otimes y,\quad \Delta(x)=1\otimes x+y\otimes x,\quad \epsilon(y)=1,\quad \epsilon(x)=0,\quad S(y)=y,\quad S(x)=-xy.$$

This Hopf algebra is quasitriangular with universal $R$-matrices

$$R_\alpha=\tfrac{1}{2}\left(1\otimes 1+1\otimes y+y\otimes 1-y\otimes y\right)+\tfrac{\alpha}{2}\left(x\otimes x-x\otimes xy+xy\otimes xy+xy\otimes x\right)\qquad \text{for}\alpha\in\mathbb{F}.$$

4. Let $q\in\mathbb{F}$ be a primitive $r$th root of unity with $r>1$ odd. Then the Hopf algebra $U_q^r(\mathfrak{sl}_2)$ from Proposition 2.3.12 is quasitriangular with universal $R$-matrix

$$R=\frac{1}{r}\sum_{i,j,k=0}^{r-1}\frac{(q-q^{-1})^k}{(k)!_{q^2}}q^{k(k-1)+2k(i-j)-2ij}\,E^kK^i\otimes F^kK^j. \tag{40}$$

If $V_m$ denotes the simple $U_q^r(\mathfrak{sl}_2)$-module of weight $\lambda=q^m$, $0\leq m<r-1$ from Theorem 3.6.4 and Remark 3.6.8, then the braiding $c_{V_m,V_n}: V_m\otimes V_n\to V_n\otimes V_m$ is given by

$$c_{V_m,V_n}(v_p\otimes v_t)=\sum_{k=0}^{\min\{n-t,p\}}(q-q^{-1})^k(k)!_{q^2}\,q^{nm}_{pt}(k,\alpha)\binom{m-p+k}{k}_{q^2}\binom{t+k}{k}_{q^2}v_{t+k}\otimes v_{p-k}$$

where $v_p$ with $0\leq p\leq m$ are the basis vectors of $V_m$ from Lemma 3.6.2, the integer $\alpha\in\mathbb{Z}$ is chosen such that $n+\alpha r$ is even and

$$q^{mn}_{pt}(k,\alpha)=q^{k(n-m)-k(m-p)-kt-pn-mt-2(k-p)(k+t)+(n+\alpha r)m/2}.$$

For $n=m=1$, we can take $\alpha=1$ and obtain:

$$c_{V_1,V_1}(v_0\otimes v_0)=\lambda q\,v_0\otimes v_0 \qquad c_{V_1,V_1}(v_0\otimes v_1)=\lambda\,v_1\otimes v_0$$
$$c_{V_1,V_1}(v_1\otimes v_0)=\lambda(v_0\otimes v_1+(q-q^{-1})v_1\otimes v_0) \qquad c_{V_1,V_1}(v_1\otimes v_1)=\lambda q\,v_1\otimes v_1$$

with $\lambda=q^{(r-1)/2}$. This is the braided vector space from Example 4.2.10, 4.

These examples show that, just like a braiding in a monoidal category, quasitriangularity is a *structure* and not a property. For a given bialgebra or Hopf algebra $B$, there may be several different universal $R$-matrices that satisfy the requirements in Definition 5.1.1, or none. We now investigate the basic properties of quasitriangular bialgebras. Our first result is an important equation on the universal $R$-matrix, the *quantum Yang-Baxter equation* (abbreviated *QYBE*) that can be viewed as the algebra counterpart of the dodecagon identity and has important applications in knot theory and mathematical physics.

**Proposition 5.1.5:** Let $(B,R)$ be a quasitriangular bialgebra.

1. Then the universal $R$-matrix satisfies $(\epsilon\otimes\mathrm{id})(R)=(\mathrm{id}\otimes\epsilon)(R)=1$ and the **quantum Yang-Baxter equation (QYBE)**: $R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12}$.
2. $(B,R_{21}^{-1})$ with $R_{21}^{-1}=\tau(R^{-1})$ is a quasitriangular bialgebra as well.

**Proof:**
1. With the defining condition on the universal $R$-matrix we compute

$$\begin{aligned}R&=(\mathrm{id}\otimes\mathrm{id})(R)=(\epsilon\otimes\mathrm{id}\otimes\mathrm{id})\circ(\Delta\otimes\mathrm{id})(R)=(\epsilon\otimes\mathrm{id}\otimes\mathrm{id})(R_{13}\cdot R_{23})\\&=(\epsilon\otimes\mathrm{id}\otimes\mathrm{id})(R_{13})\cdot(\epsilon\otimes\mathrm{id}\otimes\mathrm{id})(R_{23})=\epsilon(1)\,(\epsilon\otimes\mathrm{id}\otimes\mathrm{id})(R_{13})\cdot R=(1\otimes(\epsilon\otimes\mathrm{id})(R))\cdot R\end{aligned}$$

As $R$ is invertible, right multiplication of this equation with $R^{-1}$ yields $1\otimes(\epsilon\otimes\mathrm{id})(R) = 1\otimes 1$, and applying $\epsilon\otimes\mathrm{id}$ to this equation we obtain $(\epsilon\otimes\mathrm{id})(R) = (\epsilon\otimes\mathrm{id})(1\otimes 1) = \epsilon(1)1 = 1$. The proof of the identity $(\mathrm{id}\otimes\epsilon)(R) = 1$ is analogous. The QYBE follows directly from the defining properties of the universal $R$-matrix:

$$R_{12}R_{13}R_{23} = R_{12}\cdot(\Delta\otimes\mathrm{id})(R) = (\Delta^{op}\otimes\mathrm{id})(R)\cdot R_{12} = (\tau\otimes\mathrm{id})(R_{13}R_{23})\cdot R_{12} = R_{23}R_{13}R_{12}.$$

2. Applying the flip map $\tau: B\otimes B\to B\otimes B$, $b\otimes c\mapsto c\otimes b$ to the defining conditions in Definition 5.1.1 yields for all $b\in B$

$$\Delta(b) = \tau\circ\Delta^{op}(b) = \tau(R\cdot\Delta(b)\cdot R^{-1}) = R_{21}\cdot\Delta^{op}(b)\cdot R^{-1}_{21}$$
$$(\mathrm{id}\otimes\Delta)(R^{-1}_{21}) = (\tau\otimes\mathrm{id})\circ(\mathrm{id}\otimes\tau)\circ(\Delta\otimes\mathrm{id})(R^{-1}) = (\tau\otimes\mathrm{id})\circ(\mathrm{id}\otimes\tau)(R^{-1}_{23}R^{-1}_{13}) = (R^{-1}_{21})_{13}(R^{-1}_{21})_{12}$$
$$(\Delta\otimes\mathrm{id})(R^{-1}_{21}) = (\mathrm{id}\otimes\tau)\circ(\tau\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Delta)(R^{-1}) = (\mathrm{id}\otimes\tau)\circ(\tau\otimes\mathrm{id})(R^{-1}_{12}R^{-1}_{13}) = (R^{-1}_{21})_{13}(R^{-1}_{21})_{23}.$$

Conjugating the first equation with $R^{-1}_{21}$ shows that $R^{-1}_{21}$ is another universal $R$-matrix for $B$. □

Another way to motivate quasitriangular bialgebras that does not rely on representation theory and braided monoidal categories is from the viewpoint of module algebras and module coalgebras. As shown in Section 3.1, one can view a module coalgebra over a bialgebra as a non-commutative generalisation of a space with a group action and a module algebra as a non-commutative generalisation of the algebra of functions on this space. The compatibility between the algebra and the module structure ensures that the invariants of this module form a subalgebra which can be viewed as the algebra of functions on the orbit space.

This raises the question about non-commutative counterparts of *products* of spaces with group actions or, equivalently, noncommutative function algebras on such product spaces. The natural candidate for the product of two module algebras $(A,\rhd)$ and $(A',\rhd')$ over a bialgebra $B$ is the tensor product $A\otimes A'$ with the canonical algebra and $B$-module structure. However, one can show that in general this is *not a module algebra* over $B$ unless $B$ is cocommutative. However, if $(B,R)$ is quasitriangular, then it is possible to deform the multiplication of $A\otimes A'$ in such a way that it becomes a module algebra with the induced $B$-module structure. This is the so-called **braided tensor product of module algebras**.

**Proposition 5.1.6:** Let $(B,R)$ be a quasitriangular bialgebra and $(A,\rhd)$, $(A',\rhd')$ module algebras over $B$. Then the vector space $A\otimes A'$ with the canonical $B$-module structure

$$\rhd: B\otimes A\otimes A'\to A\otimes A',\quad b\rhd(a\otimes a') = \Sigma_{(b)}(b_{(1)}\rhd a)\otimes(b_{(2)}\rhd' a') \tag{41}$$

is a module algebra over $H$ with the twisted multiplication

$$(a\otimes a')\cdot(c\otimes c') = a(R_{(2)}\rhd c)\otimes(R_{(1)}\rhd a')c'. \tag{42}$$

**Proof:**
To show that it is an algebra, we verify unitality and associativity

$$(a\otimes a')\cdot(1_A\otimes 1_{A'}) \overset{(42)}{=} a(R_{(2)}\rhd 1_A)\otimes(R_{(1)}\rhd' a')1_{A'} = \epsilon(R_{(2)})\,a\otimes(R_{(1)}\rhd' a') \overset{5.1.5}{=} a\otimes a'$$
$$(1_A\otimes 1_{A'})\cdot(a\otimes a') \overset{(42)}{=} 1_A(R_{(2)}\rhd a)\otimes(R_{(1)}\rhd' 1_{A'})a' = \epsilon(R_{(1)})(R_{(2)}\rhd a)\otimes a' \overset{5.1.5}{=} a\otimes a'$$

$$\begin{aligned}
((a\otimes a')\cdot(c\otimes c'))\cdot(d\otimes d') &\overset{(42)}{=} (a(R_{(2)}\rhd c)\otimes(R_{(1)}\rhd' a')c')\cdot(d\otimes d')\\
(42)\quad &= a(R_{(2)}\rhd c)(R'_{(2)}\rhd' d)\otimes(R'_{(1)}\rhd'_a((R_{(1)}\rhd' a')c')d'\\
(\text{module algebra})\quad &= \Sigma_{(R')}\,a(R_{(2)}\rhd c)(R'_{(2)}\rhd d)\otimes(R'_{(1)(1)}R_{(1)})\rhd' a')(R'_{(1)(2)}\rhd' c')d'\\
((\Delta\otimes\mathrm{id})(R) = R_{13}R_{23})\quad &= a(R_{(2)}\rhd c)(R'_{(2)}R''_{(2)}\rhd d)\otimes R'_{(1)}R_{(1)}\rhd' a')(R''_{(1)}\rhd' c')d'
\end{aligned}$$

$$\begin{aligned}
(a\otimes a')\cdot((c\otimes c')\cdot(d\otimes d')) &\overset{(42)}{=} (a\otimes a')\cdot(c(R_{(2)}\rhd d)\otimes(R_{(1)}\rhd' c')d')\\
(42)\quad &= a(R'_{(2)}\rhd_A (c(R_{(2)}\rhd d)))\otimes(R'_{(1)}\rhd' a')(R_{(1)}\rhd' c')d'\\
(\text{module algebra})\quad &= \Sigma_{(R')}\, a(R'_{(2)(1)}\rhd c)(R'_{(2)(2)}R_{(2)}\rhd d)))\otimes(R'_{(1)}\rhd' a')(R_{(1)}\rhd' c')d')\\
((\mathrm{id}\otimes\Delta)(R)=R_{13}R_{12})\quad &= a(R''_{(2)}\rhd c)(R'_{(2)}R_{(2)}\rhd d)))\otimes(R'_{(1)}R''_{(1)}\rhd' a')(R_{(1)}\rhd' c')d')
\end{aligned}$$

To show that it is a module algebra, we compute

$$\begin{aligned}
b\rhd((a\otimes a')\cdot(c\otimes c')) &\overset{(42)}{=} b\rhd(a(R_{(2)}\rhd c)\otimes(R_{(1)}\rhd' a')c')\\
(41)\quad &= \Sigma_{(b)}(b_{(1)}\rhd(a\cdot(R_{(2)}\rhd c)))\otimes(b_{(2)}\rhd'((R_{(1)}\rhd' a')\cdot c'))\\
(\text{module algebra})\quad &= \Sigma_{(b)}(b_{(1)(1)}\rhd a)((b_{(1)(2)}R_{(2)})\rhd c)\otimes((b_{(2)(1)}R_{(1)})\rhd' a')(b_{(2)(2)}\rhd' c')\\
(\text{coassociativity})\quad &= \Sigma_{(b)}(b_{(1)}\rhd a)((b_{(2)}R_{(2)})\rhd c)\otimes((b_{(3)}R_{(1)})\rhd' a')(b_{(4)}\rhd' c')\\
(\text{coassociativity})\quad &= \Sigma_{(b)}(b_{(1)}\rhd a)((b_{(2)(1)}R_{(2)})\rhd c)\otimes((b_{(2)(2)}R_{(1)})\rhd' a')(b_{(4)}\rhd' c')\\
(\Delta^{op}\cdot R=R\cdot\Delta)\quad &= \Sigma_{(b)}(b_{(1)}\rhd a)((R_{(2)}b_{(2)(2)})\rhd c)\otimes((R_{(1)}b_{(2)(1)})\rhd' a')(b_{(4)}\rhd' c')\\
(\text{coassociativity})\quad &= \Sigma_{(b)}(b_{(1)}\rhd a)((R_{(2)}b_{(3)})\rhd c)\otimes((R_{(1)}b_{(2)})\rhd' a')(b_{(4)}\rhd' c')\\
(42)\quad &= \Sigma_{(b)}(b_{(1)}\rhd a\otimes b_{(2)}\rhd' a')\cdot(b_{(3)}\rhd c\otimes b_{(4)}\rhd' c')\\
(\text{coassociativity})\quad &= \Sigma_{(b)}(b_{(1)(1)}\rhd a\otimes b_{(1)(2)}\rhd' a')\cdot(b_{(2)(1)}\rhd c\otimes b_{(2)(2)}\rhd' c')\\
(41)\quad &= \Sigma_{(b)}(b_{(1)}\rhd(a\otimes a'))\cdot(b_{(2)}\rhd(c\otimes c'))
\end{aligned}$$

$$h\rhd(1_A\otimes 1_{A'}) = \Sigma_{(b)}\,(b_{(1)}\rhd 1_A)\otimes(b_{(2)}\rhd' 1_{A'}) = \Sigma_{(b)}\epsilon(b_{(1)})\epsilon(b_{(2)})\,1_A\otimes 1_{A'} = \epsilon(b)\,1_A\otimes 1_{A'}\ \square$$

If $(H,R)$ is a quasitriangular *Hopf* algebra, the quasitriangularity has important consequences for the behaviour of the antipode. It implies that the antipode is invertible and that its square is given by conjugation with an element of $H$. This can be viewed as a generalisation of Radford's formula. However, in the quasitriangular case, these properties are not obtained from integrals.

**Theorem 5.1.7:** Let $(H,R)$ be a quasitriangular Hopf algebra. Then:

1. The antipode of $(H,R)$ is invertible.
2. The universal $R$-matrix satisfies
$$(S\otimes\mathrm{id})(R)=(\mathrm{id}\otimes S^{-1})(R)=R^{-1}\qquad (S\otimes S)(R)=(S^{-1}\otimes S^{-1})(R)=R.$$
3. The **Drinfeld element** $u=S(R_{(2)})R_{(1)}$ is invertible with inverse $u^{-1}=R_{(2)}S^2(R_{(1)})$ and coproduct $\Delta(u)=(u\otimes u)\cdot(R_{21}R)^{-1}$.
4. The element $g=uS(u)^{-1}$ is grouplike.
5. One has $S^2(h)=uhu^{-1}$ and $S^4(h)=ghg^{-1}$ for all $h\in H$.

**Proof:**
1. To prove the identities $(S\otimes\mathrm{id})(R)=R^{-1}$ and $(S\otimes S)(R)=R$, we compute

$$\begin{aligned}
&(m\otimes\mathrm{id})\circ(S\otimes\mathrm{id}\otimes\mathrm{id})\circ(\Delta\otimes\mathrm{id})(R)=(1_H\epsilon\otimes\mathrm{id})(R)=1\otimes 1\\
=&(m\otimes\mathrm{id})\circ(S\otimes\mathrm{id}\otimes\mathrm{id})(R_{13}R_{23})=(S\otimes\mathrm{id})(R)\cdot R.
\end{aligned}$$

Right multiplication by $R^{-1}$ then yields $(S\otimes\mathrm{id})(R)=R^{-1}$. As $\tau\circ R^{-1}=R_{21}^{-1}$ is another universal $R$-matrix for $H$ by Proposition 5.1.5, we also obtain $(\mathrm{id}\otimes S)(R^{-1})=R$ and

$$(S\otimes S)(R)=(\mathrm{id}\otimes S)\circ(S\otimes\mathrm{id})(R)=(\mathrm{id}\otimes S)(R^{-1})=R.$$

The identities $(\mathrm{id}\otimes S^{-1})(R) = R^{-1}$ and $(S^{-1}\otimes S^{-1})(R) = R$ then follow by applying $S^{-1}\otimes S^{-1}$ to these two equations, once it is established that $S$ is invertible.

2. To prove that $S$ is invertible, we first show that $S^2(h)u = hu$ for all $h \in H$ and then use this identity to prove that $u$ is invertible with inverse $u^{-1} = R_{(2)}S^2(R_{(1)})$. It then follows that $S^2(h) = uhu^{-1}$ for all $h \in H$, and this implies that $S^2$ and $S$ are invertible. For this, we compute

$$\begin{aligned}
&S^2(h)u = S^2(h)S(R_{(2)})R_{(1)} = S^2(h_{(3)})S(R_{(2)})S(h_{(1)})h_{(2)}R_{(1)} = S^2(h_{(3)})S(h_{(1)}R_{(2)})h_{(2)}R_{(1)}\\
&= S^2(h_{(3)})S(R_{(2)}h_{(2)})R_{(1)}h_{(1)} = S^2(h_{(3)})S(h_{(2)})S(R_{(2)})R_{(1)}h_{(1)} = S^2(h_{(3)})S(h_{(2)})uh_{(1)}\\
&= S(h_{(2)}S(h_{(3)}))uh_{(1)} = uh,
\end{aligned}$$

where we used first the definition of $u$, then the identity $S(h_{(1)})h_{(2)}\otimes h_{(3)} = 1\otimes h$, then the fact that $S$ is an anti-algebra homomorphism, then the identity $\Delta^{op}(h)\cdot R = R\cdot\Delta(h)$, then again that $S$ is an anti-algebra homomorphism, the definition of $u$ and the identity $h_{(1)}\otimes h_{(2)}S(h_{(3)}) = h\otimes 1$. We now show that $u$ is invertible. For this we compute with the identity $(S\otimes S)(R) = R$

$$u \cdot R_{(2)}S^2(R_{(1)}) = S^2(R_{(2)})uS^2(R_{(1)}) = R_{(2)}uR_{(1)} = R_{(2)}S^2(R_{(1)}) \cdot u.$$

Using this equation together with the definition of $u$, the identities $(S\otimes S)(R) = R$ and $(S\otimes\mathrm{id})(R) = R^{-1}$ and the fact that $S$ is an anti-algebra homomorphism and the identity $S^2(h)u = uh$ for all $h \in H$, we then obtain

$$\begin{aligned}
&u \cdot R_{(2)}S^2(R_{(1)}) = R_{(2)}S^2(R_{(1)}) \cdot u = R_{(2)}uR_{(1)} = R_{(2)}S(R'_{(2)})R'_{(1)}R_{(1)} = S(R_{(2)})S(R'_{(2)})R'_{(1)}S(R_{(1)})\\
&= S(R'_{(2)}R_{(2)})R'_{(1)}S(R_{(1)}) = m^{op} \circ (\mathrm{id}\otimes S)(R \cdot (S\otimes\mathrm{id})(R)) = m^{op} \circ (\mathrm{id}\otimes S)(R \cdot R^{-1}) = 1.
\end{aligned}$$

This shows that $u$ is invertible with inverse $u^{-1} = R_{(2)}S^2(R_{(1)})$, that $S^2(h) = uhu^{-1}$ for all $h \in H$ and and that $S^2$ and $S$ are invertible.

3. To prove that $S^4(h) = ghg^{-1}$ with $g = uS(u)^{-1}$, we consider the quasitriangular Hopf algebra $(H^{op,cop}, R)$ with the opposite product and coproduct, the same antipode and the same $R$-matrix. Then the element $S(u) = R_{(1)}S(R_{(2)})$ takes the role of $u = S(R_{(2)})R_{(1)}$ for $(H^{op,cop}, R)$. This implies $S^2(h) = S(u)^{-1}hS(u)$ and $S^4(h) = uS^2(h)u^{-1} = uS(u)^{-1}hS(u)u^{-1} = ghg^{-1}$.

4. To prove the identity $\Delta(u) = (u\otimes u) \cdot (R_{21}R)^{-1}$ we use the identities $(\Delta\otimes\mathrm{id})(R) = R_{13}R_{23}$ and $(\mathrm{id}\otimes\Delta)(R) = R_{13}R_{12}$ to compute

$$\begin{aligned}
&\Delta(u) = \Delta(S(R_{(2)})R_{(1)}) = \Delta(S(R_{(2)})) \cdot \Delta(R_{(1)}) = \Delta(S(R_{(2)}R'_{(2)})) \cdot (R_{(1)}\otimes R'_{(1)})\\
&=(S\otimes S)(\Delta^{op}(R_{(2)}) \cdot \Delta^{op}(R'_{(2)})) \cdot (R_{(1)}\otimes R'_{(1)}) = (S\otimes S)(R_{(2)}R'_{(2)}\otimes\tilde{R}_{(2)}\tilde{R}'_{(2)}) \cdot (R_{(1)}\tilde{R}_{(1)}\otimes R'_{(1)}\tilde{R}'_{(1)})\\
&=S(R'_{(2)})S(R_{(2)})R_{(1)}\tilde{R}_{(1)}\otimes S(\tilde{R}'_{(2)})S(\tilde{R}_{(2)})R'_{(1)}\tilde{R}'_{(1)}) = S(R'_{(2)})u\tilde{R}_{(1)}\otimes S(\tilde{R}'_{(2)})S(\tilde{R}_{(2)})R'_{(1)}\tilde{R}'_{(1)})\\
&=uS^{-1}(R'_{(2)})\tilde{R}_{(1)}\otimes S(\tilde{R}'_{(2)})S(\tilde{R}_{(2)})R'_{(1)}\tilde{R}'_{(1)}) = uR'_{(2)}\tilde{R}_{(1)}\otimes S(\tilde{R}'_{(2)})S(\tilde{R}_{(2)})S(R'_{(1)})\tilde{R}'_{(1)})
\end{aligned}$$

To simplify this expression, we consider the QYBE and multiply it with from the left and from the right with $R_{12}^{-1}$, which yields $R_{13}R_{23}R_{12}^{-1} = R_{12}^{-1}R_{23}R_{13}$. In Sweedler notation, this reads

$$R_{(1)}S(R'_{(1)})\otimes\tilde{R}_{(1)}R'_{(2)}\otimes R_{(2)}\tilde{R}_{(2)} = S(R_{(1)})R'_{(1)}\otimes R_{(2)}\tilde{R}_{(1)}\otimes\tilde{R}_{(2)}R'_{(2)}.$$

Applying the map $(m\otimes\mathrm{id}) \circ \tau_{12} \circ \tau_{23} \circ (\mathrm{id}\otimes\mathrm{id}\otimes S)$ to both sides of this equation yields

$$S(\tilde{R}_{(2)})S(R_{(2)})R_{(1)}S(R'_{(1)})\otimes\tilde{R}_{(1)}R'_{(2)} = S(R'_{(2)})S(\tilde{R}_{(2)})S(R_{(1)})R'_{(1)}\otimes R_{(2)}\tilde{R}_{(1)},$$

and by inserting this equation into the last term in the expression for $\Delta(u)$, we obtain

$$\begin{aligned}\Delta(u) &= u\tilde{R}_{(1)}R'_{(2)}\otimes S(\tilde{R}_{(2)})S(\tilde{R}'_{(2)})\tilde{R}'_{(1)}S(R'_{(1)}) = u\tilde{R}_{(1)}R'_{(2)}\otimes S(\tilde{R}_{(2)})uS(R'_{(1)})\\ &= u\tilde{R}_{(1)}R'_{(2)}\otimes uS^{-1}(\tilde{R}_{(2)})S(R'_{(1)}) = uS(\tilde{R}_{(1)})R'_{(2)}\otimes u\tilde{R}_{(2)}S(R'_{(1)})\\ &= (u\otimes u)\cdot(R^{-1}R_{21}^{-1}) = (u\otimes u)\cdot(R_{21}R)^{-1}\end{aligned}$$

As the element $S(u) = R_{(1)}S(R_{(2)})$ takes the role of $u = S(R_{(2)})R_{(1)}$ for $(H^{op,cop}, R)$, its coproduct satisfies $\Delta(S(u)) = (R_{21}R)^{-1}(S(u)\otimes S(u))$. Moreover, the identities $S^2(h) = S(u)^{-1}hS(u)$ and $(S\otimes S)(R) = R$ imply

$$(S(u)^{-1}\otimes S(u)^{-1})(R_{21}R) = (S^2\otimes S^2)(R_{21}R)(S(u)^{-1}\otimes S(u)^{-1}) = (R_{21}R)\cdot(S(u)^{-1}\otimes S(u)^{-1}),$$

and by combining these equations with the expression for the coproduct of $u$, we obtain

$$\begin{aligned}\Delta(g) &= \Delta(uS(u)^{-1}) = \Delta(u)\cdot\Delta(S(u))^{-1} = (u\otimes u)(R_{21}R)^{-1}\cdot(S(u)^{-1}\otimes S(u)^{-1})\cdot(R_{21}R)\\ &= (u\otimes u)(R_{21}R)^{-1}\cdot(R_{21}R)\cdot(S(u)^{-1}\otimes S(u)^{-1}) = uS(u)^{-1}\otimes uS(u)^{-1} = g\otimes g.\end{aligned}$$

□

**Remark 5.1.8:** If $H$ is finite-dimensional and quasitriangular and $\alpha\in\mathrm{Gr}(H^*)$ and $a\in\mathrm{Gr}(H)$ are the modular elements of $H$ and $H^*$, then $S^4(h) = a^{-1}(\alpha\rhd^*_R h\lhd^*_L\alpha^{-1})\cdot a$ by Radford's formula. One can show that the element $g\in H$ from Theorem 5.1.7 is given in terms of the modular elements by $g = a^{-1}\cdot(\alpha\otimes\mathrm{id})(R) = (\alpha\otimes\mathrm{id})(R)\cdot a$.

We will now consider a systematic construction that associates a quasitriangular Hopf algebra to every finite-dimensional Hopf algebra $H$, namely the Drinfeld double or quantum double $D(H)$ from Lemma 3.5.4. Drinfeld doubles are the most important and widely used Examples of quasitriangular Hopf algebras and can be viewed as the smallest quasitriangular Hopf algebra that contains both $H$ and $H^{*cop}$ as Hopf subalgebras.

**Theorem 5.1.9:** For every finite-dimensional Hopf algebra $H$ there is a unique quasitriangular Hopf algebra structure on the vector space $H^*\otimes H$ such that the inclusion maps

$$\iota_H: H\to H^*\otimes H,\ h\mapsto 1\otimes h \qquad \iota_{H^*}: H^{*cop}\to H^*\otimes H,\ \alpha\mapsto\alpha\otimes 1$$

are homomorphisms of Hopf algebras. This is the **Drinfeld double** or **quantum double** $D(H)$ from Lemma 3.5.4 with the Hopf algebra structure

$$\begin{aligned}&(\alpha\otimes h)\cdot(\beta\otimes k) = \Sigma_{(h),(\beta)}\,\beta_{(3)}(h_{(1)})\,\beta_{(1)}(S^{-1}(h_{(3)}))\,\alpha\beta_{(2)}\otimes h_{(2)}k && 1 = 1_{H^*}\otimes 1_H\\ &\Delta(\alpha\otimes h) = \Sigma_{(h),(\alpha)}\,\alpha_{(2)}\otimes h_{(1)}\otimes\alpha_{(1)}\otimes h_{(2)} && \epsilon(\alpha\otimes h) = \epsilon_{H^*}(\alpha)\epsilon_H(h)\\ &S(\alpha\otimes h) = (1\otimes S(h))\cdot(S(\alpha)\otimes 1)\end{aligned}$$

A universal $R$-matrix for $D(H)$ is given by $R = \Sigma_{i=1}^n 1\otimes x_i\otimes\alpha^i\otimes 1$, where $(x_1,...,x_n)$ is an ordered basis of $H$ with dual basis $(\alpha^1,...,\alpha^n)$.

**Proof:**
It was already shown in Lemma 3.5.4 that above expressions define a Hopf algebra structure on $H^*\otimes H$. That the inclusion maps $\iota_H: H\to H^*\otimes H$ and $\iota_{H^*}: H^{*cop}\to H^*\otimes H$ are homomorphisms of Hopf algebras follows directly from the expressions for (co)multiplication, (co)unit and antipode. To show that $R = \Sigma_{i=1}^n 1\otimes x_i\otimes\alpha^i\otimes 1$ is a universal $R$-matrix for $D(H)$ we use the auxiliary identities

$$\Sigma_{i=1}^n\Delta(x_i)\otimes\alpha^i = \Sigma_{i,j=1}^n x_i\otimes x_j\otimes\alpha^i\alpha^j \qquad \Sigma_{i=1}^n x_i\otimes\Delta(\alpha^i) = \Sigma_{i,j=1}^n x_ix_j\otimes\alpha^i\otimes\alpha^j \tag{43}$$

which follow by evaluating the left and right hand side on elements of $H$ and $H^*$:

$$\begin{aligned}
\Sigma_{i=1}^n(\beta\otimes\gamma)(\Delta(x_i))\alpha^i(k) &= \Sigma_{i=1}(\beta\cdot\gamma)(x_i)\alpha^i(k) = (\beta\gamma)(k) = (\beta\otimes\gamma)(\Delta(k))\\
&= \Sigma_{i,j=1}^n\beta(x_i)\gamma(x_j)(\alpha^i\otimes\alpha^j)(\Delta(k)) = \Sigma_{i,j=1}^n(\beta\otimes\gamma)(x_i\otimes x_j)(\alpha^i\cdot\alpha^j)(k)\\
\Sigma_{i=1}^n\beta(x_i)\Delta(\alpha^i)(h\otimes k) &= \Sigma_{i=1}^n\beta(x_i)\alpha^i(h\cdot k) = \beta(h\cdot k) = \Delta(\beta)(h\otimes k)\\
&= \Sigma_{i,j=1}^n\Delta(\beta)(x_i\otimes x_j)\alpha^i(h)\alpha^j(k) = \Sigma_{i,j=1}^n\beta(x_ix_j)\otimes(\alpha^i\otimes\alpha^j)(h\otimes k)
\end{aligned}$$

for all $\beta,\gamma\in H^*$ and $k,h\in H$. With the identities (43) we compute

$$\begin{aligned}
(\Delta\otimes\mathrm{id})(R) &= \Sigma_{i=1}^n\Delta(1\otimes x_i)\otimes\alpha^i\otimes 1 = \Sigma_{i=1}^n 1\otimes x_{i(1)}\otimes 1\otimes x_{i(2)}\otimes\alpha^i\otimes 1 = \Sigma_{i,j=1}^n 1\otimes x_i\otimes 1\otimes x_j\otimes\alpha^i\alpha^j\otimes 1\\
&= (\Sigma_{i=1}^n 1\otimes x_i\otimes 1\otimes 1\otimes\alpha^i\otimes 1)\cdot(\Sigma_{j=1}^n 1\otimes 1\otimes 1\otimes x_j\otimes\alpha^j\otimes 1) = R_{13}\cdot R_{23}\\
(\mathrm{id}\otimes\Delta)(R) &= \Sigma_{i=1}^n 1\otimes x_i\otimes\Delta(\alpha^i\otimes 1) = \Sigma_{i=1}^n 1\otimes x_i\otimes\alpha^{i(2)}\otimes 1\otimes\alpha^{i(1)}\otimes 1 = \Sigma_{i,j=1}^n 1\otimes x_ix_j\otimes\alpha^j\otimes 1\otimes\alpha^i\otimes 1\\
&= (\Sigma_{i=1}^n 1\otimes x_i\otimes 1\otimes 1\otimes\alpha^i\otimes 1)\cdot(\Sigma_{j=1}^n 1\otimes x_j\otimes\alpha^j\otimes 1\otimes 1\otimes 1) = R_{13}\cdot R_{12},
\end{aligned}$$

and show that $R$ is invertible:

$$\begin{aligned}
R\cdot(S\otimes\mathrm{id})(R) &= (\Sigma_{i=1}^n 1\otimes x_i\otimes\alpha^i\otimes 1)\cdot(\Sigma_{i=1}^n 1\otimes S(x_j)\otimes\alpha^j\otimes 1) = \Sigma_{i,j=1}^n 1\otimes x_iS(x_j)\otimes\alpha^i\alpha^j\otimes 1\\
&= \Sigma_{i=1}^n 1\otimes x_{i(1)}S(x_{i(2)})\otimes\alpha^i\otimes 1 = \Sigma_{i=1}^n\epsilon(x_i)1\otimes 1\otimes\alpha^i\otimes 1 = 1\otimes 1\otimes 1\otimes 1.
\end{aligned}$$

The condition $R\cdot\Delta = \Delta^{op}\cdot R$ then follows again from (43) by a direct computation

$$\begin{aligned}
&R\cdot\Delta(\alpha\otimes h) = \Sigma_{(h),(\alpha)}\Sigma_{i=1}^n(1\otimes x_i\otimes\alpha^i\otimes 1)\cdot(\alpha_{(2)}\otimes h_{(1)}\otimes\alpha_{(1)}\otimes h_{(2)})\\
&= \Sigma_{(h),(\alpha)}\Sigma_{i=1}^n\alpha_{(2)(3)}(x_{i(1)})\alpha_{(2)(1)}(S^{-1}(x_{i(3)}))\alpha_{(2)(2)}\otimes x_{i(2)}h_{(1)}\otimes\alpha^i\alpha_{(1)}\otimes h_{(2)}\\
&= \Sigma_{(h),(\alpha)}\Sigma_{i,j,k=1}^n\alpha_{(4)}(x_i)\alpha_{(2)}(S^{-1}(x_k))\alpha_{(3)}\otimes x_jh_{(1)}\otimes\alpha^i\alpha^j\alpha^k\alpha_{(1)}\otimes h_{(2)}\\
&= \Sigma_{(h),(\alpha)}\Sigma_{j=1}^n\alpha_{(3)}\otimes x_jh_{(1)}\otimes\alpha_{(4)}\alpha^jS^{-1}(\alpha_{(2)})\alpha_{(1)}\otimes h_{(2)}\\
&= \Sigma_{(h),(\alpha)}\Sigma_{j=1}^n\epsilon(\alpha_{(1)})\alpha_{(2)}\otimes x_jh_{(1)}\otimes\alpha_{(3)}\alpha^j\otimes h_{(2)} = \Sigma_{(h),(\alpha)}\Sigma_{j=1}^n\alpha_{(1)}\otimes x_jh_{(1)}\otimes\alpha_{(2)}\alpha^j\otimes h_{(2)}
\end{aligned}$$

$$\begin{aligned}
&\Delta^{op}(\alpha\otimes h)\cdot R = \Sigma_{(h),(\alpha)}\Sigma_{i=1}^n(\alpha_{(1)}\otimes h_{(2)}\otimes\alpha_{(2)}\otimes h_{(1)})\cdot(1\otimes x_i\otimes\alpha^i\otimes 1)\\
&= \Sigma_{(h),(\alpha)}\Sigma_{i=1}^n\alpha^i_{(3)}(h_{(1)(1)})\alpha^i_{(1)}(S^{-1}(h_{(1)(3)}))\,\alpha_{(1)}\otimes h_{(2)}x_i\otimes\alpha_{(2)}\alpha^i_{(2)}\otimes h_{(1)(2)}\\
&= \Sigma_{(h),(\alpha)}\Sigma_{i,j,k=1}^n\alpha^k(h_{(1)})\alpha^i(S^{-1}(h_{(3)}))\,\alpha_{(1)}\otimes h_{(4)}x_ix_jx_k\otimes\alpha_{(2)}\alpha^j\otimes h_{(2)}\\
&= \Sigma_{(h),(\alpha)}\Sigma_{i,j,k=1}^n\alpha_{(1)}\otimes h_{(4)}S^{-1}(h_{(3)})x_jh_{(1)}\otimes\alpha_{(2)}\alpha^j\otimes h_{(2)}\\
&= \Sigma_{(h),(\alpha)}\Sigma_{i,j,k=1}^n\epsilon(h_{(3)})\alpha_{(1)}\otimes x_jh_{(1)}\otimes\alpha_{(2)}\alpha^j\otimes h_{(2)} = \Sigma_{(h),(\alpha)}\Sigma_{i,j,k=1}^n\alpha_{(1)}\otimes x_jh_{(1)}\otimes\alpha_{(2)}\alpha^j\otimes h_{(2)}.\quad\square
\end{aligned}$$

**Example 5.1.10:** Let $G$ be a finite group. Then the Drinfeld double $D(\mathbb{F}[G])$ is given by

$$\begin{aligned}
&(\delta_u\otimes g)\cdot(\delta_v\otimes h) = \delta_u(gvg^{-1})\delta_u\otimes gh && 1 = 1\otimes e = \Sigma_{g\in G}\,\delta_g\otimes e\\
&\Delta(\delta_u\otimes g) = \Sigma_{xy=u}\,\delta_y\otimes g\otimes\delta_x\otimes g && \epsilon(\delta_u\otimes g) = \delta_u(e)\\
&S(\delta_u\otimes g) = \delta_{g^{-1}u^{-1}g}\otimes g^{-1} && R = \Sigma_{g\in G}1\otimes g\otimes\delta_g\otimes e
\end{aligned}$$

Integrals of the Drinfeld double $D(H)$ are obtained in a simple way from the integrals of $H$ and $H^*$. The properties of the integrals in $H$ and $H^*$ and the Hopf algebra structure of $D(H)$ then imply that that the Drinfeld double $D(H)$ is always unimodular. In particular, it follows for $\mathrm{char}(\mathbb{F}) = 0$ that the Drinfeld double of any semisimple Hopf algebra $H$ is again semisimple.

**Theorem 5.1.11:** Let $H$ be a finite-dimensional Hopf algebra. Then $D(H)$ is unimodular: If $\ell \in H$ is a non-trivial right integral and $\lambda \in H^*$ a non-trivial left integral, then $\lambda \otimes \ell$ is a two-sided integral in $D(H)$.

**Proof:**
This is a direct consequence of the multiplication of the Drinfeld double and the formula for the Nakayama automorphism from Proposition 3.3.6. By Proposition 3.3.6, 4. we have for any left integral $\lambda' \in I_L(H^*)$ and $x, y \in H$

$$\begin{aligned}\Sigma_{(\lambda')}(\lambda'_{(1)} \otimes \lambda'_{(2)})(x \otimes y) &= \Delta(\lambda')(x \otimes y) = \lambda'(x \cdot y) = \lambda'(\rho(y) \cdot x) = \lambda'((\Sigma_{(y)} \alpha(y_{(1)}) S^2(y_{(2)})) \cdot x) \\ &= (\Sigma_{(\lambda')} \lambda'_{(2)} \otimes \alpha S^2(\lambda'_{(1)}))(x \otimes y),\end{aligned}$$

where $\rho : H \to H, h \mapsto \Sigma_{(h)} \alpha(h_{(1)})\, S^2(h_{(2)})$ is the Nakayama automorphism for the Frobenius form $\kappa : H \otimes H \to \mathbb{F}$, $x \otimes y \mapsto \lambda'(x \cdot y)$ and $\alpha \in \mathrm{Gr}(H^*)$ is the modular element of $H$. By using the fact that $S(\lambda) \in I_L(H^*)$ for all $\lambda \in I_R(H^*)$, $S^{\pm 1}(\alpha) = \alpha^{-1}$ and exchanging the roles of $H$ and $H^*$, we obtain the following two formulas for the integrals $\lambda \in I_R(H^*)$ and $\ell \in I_L(H)$

$$\Sigma_{(\lambda)} \lambda_{(2)} \otimes \lambda_{(1)} = \Sigma_{(\lambda)} \lambda_{(1)} \otimes S^2(\lambda_{(2)}) \alpha^{-1} \qquad \Sigma_{(\ell)} \ell_{(1)} \otimes \ell_{(2)} = \Sigma_{(\ell)} \ell_{(2)} \otimes a S^2(\ell_{(1)}),$$

where $\alpha \in \mathrm{Gr}(H^*)$ and $a \in \mathrm{Gr}(H)$ are the modular elements of $H$ and $H^*$. By applying the maps $(\mathrm{id} \otimes m) \circ (\mathrm{id} \otimes S \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta) \circ \tau$ and $(\mathrm{id} \otimes m) \circ (\mathrm{id} \otimes S \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id})$ to the first and the second equation, respectively, we obtain

$$\begin{aligned}&\Sigma_{(\lambda)} S^2(\lambda_{(3)}) \alpha^{-1} S(\lambda_{(1)}) \otimes \lambda_{(2)} = \Sigma_{(\lambda)} \lambda_{(1)} S(\lambda_{(2)}) \otimes \lambda_{(3)} = \Sigma_{(\lambda)} \epsilon(\lambda_{(1)}) 1 \otimes \lambda_{(2)} = 1 \otimes \lambda \\ &\Sigma_{(\ell)} \ell_{(2)} \otimes S(\ell_{(3)}) a S^2(\ell_{(1)}) = \Sigma_{(\ell)} \ell_{(1)} \otimes S(\ell_{(2)}) \ell_{(3)} = \Sigma_{(\ell)} \epsilon(\ell_{(2)})\, \ell_{(1)} \otimes 1 = \ell \otimes 1.\end{aligned} \tag{44}$$

With the multiplication law of the Drinfeld double, we then compute

$$\begin{aligned}(\lambda \otimes \ell) \cdot (\beta \otimes h) &= \Sigma_{(\ell)(\beta)} \beta_{(3)}(\ell_{(1)}) \beta_{(1)}(S^{-1}(\ell_{(3)}))\, \lambda \beta_{(2)} \otimes \ell_{(2)} h \\ &\overset{\lambda \in I_L(H^*)}{=} \Sigma_{(\ell)(\beta)} \beta_{(2)}(a) \beta_{(3)}(\ell_{(1)}) \beta_{(1)}(S^{-1}(\ell_{(3)}))\, \lambda \otimes \ell_{(2)} h \\ &= \Sigma_{(\ell)} \beta(S^{-1}(\ell_{(3)}) a \ell_{(1)})\, \lambda \otimes \ell_{(2)} h \\ &= \Sigma_{(\ell)} (\beta \circ S^{-2})(S(\ell_{(3)}) a S^2(\ell_{(1)}))\, \lambda \otimes \ell_{(2)} h \overset{(44)}{=} (\beta \circ S^{-2})(1)\, \lambda \otimes \ell h \\ &\overset{\ell \in I_R(H)}{=} \beta(1) \epsilon(h)\, \lambda \otimes \ell,\end{aligned}$$

$$\begin{aligned}(\beta \otimes h) \cdot (\lambda \otimes \ell) &= \Sigma_{(\lambda)(h)} \lambda_{(3)}(h_{(1)}) \lambda_{(1)}(S^{-1}(h_{(3)}))\, \beta \lambda_{(2)} \otimes h_{(2)} \ell \\ &\overset{\ell \in I_R(H)}{=} \Sigma_{(\lambda)(h)} \alpha^{-1}(h_{(2)}) \lambda_{(3)}(h_{(1)}) \lambda_{(1)}(S^{-1}(h_{(3)}))\, \beta \lambda_{(2)} \otimes \ell \\ &= \Sigma_{(\lambda)} (\lambda_{(3)} \alpha^{-1} S^{-1}(\lambda_{(1)}))(h)\, \beta \lambda_{(2)} \otimes \ell \\ &= \Sigma_{(\lambda)} (S^2(\lambda_{(3)}) \alpha^{-1} S(\lambda_{(1)}))(S^{-2}(h))\, \beta \lambda_{(2)} \otimes \ell \overset{(44)}{=} \epsilon(S^2(h))\, \beta \lambda \otimes \ell \\ &\overset{\lambda \in I_L(H^*)}{=} \epsilon(h) \beta(1)\, \lambda \otimes \ell.\end{aligned}$$

This shows that $\lambda \otimes \ell$ is a two-sided integral in $D(H)$. As $\dim_{\mathbb{F}} I_L(D(H)) = \dim_{\mathbb{F}} I_R(D(H)) = 1$ by Theorem 3.2.15, it follows that $I_L(D(H)) = I_R(D(H))$ and $D(H)$ is unimodular. □

**Corollary 5.1.12:** For a finite-dimensional Hopf algebra $H$ over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) = 0$, the following are equivalent:

(i) $H$ is semisimple.
(ii) $H^*$ is semisimple.
(iii) $D(H)$ is semisimple.

