Tensor categories

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In preparing this lecture, I used the references in the bibliography and the following textbooks and lecture notes:

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1 Monoidal categories

1.1 Monoidal categories and monoidal functors

Monoidal categories can be viewed as categories equipped with a *tensor product* that generalises the tensor product over a commutative ring k.

For this generalisation one describes the tensor product of k-modules in such a way that it involves only objects, morphisms, functors and natural transformations (cf. Exercise 1). The starting point is the observation that the tensor product of modules over a commutative ring k defines a functor $\otimes : k\text{-Mod} \times k\text{-Mod} \to k\text{-Mod}$. This suggests that one should view a tensor product in a general category $\mathcal{C}$ as a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ that satisfies certain additional conditions. These additional conditions generalise the associativity of the tensor product and the fact that the ring k acts as a unit for the tensor product.

For modules over a commutative ring k, they are encoded in the k-linear isomorphisms

$$a_{M,N,P} : (M \otimes_k N) \otimes_k P \to M \otimes_k (N \otimes_k P),\ (m \otimes n) \otimes p \mapsto m \otimes (n \otimes p)$$
$$l_M : k \otimes_k M \to M,\ \lambda \otimes m \mapsto \lambda m, \qquad r_M : M \otimes_k k \to M,\ m \otimes \lambda \mapsto \lambda m$$

for all k-modules M, N, P. If we denote by $k \times \mathrm{id} : k\text{-Mod} \to k\text{-Mod} \times k\text{-Mod}$ the functor that assigns to a k-module M the pair (k, M) and to a k-linear map $f : M \to M'$ the pair (id_k, f), then the k-module isomorphisms $l_M : k \otimes_k M \to M$ and $r_M : M \otimes_k k \to M$ relate the functors $\otimes(k \times \mathrm{id}) : k\text{-Mod} \to k\text{-Mod}$ and $\otimes(\mathrm{id} \times k) : k\text{-Mod} \to k\text{-Mod}$ to the identity functor $\mathrm{id}_{k\text{-Mod}}$. Similarly, the k-module isomorphisms $a_{M,N,P}$ relate the functors $\otimes(\otimes \times \mathrm{id})$ and $\otimes(\mathrm{id} \times \otimes)$.

The k-linear isomorphisms l_M, r_M and $a_{M,N,P}$ commute with k-linear maps. For all k-linear maps $f : M \to M'$, $g : N \to N'$ and $h : P \to P'$ we have

$$a_{M',N',P'} \circ ((f \otimes g) \otimes h) = (f \otimes (g \otimes h)) \circ a_{M,N,P}, \quad l_{M'} \circ (\mathrm{id}_k \otimes f) = f \circ l_M, \quad r_{M'} \circ (f \otimes \mathrm{id}_k) = f \circ r_M.$$

We can therefore interpret $a_{M,N,P}$, l_M and r_M as component morphisms of natural isomorphisms $a : \otimes(\otimes \times \mathrm{id}) \to \otimes(\mathrm{id} \times \otimes)$, $l : \otimes(k \times \mathrm{id}) \to \mathrm{id}$ and $r : \otimes(\mathrm{id} \times k) \to \mathrm{id}$. Note also that there are identities between composites of the maps l_M, r_M and $a_{M,N,P}$ that allow us to omit tensoring with k and the brackets in iterated tensor products.

The existence of a special object e that generalises the commutative ring k and of natural isomorphisms $a : \otimes(\otimes \times \mathrm{id}) \to \otimes(\mathrm{id} \times \otimes)$, $l : \otimes(e \times \mathrm{id}) \to \mathrm{id}$ and $r : \otimes(\mathrm{id} \times e) \to \mathrm{id}$ can be imposed in any category $\mathcal{C}$ with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. If we also take into account the identities between multiple composites of the natural isomorphisms a, l and r, we obtain the following definition that generalises tensor products over commutative rings.

Definition 1.1.1:

A **monoidal category** is a sextuple $(\mathcal{C}, \otimes, e, a, l, r)$ consisting of

- a category $\mathcal{C}$,
- a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, the **tensor product**,
- an object e in $\mathcal{C}$, the **tensor unit**,
- a natural isomorphism $a : \otimes(\otimes \times \mathrm{id}_\mathcal{C}) \to \otimes(\mathrm{id}_\mathcal{C} \times \otimes)$, the **associator**,
- natural isomorphisms $r : \otimes(\mathrm{id}_\mathcal{C} \times e) \to \mathrm{id}_\mathcal{C}$ and $l : \otimes(e \times \mathrm{id}_\mathcal{C}) \to \mathrm{id}_\mathcal{C}$, the **unit constraints**,

subject to the following two conditions:

1. **pentagon axiom**: for all objects U, V, W, X of $\mathcal{C}$ the following diagram commutes

$$\begin{tikzcd}
((U\otimes V)\otimes W)\otimes X \arrow[r, "a_{U\otimes V,W,X}"] \arrow[d, "a_{U,V,W}\otimes 1_X"'] & (U\otimes V)\otimes(W\otimes X) \arrow[r, "a_{U,V,W\otimes X}"] & U\otimes(V\otimes(W\otimes X)) \\
(U\otimes(V\otimes W))\otimes X \arrow[r, "a_{U,V\otimes W,X}"'] & U\otimes((V\otimes W)\otimes X). \arrow[ur, "1_U\otimes a_{V,W,X}"'] &
\end{tikzcd} \tag{1}$$

2. **triangle axiom**: for all objects V, W of $\mathcal{C}$ the following diagram commutes

$$\begin{tikzcd}
(V\otimes e)\otimes W \arrow[rr, "a_{V,e,W}"] \arrow[dr, "r_V\otimes 1_W"'] & & V\otimes(e\otimes W) \arrow[dl, "1_V\otimes l_W"] \\
& V\otimes W. &
\end{tikzcd} \tag{2}$$

It is called **strict** if a, r and l are identity natural transformations.

Remark 1.1.2:

- The tensor unit and the unit constraints are determined by $\otimes$ and a uniquely up to unique isomorphism.
- The functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is in general not unique. It is a choice of *structure*, not a property. A category $\mathcal{C}$ may have several different monoidal structures.
- The tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ does *not* determine the associator, not even up to isomorphisms (Example 1.1.6, Exercise 4).

Remark 1.1.3: (Exercise 3)
Equivalently, a monoidal category can be defined as a pentuple $(\mathcal{C}, \otimes, a, e, \iota)$ of a category $\mathcal{C}$, a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a natural isomorphism $a : \otimes(\otimes \times \mathrm{id}) \to \otimes(\mathrm{id} \times \otimes)$, an object e and an isomorphism $\iota : e\otimes e \to e$ such that

(i) a satisfies the pentagon axiom: diagram (1) commutes,

(ii) the functors $e\otimes - : \mathcal{C} \to \mathcal{C}$ and $-\otimes e : \mathcal{C} \to \mathcal{C}$ are equivalences of categories.

The name *monoidal category* comes from the fact that in a monoidal category, the endomorphisms of the unit object form a commutative monoid (cf. Corollary 1.2.2). Many well-known categories from algebra or topology have the structure of a monoidal category, some of them even several non-equivalent ones.

Example 1.1.4:

1. For any commutative ring k, the category k-Mod is a monoidal category with:

   - the functor $\otimes : k\text{-Mod} \times k\text{-Mod} \to k\text{-Mod}$ that assigns to a pair (M, N) of k-modules the k-module $M\otimes_k N$ and to a pair (f, g) of k-linear maps $f : M \to M'$, $g : N \to N'$ the linear map $f\otimes g : M\otimes_k N \to M'\otimes_k N'$, $m\otimes n \mapsto f(m)\otimes g(n)$,
   - the tensor unit $e = k$,
   - the associator with component isomorphisms
     $a_{M,N,P} : (M\otimes N)\otimes P \to M\otimes(N\otimes P)$, $(m\otimes n)\otimes p \mapsto m\otimes(n\otimes p)$,

- the unit constraints with component morphisms
  $r_M : M \otimes_k k \to M$, $m \otimes \lambda \mapsto \lambda m$ and $l_M : k \otimes_k M \to M$, $\lambda \otimes m \mapsto \lambda m$.

  This includes the category $\mathbb{F}$-Mod $=$ Vect$_{\mathbb{F}}$ for a field $\mathbb{F}$, $\mathbb{Z}$-Mod $=$ Ab and also the category of modules over the polynomial ring $k[X]$.

2. For every ring R, the category $(R,R)-$BiMod of (R,R)-bimodules and (R,R)-bimodule morphisms is a monoidal category with the tensor product $\otimes_R$ over R and the ring R as a bimodule over itself as the tensor unit. Associators and unit constraints are defined as in the last example, but with respect to the tensor product over R.

3. For any small category $\mathcal{C}$, the category End$(\mathcal{C})$ of endofunctors $F : \mathcal{C} \to \mathcal{C}$ and natural transformations between them is a *strict* monoidal category with:
   - the functor $\otimes : \text{End}(\mathcal{C}) \times \text{End}(\mathcal{C}) \to \text{End}(\mathcal{C})$ that assigns to a pair (F,G) of endofunctors the endofunctor $FG : \mathcal{C} \to \mathcal{C}$ and to a pair (μ, η) of natural transformations $\mu : F \to F'$, $\eta : G \to G'$ the natural transformation $\mu \otimes \eta : FG \to F'G'$ with component morphisms $(\mu \otimes \eta)_C = \mu_{G'(C)} \circ F(\eta_C) = F'(\eta_C) \circ \mu_{G(C)} : FG(C) \to F'G'(C)$,
   - the identity functor as the tensor unit: $e = \text{id}_{\mathcal{C}}$.

4. The categories Set and Top are monoidal categories with:
   - the functor $\otimes : \text{Set} \times \text{Set} \to \text{Set}$ that assigns to a pair of sets (X,Y) their cartesian product $X \times Y$ and to a pair (f,g) of maps $f : X \to X'$, $g : Y \to Y'$ the product map $f \times g : X \times Y \to X' \times Y'$,
   - the functor $\otimes : \text{Top} \times \text{Top} \to \text{Top}$ that sends a pair (X,Y) of topological spaces the product space $X \times Y$ and a pair of continuous maps $f : X \to X'$, $g : Y \to Y'$ to the product map $f \times g : X \times Y \to X' \times Y'$,
   - the one-point set $\{p\}$ and the one-point space $\{p\}$ as the tensor unit,
   - the associators with component morphisms
     $a_{X,Y,Z} : (X \times Y) \times Z \to X \times (Y \times Z)$, $((x,y),z) \mapsto (x,(y,z))$,
   - the unit constraints with component morphisms
     $r_X : X \times \{p\} \to X$, $(x,p) \mapsto x$ and $l_X : \{p\} \times X \to X$, $(p,x) \mapsto x$.

5. More generally, any category $\mathcal{C}$ with finite (co)products is a monoidal category with:
   - the functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ that sends a pair of objects to their (co)product and a pair of morphisms to the induced morphism between (co)products,
   - the empty (co)product, i. e. the final (initial) object in $\mathcal{C}$ as the tensor unit,
   - the associators induced by the universal properties of the (co)products,
   - the unit constraints induced by the universal properties of the (co)products.

   This includes:
   - any abelian category $\mathcal{A}$,
   - the category Set with the disjoint union of sets and the empty set, or with the Cartesian product of sets and the 1-point set,
   - the category Mfld$_n$ of topological or smooth n-dimensional manifolds with the disjoint union and the empty manifold,
   - the category Top with the sum of topological spaces and the empty space, or with

the product of topological spaces and the 1-point space,

- the category Top^1 of pointed topological spaces with wedge sums and the one-point space or with products of pointed spaces and the one-point space,
- the category Grp with the direct product of groups and the trivial group or with the free product of groups and the trivial group.

6. For any commutative ring k, the category $\mathrm{Ch}_{k\text{-Mod}}$ of chain complexes in k-Mod is a monoidal category with the tensor product of chain complexes

$$(A_\bullet \otimes B_\bullet)_n = \oplus_{j=0}^n A_j \otimes_k B_{n-j},\ d_n^{A\otimes B}(a\otimes b) = d_j^A(a)\otimes b + (-1)^k a\otimes d_{n-j}^B(b) \text{ for } a\in A_j, b\in B_{n-j}$$

and with the tensor product of chain maps given by

$$(f_\bullet \otimes g_\bullet)_n(a\otimes b) = f_j(a)\otimes g_{n-j}(b) \qquad \text{for} \quad a\in A_j, b\in B_{n-j}.$$

The tensor unit is the chain complex $0\to k\to 0$ and the associators and unit constraints are induced by the ones in k via the universal property of direct sums.

7. For any monoidal category $\mathcal{C}$ and small category $\mathcal{B}$, the category $\mathrm{Fun}(\mathcal{B},\mathcal{C})$ is a monoidal category with

   - the tensor product of two functors $F, G : \mathcal{B}\to\mathcal{C}$ given by $(F\otimes G)(B) = F(B)\otimes G(B)$ and $(F\otimes G)(f) = F(f)\otimes G(f)$ for all objects $B\in \mathrm{Ob}\,\mathcal{B}$ and morphisms $f : B\to B'$, and the tensor product of natural transformations η,κ given by $(\eta\otimes\kappa)_B = \eta_B\otimes\kappa_B$,
   - the constant functor $I : \mathcal{B}\to\mathcal{C}$ with $I(B) = e$ and $I(\beta) = 1_e$ for all objects B and morphisms β in $\mathcal{B}$ as the tensor unit,
   - the associator and the unit constraints induced by the associators and unit constraints in $\mathcal{C}$.

8. For any monoidal category $(\mathcal{C},\otimes,e,a,l,r)$, there is an **opposite monoidal category** $(\mathcal{C},\otimes^{op},e,a^{-1},r,l)$ with the tensor product $\otimes^{op} : \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ given by $X\otimes^{op}Y = Y\otimes X$ and $f\otimes^{op}g = g\otimes f$ for all objects X, Y and morphisms f, g in $\mathcal{C}$.

In this lecture, we are particularly interested in monoidal categories that arise in representation theory. The standard example is the category $\mathbb{F}[G]{-}\mathrm{Mod}$ of modules over a group algebra $\mathbb{F}[G]$ and its full subcategory $\mathbb{F}[G]{-}\mathrm{Mod}^{fd}$ of finite-dimensional $\mathbb{F}[G]$-modules.