**Proof:**
The equivalence of (i) and (ii) follows from Theorem 3.4.10 by Larson and Radford, which also implies that (i) and (ii) are equivalent to $S_H^2 = \mathrm{id}_H$ and $S_{H^*}^2 = \mathrm{id}_{H^*}$. As one has for all $\alpha \in H^*$ and $h \in H$

$$\begin{aligned} S^2_{D(H)}(\alpha\otimes h) &= S^2_{D(H)}((\alpha\otimes 1)(1\otimes h)) = S^2_{D(H)}(\alpha\otimes 1)S^2_{D(H)}(1\otimes h) = (S^2_{H^*}(\alpha)\otimes 1)(1\otimes S^2_H(h)) \\ &= (S^2_{H^*}(\alpha)\otimes S^2_H(h)), \end{aligned}$$

if follows that (i) and (ii) are equivalent to the condition $S^2_{D(H)} = \mathrm{id}$, which by Theorem 3.4.10 is equivalent to (iii). □

## 5.2 *Factorisable Hopf algebras

Clearly, the Drinfeld double $D(H)$ of a finite-dimensional Hopf algebra $H$ is not just a quasitriangular Hopf algebra, but a quasitriangular Hopf algebra that contains $H$ and $H^{*cop}$ as Hopf subalgebras in such a way that every element of $D(H)$ can be factorised uniquely as a product $\alpha\otimes h = (\alpha\otimes 1)\cdot(1\otimes h)$ of an element $\alpha \in H^*$ and an element $h \in H$. This becomes even simpler for the dual Hopf algebra $D(H)^* = H^{op}\otimes H^*$, which is the tensor product of the *algebras* $H^{op}$ and $H^*$. It turns out that parts of this pattern generalise to quasitriangular Hopf algebras $(H, R)$. More specifically, the universal $R$-matrix gives rise to two canonical (anti)algebra and (anti)coalgebra homomorphisms $H^* \to H$, which can be viewed as the counterparts of the projectors

$$\pi_H : D(H)^* \to H, h\otimes\alpha \mapsto \epsilon(\alpha)h \qquad\qquad \pi_{H^*} : D(H)^* \to H^*, \quad h\otimes\alpha \mapsto \epsilon(h)\,\alpha.$$

**Proposition 5.2.1:** Let $(H, R)$ be a finite-dimensional quasitriangular Hopf algebra. Then:

1. The linear map $\phi_R : H^* \to H$, $\alpha \mapsto (\mathrm{id}\otimes\alpha)(R)$ is an anti-algebra homomorphism and a coalgebra homomorphism and satisfies $\phi_R \circ S^{\pm 1} = S^{\mp 1} \circ \phi_R$.
2. The linear map $\bar\phi_R = S \circ \phi_{R_{21}^{-1}} : H^* \to H$, $\alpha \mapsto (\alpha\otimes\mathrm{id})(R)$ is an algebra homomorphism and an anti-coalgebra homomorphism and satisfies $\bar\phi_R \circ S^{\pm 1} = S^{\mp 1} \circ \bar\phi_R$.
3. The linear map $D = \bar\phi_R * \phi_R : H^* \to H$, $\alpha \mapsto (\mathrm{id}\otimes\alpha)(Q)$ with $Q = R_{21}R$ is a module homomorphism with respect to the left regular action $\rhd_L : H^*\otimes H^* \to H^*$, $\alpha \rhd \beta = \alpha\cdot\beta$ and the $H^*$-module structure $\rhd : H^*\otimes H \to H$, $\alpha \rhd h = \Sigma_{(\alpha)}\,\bar\phi_R(\alpha_{(1)})\cdot h\cdot \phi_R(\alpha_{(2)})$.
4. The element $Q = R_{21}R$ satisfies $Q\cdot\Delta(h) = \Delta(h)\cdot Q$ for all $h \in H$ and is called the **monodromy element**.

**Proof:**
The second claim follows from the first, the fact that $R_{21}^{-1}$ is another universal $R$-matrix for $H$ by Proposition 5.1.5 and the fact that the antipode is an anti-algebra and coalgebra homomorphism. The first follows by a direct computation using the properties of the universal $R$-matrix.

Using Sweedler notation for the universal $R$-matrix, we then obtain with

$$\phi_R(1_{H^*}) = \phi_R(\epsilon) = \epsilon(R_{(2)})\, R_{(1)} = m \circ (\mathrm{id}\otimes\epsilon)(R) = m \circ (1\otimes 1) = 1$$
$$\phi_R(\alpha\beta) = (\alpha\beta)(R_{(2)})\, R_{(1)} = (\alpha\otimes\beta)(\Delta(R_{(2)}))\, R_{(1)} = (\alpha\otimes\beta)(R_{(2)}\otimes R'_{(2)})R'_{(1)}R_{(1)} = \phi_R(\beta)\cdot\phi_R(\alpha)$$
$$(\phi_R\otimes\phi_R)(\Delta(\alpha)) = \Sigma_{(\alpha)}\, \alpha_{(1)}(R_{(2)})\alpha_{(2)}(R'_{(2)})\, R_{(1)}\otimes R'_{(1)} = \alpha(R_{(2)}R'_{(2)})R_{(1)}\otimes R'_{(1)}$$
$$= (\mathrm{id}\otimes\alpha)(R_{13}R_{23}) = (\mathrm{id}\otimes\alpha)(\Delta\otimes\mathrm{id})(R) = \Delta(\alpha(R_{(2)})\, R_{(1)}) = \Delta(\phi_R(\alpha))$$
$$\epsilon\circ\phi_R(\alpha) = \alpha(R_{(2)})\epsilon(R_{(1)}) = \alpha(1)1$$
$$S(\phi_R(\alpha)) = \alpha(R_{(2)})S(R_{(1)}) = \alpha(S^{-1}(R_{(2)}))R_{(1)} = \phi_R(S^{-1}(\alpha)).$$

That $\rhd : H^*\otimes H \to H$, $\alpha \rhd h = \Sigma_{(\alpha)}\, \bar\phi_R(\alpha_{(1)})\cdot h\cdot \phi_R(\alpha_{(2)})$ defines an $H^*$-module structure on $H$ follows directly from the fact that $\phi_R$ is an anti-algebra homomorphism and $\bar\phi_R$ and $\Delta$ are algebra homomorphisms. As we have $Q = R_{21}\cdot R$, the Definition of $D$ implies for all $\alpha\in H^*$

$$D(\alpha) = (\mathrm{id}\otimes\alpha)(R_{21}R) = \Sigma_{(\alpha)}\, (\mathrm{id}\otimes\alpha_{(1)})(R_{21})\cdot(\mathrm{id}\otimes\alpha_{(2)})(R) = \Sigma_{(\alpha)}\, \bar\phi_R(\alpha_{(1)})\cdot\phi_R(\alpha_{(2)}) = (\bar\phi_R * \phi_R)(\alpha).$$

Using again the fact that $\phi_R$ is an anti-algebra and $\bar\phi_R$ an algebra homomorphism, one obtains

$$D(\alpha\beta) = \bar\phi_R(\alpha_{(1)}\beta_{(1)})\cdot\phi_R(\alpha_{(2)}\beta_{(2)}) = \bar\phi_R(\alpha_{(1)})\,\phi_R(\beta_{(1)})\phi_R(\beta_{(2)})\phi_R(\alpha_{(2)}) = \bar\phi_R(\alpha_{(1)})\cdot D(\beta)\cdot\phi_R(\alpha_{(2)}).$$

The identity for the monodromy element follows directly from its definition. With the identites $Q = R_{21}R = R_{(2)}R'_{(1)}\otimes R_{(1)}R'_{(2)}$ and $R\cdot\Delta = \Delta^{op}\cdot R$, we compute for all $h\in H$

$$\Delta(h)\cdot Q = \Sigma_{(h)}(h_{(1)}\otimes h_{(2)})\cdot(R_{(2)}R'_{(1)}\otimes R_{(1)}R'_{(2)}) = \Sigma_{(h)}h_{(1)}R_{(2)}R'_{(1)}\otimes h_{(2)}R_{(1)}R'_{(2)}$$
$$= \Sigma_{(h)}R_{(2)}h_{(2)}R'_{(1)}\otimes R_{(1)}h_{(1)}R'_{(2)} = \Sigma_{(h)}R_{(2)}R'_{(1)}h_{(1)}\otimes R_{(1)}R'_{(2)}h_{(2)}$$
$$= \Sigma_{(h)}(R_{(2)}R'_{(1)}\otimes R_{(1)}R'_{(2)})\cdot(h_{(1)}\otimes h_{(2)}) = Q\cdot\Delta(h).$$ □

Clearly, if the map $D : H^* \to H$ from Proposition 5.2.1 is surjective, then every element of $H$ can be factorised as the product of an elements in the subalgebras $\bar\phi_R(H^*) \subset H$ and $\phi_R(H^*) \subset H$. As $\dim_{\mathbb{F}} H = \dim_F H^*$ for every finite-dimensional Hopf algebra $H$, this is the case if and only if the Drinfeld map $D : H \to H^*$ is a linear isomorphism.

**Definition 5.2.2:** A finite-dimensional quasitriangular Hopf algebra $(H, R)$ is called **factorisable** if the Drinfeld map $D : H^* \to H$ is a linear isomorphism.

**Example 5.2.3:** For any finite-dimensional Hopf algebra $H$, the Drinfeld double $D(H)$ is factorisable. If $(x_1, ..., x_n)$ is an ordered basis of $H$ and $(\alpha^1, ..., \alpha^n)$ its dual basis, one has for the universal $R$-matrix $R = \Sigma_{i=1}^n 1\otimes x_i\otimes\alpha^i\otimes 1$:

$$\phi_R : H\otimes H^* \to H^*\otimes H,\ h\otimes\alpha \mapsto \epsilon(\alpha)\, 1\otimes h \qquad \bar\phi_R : H\otimes H^* \to H^*\otimes H,\ h\otimes\alpha \mapsto \epsilon(h)\,\alpha\otimes 1$$
$$D : H\otimes H^* \to H^*\otimes H,\ h\otimes\alpha \mapsto \Sigma_{i,j=1}^n \alpha^j\alpha S^{-1}(\alpha^i)\otimes x_i h x_j.$$

This follows directly from the expression for the $R$-matrix

$$
\begin{aligned}
R_{21}R &= \Sigma_{i,j=1}^n(\alpha^i\otimes 1\otimes 1\otimes x_i)\cdot(1\otimes x_j\otimes\alpha^j\otimes 1) = \Sigma_{i,j=1}^n\Sigma_{(x_i)(\alpha^j)}\alpha^j_{(3)}(x_{i(1)})\alpha^j_{(1)}(S^{-1}(x_{i(3)}))\,\alpha^i\otimes x_j\otimes\alpha^j_{(2)}\otimes x_{i(2)}\\
&= \Sigma_{u,v,w,j=1}^n\alpha^j_{(3)}(x_u)\alpha^j_{(1)}(S^{-1}(x_w))\,\alpha^u\alpha^v\alpha^w\otimes x_j\otimes\alpha^j_{(2)}\otimes x_v\\
&= \Sigma_{v,j=1}^n\alpha^j_{(3)}\alpha^v S^{-1}(\alpha^j_{(1)})\otimes x_j\otimes\alpha^j_{(2)}\otimes x_v = \Sigma_{v,i,j,k=1}^n\alpha^k\alpha^v S^{-1}(\alpha^i)\otimes x_i x_j x_k\otimes\alpha^j\otimes x_v\\
\phi_R(h\otimes\alpha) &= \Sigma_{i=1}^n(h\otimes\alpha)(\alpha^i\otimes 1)\,1\otimes x_i = \Sigma_{i=1}^n\epsilon(\alpha)\alpha^i(h)1\otimes x_i = \epsilon(\alpha)1\otimes h\\
\bar\phi_R(h\otimes\alpha) &= \Sigma_{i=1}^n(h\otimes\alpha)(1\otimes x_i)\,\alpha^i\otimes 1 = \Sigma_{i=1}^n\epsilon(h)\alpha(x_i)\alpha^i\otimes 1 = \epsilon(h)\alpha\otimes 1\\
D(h\otimes\alpha) &= \Sigma_{v,i,j,k=1}^n\alpha^j(h)\alpha(x_v)\,\alpha^k\alpha^v S^{-1}(\alpha^i)\otimes x_i x_j x_k = \Sigma_{i,k=1}^n\alpha^k\alpha S^{-1}(\alpha^i)\otimes x_i h x_k\\
D^{-1}(\alpha\otimes h) &= \Sigma_{i,k=1}^n\alpha^k\alpha^l\alpha^i\otimes S(x_i)x_j x_k = \Sigma_{i,k=1}^n x_k h S^{-1}(x_i)\otimes\alpha^i\alpha\alpha^k\\
D^{-1}(D(h\otimes\alpha)) &= \Sigma_{i,k,u,v=1}^n\,x_u x_i h x_k S^{-1}(x_v)\otimes\alpha^v\alpha^k\alpha S^{-1}(\alpha^i)\alpha^u\\
&= \Sigma_{i,k=1}^n\,x_i h x_{k(2)}S^{-1}(x_{k(1)})\otimes\alpha^k\alpha S^{-1}(\alpha^i_{(2)})\alpha^i_{(1)} = \Sigma_{i,k=1}^n\,\epsilon(x_k)\epsilon(\alpha^i)x_i h\otimes\alpha^k\alpha = h\otimes\alpha
\end{aligned}
$$

Another important conceptual motivation for the Drinfeld map $D: H^*\to H$ is that it relates the *characters* of the Hopf algebra $H$, i. e. the invariants under the coadjoint action of $H$ on $H^*$, to elements of the center $C(H)$. If $H$ is factorisable, then the character algebra and the center are isomorphic.

**Theorem 5.2.4:** Let $H$ be a finite-dimensional quasitriangular Hopf algebra. Then the invariants under the coadjoint action

$$\lhd^*_{ad}: H^*\otimes H\to H^*,\quad \alpha\lhd^*_{ad}h = \Sigma_{(\alpha)}(\alpha_{(1)}S(\alpha_{(3)}))(h)\,\alpha_{(2)}.$$

form a subalgebra $C(H) = (H^*)^{\rhd^*_{ad}}\subset H^*$, the **character algebra** of $H$. The Drinfeld map induces an algebra homomorphism $D: C(H)\to Z(H)$. If $H$ is factorisable, then this is an algebra isomorphism.

**Proof:**
That $C(H)\subset H^*$ is a subalgebra follows by a direct computation: if $\alpha,\beta\in C(H)$, then one has

$$
\begin{aligned}
&\Sigma_{(h)}\,(\alpha\beta)(S(h_{(2)})kh_{(1)}) = \Sigma_{(h)}\,\alpha(S(h_{(2)(2)})k_{(1)}h_{(1)(1)})\beta(S(h_{(2)(1)})k_{(2)}h_{(1)(2)})\\
&= \Sigma_{(h)(k)}\,\alpha(S(h_{(3)})k_{(1)}h_{(1)})\beta(S(h_{(2)(2)})k_{(2)}h_{(2)(1)})\\
&\overset{\beta\in C(H)}{=} \Sigma_{(h)(k)}\,\epsilon(h_{(2)})\alpha(S(h_{(3)})k_{(1)}h_{(1)})\beta(k_{(2)}) = \Sigma_{(h)(k)}\,\alpha(S(h_{(2)})k_{(1)}h_{(1)})\beta(k_{(1)})\\
&\overset{\alpha\in C(H)}{=} \Sigma_{(k)}\,\epsilon(h)\,\alpha(k_{(1)})\beta(k_{(2)}) = \epsilon(h)(\alpha\beta)(k)
\end{aligned}
$$

for all $h,k\in H$. With the definition of the Drinfeld map from Proposition 5.2.1 and the identity in Proposition 5.2.1, 4. one then obtains

$$
\begin{aligned}
h\cdot D(\beta) &= \beta(Q_{(2)})hQ_{(1)} = \Sigma_{(h)}\,\beta(S^{-1}(h_{(3)})h_{(2)}Q_{(2)})h_{(1)}Q_{(1)}\\
&\overset{\text{Prop.5.2.1}}{=} \Sigma_{(h)}\beta(S^{-1}(h_{(3)})Q_{(2)}h_{(2)})Q_{(1)}h_{(1)} \overset{\beta\in C(H)}{=} \beta(Q_{(2)})\,Q_{(1)}h = D(\beta)h\\
D(\alpha\beta) &= \bar\phi_R(\alpha_{(1)})D(\beta)\phi_R(\alpha_{(2)}) = \bar\phi_R(\alpha_{(1)})\phi_R(\alpha_{(2)})D(\beta) = D(\alpha)D(\beta).
\end{aligned}
$$

This shows that the Drinfeld map induces an algebra homomorphism $D: C(H)\to Z(H)$. If $H$ is factorisable, then this algebra homomorphism is injective. To show that it is surjective, it is sufficient to show that $\dim_{\mathbb{F}} C(H) = \dim_{\mathbb{F}} Z(H)$.

For this, we consider a right integral $\lambda \in I_R(H^*) \setminus \{0\}$ and the associated map $\psi_\lambda : H \to H^*$, $h \mapsto \Sigma_{(\lambda)}\lambda_{(2)}(h)\,\lambda_{(1)}$, which is an isomorphism by Theorem 3.2.15. As $H$ is factorisable, it is unimodular by Theorem 5.1.11 and hence one has $\alpha = \epsilon$ for the modular element $\alpha \in \mathrm{Gr}(H^*)$ by Proposition 3.2.16. This implies that for any left integral $\lambda' \in I_L(H^*) \setminus \{0\}$ the Nakayama automorphism from Proposition 3.3.6 is given by $\rho(h) = S^2(h)$ for all $h \in H$ and hence $\lambda'(x \cdot y) = \lambda'(S^2(y) \cdot x)$ for all $x, y \in H$. As the element $S(\lambda) \in I_L(H^*) \setminus \{0\}$ is a left integral, it follows that $\lambda(x \cdot y) = \lambda(y \cdot S^2(x))$, and we obtain for all $x, y \in H$

$$\Sigma_{(x)}\psi_\lambda(h)(S^{-1}(x_{(2)})yx_{(1)}) = \Sigma_{(x)(\lambda)}\lambda_{(2)}(h)\lambda_{(1)}(S^{-1}(x_{(2)})yx_{(1)}) = \Sigma_{(x)}\lambda(S^{-1}(x_{(2)})yx_{(1)}h)$$
$$= \Sigma_{(x)}\lambda(yx_{(1)}hS(x_{(2)})) = \Sigma_{(x)}\psi_\lambda(x_{(1)}hS(x_{(2)}))(y).$$

As $\psi_\lambda$ is injective, this implies that $\psi_\lambda(h) \in C(H)$ if and only if $h \in Z(H)$ and hence $\dim_{\mathbb{F}} C(H) = \dim_{\mathbb{F}} Z(H)$. □

## 5.3 *Twisting

Clearly, the Drinfeld double $D(H)$ differs from the tensor product $H^*\otimes H$ since its multiplication is not the tensor product multiplication but a mixed multiplication law in which both, $H$ and $H^*$ act on each other. However, its coalgebra structure is simply the tensor product coalgebra structure on $H^{*cop}\otimes H$ and hence we the dual Hopf algebra $D(H)^*$ coincides with $H^{op}\otimes H^*$ as an algebra, while its comultiplication differs from the tensor product coalgebra structure. It turns out that this is a spacial case of a systematic construction that associates to a bialgebra $B$ bialgebra a new bialgebra $B$ with the same multiplication, unit and counit, but with a different coproduct. The ingredient in this construction is a so-called em 2-cocycle.

**Definition 5.3.1:** A **2-cocycle** for a bialgebra $B$ is an invertible element $F = F_{(1)}\otimes F_{(2)} \in B\otimes B$ satisfying

$$F_{12} \cdot (\Delta\otimes\mathrm{id})(F) = F_{23} \cdot (\mathrm{id}\otimes\Delta)(F) \qquad (\epsilon\otimes\mathrm{id})(F) = 1_{\mathbb{F}}\otimes 1 \qquad (\mathrm{id}\otimes\epsilon)(F) = 1\otimes 1_{\mathbb{F}}. \qquad (45)$$

**Example 5.3.2:**

1. If $R$ is a universal $R$-matrix for a bialgebra $B$, then $R$ is a 2-cocycle for $B$, since we have $R_{12}(\Delta\otimes\mathrm{id})(R) = R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} = R_{23}(\mathrm{id}\otimes\Delta)(R)$, $(\epsilon\otimes\mathrm{id})R = 1_{\mathbb{F}}\otimes 1$ and $(\mathrm{id}\otimes\epsilon)(R) = 1\otimes 1_{\mathbb{F}}$ by Proposition 5.1.5. .
2. If $G$ is a 2-cocycle for $(B, m, \eta, \Delta, \epsilon)$ and $F$ a 2-cocycle for $(B, m, \eta, \Delta_G, \epsilon)$, then $FG$ is a cocycle for $(B, m, \eta, \Delta, \epsilon)$ (Exercise).
3. One can show that if $F$ is a cocycle for $(B, m, \eta, \Delta, \epsilon)$, then $F^{-1}$ is a cocycle for $(B, m, \eta, \Delta_F, \epsilon)$ (Exercise).
4. If $F = F_{(1)}\otimes F_{(2)} \in B\otimes B$ is a 2-cocycle for a bialgebra $B$, then $G = 1\otimes F_{(1)}\otimes F_{(2)}\otimes 1$ is a 2-cocycle for $B\otimes B$ with the tensor product bialgebra structure (Exercise).

Given a cocycle $F \in B\otimes B$, one can form a new bialgebra by conjugating the comultiplication of $B$ with $F$. If $B$ is a Hopf algebra and the antipode of $B$ is twisted as well, then twisting yields another Hopf algebra and twists of quasitriangular bialgebra are again quasitriangular.

**Theorem 5.3.3:** Let $(B, m, \eta, \Delta, \epsilon)$ be bialgebra and $F = F_{(1)} \otimes F_{(2)} \in B \otimes B$ a 2-cocycle.

1. $(B, m, \eta, \Delta_F, \epsilon)$ is a bialgebra with the **twisted coproduct**
$$\Delta_F : H \to H \otimes H, \qquad h \mapsto F \cdot \Delta(h) \cdot F^{-1}$$
2. If $(B, m, \eta, \Delta, \epsilon)$ is a Hopf algebra with antipode $S : B \to B$, then $(B, m, \eta, \Delta_F, \epsilon)$ is a Hopf algebra with the **twisted antipode**
$$S_F : H \to H, \quad h \mapsto vS(h)v^{-1} \qquad \text{with} \quad v = m \circ (\mathrm{id} \otimes S)(F) = F_{(1)} S(F_{(2)})$$
3. If $(B, m, \eta, \Delta, \epsilon)$ is quasitriangular with a universal $R$-matrix $R \in B \otimes B$, then $(B, m, \eta, \Delta_F, \epsilon)$ is quasitriangular with universal $R$-matrix $R_F = F_{21} \cdot R \cdot F^{-1}$.

**Proof:**
1. It follows directly from the definition of $\Delta_F$ that $\Delta_F$ is an algebra homomorphism
$$\Delta_F(ab) = F \cdot \Delta(ab) \cdot F^{-1} = F \cdot \Delta(a) \cdot \Delta(b) \cdot F^{-1} = F \cdot \Delta(a) \cdot F^{-1} \cdot F \cdot \Delta(b) \cdot F^{-1} = \Delta_F(a) \cdot \Delta_F(b)$$
for all $a, b \in B$. The counitality of $\Delta_F$ follows from the counitality of $\Delta$ and the last two equations in the definition of the 2-cocycle
$$\begin{aligned}(\epsilon \otimes \mathrm{id}) \circ \Delta_F(b) &= (\epsilon \otimes \mathrm{id})(F \cdot \Delta(b) \cdot F^{-1}) = (\epsilon \otimes \mathrm{id})(F)(\epsilon \otimes \mathrm{id})(\Delta(b))(\epsilon \otimes \mathrm{id})(F^{-1}) \\ &= (\epsilon \otimes \mathrm{id})(F) \cdot (1 \otimes b) \cdot (\epsilon \otimes \mathrm{id})(F)^{-1} = 1 \otimes b \\ (\mathrm{id} \otimes \epsilon) \circ \Delta_F(b) &= (\mathrm{id} \otimes \epsilon)(F \cdot \Delta(b) \cdot F^{-1}) = (\mathrm{id} \otimes \epsilon)(F)(\mathrm{id} \otimes \epsilon)(\Delta(b))(\mathrm{id} \otimes \epsilon)(F^{-1}) \\ &= (\mathrm{id} \otimes \epsilon)(F) \cdot (b \otimes 1) \cdot (\mathrm{id} \otimes \epsilon)(F)^{-1} = b \otimes 1.\end{aligned}$$
The coassociativity of $\Delta_F$ follows from the coassociativity of $\Delta$ and the first equation in the definition of a 2-cocycle
$$\begin{aligned}(\Delta_F \otimes \mathrm{id}) \circ \Delta_F(b) &= (\Delta_F \otimes \mathrm{id})(F \cdot \Delta(b) \cdot F^{-1}) = (\Delta_F \otimes \mathrm{id})(F) \cdot (\Delta_F \otimes \mathrm{id}) \circ \Delta(b) \cdot (\Delta_F \otimes \mathrm{id})(F^{-1}) \\ &= F_{12} \cdot (\Delta \otimes \mathrm{id})(F) \cdot F_{12}^{-1} \cdot F_{12} \cdot (\Delta \otimes \mathrm{id}) \circ \Delta(b) \cdot F_{12}^{-1} \cdot F_{12} \cdot (\Delta \otimes \mathrm{id})(F)^{-1} \cdot F_{12}^{-1} \\ &= (F_{12} \cdot (\Delta \otimes \mathrm{id})(F)) \cdot (\Delta \otimes \mathrm{id}) \circ \Delta(b) \cdot (F_{12} \cdot (\Delta \otimes \mathrm{id})(F))^{-1}\end{aligned}$$
$$\begin{aligned}(\mathrm{id} \otimes \Delta_F) \circ \Delta_F(b) &= (\mathrm{id} \otimes \Delta_F)(F \cdot \Delta(b) \cdot F^{-1}) = (\mathrm{id} \otimes \Delta_F)(F) \cdot (\mathrm{id} \otimes \Delta_F) \circ \Delta(b) \cdot (\mathrm{id} \otimes \Delta_F)(F^{-1}) \\ &= F_{23} \cdot (\mathrm{id} \otimes \Delta)(F) \cdot F_{23}^{-1} \cdot F_{23} \cdot (\mathrm{id} \otimes \Delta) \circ \Delta(b) \cdot F_{23}^{-1} \cdot F_{23} \cdot (\mathrm{id} \otimes \Delta)(F)^{-1} \cdot F_{23}^{-1} \\ &= (F_{23} \cdot (\mathrm{id} \otimes \Delta)(F)) \cdot (\mathrm{id} \otimes \Delta) \circ \Delta(b) \cdot (F_{23} \cdot (\mathrm{id} \otimes \Delta)(F))^{-1}.\end{aligned}$$
2. To prove that $S_F$ is an antipode for $(B, m, \eta, \Delta_F, \epsilon)$ we show first that
$$v^{-1} = v' = m \circ (S \otimes \mathrm{id})(F^{-1}) = S(F^{-1}_{(1)}) F^{-1}_{(2)}$$
This follows from the first identity in (45), which also implies
$$(\Delta \otimes \mathrm{id})(F^{-1}) \cdot F_{12}^{-1} = (\mathrm{id} \otimes \Delta)(F^{-1}) \cdot F_{23}^{-1}. \tag{46}$$

Using the Definition of $v$ and $v'$, we then obtain

$$\begin{aligned}
v'v &= S(F'^{-1}_{(1)})F'^{-1}_{(2)}F_{(1)}S(F_{(2)}) = S(F'^{-1}_{(1)}(F''_{(1)}\epsilon(F''_{(2)})))F'^{-1}_{(2)}F_{(1)}S(F_{(2)})\\
&= \Sigma_{(F''_{(2)})}S(F'^{-1}_{(1)}F''_{(1)})F'^{-1}_{(2)}F_{(1)}F''_{(2)(1)}S(F''_{(2)(2)})S(F_{(2)})\\
&= \Sigma_{(F''_{(2)})}S(F'^{-1}_{(1)}F''_{(1)})F'^{-1}_{(2)}F_{(1)}F''_{(2)(1)}S(F_{(2)}F''_{(2)(2)})\\
&= m\circ(m\otimes\mathrm{id})\circ(S\otimes\mathrm{id}\otimes S)(\Sigma_{(F''_{(2)})}F'^{-1}_{(1)}F''_{(1)}\otimes F'^{-1}_{(2)}F_{(1)}F''_{(2)(1)}\otimes F_{(2)}F''_{(2)(2)})\\
&= m\circ(m\otimes\mathrm{id})\circ(S\otimes\mathrm{id}\otimes S)(F_{12}F_{23}(\mathrm{id}\otimes\Delta)(F))\\
&\overset{(45)}{=} m\circ(m\otimes\mathrm{id})\circ(S\otimes\mathrm{id}\otimes S)\circ(\Delta\otimes\mathrm{id})(F)\\
&= m((m\circ(S\otimes\mathrm{id})\circ\Delta)(F_{(1)})\otimes F_{(2)}) = m(\epsilon(F_{(1)})1\otimes F_{(2)}) = m(1\otimes 1) = 1
\end{aligned}$$

$$\begin{aligned}
vv' &= F_{(1)}S(F_{(2)})S(F'^{-1}_{(1)})F'^{-1}_{(2)} = F_{(1)}S(F'^{-1}_{(1)}F_{(2)})F'^{-1}_{(2)} = (\epsilon(F''^{-1}_{(2)})F''^{-1}_{(1)})\; F_{(1)}S(F'^{-1}_{(1)}F_{(2)})F'^{-1}_{(2)}\\
&= \Sigma_{(F''^{-1}_{(2)})}\; F''^{-1}_{(1)}F_{(1)}S(F'^{-1}_{(1)}F_{(2)})S(F''^{-1}_{(2)(1)})F''^{-1}_{(2)(2)}F'^{-1}_{(2)}\\
&= \Sigma_{(F''^{-1}_{(2)})}\; F''^{-1}_{(1)}F_{(1)}S(F''^{-1}_{(2)(1)}F'^{-1}_{(1)}F_{(2)})F''^{-1}_{(2)(2)}F'^{-1}_{(2)}\\
&= m\circ(m\otimes\mathrm{id})\circ(\mathrm{id}\otimes S\otimes\mathrm{id})(\Sigma_{(F''^{-1}_{(2)})}\; F''^{-1}_{(1)}F_{(1)}\otimes F''^{-1}_{(2)(1)}F'^{-1}_{(1)}F_{(2)}\otimes F''^{-1}_{(2)(2)}F'^{-1}_{(2)})\\
&= m\circ(m\otimes\mathrm{id})\circ(\mathrm{id}\otimes S\otimes\mathrm{id})((\mathrm{id}\otimes\Delta)(F^{-1})\cdot F^{-1}_{23}\cdot F_{12})\\
&\overset{(46)}{=} m\circ(m\otimes\mathrm{id})\circ(\mathrm{id}\otimes S\otimes\mathrm{id})((\Delta\otimes\mathrm{id})(F^{-1}))\\
&= m((m\circ(\mathrm{id}\otimes S)\circ\Delta)(F_{(1)})\otimes F_{(2)}) = m(\epsilon(F_{(1)})1\otimes F_{(2)}) = m\circ(\epsilon\otimes\mathrm{id})(F) = m(1\otimes 1) = 1.
\end{aligned}$$

This shows that $v' = v^{-1}$. With the definition of the twisted comultiplication and antipode, we then obtain for all $b\in B$

$$\begin{aligned}
&m\circ(S_F\otimes\mathrm{id})\circ\Delta_F(b) = \Sigma_{(b)}\; S_F(F_{(1)}b_{(1)}F'^{-1}_{(1)})F_{(2)}b_{(2)}F'^{-1}_{(2)}\\
&= \Sigma_{(b)}\; S_F(F'^{-1}_{(1)})S_F(b_{(1)})S_F(F_{(1)})F_{(2)}b_{(2)}F'^{-1}_{(2)} = \Sigma_{(b)}\; vS(F'^{-1}_{(1)})S(b_{(1)})S(F_{(1)})v^{-1}F_{(2)}b_{(2)}F'^{-1}_{(2)}\\
&= \Sigma_{(b)}\; \tilde F_{(1)}S(\tilde F_{(2)})S(F'^{-1}_{(1)})S(b_{(1)})S(F_{(1)})S(\tilde F'^{-1}_{(1)})\tilde F'^{-1}_{(2)}F_{(2)}b_{(2)}F'^{-1}_{(2)}\\
&= \Sigma_{(b)}\; \tilde F_{(1)}S(\tilde F_{(2)})S(F'^{-1}_{(1)})S(b_{(1)})S(\tilde F'^{-1}_{(1)}F_{(1)})(\tilde F'^{-1}_{(2)}F_{(2)})b_{(2)}F'^{-1}_{(2)}\\
&= \Sigma_{(b)}\; \tilde F_{(1)}S(\tilde F_{(2)})S(F'^{-1}_{(1)})S(b_{(1)})b_{(2)}F'^{-1}_{(2)} = \epsilon(b)\;\tilde F_{(1)}S(\tilde F_{(2)})S(F'^{-1}_{(1)})F'^{-1}_{(2)} = \epsilon(b)\; vv^{-1} = \epsilon(b)1.
\end{aligned}$$

The proof of the identity $m\circ(\mathrm{id}\otimes S_F)\circ\Delta_F = \eta\epsilon$ is analogous.

3. To prove that $(B, m, \eta, \Delta_F, \epsilon)$ is quasitriangular with universal $R$-matrix $R_F = F_{21}\cdot R\cdot F^{-1}$, we compute first for all $b\in B$

$$\begin{aligned}
&R_F\cdot\Delta_F(b)\cdot R_F^{-1} = (F_{21}\cdot R\cdot F^{-1})\cdot F\cdot\Delta(b)\cdot F^{-1}\cdot(F_{21}\cdot R\cdot F^{-1})^{-1}\\
&= F_{21}\cdot R\cdot\Delta(b)\cdot R^{-1}\cdot F_{21}^{-1} = F_{21}\cdot\Delta^{op}(b)\cdot F_{21}^{-1} = \Delta_F^{op}(b).
\end{aligned}$$

It remains to show that $(\Delta_F\otimes\mathrm{id})(R_F) = (R_F)_{13}(R_F)_{23}$ and $(\mathrm{id}\otimes\Delta_F)(R_F) = (R_F)_{13}(R_F)_{12}$. For the first of these identities, we compute with the definition of $\Delta_F$ and $R_F$, the properties of the

2-cocycle $F$ and the identity $(\Delta\otimes\mathrm{id})(R) = R_{13}R_{23}$

$$
\begin{aligned}
(\Delta_F\otimes\mathrm{id})(R_F) &= F_{12}\cdot(\Delta\otimes\mathrm{id})(F_{21}\cdot R\cdot F^{-1})\cdot F_{21}^{-1}\\
&= F_{12}\cdot(\Delta\otimes\mathrm{id})(F_{21})\cdot(\Delta\otimes\mathrm{id})(R)\cdot(\Delta\otimes\mathrm{id})(F^{-1})\cdot F_{21}^{-1}\\
&= F_{12}\cdot(\Delta\otimes\mathrm{id})(F_{21})\cdot R_{13}R_{23}\cdot(\Delta\otimes\mathrm{id})(F^{-1})\cdot F_{21}^{-1}\\
(46)\quad &= F_{12}\cdot(\Delta\otimes\mathrm{id})(F_{21})\cdot R_{13}R_{23}\cdot(\mathrm{id}\otimes\Delta)(F^{-1})\cdot F_{23}^{-1}\\
&= (\mathrm{id}\otimes\tau)\circ(\tau\otimes\mathrm{id})(F_{23}\cdot(\mathrm{id}\otimes\Delta)(F))\cdot R_{13}R_{23}\cdot(\mathrm{id}\otimes\Delta)(F^{-1})\cdot F_{23}^{-1}\\
(45)\quad &= (\mathrm{id}\otimes\tau)\circ(\tau\otimes\mathrm{id})(F_{12}\cdot(\Delta\otimes\mathrm{id})(F))\cdot R_{13}R_{23}\cdot(\mathrm{id}\otimes\Delta)(F^{-1})\cdot F_{23}^{-1}\\
&= (\mathrm{id}\otimes\tau)(F_{21}\cdot(\Delta^{op}\otimes\mathrm{id})(F))\cdot R_{13}R_{23}\cdot(\mathrm{id}\otimes\Delta)(F^{-1})\cdot F_{23}^{-1}\\
&= (\mathrm{id}\otimes\tau)(F_{21}\cdot(\Delta^{op}\otimes\mathrm{id})(F)\cdot R_{12})\cdot(R_{23}\cdot(\mathrm{id}\otimes\Delta)(F^{-1})\cdot F_{23}^{-1})\\
(R\Delta=\Delta^{op}R)\quad &= (\mathrm{id}\otimes\tau)(F_{21}\cdot R_{12}\cdot(\Delta\otimes\mathrm{id})(F))\cdot((\mathrm{id}\otimes\Delta^{op})(F^{-1})\cdot R_{23}\cdot F_{23}^{-1})\\
&= F_{31}\cdot R_{13}\cdot(\mathrm{id}\otimes\tau)(\Delta\otimes\mathrm{id})(F))\cdot(\mathrm{id}\otimes\Delta^{op})(F^{-1})\cdot R_{23}\cdot F_{23}^{-1}\\
(F^{-1}F=1)\quad &= F_{31}\cdot R_{13}\cdot F'^{-1}_{13}\cdot F''_{13}\cdot(\mathrm{id}\otimes\tau)(\Delta\otimes\mathrm{id})(F)\cdot(\mathrm{id}\otimes\Delta^{op})(F^{-1})\cdot R_{23}\cdot F_{23}^{-1}\\
&= F_{31}\cdot R_{13}\cdot F'^{-1}_{13}\cdot(\mathrm{id}\otimes\tau)(F''_{12}\cdot(\Delta\otimes\mathrm{id})(F))\cdot(\mathrm{id}\otimes\Delta^{op})(F^{-1})\cdot R_{23}\cdot F_{23}^{-1}\\
(45)\quad &= F_{31}\cdot R_{13}\cdot F'^{-1}_{13}\cdot(\mathrm{id}\otimes\tau)(F''_{23}\cdot(\mathrm{id}\otimes\Delta)(F))\cdot(\mathrm{id}\otimes\Delta^{op})(F^{-1})\cdot R_{23}\cdot F_{23}^{-1}\\
&= F_{31}\cdot R_{13}\cdot F'^{-1}_{13}\cdot(\mathrm{id}\otimes\tau)(F''_{23}\cdot(\mathrm{id}\otimes\Delta)(F)\cdot(\mathrm{id}\otimes\Delta)(F^{-1}))\cdot R_{23}\cdot F_{23}^{-1}\\
&= F_{31}\cdot R_{13}\cdot F'^{-1}_{13}\cdot(\mathrm{id}\otimes\tau)(F''_{23})\cdot R_{23}\cdot F_{23}^{-1} = F_{31}\cdot R_{13}\cdot F'^{-1}_{13}\cdot F''_{32}\cdot R_{23}\cdot F_{23}^{-1}\\
&= (R_F)_{13}(R_F)_{23}.
\end{aligned}
$$

The proof of the identity $(\mathrm{id}\otimes\Delta_F)(R_F) = (R_F)_{13}(R_F)_{12}$ is analogous. □

The results of this theorem and the statements about twists in Example 5.3.2 2. and 3. show that being related by twists defines an equivalence relation on the set of bialgebras, of Hopf algebras and of quasitriangular Hopf algebras over $\mathbb{F}$. Two bialgebras are Hopf algebras that are related by a twist are often called **twist equivalent**. One can show that the categories $B$-Mod and $B'$-Mod for two twist equivalent bialgebras $B$ and $B'$ are related by a monoidal equivalence of categories. In this sense, twist equivalent bialgebras have the same representations. It is also clear from Example 5.3.2, 1. that one can view a quasitriangular bialgebra as a bialgebra that is twist equivalent to the bialgebra with the opposite coproduct. In the case of a Drinfeld double $D(H)$ there is an even stronger statement, namely that the dual of $D(H)^*$ is twist equivalent to the tensor product Hopf algebra $H^{op}\otimes H$.

**Theorem 5.3.4:** Let $H$ be a finite-dimensional Hopf algebra. Then the Drinfeld double $D(H)$ is the dual of the Hopf algebra $(H^{op}\otimes H^*)^F$ obtained by twisting $H^{op}\otimes H^*$ with the 2-cocycle $F = \Sigma_{i=1}^n 1\otimes\alpha^i\otimes x_i\otimes 1$ from Theorem 5.1.9.

**Proof:**
We show first that $F = \Sigma_{i=1}^n 1\otimes\alpha^i\otimes x_i\otimes 1$ is a 2-cocycle for $H^{op}\otimes H$. The identities in (43) imply

$$
\begin{aligned}
&(\epsilon\otimes\mathrm{id})(F) = \Sigma_{i=1}^n \epsilon(1_H)\alpha^i(1)\otimes x_i\otimes 1_{H^*} = \Sigma_{i=1}^n 1_{\mathbb{F}}\otimes\alpha^i(1)x_i\otimes 1_{H^*} = 1_{\mathbb{F}}\otimes 1_H\otimes 1_{H^*}\\
&(\mathrm{id}\otimes\epsilon)(F) = \Sigma_{i=1}^n 1_H\otimes\alpha^i\otimes 1_{H^*}(1_H)\epsilon(x_i) = \Sigma_{i=1}^n 1_H\otimes\epsilon(x_i)\alpha^i\otimes 1_{\mathbb{F}} = 1_H\otimes 1_{H^*}\otimes 1_{\mathbb{F}}\\
&F_{12}\cdot(\Delta\otimes\mathrm{id})(F) = (\Sigma_{i=1}^n 1\otimes\alpha^i\otimes x_i\otimes 1\otimes 1\otimes 1)\cdot(\Sigma_{j=1}^n 1\otimes\alpha^j_{(1)}\otimes 1\otimes\alpha^j_{(2)}\otimes x_j\otimes 1)\\
&\overset{(43)}{=} \Sigma_{i,j,k=1}^n (1\otimes\alpha^i\otimes x_i\otimes 1\otimes 1\otimes 1)\cdot(\Sigma_{j=1}^n 1\otimes\alpha^j\otimes 1\otimes\alpha^k\otimes x_j x_k\otimes 1)\\
&= \Sigma_{i,j,k=1}^n 1\otimes\alpha^i\alpha^j\otimes x_i\otimes\alpha^k\otimes x_j x_k\otimes 1 = \Sigma_{i,j,k=1}^n (1\otimes 1\otimes 1\otimes\alpha^k\otimes x_k\otimes 1)\cdot(1\otimes\alpha^i\alpha^j\otimes x_i\otimes 1\otimes x_j\otimes 1)\\
&\overset{(43)}{=} (\Sigma_{k=1}^n 1\otimes 1\otimes 1\otimes\alpha^k\otimes x_k\otimes 1)\cdot(\Sigma_{j=1}^n 1\otimes\alpha^j\otimes x_{j(1)}\otimes 1\otimes x_{j(2)}\otimes 1) = F_{23}\cdot(\mathrm{id}\otimes\Delta)(F).
\end{aligned}
$$

A short computation analogous to the one in the proof of Theorem 5.1.9 and taking into account that the antipode of $H^{op}$ is $S^{-1}$ shows that its inverse is given by

$$F^{-1} = \Sigma_{i=1}^n 1\otimes S(\alpha^i)\otimes x_i\otimes 1 = \Sigma_{i=1}^n 1\otimes\alpha^i\otimes S^{-1}(x_i)\otimes 1.$$

Using this identity and the multiplication of $H^{op}\otimes H^*$, we obtain

$$\begin{aligned}\Delta_F(h\otimes\alpha) &= (\Sigma_{i=1}^n 1\otimes\alpha^i\otimes x_i\otimes 1)\cdot(\Sigma_{(h)(\alpha)}h_{(1)}\otimes\alpha_{(1)}\otimes h_{(2)}\otimes\alpha_{(2)})\cdot(\Sigma_{j=1}^n 1\otimes\alpha^j\otimes S^{-1}(x_j)\otimes 1\otimes 1\otimes 1)\\ &= \Sigma_{i,j=1}^n\Sigma_{(h)(\alpha)}h_{(1)}\otimes\alpha^i\alpha_{(1)}\alpha^j\otimes S^{-1}(x_j)h_{(2)}x_i\otimes\alpha_{(2)}.\end{aligned}$$

The dual algebra structure on $H^*\otimes H$ is given by

$$\begin{aligned}&((\beta\otimes x)\cdot(\gamma\otimes y))(h\otimes\alpha) = (\beta\otimes x\otimes\gamma\otimes y)(\Delta(h\otimes\alpha))\\ &= \Sigma_{i,j=1}^n\Sigma_{(h)(\alpha)}(\beta\otimes x\otimes\gamma\otimes y)(h_{(1)}\otimes\alpha^i\alpha_{(1)}\alpha^j\otimes S^{-1}(x_j)h_{(2)}x_i\otimes\alpha_{(2)})\\ &= \Sigma_{i,j=1}^n\Sigma_{(h)(\alpha)}\,\beta(h_{(1)})(\alpha^i\alpha_{(1)}\alpha^j)(x)\gamma(S^{-1}(x_j)h_{(2)}x_i)\alpha_{(2)}(y)\\ &= \Sigma_{i,j=1}^n\Sigma_{(h)(\alpha)(x)(\gamma)}\,\beta(h_{(1)})\alpha^i(x_{(1)})\alpha_{(1)}(x_{(2)})\alpha^j(x_{(3)})\gamma_{(1)}(S^{-1}(x_j))\gamma_{(2)}(h_{(2)})\gamma_{(3)}(x_i)\alpha_{(2)}(y)\\ &= \Sigma_{(x)(\gamma)}\,(\beta\gamma_{(2)})(h)\alpha(x_{(2)}y)\gamma_{(3)}(x_{(1)})\gamma_{(1)}(S^{-1}(x_{(3)}))\\ &= (\Sigma_{(x)(\gamma)}\gamma_{(3)}(x_{(1)})\gamma_{(1)}(S^{-1}(x_{(3)}))\,\beta\gamma_{(2)}\otimes x_{(2)}y)(h\otimes\alpha).\end{aligned}$$

This is the multiplication of the Drinfeld double $D(H)$. Similarly, one finds that the dual coalgebra structure on $H^*\otimes H$ is the comultiplication of $D(H)$:

$$\begin{aligned}&\Delta(\gamma\otimes y)(h\otimes\alpha\otimes k\otimes\beta) = (\gamma\otimes y)((h\otimes\alpha)\cdot(k\otimes\beta)) = (\gamma\otimes y)(kh\otimes\alpha\beta) = \gamma(kh)(\alpha\beta)(y)\\ &= \Sigma_{(\gamma)(y)}\,\gamma_{(1)}(k)\gamma_{(2)}(h)\alpha(y_{(1)})\beta(y_{(2)}) = (\Sigma_{(\gamma)(y)}\,\gamma_{(2)}\otimes y_{(1)}\otimes\gamma_{(1)}\otimes y_{(2)})(h\otimes\alpha\otimes k\otimes\beta).\end{aligned}$$

As the Hopf algebra structure of $D(H)$ is determined uniquely by its multiplication and comultiplication, this proves the claim. □

# 6 Ribbon categories and and ribbon Hopf algebras

## 6.1 Knots, links, ribbons and tangles

In the last section, we showed that a quasitriangular structure on a bialgebra $B$ is exactly the additional data needed so that its representations form a braided monoidal category $B$-Mod. This shows in particular that every quasitriangular bialgebra gives rise to representations of the braid group, which can be realised as a functor $F : \mathcal{B} \to B$-Mod. However, this did not make use of the additional data in a quasitriangular *Hopf* algebra, namely the antipode, which defines representations on the duals of the representation spaces such that the right evaluation and coevaluation are homomorphisms of representations.

In this section, we consider not only braids but *knots* and *links* and show that the antipode of quasitriangular the Hopf algebra, subject to some additional conditions, is precisely the information needed to include them into this picture.

**Definition 6.1.1:**

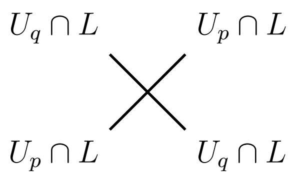
1. A **link** is a compact one-dimensional smooth submanifold of $L \subset \mathbb{R}^3$. A **knot** is a connected link.
2. A **oriented link** is a link together with a choice of an orientation on each of its connected components.
3. Two (oriented) links $L, L'$ are called **equivalent** or **ambient isotopic**[3] if there is an orientation preserving diffeomorphism $f : \mathbb{R}^3 \to \mathbb{R}^3$ with $f(L) = L'$ (and such that the orientations on $f(L)$ and $L'$ agree).

Links in $\mathbb{R}^3$ can be described by link diagrams. A link diagram is obtained by projecting a link $L \subset \mathbb{R}^3$ onto the plane with the map $P : \mathbb{R}^3 \to \mathbb{R}^2$, $(x, y, z) \mapsto (x, y)$. By applying an orientation preserving diffeomorphism to $L$ one can always achieve that the projection $P(L) \subset L$ is **generic**, i. e. satisfies the following conditions:

(i) $|P^{-1}(x) \cap L| < 3$ for all $x \in \mathbb{R}^2$,

(ii) there are only finitely many points $x \in \mathbb{R}^2$ with $|P^{-1}(x) \cap L| = 2$,

(iii) if $P^{-1}(x) = \{p, q\}$, then there are neighbourhoods $U_p, U_q \subset \mathbb{R}^3$ such that $P(U_p \cup U_q)$ can be mapped to the following diagram by an orientation preserving diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$



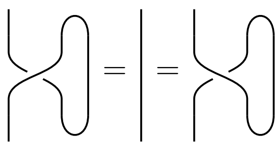

This means that each crossing point in the link projection involves exactly two strands.

[3]Note that this definition of ambient isotopy is only possible for links in $\mathbb{R}^3$ and $S^3$, due to a famous result of Cerf that states that every orientation preserving diffeomorphism $f : \mathbb{R}^3 \to \mathbb{R}^3$ or $f : S^3 \to S^3$ is isotopic to the identity. The general notion of ambient isotopy is more complicated.

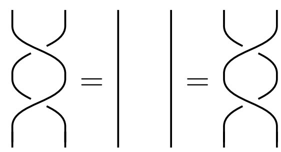
**Definition 6.1.2:** A **link diagram** for a link $L \subset \mathbb{R}^3$ is a generic link projection of $L$ together with the information which of the two points $p, q \in P^{-1}(x)$ has the greater $z$-coordinate.

The information about the $z$ coordinates in a crossing point of a link diagram is indicated in diagrams by drawing the crossing as an **overcrossing** or **undercrossing**, where the strand with greater $z$ coordinate crosses over the one with smaller $z$-coordinate. The same diagrams are used for oriented links, with the orientation of the link indicated by arrows on each connected component. A knot diagram without crossing points is called a **unknot**. One can show, see for instance [Mn, Mu], that (oriented) link diagrams capture all information about the equivalence of (oriented) links.

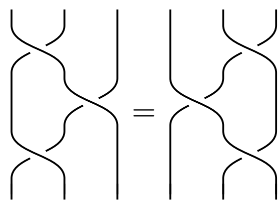
**Theorem 6.1.3:** Two (oriented) links $L, L' \subset \mathbb{R}^3$ are equivalent of and only if their (oriented) link diagrams $D_L$ and $D_{L'}$ are **equivalent**, that is related by a finite sequence of orientation preserving diffeomorphisms $f : \mathbb{R}^2 \to \mathbb{R}^2$ and the three **Reidemeister moves**:



Reidemeister move RM1



Reidemeister move RM2



Reidemeister move RM3.

These moves are understood as local moves that change only the depicted region in the link diagram and leave the rest of the link diagram invariant. They are defined analogously for oriented links.

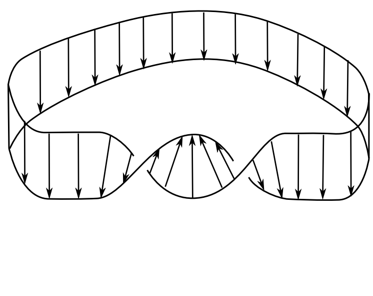
Besides the notion of a link, there is also the related concept of a *framed link* or *ribbon*, which can also be oriented. It can be viewed as a link that is thickened to a strip or ribbon, and the information needed to define the thickening is contained in the framing.

**Definition 6.1.4:**

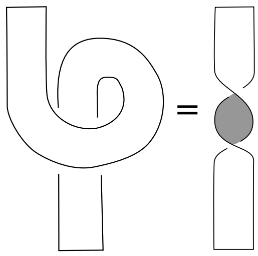
1. A **framed link** or **ribbon** is a link $L \subset \mathbb{R}^3$ together with a vector field $X$ on $L$ that is nowhere tangent to $L$, i. e. a smooth map $X : L \to \mathbb{R}^3$ with $X(l) \notin T_l L \subset \mathbb{R}^3$ for all $l \in L$.

2. Two framed links $(L, X)$ and $(L', X')$ are called **equivalent** or **ambient isotopic** if there is an orientation preserving diffeomorphism $f : \mathbb{R}^3 \to \mathbb{R}^3$ with $f(L) = L'$ and $X'(f(l)) = T_l f(X(l))$ for all $l \in L$.

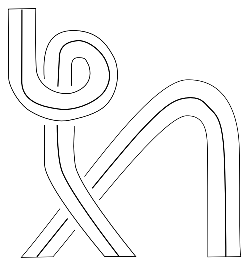
Given a link $L \subset \mathbb{R}^3$ and a vector field $X$ on $L$ that is nowhere tangent to $L$, we can thicken the link to a ribbon, that is twisted around itself only by multiples of $2\pi$. Note that this excludes Möbius strips.



If one is only interested in equivalence classes of framed links, one can therefore forget about the vector field and define a framed link as a link with an assignment of an integer $z \in \mathbb{Z}$ to each connected component that indicates how many times the connected component is twisted around itself. By using the relation

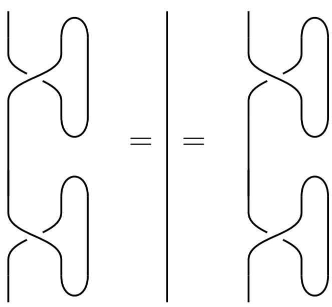


we can transform any projection of a ribbon onto the plane $\mathbb{R}^2$ into a **blackboard framed** ribbon projection that only involves twists of the type on the left but not the twist on the right. This corresponds to colouring the ribbon in $\mathbb{R}^3$ in two colours, black and white, and projecting in such a way that the white side is up in all parts of the projection. Blackboard framed links can be characterised by the same diagrams as links, where the link diagram represents the projection of a line in the middle of a ribbon, the **core**.

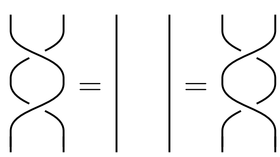


The only difference is that link diagrams that are related by the the Reidemeister move RM1 no longer describe projections of equivalent ribbons. Instead, one has the following.

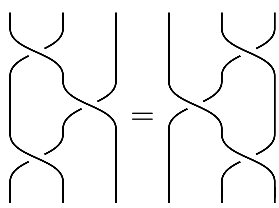
**Theorem 6.1.5:** Two framed (oriented) links $L, L' \subset \mathbb{R}^3$ are equivalent if and only if the associated link diagrams are related by a finite sequence of orientation preserving diffeomorphisms $f : \mathbb{R}^2 \to \mathbb{R}^2$ and of the three **Reidemeister moves** RM1', RM2, RM3 below. In this case, the ribbon diagrams are called **equivalent**.



Reidemeister move RM1'



Reidemeister move RM2



Reidemeister move RM3.

The central question of knot theory is to decide from two given link diagrams $D_L, D_{L'}$ if the associated links $L, L' \in \mathbb{R}^3$ or framed links in $L, L' \subset \mathbb{R}^3$ are equivalent. By Theorem 6.1.3 and 6.1.5 this is the case if and only if the associated link diagrams are related by finite sequences of orientation preserving diffeomorphisms $f : \mathbb{R}^2 \to \mathbb{R}^2$ and the three Reidemeister moves RM1, RM2, RM3 or the three Reidemeister moves RM1', RM2, RM3, respectively. However, it is not practical to address this question by the Reidemeister moves alone. Instead, one considers **link invariants** or **ribbon invariants**, which are functions from the set of links or ribbons into a commutative ring or monoid $R$ that are constant on the equivalence classes of links or framed links. A good link or ribbon invariant should be (i) easy to compute from a diagram and (ii) distinguish as many nonequivalent links or ribbons in $\mathbb{R}^3$ as possible. By Theorem 6.1.3 and 6.1.5 , we can also define a link invariant or ribbon invariant in terms of diagrams.

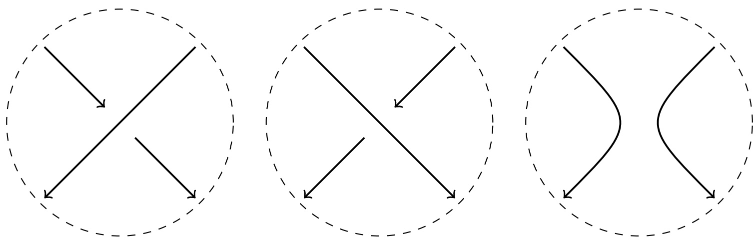
**Definition 6.1.6:** Let $R$ be a commutative monoid or a commutative ring.

1. An **(oriented) link invariant** is a map $f : \mathcal{D} \to R$ from the set $\mathcal{D}$ of (oriented) link diagrams to $R$ that is invariant under orientation preserving diffeomorphisms $f : \mathbb{R}^2 \to \mathbb{R}^2$ and the Reidemeister moves RM1, RM2, RM3.
2. An **(oriented) ribbon invariant** is a map $f : \mathcal{D} \to R$ from the set $\mathcal{D}$ of (oriented) link diagrams to $R$ that is invariant under orientation preserving diffeomorphisms $f : \mathbb{R}^2 \to \mathbb{R}^2$ and the Reidemeister moves RM1', RM2, RM3.

An obvious but not very useful invariant of a link or ribbon is the number of connected components. Two famous and important link invariants that can distinguish many links are the *HOMFLY polynomial* and the *Kauffman polynomial* of a link. For more detailed information and the proofs of the following theorems see [Ka] and [Mn].

**Theorem 6.1.7:** There is a unique invariant of oriented links with values in $\mathbb{Z}[x, x^{-1}, y, y^{-1}]$, the **HOMFLY polynomial** $H$, that satisfies the following conditions:

1. It takes the value 1 on the **unknot**, whose diagram is a circle in the plane: $H(O) = 1$.
2. If the diagrams of oriented links $L, L' \in \mathbb{R}^3$ are related by the three Reidemeister moves RM1-RM3 and orientation preserving diffeomorphisms $f : \mathbb{R}^2 \to \mathbb{R}^2$, then $H(L) = H(L')$.
3. If the diagrams of the oriented links $L_+, L_-, L_0$ are **skein related**, i. e. locally related by



$L_+$ $L_-$ $L_0$

while the rest of their diagrams coincide, then

$$x \cdot H(L_+) - x^{-1} \cdot H(L_-) = y \cdot H(L_0). \quad (47)$$

**Proof:**
That the HOMFLY polynomial is defined uniquely by these conditions follows because every

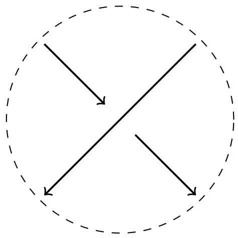
link can be transformed into an unknot by applying the skein relation and the Reidemeister relations RM1-RM3. To show that the HOMFLY polynomial is defined, it is sufficient to prove that the skein relation is compatible with the Reidemeister relations RM1-RM3. This follows by applying the skein relation to the diagrams on the left and right in the Reidemeister relations RM2 and RM3 and show that this does not give rise to any contradictions (Exercise). We will give an alternative proof in Section 6.3. □

Another important knot invariant is the *Kauffman invariant*. It is obtained from an invariant of framed knots be rescaling the invariant with the *writhe* of the knot, which describes how often each connected component of the knot twists around itself.

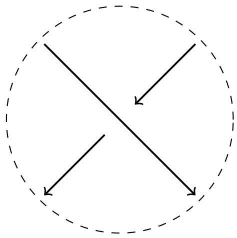
**Definition 6.1.8:** The **writhe** $w(K)$ of a knot $K$ is the sum over all crossing points $p$ in the knot diagram for $K$ over the signs of the crossing

$$w(K) = \sum_{p \in K \cap K} \text{sgn}(p),$$

where $K$ is given an arbitrary orientation and the sign $\text{sgn}(p)$ of a crossing point $p$ is



positive crossing point $p$
lower strand crosses from right to left
$\text{sgn}(p) = 1$



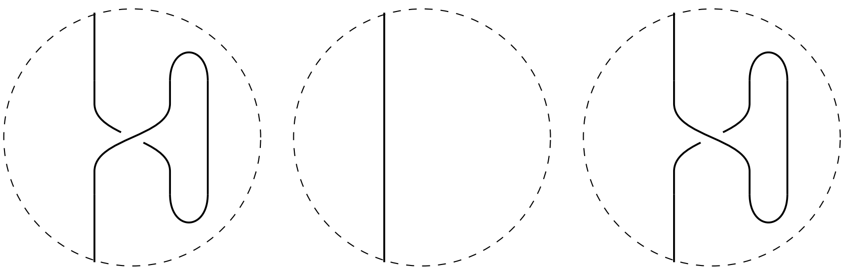
negative crossing point $p$
lower strand crosses from left to right
$\text{sgn}(p) = -1$

The **writhe** of a link is the sum of the writhes of its connected components.

Note that in the definition of the writhe, only the the *crossings of a connected component of the link with itself* are taken into account, but not the crossings involving two different connected components. This implies that the writhe does not depend on the orientation of a link. Reversing the orientation of a connected component $K \subset L$ reverses the orientation of both strands in each crossing point and hence does not change the sign of the crossing. It also follows directly from the definition that the writhe is invariant under the Reidemeister moves RM2 and RM3, since the Reidemeister move RM2 for one connected component of a link creates two additional crossings with opposite sign and the Reidemeister move RM3 does not change the number of crossings. It is also invariant under the Reidemeister move RM1', which creates or removes two crossings with opposite signs, but not under the Reidemeister move RM1, since which creates or removes a crossing point with sign 1 or -1 and hence changes the writhe by ±1. Hence, the writhe is an *invariant of framed links* but not a link invariant.

**Theorem 6.1.9:** There is a unique link invariant $P$ with values in $\mathbb{Z}[z, z^{-1}, a, a^{-1}]$, the **rescaled Kauffman polynomial**, that is given as $P(L) = a^{-w(L)}K(L)$, where $w(L)$ is the writhe of the oriented link $L$ and $K$ the **Kauffman polynomial** of $L$ defined by the following conditions:

1. It takes the value 1 on the **unknot**, whose diagram is a circle in the plane: $K(O) = 1$.
2. If the diagrams of two links $L, L' \in \mathbb{R}^3$ are related by the Reidemeister moves RM2, RM3 and orientation preserving diffeomorphisms $f : \mathbb{R}^2 \to \mathbb{R}^2$, then $K(L) = K(L')$.
3. If the diagrams of links $L'_+, L'_0, L'_-$ are locally related by

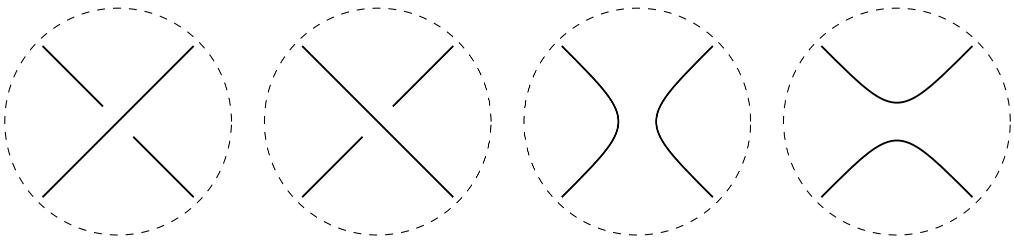


$L'_+$ $L'_0$ $L'_-$

while the rest of their diagrams coincide, then

$$a^{-1} \cdot K(L'_+) = K(L'_0) = a \cdot K(L'_-).$$

4. If the diagrams of the links $L_+, L_-, L_0, L_\infty$ are locally related by



$L_+$ $L_-$ $L_0$ $L_\infty$

while the rest of their diagrams coincide, then

$$K(L_+) + K(L_-) = z \cdot K(L_0) + z \cdot K(L_\infty).$$

**Proof:**
It is clear that the Kauffman polynomial and the rescaled Kauffman polynomial are defined uniquely by these conditions, since any link $L$ can be transformed into an unknot by removing twists as in 2. and transforming overcrossings into undercrossings and vice versa as in 4.

To show that the Kauffman polynomial and the rescaled Kauffman polynomial are well-defined and invariants of framed links, it is sufficient to show that the relations in 2. and in 4. are compatible with the Reidemeister relations RM1', RM2 and RM3. For the Reidemeister relation RM1', this follows directly from 2. For the Reidemeister relations RM2 and RM3, it follows by applying the relations in 2. and 4. to the diagrams on the left and right in the Reidemeister relations RM2 and RM3 and to show that the resulting polynomials are indeed equal. (Exercise). □

Note that the Kauffman polynomial $K(L)$ and the rescaled Kauffman polynomial $P(L)$ do not depend on the orientation of $L$ and are both invariant under the Reidemeister moves RM2 and RM3. However, the Kauffman polynomial is invariant only under the Reidemeister move RM1' and not under RM1 and hence an *invariant of framed links*, while the polynomial $P$ is invariant under the Reidemeister moves RM1 and hence a *link invariant*. More generally, it is always possible to obtain a link invariant from an invariant of framed links by rescaling with an appropriate function of the writhe.

Two other famous link invariants that can be viewed as special cases of the HOMFLY polynomial and the rescaled Kauffman polynomial are the following:

**Remark 6.1.10:**

1. The **Alexander polynomial** of an oriented link $L$ is the polynomial in $\mathbb{Z}[t]$ given by

$$A(L)(t) = H(L)(1, t^{1/2} - t^{-1/2})$$

2. The **Jones polynomial** of an oriented link $L$ is the polynomial in $\mathbb{Z}[t^{1/2}, t^{-1/2}]$ given by

$$J(L)(t) = H(L)(t^{-1}, t^{1/2} - t^{-1/2}) = -P(L)(-t^{-1/4} - t^{1/4}, t^{-3/4})$$

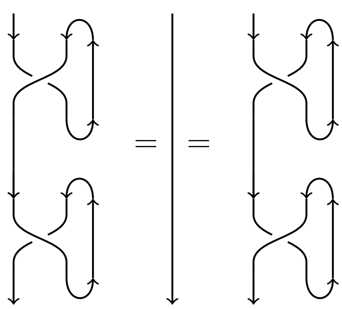
The aim is now to understand invariants of (framed, oriented) links from a representational perspective, as this was achieved for braids in Section 4.2. For this we introduce the category $\mathcal{T}$ of *ribbon tangles* that resembles the braid category $\mathcal{B}$ but takes into account orientation and contains additional morphisms that allow one to close oriented braids to oriented framed links. Just as the braid category, the category $\mathcal{T}$ is a strict monoidal category that is presented in terms of generating morphisms and defining relations given by diagrams.

**Definition 6.1.11:** The **category $\mathcal{T}$ of ribbon tangles** is the strict monoidal category with finite sequences $(\epsilon_1, ..., \epsilon_n)$ in $\mathbb{Z}/2\mathbb{Z}$ as objects, six generating morphisms $\cup, \cup', \cap, \cap', X, X^{-1}$

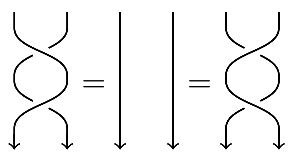
| $\cup$ : | $\cap$ : | $\cup'$ : | $\cap'$ : | $X$ : | $X^{-1}$ : |
|---|---|---|---|---|---|
| $-\otimes+ \to \emptyset$ | $\emptyset \to +\otimes-$ | $+\otimes- \to \emptyset$ | $\emptyset \to -\otimes+$ | $+\otimes+ \to +\otimes+$ | $+\otimes+ \to +\otimes+$ |

and the following relations:

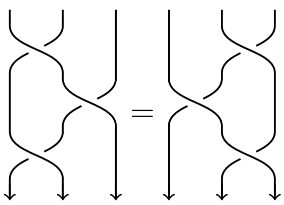
- 1. **RM1':**



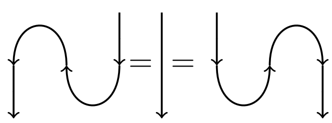
2. **RM2**

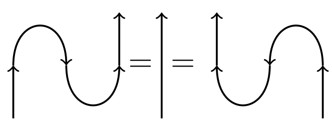


3. **RM3**

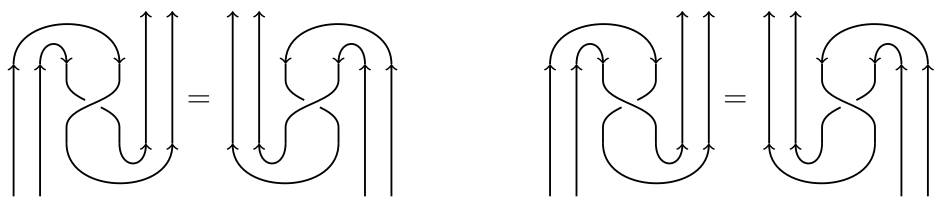


- 4. **Snake identities:**

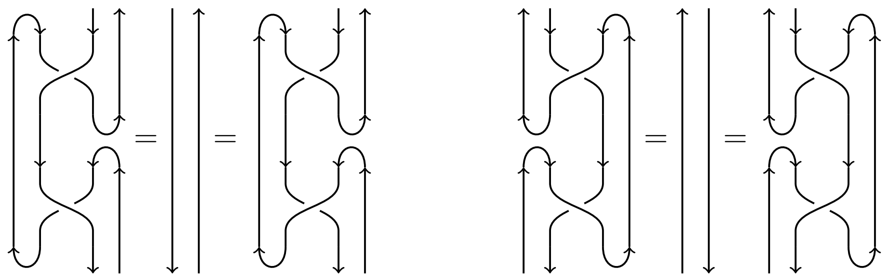




- 5. **Snaked braiding:**



- 6. **Modified RM2:**



The identity morphisms $1_+ : + \to +$ and $1_- : - \to -$ are denoted

$1_+ =$  $\qquad 1_- =$ 

A morphism $f : (\epsilon_1, ..., \epsilon_m) \to (\epsilon'_1, ..., \epsilon'_n)$ in $\mathcal{T}$ is called an $(m, n)$**-ribbon tangle**.

**Remark 6.1.12:** There is an analogous category $\mathcal{T}'$, the **tangle category**, presented by the same objects and generating morphisms as $\mathcal{T}$, with the defining relations 2-6 but with an oriented version of the Reidemeister move RM1 instead of RM1'.

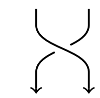
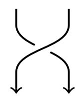
Clearly, the category $\mathcal{T}$ of ribbon tangles contains morphisms $X^{\pm 1}$ whose diagrams and defining relations resemble the ones for the morphisms $\sigma_i$ in the braid category $\mathcal{B}$. The only difference is that the lines in the morphisms $X^{\pm 1}$ are oriented. This suggests that $\mathcal{T}$ is a *braided* monoidal category with a braiding similar to the one in $\mathcal{B}$ defined in the proof of Theorem 4.2.5.

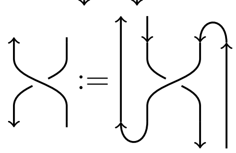
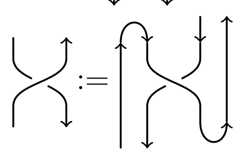
**Proposition 6.1.13:** The category $\mathcal{T}$ of ribbon tangles is a strict braided monoidal category.

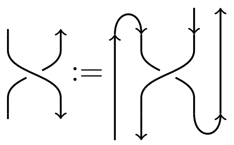
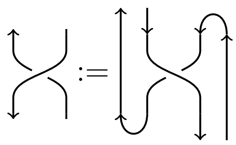
**Proof:**
By definition, the category $\mathcal{T}$ is a strict monoidal category. It remains to show that it is braided. For this, note that a braiding in $\mathcal{T}$ is defined uniquely by its component morphisms $c_{\epsilon,\epsilon'} : (\epsilon, \epsilon') \to (\epsilon', \epsilon)$ for $\epsilon, \epsilon' \in \{\pm\}$ since every object $(\epsilon_1, ..., \epsilon_n) = \epsilon_1 \otimes ... \otimes \epsilon_n$ in $\mathcal{T}$ is a multiple tensor product of the objects $\pm$, and the braiding in a strict monoidal category satisfies

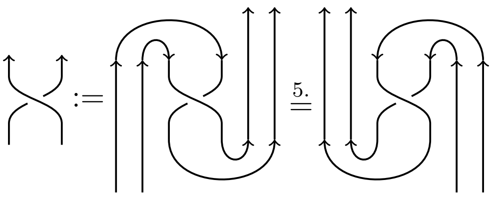
$$c_{e,U} = c_{U,e} = 1_U \qquad c_{U\otimes V,W} = (c_{U,W} \otimes 1_V) \circ (1_U \otimes c_{V,W}) \qquad c_{U,V\otimes W} = (1_V \otimes c_{U,W}) \circ (c_{U,V} \otimes 1_W)$$

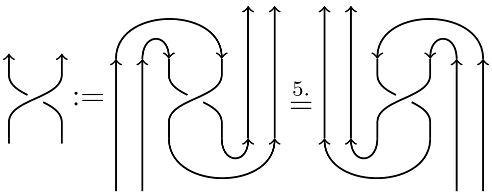
for all objects $U, V, W$ by the hexagon identity, see Remark 4.2.2. We define

$c_{+,+} := X =$  $\qquad c^{-1}_{+,+} := X^{-1} =$ 

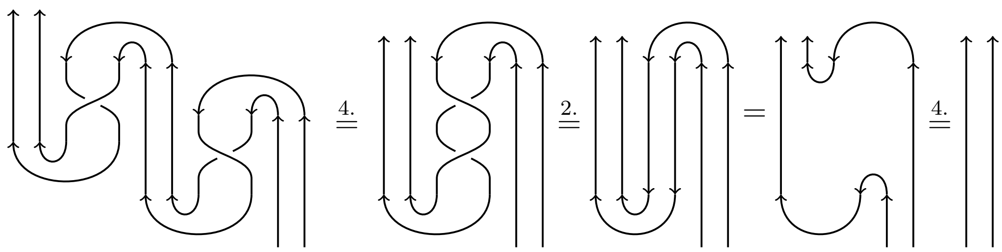
$c_{-,+} :=$  $\qquad c^{-1}_{-,+} :=$ 

$c_{+,-} :=$  $c^{-1}_{+,-} :=$ 

$c_{-,-} :=$ 

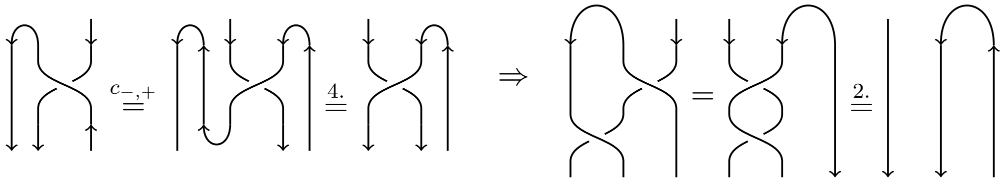
$c^{-1}_{-,-} :=$ 

1. We show that the braiding is invertible: The identities $c^{\mp 1}_{+,+} \circ c^{\pm 1}_{+,+} = 1_{+\otimes +}$ follow directly from the defining relation 2, and the identities $c^{\mp 1}_{+,-} \circ c^{\pm 1}_{+,-} = 1_{\pm\otimes\mp}$ and $c^{\mp 1}_{-,+} \circ c^{\pm 1}_{-,+} = 1_{\mp\otimes\pm}$ follow from the defining relation 6. The identity $c_{-,-} \circ c^{-1}_{-,-} = 1_{-\otimes -}$ is then obtained as follows

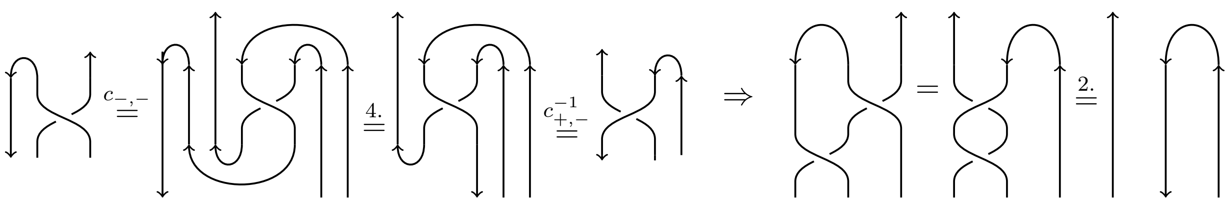


and the identity $c^{-1}_{-,-} \circ c_{-,-} = 1_{-\otimes -}$ follows analogously. This shows that $c_{\epsilon,\epsilon'} : \epsilon\otimes\epsilon' \to \epsilon'\otimes\epsilon$ for $\epsilon, \epsilon' \in \{\pm\}$ are isomorphisms.

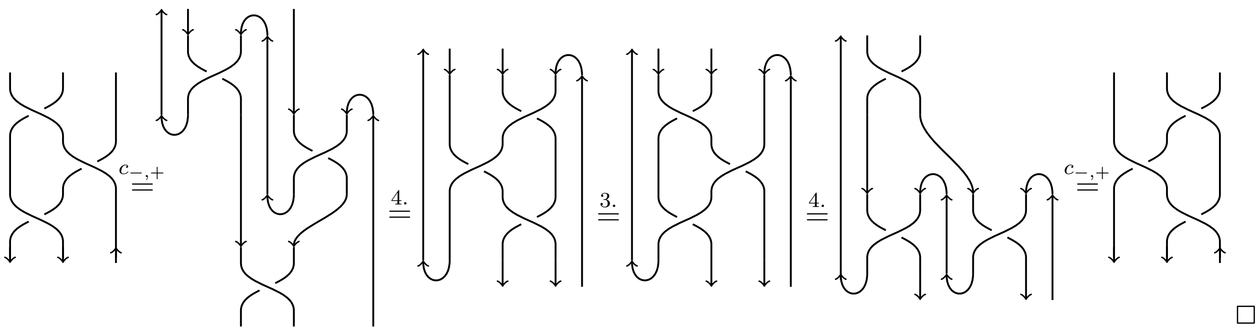
2. It remains to prove the naturality of the braiding. As the morphisms $\cap$, $\cap$, $\cup'$, $\cap'$ and $X^{\pm 1}$ generate $\mathcal{T}$, it is sufficient to prove naturality for $\cap$, $\cap$, $\cup'$, $\cap'$ and $X$ . To prove the naturality for $\cap$ we note that the definition of $c_{-,+}$ and the snake identity imply



This proves the naturality for $\cap$ for a line that is oriented downwards and crosses under the strands $\cap$. The corresponding identity for a line that is oriented upwards and crosses under the strands of $\cap$ follows in a similar way from the definition of $c_{-,-}$ and $c^{-1}_{+,-}$, which imply



The corresponding identities where the line crosses over the strands of $\cap$ and the identities for $\cap'$, $\cup$ and $\cup'$ follow analogously. This proves the naturality of the braiding with respect to the morphisms $\cap, \cup, \cap', \cup'$. It remains to prove the naturality of the braiding with respect to the morphism $X$. For a line that is oriented downwards and crosses over or under the strands of $X$ this follows directly from the defining relation 3. For a line that is oriented upwards it follows from the definition of $c_{-,+}$, the snake identity 4. and the Reidemeister move 3:



From the definition of the category of ribbon tangles, it is apparent that one can represent oriented link diagrams by ribbon tangles. By applying an orientation preserving diffeomorphism $f:\mathbb{R}^2\to\mathbb{R}^2$ to an oriented link diagram, one can always achieve that the intersection of the link diagram with any horizontal line is a a discrete subset of the line. Such an oriented link diagram defines a unique $(0,0)$-ribbon tangle, with morphisms $c_{\epsilon,\epsilon'}$ at each crossing, as defined above. The defining relations of the category $\mathcal{T}$ are chosen in precisely such a way that oriented link diagrams that are related by orientation preserving diffeomorphisms $f:\mathbb{R}^2\to\mathbb{R}^2$ and the Reidemeister moves RM1', RM2 and RM3 define the same ribbon tangles. Hence, equivalence classes of framed oriented links are in bijection with $(0,0)$-ribbon tangles.

If $\mathcal{C}$ is a strict braided monoidal category, then a braided monoidal functor $F:\mathcal{T}\to\mathcal{C}$ assigns to a $(0,0)$-ribbon tangle $t:\emptyset\to\emptyset$ an endomorphism $F(t):e\to e$ in the commutative monoid $\mathrm{End}_{\mathcal{C}}(e)$, and this morphism is an invariant of framed links due to the defining relations in $\mathcal{T}$. Hence, to define an invariant of framed links, we require a functor $F:\mathcal{T}\to\mathcal{C}$ into a braided monoidal category $\mathcal{C}$. Such a functor can only exist if $\mathcal{C}$ contains additional structure that corresponds to orientation reversal of lines and to the morphisms $\cap, \cup, \cap', \cup'$. In the next section, we will investigate systematically which additional structure in a braided monoidal category guarantees the existence of such a functor and which additional structure on a quasitriangular Hopf algebra $H$ equips its representation category $H$-Mod with this structure.

## 6.2 Dualities and traces

In this section, we investigate the additional structure in a monoidal category that corresponds to the orientation of lines and to the morphisms $\cap, \cup, \cap'$ and $\cup'$ in the category $\mathcal{T}$ of ribbon tangles. For this, note that if $F:\mathcal{T}\to\mathcal{C}$ is a monoidal functor with $V=F(+)$, then there is another object $V^*:=F(-)$ in $\mathcal{C}$ and morphisms $F(\cap):e\to V\otimes V^*$ and $F(\cup):V^*\otimes V\to e$ as well as morphisms $F(\cap'):e\to V^*\otimes V$ and $F(\cup'):V^*\otimes V\to e$ and $F(\cap'):e\to V\otimes V^*$ such that both pairs of morphisms satisfy the snake identity. It is also clear that for $\mathcal{C}=\mathrm{Vect}_{\mathbb{F}}^{fin}$ we can take for $V^*$ the dual vector space and set $F(\cap)=\mathrm{coev}_V^R:\mathbb{F}\to V\otimes V^*$, $F(\cup)=\mathrm{ev}_V^R:V^*\otimes V\to\mathbb{F}$, $F(\cap')=\mathrm{coev}_V^L:\mathbb{F}\to V^*\otimes V$, $F(\cup')=\mathrm{ev}_V^L:V\otimes V^*\to\mathbb{F}$. For a general monoidal category, one should consider the pairs $(\cap,\cup)$ and $(\cap',\cup')$ separately, and require that the corresponding objects and morphisms exist for all objects $V$ in $\mathcal{C}$.