Example 1.1.5: Let G be a group and $\mathbb{F}[G]$ its group algebra over $\mathbb{F}$. Then $\mathbb{F}[G]$-Mod is a monoidal category with

- the tensor product $M\otimes_{\mathbb{F}}N$ with the $\mathbb{F}[G]$-module structure $g\rhd(m\otimes n) = (g\rhd m)\otimes(g\rhd n)$ as tensor product of $\mathbb{F}[G]$-modules,
- the tensor product $f\otimes g : M\otimes_{\mathbb{F}}N\to M\otimes_{\mathbb{F}}N$, $m\otimes n\mapsto f(m)\otimes g(n)$ for $\mathbb{F}[G]$-linear maps,
- the tensor unit $\mathbb{F}$ with the trivial $\mathbb{F}[G]$-module structure: $g\rhd\lambda = \lambda$ for all $g\in G$, $\lambda\in\mathbb{F}$,
- the associators and unit constraints from $\mathrm{Vect}_{\mathbb{F}}$, which become $\mathbb{F}[G]$-linear.

With this structure, the full subcategory $\mathbb{F}[G]{-}\mathrm{Mod}^{fd}$ becomes monoidal as well.

It is clear that Example 1.1.5 does not generalise to the category $A-\text{Mod}$ of modules over a general algebra A over $\mathbb{F}$. This requires additional structure, namely an algebra homomorphism $\Delta : A \to A \otimes A$ to define an A-module structure on the tensor product of two A-modules over $\mathbb{F}$ and an algebra homomorphism $\epsilon : A \to \mathbb{F}$ to define an A-module structure on $\mathbb{F}$. The pentagon and the triangle axiom then impose additional conditions on these algebra homomorphisms, and this leads to the notion of a bialgebra. We will investigate bialgebras in Section 5.

Besides the category $\mathbb{F}[G]-\text{Mod}$, there is another monoidal category associated to a group G and a field $\mathbb{F}$, namely the category of G-graded vector spaces. This category can be deformed with additional data, namely a multiplicative 3-cocycle for the group G. The category of G-graded vector spaces is often useful to build examples and counterexamples.

Example 1.1.6 (Exercise 4)**:** Let G be a group, $\mathbb{F}$ a field and $\omega : G \times G \times G \to \mathbb{F}^\times$ a **3-cocycle**, a map that satisfies $\omega(gh, k, l)\omega(g, h, kl) = \omega(g, h, k)\omega(g, hk, l)\omega(h, k, l)$ for all $g, h, k, l \in G$.

The category Vect_G^ω of G-**graded vector spaces** over $\mathbb{F}$ has

- vector spaces over $\mathbb{F}$ with a decomposition $V = \oplus_{g \in G} V_g$ as objects,
- $\mathbb{F}$-linear maps $f : V \to W$ with $f(V_g) \subset W_g$ for all $g \in G$ as morphisms.

It is a monoidal category with the tensor product

$$V \otimes W = \oplus_{g \in G} (V \otimes W)_g \qquad (V \otimes W)_g = \oplus_{x \in G} V_x \otimes_{\mathbb{F}} W_{x^{-1}g},$$

and the associator given by the linear maps

$$a_{U_g, V_h, W_k} : (U_g \otimes_{\mathbb{F}} V_h) \otimes_{\mathbb{F}} W_k \to U_g \otimes_{\mathbb{F}} (V_h \otimes_{\mathbb{F}} W_k),\; (u \otimes v) \otimes w \mapsto \omega(g, h, k) u \otimes (v \otimes w).$$

Note that if $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ the notion of 3-cocycle in this Example is directly related to the additive notion of a 3-cocycle from group cohomology with values in the trivial $\mathbb{Z}[G]$-module $\mathbb{F}$. The latter is defined as a map $\tau : G \times G \times G \to \mathbb{F}$ with

$$\tau(h, k, l) - \tau(gh, k, l) + \tau(g, hk, l) - \tau(g, h, kl) + \tau(g, h, k) = 0 \qquad \forall g, h, k, l \in G.$$

Setting $\omega(g, h, k) = \exp(\tau(g, h, k))$ yields a multiplicative 3-cocycle as in Example 1.1.6. Note also that one can always choose the trivial 3-cocycle $\omega : G \times G \times G \to \mathbb{F}$, $(g, h, k) \mapsto 1$. In this case, the index ω is omitted and one denotes the category from Example 1.1.6 by $\text{Vect}_G^{\mathbb{F}}$.

Besides the standard examples and representation theoretical examples of monoidal categories, there are also combinatorial examples. An important examples is the simplex category, which plays an important role in homological algebra, including group (co)homology, Hochschild (ho)mology and singular (co)homology. Functors from this category in another category $\mathcal{C}$ allow one to defined homology theories for $\mathcal{C}$.

Example 1.1.7: The **simplex category** Δ has

- as objects finite **ordinal numbers** $[0] = \emptyset$ and $[n] = \{0, 1, ..., n-1\}$ for $n \in \mathbb{N}$,
- as morphisms $f : [n] \to [m]$ monotonic maps maps $f : \{0, ..., n-1\} \to \{0, ..., m-1\}$ with $f(i) \leq f(j)$ for all $0 \leq i \leq j < n$.

It is a strict monoidal category with:

- the functor $\otimes : \Delta \times \Delta \to \Delta$ that assigns

– to a pair $([m],[n])$ of ordinals the ordinal $[m+n]$,
– to a pair (f,g) of monotonic maps $f:[m]\to[m']$, $g:[n]\to[n']$ the monotonic map

$$f\otimes g:[m+n]\to[m'+n'],\quad i\mapsto\begin{cases} f(i) & 0\leq i\leq m-1,\\ m'+g(i-m) & m\leq i\leq n+m-1,\end{cases}$$

- the ordinal $[0]=\emptyset$ as the tensor unit.

Another important class of examples are monoidal categories constructed from families of groups that are related by group homomorphisms. If these group homomorphisms satisfy certain consistency conditions, they give rise to strict monoidal categories.

Example 1.1.8: Suppose $(G_n)_{n\in\mathbb{N}_0}$ is a family of groups with $G_0=\{e\}$ and $(\rho_{m,n})_{m,n\in\mathbb{N}_0}$ a family of group homomorphisms $\rho_{m,n}:G_m\times G_n\to G_{m+n}$ such that $\rho_{0,m}:\{e\}\times G_m\to G_m$, $(e,g)\mapsto g$ and $\rho_{m,0}:G_m\times\{e\}\to G_m$, $(g,e)\mapsto g$ and

$$\rho_{m+n,p}\circ(\rho_{m,n}\times\mathrm{id}_{G_p})=\rho_{m,n+p}\circ(\mathrm{id}_{G_m}\times\rho_{n,p})\quad\forall m,n,p\in\mathbb{N}_0. \tag{3}$$

Then one obtains a strict monoidal category $\mathcal{C}$ with non-negative integers $n\in\mathbb{N}_0$ as objects,

$$\mathrm{Hom}_{\mathcal{C}}(m,n)=\begin{cases}\emptyset & n\neq m\\ G_n & n=m,\end{cases}$$

and the tensor product $\otimes:\mathcal{C}\times\mathcal{C}\to\mathcal{C}$ given by $m\otimes n=m+n$ for all $n,m\in\mathbb{N}_0$ and $f\otimes g=\rho_{m,n}(f,g)$ for all morphisms $f\in G_m$, $g\in G_n$ and the tensor unit $e=0$.

In particular, the construction from Example 1.1.8 can be applied to the braid groups and permutation groups. Both of these groups play an important role in many areas of mathematics. Braid groups can be viewed as generalisations of permutation groups. For this, recall that the symmetric group S_n is presented with generators $\sigma_1,...,\sigma_{n-1}$ and relations

$$\begin{aligned}&\sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1} && \forall i\in\{1,...,n-2\}\\ &\sigma_i\sigma_j=\sigma_j\sigma_i && \forall i,j\in\{1,...,n-1\}\text{ with }|i-j|>1\\ &\sigma_i^2=1 && \forall i\in\{1,...,n-1\},\end{aligned} \tag{4}$$

where $\sigma_i=(i,i+1)$ stands for the elementary transpositions. Omitting the relations in the last line in (4) yields the presentation of the braid group B_n. As the symmetric group S_n is obtained from the braid group by imposing additional relations, there is a canonical group homomorphism $\Pi_n:B_n\to S_n$, $\sigma_i\mapsto\sigma_i$.

Definition 1.1.9: For $n\in\mathbb{N}$ the **braid group** B_n on n strands is the group presented by generators $\sigma_1,..,\sigma_{n-1}$ and relations

$$\begin{aligned}&\sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1} && \forall i\in\{1,...,n-2\}\\ &\sigma_i\sigma_j=\sigma_j\sigma_i && \forall i,j\in\{1,...,n-1\}\text{ with }|i-j|>1.\end{aligned} \tag{5}$$

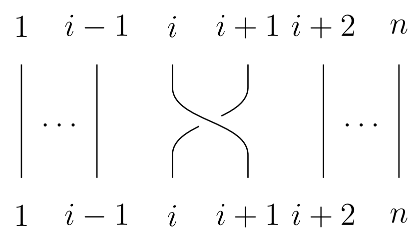
To apply the construction from Example 1.1.8 to the families $(B_n)_{n\in\mathbb{N}_0}$ and $(S_n)_{n\in\mathbb{N}_0}$, one considers the following group homomorphisms that satisfy the condition (3)

$$\rho_{m,n}:B_m\times B_n\to B_{n+m},\ (\sigma_i,\sigma_j)\mapsto\sigma_i\circ\sigma_{m+j}\qquad \rho'_{m,n}:S_m\times S_n\to S_{m+n},\ (\sigma_i,\sigma_j)\mapsto\sigma_i\circ\sigma_{m+j}.$$

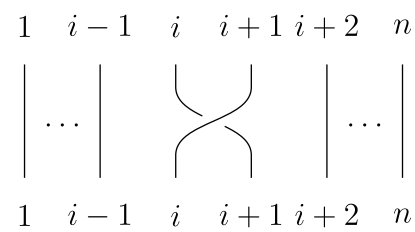
Definition 1.1.10:

1. The **braid category** $\mathcal{B}$ is the strict monoidal category from Example 1.1.8 associated with the family $(B_n)_{n\in\mathbb{N}_0}$ of braid groups and the group homomorphisms $\rho_{m,n}: B_m \times B_n \to B_{n+m}$, $(\sigma_i, \sigma_j) \mapsto \sigma_i \circ \sigma_{m+j}$.
2. The **permutation category** $\mathcal{S}$ is the strict monoidal category from Example 1.1.8 associated with the family $(S_n)_{n\in\mathbb{N}_0}$ of permutation groups and the group homomorphisms $\rho'_{m,n}: S_m \times S_n \to S_{n+m}$, $(\sigma_i, \sigma_j) \mapsto \sigma_i \circ \sigma_{m+j}$.

The name *braid group* is due to the fact that elements $\sigma \in B_n$ can be visualised by braid diagrams, which involve two parallel horizontal lines with n marked points, labelled from the left to the right by $1, ..., n$. The diagram for an element $\sigma \in B_n$ is obtained by drawing n smooth curves that are nowhere horizontal and connect the point i on the upper line to the point $\Pi_n(\sigma)(i)$ on the lower line. The intersection points of these curves are then changed to overcrossings and undercrossings, in such a way that exactly one curve crosses over another at each intersection point. The diagrams for the generators σ_i and their inverses σ_i^{-1} are

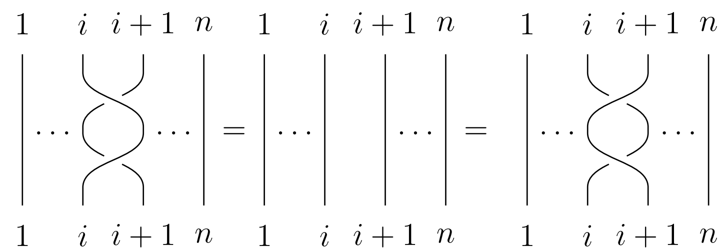


The generator $\sigma_i \in B_n$.



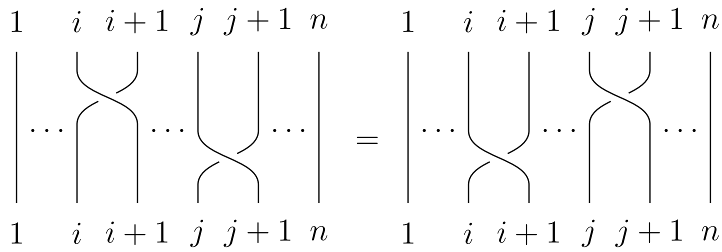
The generator $\sigma_i^{-1} \in B_n$.

The group multiplication in B_n is given by the vertical composition of diagrams. The diagram for the group element $\tau \circ \sigma \in B_n$ is obtained by putting the diagram for τ below the one for σ such that the points on the horizontal lines match, erasing the middle lines, connecting the strands of the two diagrams and tightening them to remove unnecessary crossings. This corresponds to applying the relations $\sigma_i^{\mp1} \circ \sigma_i^{\pm1} = 1$:

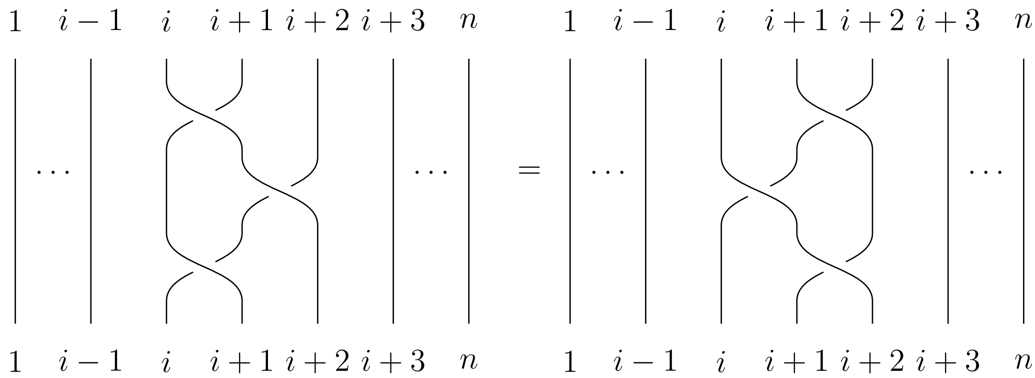


The relation $\sigma_i^{-1} \circ \sigma_i = 1 = \sigma_i \circ \sigma_i^{-1}$.

The remaining relations of the braid group in Definition 1.1.9 correspond to sliding two crossings that do not share a strand past each other and to sliding one crossing point in a triple crossing below the remaining strand:

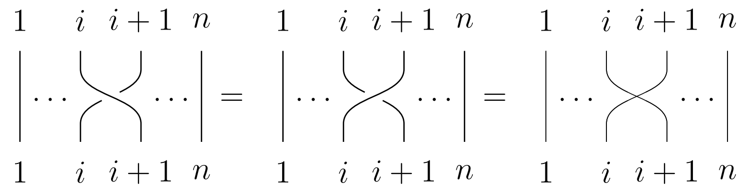


The relation $\sigma_i \circ \sigma_j = \sigma_j \circ \sigma_i$ for $|i-j| > 1$.