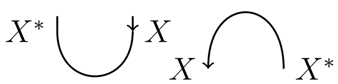
**Definition 6.2.1:** Let $\mathcal{C}$ be a tensor category.

1. An object $X$ of $\mathcal{C}$ is called **right dualisable** if there is an object $X^*$, the **right dual** of $X$, and morphisms
$$\mathrm{ev}_X^R : X^*\otimes X \to e \qquad \mathrm{coev}_X^R : e \to X\otimes X^*$$
such that the following diagrams commute
$$\begin{array}{ccccc} X & \xrightarrow{l_X^{-1}} & e\otimes X & \xrightarrow{\mathrm{coev}_X^R\otimes 1_X} & (X\otimes X^*)\otimes X \\ {\scriptstyle 1_X}\downarrow & & & & \downarrow{\scriptstyle a_{X,X^*,X}} \\ X & \xleftarrow[r_X]{} & X\otimes e & \xleftarrow[1_X\otimes \mathrm{ev}_X^R]{} & X\otimes(X^*\otimes X) \end{array} \qquad \begin{array}{ccccc} X^* & \xrightarrow{r_{X^*}^{-1}} & X^*\otimes e & \xrightarrow{1_{X^*}\otimes \mathrm{coev}_X^R} & X^*\otimes(X\otimes X^*) \\ {\scriptstyle 1_{X^*}}\downarrow & & & & \downarrow{\scriptstyle a^{-1}_{X^*,X,X^*}} \\ X^* & \xleftarrow[l_{X^*}]{} & e\otimes X^* & \xleftarrow[\mathrm{ev}_X^R\otimes 1_{X^*}]{} & (X^*\otimes X)\otimes X^*. \end{array} \qquad (48)$$

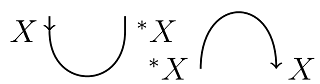
2. An object $X$ of $\mathcal{C}$ is called **left dualisable** if there is an object ${}^*X$, the **left dual** of $X$, and morphisms
$$\mathrm{ev}_X^L : X\otimes {}^*X \to e \qquad \mathrm{coev}_X^L : e \to {}^*X\otimes X$$
such that the following diagrams commute
$$\begin{array}{ccccc} X & \xrightarrow{r_X^{-1}} & X\otimes e & \xrightarrow{1_X\otimes \mathrm{coev}_X^L} & X\otimes({}^*X\otimes X) \\ {\scriptstyle 1_X}\downarrow & & & & \downarrow{\scriptstyle a^{-1}_{X,{}^*X,X}} \\ X & \xleftarrow[l_X]{} & e\otimes X & \xleftarrow[\mathrm{ev}_X^L\otimes 1_X]{} & (X\otimes {}^*X)\otimes X \end{array} \qquad \begin{array}{ccccc} {}^*X & \xrightarrow{l_{{}^*X}^{-1}} & e\otimes {}^*X & \xrightarrow{\mathrm{coev}_X^L\otimes 1_{{}^*X}} & ({}^*X\otimes X)\otimes {}^*X \\ {\scriptstyle 1_{{}^*X}}\downarrow & & & & \downarrow{\scriptstyle a_{{}^*X,X,{}^*X}} \\ {}^*X & \xleftarrow[r_{{}^*X}]{} & {}^*X\otimes e & \xleftarrow[1_{{}^*X}\otimes \mathrm{ev}_X^L]{} & {}^*X\otimes(X\otimes {}^*X). \end{array} \qquad (49)$$

3. The category $\mathcal{C}$ is called **right rigid** if every object in $\mathcal{C}$ is right dualisable, **left rigid** if every object of $\mathcal{C}$ is left dualisable, and **rigid** if it is both right and left rigid.

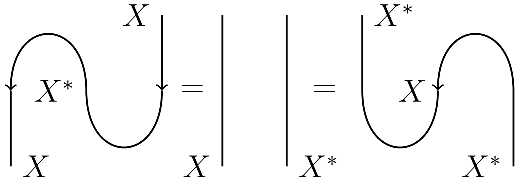
Inspired by the diagrams for the morphisms $\cap, \cup, \cap', \cup'$ in the category $\mathcal{T}$ of ribbon tangles, we represent right and left (co)evaluation for right and left dualisable objects in a monoidal category by diagrams. The commuting diagrams (48) and (49) in Definition 6.2.1 then generalise the snake identity in $\mathcal{T}$ from Definition 6.1.11.



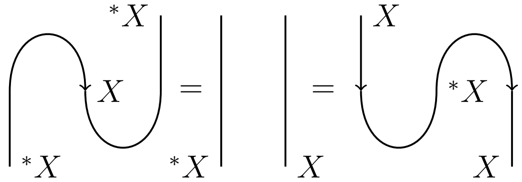

right evaluation
and right coevaluation




left evaluation
and left coevaluation



**snake identities**:
commuting diagrams (48) for the
right evaluation and coevaluation



**snake identities**:
commuting diagrams (49) for the
left evaluation and coevaluation.

**Example 6.2.2:**

1. The category $\mathrm{Vect}_{\mathbb{F}}^{fin}$ of finite-dimensional vector spaces over $\mathbb{F}$ is rigid with the dual vector space $V^* = {}^*V$ as the left and right dual and the left and right evaluation and coevaluation maps

$$\begin{aligned} &\mathrm{coev}_V^R : \mathbb{F} \to V \otimes V^*, \quad \lambda \mapsto \lambda \Sigma_{i=1}^n v_i \otimes \alpha^i && \mathrm{ev}_V^R : V^* \otimes V \to \mathbb{F}, \quad \alpha \otimes v \mapsto \alpha(v) \\ &\mathrm{coev}_V^L : \mathbb{F} \to V^* \otimes V, \quad \lambda \mapsto \lambda \Sigma_{i=1}^n \alpha^i \otimes v_i && \mathrm{ev}_V^L : V \otimes V^* \to \mathbb{F}, \quad v \otimes \alpha \mapsto \alpha(v), \end{aligned} \tag{50}$$

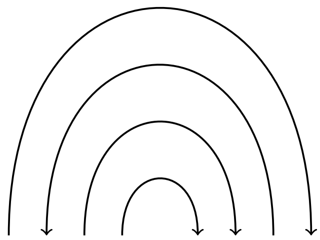
where $(v_1, ..., v_n)$ is a basis of $V$ and $(\alpha^1, ..., \alpha^n)$ the dual basis of $V^*$. To show that $\mathrm{Vect}_{\mathbb{F}}^{fin}$ is right rigid, we verify the snake identities

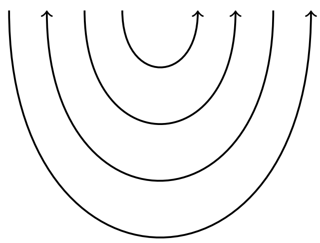
$$r_V \circ (\mathrm{id}_V \otimes \mathrm{ev}_V^R) \circ (\mathrm{coev}_V^R \otimes \mathrm{id}_V) \circ l_V^{-1}(v) = r_V \circ (\mathrm{id}_V \otimes \mathrm{ev}_V^R)(\Sigma_{i=1}^n v_i \otimes \alpha^i \otimes v) = \Sigma_{i=1}^n \alpha^i(v) v_i = v$$
$$l_V \circ (\mathrm{ev}_V^R \otimes \mathrm{id}_{V^*}) \circ (\mathrm{id}_{V^*} \otimes \mathrm{coev}_V^R) \circ r_V^{-1}(\beta) = l_V \circ (\mathrm{ev}_V^R \otimes \mathrm{id}_{V^*})(\Sigma_{i=1}^n \beta \otimes v_i \otimes \alpha^i) = \Sigma_{i=1}^n \beta(v_i) \alpha^i = \beta$$

for all $v \in V$, $\beta \in V^*$. A similar computation shows that $\mathrm{Vect}_{\mathbb{F}}^{fin}$ is left rigid.

2. The category $\mathcal{T}$ of ribbon tangles is rigid with $(\epsilon_1, ..., \epsilon_n)^* = {}^*(\epsilon_1, ..., \epsilon_n) = (-\epsilon_n, ..., -\epsilon_1)$ and evaluation and coevaluation maps given as composites of the morphisms $\cap, \cup, \cap', \cup'$

$\mathrm{coev}^R_{(-,+,-,-)} = \mathrm{coev}^L_{(+,+,-,+)}$ $\qquad$ $\mathrm{ev}^R_{(-,+,-,-)} = \mathrm{ev}^L_{(+,+,-,+)}$





Although in the two examples considered so far left dual objects are right dual and vice versa, it is necessary to distinguish left and right dual objects in general. The main example that shows the need for this distinction is the category $H$-Mod$^{fin}$ of finite-dimensional representations for a Hopf algebra $H$. As discussed in Section 2.2, the properties of the antipode ensure that the *right* evaluation and coevaluation from (50) are homomorphisms of representations if the dual $V^*$ of a finite-dimensional $H$-module $V$ is equipped with the $H$-module structure $\rhd : H \otimes V^* \to V^*$ with $h \rhd \alpha(v) = \alpha(S(h) \rhd v)$, while the *left* evaluation and coevaluation from (50) are homomorphisms of representations if $S$ is invertible and the vector space $V^*$ is equipped with the $H$-module structure $\rhd : H \otimes V^* \to V^*$ with $h \rhd \alpha(v) = \alpha(S^{-1}(h) \rhd v)$.

**Theorem 6.2.3:** Let $H$ be a Hopf algebra over $\mathbb{F}$. Then the tensor category $H$-Mod$_{fin}$ of finite-dimensional $H$-modules is right rigid. If the antipode of $H$ is invertible, then the category $H$-Mod$_{fin}$ is rigid.

**Proof:**
It was already shown at the beginning of Section 2.2 that for every finite-dimensional module $(V, \rhd)$ over a Hopf algebra $H$, the *right* evaluation and coevaluation maps $\mathrm{ev}_V^R : V^* \otimes V \to \mathbb{F}$, $\alpha \otimes v \mapsto \alpha(v)$ and $\mathrm{coev}_V^R : \mathbb{F} \to V \otimes V^*$, $\lambda \mapsto \Sigma_{i=1}^n v_i \otimes \alpha^i$, where $(v_1, ..., v_n)$ is an ordered basis of $V$ and $(\alpha^1, ..., \alpha^n)$ the dual basis of $V^*$, are $H$-module morphisms when $V^*$ is equipped with the

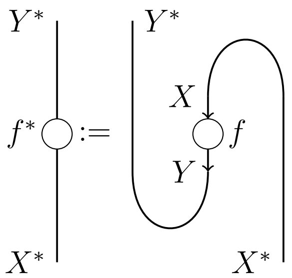
$H$-module structure $\triangleright : H\otimes V^* \to V^*$, $h \triangleright_{V^*} \alpha = \alpha \circ S(h)$. Together with Lemma 2.2.6, 1. the discussion at the beginning of Section 2.2 also implies that in the case of a Hopf algebra with an invertible antipode the *left* evaluation and coevaluation maps $\mathrm{ev}_V^L : V\otimes V^* \to \mathbb{F}$, $v\otimes\alpha \mapsto \alpha(v)$ and $\mathrm{coev}_V^L : \mathbb{F} \to V^*\otimes V$, $\lambda \mapsto \Sigma_{i=1}^n \alpha^i \otimes v_i$ are $H$-module morphisms when $V^*$ is equipped with the $H$-module structure $\triangleright : H\otimes V^* \to V^*$, $h \triangleright_{V^*} \alpha = \alpha \circ S^{-1}(h)$. As the right and left evaluation and coevaluation satisfy the snake identities by Example 6.2.2, 1. the claim follows. □

There is also a more conceptual interpretation of right and left duals in a right or left rigid monoidal category, which makes it clear that such duals can be viewed as a symmetry. The appropriate notion of a symmetry for a monoidal category $\mathcal{C}$ is that of a monoidal functor $* : \mathcal{C} \to \mathcal{C}^{(op)}$, which may reverse the composition of morphisms or the tensor product. As the right and left duals in a left or right rigid monoidal category correspond to a 180° rotation of the associated diagrams in the plane, the associated monoidal functor reverses both, the composition of morphisms and the tensor product in $\mathcal{C}$.

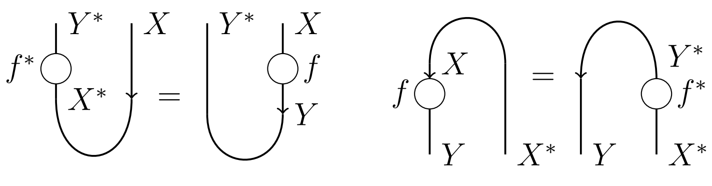
**Theorem 6.2.4:** Let $\mathcal{C}$ be a right rigid monoidal category. Then the right duals define a monoidal functor $* : \mathcal{C} \to \mathcal{C}^{op,op}$ where $\mathcal{C}^{op,op}$ is the category with the opposite composition and the opposite tensor product.

**Proof:**
We define $*$ on morphisms by setting

$$f^* := l_X \circ (\mathrm{ev}_Y^R \otimes 1_{X^*}) \circ ((1_{Y^*}\otimes f)\otimes 1_{X^*}) \circ a^{-1}_{Y^*,X,X^*} \circ (1_{Y^*}\otimes \mathrm{coev}_X^R) \circ r^{-1}_{Y^*} \qquad (51)$$

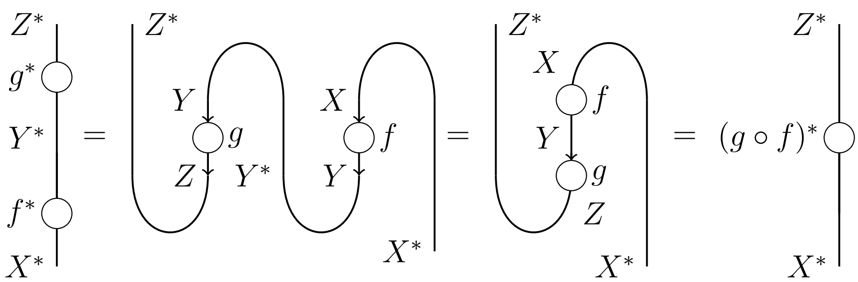



With the snake identity (48), this implies



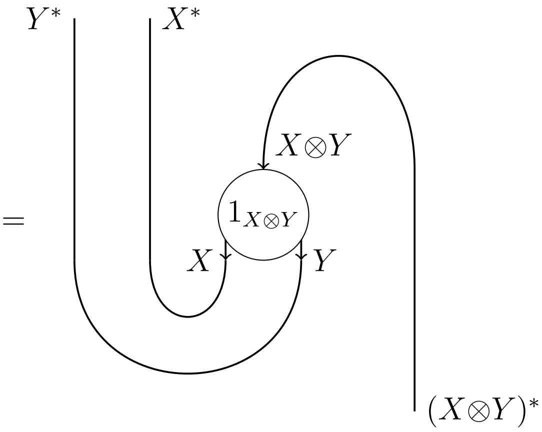

(52)

for all objects $X$, $Y$ and morphisms $f : X \to Y$. The snake identity (48) also implies $1_X^* = 1_X$ for all objects $X$ and $(g\circ f)^* = f^* \circ g^*$ for all morphisms $g : Y \to Z$:



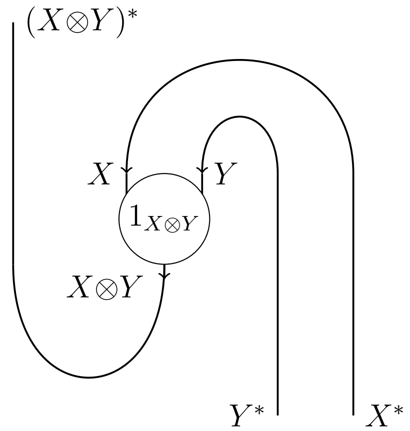
This shows that $* : \mathcal{C} \to \mathcal{C}^{op}$ is a functor. To prove that it is a *monoidal* functor, we need to construct an isomorphism $\phi^e : e \to e^*$ and a natural isomorphism $\phi^\otimes : \otimes^{op}(* \times *) \to *\otimes$ that satisfy the compatibility conditions with the associator and unit constraints from Definition 4.1.11. For this, we define $\phi^e := l_{e^*} \circ \mathrm{coev}^R_e : e \to e^*$ and for all objects $X, Y$

$\phi^\otimes_{X,Y} =$




By combining the commuting diagram in Definition 6.2.1, 1. with Lemma 4.1.3, one finds that $\phi_e$ is invertible, with inverse $\phi^{e\,-1} = r_e \circ (1_e \otimes \mathrm{ev}^R_e) \circ l^{-1}_{e^* \otimes e} \circ r_e^{-1}$ (Exercise). The snake identity implies that $\phi_{X,Y}$ in invertible with inverse

$\phi^{\otimes\,-1}_{X,Y} =$




The naturality of $\phi^\otimes$ follows from the identity (52), together with the fact that $\otimes$ is a functor and the naturality of the associator and the unit constraints. The compatibility condition from Definition 4.1.11 with the associator follows directly from the definition of $\phi^\otimes$ and $\phi^e$ and the coherence theorem, and the same holds for the compatibility condition with the unit constraints. This shows that $*$ is monoidal. □

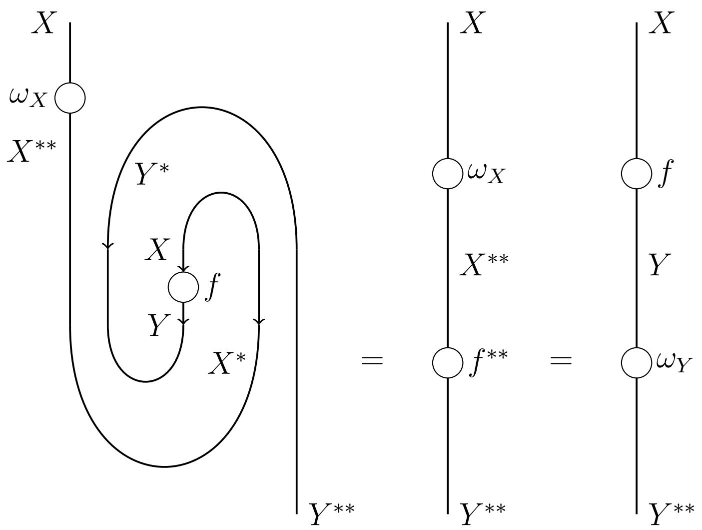
Clearly, there is an analogous definition of a duality functor for a left rigid monoidal category, which is obtained by replacing all right dual objects by left dual objects and right (co)evaluations by left(co)evaluations. It is also clear from the diagrammatic definition of the dual morphisms in (51) that taking multiple right or left duals corresponds to wrapping the lines representing the objects $X$ and $Y$ around the morphism $f : X \to Y$. However, since taking duals should correspond to a 180° rotations of the diagrams in the plane, there should be a way to unwrap morphisms in a consistent way. This amounts to a natural isomorphism from the identity functor $\mathrm{id}_\mathcal{C} : \mathcal{C} \to \mathcal{C}$ to the dual functor $** : \mathcal{C} \to \mathcal{C}$.

**Definition 6.2.5:** Let $\mathcal{C}$ be a right rigid monoidal category. A **pivotal structure** on $\mathcal{C}$ is a monoidal natural isomorphism $\omega : \mathrm{id}_{\mathcal{C}} \to **$. A **pivotal category** is a pair $(\mathcal{C}, \omega)$ of a right rigid monoidal category $\mathcal{C}$ and a pivotal structure $\omega$.

**Example 6.2.6:**

1. The rigid monoidal category $\mathrm{Vect}_{\mathbb{F}}^{fin}$ is pivotal with the pivot $\omega$ given by the canonical isomorphisms $\omega_V = \mathrm{can}_V : V \to V^{**}$, $v \mapsto f_v$ with $f_v(\alpha) = \alpha(v)$ for $\alpha \in V^*$, $v \in V$.
2. The category $\mathcal{T}$ of ribbon tangles is pivotal with the identity natural transformation on the functor $\mathrm{id}_{\mathcal{T}}$ as the pivot (Exercise).

A indicated by the discussion above, the pivot of a pivotal category is represented diagrammatically as a morphism that unwraps the double dual of each morphism in $\mathcal{C}$. The naturality of the pivot states that morphisms $\omega_X$ can be moved through any morphism $f : X \to Y$, provided that suitable double duals are taken and the argument of the pivot is adjusted:



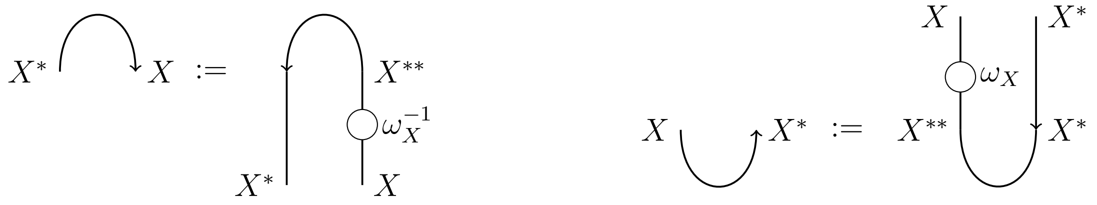
(53)

It turns out that the pivot of a pivotal category guarantees that right dual objects are also left dual objects and vice versa, just as in the pivotal category finite-dimensional vector spaces. More specifically, one can use the pivot to define the left evaluation and coevaluation for an object $X$ by composing it with the right evaluation and coevaluation of its right dual $X^*$.

**Proposition 6.2.7:** Every pivotal category is left rigid, and right dual objects and left dual objects in a pivotal category coincide.

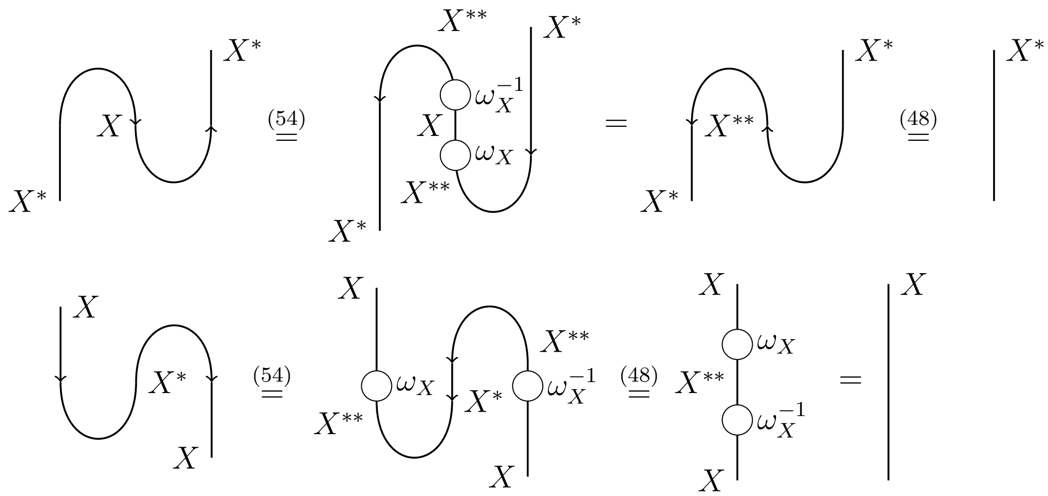
**Proof:**
Let $\mathcal{C}$ be a pivotal category. Define ${}^*X := X^*$ for all objects $X$ of $\mathcal{C}$ and define the morphisms $\mathrm{ev}_X^L : X \otimes {}^*X \to e$ and $\mathrm{coev}_X^L : e \to {}^*X \otimes X$ in terms of the morphisms $\mathrm{ev}_X^R : X^* \otimes X \to e$,

$\mathrm{coev}_X^R : e \to X \otimes X^*$ and the pivot as



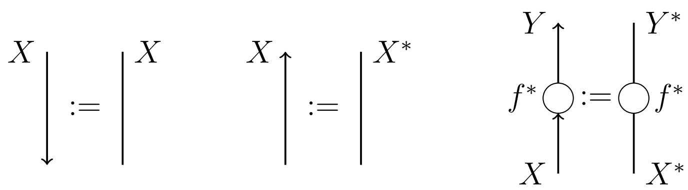

(54)

To show that the snake identities are satisfied for this left evaluation and coevaluation morphisms, we compute graphically with the definition of the left evaluation and coevaluation and the snake identities for the right evaluation and coevaluation:




□

Proposition 6.2.7 allows one to simplify the graphical calculus for a pivotal category. As left and right dual objects in a pivotal category coincide, we can denote the object $X^* = {}^*X$ diagrammatically by an arrow labelled with $X$ that points upward and the object $X$ by an arrow labelled by $X$ that points downwards. The left evaluation can be represented by the diagrams in (54) and all labels corresponding to left and right duals can be omitted.

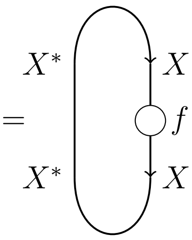


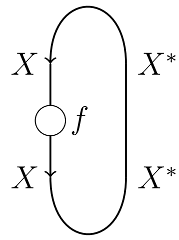

Another fundamental benefit of a pivotal structure is that any pivotal category is equipped with the notion of a left trace and right trace that generalises the notion of a trace in the category $\mathrm{Vect}_{\mathbb{F}}^{fin}$ of finite-dimensional vector spaces over $\mathbb{F}$. Just as the trace in the category $\mathrm{Vect}_{\mathbb{F}}^{fin}$ assigns to each endomorphism $f : V \to V$ a number, i. e. an endomorphism of the unit object $\mathbb{F}$, the left and right trace in a pivotal category $(\mathcal{C}, \omega)$ assign to each endomorphism $f : X \to X$ an endomorphism of the unit object $e$, i. e. an element of the commutative monoid $\mathrm{End}_{\mathcal{C}}(e)$. In particular, this yields a generalised notion of dimension for each object $X$, namely the left and right traces of the identity morphism $1_X : X \to X$. The only difference is that in a general pivotal category it is not guaranteed that left and right traces coincide. A pivotal category with this property is called *spherical*, because the diagrams for left and right traces can be deformed onto each other if they are drawn on a sphere $S^2$ by pulling the left strand of the left trace behind the sphere and making it reappear on the right.

**Definition 6.2.8:** Let $\mathcal{C}$ be a pivotal category, equipped with the left evaluation and coevaluation from Proposition 6.2.7, $X$ an object in $\mathcal{C}$ and $f : X \to X$ a morphism.

1. The **left** and **right trace** of $f$ are defined as

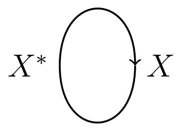
$$\mathrm{tr}_L(f) = \mathrm{ev}^R_X \circ (1_{X^*} \otimes f) \circ \mathrm{coev}^L_X \qquad \mathrm{tr}_R(f) = \mathrm{ev}^L_X \circ (f \otimes 1_{X^*}) \circ \mathrm{coev}^R_X$$

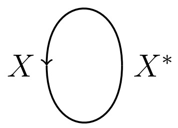
$\mathrm{tr}_L(f) =$ 


$\mathrm{tr}_R(f) =$ 


2. The **left** and **right dimension** of $X$ are defined as

$$\dim_L(X) = \mathrm{tr}_L(1_X) = \mathrm{ev}^R_X \circ \mathrm{coev}^L_X \qquad \dim_R(X) = \mathrm{tr}_R(1_X) = \mathrm{ev}^L_X \circ \mathrm{coev}^R_X$$

$\dim_L(X) =$ 


$\dim_R(X) =$ 

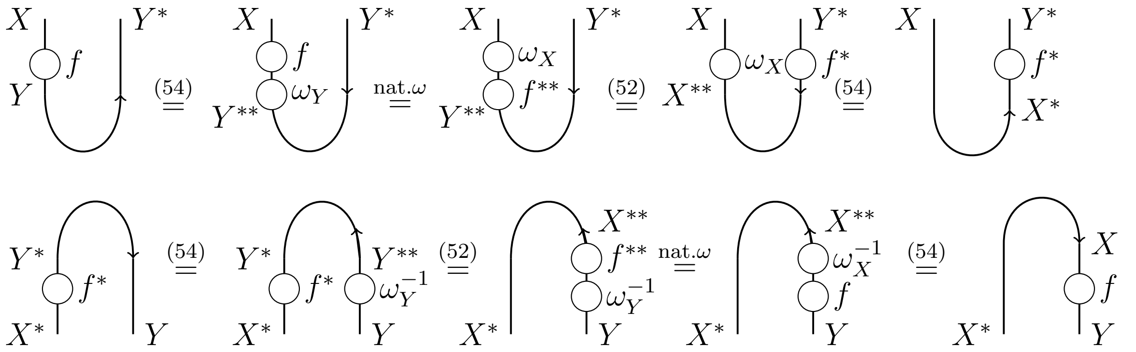

3. The category $\mathcal{C}$ is called **spherical** if $\mathrm{tr}_L(f) = \mathrm{tr}_R(f)$ for all endomorphisms $f$ in $\mathcal{C}$.

The left and right traces in a pivotal category have many properties that are familiar from the traces in $\mathrm{Vect}^{fin}_{\mathbb{F}}$ such as cyclic invariance and compatibility with duality. They are also compatible with tensor products, provided the morphisms satisfy a mild addition assumption. In particular, this implies that the left and right dimensions of objects in $\mathcal{C}$ behave in a way that is very similar to the dimensions of vector spaces.

**Lemma 6.2.9:** Let $\mathcal{C}$ be a pivotal category. The traces in $\mathcal{C}$ have the following properties:

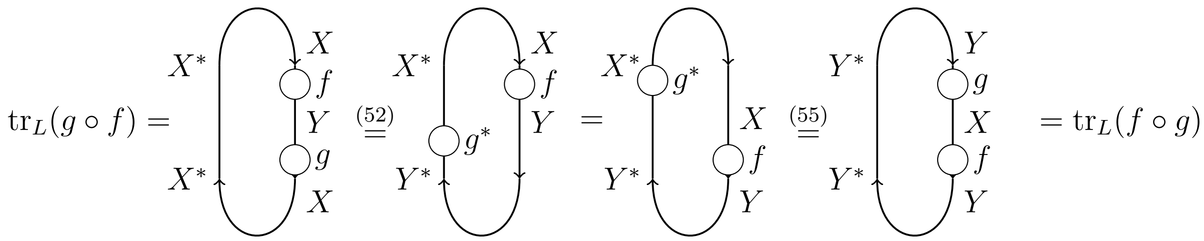
1. **cyclic invariance:** $\mathrm{tr}_{L,R}(g \circ f) = \mathrm{tr}_{L,R}(f \circ g)$ for all morphisms $f : X \to Y$, $g : Y \to X$.
2. **duality:** $\mathrm{tr}_{L,R}(f) = \mathrm{tr}_{R,L}(f^*) = \mathrm{tr}_{L,R}(f^{**})$ for all endomorphisms $f : X \to X$.
3. **compatibility with tensor products:**
If $r_e \circ (1_X \otimes h) \circ r_e^{-1} = l_e \circ (h \otimes 1_X) \circ l_e^{-1}$ for all endomorphisms $h : e \to e$ and objects $X$, then $\mathrm{tr}_{L,R}(f \otimes g) = \mathrm{tr}_{L,R}(g) \cdot \mathrm{tr}_{L,R}(f)$ for all endomorphisms $f : X \to X$, $g : Y \to Y$.

**Proof:**
We prove these identities graphically for the left traces. The proofs for the right traces are analogous. The definition of the left evaluation and coevaluation and the pivot implies:

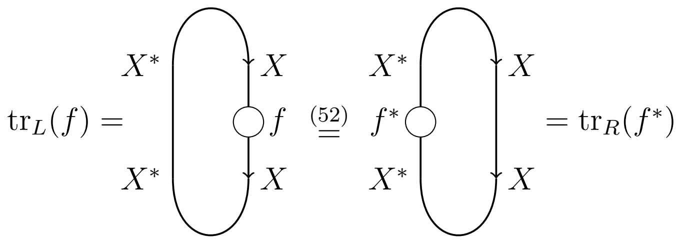


(55)

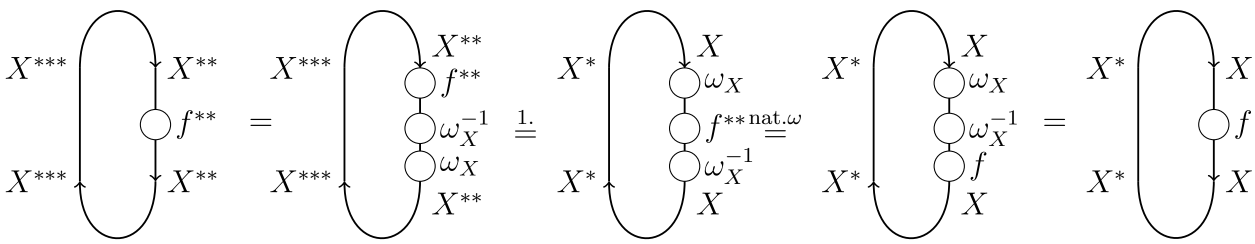
By combining these identities with the corresponding identities for the right evaluation and coevaluation in (52), we obtain:



2. Similarly, we obtain for all endomorphisms $f : X \to X$

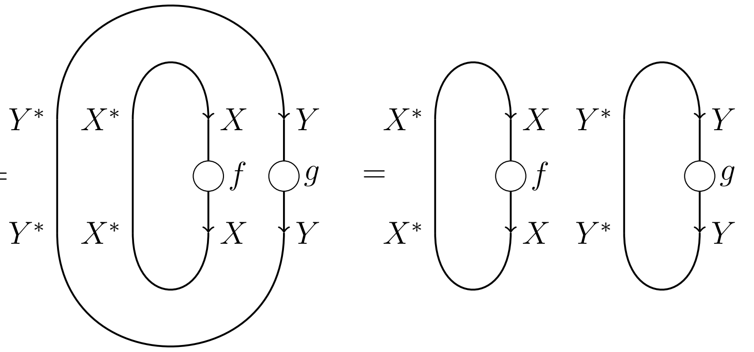


To prove the identity $\mathrm{tr}_L(f) = \mathrm{tr}_L(f^{**})$, we insert $\omega_X^{-1} \circ \omega_X$ in the diagram for $\mathrm{tr}_L(f^{**})$ and use the cyclic invariance of the trace and the properties of the pivot:



3. The condition $r_e \circ (1_X \otimes h) = l_e \circ (h \otimes 1_X)$ for all objects $X$ and endomorphisms $h : e \to e$ implies that we can move $\mathrm{tr}_L(f) : e \to e$ to the left of the endomorphism $1_{Y^*}$ in the picture for $\mathrm{tr}_L(f \otimes g)$, as shown below. As the endomorphisms of $e$ form a commutative monoid by Corollary 4.1.4, the claim follows:

$\mathrm{tr}_L(f\otimes g) =$




$= \mathrm{tr}_L(f) \cdot \mathrm{tr}_L(g)$

□

**Remark 6.2.10:** The condition $r_e \circ (1_X \otimes h) = l_e \circ (h \otimes 1_X)$ for all objects $X$ and endomorphisms $h : e \to e$ holds in the category $H$-Mod$_{fin}$ for any Hopf algebra $H$ and for all braided pivotal categories since $r_X \circ c_{e,X} = l_X \circ c_{X,e} = 1_X$.

**Corollary 6.2.11:** Let $\mathcal{C}$ be a pivotal category. Then:

1. $X \cong Y$ implies $\dim_{L,R}(X) = \dim_{L,R}(Y)$.
2. $\dim_{L,R}(X) = \dim_{R,L}(X^*) = \dim_{L,R}(X^{**})$ for all objects $X$.
3. If $r_e \circ (1_X \otimes h) \circ r_e^{-1} = l_e \circ (h \otimes 1_X) \circ l_e^{-1}$ for all endomorphisms $h : e \to e$ and objects $X$, then $\dim_{L,R}(X \otimes Y) = \dim_{L,R}(X) \cdot \dim_{L,R}(Y)$ for all objects $X, Y$.
4. $\dim_L(e) = \dim_R(e) = 1_e$.

**Proof:**
1. If $X \cong Y$, then there are morphisms $f : X \to Y$ and $g : Y \to X$ with $g \circ f = 1_X$ and $f \circ g = 1_Y$. With the cyclic invariance of the trace one obtains

$$\dim_L(X) = \mathrm{tr}_L(1_X) = \mathrm{tr}_L(g \circ f) = \mathrm{tr}_L(f \circ g) = \mathrm{tr}_L(1_Y) = \dim_L(Y).$$

2. Follows directly from Lemma 6.2.9, 2. by setting $f = 1_X$ and using the identity $1_{X^*} = 1_X^*$ and $1_{X^{**}} = 1_X^{**}$ which follow from the fact that $* : \mathcal{C} \to \mathcal{C}^{op}$ is a functor. Similarly, 3. is obtained from Lemma 6.2.9, 3. by setting $f = 1_X$ and $g = 1_Y$.

4. By Lemma 4.1.3 we have $r_e = l_e$, and the naturality of the unit constraints implies $r_e \circ (1_e \otimes h) \circ r_e^{-1} = h = l_e \circ (h \otimes 1_e) \circ l_e^{-1}$. With the cyclic invariance of the trace from Lemma 6.2.9, 1. this yields

$$\mathrm{tr}_{L,R}(h) \cdot \dim_{L,R}(e) = \mathrm{tr}_{L,R}(h \otimes 1_e) = \mathrm{tr}_{L,R}(r_e^{-1} \circ h \circ r_e) = \mathrm{tr}_{L,R}(h)$$

for all endomorphisms $h : e \to e$ and hence $\dim_{L,R}(e) = 1$. □

We will now investigate which additional structures on a Hopf algebra $H$ are needed in order to make its representation category $H$-Mod$_{fin}$ is pivotal or spherical. It turns out that this the appropriate condition for the former is that the square of the antipode is given by conjugation with a grouplike element and that the latter depends on the way this grouplike element acts on the finite-dimensional representations of $H$.

**Definition 6.2.12:**

1. A **pivotal Hopf algebra** is a Hopf algebra $H$ over $\mathbb{F}$ together with an element $g \in \mathrm{Gr}(H)$, the **pivot**, such that $S^2(h) = g \cdot h \cdot g^{-1}$ for all $h \in H$.

2. A pivotal Hopf algebra $(H, \omega)$ is called **spherical** if for all finite-dimensional representations $(V, \rho_V)$ of $H$ and all homomorphisms of representations $\phi : V \to V$ one has

$$\mathrm{Tr}\,(\phi \circ \rho_V(g)) = \mathrm{Tr}\,(\phi \circ \rho_V(g^{-1})).$$

**Example 6.2.13:**

1. Every *finite-dimensional semisimple* Hopf algebra $H$ over a field of characteristic zero is spherical with pivot $1 \in H$. This follows from the Theorem of Larson and Radford.
2. By Theorem 5.1.7, 4. every *triangular* Hopf algebra is pivotal with the Drinfeld element $u = S(R_{(2)})R_{(1)}$ as the pivot, since triangularity implies $R_{21}R = 1$ and $u$ grouplike. Note that this does not hold for *quasitriangular* Hopf algebras, since their antipode satisfies $S^2(h) = uhu^{-1}$ for all $h \in H$, but the element $u \in H$ is not necessarily grouplike.
3. In general, the pivot of a pivotal Hopf algebra is not unique. If $g \in \mathrm{Gr}(H)$ is a pivot for $H$ and $v \in \mathrm{Gr}(H)$ a central grouplike element, then $gv$ is another pivot for $H$.

**Proposition 6.2.14:**

1. If $(H, g)$ is a pivotal Hopf algebra, then the representation category $H\text{-Mod}_{fin}$ is pivotal with $\omega_V = \mathrm{can}_V \circ \rho_V(g)$ for all finite-dimensional $H$-modules $(V, \rho_V)$. If $H$ is spherical, then $H\text{-Mod}_{fin}$ is spherical.

2. If $H$ is a finite-dimensional Hopf algebra and $H\text{-Mod}_{fin}$ is pivotal with pivot $\omega : \mathrm{id}_{\mathcal{C}} \Rightarrow **$, then $H$ is a pivotal Hopf algebra with pivot $g = \mathrm{can}_H^{-1} \circ \omega_H(1_H)$. If $H\text{-Mod}_{fin}$ is spherical, then $(H, g)$ is spherical.

**Proof:**
1. Let $(H, g)$ be a pivotal Hopf algebra. Then $H\text{-Mod}_{fin}$ is rigid by Theorem 6.2.3. The functor $** : H\text{-Mod}_{fin} \to H\text{-Mod}_{fin}$ assigns to each $H$-module $(V, \rho_V)$ the $H$-module $(V^{**}, \rho_{V^{**}})$ with

$$(\rho_{V^{**}}(h)x)(\alpha) = (x \circ \rho_{V^*}(S(h)))(\alpha) = x(\rho_{V^*}(S(h))\alpha) = x(\alpha \circ \rho_V(S^2(h)))$$

for all $x \in V^{**}$, $\alpha \in V^*$. This implies $\rho_{V^{**}}(h) \circ \mathrm{can}_V = \mathrm{can}_V \circ \rho_V(S^2(h))$ and

$$\begin{aligned}\rho_{V^{**}}(h) \circ \omega_V &= \rho_{V^{**}}(h) \circ \mathrm{can}_V \circ \rho_V(g) = \mathrm{can}_V \circ \rho_V(S^2(h)) \circ \rho_V(g) = \mathrm{can}_V \circ \rho_V(S^2(h)g)\\ &= \mathrm{can}_V \circ \rho_V(gh) = \mathrm{can}_V \circ \rho(g) \circ \rho_V(h) = \omega_V \circ \rho_V(h)\end{aligned}$$

for all $h \in H$. This shows that $\omega_V : V \to V^{**}$ is a homomorphism of representations. Clearly, $\omega_V$ is invertible with inverse $\omega_V^{-1} = \rho_V(g^{-1}) \circ \mathrm{can}_V^{-1}$. The naturality of $\omega$ follows from the naturality of can, which implies for all $H$-module homomorphisms $f : V \to W$

$$f^{**} \circ \omega_V = f^{**} \circ \mathrm{can}_V \circ \rho_V(g) = \mathrm{can}_W \circ f \circ \rho_V(g) = \mathrm{can}_W \circ \rho_W(g) \circ f = \omega_W \circ f.$$

That $\omega$ is a *monoidal* natural isomorphism follows from the fact that $g$ is grouplike

$$\omega_{V \otimes V} = \mathrm{can}_{V \otimes V}(\rho_{V \otimes V}(g)) = (\mathrm{can}_V \otimes \mathrm{can}_V)(\rho_V(g) \otimes \rho_V(g)) = \omega_V \otimes \omega_V.$$

This shows that $H\text{-Mod}_{fin}$ is pivotal with $\omega_V = \mathrm{can}_V \circ \rho_V(g)$.

Theorem 6.2.3 and the definition of the left duals in (54) imply that the left and right evaluation and coevaluation maps for a finite-dimensional $H$-module $(V, \rho_V)$ with basis $(x_1, ..., x_n)$ and dual basis $(\alpha^1, ..., \alpha^n)$ are given by

$$\begin{aligned} &\mathrm{ev}_V^R : V^*\otimes V \to \mathbb{F}, \alpha\otimes v \mapsto \alpha(v) && \mathrm{coev}_V^R : \mathbb{F} \to V\otimes V^*, \lambda \mapsto \lambda\Sigma_{i=1}^n x_i\otimes\alpha^i \\ &\mathrm{ev}_V^L : V\otimes V^* \to \mathbb{F}, v\otimes\alpha \mapsto \alpha(\rho_V(g)v) && \mathrm{coev}_V^L : \mathbb{F} \to V^*\otimes V, \lambda \mapsto \lambda\Sigma_{i=1}^n \alpha^i\otimes\rho_V(g^{-1})x_i. \end{aligned} \tag{56}$$

With the expressions for the left and right traces in Definition 6.2.8 this implies

$$\mathrm{tr}_L(f) = \mathrm{tr}_V(f \circ \rho_V(g^{-1})) \qquad \mathrm{tr}_R(f) = \mathrm{tr}_V(f \circ \rho_V(g)). \tag{57}$$

and hence $H$-Mod$_{fin}$ is spherical if $(H, g)$ is spherical.

2. Let $H$ be a finite-dimensional Hopf algebra such that $H$-Mod$_{fin}$ is a pivotal category and define $g = \mathrm{can}_H^{-1}(\omega_H(1_H)) \in H$. Then for all $h \in H$, the linear map $\phi_h : H \to H$, $k \mapsto kh$ is an $H$-module homomorphism and by the naturality of $\omega$, one has $\omega_H \circ \phi_h = \phi_h^{**} \circ \omega_H$. Using the identity $\rho_{H^{**}}(h)\mathrm{can}_H(k) = \mathrm{can}_H(S^2(h) \cdot k)$, which follows from 1., we then obtain

$$\begin{aligned} &\mathrm{can}_H(S^2(h) \cdot g) = \mathrm{can}_H(\rho_H(S^2(h))g) = \rho_{H^{**}}(h)\mathrm{can}_H(g) = \rho_{H^{**}}(h)\omega_H(1_H) = \omega_H(\rho_H(h)1_H) \\ &= \omega_H(h) = \omega_H(\phi_h(1_H)) = \phi_h^{**}(\omega_H(1_H)) = \phi_h^{**}(\mathrm{can}_H(g)) = \mathrm{can}_H(\phi_h(g)) = \mathrm{can}_H(g \cdot h), \end{aligned}$$

where we used that $\omega_H : H \to H^{**}$ is a module homomorphism in the first line and then the condition $\omega_H \circ \phi_h = \phi_h^{**} \circ \omega_H$ together with the definition of $g$. As $\mathrm{can}_H : H \to H^{**}$ is a linear isomorphism, this shows that $S^2(h) \cdot g = g \cdot h$ for all $h \in H$. To show that $g$ is grouplike, we note that $\Delta : H \to H\otimes H$ is a module homomorphism, which implies $\omega_{H\otimes H} \circ \Delta = \Delta^{**} \circ \omega_H$ and that $\omega$ is a *monoidal* natural isomorphism, which implies $\omega_{H\otimes H} = \omega_H\otimes\omega_H$. Identifying the vector spaces $(H\otimes H)^{**} \cong H^{**}\otimes H^{**}$, we then obtain

$$\begin{aligned} &\mathrm{can}_H(g)\otimes\mathrm{can}_H(g) = \omega_H(1_H)\otimes\omega_H(1_H) = \omega_{H\otimes H}(1_H\otimes 1_H) = \omega_{H\otimes H} \circ \Delta(1_H) \\ &= \Delta^{**}(\omega_H(1_H)) = \Delta^{**}(\mathrm{can}_H(g)) = \mathrm{can}_{H\otimes H}(\Delta(g)) = (\mathrm{can}_H\otimes\mathrm{can}_H)(\Delta(g)) \end{aligned}$$

This shows that $\Delta(g) = g\otimes g$ and hence $g$ is grouplike and $(H, g)$ a pivotal Hopf algebra. If $H$-Mod$_{fin}$ is spherical, then it follows from (57) that $(H, g)$ is spherical as well. □

## 6.3 Ribbon categories and ribbon Hopf algebras

The notion of a pivotal category captures all geometrical content of the category $\mathcal{T}$ of ribbon tangles except the one related to the braiding. The pivot guarantees the existence of dual objects, given by diagrams with the same labels but lines that are oriented upwards, and of left and right (co)evaluations that satisfy snake identities analogous to the ones in $\mathcal{T}$.

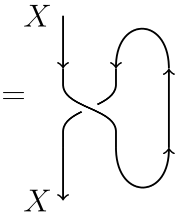
By the proof of Proposition 6.1.13, the braiding of $\mathcal{T}$ is determined uniquely by the morphism $X$ in $\mathcal{T}$, its inverse $X^{-1}$ and their duals defined in terms of $\cup, \cap, \cup', \cap'$. The results in Section 4.2, in particular Propositions 4.2.8 and 4.2.11, imply that if $\mathcal{C}$ is a *braided* pivotal category, then any object in $\mathcal{C}$ is a Yang-Baxter operator and one obtains analogues of the morphism $X$ that satisfy analogues of the Reidemeister relations RM2 and RM3 in Definition 6.1.11.

It is also apparent that combining the braiding of an object $X$ in $\mathcal{C}$ with itself with the coevaluation and evaluation morphisms yields analogues of the twists that arise in the Reidemeister relation RM1' in Definition 6.1.11. However, it is not guaranteed that these analogues satisfy analogues of the Reidemeister relation RM1'. We will now show that an analogue of the the Reidemeister relation RM1' in Definition 6.1.11 holds for an object $X$ in $\mathcal{C}$ if and only if the associated twist morphism is self-dual. If this is the case for all objects in $\mathcal{C}$, then the category $\mathcal{C}$ is called a *ribbon category*.

**Definition 6.3.1:** Let $\mathcal{C}$ be a braided pivotal category.

1. For any object $X$ in $\mathcal{C}$ the **twist** on $X$ is the morphism

$$\theta_X = r_X \circ (1_X \otimes \mathrm{ev}_X^L) \circ a_{X,X,X^*} \circ (c_{X,X} \otimes 1_{X^*}) \circ a^{-1}_{X,X,X^*} \circ (1_X \otimes \mathrm{coev}_X^R) \circ r_X^{-1} \tag{58}$$

$\theta_X =$ 

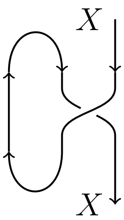

2. A **ribbon category** is a braided pivotal category in which all twists are self-dual: $\theta_X^* = \theta_{X^*}$ for all objects $X$ in $\mathcal{C}$.

A more detailed investigation of the properties of the twist shows that the twists define a natural isomorphism $\theta : \mathrm{id}_\mathcal{C} \to \mathrm{id}_\mathcal{C}$, but that this natural isomorphism is in general *not monoidal*. It also follows directly from the graphical representation of the twists and the duals that the twists are self-dual if and only if they satisfy an analogue of the Reidemeister relation RM1' in Definition 6.1.11. More specifically, we have the following Lemma.

**Lemma 6.3.2:** Let $\mathcal{C}$ be a braided pivotal category.

1. The twist is invertible with inverse

$$\theta_X^{-1} = l_X \circ \mathrm{ev}_X^R \circ a^{-1}_{X^*,X,X} \circ (1 \otimes c^{-1}_{X,X}) \circ a_{X^*,X,X} \circ \mathrm{coev}_X^L \circ l_X^{-1}$$

$\theta_X^{-1} =$ 

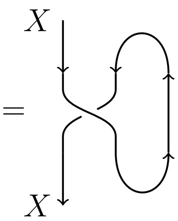
(59)

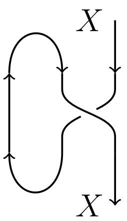
2. The twist satisfies $\theta_e = 1_e$ and $\theta_{X\otimes Y} = c_{Y,X} \circ c_{X,Y} \circ (\theta_X \otimes \theta_Y) = (\theta_X \otimes \theta_Y) \circ c_{Y,X} \circ c_{X,Y}$.

3. The twist is natural: $f \circ \theta_X = \theta_Y \circ f$ for all morphisms $f : X \to Y$.

4. $\mathcal{C}$ is ribbon if and only if for all objects $X$ one has

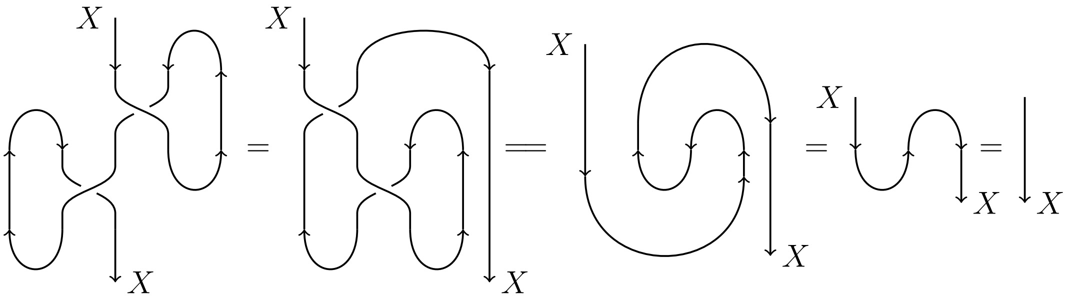
$$\theta_X = \theta'_X := r_X \circ (\mathrm{ev}_X^R \otimes 1_X) \circ a^{-1}_{X^*,X,X} \circ (1_{X^*} \otimes c_{X,X}) \circ a_{X^*,X,X} \circ (\mathrm{coev}_X^L \otimes 1_X) \circ l_X^{-1}$$

$\theta_X =$ 

$=$ 

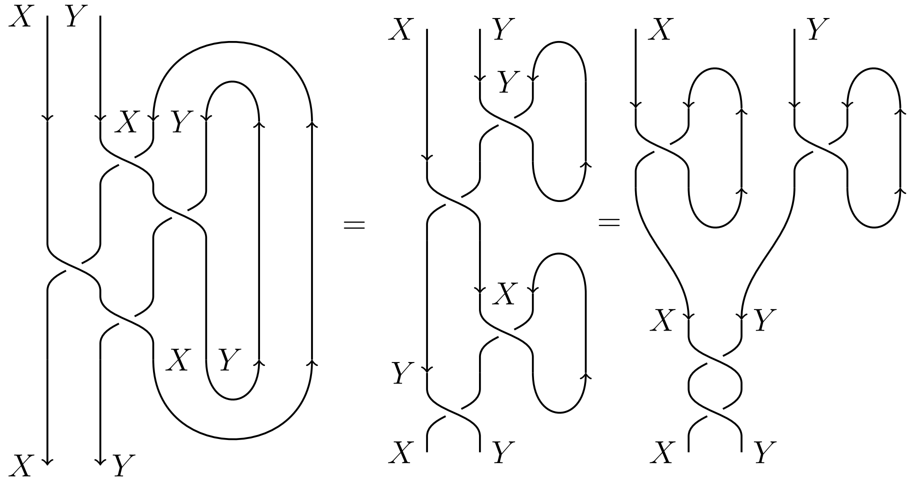
$= \theta'_X$ (60)

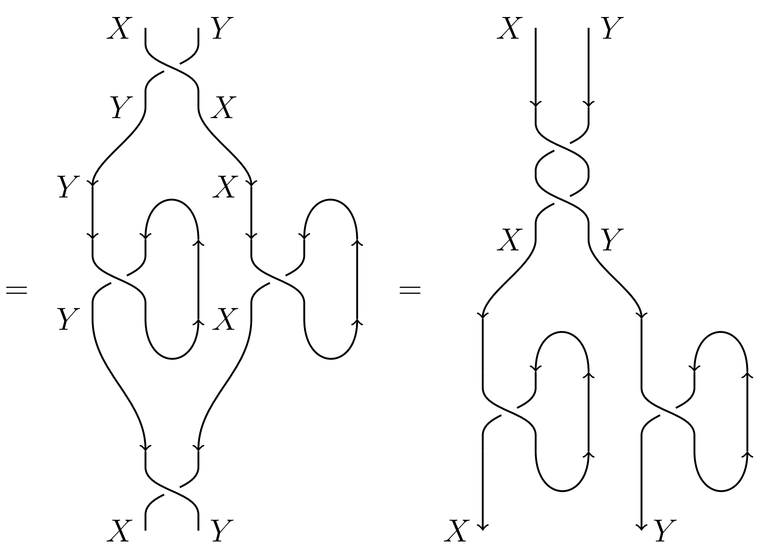
**Proof:**
1. We prove graphically that $\theta_X^{-1} \circ \theta_X = 1_X$:




where we used the naturality of the braiding with respect to the twist in the first setp, the naturality of the braiding with respect to $\cup$ in the second step and then the snake identity in the third and fourth step. The graphical proof that $\theta_X \circ \theta_X^{-1} = 1$ is analogous (Exercise).

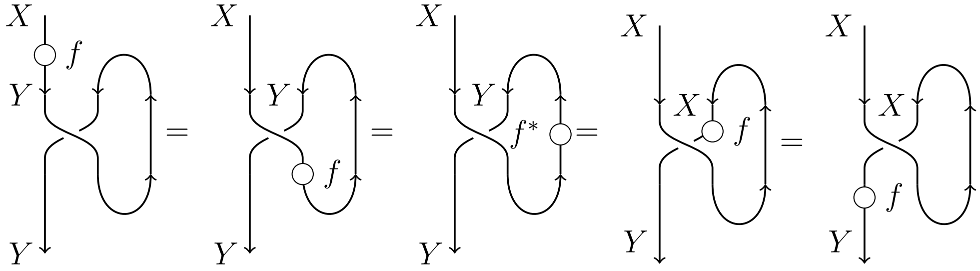
2. That $\theta_e = 1_e$ follows directly from the identities $c_{e,e} = 1_e$ and $\mathrm{tr}_L(e) = \mathrm{tr}_R(e) = 1$. The identities for $\theta_{X\otimes Y}$ can be proved graphically. From the diagram for the twist $\theta_{X\otimes Y}$ we obtain



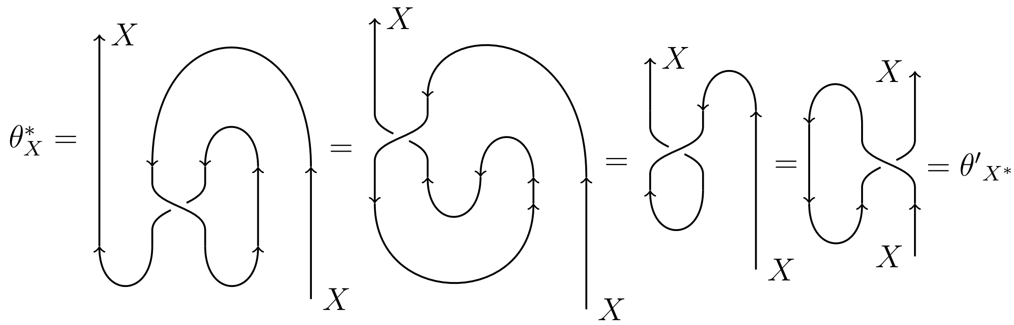




where we used the naturality of the braiding with respect to the twist on $Y$ and the Reidemeister identity RM3 in the first step and then twice the naturality of the braiding with respect to the twist.

3. The naturality of the twist follows from the naturality of the braiding and the identities (52) and (55) for the left and right evaluation and coevaluation. This yields:




4. To prove 4. we use the definition of the dual morphism $\theta_X^*$ in (51) and compute

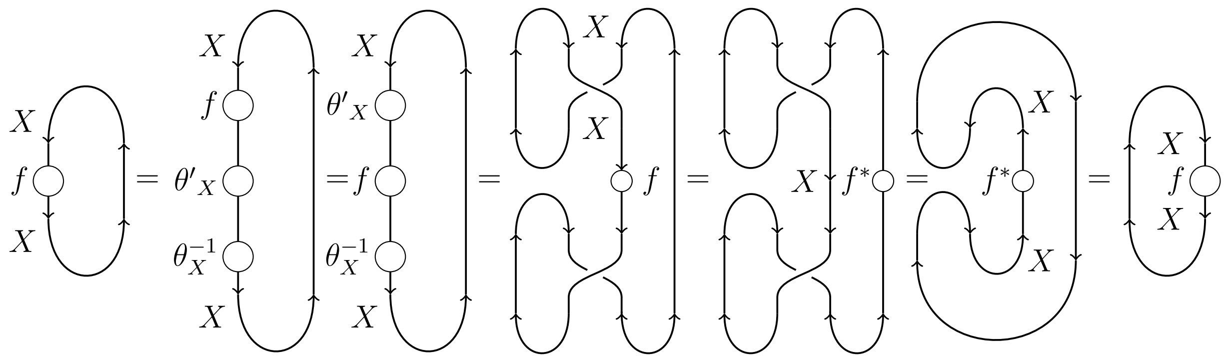



where we used the naturality of the braiding and the Reidemeister identity RM2 in the first step, the snake identity in the second step, then the Reidemeister relation RM2 and the naturality if the braiding with respect to $\cap$ in the third step and then the definition of $\theta_{X^*}$. Hence we have $\theta_X^* = \theta_{X^*}$ if and only if $\theta_X = \theta'_X$. □

By inserting the graphical identity in (60) into the Reidemeister relation RM1' from Definition 6.1.11 together with expression (58) for $\theta_X$ and (59) for $\theta_X^{-1}$, we find that a braided pivotal category $\mathcal{C}$ is a ribbon category if and only if the diagrams for the twist morphism on each object $X$ satisfy an analogue of the Reidemeister relation RM1' from Definition 6.1.11. In particular, this relation allows one to transform left traces of morphisms into right traces and vice versa.

**Corollary 6.3.3:** Every ribbon category is spherical.

**Proof:**



where we used the identity $\theta_X = \theta'_X$ in the first step, the naturality of $\theta'$ in the second step, the definition of $\theta_X$ and $\theta'_X$ in the third step, then the naturality of the braiding with respect to $f$ and the pivotality of $\mathcal{C}$, the Reidemeister relation RM2 and the naturality of the braiding with respect to $\cup'$, $\cap$ and $f$ in the fifth step and the snake identity and the pivotality in the last step. □

To construct interesting examples of ribbon categories, we need to determine the additional structure on a quasitriangular Hopf algebra $(H, R)$ that is required to ensure that the category $H\text{-Mod}_{fin}$ of its finite-dimensional representations is a ribbon category. It turns out that this requires a central invertible element, the so-called ribbon element that is related to the Drinfeld element $u = S(R_{(2)})R_{(1)}$.

**Definition 6.3.4:** A **ribbon Hopf algebra** is a quasitriangular Hopf algebra $(H, R)$ together with an invertible central element $\nu \in H$, the **ribbon element**, such that

$$uS(u) = \nu^2 \qquad\qquad \Delta(\nu) = (\nu\otimes\nu)\cdot(R_{21}R)^{-1}.$$

**Remark 6.3.5:**

1. A ribbon element is unique only up to right multiplication with a central grouplike element $g \in H$ satisfying $g^2 = 1$.
2. One can show that any ribbon element satisfies $\epsilon(\nu) = 1$ and $S(\nu) = \nu$.

**Example 6.3.6:**

1. If $H$ is quasitriangular with $S^2 = \mathrm{id}_H$, then $H$ is a ribbon Hopf algebra with ribbon element $u$. By the Larson-Radford theorem, this holds in particular for finite-dimensional semisimple quasitriangular Hopf algebras over fields of characteristic zero.

   The identity $S^2 = \mathrm{id}_H$ implies $u = S(R_{(2)})R_{(1)} = R_{(2)}S(R_{(1)})$, $u^{-1} = R_{(2)}R_{(1)}$ and $S(u) = S(R_{(1)})R_{(2)} = R_{(1)}S(R_{(2)})$. This implies

   $$S(u)u^{-1} = R_{(1)}S(R_{(2)})R'_{(2)}R'_{(1)} = 1 = R'_{(2)}R'_{(1)}S(R_{(1)})R_{(2)} = u^{-1}S(u)$$

   This shows that $u = S(u)$ and $uS(u) = u^2$. The remaining conditions follows from Theorem 5.1.7, which implies in particular that $uhu^{-1} = S^2(h) = h$ for all $h \in H$ and hence $u$ is central.

2. The Drinfeld double of any finite-dimensional semisimple Hopf algebra $H$ over a field of characteristic zero is a ribbon Hopf algebra with ribbon element $\nu = u = S(R_{(2)})R_{(1)}$.

   By corollary 5.1.12, the Drinfeld double $D(H)$ of a finite-dimensional semisimple Hopf algebra $H$ is semisimple as well, and its antipode satisfies $S^2 = \mathrm{id}_H$. Hence, $D(H)$ is a ribbon Hopf algebra by 1.

3. Let $q \in \mathbb{F}$ be a primitive $r$th root of unity with $r > 1$ odd. Then the Hopf algebra $U_q^r(\mathfrak{sl}_2)$ from Proposition 2.3.12 is ribbon with ribbon element $\nu = K^{-1}u = uK^{-1}$ where $u = S(R_{(2)})R_{(1)}$ for the $R$-matrix (40).

   That $\nu$ is central follows from the identity $S^2(h) = KhK^{-1} = uhu^{-1}$ for all $h \in U_q^r(\mathfrak{sl}_2)$. The identity $\Delta(\nu) = (\nu\otimes\nu)(R_{21}R)^{-1}$ follows from the identity $\Delta(K^{\pm 1}) = K^{\pm 1}\otimes K^{\pm 1}$ and the corresponding identity for $u$ in Theorem 5.1.7.

**Proposition 6.3.7:**

1. If $(H, R, \nu)$ is a ribbon Hopf algebra, then the category $H-\mathrm{Mod}_{fin}$ is a ribbon category.
2. If $(H, R, g)$ is a finite-dimensional quasitriangular pivotal Hopf algebra and $\nu \in H$ such that the linear maps $\theta_V : V \to V$, $v \mapsto \nu^{-1} \rhd v$ define a twist on $H$-mod$_{fin}$, then $\nu = g^{-1}u$ and $\nu$ is a ribbon element.

**Proof:**
1. If $\nu \in H$ is invertible and central with $uS(u) = \nu^2$ and $\Delta(\nu) = (\nu \otimes \nu)(R_{21}R)^{-1}$, then its inverse satisfies $\Delta(\nu^{-1}) = (\nu^{-1} \otimes \nu^{-1})(R_{21}R)$. It follows that the element $g := u\nu^{-1}$ is grouplike since we obtain with the coproduct of $u$ from Theorem 5.1.7

$$\Delta(g) = \Delta(u) \cdot \Delta(\nu^{-1}) = (u \otimes u)(R_{21}R)^{-1} \cdot (\nu^{-1} \otimes \nu^{-1})(R_{21}R) = u\nu^{-1} \otimes u\nu^{-1} = g \otimes g.$$

Moreover, since $\nu$ is central, we have $ghg^{-1} = (u\nu^{-1})h(\nu u^{-1}) = uhu^{-1} = S^2(h)$ for all $h \in H$. This shows that $g = u\nu^{-1}$ is a pivot for $H$ and hence $H - \mathrm{Mod}^{fin}$ is a braided pivotal category by Theorem 5.1.2 and Proposition 6.2.14.

It remains to verify that the twist in $H$-Mod$_{fin}$ satisfies the condition in Lemma 6.3.2, 4. Let $V$ be a finite-dimensional $H$-module with basis $(x_1, ..., x_n)$ and dual basis $(\alpha^1, ..., \alpha^n)$ of $V^*$. With formula (56) for the evaluation and coevaluation maps, formula (39) for the braiding and the formulas from Theorem 5.1.7 we compute

$$\begin{aligned}
\theta_V(v) &= \Sigma_{i=1}^n \alpha^i((gR_{(1)}) \rhd v)\, R_{(2)} \rhd x_i = (R_{(2)} g R_{(1)}) \rhd v = (R_{(2)} S^2(R_{(1)}) g) \rhd v = (u^{-1}g) \rhd v \\
&= \nu^{-1} \rhd v \\
\theta'_V(v) &= \Sigma_{i=1}^n \alpha^i(R_{(2)}) \rhd v)(R_{(1)} g^{-1}) \rhd x_i = (R_{(1)} g^{-1} R_{(2)}) \rhd v = (g^{-1} S^2(R_{(1)}) R_{(2)}) \rhd v \\
&= (\nu u^{-1} S(u^{-1})) \rhd v = (\nu (S(u)u)^{-1}) \rhd v = (\nu(uS(u))^{-1}) \rhd v = (\nu\nu^{-2}) \rhd v = \nu^{-1} \rhd v.
\end{aligned}$$

This shows that the condition from Lemma 6.3.2 is satisfied and $H$-Mod$_{fin}$ is a ribbon category.

2. Let $H$ be a finite-dimensional quasitriangular pivotal Hopf algebra with pivot $g \in H$ and $\mu \in H$ such that $\theta_V : V \to V$, $v \mapsto \nu^{-1} \rhd v$ is a twist on $H$-Mod$_{fin}$. Then by Lemma 6.3.2, 4. and the computation above we have

$$u^{-1}g = \theta_H(1) = \nu^{-1} = \theta'_H(1) = g^{-1}S(u)^{-1}$$

and this implies $\nu^{-2} = (u^{-1}g)(g^{-1}S(u)^{-1}) = (S(u)u)^{-1} = (uS(u))^{-1}$ and $\nu^2 = uS(u)$. By Lemma 6.3.2, 2, we also have

$$\Delta(\nu^{-1}) = \Delta(\nu)^{-1} = \theta_{H\otimes H}(1) = c_{H,H} \circ c_{H,H} \circ (\theta_H \otimes \theta_H)(1) = (R_{21}R)(\nu^{-1} \otimes \nu^{-1}),$$

which implies that $\Delta(\nu) = (\nu \otimes \nu)(R_{21}R)^{-1}$. As $\theta_H : H \to H$ is a $H$-module homomorphism, we have $\nu^{-1}h = \theta_H(h) = \theta_H(h \rhd 1) = h \rhd \theta_H(1) = h\nu^{-1}$ for all $h \in H$ and hence $\nu$ is central in $H$. □

This Proposition shows that the different Hopf algebras in Example 6.3.6 give interesting and non-trivial examples of ribbon categories. We will now prove that any object in a ribbon category $\mathcal{C}$ defines a *braided* monoidal functor $F : \mathcal{T} \to \mathcal{C}$ from the category $\mathcal{T}$ of ribbon tangles to $\mathcal{C}$. This allows us to obtain invariants of framed links or ribbons based on from the finite-dimensional representations of ribbon Hopf algebras.

**Theorem 6.3.8:** Let $\mathcal{C}$ be a ribbon category and $V \in \mathrm{Ob}\,\mathcal{C}$. Then there is a braided monoidal functor $F_V : \mathcal{T} \to \mathcal{C}$, unique up to natural isomorphisms composed of associators and unit constraints in $\mathcal{C}$, with $F_V(+) = V$, $F_V(-) = V^*$ and

$$F_V(\cup) = \mathrm{ev}_V^R : V^* \otimes V \to e \qquad F_V(\cap) = \mathrm{coev}_V^R : e \to V \otimes V^*$$
$$F_V(\cup') = \mathrm{ev}_V^L : V \otimes V^* \to e, \qquad F_V(\cap') = \mathrm{coev}_V^L : e \to V^* \otimes V.$$

**Proof:**
We prove the claim for the case where $\mathcal{C}$ is a strict braided monoidal category and $F_V : \mathcal{T} \to \mathcal{C}$ is a strict braided monoidal functor. If $F_V : \mathcal{T} \to \mathcal{C}$ is a strict braided monoidal functor with $F_V(+) = V$ and $F_V(-) = V^*$, then one has $F_V(\emptyset) = F_V(e)$ and $F(\epsilon_1, ..., \epsilon_n) = F_V(\epsilon_1) \otimes ... \otimes F_V(\epsilon_n)$ for all $n \in \mathbb{N}$ and $\epsilon_1, ..., \epsilon_n \in \{\pm\}$ and hence $F_V$ is defined uniquely on the objects by $F_V(+) = V$ and $F_V(-) = V^*$. As $F_V : \mathcal{T} \to \mathcal{C}$ is braided, one needs to have $F_V(X^{\pm 1}) = F_V(c^{\pm 1}_{+,+}) = c^{\pm 1}_{V,V} : V \otimes V \to V \otimes V$, and by assumption one has $F_V(\cup) = \mathrm{ev}_V^R$, $F_V(\cap) = \mathrm{coev}_V^R$, $F_V(\cup') = \mathrm{ev}_V^L$, $F_V(\cap') = \mathrm{coev}_V^L$. As the morphisms $\cup, \cap, \cup', \cap', X^{\pm 1}$ generate $\mathcal{T}$, this defines $F_V$ uniquely on the morphisms. To show that this defines a braided monoidal functor $F_V : \mathcal{T} \to \mathcal{C}$, we have to show that these assignments are compatible with the defining relations in $\mathcal{T}$, i. e. that the morphisms $F_V(\cup)$, $F_V(\cap)$, $F_V(\cup')$, $F_V(\cap')$ and $F_V(X^{\pm 1})$ satisfy analogues of the defining relations from Definition 6.1.11.

The Reidemeister relation RM1' follows directly from the fact condition $\theta_V = \theta'_V$ in Lemma 6.3.2 and the associated diagrams (58), (59) and (60) for the component morphisms of the twist $\theta$ and its dual. The Reidemeister relations RM2 and RM3 follow directly from the fact that $\mathcal{C}$ is braided and $F_V(X) = c_{V,V}$. The snake identities for $F_V(\cup)$, $F_V(\cap)$, $F_V(\cup')$ and $F_V(\cap')$ follow from the fact that $\mathcal{C}$ is pivotal and hence left and right rigid, see the corresponding diagrams after Example 6.2.2. The snaked braiding identities in Definition 6.1.11 follow from the identities (52) and (55), applied to the braiding in $\mathcal{C}$ and the snake identities in $\mathcal{C}$. The modified RM2 relations in Definition 6.1.11 follow again from the identities (52) and (55), applied to the braiding, and the snake identities. This shows that the assgnments are compatible with the relations in $\mathcal{T}$ and define a monoidal functor $F_V : \mathcal{T} \to \mathcal{C}$. This functor is braided, since it is compatible with the relations for $X$ and sends $X$ to $c_{V,V}$. As the braiding in $\mathcal{T}$ is determined uniquely by the morphism $X$, it follows that $F_V$ is braided. □

To see how this theorem gives rise to invariants of framed links, recall that any framed link $L \subset \mathbb{R}^3$ projects to a generic link diagram $D_L$ and two framed links $L, L' \in \mathbb{R}^3$ are equivalent if and only if their link diagrams $D_L$ and $D_{L'}$ are related by orientation preserving diffeomorphisms $f : \mathbb{R}^2 \to \mathbb{R}^2$ and the three Reidemeister moves RM1', RM2 and RM3 for ribbons. As every generic link diagram describes a $(0,0)$-ribbon tangle and the ribbon tangles of links that are related by orientation preserving diffeomorphisms $f : \mathbb{R}^2 \to \mathbb{R}^2$ and the three Reidemeister moves RM1', RM2 and RM3 are equal, $(0,0)$-ribbon tangles are in bijection with equivalence classes of framed oriented links in $\mathbb{R}^3$. The functor $F_V$ from Theorem 6.3.8 assigns to each $(0,0)$-ribbon tangle $t$ an endomorphism $F(t) \in \mathrm{End}_{\mathcal{C}}(e)$. This defines a map $F_V : \mathcal{F}_O \to \mathrm{End}_{\mathcal{C}}(e)$ from the set of oriented framed links in $\mathbb{R}^3$ to the commutative monoid $\mathrm{End}_{\mathcal{C}}(e)$ that is constant on equivalence classes of framed links.

**Corollary 6.3.9:** Let $\mathcal{C}$ be a ribbon category and $V \in \mathrm{Ob}\,\mathcal{C}$. Assign to every oriented framed link $L \subset \mathbb{R}^3$ the endomorphism $F_V(D_L) : e \to e$ obtained by projecting $L$ to a generic oriented link diagram $D_L$ and applying the functor $F_V : \mathcal{T} \to \mathcal{C}$ from Theorem 6.3.8. Then this defines an invariant of framed oriented links.

Just as the construction of a functor $F : \mathcal{B} \to \mathcal{C}$ from the braid category $\mathcal{B}$ into a monoidal category $\mathcal{C}$ required only a Yang-Baxter operator $(V, \sigma)$ in $\mathcal{C}$, it is sufficient to have one object $V$ in a braided tensor category $\mathcal{C}$, together with left and right coevaluation morphisms that satisfy the axioms of a ribbon category to obtain a braided tensor functor $F : \mathcal{T} \to \mathcal{C}$.

**Example 6.3.10:** Suppose that $q \in \mathbb{F}$ is not a root of unity, and consider the braided vector space $V = \text{span}_{\mathbb{F}}\{v_0, v_1\}$ from Example 4.2.10. Then the full monoidal subcategory of $\text{Vect}_{\mathbb{F}}^{fin}$ that is generated by $V$ and $V^*$ with the braiding, left and right (co)evaluation

$$
\begin{aligned}
&c_{V,V}(v_0 \otimes v_0) = \lambda q \, v_0 \otimes v_0 && c_{V,V}(v_1 \otimes v_1) = \lambda q \, v_1 \otimes v_1 \\
&c_{V,V}(v_0 \otimes v_1) = \lambda v_1 \otimes v_0 && c_{V,V}(v_1 \otimes v_0) = \lambda \, v_0 \otimes v_1 + \lambda(q - q^{-1}) v_1 \otimes v_0 \\
&\text{ev}_V^R(\alpha^i \otimes v_j) = \delta^i_j && \text{coev}_V^R(1_{\mathbb{F}}) = v_0 \otimes \alpha^0 + v_1 \otimes \alpha^1 \\
&\text{ev}_V^L(\alpha^i \otimes v_j) = q \, \delta^i_0 \delta^0_j + q^{-1} \delta^i_1 \delta^1_j && \text{coev}_V^L(1_{\mathbb{F}}) = q^{-1} \, \alpha^0 \otimes v_0 + q \, \alpha^1 \otimes v_1
\end{aligned}
$$

and the induced morphisms for $V^*$ is a ribbon category. The inverse of the braiding is given by

$$
\begin{aligned}
&c^{-1}_{V,V}(v_0 \otimes v_0) = \lambda^{-1} q^{-1} \, v_0 \otimes v_0 && c_{V,V}(v_1 \otimes v_1) = \lambda^{-1} q^{-1} \, v_1 \otimes v_1 \\
&c^{-1}_{V,V}(v_0 \otimes v_1) = \lambda^{-1} \, v_1 \otimes v_0 - \lambda^{-1}(q - q^{-1}) v_0 \otimes v_1 && c_{V,V}(v_1 \otimes v_0) = \lambda^{-1} \, v_0 \otimes v_1,
\end{aligned}
$$

and this implies

$$
\lambda c^{-1}_{V,V} - \lambda^{-1} c_{V,V} = (q^{-1} - q) \, \text{id}_{V \otimes V}
$$

which is precisely the skein relation (47) for the HOMFLY polynomial if we set $x = \lambda$ and $y = q^{-1} - q$. A short computation shows that the twist is given by

$$
\theta_V(v_0) = \theta'_V(v_0) = \lambda q^2 v_0 \qquad \theta_V(v_1) = \theta'_V(v_1) = \lambda q^2 v_1.
$$

Hence, we have a ribbon category and obtain invariants of oriented links. If we choose $\lambda = q^{-2}$, then $\theta_V = \theta'_V = \text{id}_V$, and we obtain an invariant of oriented links whose value on the unknot is given by $\dim_q(V) = \text{ev}_V^L \circ \text{coev}_V^R = \text{ev}_V^R \circ \text{coev}_V^L = q + q^{-1}$. Up to normalisation, this is the HOMFLY polynomial for $x = \lambda = q^{-2}$ and $y = q^{-1} - q$.

# 7 Exercises

## 7.1 Exercises for Chapter 1

**Exercise 1:** Let $V, W$ be finite-dimensional vector spaces over $\mathbb{F}$. Prove that the vector space $\mathrm{Hom}_{\mathbb{F}}(V, W)$ of linear maps $\phi : V \to W$ is isomorphic to $V^* \otimes W$.

**Exercise 2:** Let $V, V', W, W'$ be vector spaces over $\mathbb{F}$. Use the universal property of the tensor product to prove:

(a) For every pair of linear maps $f : V \to V'$ and $g : W \to W'$ there is a unique linear map $f \otimes g : V \otimes W \to V' \otimes W'$ with $(f \otimes g) \circ \tau_{V \otimes W} = \tau_{V' \otimes W'} \circ (f \times g)$, where

$$\begin{aligned} f \times g &: V \times W \to V' \times W', & (v, w) &\mapsto f(v) \otimes g(w) \\ \tau_{V \otimes W} &: V \times W \to V \otimes W, & (v, w) &\mapsto v \otimes w \\ \tau_{V' \otimes W'} &: V' \times W' \to V' \otimes W', & (v', w') &\mapsto v' \otimes w'. \end{aligned}$$

(b) One has $f \otimes g = (\mathrm{id}_{V'} \otimes g) \circ (f \otimes \mathrm{id}_W) = (f \otimes \mathrm{id}_{W'}) \circ (\mathrm{id}_V \otimes g)$

**Exercise 3:** Let $\mathbb{F}$ be a field, viewed as a vector space over itself.

(a) Determine a basis of the tensor algebra $T(\mathbb{F})$.

(b) Show that the tensor algebra $T(\mathbb{F})$ is isomorphic to the algebra $\mathbb{F}[x]$ of polynomials with coefficients in $\mathbb{F}$.

**Exercise 4:** Let $V$ and $W$ be vector spaces over $\mathbb{F}$. Show with the universal property of the tensor algebra that every linear map $\phi : V \to W$ induces a linear map $T(\phi) : T(V) \to T(W)$ such that the following diagram commutes

$$\begin{array}{ccc} V & \xrightarrow{\phi} & W \\ {\scriptstyle \iota_V}\downarrow & & \downarrow{\scriptstyle \iota_W} \\ T(V) & \xrightarrow[T(\phi)]{} & T(W). \end{array}$$

**Exercise 5:** Let $V$ be a vector space over $\mathbb{F}$. Show that the tensor algebra $T(V)$ is unique up to unique isomorphism:

If $T'(V)$ is an algebra over $\mathbb{F}$ and $\iota'_V : V \to T'(V)$ a linear map such that for every linear map $\phi : V \to W$ there is a unique algebra homomorphism $\phi' : T'(V) \to W$ with $\phi' \circ \iota'_V = \phi$, then there is a unique algebra isomorphism $f : T(V) \to T'(V)$ with $f \circ \iota_V = \iota'_V$.

**Hint:** Use the universal properties of $T(V)$ and $T'(V)$ to construct algebra isomorphisms between $T(V)$ and $T'(V)$.

**Exercise 6:** Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$. Show that the universal enveloping algebra $U(\mathfrak{g})$ is unique up to unique isomorphism:

If $U'(\mathfrak{g})$ is an algebra over $\mathbb{F}$ and $\iota'_{\mathfrak{g}} : \mathfrak{g} \to U'(\mathfrak{g})$ a Lie algebra homomorphism such that for every Lie algebra homomorphism $\phi : \mathfrak{g} \to A$ into an algebra $A$ over $\mathbb{F}$, there is a unique algebra homomorphism $\phi' : U'(\mathfrak{g}) \to A$ with $\phi' \circ \iota'_{\mathfrak{g}} = \phi$, then there is a unique algebra isomorphism $f : U(\mathfrak{g}) \to U'(\mathfrak{g})$ with $f \circ \iota_{\mathfrak{g}} = \iota'_{\mathfrak{g}}$.

**Hint:** Use the universal properties of $U(\mathfrak{g})$ and $U'(\mathfrak{g})$ to construct algebra isomorphisms between $U(\mathfrak{g})$ and $U'(\mathfrak{g})$.

**Exercise 7:** Let $(\mathfrak{g}, [\,,\,]_{\mathfrak{g}})$ and $(\mathfrak{h}, [\,,\,]_{\mathfrak{h}})$ be Lie algebras over $\mathbb{F}$ and $\mathfrak{g} \oplus \mathfrak{h}$ their direct sum, i. e. the vector space $\mathfrak{g} \oplus \mathfrak{h}$ with the Lie bracket

$$[x + y, x' + y'] = [x, x]_{\mathfrak{g}} + [y, y]_{\mathfrak{h}} \qquad \forall x, x' \in \mathfrak{g}, y, y' \in \mathfrak{h}$$

Prove that the universal enveloping algebra $U(\mathfrak{g} \oplus \mathfrak{h})$ is isomorphic to the algebra $U(\mathfrak{g}) \otimes U(\mathfrak{h})$.

**Exercise 8:** Give a presentation of the following algebras in terms of generators and relations

(a) the algebra $\mathrm{Mat}(n \times n, \mathbb{F})$ of $(n \times n)$-matrices with entries in $\mathbb{F}$,
(b) the algebra $\mathrm{Diag}(n \times n, \mathbb{F})$ of diagonal $(n \times n)$-matrices with entries in $\mathbb{F}$
(c) the group algebra $\mathbb{F}[G]$ for $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

**Exercise 9:** Show that the algebra over $\mathbb{F}$ with generators $x, y, z$ and the relation $z - xy + y^3 x$ is isomorphic to $T(\mathbb{F}^2)$.

**Exercise 10:** Show that the algebra $\mathrm{Mat}(2 \times 2, \mathbb{R})$ can be presented with generators $x, y, z$ and relations $x^2 = y^2 = 1$, $z^2 = -1$, $xy = -yx = z$, $xz = -zx = y$, $yz = -zy = x$.