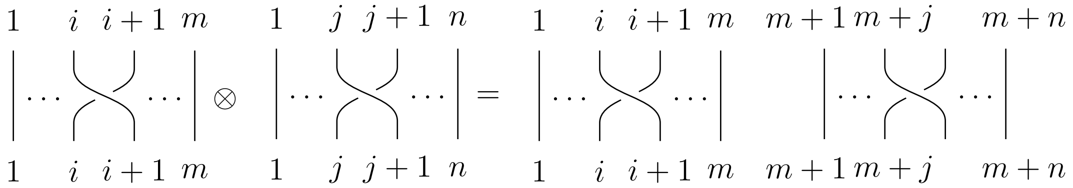
The relation $\sigma_i \circ \sigma_{i+1} \circ \sigma_i = \sigma_{i+1} \circ \sigma_i \circ \sigma_{i+1}$.

Elements of the permutation group S_n are represented by the same diagrams, but with crossings instead of overcrossings and undercrossings. This corresponds to the additional relations $\sigma_i^2 = 1$ for all $i \in \{1, ..., n-1\}$, which identify over- and undercrossings. This implies that the group homomorphism $\Pi_n : B_n \to S_n$, $\sigma_i \mapsto \sigma_i$ is represented graphically by changing each overcrossing or undercrossing in a braid diagram to a crossing:



The relation $\sigma_i^2 = 1$ in S_n.

Elements of the braid category and the permutation category are visualised by the same diagrams. The only difference is that in addition to the *vertical composition* of diagrams that corresponds to the composition of morphisms, there is also a *horizontal composition* corresponding to the tensor product. The tensor product $f \otimes g : m+n \to m+n$ of two morphisms $f : m \to m$ and $g : n \to n$ is obtained by putting the diagram for g with n strands to the right of the diagram for f and adding m to each number in the diagram for g.



After introducing monoidal categories and investigating examples, we will now consider their interaction with functors and natural transformations. The strictest possible compatibility requirement for a functor $F : \mathcal{C} \to \mathcal{D}$ with their monoidal structures on $\mathcal{C}$ and $\mathcal{D}$ would be to impose that $\otimes_{\mathcal{D}}(F \times F) = F \otimes_{\mathcal{C}}$ and $F(e_{\mathcal{C}}) = e_{\mathcal{D}}$. However, there are very few functors that satisfy these strict requirements. It is more natural to impose that these relations hold up to isomorphisms, namely a natural isomorphism $\phi^{\otimes} : \otimes_{\mathcal{D}}(F \times F) \to F \otimes_{\mathcal{C}}$ and an isomorphism $\phi^e : e_{\mathcal{D}} \to F(e_{\mathcal{C}})$.

Definition 1.1.11:
Let $(\mathcal{C}, \otimes_{\mathcal{C}}, e_{\mathcal{C}}, a^{\mathcal{C}}, l^{\mathcal{C}}, r^{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, e_{\mathcal{D}}, a^{\mathcal{D}}, l^{\mathcal{D}}, r^{\mathcal{D}})$ be monoidal categories.

1. A **monoidal functor** from $\mathcal{C}$ to $\mathcal{D}$ is a triple $(F, \phi^e, \phi^{\otimes})$ of
   - a functor $F : \mathcal{C} \to \mathcal{D}$,
   - an isomorphism $\phi^e : e_{\mathcal{D}} \to F(e_{\mathcal{C}})$ in $\mathcal{D}$,
   - a natural isomorphism $\phi^{\otimes} : \otimes_{\mathcal{D}}(F \times F) \to F \otimes_{\mathcal{C}}$,

   that satisfy the following axioms:

   (a) **compatibility with the associativity constraint**:
   for all objects U, V, W of $\mathcal{C}$ the following diagram commutes

$$\begin{tikzcd}
(F(U)\otimes F(V))\otimes F(W) \arrow[r, "a^{\mathcal{D}}_{F(U),F(V),F(W)}"] \arrow[d, "\phi^{\otimes}_{U,V}\otimes 1_{F(W)}"'] & F(U)\otimes(F(V)\otimes F(W)) \arrow[d, "1_U\otimes\phi^{\otimes}_{V,W}"] \\
F(U\otimes V)\otimes F(W) \arrow[d, "\phi^{\otimes}_{U\otimes V,W}"'] & F(U)\otimes F(V\otimes W) \arrow[d, "\phi^{\otimes}_{U,V\otimes W}"] \\
F((U\otimes V)\otimes W) \arrow[r, "F(a^{\mathcal{C}}_{U,V,W})"'] & F(U\otimes(V\otimes W)).
\end{tikzcd} \tag{6}$$

   (b) **compatibility with the unit constraints**:
   for all objects V of $\mathcal{C}$ the following diagrams commute

$$\begin{tikzcd}
e_{\mathcal{D}}\otimes F(V) \arrow[r, "\phi^e\otimes 1_{F(V)}"] \arrow[d, "l^{\mathcal{D}}_{F(V)}"'] & F(e_{\mathcal{C}})\otimes F(V) \arrow[d, "\phi^{\otimes}_{e_{\mathcal{C}},V}"] \\
F(V) & F(e_{\mathcal{C}}\otimes V) \arrow[l, "F(l^{\mathcal{C}}_V)"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F(V)\otimes e_{\mathcal{D}} \arrow[r, "1_{F(V)}\otimes\phi^e"] \arrow[d, "r^{\mathcal{D}}_{F(V)}"'] & F(V)\otimes F(e_{\mathcal{C}}) \arrow[d, "\phi^{\otimes}_{V,e_{\mathcal{C}}}"] \\
F(V) & F(V\otimes e_{\mathcal{C}}). \arrow[l, "F(r^{\mathcal{C}}_V)"]
\end{tikzcd} \tag{7}$$

   A monoidal functor $(F, \phi^e, \phi^{\otimes})$ is called **strict** if $\phi^e = 1_{e_{\mathcal{D}}}$ and $\phi^{\otimes} = \mathrm{id}_{F\otimes_{\mathcal{C}}}$ is the identity natural transformation. It is called a **monoidal equivalence** if $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories, and in this case, $\mathcal{C}$ and $\mathcal{D}$ are called **monoidally equivalent**.

2. Let $(F, \phi^e, \phi^{\otimes}), (F', \phi'^e, \phi'^{\otimes}) : \mathcal{C} \to \mathcal{D}$ be monoidal functors. A **monoidal natural transformation** from F to F' is a natural transformation $\eta : F \to F'$ that satisfies:

   (a) **compatibility with ϕ^e and ϕ'^e**: the following diagram commutes

$$\begin{tikzcd}
F(e_{\mathcal{C}}) \arrow[rr, "\eta_{e_{\mathcal{C}}}"] & & F'(e_{\mathcal{C}}) \\
& e_{\mathcal{D}}. \arrow[ul, "\phi^e"] \arrow[ur, "\phi'^e"'] &
\end{tikzcd} \tag{8}$$

(b) **compatibility with $\phi^\otimes$ and $\phi'^\otimes$**: For all objects V, W of $\mathcal{C}$ the diagram

$$\begin{tikzcd} F(V)\otimes F(W) \arrow[r, "\eta_V\otimes\eta_W"] \arrow[d, "\phi^\otimes_{V,W}"'] & F'(V)\otimes F'(W) \arrow[d, "\phi'^\otimes_{V,W}"] \\ F(V\otimes W) \arrow[r, "\eta_{V\otimes W}"'] & F'(V\otimes W). \end{tikzcd} \tag{9}$$

commutes. A monoidal natural transformation $\eta: F\to F'$ is called **monoidal isomorphism** if for all objects V of $\mathcal{C}$ the morphism $\eta_V: F(V)\to F(V')$ is an isomorphism.

Remark 1.1.12: (Exercise 6) Alternatively, a monoidal functor can be defined as a pair $(F, \phi^\otimes)$ of a functor $F:\mathcal{C}\to\mathcal{D}$ and a natural isomorphism $\phi^\otimes: \otimes_\mathcal{D}(F\times F)\to F\otimes_\mathcal{C}$ such that

(i) $\phi^\otimes$ satisfies the compatibility condition (6) with the associators,
(ii) $F(e_\mathcal{C})\cong e_\mathcal{D}$.

A monoidal natural transformation from $(F, \phi^\otimes)$ to $(F', \phi'^\otimes)$ can be defined equivalently as a natural transformation $\eta: F\to F'$ such that $\eta_e: F(e)\to F'(e)$ is an isomorphism and diagram (9) commutes.

Remark 1.1.13:

1. The isomorphism ϕ^e and the natural isomorphism $\phi^\otimes$ are called the **coherence data** of the monoidal functor $(F, \phi^e, \phi^\otimes)$. The natural isomorphism $\phi^\otimes$ determines ϕ^e uniquely (Exercise 6), and it is a choice of structure, not a property. A given functor between monoidal categories may have several different monoidal functor structures, or none of them (Exercise 7).

2. If $F:\mathcal{C}\to\mathcal{D}$ is a monoidal equivalence, there is monoidal equivalence $G:\mathcal{D}\to\mathcal{C}$ such that $FG:\mathcal{D}\to\mathcal{D}$ and $GF:\mathcal{C}\to\mathcal{C}$ are isomorphic to the identity functors by monoidal isomorphisms. This gives an alternative definition of monoidal equivalence (Exercise 10).

Example 1.1.14:

1. The forgetful functor $F:\mathrm{Top}\to\mathrm{Set}$ is a strict monoidal functor, when Top and Set are equipped with the monoidal structures defined by their products or coproducts.

2. For any group G and field $\mathbb{F}$ the forgetful functor $F:\mathbb{F}[G]-\mathrm{Mod}\to\mathrm{Vect}_\mathbb{F}$ is a strict monoidal functor.

3. The functor $F:\mathrm{Set}\to k\text{-Mod}$ that assigns to a set X the free k-module $F(X)=\langle X\rangle_k$ and to a map $f: X\to Y$ the unique k-linear map $F(f):\langle X\rangle_k\to\langle Y\rangle_k$ with $F(f)\circ\iota_X=\iota_Y\circ f$ is a monoidal functor, when Set is equipped with the product monoidal structure. Its coherence data is given by the maps $\phi^e: k\to\langle p\rangle_k$, $\lambda\mapsto\lambda p$ and $\phi^\otimes_{X,Y}:\langle X\rangle_k\otimes_k\langle Y\rangle_k\to\langle X\times Y\rangle_k$, $x\otimes y\mapsto(x,y)$.

4. For any group homomorphism $\alpha: G\to H$, the functor $\alpha^*:\mathbb{F}[H]-\mathrm{Mod}\to\mathbb{F}[G]-\mathrm{Mod}$ that assigns to an $\mathbb{F}[H]$-module $(M,\rhd)$ the $\mathbb{F}[G]$-module $(M,\rhd_\alpha)$ with $g\rhd_\alpha m=\alpha(g)\rhd m$ and every $\mathbb{F}[H]$-linear map to itself is a strict monoidal functor.

5. For any ring R the functor $F : (R,R)\text{−BiMod} \to \text{Fun}(R\text{−Mod}, R\text{−Mod})$ that assigns
   - to an (R,R)-bimodule M the functor $L_M = M \otimes_R - := R\text{−Mod} \to R\text{−Mod}$,
   - to an (R,R)-bimodule morphism $f : M \to N$ the natural transformation $L_f = f \otimes_R - : M \otimes_R - \to N \otimes_R -$,

   is a monoidal functor. Its coherence data is
   - the natural isomorphism $\phi^e : \text{id}_{R-\text{Mod}} \to L_R$ with $\phi^e_M : M \to R \otimes_R M$, $m \mapsto 1 \otimes m$,
   - the natural isomorphism $\phi^\otimes : \otimes(L \times L) \to L \otimes$ whose component morphisms are the natural isomorphisms $\phi^\otimes_{M,N} : L_M L_N \to L_{M \otimes N}$ with component morphisms $(\phi^\otimes_{M,N})_P = a^{-1}_{M,N,P} : M \otimes_R (N \otimes_R P) \to (M \otimes_R) \otimes_R P$, $m \otimes (n \otimes p) \to (m \otimes n) \otimes p$.

6. Let $\mathcal{D}, \mathcal{E}$ be small categories, $\mathcal{C}$ a monoidal category and equip $\text{Fun}(\mathcal{D}, \mathcal{C})$ and $\text{Fun}(\mathcal{E}, \mathcal{C})$ with the monoidal structures from from Example 1.1.4, 6.

   • Pre-composition with a given functor $F : \mathcal{E} \to \mathcal{D}$ defines a monoidal functor $F^* : \text{Fun}(\mathcal{D}, \mathcal{C}) \to \text{Fun}(\mathcal{E}, \mathcal{C})$ that sends a functor $G : \mathcal{D} \to \mathcal{C}$ to GF and a natural transformation $\eta : G \to G'$ to $\eta F : GF \to GF'$.

   • Pre-composition with a natural transformation $\eta : F \to F'$ defines a monoidal natural transformation $\eta^* : F^* \to F'^*$ with component morphisms $\eta^*_G = G\eta : GF \to GF'$ for all functors $G : \mathcal{D} \to \mathcal{C}$ (Exercise).

1.2 Strictification and coherence

Computations and proofs in monoidal categories can be quite complicated due to the amount of coherence data. Many arguments that are fairly straightforward in a strict monoidal category become a maze of commuting pentagon axioms, triangle axioms and diagrams that encode the naturality of the associator and unit constraints in a non-strict monoidal category.

Mac Lane's famous strictification and coherence theorem allow one to largely ignore these complications. It states that any monoidal category is monoidally equivalent to a strict monoidal category. The coherence theorem is a consequence of the strictification theorem and implies that any two morphisms between given objects in a monoidal category that are composed of unit morphisms, associators and left and right unit constraints are equal.

This allows one to perform computations as in a strict monoidal category, then choose appropriate bracketings of the tensor products and insert units, and finally add in coherence data such that all the morphisms have the right start and target object. As a consequence of the coherence theorem, any way of doing this will yield the same result. In fact, this result is already used implicitly in elementary computations with tensor products of vector spaces, in which the bracketings are usually omitted.

To prove Mac Lane's strictification and coherence result, we need additional results about the properties of the tensor unit in a monoidal category. The first result related the left and right unit constraint for a tensor product of objects to the left and right unit constraints of the individual objects and can be viewed as a generalisation of the triangle axiom.

Lemma 1.2.1: **(properties of the unit constraints)** Let $(\mathcal{C}, \otimes, e, a, l, r)$ be a monoidal category. Then the diagrams

$$\begin{tikzcd} (e\otimes V)\otimes W \arrow[rr, "a_{e,V,W}"] \arrow[dr, "l_V\otimes 1_W"'] & & e\otimes(V\otimes W) \arrow[dl, "l_{V\otimes W}"] \\ & V\otimes W & \end{tikzcd} \qquad \begin{tikzcd} (V\otimes W)\otimes e \arrow[rr, "a_{V,W,e}"] \arrow[dr, "r_{V\otimes W}"'] & & V\otimes(W\otimes e) \arrow[dl, "1_V\otimes r_W"] \\ & V\otimes W & \end{tikzcd}$$

commute for all objects V, W, and one has $l_{e\otimes V} = 1_e\otimes l_V$, $r_{V\otimes e} = r_V\otimes 1_e$ and $l_e = r_e$.

Proof:
1. We consider for objects U, V, W of $\mathcal{C}$ the diagram

$$\begin{tikzcd} ((U\otimes e)\otimes V)\otimes W \arrow[rr, "a_{U,e,V}\otimes 1_W"] \arrow[dr, "(r_U\otimes 1_V)\otimes 1_W"] \arrow[dd, "a_{U\otimes e,V,W}"'] & & (U\otimes(e\otimes V))\otimes W \arrow[dl, "(1_U\otimes l_V)\otimes 1_W"'] \arrow[dd, "a_{U,e\otimes V,W}"] \\ & (U\otimes V)\otimes W \arrow[d, "a_{U,V,W}"] & \\ (U\otimes e)\otimes(V\otimes W) \arrow[r, "r_U\otimes 1_{V\otimes W}"] \arrow[dr, "a_{U,e,V\otimes W}"'] & U\otimes(V\otimes W) & U\otimes((e\otimes V)\otimes W) \arrow[l, "1_U\otimes(l_V\otimes 1_W)"'] \arrow[dl, "1_U\otimes a_{e,V,W}"] \\ & U\otimes(e\otimes(V\otimes W)). \arrow[u, "1_U\otimes l_{V\otimes W}"'] & \end{tikzcd}$$

The outer pentagon in this diagram commutes by the pentagon axiom, the upper triangle and the lower left triangle commute by the triangle axiom and the two quadrilaterals by the naturality of $a : \otimes(\otimes\times\mathrm{id}_\mathcal{C}) \to \otimes(\mathrm{id}_\mathcal{C}\times\otimes)$. As all arrows in this diagram are isomorphisms, it follows that the lower right triangle commutes as well. To show that this implies the commutativity of the first triangle in the lemma, we choose $U = e$ and use the naturality of $l : \otimes(e\times\mathrm{id}_\mathcal{C}) \to \mathrm{id}_\mathcal{C}$, which implies $f = g$ for all morphisms $f, g : X \to Y$ with $1_e\otimes f = 1_e\otimes g$:

$$\begin{tikzcd} X \arrow[rrr, bend left, "f"] \arrow[rrr, bend right, "g"'] & e\otimes X \arrow[l, "l_X"'] \arrow[r, shift left, "1_e\otimes f"] \arrow[r, shift right, "1_e\otimes g"'] & e\otimes Y \arrow[r, "l_Y"] & Y. \end{tikzcd}$$

This shows that the first triangle commutes, and the proof for the second triangle is similar.

2. To prove the last three identities in the lemma, we consider the commutative diagrams

$$\begin{tikzcd} e\otimes(e\otimes V) \arrow[r, "l_{e\otimes V}"] \arrow[d, "1_e\otimes l_V"'] & e\otimes V \arrow[d, "l_V"] \\ e\otimes V \arrow[r, "l_V"'] & V \end{tikzcd} \qquad \begin{tikzcd} & (e\otimes e)\otimes e \arrow[dl, "r_e\otimes 1_e"'] \arrow[d, "a_{e,e,e}"] \arrow[dr, "l_e\otimes 1_e"] & \\ e\otimes e \arrow[rr, bend right, "1_{e\otimes e}"'] & e\otimes(e\otimes e) \arrow[l, "1\otimes l_e"'] \arrow[r, "l_{e\otimes e}"] & e\otimes e \end{tikzcd}$$

The first diagram commutes by the naturality of $l : \otimes(e\times\mathrm{id}_\mathcal{C}) \to \mathrm{id}_\mathcal{C}$. Because $l_V : e\otimes V \to V$ is an isomorphism, it follows that $l_{e\otimes V} = 1_e\otimes l_V$. The proof of the identity $r_{V\otimes e} = r_V\otimes 1_e$ is analogous. In the second diagram, the lower triangle is the identity $l_{e\otimes e} = 1_e\otimes l_e$, which follows from the first diagram with $V = e$, the left triangle commutes by the triangle axiom and the right triangle commutes by 1. Hence, the outer triangle commutes as well and $l_e \otimes 1_e = r_e \otimes 1_e$. By the same argument as in 1., this implies $r_e = l_e$. □

Corollary 1.2.2: The endomorphisms of the tensor unit in a monoidal category form a commutative monoid.

Proof:
In any category $\mathcal{C}$ and for any object C the set $\mathrm{Hom}_{\mathcal{C}}(C, C)$ is a monoid with the composition of morphisms. To show that $\mathrm{Hom}_{\mathcal{C}}(e, e)$ is commutative, we consider the diagram

$$\begin{tikzcd}
e \arrow[rrr, "g"] \arrow[ddd, "f"'] & & & e \arrow[ddd, "f"] \\
& e\otimes e \arrow[ul, "r_e"', "l_e"] \arrow[r, "1_e\otimes g"] \arrow[d, "f\otimes 1_e"'] & e\otimes e \arrow[ur, "r_e", "l_e"'] \arrow[d, "f\otimes 1_e"] & \\
& e\otimes e \arrow[r, "1_e\otimes g"'] \arrow[dl, "l_e"', "r_e"] & e\otimes e \arrow[dr, "l_e", "r_e"'] & \\
e \arrow[rrr, "g"] & & & e
\end{tikzcd}$$

where we used the identity $l_e = r_e$ from Lemma 1.2.1. The inner rectangle commutes because $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a functor. The inner quadrilaterals commute by the naturality of r and l, and hence the outer square commutes as well. □

This last result motivates the name *monoidal category.* Using these results, we can prove Mac Lane's coherence theorem. Instead of MacLane's original proof, which can be found for instance in [Ka, Section XI.5] and in Exercise 11, we give an alternative proof from [EGNO, Chapter 2.8]. As usual, we ignore largeness and smallness issues in categories.

The construction in this proof can be viewed as the analogue of the right-module structure of a ring R over itself by right multiplication, but for a monoidal category instead of a ring. Consider a ring R as a right module over itself. Then R-right module endomorphisms of R are maps $f : R \to R$ with $f(s \cdot r) = f(s) \cdot r$ for all $r, s \in R$. This implies that any R-right module endomorphism $f : R \to R$ satisfies $f(r) = f(1 \cdot r) = f(1) \cdot r$ and hence is determined by $f(1) \in R$. It follows that the map $L : R \to \mathrm{End}_{R^{op}}(R)$, $r \mapsto L_r$ with $L_r(s) = r \cdot s$ for all $s \in S$ is a ring isomorphism.

In the following we consider the categorical counterpart of this statement. Instead of a ring R we consider a monoidal category $\mathcal{C}$, whose tensor product replaces the ring multiplication. Instead of R-right module endomorphisms $f : R \to R$, we consider endofunctors $F : \mathcal{C} \to \mathcal{C}$ together with a natural isomorphism $c : \otimes(F \times \mathrm{id}_{\mathcal{C}}) \to F\otimes$. To obtain a category, we take as morphisms natural transformations between such endofunctors. After imposing compatibility conditions involving the isomorphisms c and the associators, we then obtain a strict monoidal category $\mathcal{C}'$, in which the tensor product is given by the composition of functors.

Instead of the R-right module endomorphisms $L_r : R \to R$, $s \mapsto r \cdot s$, we then consider the functors $L_X = x\otimes - : \mathcal{C} \to \mathcal{C}$ for objects X in $\mathcal{C}$ and instead of the ring isomorphism $L : R \to \mathrm{End}_R(R)$, we obtain a monoidal equivalence $L : \mathcal{C} \to \mathcal{C}'$.

Theorem 1.2.3: (**strictification of monoidal categories**)
Any monoidal category is monoidally equivalent to a strict monoidal category.

Proof:
1. Let $(\mathcal{C}, \otimes, e, a, l, r)$ be a monoidal category. Construct a strict monoidal category $(\mathcal{C}', \otimes', e')$:

- Objects of $\mathcal{C}'$ are pairs (F, c) of an endofunctor $F : \mathcal{C} \to \mathcal{C}$ and a natural isomorphism $c : \otimes(F \times \mathrm{id}) \to F\otimes$ such that the following diagram commutes for all $X, Y, Z \in \mathrm{Ob}\mathcal{C}$

$$\begin{tikzcd}
(F(X)\otimes Y)\otimes Z \arrow[r, "a_{F(X),Y,Z}"] \arrow[d, "c_{X,Y}\otimes 1_Z"'] & F(X)\otimes(Y\otimes Z) \arrow[r, "c_{X,Y\otimes Z}"] & F(X\otimes(Y\otimes Z)) \\
F(X\otimes Y)\otimes Z \arrow[r, "c_{X\otimes Y,Z}"'] & F((X\otimes Y)\otimes Z). \arrow[ur, "F(a_{X,Y,Z})"'] &
\end{tikzcd} \tag{10}$$

- Morphisms $\nu : (F, c) \to (F', c')$ in $\mathcal{C}'$ are natural transformations $\nu : F \to F'$ such that the following diagram commutes for all $X, Y \in \mathrm{Ob}\mathcal{C}$

$$\begin{tikzcd}
F(X)\otimes Y \arrow[r, "c_{X,Y}"] \arrow[d, "\nu_X\otimes 1_Y"'] & F(X\otimes Y) \arrow[d, "\nu_{X\otimes Y}"] \\
F'(X)\otimes Y \arrow[r, "c'_{X,Y}"'] & F'(X\otimes Y).
\end{tikzcd} \tag{11}$$

  The composition of morphisms is the composition of natural transformations, and the identity morphisms are identity natural transformations $1_{(F,c)} = \mathrm{id}_F : (F, c) \to (F, c)$.

- The tensor product $\otimes' : \mathcal{C}'\otimes\mathcal{C}' \to \mathcal{C}'$ is given by $(F, c)\otimes'(G, d) = (FG, cd)$ on the objects, where $cd : \otimes(FG \times \mathrm{id}) \to FG\otimes$ is the natural isomorphism with component morphisms $(cd)_{X,Y} = F(d_{X,Y}) \circ c_{G(X),Y} : FG(X)\otimes Y \to FG(X\otimes Y)$.

  The tensor product of morphisms $\mu : (F, c) \to (F', c')$ and $\nu : (G, d) \to (G', d')$ in $\mathcal{C}'$ is the natural transformation $\mu\otimes'\nu = (\mu G') \circ (F\nu) = (F'\nu) \circ (\mu G) : FG \to F'G'$.

To see that $\mathcal{C}'$ is a category, note that the identity natural transformation $\mathrm{id}_F : F \to F$ makes the diagram (11) for $c = c'$ and $F = F'$ commute. By stacking the diagrams (11) for morphisms $\nu : (F, c) \to (F', c')$ and $\nu' : (F', c') \to (F'', c'')$ vertically, one sees that the composite natural transformation $\nu' \circ \nu : F \to F''$ makes the diagram (11) for c and c'' commute. Hence, the composition of morphisms is well-defined.

To show that the tensor product $\otimes'$ is well-defined on the objects, we verify that the diagram (10) commutes for $(F, c)\otimes'(G, d) = (FG, cd)$ for all objects (F, c) and (G, d) in $\mathcal{C}'$. By subdividing it, we obtain the diagram

$$\begin{tikzcd}
(FG(X)\otimes Y)\otimes Z \arrow[r, "a_{FG(X),Y,Z}"] \arrow[d, "c_{G(X),Y}\otimes 1_Z"] \arrow[dd, bend right=60, "(cd)_{X,Y}\otimes 1_Z"'] & FG(X)\otimes(Y\otimes Z) \arrow[r, "(cd)_{X,Y\otimes Z}"] \arrow[dr, "c_{G(X),Y\otimes Z}"] & FG(X\otimes(Y\otimes Z)) \\
F(G(X)\otimes Y)\otimes Z \arrow[r, "c_{G(X)\otimes Y,Z}"] \arrow[d, "F(d_{X,Y})\otimes 1_Z"] & F((G(X)\otimes Y)\otimes Z) \arrow[r, "F(a_{G(X),Y,Z})"] \arrow[d, "F(d_{X,Y}\otimes 1_Z)"'] & F(G(X)\otimes(Y\otimes Z)) \arrow[u, "F(d_{X,Y\otimes Z})"'] \\
FG(X\otimes Y)\otimes Z \arrow[r, "c_{G(X\otimes Y),Z}"'] \arrow[rr, bend right=20, "(cd)_{X\otimes Y,Z}"'] & F(G(X\otimes Y)\otimes Z) \arrow[r, "F(d_{X\otimes Y,Z})"'] & FG((X\otimes Y)\otimes Z). \arrow[uu, bend right=60, "FG(a_{X,Y,Z})"']
\end{tikzcd}$$

in which the triangles commute by definition of $cd : \otimes(FG \times \mathrm{id}_{\mathcal{C}}) \to FG\otimes$, the lower pentagon, because it is the image of diagram (10) for (G, d) under F, the upper pentagon, because it is the diagram (10) for (F, c), and the quadrilateral by naturality of c. This shows that (FG, cd) is again an object in $\mathcal{C}'$ and $\otimes'$ is well-defined on the objects of $\mathcal{C}'$.