**Hint:** Find a basis $B$ of $\mathrm{Mat}(2 \times 2, \mathbb{R})$ that contains only diagonal and antidiagonal matrices and such that each matrix $M \in B$ satisfies $M^2 \in \{1, -1\}$.

**Exercise 11:** Let $G, H$ be groups and $\mathbb{F}[G]$, $\mathbb{F}[H]$ their group algebra over a field $\mathbb{F}$.

(a) The *left regular representation* of $G$ on $\mathbb{F}[G]$ is defined by $\rho : G \to \mathrm{Aut}_{\mathbb{F}}(G)$, $g \mapsto \rho(g)$ with $\rho(g)(\Sigma_{h\in G}\lambda_h\, h) = \Sigma_{h\in G}\lambda_h\, gh$. Show that this is a representation of $G$.
(b) The *right regular representation* of $G$ on $\mathbb{F}[G]$ is defined by $\rho : G \to \mathrm{Aut}_{\mathbb{F}}(G)$, $g \mapsto \rho(g)$ with $\rho(g)(\Sigma_{h\in G}\lambda_h\, h) = \Sigma_{h\in G}\lambda_h\, hg^{-1}$. Show that this is a representation of $G$.
(c) A *group action* of $G$ on $H$ is a map $\rho : G \times H \to H$ with $\rho(g \cdot g', h) = \rho(g, \rho(g', h))$ and $\rho(1, h) = h$ for all $g, g' \in G$ and $h \in H$. Show that every group action of $G$ on $H$ defines a representation of $G$ on $\mathbb{F}[H]$.

**Exercise 12:** An $(n \times n)$-*permutation matrix* with entries in $\mathbb{F}$ is an $(n \times n)$-matrix with entries in $\mathbb{F}$ that has a single entry 1 in each row and column and zeros in all other entries. Show that the $(n \times n)$-permutation matrices define a representation of the group $S_n$ on $\mathbb{F}^n$.

**Exercise 13:** Let $G$ be a group and $\rho_V : G \to \mathrm{Aut}_{\mathbb{F}}(V)$, $\rho_W : G \to \mathrm{Aut}_{\mathbb{F}}(W)$ representations of $G$ on vector spaces $V, W$ over $\mathbb{F}$. Prove that the following are representations of $G$:

(a) the *trivial representation* on $\mathbb{F}$: $\rho_{\mathbb{F}} : G \to \mathrm{Aut}_{\mathbb{F}}(\mathbb{F})$, $g \mapsto \mathrm{id}_{\mathbb{F}}$.
(b) the *tensor product of representations* $\rho_{V\otimes W} : G \to \mathrm{Aut}_{\mathbb{F}}(V \otimes_{\mathbb{F}} W)$, $g \mapsto \rho_V(g) \otimes W(g)$.

(c) the *dual representation*: $\rho_{V^*}: G \to \mathrm{Aut}_{\mathbb{F}}(V^*)$, $g \mapsto \rho_{V^*}(g)$ with $\rho_{V^*}(g)\alpha = \alpha \circ \rho(g)$ for all $\alpha \in V^*$.

**Exercise 14:** Let $G$ be a group, $U, V, W$ vector spaces over $\mathbb{F}$ and $\rho_U : G \to \mathrm{Aut}_{\mathbb{F}}(U)$, $\rho_V : G \to \mathrm{Aut}_{\mathbb{F}}(V)$, $\rho_W : G \to \mathrm{Aut}_{\mathbb{F}}(W)$ representations of $G$. Prove the following statements.

(a) the *associativity isomorphism* $a_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$, $(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w)$ is an isomorphism of representations from $\rho_{(U\otimes V)\otimes W}$ to $\rho_{U\otimes(V\otimes W)}$.

(b) the *unit isomorphisms* $r_V : V \otimes \mathbb{F} \to V$, $v \otimes \lambda \mapsto \lambda v$ and $l_V : \mathbb{F} \otimes V \to V$, $\lambda \otimes v \mapsto \lambda v$ are isomorphisms of representations from $\rho_{V\otimes\mathbb{F}}$ and $\rho_{\mathbb{F}\otimes V}$ to $\rho_V$.

(c) the *flip isomorphism* $c_{U,V} : U \otimes V \to V \otimes U$, $u \otimes v \mapsto v \otimes u$ is an isomorphism of representations from $\rho_{U\otimes V}$ to $\rho_{V\otimes U}$.

(d) the *evaluation maps* $\mathrm{ev}^L_V : V \otimes V^* \to \mathbb{F}$, $v \otimes \alpha \mapsto \alpha(v)$, $\mathrm{ev}^R_V : V^* \otimes V \to \mathbb{F}$, $\alpha \otimes v \mapsto \alpha(v)$ are homomorphisms of representations from $\rho_{V\otimes V^*}$ and $\rho_{V^*\otimes V}$ to $\rho_{\mathbb{F}}$.

(e) If $\dim_{\mathbb{F}}(V) < \infty$, the *coevaluation maps* $\mathrm{coev}^L_V : \mathbb{F} \to V^* \otimes V$, $\lambda \mapsto \lambda \Sigma_{i=1}^n \beta^i \otimes b_i$ and $\mathrm{coev}^R_{V^*} : \mathbb{F} \to V \otimes V^*$, $\lambda \mapsto \lambda \Sigma_{i=1}^n b_i \otimes \beta^i$ are homomorphisms of representations from $\rho_{\mathbb{F}}$ to $\rho_{V^*\otimes V}$ and $\rho_{V\otimes V^*}$. Here, $B = (b^1, .., b^n)$ is an ordered basis of $V$ and $B^* = (\beta^1, ..., \beta^n)$ the dual basis of $V^*$.

**Exercise 15:** Let $A$ be an algebra over $\mathbb{F}$, $(V, \rhd)$ and $A$-module and $W \subset V$ an *submodule*, i. e. a linear subspace $W \subset V$ with $a \rhd w \in W$ for all $w \in W$. Show that the $A$-module structure on $V$ induces an $A$-module structure on the quotient space $V/W$.

**Exercise 16:** Let $G$ be a group, $N = \langle R \rangle \subset G$ a normal subgroup and $G/N$ the factor group.

(a) Prove that for any field $\mathbb{F}$ the group algebra $\mathbb{F}[G/N]$ is isomorphic to the quotient algebra $\mathbb{F}[G]/I$, where $I$ is the two-sided ideal generated by the set $\{r - e \,|\, r \in R\}$.

(b) Show that every representation $\rho : G \to \mathrm{Aut}_{\mathbb{F}}(V)$ with $\rho(n) = \mathrm{id}_V$ for all $n \in N$ induces a representation of $G/N$ on $V$.

**Exercise 17:** True or false? Prove the claim or give a counterexample.

(a) If $G$ is a finite group and $\rho : G \to \mathrm{Aut}_{\mathbb{F}}(V)$ a representation of $G$ on $V$, then there is an $n \in \mathbb{N}$ with $\rho(g)^n = \rho(g) \circ ... \circ \rho(g) = \mathrm{id}_V$ for all $g \in G$.

(b) The only representations of the permutation group $S_n$ on $\mathbb{C}$ are the trivial representation and the representation $\rho : S_n \to \mathrm{Aut}_{\mathbb{C}}(V)$, $\pi \mapsto \mathrm{sgn}(\pi)$

(c) If $G$ is an abelian group, then for every representation $\rho : G \to \mathrm{Aut}_{\mathbb{C}}(V)$ on a complex vector space $V$ there is a one-dimensional subspace $U \subset V$ with $\rho(g)u \in U$ for all $g \in G$, $u \in U$.

(d) If $G$ is an abelian group, then for every representation $\rho : G \to \mathrm{Aut}_{\mathbb{C}}(V)$ on a finite-dimensional complex vector space $V$ there is a one-dimensional subspace $U \subset V$ with $\rho(g)u \in U$ for all $g \in G$, $u \in U$.

(e) If $G$ is an abelian group, then for every representation $\rho : G \to \mathrm{Aut}_{\mathbb{R}}(V)$ on a finite-dimensional real vector space $V$ there is a one-dimensional subspace $U \subset V$ with $\rho(g)u \in U$ for all $g \in G$, $u \in U$.

## 7.2 Exercises for Chapter 2

**Exercise 18:** Let $(C, \Delta, \epsilon)$ and $(C', \Delta', \epsilon')$ be coalgebras over $\mathbb{F}$, $I \subset C$ a linear subspace and $\pi : C \to C/I$, $c \mapsto c + I$ the canonical surjection. Prove the following:

(a) The counit is unique: If $\epsilon'' : C \to \mathbb{F}$ is a linear map that satisfies for all $c \in C$ $(\mathrm{id}_C \otimes \epsilon'') \circ \Delta(c) = (\epsilon'' \otimes \mathrm{id}_C) \circ \Delta(c) = 1_{\mathbb{F}} \otimes c$, then $\epsilon'' = \epsilon$.

(b) For every coalgebra homomorphism $\phi : C \to C'$, the kernel $\ker(\phi) \subset C$ is a coideal in $C$ and the image $\mathrm{im}(\phi') \subset C'$ is a subcoalgebra of $C'$.

(c) $\delta : C/I \to C \otimes C/I$, $c + I \mapsto (\mathrm{id} \otimes \pi) \circ \Delta(c)$ defines a $C$-left comodule structure on $C/I$ if and only if $I$ is a left coideal.

(d) If $I \subset C$ is a left coideal, a right coideal or a subcoalgebra of $C$, then $\epsilon(I) = \{0\}$ if and only if $I = \{0\}$.

**Exercise 19:** Show that the tensor product coalgebra $C \otimes C'$ for two coalgebras $(C, \Delta, \epsilon)$ and $(C', \Delta', \epsilon')$ has the following universal property:

The projection maps $\pi : C \otimes C' \to C$, $c \otimes c' \mapsto \epsilon'(c')\, c$ and $\pi' : C \otimes C' \to C'$, $c \otimes c' \mapsto \epsilon(c)\, c'$ are coalgebra homomorphisms. For every *cocommutative* coalgebra $D$ and every pair of coalgebra homomorphisms $f : D \to C$, $f' : D \to C'$ there is a unique coalgebra homomorphism $\tilde{f} : D \to C \otimes C'$ with $\pi \circ \tilde{f} = f$ and $\pi' \circ \tilde{f} = f'$.

**Exercise 20:** Consider the vector space $\mathbb{F}[x]$ of polynomials with coefficients in $\mathbb{F}$ with the multiplication $m : \mathbb{F}[x] \otimes \mathbb{F}[x] \to \mathbb{F}[x]$ and comultiplication $\Delta : \mathbb{F}[x] \to \mathbb{F}[x] \otimes \mathbb{F}[x]$ given by

$$m(x^n \otimes x^m) = \binom{n+m}{n} x^{m+n} \qquad \Delta(x^m) = \sum_{n=0}^{m} x^n \otimes x^{m-n} \qquad \forall n, m \in \mathbb{N}_0.$$

Show that these maps define a Hopf algebra structure on $\mathbb{F}[x]$.

**Exercise 21:** In this exercise, we show that for the vector space $\mathbb{F}[x]$ of polynomials with coefficients in $\mathbb{F}$ we have $\mathbb{F}[x]^* \otimes \mathbb{F}[x]^* \subsetneq (\mathbb{F}[x] \otimes \mathbb{F}[x])^*$.

Recall that $\mathbb{F}[x] \otimes \mathbb{F}[x] \cong \mathbb{F}[x, y]$ as a vector space. Consider for $n \in \mathbb{N}_0$ the linear subspaces

$$\mathbb{F}[x]_{\leq n} = \mathrm{span}_{\mathbb{F}}\{x^k \,|\, 0 \leq k \leq n\} \subset \mathbb{F}[x] \qquad \mathbb{F}[x, y]_{\leq n} = \mathrm{span}_{\mathbb{F}}\{x^k y^l \,|\, 0 \leq k, l \leq n\} \subset \mathbb{F}[x, y],$$

and the linear maps $\delta_n : \mathbb{F}[x, y]_{\leq n} \to \mathbb{F}$, $\delta : \mathbb{F}[x, y] \to \mathbb{F}$ with $\delta_n(x^k y^l) = \delta(x^k y^l) = \delta_{kl}$.

(a) Show that for $f = \Sigma_{i=1}^m \alpha^i \otimes \beta^i \in \mathbb{F}_{\leq n}[x, y]^*$ with $\alpha^i \in \mathbb{F}[x]^*_{\leq n}$ and $\beta^i \in \mathbb{F}[y]^*_{\leq n}$ the matrix with entries $f(x^{i-1} y^{j-1})$ for $i, j \in \{1, ..., n+1\}$ has rank at most $m$.

(b) Conclude that if $\delta_n = \Sigma_{i=1}^m \alpha^i \otimes \beta^i$ with $\alpha^i \in \mathbb{F}[x]^*_{\leq n}$ and $\beta^i \in \mathbb{F}[y]^*_{\leq n}$, then $m \geq n$.

(c) Suppose that $\delta = \Sigma_{i=1}^m \alpha^i \otimes \beta^i$ with $\alpha^i \in \mathbb{F}[x]^*$ and $\beta^i \in \mathbb{F}[y]^*$ and restrict $\delta$ to $\mathbb{F}_{\leq n}[x, y]$ to obtain a contradiction.

**Exercise 22:** Let $(A, m, \eta)$ be an algebra over $\mathbb{F}$, $m^* : A^* \to (A \otimes A)^*$ and $\eta^* : \mathbb{F} \to A^*$ the dual maps of $m : A \otimes A \to A$ and $\eta : \mathbb{F} \to A$ and $A^\circ = \{\alpha \in A^* \,|\, m^*(\alpha) \in A^* \otimes A^*\}$.

Prove that $(A^\circ, m^*|_{A^\circ}, \eta^*|_{A^\circ})$ is a coalgebra over $\mathbb{F}$:

(a) Prove first the following fact from linear algebra:

Let $V$ be a vector space over $\mathbb{F}$ and $\alpha^1, ..., \alpha^n \in V^*$ linearly independent. Then for all $\lambda_1, ..., \lambda_n \in \mathbb{F}$, there is a vector $v \in V$ with $\alpha^i(v) = \lambda_i$ for $i = 1, ..., n$.

(b) Use (a) and the coassociativity of $m^* : A^* \to (A \otimes A)^*$ to prove that for every element $\alpha \in A^\circ$, one has $m^*(\alpha) \in A^\circ \otimes A^\circ$.

**Exercise 23:** Let $(H, m, \eta, \Delta, \epsilon, S)$ be a Hopf algebra and consider the algebra $\mathrm{End}_{\mathbb{F}}(H)$ with the convolution product. We define $\psi^n \in \mathrm{End}_{\mathbb{F}}(H)$ by

$$\psi^n = \begin{cases} \mathrm{id}_H^{*n} = \underbrace{\mathrm{id}_H * .... * \mathrm{id}_H}_{n\times} & n \in \mathbb{N} \\ S^{*n} = \underbrace{S * ... * S}_{n\times} & -n \in \mathbb{N} \\ \eta \circ \epsilon & n = 0. \end{cases}$$

(a) Give a formula for $\psi^n(h)$ in Sweedler notation for all $n \in \mathbb{Z}$.

(b) Determine $\psi^n$ for the group algebra $\mathbb{F}[G]$ of a finite group $G$ and for its dual $\mathrm{Fun}(G) = \mathbb{F}[G]^*$.

(c) Determine $\psi^n$ for the Hopf algebra from Exercise 20.

(d) Show that if $H$ is commutative, then $\psi^n : H \to H$ is an algebra homomorphism, and if $H$ is cocommutative, then $\psi^n : H \to H$ is a coalgebra homomorphism for all $n \in \mathbb{Z}$.

**Exercise 24:** Let $(B, m, \eta, \Delta, \epsilon)$ and $(B', m', \eta', \Delta', \epsilon')$ be finite-dimensional bialgebras over $\mathbb{F}$. Show that the vector space $\mathrm{Hom}_{\mathbb{F}}(B, B')$ becomes a bialgebra when eqipped with

$$\tilde{m} : \mathrm{Hom}_{\mathbb{F}}(B, B') \otimes \mathrm{Hom}_{\mathbb{F}}(B, B') \to \mathrm{Hom}_{\mathbb{F}}(B, B'), \quad f \otimes g \mapsto f * g = m' \circ (f \otimes g) \circ \Delta$$
$$\tilde{\Delta} : \mathrm{Hom}_{\mathbb{F}}(B, B') \to \mathrm{Hom}_{\mathbb{F}}(B, B') \otimes \mathrm{Hom}_{\mathbb{F}}(B, B'), \quad f \mapsto \Delta' \circ f \circ m.$$

where we identify the vector spaces $\mathrm{Hom}_{\mathbb{F}}(B, B') \otimes \mathrm{Hom}_{\mathbb{F}}(B, B') \cong \mathrm{Hom}_{\mathbb{F}}(B \otimes B, B' \otimes B')$. Show that if both $B$ and $B'$ are Hopf algebras, then the bialgebra $\mathrm{Hom}_{\mathbb{F}}(B, B')$ is a Hopf algebra.

**Exercise 25:** Let $B$ be a finite-dimensional bialgebra over $\mathbb{F}$. Prove the following:

(a) The field $\mathbb{F}$ has a canonical right comodule structure over $B$.

(b) For all right comodules $(V, \delta_V)$, $(W, \delta_W)$ over $B$ the tensor product $V \otimes W$ has a canonical left comodule structure over $B$.

(c) If $B$ is a Hopf algebra, then for all right comodules $(V, \delta_V)$ over $B$, the dual vector space $V^*$ has a canonical right comodule structure over $B$.

**Exercise 26:** Let $V$ be a vector space over $\mathbb{F}$ and $T(V)$ the tensor algebra over $V$ with the Hopf algebra structure from Example 2.3.1. Show that the Hopf algebra structure of $T(V)$ induces a Hopf algebra structure on the symmetric algebra $S(V) = T(V)/(x \otimes y - y \otimes x)$.

**Exercise 27:** Prove the $q$-**Chu-Vandermonde formula** for the $q$-binomials:

$$\binom{m+n}{p}_q = \sum_{k=0}^{p} q^{(m-k)(p-k)} \binom{m}{k}_q \binom{n}{p-k}_q \qquad \forall 0 \leq p \leq n, m.$$

**Exercise 28:** Let $\mathbb{F}$ be a field and $q \in \mathbb{F} \setminus \{0, 1, -1\}$.
The $q$**-deformed universal enveloping algebra** $U_q(\mathfrak{sl}_2)$ is the algebra over $\mathbb{F}$ with generators $E, F, K, K^{-1}$ and relations

$$K^{\pm 1} K^{\mp 1} = 1, \quad KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}. \tag{61}$$

Show that $U_q(\mathfrak{sl}_2)$ is a Hopf algebra with the comultiplication, counit and antipode

$$\begin{aligned} &\Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1}, && \Delta(E) = 1 \otimes E + E \otimes K, && \Delta(F) = F \otimes 1 + K^{-1} \otimes F \\ &\epsilon(K^{\pm 1}) = 1, && \epsilon(E) = 0, && \epsilon(F) = 0 \\ &S(K^{\pm 1}) = K^{\mp 1} && S(E) = -EK^{-1} && S(F) = -KF. \end{aligned} \tag{62}$$

**Hint:** The proof is analogous to the one for Taft's example and for the Hopf algebra $\mathrm{SL}_q(2, \mathbb{F})$.

**Exercise 29:** Let $\mathbb{F}$ be a field, $q \in \mathbb{F} \setminus \{0, 1, -1\}$ and $U_q(\mathfrak{sl}_2)$ the $q$-deformed universal enveloping algebra from Example 2.3.9. Prove the following:

(a) The **quantum Casimir** element

$$C_q = EF + \frac{q^{-1}K + qK^{-1}}{(q - q^{-1})^2} = FE + \frac{qK + q^{-1}K^{-1}}{(q - q^{-1})^2}$$

is in the centre of $U_q(\mathfrak{sl}_2)$: $C_q \cdot X = X \cdot C_q$ for all $X \in U_q(\mathfrak{sl}_2)$.

(b) The antipode of $U_q(\mathfrak{sl}_2)$ is invertible.

(c) For all elements $X \in U_q(\mathfrak{sl}_2)$ one has $S^2(X) = KXK^{-1}$.

(d) There is a unique Hopf algebra isomorphism $\phi : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2)^{cop}$ with

$$\phi(E) = F \qquad \phi(F) = E \qquad \phi(K) = K^{-1}.$$

It is called the **Cartan automorphism** of $U_q(\mathfrak{sl}_2)$.

**Exercise 30:** Let $q$ be a primitive $n$th root of unity and let $H$ be Taft's Hopf algebra with generators $x, y$ and relations $xy = qyx$, $x^n = 0$ and $y^n = 1$.

(a) Show that Taft's Hopf algebra is isomorphic as an algebra to a semidirect product $\mathbb{F}[\mathbb{Z}/n\mathbb{Z}] \ltimes \mathbb{F}[x]/(x^n)$,.i. e. to the vector space $\mathbb{F}[\mathbb{Z}/n\mathbb{Z}] \otimes \mathbb{F}[x]/(x^n)$ with the multiplication

$$(\bar{k} \otimes [x]) \cdot (\bar{m} \otimes [x']) = \overline{k + m} \otimes ([x] + \rho(\bar{k}) \rhd [x'])$$

with a group homomorphism $\rho : \mathbb{Z}/n\mathbb{Z} \to \mathrm{Aut}_{\mathbb{F}}(\mathbb{F}[x]/(x^n))$. Determine $\rho$.

(b) Show that the dual Hopf algebra $H^*$ is isomorphic to Taft'sHopf algebra $H$ as a Hopf algebra: Consider the linear maps $\alpha, \beta : H \to \mathbb{F}$ defined by $\alpha(y^k x^j) = q^{-k} \delta_{j0}$ and $\beta(y^k x^j) = \delta_{j1}$ for all $k, j \in \{0, 1, .., n-1\}$ and show that there is a unique Hopf algebra isomorphism $\phi : H \to H^*$ with $\phi(y) = \alpha$, $\phi(x) = \beta$.

**Exercise 31:** Let $\mathbb{F}$ be a field of prime characteristic $\mathrm{char}(\mathbb{F}) = p$. A **restricted Lie algebra** over $\mathbb{F}$ is a Lie algebra $(\mathfrak{g}, [\,,\,])$ over $\mathbb{F}$ together with a map $\phi : \mathfrak{g} \to \mathfrak{g}$, $x \mapsto x^{[p]}$ that satisfies

$$(\lambda x)^{[p]} = \lambda^p x^{[p]}, \quad \mathrm{ad}_{x^{[p]}} = \mathrm{ad}_x^p = \mathrm{ad}_x \circ ... \circ ad_x, \quad (x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{k=1}^{p-1} \frac{\sigma_k(x,y)}{k}$$

where $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$, $y \mapsto [x,y]$ is the adjoint action and $\sigma_k(x,y)$ is given by

$$\mathrm{ad}^{p-1}_{\lambda x+y}(x) = [\lambda x + y, [\lambda x + y, [\cdots[\lambda x + y, [\lambda x + y, x]]\cdots]]] = \Sigma_{k=1}^{p-1} \lambda^{k-1} \sigma_k(x,y)$$

(a) Show that any algebra $A$ over $\mathbb{F}$ is a restricted Lie algebra with the commutator as the Lie bracket and the map $\phi : A \to A$, $a \mapsto a^p$.

(b) Let $(g, [\,,\,], \phi)$ be a restricted Lie algebra and $\mathcal{U} = U(\mathfrak{g})/(x^p - \phi(x))$ the quotient of its universal enveloping algebra by the two-sided ideal $(x^p - \phi(x))$ in $U(\mathfrak{g})$ generated by the elements $x^p - \phi(x)$ for $x \in \mathfrak{g}$. Denote by $\pi : U(\mathfrak{g}) \to \mathcal{U}$, $x \mapsto [x]$ the canonical surjection, by $\iota_{\mathfrak{g}} : \mathfrak{g} \to U(\mathfrak{g})$ the canonical inclusion and set and $\tau = \pi \circ \iota_{\mathfrak{g}} : \mathfrak{g} \to \mathcal{U}$. Show that $\mathcal{U}$ is a Hopf algebra with the comultiplication, counit and antipode given by

$$\Delta(\tau(x)) = 1 \otimes \tau(x) + \tau(x) \otimes 1 \qquad \epsilon(\tau(x)) = 0 \qquad S(\tau(x)) = -\tau(x).$$

(c) Show that $\mathcal{U}$ is a finite-dimensional cocommutative Hopf algebra, but that it is *not* isomorphic to the group algebra of a finite group.

## 7.3 Exercises for Chapter 3

**Exercise 32:** Let $B$ be a bialgebra and $H$ a Hopf algebra over $\mathbb{F}$. Prove the following:

(a) $H$ is a module algebra over itself with the **adjoint action**

$$\rhd_{ad} : H \otimes H \to H, \; h \otimes k \mapsto h \rhd_{ad} k \quad \text{with} \quad h \rhd_{ad} k = \Sigma_{(h)}\, h_{(1)} \cdot k \cdot S(h_{(2)}).$$

(b) The dual algebra $(B^*, \Delta^*, \epsilon^*)$ is a module algebra over $B$ with the **right dual action**

$$\rhd_R^* : B \otimes B^* \to B^*, \; b \otimes \alpha \mapsto b \rhd_R^* \alpha \quad \text{with} \quad b \rhd_R^* \alpha = \Sigma_{(\alpha)} \alpha_{(2)}(b) \otimes \alpha_{(1)}.$$

and a module algebra over $B^{op}$ with the **left dual action**

$$\rhd_L^* : B^{op} \otimes B^* \to B^*, \; b \otimes \alpha \mapsto b \rhd_L^* \alpha \quad \text{with} \quad b \rhd_L^* \alpha = \Sigma_{(\alpha)} \alpha_{(1)}(b)\, \alpha_{(2)}$$

This gives $B^*$ the structure of a $B \otimes B^{op}$-module algebra.

(c) If $V$ is a comodule (co)algebra over $B$ with $\delta : V \to B \otimes V$, then $V$ is a $B^{*op}$ module (co)algebra with $\rhd : B^* \otimes V \to V$, $\beta \rhd v = (\beta \otimes \mathrm{id}) \circ \delta(v)$.

(d) If $B$ is finite-dimensional and $(A, m_A, \eta_A)$ is a finite-dimensional module algebra over $B$, then $(A^*, m_A^*, \eta_A^*)$ is a comodule coalgebra over $B^*$ and vice versa.

**Exercise 33:** Let $C$ be a module coalgebra and $D$ a comodule coalgebra over a Hopf algebra $H$. Do the invariants of $C$ and the coinvariants of $D$ form cosubalgebras of $C$ and $D$?

**Exercise 34:** Consider for $q \in \mathbb{F} \setminus \{0, \pm 1\}$ the quantum plane $\mathbb{F}_q[x,y]$ as a module algebra over $U_q(\mathfrak{sl}_2)$ with

$$E \rhd (y^m x^n) = q^{1-n}(n)_{q^2} y^{m+1} x^{n-1} \qquad F \rhd (y^m x^n) = q^{1-m}(m)_{q^2} y^{m-1} x^{n+1}$$
$$K \rhd (y^m x^n) = q^{m-n} y^m x^n \qquad K^{-1} \rhd (y^m x^n) = q^{n-m} y^m x^n.$$

(a) Determine the action of the quantum Casimir $C_q = EF + \frac{q^{-1}K + qK^{-1}}{(q-q^{-1})^2}$ on $\mathbb{F}_q[x,y]$.

(b) Determine the invariants of the $U_q(\mathfrak{sl}_2)$-module algebra $\mathbb{F}_q[x,y]$ for (i) the case where $q$ is not a root of unity and for (ii) the case where $q$ is a primitive $n$th root of unity.

(c) Show that the subalgebra $H$ of $U_q(\mathfrak{sl}_2)$ generated by $K^{\pm 1}$ is a Hopf subalgebra of $U_q(\mathfrak{sl}_2)$. Determine the the invariants of $\mathbb{F}_q[x,y]$ as a module algebra over $H$ for (i) the case where $q$ is not a root of unity and (ii) the case where $q$ is a primitive $n$th root of unity.

(d) Consider for $q = 1$ the plane $\mathbb{F}[x,y]$ as a module algebra over $U_1'(\mathfrak{sl}_2) \cong U(\mathfrak{sl}_2)$. Determine the invariants of $\mathbb{F}[x,y]$ as a module over the Hopf subalgebras generated by the elements $e$, $f$ and $l$, respectively.

**Hint:** In (b) it is helpful to consider $r := n$ for $n$ odd and $r := n/2$ for $n$ even.

**Exercise 35:** Let $G$ be a group and $\mathbb{F}$ a field. Prove that an algebra $A$ over $\mathbb{F}$ has the structure of a comodule algebra over the group algebra $\mathbb{F}[G]$ if and only if $A$ it is a $G$-**graded algebra**, i. e. it is given as a direct sum $A = \oplus_{g \in G} A_g$ with linear subspaces $A_g \subset A$ such that $1_A \in A_e$ and $A_g \cdot A_h = A_{gh}$ for all $g, h \in G$.

**Exercise 36:** Let $H$ be a Hopf algebra over $\mathbb{F}$ with an invertible antipode. A left (right) **integral** for $H$ is an invariant of the left (right) action of $H$ on itself by left (right) multiplication, i. e. an element $\ell \in H$ with $h \cdot \ell = \epsilon(h)\,\ell$ (with $\ell \cdot h = \epsilon(h)\,\ell$) for all $h \in H$.

(a) Show that for every left integral $\ell \in H$ one has

$$\Sigma_{(\ell)}\, S(\ell_{(1)})h \otimes \ell_{(2)} = \Sigma_{(\ell)}\, S(\ell_{(1)}) \otimes S^{-2}(h)\,\ell_{(2)} \qquad \Sigma_{(\ell)}\, h\ell_{(1)} \otimes S(\ell_{(2)}) = \Sigma_{(\ell)}\, \ell_{(1)} \otimes S(\ell_{(2)})h$$

(b) Show that for every right integral $\ell \in H$ one has

$$\Sigma_{(\ell)}\, hS(\ell_{(1)}) \otimes \ell_{(2)} = \Sigma_{(\ell)}\, S(\ell_{(1)}) \otimes \ell_{(2)} h \qquad \Sigma_{(\ell)} \ell_{(1)} h \otimes S(\ell_{(2)}) = \Sigma_{(\ell)} \ell_{(1)} \otimes S^2(h) S(\ell_{(2)}).$$

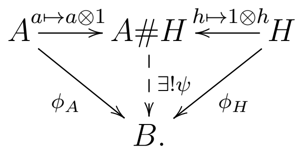
**Exercise 37:** Let $H$ be a Hopf algebra over $\mathbb{F}$ and $A$ a module algebra over $H$. We consider the cross product $A \# H$ with multiplication law

$$(a \otimes h) \cdot (b \otimes k) = \Sigma_{(h)} a(h_{(1)} \rhd b) \otimes h_{(2)} k.$$

Show that the cross product has the following universal property: for every triple $(B, \phi_A, \phi_H)$ of an algebra $B$ over $\mathbb{F}$ and algebra homomorphisms $\phi_A : A \to B$, $\phi_H : H \to B$ with

$$\phi_A(h \rhd a) = \Sigma_{(h)}\, \phi_H(h_{(1)}) \cdot \phi_A(a) \cdot \phi_H(S(h_{(2)}))$$

there is a unique algebra homomorphism $\psi : A \# H \to B$ such that the following diagram commutes



**Exercise 38:**

(a) Determine the vector spaces of left and right integrals for:

  (i) The Hopf algebras $\mathbb{F}[G]$ and $\mathrm{Fun}(G, \mathbb{F})$ for a finite group $G$,
  (ii) the Taft algebra,
  (iii) the $q$-deformed universal enveloping algebra $U_q^r(\mathfrak{sl}_2)$ at a root of unity,
  (iv) the tensor algebra $T(V)$ and the universal enveloping algebra $U(\mathfrak{g})$.

(b) Determine the modular element for (i), (ii) and (iii).

**Exercise 39:** Let $H$ be a finite-dimensional Hopf algebra. Show the following:

(a) If there is a left integral $\ell \in H$ with $\epsilon(\ell) \neq 0$, then $H$ is unimodular.

(b) The converse of this statement is false.

**Exercise 40:** Let $(A, m, \eta)$ be an algebra over $\mathbb{F}$ and $(A^\circ, m^*, \eta^*)$ its finite dual coalgebra. Let $\mathrm{Gr}(A^\circ) \subset A^\circ$ be the set of algebra homomorphism $\alpha : A \to \mathbb{F}$ and define for $\alpha \in \mathrm{Gr}(A^\circ)$

$$L_\alpha = \{\ell \in A \,|\, a \cdot \ell = \alpha(a)\, \ell \;\forall a \in A\} \qquad R_\alpha = \{\ell \in A \,|\, \ell \cdot a = \alpha(a)\, \ell \;\forall a \in A\}.$$

Prove the following:

(a) $L_\alpha$ and $R_\alpha$ are two-sided ideals in $A$ for all $\alpha \in \mathrm{Gr}(A^\circ)$. Any element of $A$ that generates a one-dimensional left (right) ideal in $A$ is contained in $L_\alpha$ (in $R_\alpha$) for some $\alpha \in \mathrm{Gr}(A^\circ)$.

(b) If $L_\alpha = L_\beta \neq \{0\}$ or $R_\alpha = R_\beta \neq \{0\}$ then $\alpha = \beta$. If $\alpha(L_\alpha) \neq \{0\}$ or $\alpha(R_\alpha) \neq \{0\}$ then $R_\alpha L_\alpha \neq \{0\}$.

(c) For all $\alpha, \beta \in \mathrm{Gr}(A^\circ)$ with $R_\beta L_\alpha \neq \{0\}$ the following hold:

  (i) $L_\alpha = R_\beta$,
  (ii) $\dim_{\mathbb{F}} L_\alpha = \dim_{\mathbb{F}} R_\beta = 1$,
  (iii) $L_\alpha$ is generated by an idempotent, i. e. an element $\ell \in L_\alpha$ with $\ell \cdot \ell = \ell$.

(d) For all algebra isomorphisms $f : A \to A$ and $g : A \to A^{op}$ one has $f(L_\alpha) = L_{\alpha \circ f^{-1}}$, $f(R_\alpha) = R_{\alpha \circ f^{-1}}$ and $g(L_\alpha) = R_{\alpha \circ g^{-1}}$, $g(R_\alpha) = L_{\alpha \circ g^{-1}}$ for all $\alpha \in \mathrm{Gr}(A^\circ)$.

**Exercise 41:** Let $H$ be a finite-dimensional Hopf algebra. The **coadjoint action** is given by

$$\rhd^*_{ad} : H \otimes H^* \to H^* \qquad h \rhd^*_{ad} \alpha = \Sigma_{(\alpha)} \, (\alpha_{(3)} \cdot S(\alpha_{(1)}))(h)\, \alpha_{(2)}.$$

Prove that this defines a left action of $H$ on $H^*$ whose invariants are given by

$$H^{*ad} = \{\alpha \in H^* \,|\Delta(\alpha) = \Delta^{op}(\alpha)\}.$$

**Hint:** Show first that $\Delta(\alpha) = \Delta^{op}(\alpha)$ implies that $(\Delta \otimes \mathrm{id}) \circ \Delta(\alpha)$ is invariant under cyclic permutations of the factors in the tensor product and vice versa.

**Exercise 42:** Let $H$ be a finite-dimensional Hopf algebra over $\mathbb{F}$. We consider the convolution Hopf algebra $C = \mathrm{Hom}_{\mathbb{F}}(H, H)$ with the Hopf algebra structure from Exercise 24

$$f * g = m_H \circ (f \otimes g) \circ \Delta_H \qquad \Delta_H(f) = \Delta_H \circ f \circ m_H \qquad S(f) = S_H \circ f \circ S_H$$
$$1 = \eta_H \circ \epsilon_H \qquad \epsilon(f) = \epsilon_H(f(1_H)).$$

(a) Prove that $f \in I_L(C)$ if and only if $f$ is of the form $f : H \to H$, $h \mapsto \eta(h)\,\ell$ with left integrals $\eta \in I_L(H^*)$ and $\ell \in I_L(H)$.

(b) Let $M$ be a finite-dimensional vector space over $\mathbb{F}$. Show that $C$-left module structures on $M$ are in bijection with pairs $(\rhd, \delta)$, where $\rhd : H \otimes M \to M$ is a $H$-left module structure on $M$ and $\delta : M \to M \otimes H$, $m \mapsto \Sigma_{(m)} m_{(0)} \otimes m_{(1)}$ a $H$-right comodule structure on $M$ such that $\delta(h \rhd m) = \Sigma_{(m)} h \rhd m_{(0)} \otimes m_{(1)}$ for all $h \in H$ and $m \in M$.

(c) Let $M$ be a finite-dimensional left module over $C$. Show that $M^C = M^H \cap M^{coH}$, where $M^H$ is the set of invariants for $\rhd$ and $M^{coH}$ the set of coinvariants for $\delta$ in (b).

**Exercise 43:** Consider the algebra $\mathrm{Mat}(n \times n, \mathbb{F})$ with the linear map $\lambda_A : \mathrm{Mat}(n \times n, \mathbb{F}) \to \mathbb{F}$, $\lambda_A(M) = \mathrm{tr}(M \cdot A)$ for a fixed matrix $A \in \mathrm{GL}(n, \mathbb{F})$. Determine a set of dual bases for $\lambda_A$.

**Exercise 44:** Let $G$ be a finite group and $\mathbb{F}$ a field. We consider the bilinear forms

$$\kappa : \mathbb{F}[G] \otimes \mathbb{F}[G] \to \mathbb{F} \qquad \kappa(g \otimes h) = \delta_e(gh) \quad \forall g, h \in G$$
$$\gamma : \mathrm{Fun}(G, \mathbb{F}) \otimes \mathrm{Fun}(G, \mathbb{F}) \to \mathbb{F} \qquad \gamma(f_1, f_2) = \Sigma_{g \in G} f_1(g) f_2(g) \quad \forall f_1, f_2 \in \mathrm{Fun}(G, \mathbb{F})$$

(a) Show that $(\mathbb{F}[G], \kappa)$ and $(\mathrm{Fun}(G, \mathbb{F}), \gamma)$ are Frobenius algebras.

(b) Determine for both Frobenius algebras the dual bases $\{r_1, ..., r_n\}$ and $\{l_1, ..., l_n\}$ that arise from integrals in $\mathbb{F}[G]$ and $\mathrm{Fun}(G, \mathbb{F})$. Show by an explicit calculation that they are indeed dual bases.

**Exercise 45:** A $(\Delta, \epsilon)$**-Frobenius algebra** is a pentuple $(A, m, \eta, \Delta, \epsilon)$ such that $(A, m, \eta)$ is an algebra, $(A, \Delta, \epsilon)$ is a coalgebra and $\Delta : A \to A \otimes A$ is a morphism of $A \otimes A^{op}$ modules, i. e. $\Delta(a \cdot b) = (a \otimes 1) \cdot \Delta(b) = \Delta(a) \cdot (1 \otimes b)$ for all $a, b \in A$.

(a) Show that every Frobenius algebra $(A, \kappa)$ has a $(\Delta, \epsilon)$-Frobenius algebra structure with

$$\epsilon = \kappa \circ (\mathrm{id}_A \otimes 1_A) \qquad \Delta = (\chi_\kappa^{-1} \otimes \chi_\kappa^{-1}) \circ m^{op*} \circ \chi_\kappa,$$

where $m^* : A^* \to A^* \otimes A^*$ is the dual of $m : A \otimes A \to A$ and $\chi_\kappa : A \to A^*$, $a \mapsto \kappa_a$ is given by $\kappa_a(b) = \kappa(b \otimes a)$ for all $a, b \in A$.

(b) Show that every $(\Delta, \epsilon)$-Frobenius algebra is Frobenius algebra with $\kappa = \epsilon \circ m$.

(c) Show that if $(A, m, \eta, \Delta, \epsilon)$ is both, a bialgebra and a $(\Delta, \epsilon)$-Frobenius algebra over $\mathbb{F}$, then $A \cong \mathbb{F}$.

**Exercise 46:** An algebra $A$ over $\mathbb{F}$ is called **separable** if the multiplication $m : A \otimes A \to A$ has a right inverse in the category of $A \otimes A^{op}$-modules, i. e. there is a linear map $\phi : A \to A \otimes A$ with $m \circ \phi = \mathrm{id}_A$ that is a homomorphism of $A \otimes A^{op}$-modules.

(a) Show that $A$ is separable if and only if there is an element $e \in A \otimes A^{op}$, the **separability idempotent**, with $e^2 = e$, $m(e) = 1$ and $(a \otimes 1) \cdot e = e \cdot (1 \otimes a)$ for all $a \in A$.

(b) Let $H$ be a finite-dimensional semisimple Hopf algebra. Show that $H$ is separable by specifying a separability idempotent for $H$.

**Exercise 47:** Let $H$ be a Hopf algebra over $\mathbb{F}$ and $(M, \rho_M)$ and $(N, \rho_N)$ finite-dimensional modules over $H$. We denote by $(M \oplus N, \rho_{M\oplus N})$ the $H$-module structure on $M \oplus N$ given by $\rho_{M\oplus N}(h)m = \rho_M(h)m$ and $\rho_{M\oplus N}(h) = \rho_N(h)n$ for all $m \in M$, $n \in N$ and by $(M\otimes N, \rho_{M\otimes N})$ the $H$-module structure on $M\otimes N$. The **character** of $(M, \rho_M)$ is the linear map

$$\chi_M : H \to \mathbb{F}, \quad h \mapsto \mathrm{tr}(\rho_M(h))$$

(a) Prove that $\chi_M(1_H) = \dim_{\mathbb{F}}(M)$, $\chi_{M\oplus N} = \chi_M + \chi_N$ and $\chi_{M\otimes N} = \chi_M \cdot \chi_N$ for all $H$-modules $M, N$, where $\cdot$ denotes the multiplication of $H^*$.

(b) Prove that $\chi_M = \chi_N$ if the $H$-modules $M$ and $N$ are isomorphic.

(c) Suppose that $H$ is finite-dimensional and consider $H$ as a left module over itself with the left multiplication. Show that $\chi_H^2 = \chi_H \cdot \dim_{\mathbb{F}} H$ and $\chi_H \circ S^2 = \chi_H$.

**Exercise 48:** Let $\mathbb{F}$ be a field with $\mathrm{char}(\mathbb{F}) = 0$. The algebra $H_8$ is the algebra with generators $x, y, z$ and relations

$$xy = yx \qquad zx = yz, \qquad zy = xz, \qquad x^2 = y^2 = 1, \qquad z^2 = \tfrac{1}{2}(1 + x + y - xy)$$

(a) Show that

$$\begin{aligned} &\Delta(x) = x\otimes x && \Delta(y) = y\otimes y && \Delta(z) = \tfrac{1}{2}(z\otimes z + yz\otimes z + z\otimes xz - yz\otimes xz) \\ &\epsilon(x) = 1 && \epsilon(y) = 1, && \epsilon(z) = 1. \end{aligned}$$

defines a Hopf algebra structure on $H_8$ and determine its antipode.