To show that $\otimes'$ is well-defined on the morphisms of $\mathcal{C}'$, we show that the diagram (11) commutes for the natural transformation $\mu\otimes'\nu : FG \to F'G'$ for all morphisms $\mu : (F, c) \to (F', c')$ and $\nu : (G, d) \to (G', d')$. This is the outer rectangle in the diagram

$$\begin{tikzcd}
FG(X)\otimes Y \arrow[r, "c_{G(X),Y}"] \arrow[d, "F(\nu_X)\otimes 1_Y"] \arrow[rr, bend left, "(cd)_{X,Y}"] \arrow[dd, bend right=60, "(\mu\otimes'\nu)_X\otimes 1_Y"'] & F(G(X)\otimes Y) \arrow[r, "F(d_{X,Y})"] \arrow[d, "F(\nu_X\otimes 1_Y)"'] & FG(X\otimes Y) \arrow[d, "F(\nu_{X\otimes Y})"'] \arrow[dd, bend left=60, "(\mu\otimes'\nu)_{X\otimes Y}"] \\
FG'(X)\otimes Y \arrow[r, "c_{G'(X),Y}"] \arrow[d, "\mu_{G'(X)}\otimes 1_Y"] & F(G'(X)\otimes Y) \arrow[r, "F(d'_{X,Y})"] \arrow[d, "\mu_{G'(X)\otimes Y}"'] & FG'(X\otimes Y) \arrow[d, "\mu_{G'(X\otimes Y)}"'] \\
F'G'(X)\otimes Y \arrow[r, "c'_{G'(X),Y}"'] \arrow[rr, bend right, "(c'd')_{X,Y}"'] & F'(G'(X)\otimes Y) \arrow[r, "F'(d'_{X,Y})"'] & F'G'(X\otimes Y),
\end{tikzcd}$$

where the upper triangle commutes by definition of cd, the lower triangle by definition of $c'd'$, the left and right triangles by definition of $\mu\otimes'\nu$, the upper left rectangle by naturality of c, the lower right rectangle by naturality of μ, the lower left rectangle, because it is the diagram (11) for μ, c and c' and the upper right rectangle, because it is the image of the diagram (11) for ν, d and d' under F. This shows that $\mu\otimes'\nu : (FG, cd) \to (F'G', c'd')$ is a morphism in $\mathcal{C}'$.

The tensor product of $\mathcal{C}'$ is a functor $\otimes' : \mathcal{C}' \times \mathcal{C}' \to \mathcal{C}'$, since we have for all natural transformations $\mu : F \to F'$, $\mu' : F' \to F''$, $\nu : G \to G'$, $\nu' : G' \to G''$ and $X \in \mathrm{Ob}\,\mathcal{C}$

$$\begin{aligned}
&(\mathrm{id}_F\otimes'\mathrm{id}_G)_X = (\mathrm{id}_F)_{G(X)} \circ F(\mathrm{id}_{G(X)}) = \mathrm{id}_{FG(X)}\\
&((\mu'\otimes'\nu') \circ (\mu\otimes'\nu))_X = \mu'_{G''(X)} \circ F'(\nu'_X) \circ \mu_{G'(X)} \circ F(\nu_X) = \mu'_{G''(X)} \circ \mu_{G''(X)} \circ F(\nu'_X) \circ F(\nu_X)\\
&\qquad\qquad\qquad\qquad\quad = (\mu' \circ \mu)_{G''(X)} \circ F((\nu' \circ \nu)_X) = ((\mu' \circ \mu)\otimes'(\nu' \circ \nu))_X.
\end{aligned}$$

The tensor product $\otimes'$ is strictly associative, since this holds for the composition of functors, and we have for all $(E, b), (F, c), (G, d) \in \mathrm{Ob}\,\mathcal{C}'$ and $X, Y \in \mathrm{Ob}\,\mathcal{C}$

$$\begin{aligned}
((bc)d)_{X,Y} &= EF(d_{X,Y}) \circ (bc)_{G(X),Y} = EF(d_{X,Y}) \circ E(c_{G(X),Y}) \circ b_{FG(X),Y}\\
&= E((cd)_{X,Y}) \circ b_{FG(X),Y} = (b(cd))_{X,Y}.
\end{aligned}$$

That it is strictly associative on the morphisms follows from the identities

$$\begin{aligned}
(\mu\otimes'\nu)\otimes'\rho &= (\mu\otimes'\nu)H' \circ (FG\rho) = (\mu G'H' \circ F\nu H') \circ (FG\rho) = (\mu G'H') \circ (F\nu H' \circ FG\rho)\\
&= (\mu G'H') \circ F(\nu\otimes'\rho) = (\mu G'H')\otimes'(\nu'\otimes'\rho)
\end{aligned}$$

for all natural transformations $\mu : F \to F'$, $\nu : G \to G'$ and $\rho : H \to H'$.

It is strictly unital with unit $(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_\otimes)$, since $F\mathrm{id}_{\mathcal{C}} = \mathrm{id}_{\mathcal{C}}F = F$ for all functors $F : \mathcal{C} \to \mathcal{C}$ and for all natural isomorphisms $c : \otimes(F \times \mathrm{id}_{\mathcal{C}}) \to \otimes F$ and $X, Y \in \mathrm{Ob}\,\mathcal{C}$

$$(\mathrm{id}_\otimes c)_{X,Y} = c_{X,Y} \circ 1_{F(X)\otimes Y} = c_{X,Y} = 1_{F(X\otimes Y)} \circ c_{X,Y} = (c\,\mathrm{id}_\otimes)_{X,Y}.$$

2. We construct a monoidal functor $(L, \phi^\otimes, \phi^e) : \mathcal{C} \to \mathcal{C}'$. Let $L : \mathcal{C} \to \mathcal{C}'$ send

- an object C in $\mathcal{C}$ to the the pair $(L_C, a_{C,-,-})$ of the endofunctor $L_C = C\otimes - : \mathcal{C} \to \mathcal{C}$ and the natural isomorphism $a_{C,-,-} : \otimes(L_C \times \mathrm{id}_\mathcal{C}) \to L_C\otimes$ with component morphisms $a_{C,Y,Z} : (C\otimes Y)\otimes Z \to C\otimes(Y\otimes Z)$,
- a morphism $f : C \to C'$ in $\mathcal{C}$ to the natural transformation $L(f) : L_C \to L_{C'}$ with component morphisms $L(f)_X = (f\otimes 1_X) : C\otimes X \to C'\otimes X$.

The pentagon axiom for $\mathcal{C}$ ensures that the diagram (10) commutes for $F = L_C = C\otimes - : \mathcal{C} \to \mathcal{C}$ and the natural isomorphisms $a_{C,-,-} : \otimes(L_C \times \mathrm{id}_\mathcal{C}) \to L_C\otimes$, so L is well-defined on the objects. The naturality of the associator guarantees that (11) commutes for $F = L_C$, $c_{X,Y} = a_{C,X,Y}$, $F' = L_{C'}$, $c'_{X,Y} = a_{C',X,Y}$ and $\nu_X = f\otimes 1_X$ and hence L is well-defined on the morphisms.

That L is a functor follows because $L(1_C) = \mathrm{id}_{L_C}$ and one has for all morphisms $f : C \to C'$ and $f' : C' \to C''$ and $X \in \mathrm{Ob}\,\mathcal{C}$

$$(L(f') \circ L(f))_X = (f'\otimes 1_X) \circ (f\otimes 1_X) = ((f' \circ f)\otimes 1_X) = L(f' \circ f)_X.$$

To show that L is monoidal, we construct an isomorphism $\phi^e : (\mathrm{id}_\mathcal{C}, \mathrm{id}_\otimes) \to (L_e, a_{e,-,-})$ in $\mathcal{C}'$ and a natural isomorphism $\phi^\otimes : \otimes'(L \times L) \to L\otimes$.

We define $\phi^e = l^{-1} : \mathrm{id}_\mathcal{C} \to L_e$ with component morphisms $\phi^e_X = l_X^{-1} : X \to e\otimes X$. Lemma 1.2.1 implies that the diagram (11) commutes for $F = \mathrm{id}_\mathcal{C}$, $F' = L_e$, $c_{X,Y} = 1_{X\otimes Y}$, $c'_{X,Y} = a_{e,X,Y}$ and $\nu = l^{-1}$, and hence $\phi^e = l^{-1}$ is indeed an isomorphism in $\mathcal{C}'$.

We define the component morphisms of the natural isomorphism $\phi^\otimes : \otimes'(L \times L) \to L\otimes$ as $\phi^\otimes_{C,C'} = a^{-1}_{C,C',-} : L_C L_{C'} \to L_{C\otimes C'}$. Their naturality follows from the naturality of the associator in the last argument. The naturality of $\phi^\otimes$ follows from its naturality in the first two arguments. The pentagon axiom in $\mathcal{C}$ implies that diagram (11) commutes for

$$(F, c) = (L_C, a_{C,-,-})\otimes'(L_{C'}, a_{C',-,-}), \quad (F', c') = (L_{C\otimes C'}, a_{C\otimes C',-,-}), \quad \nu = \phi^\otimes_{C,C'} = a^{-1}_{C,C',-}.$$

This shows that we constructed a natural isomorphism $\phi^\otimes_{C,C'} = a^{-1}_{C,C',-} : L_C L_{C'} \to L_{C\otimes C'}$.

It remains to show that the isomorphism $\phi^e = l^{-1} : \mathrm{id}_\mathcal{C} \to L_e$ and the natural isomorphism $\phi^\otimes : \otimes'(L \times L) \to L\otimes$ satisfy the conditions from Definition 1.1.11. As $\mathcal{C}'$ is strict, these conditions are equivalent to the commutativity of the following diagrams of endofunctors and natural transformations

$$\begin{tikzcd}
L_C \arrow[r, "\phi^e L_C"] \arrow[d, "\mathrm{id}_{L_C}"'] & L_e L_C \arrow[d, "\phi^\otimes_{e,C}"] \\
L_C & L_{e\otimes C} \arrow[l, "L(l_C)"]
\end{tikzcd}
\qquad
\begin{tikzcd}
L_C \arrow[r, "L_C \phi^e"] \arrow[d, "\mathrm{id}_{L_C}"'] & L_C L_e \arrow[d, "\phi^\otimes_{C,e}"] \\
L_C & L_{C\otimes e} \arrow[l, "L(r_C)"]
\end{tikzcd}
\qquad
\begin{tikzcd}
L_U L_V L_W \arrow[r, "L_U \phi^\otimes_{V,W}"] \arrow[d, "\phi^\otimes_{U,V} L_W"'] & L_U L_{V\otimes W} \arrow[drr, "\phi^\otimes_{U,V\otimes W}"] & & \\
L_{U\otimes V} L_W \arrow[r, "\phi^\otimes_{U\otimes V,W}"'] & L_{(U\otimes V)\otimes W} \arrow[rr, "L(a_{U,V,W})"'] & & L_{U\otimes(V\otimes W)}.
\end{tikzcd}$$

Evaluating them on an object X and inserting the component morphisms of ϕ^e and $\phi^\otimes$ shows that the first diagram commutes by Lemma 1.2.1 , the second by the triangle axiom and the third by the pentagon axiom for $\mathcal{C}$. This shows that $(L, \phi^\otimes, \phi^e) : \mathcal{C} \to \mathcal{C}'$ is a monoidal functor.

3. We show that $L : \mathcal{C} \to \mathcal{C}'$ is essentially surjective and fully faithful and hence a monoidal equivalence. For essential surjectivity, note that any object (F, c) in $\mathcal{C}'$ is isomorphic to $(L_{F(e)}, a_{F(e),-,-})$. The isomorphisms $\nu_X = F(l_X) \circ c_{e,X} : F(e)\otimes X \to F(X)$ are natural in X and

define a natural isomorphism $\nu : L_{F(e)} \to F$. It makes the diagram (11) for $(L_{F(e)}, a_{F(e),-,-})$ and (F, c) commute, since this is the outer rectangle in the diagram

$$\begin{tikzcd}
(F(e)\otimes X)\otimes Y \arrow[rr, "a_{F(e),X,Y}"] \arrow[d, "c_{e,X}\otimes 1_Y"] \arrow[dd, bend right=60, "\nu_X\otimes 1_Y"'] & & F(e)\otimes(X\otimes Y) \arrow[d, "c_{e,X\otimes Y}"'] \arrow[dd, bend left=60, "\nu_{X\otimes Y}"] \\
F(e\otimes X)\otimes Y \arrow[r, "c_{e\otimes X,Y}"] \arrow[d, "F(l_X)\otimes 1_Y"] & F((e\otimes X)\otimes Y) \arrow[r, "F(a_{e,X,Y})"] \arrow[dr, "F(l_X\otimes 1_Y)"'] & F(e\otimes(X\otimes Y)) \arrow[d, "F(l_{X\otimes Y})"'] \\
F(X)\otimes Y \arrow[rr, "c_{X,Y}"'] & & F(X\otimes Y),
\end{tikzcd}$$

where the left and right triangle commute by definition of ν, the lower middle triangle by Lemma 1.2.1, the upper rectangle by (10) for (F, c) and the lower left quadrilateral by naturality of c. This shows that $\nu : (L_{F(e)}, a_{F(e),-,-}) \to (F, c)$ is an isomorphism in $\mathcal{C}'$.

To see that L is faithful suppose that $L(f) = L(g)$ for morphisms $f, g : C \to C'$ in $\mathcal{C}$. Then $f\otimes e = L(f)\otimes e = L(g)\otimes e = g\otimes e$, and by naturality of the right unit constraints one has

$$f = r_{C'} \circ (f\otimes e) \circ r_C^{-1} = r_{C'} \circ (g\otimes e) \circ r_C^{-1} = g.$$

To show that L is full, let $\nu : L(C) \to L(C')$ be a morphism in $\mathcal{C}'$. Then $\nu = L(f)$, where $f = r_{C'} \circ \nu_e \circ r_C^{-1}$, since we have for all $X \in \mathrm{Ob}\,\mathcal{C}$ the commuting diagram

$$\begin{tikzcd}
C\otimes X \arrow[d, "L(f)_X"', "=f\otimes 1_X"] \arrow[rrr, bend left, "1_{C\otimes X}"] & (C\otimes e)\otimes X \arrow[l, "r_C\otimes 1_X"'] \arrow[r, "a_{C,e,X}"'] \arrow[d, "\nu_e\otimes 1_X"] & C\otimes(e\otimes X) \arrow[r, "1_C\otimes l_X"'] \arrow[d, "\nu_{e\otimes X}"] & C\otimes X \arrow[d, "\nu_X"] \\
C'\otimes X \arrow[rrr, bend right, "1_{C'\otimes X}"'] & (C'\otimes e)\otimes X \arrow[l, "r_{C'}\otimes 1_X"] \arrow[r, "a_{C',e,X}"] & C'\otimes(e\otimes X) \arrow[r, "1_{C'}\otimes l_X"] & C'\otimes X,
\end{tikzcd}$$

in which the top and bottom triangle commute by the triangle axiom, the left rectangle by definition of f, the middle rectangle by (11) for for ν and the right rectangle by naturality of ν. This shows that L is fully faithful and hence an equivalence of categories. $\square$

As a consequence of the strictification theorem, we obtain the coherence theorem for monoidal categories that allow us to control associators and unit constraints. It justifies ignoring coherence data and performing proofs in strict monoidal categories.

Theorem 1.2.4: (**coherence for monoidal categories**)
Let $\mathcal{C}$ be a monoidal category, $X_1, \ldots, X_n \in \mathrm{Ob}\,\mathcal{C}$ and $X, Y \in \mathrm{Ob}\,\mathcal{C}$ parenthesized products of $X_1, \ldots, X_n$, in this order, and with insertions of the tensor unit. Let $f, g : X \to Y$ be morphisms in $\mathcal{C}$ that are are composites via $\circ$ and $\otimes$ of the associativity and unit constraints, their inverses and identity morphisms. Then $f = g : X \to Y$.

Sketch of Proof:
By Theorem 1.2.3 there is a monoidal equivalence $(L, \phi^\otimes, \phi^e) : \mathcal{C} \to \mathcal{C}'$ into a strict monoidal category $\mathcal{C}'$. Express f and g as a composites $f = f_n \circ \ldots \circ f_1$ and $g = g_m \circ \ldots \circ g_1$ of morphisms

f_i and g_j that are tensor products of exactly one associator, unit constraint or their inverses with identity morphisms. We need to show that the diagram

$$\begin{tikzcd}
& X_1 \arrow[r, "f_2"] & \dots \arrow[r, "f_{n-1}"] & X_{n-1} \arrow[dr, "f_n"] & \\
X \arrow[ur, "f_1"] \arrow[dr, "g_1"'] & & & & Y, \\
& Y_1 \arrow[r, "g_2"'] & \dots \arrow[r, "g_{m-1}"'] & Y_{m-1} \arrow[ur, "g_m"'] &
\end{tikzcd} \tag{12}$$

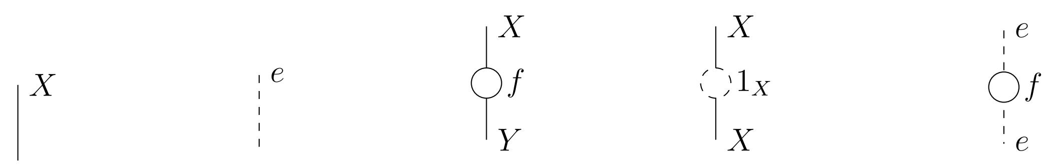
commutes. For this, take its image under L. Build a prism by attaching

- to each arrow $L(f_i)$ and $L(g_i)$ that contains an image of the associator under L the commutative diagram (6) from Definition 1.1.11,
- to each arrow $L(f_i)$ and $L(g_i)$ that contains the image of a left (right) unit constraint the corresponding diagram (7) from Definition 1.1.11.

Then the diagram at the top of the prism contains only identity morphisms due to the strictness of $\mathcal{C}'$, and all the side faces of the prism commute. This implies that the top face can be contracted to a point, and one obtains a cone with commuting side faces. It follows that the bottom face of the cone commutes as well. This is the image of diagram (12) under L, and hence $L(f) = L(g)$. As L is an equivalence of categories, it follows that $f = g$. □

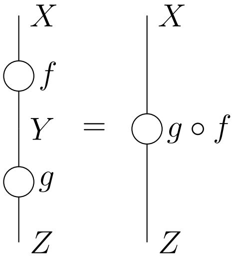
The coherence theorem allows one to perform computations in a monoidal category by assuming he category is strict. It also allows one to introduce a graphical calculus for monoidal categories. Its usefulness will become apparent when we consider monoidal categories with more additional structure.

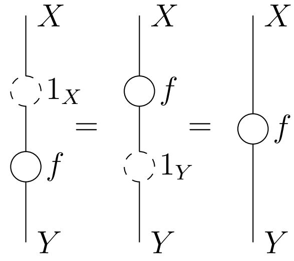
In this graphical calculus, objects in a monoidal category $\mathcal{C}$ are represented by vertical lines labelled with the object. The unit object e is represented by the empty line, i. e. not drawn in the diagrams. A morphism $f : X \to Y$ is represented as a vertex on a vertical line that divides the line into an upper part labelled by X and a lower part labelled by Y. Unit morphisms in $\mathcal{C}$ are not represented by vertices in the diagrams.



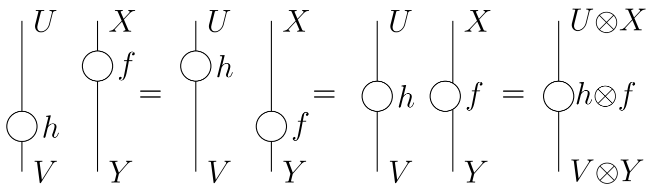
the object X | the tensor unit e | a morphism $f : X \to Y$ | identity morphism $1_X : X \to X$ | an endomorphism $f : e \to e$

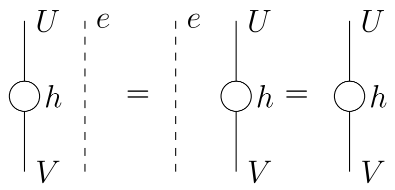
The composition of morphisms is given by the *vertical composition* of diagrams, whenever the object at the bottom of one diagram matches the object at the top of the other. More precisely, the composite $g \circ f : X \to Z$ of two morphisms $f : X \to Y$ and $g : X \to Z$ is obtained by putting the diagram for g below the one for f. The associativity of the composition of morphisms and the properties of the unit morphisms ensure that this is consistent for multiple composites and that it is possible to omit identity morphisms.





Tensor products of objects and morphisms are given by the *horizontal composition* of diagrams. The diagram for the tensor product $U{\otimes}X$ involves two parallel vertical lines, the one on the left labelled by U and the one on the right labelled by X. The tensor product of morphisms is represented by vertices on such lines. The condition that $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a functor allows one to slide these vertices past each other and to compose them individually on each line:





(13)

Just as the tensor unit and the identity morphisms, the component morphisms of the associator and of the left and right unit constraints are not represented in this diagrammatical calculus. This is consistent because of Theorem 1.2.4. Any two objects represented by the same diagram are related by a unique isomorphism composed of associators and left and right unit constraints. Consequently, any two morphisms represented by the same diagrams are related by left and right composition with the isomorphisms that relate their source and target objects.

Note also that the diagrammatic calculus for monoidal categories generalises the diagrammatic representation of morphisms in the braid category $\mathcal{B}$ and permutation category $\mathcal{S}$ from Definition 1.1.10. In these categories, all objects are tensor products of a single object 1, and all morphisms are composites of tensor products of one elementary transposition and its inverse. This allows one to omit the labelling of the objects and morphisms in the diagrammatic calculus for the categories $\mathcal{B}$ and $\mathcal{S}$.

2 Duals in monoidal categories

2.1 Rigid monoidal categories

In this section, we consider monoidal categories with additional structure, namely left and right dual objects. These duals will play an important role in the construction of knot and manifold invariants in the following section and give rise to a notion of trace in a monoidal category. They generalise the dual V^* of a finite-dimensional vector space over $\mathbb{F}$ and the associated linear maps $\mathrm{ev} : V^* \otimes V \to \mathbb{F}$, $\alpha \otimes v \mapsto \alpha(v)$ ad $\mathrm{coev} : \mathbb{F} \to V \otimes V^*$, $\lambda \mapsto \Sigma_{i=1}^n v_i \otimes \alpha^i$ for a basis $(v_1, ..., v_n)$ of V and the associated dual basis $(\alpha^1, ..., \alpha^n)$ of V^*.

For a finite-dimensional vector space V, it is merely a matter of convention if one defines the evaluation map for V as a map $\mathrm{ev} : V^* \otimes V \to \mathbb{F}$, $\alpha \otimes v \mapsto \alpha(v)$ or as map $\mathrm{ev} : V \otimes V^* \to \mathbb{F}$, $v \otimes \alpha \mapsto \alpha(v)$ and similarly for the coevaluations. These choices are related a trivial flip of the factors in the tensor product. However, in a general monoidal category there is no structure that exchanges the factors in a tensor product. For this reason, it is important to distinguish these definitions and to introduce left and right duals.

Definition 2.1.1: Let $(\mathcal{C}, \otimes, e, a, l, r)$ be a monoidal category.

1. An object X of $\mathcal{C}$ is called **right dualisable** if there is an object X^*, a **right dual** of X, and morphisms
$$\mathrm{ev}_X^R : X^* \otimes X \to e \qquad \mathrm{coev}_X^R : e \to X \otimes X^*$$
such that the following diagrams commute
$$\begin{tikzcd} X \arrow[r, "l_X^{-1}"] \arrow[d, "1_X"'] & e\otimes X \arrow[r, "\mathrm{coev}_X^R\otimes 1_X"] & (X\otimes X^*)\otimes X \arrow[d, "a_{X,X^*,X}"] \\ X & X\otimes e \arrow[l, "r_X"] & X\otimes(X^*\otimes X) \arrow[l, "1_X\otimes \mathrm{ev}_X^R"] \end{tikzcd} \qquad \begin{tikzcd} X^* \arrow[r, "r_{X^*}^{-1}"] \arrow[d, "1_{X^*}"'] & X^*\otimes e \arrow[r, "1_{X^*}\otimes \mathrm{coev}_X^R"] & X^*\otimes(X\otimes X^*) \arrow[d, "a^{-1}_{X^*,X,X^*}"] \\ X^* & e\otimes X^* \arrow[l, "l_{X^*}"] & (X^*\otimes X)\otimes X^*. \arrow[l, "\mathrm{ev}_X^R\otimes 1_{X^*}"] \end{tikzcd} \tag{14}$$

2. An object X of $\mathcal{C}$ is called **left dualisable** if there is an object ${}^*X$, a **left dual** of X, and morphisms
$$\mathrm{ev}_X^L : X \otimes {}^*X \to e \qquad \mathrm{coev}_X^L : e \to {}^*X \otimes X$$
such that the following diagrams commute
$$\begin{tikzcd} X \arrow[r, "r_X^{-1}"] \arrow[d, "1_X"'] & X\otimes e \arrow[r, "1_X\otimes \mathrm{coev}_X^L"] & X\otimes({}^*X\otimes X) \arrow[d, "a^{-1}_{X,{}^*X,X}"] \\ X & e\otimes X \arrow[l, "l_X"] & (X\otimes {}^*X)\otimes X \arrow[l, "\mathrm{ev}_X^L\otimes 1_X"] \end{tikzcd} \qquad \begin{tikzcd} {}^*X \arrow[r, "l_{{}^*X}^{-1}"] \arrow[d, "1_{{}^*X}"'] & e\otimes {}^*X \arrow[r, "\mathrm{coev}_X^L\otimes 1_{{}^*X}"] & ({}^*X\otimes X)\otimes {}^*X \arrow[d, "a_{{}^*X,X,{}^*X}"] \\ {}^*X & {}^*X\otimes e \arrow[l, "r_{{}^*X}"] & {}^*X\otimes(X\otimes {}^*X). \arrow[l, "1_{{}^*X}\otimes \mathrm{ev}_X^L"] \end{tikzcd} \tag{15}$$

The category $\mathcal{C}$ is called **right rigid** if every object in $\mathcal{C}$ is right dualisable, **left rigid** if every object of $\mathcal{C}$ is left dualisable, and **rigid** if it is both right and left rigid.

Note that Definition 2.1.1 implies that an object Y in $\mathcal{C}$ is a right dual of an object X in $\mathcal{C}$ if and only if X is a left dual of Y. Note also that the evaluation and the coevaluation are part of the duality. We will see below that right and left duals may coincide as objects but differ in their evaluations and coevaluations.

Although the duals are not defined via a universal property, it turns out that they are unique up to unique isomorphism. We will thus speak of *the* left or right dual of a given object.

Proposition 2.1.2: Let $\mathcal{C}$ be a monoidal category. If a left or right dual of an object X in $\mathcal{C}$ exists, it is unique up to unique isomorphism.

Proof:
We prove the claim for right duals.

We suppose that X_1^* and X_2^* are right duals of X with right evaluations and coevaluations $e_i : X_i^* \otimes X \to e$ and $c_i : e \to X \otimes X_i^*$ and show that there is a unique isomorphism $\phi_{ij} : X_i^* \to X_j^*$ with $e_j \circ (\phi_{ij} \otimes 1_X) = e_i$ and $(1_X \otimes \phi_{ij}) \circ c_i = c_j$.

For this, we consider for $i \neq j \in \{1, 2\}$ the morphisms

$$\phi_{ij} : X_i^* \xrightarrow{r^{-1}_{X_i^*}} X_i^* \otimes e \xrightarrow{1_{X_i^*} \otimes c_j} X_i^* \otimes (X \otimes X_j^*) \xrightarrow{a^{-1}_{X_i^*, X, X_j^*}} (X_i^* \otimes X) \otimes X_j^* \xrightarrow{e_i \otimes 1_{X_j^*}} e \otimes X_j^* \xrightarrow{l_{X_j^*}} X_j^* \qquad (16)$$

and show that ϕ_{ij} is inverse to ϕ_{ji}. For notational simplicity we restrict attention to the case where $\mathcal{C}$ is strict and consider the diagram

$$\begin{tikzcd}
X_i^* \arrow[r, "1\otimes c_i"] \arrow[d, "1\otimes c_j"] \arrow[dd, bend right=60, "\phi_{ij}"'] & X_i^* \otimes X \otimes X_i^* \arrow[d, "1\otimes c_j \otimes 1 \otimes 1"] \arrow[dr, "1\otimes 1\otimes 1"] & \\
X_i^* \otimes X \otimes X_j^* \arrow[r, "1\otimes 1\otimes 1\otimes c_i"] \arrow[d, "e_i \otimes 1"] & X_i^* \otimes X \otimes X_j^* \otimes X \otimes X_i^* \arrow[r, "1\otimes 1\otimes e_j \otimes 1"] \arrow[d, "e_i \otimes 1 \otimes 1 \otimes 1"] & X_i^* \otimes X \otimes X_i^* \arrow[d, "e_i \otimes 1"'] \\
X_j^* \arrow[r, "1\otimes c_i"] \arrow[rr, bend right=20, "\phi_{ji}"'] & X_j^* \otimes X \otimes X_i^* \arrow[r, "e_j \otimes 1"] & X_i^*,
\end{tikzcd}$$

in which the upper triangle commutes by (14), the curved triangles by definition of ϕ_{ij} and the rectangles by functoriality of the tensor product. This shows that $\phi_{ji} \circ \phi_{ij} = (e_i \otimes 1) \circ (1 \otimes c_i)$. By applying the snake identity (14), we obtain $\phi_{ji} \circ \phi_{ij} = 1_{X_i^*}$. This shows that $\phi_{ij} : X_i^* \to X_j^*$ are isomorphisms. To show the remaining identities, we consider the diagrams

$$\begin{tikzcd}
X_i^* \otimes X \arrow[r, "1\otimes c_j \otimes 1"] \arrow[d, "1\otimes 1"'] \arrow[rr, bend left=30, "\phi_{ij} \otimes 1"] & X_i^* \otimes X \otimes X_j^* \otimes X \arrow[r, "e_i \otimes 1 \otimes 1"] \arrow[dl, "1\otimes 1\otimes e_j"] & X_j^* \otimes X \arrow[dl, "e_j"] \\
X_i^* \otimes X \arrow[r, "e_i"'] & e &
\end{tikzcd}
\qquad
\begin{tikzcd}
e \arrow[r, "c_j"] \arrow[d, "c_i"] & X \otimes X_j^* \arrow[d, "c_i \otimes 1 \otimes 1"'] \arrow[dr, "1\otimes 1"] & \\
X \otimes X_i^* \arrow[r, "1\otimes 1\otimes c_j"] \arrow[rr, bend right=20, "1\otimes \phi_{ij}"'] & X \otimes X_i^* \otimes X \otimes X_j^* \arrow[r, "1\otimes e_i \otimes 1"] & X \otimes X_j^*,
\end{tikzcd}$$

in which the triangles commute by (14) and the rectangles by functoriality of the tensor product. This shows that $e_j \circ (\phi_{ij} \otimes 1_X) = e_i$ and $(1_X \otimes \phi_{ij}) \circ c_i = c_j$ for ϕ_{ij} from (16). It also shows that any morphism ϕ_{ij} that satisfies these identities is given by (16). $\square$

Example 2.1.3:

1. In the category $\mathcal{C} = \mathrm{Fun}(\mathcal{D}, \mathcal{D})$ an object $F : \mathcal{D} \to \mathcal{D}$ is left (right) dualisable, if and only if it has a right (left) adjoint (Exercise 12).