(b) Determine a basis of $H_8$ and its dimension as a vector space over $\mathbb{F}$.

(c) Show that $\mathrm{Gr}(H_8) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and determine $\mathrm{Pr}(H_8)$.

(d) Determine the left and right integrals and cointegrals of $H_8$.

(e) Show that the Hopf algebra $H_8$ is semisimple.

**Remark:** $H_8$ is an important example, because it is the lowest-dimensional semisimple Hopf algebra that is not a group algebra of a finite group

**Exercise 49:** We consider a directed graph $\Gamma$. We denote by $E$ the set of oriented edges of $\Gamma$ and by $V$ the set of vertices of $\Gamma$ and use the same letters for their cardinalities. For an edge $e \in E$ we denote by $s(e)$ the starting vertex and by $t(e)$ the target vertex of $e$. We denote by $e^{-1}$ the edge with the reversed orientation and set $s(e^{-1}) = t(e)$ and $t(e^{-1}) = s(e)$.

A **path** in $\Gamma$ is either a vertex $v \in V$ or a finite sequence of the form $p = (e_n^{\epsilon_n}, ..., e_1^{\epsilon_1})$ with $n \in \mathbb{N}$, $\epsilon_i \in \{\pm 1\}$ and $e_i \in E$, subject to the condition $s(e_{i+1}^{\epsilon_{i+1}}) = t(e_i^{\epsilon_i})$ for all $i \in \{1, ..., n-1\}$. The starting and target vertex of a path are given by $s(p) = s(e_1^{\epsilon_1})$ and $t(p) = t(e_n^{\epsilon_n})$ and $s(v) = t(v) = v$. If $p, q$ are paths with $s(p) = t(q)$, then $p$ and $q$ are called **composable** and their composite is defined as

$$p \bullet q = \begin{cases} (e_n^{\epsilon_n}, ..., e_1^{\epsilon_1}, f_m^{\tau_m}, ..., f_1^{\tau_1}) & \text{if } p = (e_n^{\epsilon_n}, ..., e_1^{\epsilon_1}), q = (f_m^{\tau_m}, ..., f_1^{\tau_1}) \quad \text{with} \quad s(e_1^{\epsilon_1}) = t(f_m^{\tau_m}) \\ p & \text{if } q = s(p) \\ q & \text{if } p = t(q) \end{cases}$$

Suppose that $H$ is a finite-dimensional semisimple Hopf algebra over a field of characteristic zero and consider the tensor product $H^{\otimes E}$, where each copy of $H$ is assigned to an oriented edge $e \in E$. A **holonomy** for $\Gamma$ is a map

$$\mathrm{Hol}: \{\text{paths in } \Gamma\} \to \mathrm{Hom}_{\mathbb{F}}(H^{\otimes E}, H), \quad p \mapsto \phi_p$$

that assigns to each path $p$ in $\Gamma$ a linear map $\phi_p : H^{\otimes E} \to H$ such that the following conditions are satisfied for all $e \in E$, $v \in V$ and composable paths $p, q$ in $\Gamma$

$$\phi_v(h^1 \otimes ... \otimes h^E) = \Pi_{e\in E}\epsilon(h^e)1 \qquad \phi_e(h^1 \otimes ... \otimes h^E) = \Pi_{f\neq e}\epsilon(h^f)\, h^e$$
$$\phi_{e^{-1}} = S \circ \phi_e \qquad \phi_{p\bullet q} = \phi_p \star \phi_q,$$

with a fixed linear map $\star : \mathrm{Hom}_{\mathbb{F}}(H^{\otimes E}, H) \otimes \mathrm{Hom}_{\mathbb{F}}(H^{\otimes E}, H) \to \mathrm{Hom}_{\mathbb{F}}(H^{\otimes E}, H)$.

(a) Show that the conditions on the holonomy for general graphs $\Gamma$ imply that the linear map $\star$ must be associative and that $\phi_v$ must be a unit for $\star$.
(b) Show that choosing for $\star$ the convolution product with respect to the multiplication of $H$ and the tensor product coalgebra structure on $H^{\otimes E}$ defines a holonomy for $\Gamma$.
(c) Determine the holonomy maps $\phi_p$ for the case where $H = \mathbb{F}[G]$ is the group algebra of a finite group and interpret them geometrically.
(d) We consider the equivalence relation on the set of paths in $\Gamma$ induced by $e^{-1} \bullet e \sim s(e)$ and $e \bullet e^{-1} \sim t(e)$ for all edges $e \in E$. Show that the holonomy from (b) satisfies $\phi_p = \phi_{p'}$ for all paths $p \sim p'$.
(e) For a path $p = (e_n^{\epsilon_n}, .., e_1^{\epsilon_1})$ the reversed path is defined as $p^{-1} = (e_1^{-\epsilon_1}, ..., e_n^{-\epsilon_n})$ and for $p = v \in V$ one sets $p^{-1} = v$. Show that for every path $p = (e_n^{\epsilon_n}, .., e_1^{\epsilon_1})$ with $e_i \neq e_j$ for $i \neq j$ one has $\phi_{p^{-1}} = S \circ \phi_p$. Does this hold in general? Does this still hold if one drops the assumption that $H$ is semisimple?

**Exercise 50:** Suppose that $q \in \mathbb{C}$ is not a root of unity and let $V_{\epsilon,n}$ denote the (n+1)-dimensional highest weight module over $U_q(\mathfrak{sl}_2)$ of weight $\lambda = \epsilon q^n$ with $\epsilon = \{\pm 1\}$. Show that $V_{\epsilon,n} \cong V_{\epsilon,0} \otimes V_{1,n} \cong V_{1,n} \otimes V_{\epsilon,0}$ as $U_q(\mathfrak{sl}_2)$-modules.

**Exercise 51:** Suppose that $q \in \mathbb{C}$ is not a root of unity and $n, m \in \mathbb{N}_0$ with $n \geq m$. Let $V_{1,n}$ denote the (n+1)-dimensional highest weight module over $U_q(\mathfrak{sl}_2)$ of weight $\lambda = q^n$ and $v^{(n)} \in V_{1,n}$ and $v^{(m)} \in V_{1,m}$ highest weight vectors and set $v_p = q^{p(p-1)}/(p)!_{q^2}\, F^p \rhd v$ for $p \in \{0, ..., n\}$ and a highest weight vector $v \in V_{1,n}$. Show that for all $p \in \{0, ..., m\}$ the vector

$$v^{(n+m-2p)} = \sum_{j=0}^{p} (-1)^j \frac{(m-p+j)!_{q^2}(n-j)!_{q^2}}{(m-p)!_{q^2}(n)!_{q^2}} q^{-j(2m-3p-n+2j+1)}\, v_j^{(n)} \otimes v_{p-j}^{(m)}$$

is a highest weight vector in $V_{1,n} \otimes V_{1,m}$ of weight $q^{n+m-2p}$.

**Exercise 52:** Suppose that $q \in \mathbb{C}$ is not a root of unity, $n, m \in \mathbb{N}_0$ with $n \geq m$ and denote by $V_{1,n}$ denote the (n+1)-dimensional highest weight module over $U_q(\mathfrak{sl}_2)$ of weight $\lambda = q^n$.

(a) Show that for all $p \in \{0, ..., m\}$, there is an injective $U_q(\mathfrak{sl}_2)$-module homomorphism $\phi_p : V_{1,n+m-2p} \to V_{1,n} \otimes V_{1,m}$.

(b) Conclude that $V_{1,n} \otimes V_{1,m} \cong V_{1,n+m} \oplus V_{1,n+m-2} \oplus ... \oplus V_{1,n-m+2} \oplus V_{1,n-m}$ as $U_q(\mathfrak{sl}_2)$-modules.

**Exercise 53:** Suppose that $q \in \mathbb{C}$, $n \in \mathbb{N}_0$ and denote by $V_{\epsilon,n}$ denote the (n+1)-dimensional highest weight module over $U_q(\mathfrak{sl}_2)$ of weight $\lambda = \epsilon q^n$ with $\epsilon = \{\pm 1\}$. Show that for all $n \in \mathbb{N}_0$ there is an $U_q(\mathfrak{sl}_2)$-module isomorphism between the module $V_{\epsilon,n}$ and the dual module $V_{\epsilon,n}^*$.

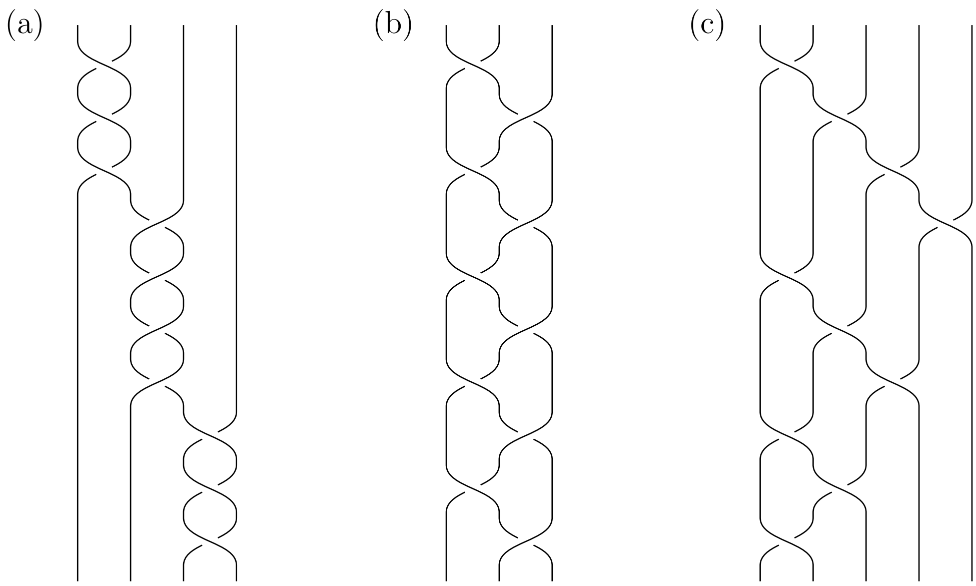
## 7.4 Exercises for Chapter 4

**Exercise 54:** We consider a group $G$ as a category with a single object, with group elements $g \in G$ as morphisms, the group multiplication as the composition of morphisms and the unit $e \in G$ as the identity morphism.

(a) Show that functors $F : G \to \mathrm{Vect}_{\mathbb{F}}$ correspond to representations of $G$ on vector spaces over $\mathbb{F}$ and natural transformations $\eta : F \to F'$ between functors $F, F' : G \to \mathrm{Vect}_{\mathbb{F}}$ to homomorphisms of representations.
(b) Characterise all functors $F : G \to H$ for groups $G$, $H$ viewed as categories with a single object and all natural transformation between such functors.
(c) Describe the functor category $\mathrm{Fun}(G, H)$ for two groups $G, H$ and the monoidal category $\mathrm{End}(G)$ for a group $G$.

**Exercise 55:** Let $G$ be a group and $\mathbb{F}$ a field. A **$G$-graded vector space** over $\mathbb{F}$ is a vector space $V$ over $\mathbb{F}$ with a direct sum decomposition $V = \oplus_{g\in G} V_g$ and a **homomorphism of $G$-graded vector spaces** is a linear map $f : V \to W$ with $f(V_g) \subset W_g$ for all $g \in G$. Show that $G$-graded vector spaces and homomorphisms of $G$-graded vector spaces form a monoidal category.

**Exercise 56:** Express the following elements of the braid groups $B_n$ as a product in the generators $\sigma_i$ for $i \in \{1, ..., n-1\}$ and their inverses:




Determine the the images of these elements under the group homomorphism $\Pi_n : B_n \to S_n$, $\sigma_i \to \pi_i$ and express them in terms of elementary transpositions $\pi_i$ for $i \in \{1, ..., n\}$ using as few elementary transpositions as possible.

**Exercise 57:** Let $(G_n)_{n\in\mathbb{N}_0}$ be a family of groups with $G_0 = \{e\}$ and $(\rho_{m,n})_{m,n\in\mathbb{N}_0}$ a family of group homomorphisms $\rho_{m,n} : G_m \times G_n \to G_{m+n}$ such that $\rho_{0,m}$ and $\rho_{m,0}$ are given by $\rho_{0,m} : \{e\} \times G_m \to G_m$, $(e, g) \mapsto g$ and $\rho_{m,0} : G_m \times \{e\} \to G_m$, $(g, e) \mapsto g$ and

$$\rho_{m+n,p} \circ (\rho_{m,n} \times \mathrm{id}_{G_p}) = \rho_{m,n+p} \circ (\mathrm{id}_{G_m} \times \rho_{n,p}) \quad \forall m, n, p \in \mathbb{N}_0. \tag{63}$$

(a) Show that this defines a *strict* tensor category $\mathcal{G}$ with objects $n \in \mathbb{N}_0$, morphisms

$$\mathrm{Hom}_{\mathcal{G}}(n,m) = \begin{cases} \emptyset & n \neq m \\ G_n & n = m \end{cases}$$

such that the composition of morphisms is given by the group multiplication, the tensor product by $m \otimes n := m + n$ on objects $m, n \in \mathbb{N}_0$ and by $f \otimes g := \rho_{m,n}(f \times g)$ on morphisms $f \in G_m$, $g \in G_n$ and the tensor unit by $e = 0$.

(b) Let $((G_n)_{n\in\mathbb{N}_0}, (\rho_{m,n})_{m,n\in\mathbb{N}_0})$ and $((H_n)_{n\in\mathbb{N}_0}, (\tau_{m,n})_{m,n\in\mathbb{N}_0})$ be two families of groups and group homomorphisms that satisfy the conditions above and $\mathcal{G}$ and $\mathcal{H}$ the associated strict tensor categories. Show that a family $(\mu_n)_{n\in\mathbb{N}}$ of group homomorphisms $\mu_n : G_n \to H_n$ defines a strict tensor functor $F : \mathcal{G} \to \mathcal{H}$ if and only if

$$\tau_{m,n} \circ (\mu_m \times \mu_n) = \mu_{n+m} \circ \rho_{m,n} \qquad \forall m, n \in \mathbb{N}_0.$$

(c) Use (b) to construct a strict tensor functor $F : \mathcal{B} \to \mathcal{S}$ from the braid category $\mathcal{B}$ to the permutation category $\mathcal{S}$.

(d) Consider the family of groups $(\mathrm{GL}(n,\mathbb{F}))_{n\in\mathbb{N}_0}$, where $\mathrm{GL}(n,\mathbb{F})$ is the group of invertible $(n \times n)$-matrices with entries in $\mathbb{F}$. Define a family $(\nu_{m,n})_{m,n\in\mathbb{N}_0}$ of group homomorphisms $\nu_{m,n} : \mathrm{GL}(m,\mathbb{F}) \times \mathrm{GL}(n,\mathbb{F}) \to \mathrm{GL}(m+n,\mathbb{F})$ with the properties above and construct the associated strict tensor category $\mathcal{GL}(\mathbb{F})$. Construct a strict tensor functor $F : \mathcal{S} \to \mathcal{GL}(\mathbb{F})$.

**Exercise 58:** Let $H$ be a Hopf algebra and $g \in \mathrm{Gr}(H)$ a grouplike element.

(a) Show that for every $H$-module $(M, \rhd)$, one obtains an $H$-module $(M, \rhd_g)$ by setting $h \rhd_g m = (ghg^{-1}) \rhd m$ for all $h \in H$, $m \in M$ and that every $H$-module map $f : (M, \rhd) \to (N, \rhd')$ is also an $H$-module map $f : (M, \rhd_g) \to (N, \rhd'_g)$.

(b) Show that this defines a tensor functor $F_g : H\text{-Mod} \to H\text{-Mod}$. Is it strict?

(c) Show that the maps $\mu_g : M \to M$, $m \mapsto g \rhd m$ define a monoidal natural isomorphism $\mu : \mathrm{id}_{H\text{-Mod}} \to F_g$.

**Exercise 59:** A **$(\Delta, \epsilon)$-Frobenius algebra** over $\mathbb{F}$ is a pentuple $(A, m, \eta, \Delta, \epsilon)$ such that $(A, m, \eta)$ is an algebra, $(A, \Delta, \epsilon)$ is a coalgebra over $\mathbb{F}$ and $\Delta : A \to A \otimes A$ is a morphism of $A \otimes A^{op}$ modules, i. e. $\Delta(a \cdot b) = (a \otimes 1) \cdot \Delta(b) = \Delta(a) \cdot (1 \otimes b)$ for all $a, b \in A$.

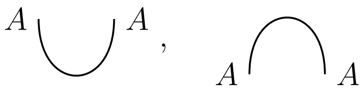
A **Frobenius algebra** over $\mathbb{F}$ is a quadruple $(A, m, \eta, \kappa)$ such that $(A, m, \eta)$ is an algebra over $\mathbb{F}$ and $\kappa : A \otimes A \to \mathbb{F}$ a non-degenerate linear map with $\kappa(a \cdot b, c) = \kappa(a, b \cdot c)$ for all $a, b, c \in A$.

(a) Generalise the concepts of a Frobenius algebra and a $(\Delta, \epsilon)$-Frobenius algebra to a Frobenius algebra and an $(\Delta, \epsilon)$-Frobenius algebra in a general monoidal category $\mathcal{C}$ and describe their defining properties by diagrams.

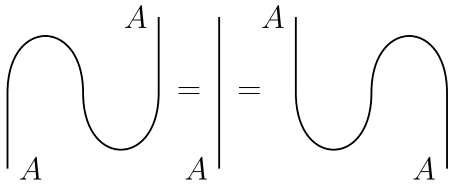
(b) Show with a *diagrammatical proof* that every that every $(\Delta, \epsilon)$-Frobenius algebra is a Frobenius algebra with $\kappa = \epsilon \circ m$.

(c) Show with a *diagrammatical proof* that every that every Frobenius algebra is a $(\Delta, \epsilon)$-Frobenius algebra.

**Hint:** In (a) the non-degeneracy of the Frobenius form $\kappa$ implies that there are diagrams of the form




such that




**Exercise 60:**
Let $\mathcal{C}$ be a category with finite products and $\mathcal{D}$ a category with finite coproducts.

(a) Show that $\mathcal{C}$ and $\mathcal{D}$ have the structure of monoidal categories with the tensor product given on objects by their (co)product and the empty (co)product as the tensor unit.

(b) Show that every object in $\mathcal{C}$ has the structure of a coalgebra object and every object in $\mathcal{D}$ the structure of an algebra object.

**Exercise 61:** Let $(\mathcal{C}, \otimes, e, a, l, r)$ be a monoidal category. Show that if $c : \otimes \to \otimes^{op}$ is a braiding, then $c' : \otimes \to \otimes^{op}$ with $c'_{U,V} = c^{-1}_{V,U} : U \otimes V \to V \otimes U$ is also a braiding.

**Exercise 62:** Let $q, \lambda \in \mathbb{F} \setminus \{0\}$ and $V$ a vector space over $\mathbb{F}$ with an ordered basis $(v_1, ..., v_n)$. Show that the linear map $\sigma : V \otimes V \to V \otimes V$ with

$$\sigma(v_i \otimes v_j) = \begin{cases} \lambda\, v_j \otimes v_i & i < j \\ \lambda q\, v_i \otimes v_i & i = j \\ \lambda\, v_j \otimes v_i + \lambda(q - q^{-1}) v_i \otimes v_j & i > j \end{cases}$$

gives $V$ the structure of a braided vector space.

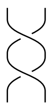
**Exercise 63:** Let $G$ be a group. A **crossed $G$-set** is a triple $(X, \lhd, \mu)$ of a set $X$, a right action $\lhd : X \times G \to X$ and a map $\mu : X \to G$ such that $\mu(x \lhd g) = g^{-1} \mu(x) g$ for all $x \in X$ and $g \in G$. A **morphism of crossed $G$-sets** $f : (X, \lhd_X, \mu_X) \to (Y, \lhd_Y, \mu_Y)$ is a map $f : X \to Y$ with $\mu_Y \circ f = \mu_X$ and $f(x \lhd_X g) = f(x) \lhd_Y g$ for all $x \in X$ and $g \in G$.

(a) Show that crossed $G$-sets and morphisms of crossed $G$-sets form a monoidal category $G$-Set with the tensor product $(X, \lhd_X, \mu_X) \otimes (Y, \lhd_Y, \mu_Y) = (X \times Y, \lhd, \mu)$ with $(x, y) \lhd g = (x \lhd_X g, y \lhd_Y g)$ and $\mu(x, y) = \mu_X(x) \cdot \mu_Y(y)$.

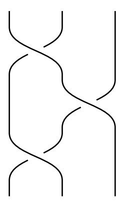
(b) Show that $c_{X,Y} : X \times Y \to Y \times X$, $(x, y) \mapsto (y, x \lhd_X \mu_Y(y))$ defines a braiding in $G$-Set.

(c) Determine the morphisms $c_{X^{\times m}, X^{\times n}} : X^{\times m} \otimes X^{\times n} \to X^{\times n} \otimes X^{\times m}$ for a crossed $G$-set $(X, \lhd, \mu)$ and $n, m \in \mathbb{N}_0$ and the images of the following braids under the braided monoidal functor $F_X : \mathcal{B} \to G$-Set with $F(1) = X$.

(i)



(ii)



(d) Show that there is a terminal object in $G$-set, i. e. a crossed $G$-set $(T, \lhd_T, \mu_T)$ such that for every crossed $G$-set $(X, \lhd_X, \mu_X)$, there is a unique morphism of crossed $G$-sets $f : (X, \lhd_X, \mu_X) \to (T, \lhd_T, \mu_T)$.

**Exercise 64:** Let $\mathcal{B}$ be the braid category and $\mathcal{C}$ a strict monoidal category.
A **Yang-Baxter operator** in $\mathcal{C}$ is a pair $(X, \sigma)$ of an object $X$ in $\mathcal{C}$ and a morphism $\sigma : X \otimes X \to X \otimes X$ that satisfies the dodecagon identity. A **morphism of Yang-Baxter operators** $f : (X, \sigma) \to (Y, \tau)$ is a morphism $f : X \to Y$ with $\tau \circ (f \otimes f) = (f \otimes f) \circ \sigma$.

(a) Show that Yang-Baxter operators in $\mathcal{C}$ and morphisms of Yang-Baxter operators in $\mathcal{C}$ form a category $\mathcal{YB}(\mathcal{C})$.
(b) For a Yang-Baxter operator $(X, \sigma)$ in $\mathcal{C}$, denote by $F_{(x,\sigma)} : \mathcal{B} \to \mathcal{C}$ the strict tensor functor with $F_{(X,\sigma)}(1) = X$ and $F_{(X,\sigma)}(c_{1,1}) = \sigma$. Show that monoidal natural transformations $\eta : F_{(X,\sigma)} \to F_{(Y,\tau)}$ are in bijection with morphisms $f : (X, \sigma) \to (Y, \tau)$ of Yang-Baxter operators.
(c) Denote by $\mathrm{Fun}_{\otimes}(\mathcal{B}, \mathcal{C})$ the category of strict monoidal functors $F : \mathcal{B} \to \mathcal{C}$ and monoidal natural transformations between them. Show that the categories $\mathrm{Fun}_{\otimes}(\mathcal{B}, \mathcal{C})$ and $\mathcal{YB}(\mathcal{C})$ are equivalent.
(d) Suppose now that $\mathcal{C}$ is *braided* and denote by $\mathrm{Fun}_{br}(\mathcal{B}, \mathcal{C})$ the category of strict braided monoidal functors $F : \mathcal{B} \to \mathcal{C}$ and monoidal natural transformations between them. Show that the categories $\mathrm{Fun}_{br}(\mathcal{B}, \mathcal{C})$ and $\mathcal{C}$ are equivalent.

**Exercise 65:** The **centre construction** associates to every strict monoidal category $\mathcal{C}$ a strict *braided* monoidal category $\mathcal{Z}(\mathcal{C})$, defined as follows:

- **objects** of $\mathcal{Z}(\mathcal{C})$ are pairs $(Z, c_{-,Z})$ of an object $Z \in \mathrm{Ob}(\mathcal{C})$ and a family of isomorphisms $c_{Y,Z} : Y \otimes Z \to Z \otimes Y$ for all objects $Y$ in $\mathcal{C}$ such that:
  (i) $c_{X \otimes Y, Z} = (c_{X,Z} \otimes 1_Y) \circ (1_X \otimes c_{Y,Z})$ for all $X, Y \in \mathrm{Ob}\,\mathcal{C}$.
  (ii) The family of morphisms $c_{-,Z}$ is natural in the first argument: for all morphisms $f : X \to Y$ the following diagram commutes

$$\begin{array}{ccc} X \otimes Z & \xrightarrow{c_{X,Z}} & Z \otimes X \\ {\scriptstyle f \otimes 1_Z}\downarrow & & \downarrow{\scriptstyle 1_Z \otimes f} \\ Y \otimes Z & \xrightarrow[c_{Y,Z}]{} & Z \otimes Y \end{array}$$

- **morphisms** $f : (Y, c_{-,Y}) \to (Z, c_{-,Z})$ of $\mathcal{Z}(\mathcal{C})$ are morphisms $f : Y \to Z$ in $\mathcal{C}$ for which the following diagram commutes for all $X \in \mathrm{Ob}\,\mathcal{C}$

$$\begin{array}{ccc} X \otimes Y & \xrightarrow{c_{X,Y}} & Y \otimes X \\ {\scriptstyle 1_X \otimes f}\downarrow & & \downarrow{\scriptstyle f \otimes 1_X} \\ X \otimes Z & \xrightarrow[c_{X,Z}]{} & Z \otimes X \end{array}$$

(a) Show that $\mathcal{Z}(\mathcal{C})$ is a strict monoidal category with tensor unit $(e, \mathrm{id}_-)$ and tensor product $(Y, c_{-,Y}) \otimes (Z, c_{-,Z}) = (Y \otimes Z, c_{-,Y \otimes Z})$, where

$$c_{X, Y \otimes Z} := (1_Y \otimes c_{X,Z}) \circ (c_{X,Y} \otimes 1_Z) : X \otimes Y \otimes Z \to Y \otimes Z \otimes X$$

(b) Show that the strict monoidal category $\mathcal{Z}(\mathcal{C})$ is braided with the braiding

$$c_{(Y,c_{-,Y}),(Z,c_{-,Z})} = c_{Y,Z} : Y \otimes Z \to Z \otimes Y.$$

(c) Let $\mathcal{A}$ be a strict braided monoidal category and denote by $\Pi : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ the strict tensor functor with $F((Z, c_{-,Z})) = Z$ and $F(f) = f : Y \to Z$ for all objects $(Z, c_{-,Z})$ and morphisms $f : (Y, c_{-,Y}) \to (Z, c_{-,Z})$ in $\mathcal{Z}(\mathcal{C})$. Suppose that $F : \mathcal{A} \to \mathcal{C}$ is a strict monoidal functor that is bijective on the objects and surjective on the morphisms. Show that there is a unique *braided* monoidal functor $\tilde{F} : \mathcal{A} \to \mathcal{Z}(\mathcal{C})$ with $\Pi\tilde{F} = F$.

**Exercise 66:** Use the presentation of the cobordism category $\mathrm{Cob}_{1,0}$ to classify all 1d oriented topological quantum field theories $Z : \mathrm{Cob}_{1,0} \to \mathrm{Vect}_{\mathbb{F}}^{fin}$ with $Z* = *Z$.

**Exercise 67:** Let $G$ be a group, $\mathbb{F}$ a field and $\lambda \in \mathbb{F} \setminus \{0\}$. We consider the commutative Frobenius algebra $\mathrm{Fun}(G, \mathbb{F})$ with the pointwise product and the Frobenius form

$$\kappa(f \otimes h) = \lambda \sum_{g \in G} f(g)h(g)$$

and the associated 2d topological quantum field theory $Z : \mathrm{Cob}_{2,1} \to \mathrm{Vect}_{\mathbb{F}}^{fin}$.

(a) Determine the corresponding $(\Delta, \epsilon)$-Frobenius structure on $\mathrm{Fun}(G, \mathbb{F})$.

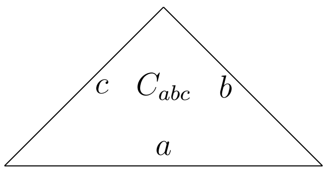
(b) Compute $Z(S)$ for the case where $S$ is an oriented surface of genus $g \in \mathbb{N}_0$.

(c) Compute $Z(S)$ for the case where $S$ is the disjoint union of $n \in \mathbb{N}$ tori.

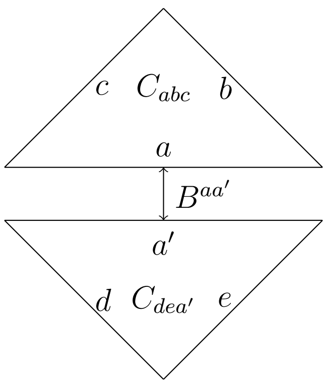
**Exercise 68:** (**Fukuma-Hosono-Kawai model**)

Let $\Sigma$ be a triangulated oriented surface, i. e. an oriented surface obtained by gluing finitely many triangles along their edges, $I$ a finite set and $\mathbb{F}$ a field. We associate to $\Sigma$ a number $Z(\Sigma) \in \mathbb{F}$ defined as follows:

1. Assign to each edge in an oriented triangle $t$ an element of $I$, and to the oriented triangle $t = (abc)$ formed by edges labelled with $a, b, c \in I$ a number $C_{abc} \in \mathbb{F}$, the **triangle constant**, satisfying $C_{abc} = C_{cab} = C_{bca}$.




2. Each edge in the surface $\Sigma$ occurs in two adjacent triangles $t, t'$ and carries two labels $a, a' \in I$. Assign to such an edge a number $B^{aa'} \in \mathbb{F}$, the **gluing constant**, such that the matrix $B = (B^{aa'})_{a,a' \in I}$ is symmetric and invertible.

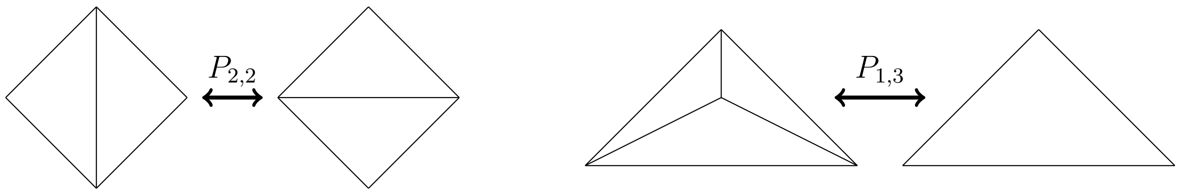


3. Assign to the triangulated surface $\Sigma$ the number

$$Z(\Sigma) = R^{-V} \sum_{f:E\to I\times I} \prod_{e\in E} B^{ee'} \prod_{t=(efg)\in T} C_{efg}$$

where $R \in \mathbb{F}\setminus\{0\}$, $V$ is the number of vertices, $E$ the set of edges and $T$ the set of triangles of the triangulated surface $\Sigma$, the sum runs over all assignments $f : E \to I\times I$ of pairs $(a,b) \in I\times I$ to the edges in $E$ and the products are taken over all labelled edges and labelled triangles.

One can show that two triangulated surfaces are homeomorphic if and only if the triangulations are related by a finite sequence of the two **Pachner moves**




If $Z(\Sigma)$ is invariant under the two Pachner moves $P_{2,2}$ and $P_{1,3}$, then $Z(\Sigma)$ depends only on the homeomorphism class of the oriented surface $\Sigma$ and not on the choice of the triangulation, i. e. $Z(\Sigma)$ is a topological invariant.

Show that $Z(\Sigma)$ is a topological invariant if and only if the constants $C_{abc}$ and $B^{aa'}$ define a Frobenius algebra $(A, m, 1, \Delta, \epsilon)$ over $\mathbb{F}$ with $\epsilon\circ m\circ\Delta = R\epsilon$ and $\epsilon(1) = R^{-1}\,|I|$. Proceed as follows:

(a) Derive the conditions on the constants $C_{abc}$, $B^{ab}$, $B_{cd}$ that are equivalent to the statement that $Z(\Sigma)$ is invariant under the two Pachner moves.

(b) Consider the free vector space $A = \langle I\rangle_{\mathbb{F}}$ with basis $I$, define a linear map $\kappa : A\otimes A \to \mathbb{F}$ and a multiplication map $m : A\otimes A \to A$ by

$$a\cdot b = m(a\otimes b) = \sum_{c,d\in I} C_{abc}B^{cd}d \qquad \kappa(c\otimes d) = B_{cd} \qquad \forall a,b,c,d\in I,$$

where $B_{aa'}$ are the coefficients of the inverse matrix $B^{-1} = (B_{aa'})_{a,a'\in I}$. Show that the conditions on the coefficients $C_{abc}$, $B^{ab}$, $B_{cd}$ from (a) guarantee that (i) $\cdot$ is associative, (ii) $\kappa$ satisfies $\kappa(a\cdot b\otimes c) = \kappa(a\otimes b\cdot c)$ and (iii) $1 = R^{-1}\sum_{a,b,c,d\in I} C_{abc}B^{ab}B^{cd}d$ is a unit for the multiplication $\cdot$.

(c) Show that the $(\Delta,\epsilon)$-Frobenius algebra associated with the Frobenius algebra in (b) satisfies the conditions $\epsilon\circ m\circ\Delta = R\,\epsilon$ and $\epsilon(1) = R^{-1}\,|I|$.

(d) Show that any $(\Delta,\epsilon)$-Frobenius algebra that satisfies the two conditions in (c) gives rise to a finite set $I$, triangle constants $C_{abc}$ and gluing constants $B^{ab}$ such that $Z(\Sigma)$ is a topological invariant.

## 7.5 Exercises for Chapters 5 and 6

**Exercise 69:** Let $G$ be a finite group and $\mathbb{F}[G]$ its group algebra over $\mathbb{F}$. Verify that the Drinfeld double $D(\mathbb{F}[G])$ is given by

$$(\delta_u \otimes g) \cdot (\delta_v \otimes h) = \delta_u(gvg^{-1})\delta_u \otimes gh \qquad 1 = 1 \otimes e = \Sigma_{g\in G}\, \delta_g \otimes e$$
$$\Delta(\delta_u \otimes g) = \Sigma_{xy=u}\, \delta_y \otimes g \otimes \delta_x \otimes g \qquad \epsilon(\delta_u \otimes g) = \delta_u(e)$$
$$S(\delta_u \otimes g) = \delta_{g^{-1}u^{-1}g} \otimes g^{-1}$$

and that $R = \Sigma_{g\in G} 1 \otimes g \otimes \delta_g \otimes e$ is a universal $R$-matrix for $D(\mathbb{F}[G])$.

**Exercise 70:** Let $G$ be a finite group.

(a) Show that a module $V$ over the Drinfeld double $D(\mathbb{F}[G])$ is a module $(V, \rhd)$ over $\mathbb{F}[G]$ together with a decomposition $V = \oplus_{g\in G} V_g$ such that $h \rhd V_g \subset V_{hgh^{-1}}$ for all $g, h \in G$.

(b) Let $V = \oplus_{g\in G} V_g$ and $W = \oplus_{h\in G} W_h$ be modules over $D(\mathbb{F}[G])$. Show that the maps $c_{V,W} : V \otimes W \to W \otimes V$ defined by $c_{V,W}|_{V_g \otimes W_h} : V_g \otimes W_h \to W_h \otimes V_{hgh^{-1}}$, $v \otimes w \mapsto w \otimes h \rhd v$ for all $g, h \in G$, $v \in V$ and $w \in W$ define a braiding for $D(\mathbb{F}[G])$-Mod.

**Exercise 71:** A **matched pair of groups** is a pair $(H, K)$ of groups together with a left action $\rhd : K \times H \to H$ and a right action $\lhd : K \otimes H \to K$ such that

$$(k \cdot k') \lhd h = (k \lhd (k' \rhd h)) \cdot (k' \lhd h) \qquad e_K \lhd h = e_K$$
$$k \rhd (h \cdot h') = (k \rhd h) \cdot ((k \lhd h) \rhd h') \qquad k \rhd e_H = e_H$$

for all $h, h' \in H$ and $k, k' \in K$. Prove the following:

(a) If $(H, K)$ is a matched pair of groups, then there is a unique group structure on the set $H \times K$, the **bicrossproduct** $H \bowtie K$, such that

$$(h, k) \cdot (h', k') = (h(k \rhd h'), (k \lhd h')k').$$

(b) If $G$ is a group and $H, K \subset G$ subgroups such that $\mu|_{H\times K} : H \times K \to G$ is a bijection, then $(H, K)$ is a matched pair of groups and $G$ is isomorphic as a group to $H \bowtie K$.

(c) Products and semidirect products of groups are special cases of bicrossproducts of groups.

**Exercise 72:** A **matched pair of bialgebras** is a pair $(H, K)$ of bialgebras together with a $K$-left module coalgebra structure $\rhd : K \times H \to H$ on $H$ and a $H$-right module coalgebra structure $\lhd : K \otimes H \to K$ on $K$ such that

$$(k \cdot k') \lhd h = \Sigma_{(h),(k')}\, (k \lhd (k'_{(1)} \rhd h_{(1)})) \cdot (k'_{(2)} \lhd h_{(2)}) \qquad 1_K \lhd h = \epsilon(h)\, 1_K$$
$$k \rhd (h \cdot h') = \Sigma_{(h),(k)}\, (k_{(1)} \rhd h_{(1)}) \cdot ((k_{(2)} \lhd h_{(2)}) \rhd h') \qquad k \rhd 1_H = \epsilon(k)\, 1_H$$
$$\Sigma_{(h)(k)}\, (k_{(1)} \lhd h_{(1)}) \otimes (k_{(2)} \rhd h_{(2)}) = \Sigma_{(h)(k)}\, (k_{(2)} \lhd h_{(2)}) \otimes (k_{(1)} \rhd h_{(1)})$$

for all $h, h' \in H$ and $k, k' \in K$.

(a) Show that if $(H, K)$ is a matched pair of bialgebras, then the following defines a bialgebra structure on $H \otimes K$

$$(h \otimes k) \cdot (h' \otimes k') = \Sigma_{(k)(h')} h(k_{(1)} \rhd h'_{(1)}) \otimes (k_{(2)} \lhd h'_{(2)})k'$$
$$\Delta(h \otimes k) = \Sigma_{(h),(k)}\, h_{(1)} \otimes k_{(1)} \otimes h_{(2)} \otimes k_{(2)}.$$

This bialgebra is called the **bicrossproduct** of $H \bowtie K$.

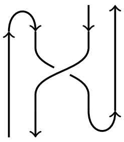
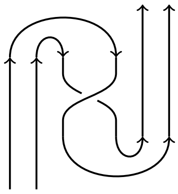
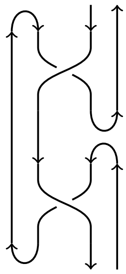
(b) Show that if $H$ and $K$ are Hopf algebras, then the bicrossproduct $H \bowtie K$ is a Hopf algebra as well.

(c) Show that every $x \in H \bowtie K$ can be factorised uniquely as $x = (h \otimes 1) \cdot (1 \otimes k) = (1 \otimes k') \cdot (h' \otimes 1)$. Determine $k', h'$ in terms of $h, k$ and $h, k$ in terms of $h', k'$.

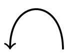
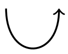
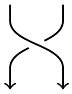
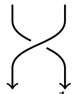
(d) Show that tensor products of bialgebras and cross products of bialgebras are examples of bicrossproduct bialgebras.

(e) Show that the Drinfeld double $D(H)$ of a finite-dimensional Hopf algebra $H$ is a bicrossproduct $H$ and $H^{*cop}$.

**Exercise 73:** Express the following morphisms in the category $\mathcal{T}$ of ribbon tangles

(a)  (b)  (c) 

in terms of the generators

| | | | | | |
|---|---|---|---|---|---|
|  |  |  |  |  |  |
| $\cup$ : | $\cap$ : | $\cup'$ : | $\cap'$ : | $X$ : | $X^{-1}$ : |
| $-\otimes+ \to \emptyset$ | $\emptyset \to +\otimes-$ | $+\otimes- \to \emptyset$ | $\emptyset \to -\otimes+$ | $+\otimes+ \to +\otimes+$ | $+\otimes+ \to +\otimes+$ |

# A Algebraic background

## A.1 Modules over rings

In this section, we assemble without proofs some basic facts about modules over rings. The notion of a module over a ring $R$ unites concepts such as vector spaces (modules over fields), abelian groups (modules over $\mathbb{Z}$), group representations (modules over the group algebra) and representations of algebras (modules over an algebra) in a common framework.

**Definition 1.1.1:**

1. A **left module** $(M, \rhd)$ over a ring $R$ is an abelian group $(M, +)$ together with a map $\rhd : R \times M \to M$, $(r, m) \mapsto r \rhd m$ that satisfies for all $r, r' \in R$ and $m, m' \in M$

   $$r \rhd (m+m') = (r \rhd m) + (r \rhd m') \quad (r+r') \rhd m = (r \rhd m) + (r' \rhd m) \quad (r \cdot r') \rhd m = r \rhd (r' \rhd m).$$

   If $R$ is unital, one also requires $1 \rhd m = m$ for all $m \in M$.

2. A **morphism of left modules** or an $R$**-linear map** from $(M, \rhd_M)$ to $(N, \rhd_N)$ is a group homomorphism $\phi : M \to N$ with $\phi(r \rhd_M m) = r \rhd_N \phi(m)$ for all $m \in M$ and $r \in R$. If $\phi$ is bijective it is called a **module isomorphism** and the modules are called **isomorphic**.

**Remark 1.1.2:**

1. Analogously one defines a **right module** over $R$ as an abelian group $(M, +)$ together with a map $\lhd : M \times R \to M$, $(m, r) \mapsto m \lhd r$ that satisfies for all $r, r' \in R$ and $m, m' \in M$

   $$(m+m') \lhd r = (m \lhd r) + (m' \lhd r) \quad m \lhd (r+r') = (m \lhd r) + (m \lhd r') \quad m \lhd (r \cdot r') = (m \lhd r) \lhd r'.$$

   If $R$ is unital, one also requires $m \lhd 1 = m$ for all $m \in M$.

2. Every right module $(M, \lhd)$ over $R$ has a canonical left module structure over the ring $R^{op}$ with the opposite multiplication given by $r \rhd_{op} m = m \lhd r$, and every left module $(M, \rhd)$ over $R$ is a right module over $R^{op}$ with $m \lhd_{op} r := r \rhd m$.