2. The category $\mathrm{Vect}_{\mathbb{F}}^{fd}$ of finite-dimensional vector spaces over $\mathbb{F}$ is rigid with the dual vector space $V^*={}^*V$ as the left and right dual and the evaluations and coevaluations

$$\begin{aligned}&\mathrm{coev}_V^R:\mathbb{F}\to V\otimes V^*, \quad \lambda\mapsto\lambda\Sigma_{i=1}^n v_i\otimes\alpha^i \qquad \mathrm{ev}_V^R: V^*\otimes V\to\mathbb{F},\quad \alpha\otimes v\mapsto\alpha(v)\\ &\mathrm{coev}_V^L:\mathbb{F}\to V^*\otimes V, \quad \lambda\mapsto\lambda\Sigma_{i=1}^n \alpha^i\otimes v_i \qquad \mathrm{ev}_V^L: V\otimes V^*\to\mathbb{F},\quad v\otimes\alpha\mapsto\alpha(v),\end{aligned} \tag{17}$$

where $(v_1,...,v_n)$ is a basis of V and $(\alpha^1,...,\alpha^n)$ the dual basis of V^*. To show that $\mathrm{Vect}_{\mathbb{F}}^{fd}$ is right rigid, we verify the identities (14)

$$\begin{aligned}&r_V\circ(\mathrm{id}_V\otimes\mathrm{ev}_V^R)\circ a_{V,V^*,V}\circ(\mathrm{coev}_V^R\otimes\mathrm{id}_V)\circ l_V^{-1}(v)=\Sigma_{i=1}^n\alpha^i(v)v_i=v\\ &l_V\circ(\mathrm{ev}_V^R\otimes\mathrm{id}_{V^*})\circ a^{-1}_{V,V^*,V}\circ(\mathrm{id}_{V^*}\otimes\mathrm{coev}_V^R)\circ r_V^{-1}(\beta)=\Sigma_{i=1}^n\beta(v_i)\alpha^i=\beta\end{aligned}$$

for all $v\in V$, $\beta\in V^*$. A similar computation shows that $\mathrm{Vect}_{\mathbb{F}}^{fd}$ is left rigid.

3. One may modify the left evaluation and coevaluation in $\mathrm{Vect}_{\mathbb{F}}^{fd}$ by setting

$$\mathrm{ev}_V'^L=\mathrm{ev}^L\circ(\mu^{-1}\otimes\mathrm{id}_{V^*}):V\otimes V^*\to\mathbb{F} \qquad \mathrm{coev}_V'^L=(\mathrm{id}_{V^*}\otimes\mu)\circ\mathrm{coev}_V^L:\mathbb{F}\to V^*\otimes V$$

for an isomorphism $\mu:V\to V$. A direct computation shows that $\mathrm{ev}_V'^L$ and $\mathrm{ev}_V'^L$, again satisfy (15). An analogous procedure can be applied to the right evaluation and coevaluation.

4. The category $\mathbb{F}[G]-\mathrm{Mod}^{fd}$ of finite-dimensional modules over a group algebra $\mathbb{F}[G]$ from Example 1.1.5 is rigid.

   The left and right dual of an $\mathbb{F}[G]$-module $(V,\rhd)$ is $(V^*,\rhd^*)$ with the $\mathbb{F}[G]$-module structure $g\rhd^*\alpha=\alpha\circ(g^{-1}\rhd -)$. The evaluations and coevaluations are again given by (17) and become $\mathbb{F}[G]$-linear with this dual module structure.

5. For any finite group G and 3-cocycle $\omega:G\times G\times G\to\mathbb{F}^\times$, the monoidal category $\mathrm{Vect}_G^{\omega\, fd}$, the full subcategory of the category Vect_G^ω in Example 1.1.6 with only *finite-dimensional* vector spaces as objects, is rigid (Exercise 13).

6. If X is a right (left) dualisable object in $\mathcal{C}$ and $F:\mathcal{C}\to\mathcal{D}$ a monoidal functor, then $F(X)$ is right (left) dualisable. It follows that if $\mathcal{C}$ is right (left) rigid and F essentially surjective, then $\mathcal{D}$ is right (left) rigid (Exercise 14).

Note that the restriction to finite-dimensional vector spaces and finite-dimensional $\mathbb{F}[G]$-modules in the second, third and fourth example is necessary to ensure the existence of the coevaluation. There is no way of extending it consistently to infinite-dimensional vector spaces. Note also that although it is defined in terms of a basis, the coevaluation in the second example does not depend on the choice of this basis (Exercise).

The first example and third example in Example 2.1.3 show that left and right duals in a rigid monoidal category need not coincide, and the existence of left (right) dual does not imply the existence of a right (left) dual. In fact, one can characterise the existence of left or right duals in a monoidal category $\mathcal{C}$ in terms of left or right adjoint functors.

Proposition 2.1.4: Let $\mathcal{C}$ be a monoidal category.

1. If $C\in\mathrm{Ob}\,\mathcal{C}$ is right dualisable, then the functor $L_{C^*}=C^*\otimes -:\mathcal{C}\to\mathcal{C}$ is left adjoint to $L_C=C\otimes -:\mathcal{C}\to\mathcal{C}$ and $R_{C^*}=-\otimes C^*:\mathcal{C}\to\mathcal{C}$ is right adjoint to $R_C=-\otimes C:\mathcal{C}\to\mathcal{C}$.

2. If $C \in \mathrm{Ob}\,\mathcal{C}$ is left dualisable, then the functor $L_{{}^*C} = {}^*C\otimes - : \mathcal{C} \to \mathcal{C}$ is right adjoint to $L_C = C\otimes - : \mathcal{C} \to \mathcal{C}$ and $R_{{}^*C} = -\otimes{}^*C : \mathcal{C} \to \mathcal{C}$ is left adjoint to $R_C = -\otimes C : \mathcal{C} \to \mathcal{C}$.

If $\mathcal{C}$ is rigid all functors $L_C = C\otimes -$ and $R_C = -\otimes C$ have left and right adjoints.

Proof:
We prove the claim for the functor $L_{C^*} = C^*\otimes - : \mathcal{C} \to \mathcal{C}$. The proofs for the functors $L_{{}^*C}, R_{C^*}, R_{{}^*C} : \mathcal{C} \to \mathcal{C}$ are analogous.

The unit $\epsilon : L_{C^*}L_C \to \mathrm{id}_{\mathcal{C}}$ and counit $\eta : \mathrm{id}_{\mathcal{C}} \to L_C L_{C^*}$ of the adjunction $L_{C^*} \dashv L_C$ are given by the right and left coevaluation and evaluation for C and have component morphisms

$$\epsilon_M : C^*\otimes(C\otimes M) \xrightarrow{a^{-1}_{C^*,C,M}} (C\otimes C^*)\otimes M \xrightarrow{\mathrm{ev}^R_C\otimes 1_M} e\otimes M \xrightarrow{l_M} M$$
$$\eta_M : M \xrightarrow{l^{-1}_M} e\otimes M \xrightarrow{\mathrm{coev}^R_C\otimes 1_M} (C\otimes C^*)\otimes M \xrightarrow{a_{C,C^*,M}} C\otimes(C^*\otimes M).$$

Due to the naturality of the associator and the unit constraints, these morphisms are natural in M and hence define natural transformations $\epsilon : L_{C^*}L_C \to \mathrm{id}_{\mathcal{C}}$ and $\eta : \mathrm{id}_{\mathcal{C}} \to L_C L_{C^*}$.

That they satisfy the conditions $(L_C\epsilon)\circ(\eta L_C) = \mathrm{id}_{L_C}$ and $(\epsilon L_{C^*})\circ(L_{C^*}\eta) = \mathrm{id}_{L_{C^*}}$ from Proposition B.20 follows from the commuting diagrams

$$\begin{tikzcd}
C\otimes M \arrow[r, "1_{C\otimes M}"] \arrow[d, "l^{-1}_{C\otimes M}"'] & C\otimes M \arrow[r, "1_{C\otimes M}"] \arrow[d, "l^{-1}_C\otimes 1"'] & C\otimes M \arrow[r, "1_{C\otimes M}"] & C\otimes M \\
e\otimes(C\otimes M) \arrow[d, "\mathrm{coev}^R_C\otimes(1\otimes 1)"'] & (e\otimes C)\otimes M \arrow[l, "a"'] \arrow[d, "(\mathrm{coev}^R_C\otimes 1)\otimes 1"'] & (C\otimes e)\otimes M \arrow[r, "a"] \arrow[u, "r_C\otimes 1"'] & C\otimes(e\otimes M) \arrow[u, "1\otimes l_M"'] \\
(C\otimes C^*)\otimes(C\otimes M) \arrow[dr, "a"'] & ((C\otimes C^*)\otimes C)\otimes M \arrow[l, "a"'] \arrow[r, "a\otimes 1"] & (C\otimes(C^*\otimes C))\otimes M \arrow[r, "a"] \arrow[u, "(1\otimes\mathrm{ev}^R_C)\otimes 1"'] & C\otimes((C^*\otimes C)\otimes M) \arrow[u, "1\otimes(\mathrm{ev}^R_C\otimes 1)"'] \\
 & C\otimes(C^*\otimes(C\otimes M)) \arrow[urr, "1\otimes a^{-1}"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
C^*\otimes M \arrow[r, "1_{C^*\otimes M}"] \arrow[d, "1\otimes l^{-1}_M"'] & C^*\otimes M \arrow[r, "1_{C^*\otimes M}"] \arrow[d, "r^{-1}_{C^*}\otimes 1"'] & C^*\otimes M \arrow[r, "1_{C^*\otimes M}"] & C^*\otimes M \\
C^*\otimes(e\otimes M) \arrow[d, "1\otimes(\mathrm{coev}^R_C\otimes 1)"'] & (C^*\otimes e)\otimes M \arrow[l, "a"'] \arrow[d, "(1\otimes\mathrm{coev}^R_C)\otimes 1"'] & (e\otimes C^*)\otimes M \arrow[r, "a"] \arrow[u, "l_{C^*}\otimes 1"'] & e\otimes(C^*\otimes M) \arrow[u, "l_{C^*\otimes M}"'] \\
C^*\otimes((C\otimes C^*)\otimes M) \arrow[dr, "1\otimes a"'] & (C^*\otimes(C\otimes C^*))\otimes M \arrow[l, "a"'] & ((C^*\otimes C)\otimes C^*)\otimes M \arrow[l, "a\otimes 1"'] \arrow[r, "a"] \arrow[u, "(\mathrm{ev}^R_C\otimes 1)\otimes 1"'] & (C^*\otimes C)\otimes(C^*\otimes M) \arrow[u, "\mathrm{ev}^R_C\otimes(1\otimes 1)"'] \arrow[dll, "a"] \\
 & C^*\otimes(C\otimes(C^*\otimes M)). & &
\end{tikzcd}$$

The outer paths from $C\otimes M$ to $C\otimes(C^*\otimes(C\otimes M))$ and from $C\otimes(C^*\otimes(C\otimes M))$ to $C\otimes M$ in the first diagram are the component morphisms $(\eta L_C)_M$ and $(L_C\epsilon)_M$. The outer paths from $C^*\otimes M$ to $C^*\otimes(C\otimes(C^*\otimes M))$ and from $C^*\otimes(C\otimes(C^*\otimes M))$ to $C^*\otimes M$ in the second diagram are the component morphisms $(L_{C^*}\eta)_M$ and $(\epsilon L_{C^*})_M$. In both diagrams, the pentagons at the bottom commute by the pentagon axiom, the upper left and upper right rectangles by Lemma 1.2.1 and the triangle axiom, the lower left and lower right rectangles by naturality of the associator and the hexagons in the middle by the snake identity (14) for the right duals. $\Box$

Left and right duals in a monoidal category can be included in the graphical calculus. We represent the right and left (co)evaluation for right and left dualisable objects in a monoidal category by the following diagrams

$$\mathrm{ev}_X^R : X^*\otimes X \to e \qquad \mathrm{coev}_X^R : e \to X\otimes X^* \qquad \mathrm{ev}_X^L : X\otimes {}^*X \to e \qquad \mathrm{coev}_X^L : e \to {}^*X\otimes X.$$

The commuting diagrams (14) and (15) in Definition 2.1.1 then take the form

(18)

For this reason they are often called the **snake identities** or **zigzag identities**.

The diagrammatic representation of the duals also suggests a way of extending duals from objects to morphisms. Given a morphism $f : X \to Y$ and representing it diagrammatically, we may use pictures as the ones in the snake identity to construct a morphism $f^* : Y^* \to X^*$ from the right duals or a morphism ${}^*f : {}^*X \to {}^*Y$ from the left duals:

(19)

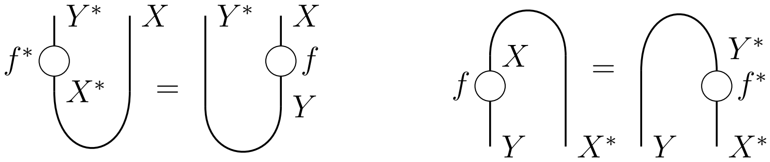
This suggests that one could extend the right or left duals to a functor. As the construction in (19) reverses the source and target objects and the tensor products, it should correspond to a monoidal functor $* : \mathcal{C} \to \mathcal{C}^{op,op}$, where $\mathcal{C}^{op,op}$ is the category with the opposite composition and the opposite tensor product. The following proposition shows that this is indeed the case.