3. An $(R, S)$-**bimodule** is a abelian group $(M, +)$ together with an $R$-left module structure $\rhd : R \times M \to M$ and an $S$-right module structure $\lhd : M \times S \to M$ such that $(r \rhd m) \lhd s = r \rhd (m \lhd s)$ for all $r \in R$, $s \in S$ and $m \in M$. Alternatively, we can view an $(R, S)$-bimodule as a left module over $R \times S^{op}$.

4. If $R$ is a commutative ring, then every $R$-left module is an $R$-right module and an $(R, R)$-bimodule and vice versa.

In the following, we take the term *module* to mean left module. A right module over a ring is either interpreted as a left module over $R^{op}$ or explicitly referred to as a right module.

**Example 1.1.3:**

1. Every ring unital ring $R$ is a left module over itself with the left multiplication $\rhd : R \times R \to R$, $(r, s) \mapsto r \cdot s$ and a right module over itself with the right multiplication $\lhd : R \times R \to R$, $(s, r) \mapsto s \cdot r$. Together, they give $R$ the structure of an $(R, R)$-bimodule.

2. For any ring $R$ and set $M$, the set of maps $f : M \to R$ has the structure of an $R$-left module with the pointwise addition and pointwise left multiplication by $R$

$$(f + f')(m) = f(m) + f'(m) \qquad (r \rhd f)(m) = r \cdot f(m).$$

3. For every ring $R$ and $R$-left module $M$, the set $\mathrm{End}_R(M) = \mathrm{Hom}_R(M, M)$ of module endomorphisms $f : M \to M$ is a unital ring with the pointwise addition and composition. This is called the **endomorphism ring** of $M$. The $R$-left module structures on an abelian group $(M, +)$ correspond bijectively to ring homomorphisms $R \to \mathrm{End}_{\mathbb{Z}}(M)$.

4. If $\phi : R \to S$ is a unital ring homomorphism, then every $S$-module becomes an $R$-module with the module structure given by $r \rhd m := \phi(r) \rhd m$. This is called the **pullback** of the $S$-module structure on $M$ by $\phi$.

5. In particular, for any subring $U \subset R$, the inclusion map $\iota : U \to R$, $u \mapsto u$ is a unital ring homomorphism and induces a $U$-module structure on any $R$-module $M$. This is called the **restriction** of the $R$-module structure to $U$

6. A module over the ring $\mathbb{Z}$ is an abelian group. Any abelian group $(M, +)$ has a unique $\mathbb{Z}$-module structure given by $\rhd : \mathbb{Z} \times M \to M$, $(z, m) \mapsto z \rhd m$, where $z \rhd m = (m + ... + m)$ is the $z$-fold sum of $m$ with itself for $z \in \mathbb{N}$, $0 \rhd m = 0$ and $z \rhd m = -(m + ... + m)$ for $-z \in \mathbb{N}$.

7. A module over a field $\mathbb{F}$ is a vector space $V$ over $\mathbb{F}$. The $\mathbb{F}$-action is given by the scalar multiplication: $\lambda \rhd v = \lambda v$ for all $v \in V$ and $\lambda \in \mathbb{F}$.

The concepts of a *linear subspace* and of a *quotient vector space* of a vector space can be generalised to modules over a ring. Their counterpart for modules over rings are *submodules* and *quotients of modules.*

**Definition 1.1.4:** Let $R$ be a unital ring and $M$ an $R$-module. A **submodule** of $M$ is a subgroup $(U, +) \subset (M, +)$ that is an $R$-module with the restriction of the action map, i. e. a subgroup with $r \rhd u \in U$ for all $u \in U$ and $r \in R$.

**Example 1.1.5:**

1. For every $R$-module homomorphism $\phi : M \to N$, the kernel $\ker(\phi) = \phi^{-1}(0) \subset M$ and the image $\mathrm{Im}(\phi) \subset N$ are submodules of $M$ and $N$.

2. For any subset $U \subset M$ of an $R$-module $M$, the set

$$\langle U \rangle_M = \{\Sigma_{i=1}^n r_i \rhd u_i \,|\, n \in \mathbb{N}, r_i \in R, u_i \in U\} \subset M$$

is a submodule of $M$, the **submodule generated by** $U$. For any submodule $U \subset N \subset M$, one has $\langle U \rangle_M \subset N$. If $\langle U \rangle_M = N$ one says that $U$ **generates** $N$.

3. If $R = \mathbb{F}$ is a field, a submodule of an $\mathbb{F}$-module $V$ is a linear subspace $U \subset V$.

4. If $R = \mathbb{Z}$, a submodule of an $\mathbb{Z}$-module $A$ is a subgroup of the abelian group $A$.

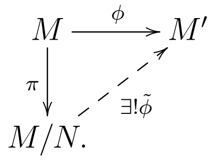
5. If we consider a ring $R$ as a left, right or bimodule over itself, then a submodule $I \subset R$ is a left, right or two-sided ideal in $R$.

The quotient of an module $M$ by a submodule $N \subset M$ is constructed in a similar way as the quotient of a vector space by a linear subspace. It has a canonical module structure and is characterised by a universal property that generalises the universal property of quotients vector spaces. In particular, the following proposition implies that every module homomorphism $\phi : M \to M'$ induces a canonical isomorphism $\phi : M/\ker(\phi) \xrightarrow{\sim} \mathrm{Im}(\phi)$. Similarly, for submodules $U \subset V \subset M$, one obtains a canonical isomorphism $(M/U)/(V/U) \xrightarrow{\sim} M/V$. (Exercise)

**Proposition 1.1.6:** Let $R$ be a unital ring, $M$ an $R$-module and $N \subset M$ a submodule.

1. There is a unique $R$-module structure on the group $(M/N, +)$ for which the canonical surjection $\pi : M \to M/N$, $m \mapsto [m]$ is a module homomorphism. The group $(M/N, +)$ with this module structure is called the **quotient module** of $M$ by $N$.

2. It has the following **universal property**: for any module homomorphism $\phi : M \to M'$ with $\ker(\phi) \subset N$, there is a unique module homomorphism $\tilde{\phi} : M/N \to M'$ such that the following diagram commutes




**Example 1.1.7:**
1. If $R = \mathbb{F}$ is a field, a submodule of an $\mathbb{F}$-module $V$ is a linear subspace $U \subset V$ and the quotient module is the quotient vector space $V/U$.
2. If we consider a ring $R$ as a left, right or bimodule over itself, then a submodule $I \subset R$ is a left, right or two-sided ideal in $R$, and the quotient module is the abelian group $R/I$ with the induced module structure.

Although modules over general rings share many features with vector spaces, a fundamental difference is that a module over a ring does not necessarily have a basis. Although the definition of a basis can be generalised straightforwardly to modules over rings, the existence of a basis for a given module is not guaranteed. Modules over a ring that have a basis are called *free*. For every ring $R$ and set $M$, there is a module with basis $M$, the free $R$-module generated by $M$.

**Definition 1.1.8:** Let $R$ be a unital ring and $M$ a set. The **free $R$-module** generated by $M$ is the set of maps $\langle M \rangle_R = \{f : M \to R \,|\, f(m) = 0 \text{ for almost all } m \in M\}$ with the pointwise addition and the module structure $\rhd : R \times M \to M$, $(r \rhd f)(m) = r \cdot f(m)$.

**Remark 1.1.9:**

1. The maps $\delta_m : M \to R$ with $\delta_m(m) = 1$ and $\delta_m(m') = 0$ for $m' \neq m$ generate $\langle M \rangle_R$, because any map $f : M \to R$ with $f(m) = 0$ for almost all $m \in M$ can be written as a finite sum $f = \Sigma_{m \in M} f(m) \rhd \delta_m$.

2. The free $R$-module $\langle M\rangle_R$ generated by a set $M$ has the following **universal property**:

   For every map $f : M \to N$ into an $R$-module $N$, there is a unique module homomorphism $\tilde{\phi} : \langle M\rangle_R \to N$ with $\tilde{\phi} \circ \iota = \phi$, where $\iota : M \to \langle M\rangle_R$, $m \mapsto m$ is the inclusion map.

3. If $R = \mathbb{F}$ is a field, then $\langle M\rangle_R$ is the free $\mathbb{F}$-vector space generated by $M$. If $R = \mathbb{Z}$, then $\langle M\rangle_\mathbb{Z}$ is the free abelian group generated by $M$.

The free module generated by a set is a specific case of a more general construction, namely the direct sum of modules. Direct sums of and products of modules over a ring are defined analogously to direct sums and products of vector spaces. For a family $(M_i)_{i\in I}$ of modules over a ring $R$ we consider the disjoint union $\dot{\cup}_{i\in I} M_i$. Its elements are families $(m_i)_{i\in I}$ of elements $m_i \in M_i$. The $R$-module structure of $\dot{\cup}_{i\in I} M_i$ is given by the module structure on $M_i$, and the elements $(m_i)_{i\in I}$ with $m_i = 0$ for almost all $i \in I$ form a submodule.

**Definition 1.1.10:**
Let $(M_i)_{i\in I}$ be a family of modules over a ring $R$, indexed by a set $I$. Then

$$(m_i)_{i\in I} + (m_i')_{i\in I} := (m_i + m_i')_{i\in I} \qquad r \rhd (m_i)_{i\in I} := (r \rhd m_i)_{i\in I}$$

defines an $R$-module structure on the sets

$$M := \{(m_i)_{i\in I} : m_i \in M_i\} \qquad M' = \{(m_i)_{i\in I} : m_i \in M_i, m_i = 0 \text{ for almost all } i \in I\} \subset M.$$

The set $M$ with this $R$-module structure is called the **product** of the modules $M_i$ and denoted $\Pi_{i\in I} M_i$. The set $M'$ with this $R$-module structure is called the **direct sum** of the modules $M_i$ and denoted $\oplus_{i\in I} M_i$.

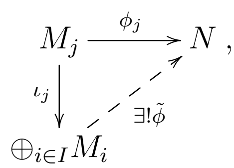
Just as in the case of vector spaces, direct sums and product of modules over a ring $R$ have universal properties that characterise them in terms of the inclusion and projection maps

$$\iota_j : M_j \to \oplus_{i\in I} M_i,\ m \mapsto (m\delta_{ij})_{i\in I} \qquad \pi_j : \Pi_{i\in I} M_i \to M_j,\ (m_i)_{i\in I} \mapsto m_j.$$

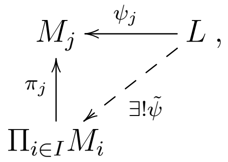
This allows one to efficiently construct module homomorphisms between direct sums and products, without defining them explicitly.

**Proposition 1.1.11:** Let $(M_i)_{i\in I}$ be a family of modules over a ring $R$. Then the direct sum $\oplus_{i\in I} M_i$ and the product $\Pi_{i\in I} M_i$ have the following **universal properties**:

1. The inclusion maps $\iota_j : M_j \to \oplus_{i\in I} M_i$, $m \mapsto (m\delta_{ij})_{i\in I}$ are module homomorphisms. For every family $(\phi_i)_{i\in I}$ of module homomorphisms $\phi_i : M_i \to N$ there is a unique module homomorphism $\tilde{\phi} : \oplus_{i\in I} M_i \to N$ such that the following diagram commutes for all $j \in I$



2. The projection maps $\pi_j : \Pi_{i\in I} M_i \to M_j,\ (m_i)_{i\in I} \mapsto m_j$ are module homomorphisms. For every family $(\psi_i)_{i\in I}$ of module homomorphisms $\psi_i : L \to M_i$ there is a unique module homomorphism $\tilde\psi : L \to \Pi_{i\in I} M_i$ such that the following diagram commutes for all $j \in I$




**Remark 1.1.12:**

1. If $R = \mathbb{F}$, then the direct sum and the product of $R$-modules are simply the direct sum and product of vector spaces.
2. For $R = \mathbb{Z}$, the direct sum and product of $R$-modules coincide with the direct sum and product of abelian groups.
3. If $I$ is finite, then $\Pi_{i\in I} M_i = \oplus_{i\in I} M_i$, but if $I$ is infinite, then $\pi_{i\in I} M_i$ and $\oplus_{i\in I} M_i$ are *not isomorphic*.
4. The direct sum $\oplus_{i\in I} R$ is canonically isomorphic to the free $R$-module $\langle I\rangle_R$.

The last important construction that generalises directly from vector spaces to modules over a general ring are *tensor products*. However, unless $R$ is commutative, we cannot take the tensor product of two left modules over $R$ and obtain another left module over $R$. Instead, we combine a *left module* over $R$ with a *right module* over $R$ and obtain an *abelian group*. Otherwise, the construction is analogous to the one for vector spaces.

**Definition 1.1.13:** Let $R$ be a ring, $M$ a right module and $N$ a left module over $R$. The **tensor product** $M \otimes_R N$ is the quotient $M \otimes_R N = \langle M \times N\rangle_{\mathbb{Z}}/A$, where $A \subset \langle M \times N\rangle_{\mathbb{Z}}$ is the subgroup of the free abelian group $\langle M \times N\rangle_{\mathbb{Z}}$ that is generated by the elements

$$\delta_{m,n} + \delta_{m',n} - \delta_{m+m',n},\quad \delta_{m,n} + \delta_{m,n'} - \delta_{m,n+n'},\quad \delta_{m,r\rhd n} - \delta_{m\lhd r,n} \qquad \forall m, m' \in M,\ n, n' \in N,\ r \in R.$$

The equivalence class of $\delta_{m,n} : M \times N \to R$ in $M \otimes_R N$ is denoted $m \otimes n$.

**Remark 1.1.14:**

1. Note that the tensor product $M \otimes_R N$ of an $R$-right module $M$ and an $R$-left module $N$ is an abelian group, but in general *not* an $R$-module. However, if $N$ is an $(R, S)$-bimodule or $M$ an $(P, R)$-bimodule, then $M \otimes_R N$ is an $(P, S)$-bimodule.

2. For a commutative ring $R$, every left or right module over $R$ is an $(R, R)$-bimodule and hence the tensor product $M \otimes_R N$ is an $(R, R)$-bimodule. For left modules $M_1, ..., M_n$ over $R$, we can therefore define multiple tensor products by

$$M_1 \otimes_R ... \otimes_R M_n := (...(M_1 \otimes_R M_2) \otimes_R M_3) \otimes_R \ldots \otimes_R M_{n-1}) \otimes_R M_n.$$

   This holds in particular for tensor products of vector spaces and of abelian groups.

3. If $R = \mathbb{F}$ is a field, the tensor product of $R$-modules is the tensor product of vector spaces. In this case, for any bases $B$ of $M$ and $C$ of $N$, the set $D = \{b \otimes c \,|\, b \in B, c \in C\}$ is a basis of $M \otimes_{\mathbb{F}} N$. This implies in particular $\dim_{\mathbb{F}}(M \otimes_{\mathbb{F}} N) = \dim_{\mathbb{F}}(M) \cdot \dim_{\mathbb{F}}(N)$, where we take $\infty \cdot n = n \cdot \infty = \infty$ for $n \in \mathbb{N}$ and $0 \cdot \infty = \infty \cdot 0 = 0$.

4. As every module over a ring $R$ exhibits a unique $\mathbb{Z}$-module structure, one can also form the tensor product $M \otimes_{\mathbb{Z}} N$ for any right module $M$ and left module $N$ over a ring $R$. However, this is general very different from the tensor product $M \otimes_R N$ and the two should not be confused.

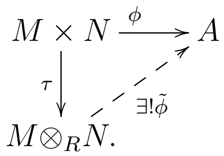
Tensor products of modules can be characterised by a universal property that resembles the one for tensor products of vector spaces. This universal property states that the $R$-bilinear maps $M \times N \to A$ for an $R$-left module $N$, an $R$-right module $M$ and an abelian group $A$ are in bijection with group homomorphisms $M \otimes_R N \to A$.

**Proposition 1.1.15:** Let $R$ be a ring, $M$ a right module and $N$ a left module over $R$. Then the **tensor product** $M \otimes_R N$ has the following universal property:

The map $\tau : M \times N \to M \otimes_R N$, $(m, n) \mapsto m \otimes n$ is $R$-**bilinear**:

$$\tau(m+m', n) = \tau(m, n)+\tau(m', n), \quad \tau(m, n+n') = \tau(m, n)+\tau(m, n'), \quad \tau(m \lhd r, n) = \tau(m, r \rhd n)$$

for all $m, m' \in M$, $n, n' \in N$, $r \in R$. For every $R$-bilinear map $\phi : M \times N \to A$ into an abelian group $A$ there is a unique group homomorphism $\tilde{\phi} : M \otimes_R N \to A$ such that the following diagram commutes




**Remark 1.1.16:** The universal property of tensor allows one to define a tensor product of module homomorphisms. If $R$ is a ring, $M, M'$ are all right modules and $N, N'$ left modules over $R$, then the maps $\tau : M \times N \to M \otimes_R N$ and $\tau' : M' \times N' \to M' \otimes_R N'$ from Proposition 1.1.15 are $R$-bilinear. Moreover, for each pair of module homomorphisms $f : M \to M'$ and $g : N \to N'$, the map $\tau' \circ (f \times g) : M \times N \to M' \otimes_R N'$ is $R$-bilinear. By the universal property of the tensor product, there is a unique linear map $f \otimes g : M \otimes_R N \to M' \otimes_R N'$ with $(f \otimes g) \circ \tau = \tau' \circ (f \times g)$. This map is called the **tensor product** of $f$ and $g$ and satisfies

$$f \otimes g = (f \otimes \mathrm{id}_{N'}) \circ (\mathrm{id}_M \otimes g) = (\mathrm{id}_{M'} \otimes g) \circ (f \otimes \mathrm{id}_N).$$

Besides the universal properties of tensor products, some canonical linear maps between multiple tensor products of vector spaces also have analogues for modules over a general ring. They are obtained by replacing the field with the ring $R$ and linear maps with module homomorphisms. The most important ones are the following.

**Proposition 1.1.17:** Let $R, S$ be rings, $M, M_i$ right modules over $R$, $N, N_i$ left modules over $R$ for all $i \in I$, $P$ an $(R, S)$-bimodule and $Q$ a left module over $S$. Then $0 \otimes_R N = M \otimes_R N = 0$, and the following are module isomorphisms:

- the maps $M \otimes_R R \to M$, $m \otimes r \mapsto m \triangleleft r$ and $R \otimes_R N \to N$, $r \otimes n \mapsto r \triangleright n$,
- the maps
$$(\oplus_{i\in I} M_i) \otimes_R N \to \oplus_{i\in I} M_i \otimes_R N, \quad (m_i)_{i\in I} \otimes n \mapsto (m_i \otimes n)_{i\in I}$$
$$M \otimes_R (\oplus_{i\in I} N_i) \to \oplus_{i\in I} M \otimes_R N_i, \quad m \otimes (n_i)_{i\in I} \mapsto (m \otimes n_i)_{i\in I},$$
- the map $(M \otimes_R P) \otimes_S Q \to M \otimes_R (P \otimes_S Q)$, $(m \otimes p) \otimes q \mapsto m \otimes (p \otimes q)$.

## A.2 Categories and functors

In this section, we summarise the basics about categories, functors and natural transformations. For more information we refer to the book [McL] and [Ka, Chapter XI].

**Definition 1.2.1:** A **category** $\mathcal{C}$ consists of:

- a class $\mathrm{Ob}\,\mathcal{C}$ of **objects**,
- for each pair $(X, Y)$ of objects a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of **morphisms**,
- for each triple $(X, Y, Z)$ ob objects a **composition map**

$$\circ : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z), \quad (f, g) \mapsto f \circ g$$

such that the following axioms are satisfied:

(C1) The morphism sets $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ are pairwise disjoint.

(C2) The composition is associative: $f \circ (g \circ h) = (f \circ g) \circ h$ for all objects $W, X, Y, Z$ of $\mathcal{C}$ and morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$, $g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, $h \in \mathrm{Hom}_{\mathcal{C}}(W, X)$.

(C3) For each object $Y$ there is a morphism $1_Y \in \mathrm{Hom}_{\mathbb{C}}(Y, Y)$, the **identity morphism** on $Y$, with $f \circ 1_Y = f$ and $1_Y \circ g = g$ for all morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

Instead of $f \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$ one also writes $f : Y \to Z$. The object $Y$ is called the **source** and the object $Z$ the **target** of $f$.

A morphism $f : Y \to Z$ is called an **isomorphism** if there is a morphism $f^{-1} : Z \to Y$ with $f^{-1} \circ f = 1_Y$ and $f \circ f^{-1} = 1_Z$. Two objects $Y, Z$ in $\mathcal{C}$ are called **isomorphic** $Y \cong Z$, if there is an isomorphism $f : Y \to Z$. A category in which all morphisms are isomorphisms is called a **groupoid**.

The reason why it is not required in Definition 1.2.1 that the objects of a category form a set is that one wants to consider the category Set whose objects are sets and whose morphisms $f : X \to Y$ are maps from $X$ to $Y$. The requirement that objects of a category form a set would then force one to the consider the *set of all sets*, which does not exist. A category whose objects form a set is called a **small category**. Important examples of categories are given in Table 1. Additional examples are the following.

**Example 1.2.2:**

1. Every group $G$ can be viewed as a groupoid with a single object $\bullet$, elements $g \in G$ as morphisms, the group multiplication as the composition of morphisms and the identity morphism $1_\bullet = e \in G$.

| **category** | **objects** | **morphisms** | **isomorphisms** |
|---|---|---|---|
| Set | sets | maps | bijections |
| $\text{Set}^{fin}$ | finite sets | maps | bijections |
| Grp | groups | group homomorphisms | group isomorphisms |
| Ab | abelian groups | group homomorphisms | group isomorphisms |
| URing | unital rings | unital ring homomorphisms | unital ring isomorphisms |
| Field | fields | field homomorphisms | field isomorphisms |
| $\text{Vect}_{\mathbb{F}}$ | vector spaces over $\mathbb{F}$ | $\mathbb{F}$-linear maps | $\mathbb{F}$-linear isomorphisms |
| $\text{Vect}_{\mathbb{F}}^{fin}$ | finite-dimensional vector spaces over $\mathbb{F}$ | $\mathbb{F}$-linear maps | $\mathbb{F}$-linear isomorphisms |
| $\text{Alg}_{\mathbb{F}}$ | algebras over $\mathbb{F}$ | algebra homomorphisms | algebra isomorphisms |
| $A$-Mod | left modules over $A$ | left module maps | left module isomorphisms |
| Mod-$A$ | right modules over $A$ | right module maps | right module isomorphisms |
| $A-\text{Mod}^{fin}$ | finite-dimensional left modules over $A$ | left module maps | left module isomorphisms |
| $\text{Rep}_{\mathbb{F}}(G)$ | representations of $G$ over $\mathbb{F}$ | homomorphisms of representations | isomorphisms of representations |
| Top | topological spaces | continuous maps | homeomorphisms |
| Mfld | smooth manifolds | smooth maps | diffeomorphisms |

Table 1: Examples of categories

2. The category Ord that has **ordinal numbers** $\underline{0} = \emptyset$ and $\underline{n} = \{0, 1, ..., n-1\}$ for $n \in \mathbb{N}$ as objects and maps $f : \{0, 1, ..., m-1\} \to \{0, 1, ..., n-1\}$ as morphisms $f : \underline{n} \to \underline{m}$.

Many constructions that are familiar in the context of sets, groups or rings have analogues for categories. Examples are the following.

**Definition 1.2.3:** Let $\mathcal{C}, \mathcal{D}$ be categories.

1. The **opposite category** $\mathcal{C}^{op}$ is the category with the same objects as $\mathcal{C}$, morphism sets $\mathrm{Hom}_{\mathcal{C}^{op}}(Y, X) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and the reversed composition of morphisms $g \circ_{op} f = f \circ g$ for all $f \in \mathrm{Hom}_{\mathcal{C}^{op}}(Z, Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}^{op}}(Y, X)$.
2. The **cartesian product** $\mathcal{C} \times \mathcal{D}$ is the category that has as objects pairs $(C, D)$ of objects $C \in \mathrm{Ob}\,\mathcal{C}$ and $D \in \mathrm{Ob}\,\mathcal{D}$, whose morphism sets are given by $\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((C, D), (C', D')) = \mathrm{Hom}_{\mathcal{C}}(C, C') \times \mathrm{Hom}_{\mathcal{D}}(D, D')$ and whose composition of morphisms is given by $(f, h) \circ (g, k) = (f \circ g, h \circ k)$.
3. A **subcategory** of $\mathcal{C}$ is a category $\mathcal{C}'$ whose objects form a subclass $\mathrm{Ob}\,\mathcal{C}' \subset \mathrm{Ob}\,\mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}'}(X, Y) \subset \mathrm{Hom}_{\mathcal{C}}(X, Y)$ for all $X, Y \in \mathrm{Ob}\,C'$ and the composition in $\mathcal{C}'$ agrees with the one in $\mathcal{C}$. A subcategory $\mathcal{C}'$ of $\mathcal{C}$ is called **full** if $\mathrm{Hom}_{\mathcal{C}'}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$ for all $X, Y \in \mathrm{Ob}\,C'$.

As a category consists of both, objects and morphisms between them, the appropriate way to relate different categories must take into account both, objects and morphisms, as well as the category axioms. this forces one to consider assignments of objects and maps between the morphisms sets that are compatible with the composition of morphisms and send identity morphisms to identity morphisms.

**Definition 1.2.4:** Let $\mathcal{B}, \mathcal{C}, \mathcal{D}$ be categories. A **functor** $F : \mathcal{C} \to \mathcal{D}$ consists of

- an assignment of an object $F(C) \in \mathrm{Ob}\,\mathcal{D}$ to each object $C \in \mathrm{Ob}\,\mathcal{C}$,
- for each pair of objects $C, C' \in \mathrm{Ob}\,\mathcal{C}$ a map

$$F : \mathrm{Hom}_{\mathcal{C}}(C, C') \to \mathrm{Hom}_{\mathcal{D}}(F(C), F(C')), \quad f \mapsto F(f)$$

that are compatible with the composition of morphisms and the identity morphisms:

$$F(f \circ g) = F(f) \circ F(g) \quad \forall f \in \mathrm{Hom}_{\mathcal{C}}(C', C''), g \in \mathrm{Hom}_{\mathcal{C}}(C, C') \qquad F(1_C) = 1_{F(C)} \quad \forall C \in \mathrm{Ob}\,\mathcal{C}.$$

A functor $F : \mathcal{C} \to \mathcal{C}$ is called an **endofunctor** of $\mathcal{C}$ and a functor $F : \mathcal{C}^{op} \to \mathcal{D}$ a **contravariant functor** from $\mathcal{C}$ to $\mathcal{D}$. The **composite** of functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{B} \to \mathcal{C}$ is the functor $FG : \mathcal{B} \to \mathcal{D}$ with the assignment $B \mapsto F(G(B))$ for all objects $B \in \mathrm{Ob}\,\mathcal{B}$ and the maps

$$FG : \mathrm{Hom}_{\mathcal{B}}(B, B') \to \mathrm{Hom}_{\mathcal{D}}(F(G(B)), F(G(B'))), \quad f \mapsto F(G(f)).$$

It follows from the definition that any functor $F : \mathcal{C} \to \mathcal{D}$ send isomorphisms in $\mathcal{C}$ to isomorphisms in $\mathcal{D}$. This is essential since in many contexts one is not interested in the objects of a category but only in *isomorphism classes* of objects, such as vector spaces up to linear isomorphisms, groups up to group isomorphisms, modules up to module isomorphisms etc. The fact that a functor maps isomorphisms to isomorphisms ensures that this is still consistent when one considers relations between categories.

**Example 1.2.5:**

1. For every category $\mathcal{C}$, there is a **identity functor** $\mathrm{id}_\mathcal{C} : \mathcal{C} \to \mathcal{C}$ that assigns every object and morphism in $\mathcal{C}$ to itself is an endofunctor.

2. The functors $\mathrm{Vect}_\mathbb{F} \to \mathrm{Set}$, $\mathrm{Grp} \to \mathrm{Set}$, $\mathrm{Top} \to \mathrm{Set}$ that assign to a vector space, a group or a topological space the underlying set and to a an $\mathbb{F}$-linear map, a group homomorphism or a continuous map the underlying map. The functors $\mathrm{Alg}_\mathbb{F} \to \mathrm{Vect}_\mathbb{F}$, $A{-}\mathrm{Mod} \to \mathrm{Vect}_\mathbb{F}$ that assign to an algebra over $\mathbb{F}$ or to a module over an $\mathbb{F}$-algebra $A$ the underlying vector space. Functors of this type are called **forgetful functors**.

3. The functor $* : \mathrm{Vect}_\mathbb{F} \to \mathrm{Vect}_\mathbb{F}$ that assigns to every vector space $V$ over $\mathbb{F}$ the dual vector space $V^*$ and to every $\mathbb{F}$-linear map $f : V \to W$ the dual map $f^* : W^* \to V^*$, $\alpha \mapsto \alpha \circ f$ is a contravariant endofunctor of $\mathrm{Vect}_\mathbb{F}$.

4. The functor $F : \mathrm{Set} \to \mathrm{Vect}_\mathbb{F}$ that assigns to a set $X$ the free vector space $F(X) = \langle X \rangle_\mathbb{F}$ with basis $X$ and to a map $f : X \to Y$ the unique $\mathbb{F}$-linear map $F(f) : \langle X \rangle_\mathbb{F} \to \langle Y \rangle_\mathbb{F}$ with $F(f)|_X = f$.

5. Every algebra homomorphism $\phi : A \to B$ defines a functor $F : B\text{-Mod} \to A\text{-Mod}$ that assigns to a $B$-module $(V, \rhd)$ the $A$-module $(A, \rhd')$ with $a \rhd' v = \phi(a) \rhd v$ and to a $B$-module morphism $\phi : (V, \rhd_V) \to (W, \rhd_W)$ the associated $A$-module morphism $\phi : (V, \rhd'_V) \to (W, \rhd'_W)$.

6. For every category $\mathcal{C}$ and object $C \in \mathrm{Ob}\,\mathcal{C}$ there is a functor $\mathrm{Hom}(C, -) : \mathcal{C} \to \mathrm{Set}$ that assigns to an object $C'$ in $\mathcal{C}$ the set $\mathrm{Hom}_\mathcal{C}(C, C')$ and to a morphism $f : C' \to C''$ the map $\mathrm{Hom}(C, f) : \mathrm{Hom}_\mathcal{C}(C, C') \to \mathrm{Hom}_\mathcal{C}(C, C'')$, $g \mapsto f \circ g$. Similarly, one has a functor $\mathrm{Hom}(-, C) : \mathcal{C}^{op} \to \mathrm{Set}$ that assigns to an object $C' \in \mathrm{Ob}\,\mathcal{C}$ the set $\mathrm{Hom}_\mathcal{C}(C', C)$ and to a morphism $f : C' \to C''$ the map $\mathrm{Hom}(f, C) : \mathrm{Hom}_\mathcal{C}(C'', C) \to \mathrm{Hom}_\mathcal{C}(C', C)$, $g \mapsto g \circ f$. These functors are called **Hom-functors**.

7. Let $G$ be a group, viewed as a category with a single object as in Example 1.2.2, 1. Then a functor $F : G \to \mathrm{Set}$ is a pair $(X, \rhd)$ of a set $X$ with a group-action $\rhd : G \times X \to X$ and a functor $F : G \to \mathrm{Vect}_\mathbb{F}$ is a representation $(V, \rho)$ of $G$.

Just as a functor $F : \mathcal{C} \to \mathcal{D}$ relates the categories $\mathcal{C}$ and $\mathcal{D}$, there is another mathematical structure that relates different functors $F : \mathcal{C} \to \mathcal{D}$. As a functor $F\mathcal{C} \to \mathcal{D}$ assigns to each object in $\mathcal{C}$ an object in $\mathcal{D}$ a mathematical structure that relates functors $F, G : \mathcal{C} \to \mathcal{D}$ must relate the objects $F(C)$ and $G(C)$ for each object $C$ in $\mathcal{C}$. Hence, we require a collection of morphisms $\eta_C : F(C) \to G(C)$ indexed by the objects in $\mathcal{C}$. Two morphisms $\eta_C : F(C) \to G(C)$ and $\eta_{C'} : F(C') \to G(C')$ in this collection can be composed with the images $F(f) : F(C) \to F(C')$ and $G(f) : G(C) \to G(C')$ of a morphism $f : C \to C'$, either on the left or on the right. This yields two morphisms $G(f) \circ \eta_C : F(C) \to G(C')$ and $\eta_{C'} \circ F(f) : F(C) \to G(C')$, and it is sensible to impose that these morphisms agree for all morphisms $f : C \to C'$. This is called the *naturality condition.*

**Definition 1.2.6:** Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A **natural transformation** $\eta : F \to G$ is an assignment of a morphism $\eta_C : F(C) \to G(C)$ to each object $C \in \mathrm{Ob}\,\mathcal{C}$ such that the following diagram commutes for all objects $C, C' \in \mathrm{Ob}\,\mathcal{C}$ and morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(C, C')$

$$\begin{array}{ccc} F(C) & \xrightarrow{\eta_C} & G(C) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(C') & \xrightarrow{\eta_{C'}} & G(C'). \end{array}$$

The morphisms $\eta_C : F(C) \to F(C')$ are called **component morphisms** of $\eta$. A natural transformation $\eta : F \to G$ for which all component morphisms $\eta_C : F(C) \to G(C)$ are isomorphisms is called a **natural isomorphism**.

**Example 1.2.7:**

1. For every functor $F : \mathcal{C} \to \mathcal{C}$ there is a natural transformation $\mathrm{id}_F : F \to F$ with component morphisms $(\mathrm{id}_F)_C = 1_{F(C)} : F(C) \to F(C)$. This is called the **identity natural transformation** on $F$.

2. Let $* : \mathrm{Vect}_{\mathbb{F}} \to \mathrm{Vect}_{\mathbb{F}}$ be the contravariant functor from Example 1.2.5, 3. Then the morphisms $\mathrm{can}_V : V \to V^{**}$, $v \mapsto f_v$ with $f_v(\alpha) = \alpha(v)$ for all $\alpha \in V^*$ define a natural transformation $\mathrm{can} : \mathrm{id}_{\mathrm{Vect}_{\mathbb{F}}} \to **$. It is not a natural isomorphism, but the associated natural transformation $\mathrm{can} : \mathrm{id}_{\mathrm{Vect}_{\mathbb{F}}^{fin}} \to **$ is a natural isomorphism.

3. Let $G$ be a group and $g \in G$ fixed. Then one has a functor $F_g : \mathrm{Rep}_{\mathbb{F}}(G) \to \mathrm{Rep}_{\mathbb{F}}(G)$ that sends a representation $(V, \rho)$ to $(V, \rho')$ with $\rho'(h) = \rho(ghg^{-1}) : V \to V$ for all $h \in H$ and each homomorphism of representations to itself. Then the homomorphisms of representations $\eta_{(\rho,V)} = \rho(g) : V \to V$ define a natural isomorphism $\eta : \mathrm{id}_{\mathrm{Rep}_{\mathbb{F}}(G)} \to F_g$.

4. Consider for $n \in \mathbb{N}$ the functor $\mathrm{GL}_n : \mathrm{Field} \to \mathrm{Grp}$ that assigns to $\mathbb{F}$ the group $\mathrm{GL}(n, \mathbb{F})$ of invertible $(n \times n)$-matrices with entries in $\mathbb{F}$ and to a field homomorphism $f : \mathbb{F} \to \mathbb{G}$ the group homomorphism $F(f) : \mathrm{GL}(n, \mathbb{F}) \to \mathrm{GL}(n, \mathbb{G})$ obtained by applying $f$ to all entries of a matrix. Then the determinant defines a natural transformation $\det : \mathrm{GL}_n \to \mathrm{id}_{\mathrm{Field}}$.

Natural transformations $\eta : G \to H$ between functors $G, H : \mathcal{C} \to \mathcal{D}$ can be composed with both, functors $F : \mathcal{B} \to \mathcal{C}$ and functors $L : \mathcal{D} \to \mathcal{E}$ as well as with natural transformations $\mu : H \to K$. These composites are obtained by applying the relevant functors to either the component morphisms or the objects indexing the component morphisms and by composing the component morphisms of the natural transformations.

**Lemma 1.2.8:** Let $F : \mathcal{B} \to \mathcal{C}$, $G, H, K : \mathcal{C} \to \mathcal{D}$ and $L : \mathcal{D} \to \mathcal{E}$ be functors and $\eta : G \to H$, $\mu : H \to K$ natural transformations. Then:

1. The morphisms $(L\eta)_C := L(\eta_C) : LG(C) \to LH(C)$ define a natural transformation $L\eta : LG \to LH$.

2. The morphisms $(\eta F)_B := \eta_{F(B)} : GF(B) \to HF(B)$ define a natural transformation $\eta F : GF \to HF$.

3. The morphisms $(\mu \circ \eta)_C := \mu_C \circ \eta_C : G(C) \to K(C)$ define a natural transformation $\mu \circ \eta : G \to K$.

**Proof:**
One only has to check the naturality of these collections of morphisms, i. e. that the diagram in Definition 1.2.6 commutes. In the first case, this follows by applying the functor $L$ to the diagram in Definition 1.2.6 and using the compatibility of $L$ with the composition of morphisms. In 2. this follows by restricting the commuting diagram for $\eta$ to objects $F(B)$ and morphisms $F(f) : F(B) \to F(B')$ for objects $B \in \mathrm{Ob}\,\mathcal{B}$ and morphisms $f : B \to B'$. The third claim follows by composing the commuting diagrams for $\eta$ and $\mu$:

$$\begin{array}{ccccc} G(C) & \xrightarrow{\eta_C} & H(C) & \xrightarrow{\mu_C} & K(C) \\ \downarrow{\scriptstyle G(f)} & & \downarrow{\scriptstyle H(f)} & & \downarrow{\scriptstyle K(f)} \\ G(C') & \xrightarrow[\eta_{C'}]{} & H(C') & \xrightarrow[\mu_{C'}]{} & K(C') \end{array}$$

□

The fact that the composition of the component morphisms of two natural transformations is associative implies that this also holds for the composition of the natural transformations. Similarly, the fact that the identity natural transformation on $F$ has as component morphisms the identity morphisms implies that composing a natural transformation with an identity natural transformation does not change the natural transformation. This shows that functors between two fixed categories and natural transformations between them form again a category.

**Corollary 1.2.9:** Let $\mathcal{C}$ be a small category and $\mathcal{D}$ a category. Then the functors $F : \mathcal{C} \to \mathcal{D}$ and natural transformations between them form a category with the composition of natural transformations as composition of morphisms and the identity natural transformations as identity morphisms. This category is called the **functor category** and denoted $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

As a functor relates different categories it is natural to ask what is the appropriate concept of a reversible functor and under what criteria a functor can be reversed. The naive approach to this would be to define a reversible functor as an *invertible functor* $F : \mathcal{C} \to \mathcal{D}$, i. e. a functor such that there is a functor $G : \mathcal{D} \to \mathcal{C}$ with $GF = \mathrm{id}_\mathcal{C}$ and $FG = id_\mathcal{D}$. However, it turns out that there are very few examples of such functors and that this concept is of limited use. Moreover, it is unnecessary to impose that $GF$ and $FG$ are *equal* to the identity functors $\mathrm{id}_\mathcal{C}$ and $\mathrm{id}_\mathcal{D}$ since there are *structures* that can be used to *relate* them to identity functors. These structures are natural isomorphisms and lead to the following concept of a reversible functor.

**Definition 1.2.10:** A functor $F : \mathcal{C} \to \mathcal{D}$ is called **equivalence of categories** if there is a functor $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\eta : \mathrm{id}_\mathcal{C} \to GF$ and $\epsilon : FG \to \mathrm{id}_\mathcal{D}$. In this case, the categories $\mathcal{C}$ and $\mathcal{D}$ are called **equivalent**.

This definition is well motivated, since it establishes the most general notion of a reversible functor that can be defined with the concepts at hand. However, it is difficult to handle and does not give a useful criterion under which conditions a functor is an equivalence of categories. such a criterion is provided by the following theorem whose proof maxes use of the axiom of choice and can be found in [Ka, Prop.XI.1.15].

**Theorem 1.2.11:** A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if it is

- **essentially surjective**: for every object $D$ in $\mathcal{D}$ there is an object $C$ in $\mathcal{C}$ with $D \cong F(C)$.
- **fully faithful**: for all objects $C, C'$ in $\mathcal{C}$ the map $F : \mathrm{Hom}_{\mathcal{C}}(C, C') \to \mathrm{Hom}_{\mathcal{D}}(F(C), F(C'))$ is a bijection.

In each of the following Examples, it is much easier to prove the equivalence of the given categories by guessing one the functors that forms the equivalence of categories and then arguing that it is essentially surjective and fully faithful (Exercise).

**Example 1.2.12:**

1. The category $\mathrm{Set}^{fin}$ is equivalent to the category Ord from Example 1.2.2.
2. The category $\mathrm{Vect}_{\mathbb{F}}^{fin}$ is equivalent to the category that has as objects numbers $n \in \mathbb{N}_0$ and as morphisms $f : m \to n$ matrices in $\mathrm{Mat}(m \times n, \mathbb{F})$ with the matrix multiplication as composition of morphisms and the unit matrices as identity morphisms.
3. For any pair of rings $R, S$ the category $(R \times S)$-Mod is equivalent to $R$-Mod $\times$ $S$-Mod.
4. A **skeleton** of a category $\mathcal{C}$ is a full subcategory $\mathcal{D}$ of $\mathcal{C}$ such that every object of $\mathcal{D}$ is isomorphic to a unique object in $\mathcal{C}$. If $\mathcal{D}$ is a skeleton of $\mathcal{C}$, then the inclusion functor $\iota : \mathcal{D} \to \mathcal{C}$ that sends every object and morphism of $\mathcal{D}$ to the corresponding object and morphism in $\mathcal{C}$ is an equivalence of categories.

The notion of a skeleton and the associated equivalence of categories capture the notion of a mathematical *classification problem*. Classifying the objects of a category $\mathcal{C}$ usually means classifying them up to isomorphisms in $\mathcal{C}$, e. g. vector spaces up to linear isomorphisms, groups up to group isomorphisms, $A$-modules up to module isomorphisms, finite abelian groups up to group isomorphisms etc. This amounts to giving a list of objects such that each object in $\mathcal{C}$ is isomorphic in $\mathcal{C}$ to exactly one object in this list. This amounts to the construction of a skeleton of $\mathcal{C}$. One can show (Exercise) that if $\mathcal{C}'$ is a skeleton of $\mathcal{C}$ and $\mathcal{D}'$ a skeleton of $\mathcal{D}$, then $\mathcal{C}$ and $\mathcal{D}$ are equivalent if and only if $\mathcal{C}'$ and $\mathcal{D}'$ are isomorphic.

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