Proposition 2.1.5: Let $\mathcal{C}$ be a right (left) rigid monoidal category. Then the right (left) duals define a monoidal functor $* : \mathcal{C} \to \mathcal{C}^{op,op}$.

Proof:
We prove the claim for right rigid monoidal categories. The proof for left rigid monoidal categories is analogous. For this, we define $*$ on morphisms by setting for each morphism $f : X \to Y$

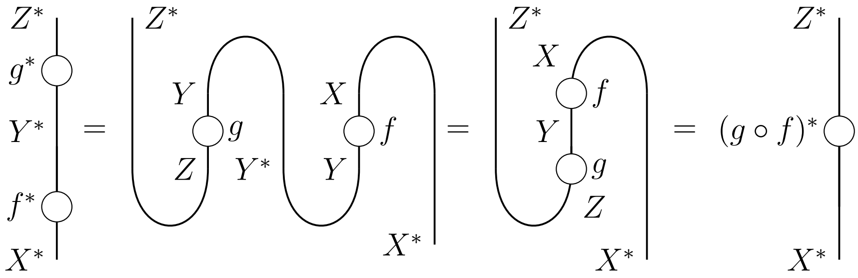
$$\begin{aligned} f^* : Y^* &\xrightarrow{r_{Y^*}^{-1}} Y^*\otimes e \xrightarrow{1_{Y^*}\otimes \mathrm{coev}_X^R} Y^*\otimes(X\otimes X^*) \xrightarrow{a_{Y^*,X,X^*}^{-1}} (Y^*\otimes X)\otimes X^* \xrightarrow{(1_{Y^*}\otimes f)\otimes 1_{X^*}} (Y^*\otimes Y)\otimes X^* \\ &\xrightarrow{\mathrm{ev}_Y^R\otimes 1_{X^*}} e\otimes X^* \xrightarrow{l_{X^*}} X^*. \end{aligned} \tag{20}$$

Diagrammatically, this morphism is given by the left picture in (19). Using this diagrammatic representation and the diagrammatic representation of the snake identities in (18), we obtain for all objects X, Y and morphisms $f : X \to Y$



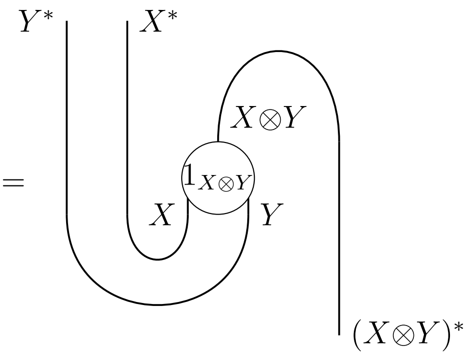
(21)

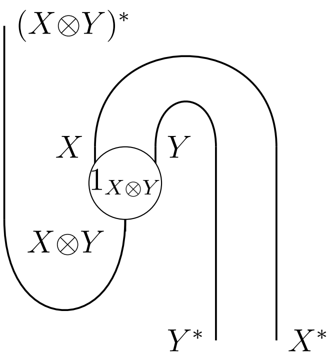
The snake identities also imply $1_X^* = 1_{X^*}$ for all objects X and $(g \circ f)^* = f^* \circ g^*$ for all morphisms $g : Y \to Z$:



This shows that $* : \mathcal{C} \to \mathcal{C}^{op}$ is a functor.

To prove that it is a *monoidal* functor, we construct an isomorphism $\phi^e : e \to e^*$ and a natural isomorphism $\phi^\otimes : \otimes^{op}(* \times *) \to *\otimes$ that satisfy the compatibility conditions with the associator and unit constraints from Definition 1.1.11. For this, we define $\phi^e := l_{e^*} \circ \mathrm{coev}_e^R : e \to e^*$. By combining the snake identity (14) with Lemma 1.2.1, one finds that ϕ_e is invertible, with inverse $\phi^{e\,-1} = \mathrm{ev}_e^R \circ r_{e^*}^{-1} : e^* \to e$. We also define for all objects X, Y

$\phi^{\otimes}_{X,Y} =$ 

$\phi^{\otimes\,-1}_{X,Y} =$ 

The snake identity then implies directly that $\phi^{-1}_{X,Y}$ is inverse to $\phi^{\otimes}_{X,Y}$. The naturality of $\phi^\otimes$ follows from the identity (21), together with the fact that $\otimes$ is a functor and the naturality of the associator and the unit constraints. The compatibility conditions from Definition 1.1.11 that involve the the associator and the unit constraints follow directly from the definition of $\phi^\otimes$ and ϕ^e and the coherence theorem. This shows that $*$ is monoidal. □

Remark 2.1.6: (Exercise 15)

1. One can show that for a rigid monoidal category $\mathcal{C}$ the functors $*^R : \mathcal{C} \to \mathcal{C}^{op,op}$ and $*^L : \mathcal{C} \to \mathcal{C}^{op,op}$ defined by the right and left duals are equivalences of categories: The functors $*^L *^R : \mathcal{C} \to \mathcal{C}$ and $*^R *^L : \mathcal{C} \to \mathcal{C}$ are naturally isomorphic to $\mathrm{id}_\mathcal{C}$.
2. For any monoidal functor $F : \mathcal{C} \to \mathcal{D}$ between right (left) rigid monoidal categories $\mathcal{C}$ and $\mathcal{D}$, the functors $*_\mathcal{D} F : \mathcal{C} \to \mathcal{D}^{op,op}$ and $F *_\mathcal{C} : \mathcal{C} \to \mathcal{D}^{op,op}$ are naturally isomorphic.
3. Statements 1. and 2. imply that if $\mathcal{C}$ and $\mathcal{D}$ are rigid, any monoidal natural transformation $\eta : F \to F'$ is a monoidal isomorphism.

2.2 Pivotal categories and traces

Proposition 2.1.5 shows that right (left) duals in a right (left) rigid monoidal category $\mathcal{C}$ define a functor $* : \mathcal{C} \to \mathcal{C}^{op,op}$. This functor generalises the functor $* : \mathrm{Vect}_\mathbb{F}^{fd} \to \mathrm{Vect}_\mathbb{F}^{fd\; op,op}$ that assigns to a vector space V its dual V^* and to a linear map $f : V \to W$ the linear map $f^* : W^* \to V^*$, $\alpha \mapsto \alpha \circ f$. The question is what happens if we apply this functor twice.

In the category $\mathrm{Vect}_{\mathbb{F}}^{fd}$, there is a natural isomorphism $\mathrm{can} : \mathrm{id} \to **$. Its component morphisms are the canonical maps $\mathrm{can}_V : V \to V^{**}$ that assign to a vector $v \in V$ the linear map $f_v : V^* \to \mathbb{F}$ with $f_v(\alpha) = \alpha(v)$ for all $\alpha \in V^*$. Hence, up to this canonical natural isomorphism, taking double duals has no effect.

However, this is not inherent in the diagrammatic calculus. The diagrammatic definition of the dual morphisms in (20) shows that that taking multiple right or left duals of a morphism $f : X \to Y$ wraps the lines for the objects X and Y around the circle representing f. To undo this wrapping, one would require a natural isomorphism $\omega : \mathrm{id}_{\mathcal{C}} \to **$. This would allow one to interpret the functor $* : \mathcal{C} \to \mathcal{C}^{op,op}$ as a symmetry of the diagrams, namely a 180 degree rotation in the plane.

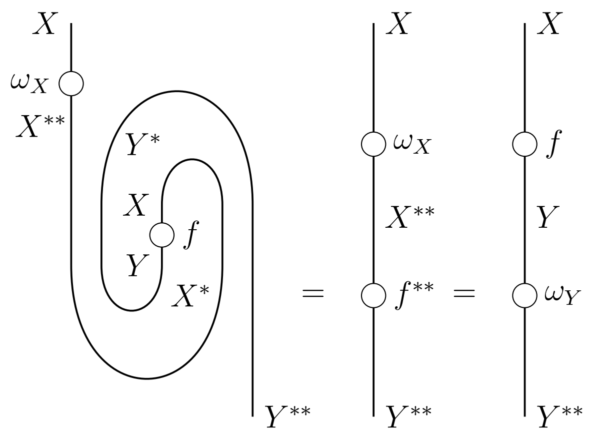
Definition 2.2.1: Let $\mathcal{C}$ be a right rigid monoidal category. A **pivotal structure** on $\mathcal{C}$ is a monoidal natural isomorphism $\omega : \mathrm{id}_{\mathcal{C}} \to **$. A **pivotal category** is a pair $(\mathcal{C}, \omega)$ of a right rigid monoidal category $\mathcal{C}$ and a pivotal structure ω.

Example 2.2.2:

1. The rigid monoidal category $\mathrm{Vect}_{\mathbb{F}}^{fd}$ is pivotal with the pivot ω given by the canonical isomorphisms $\omega_V = \mathrm{can}_V : V \to V^{**}$, $v \mapsto f_v$ with $f_v(\alpha) = \alpha(v)$ for $\alpha \in V^*$, $v \in V$.

2. For any group G, the rigid monoidal category $\mathbb{F}[G]{-}\mathrm{Mod}^{fd}$ is pivotal with the pivot ω given by the canonical isomorphisms $\omega_V = \mathrm{can}_V : V \to V^{**}$, $v \mapsto f_v$ with $f_v(\alpha) = \alpha(v)$ for $\alpha \in V^*$, $v \in V$. As the $\mathbb{F}[G]$-module structure $\rhd^*$ on V^* is defined by $g \rhd^* \alpha = \alpha \circ (g^{-1} \rhd -)$, these morphisms are $\mathbb{F}[G]$-linear.

A indicated in (22), the pivot of a pivotal category is represented diagrammatically as a morphism that unwraps the double dual of each morphism in $\mathcal{C}$. The naturality of the pivot states that morphisms ω_X can be moved through any morphism $f : X \to Y$, provided that suitable double duals are taken and the argument of the pivot is adjusted:



(22)

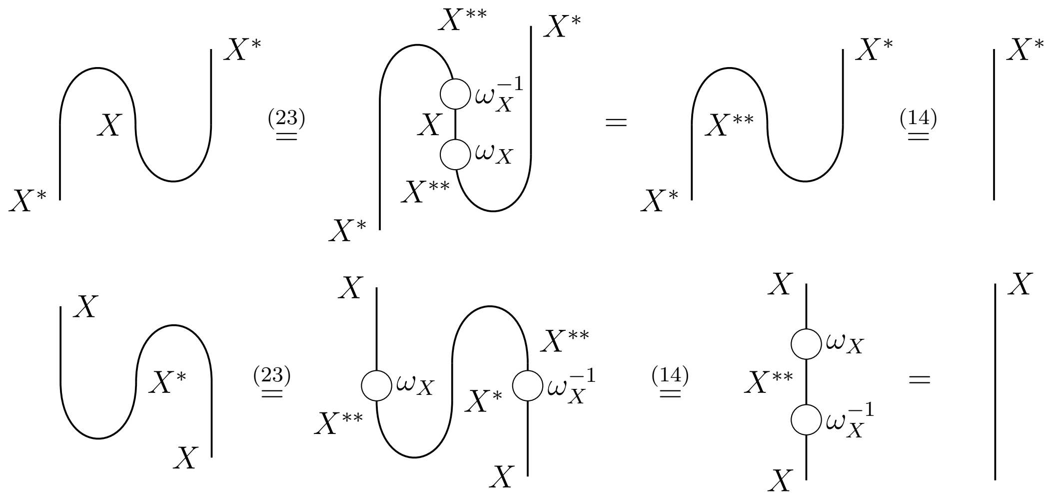
A pivot also guarantees that right dual objects are left dual objects and vice versa, just as in the pivotal category finite-dimensional vector spaces. The left evaluation and coevaluation of an object X in a right rigid monoidal category can be defined by composing the pivot with the right evaluation and coevaluation of its right dual X^*.

Proposition 2.2.3: Every pivotal category is left rigid, and right dual objects in a pivotal category are left dual objects.

Proof:
Let $\mathcal{C}$ be a pivotal category. Define ${}^*X := X^*$ for all objects X of $\mathcal{C}$ and

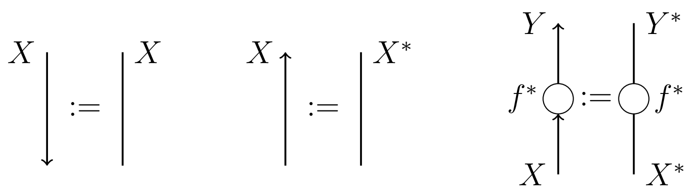
$$\mathrm{ev}^L_X : X \otimes X^* \xrightarrow{\omega \otimes 1_{X^*}} X^{**} \otimes X^* \xrightarrow{\mathrm{ev}^R_{X^*}} e \qquad \mathrm{coev}^L_X : e \xrightarrow{\mathrm{coev}^R_{X^*}} X^* \otimes X^{**} \xrightarrow{1_{X^*} \otimes \omega^{-1}} X^* \otimes X \tag{23}$$

That this left evaluation and coevaluation satisfy the snake identities follows by a direct computation from their definition and the snake identities for the right evaluation and coevaluation:



□

Proposition 2.2.3 allows one to simplify the graphical calculus for a pivotal category. As left and right dual objects in a pivotal category coincide, we can denote the object $X^* = {}^*X$ diagrammatically by an arrow labelled with X that points upward and the object X by an arrow labelled by X that points downwards. The left evaluation can be represented by the diagrams in (23) and all labels corresponding to left and right duals can be omitted.



Another benefit of a pivotal structure is that any pivotal category is equipped with the notion of a left trace and right trace that generalises the trace in the category $\mathrm{Vect}^{fd}_{\mathbb{F}}$. The trace in $\mathrm{Vect}^{fd}_{\mathbb{F}}$

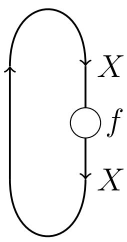
assigns to each endomorphism $f : V \to V$ a number, i. e. an endomorphism of the unit object $\mathbb{F}$. Similarly, the left and right trace in a pivotal category $(\mathcal{C}, \omega)$ assign to each endomorphism $f : X \to X$ an endomorphism of the unit object in $\mathcal{C}$, i. e. an element of the commutative monoid $\mathrm{End}_{\mathcal{C}}(e)$.

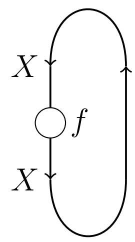
In particular, this yields a generalised notion of dimension for each object X, namely the left and right traces of the identity morphism $1_X : X \to X$. The only difference is that in a general pivotal category it is not guaranteed that left and right traces coincide. A pivotal category with this property is called *spherical*, because the diagrams for left and right traces can be deformed into each other if they are drawn on a sphere S^2.

Definition 2.2.4: Let $\mathcal{C}$ be a pivotal category, equipped with the left evaluation and coevaluation from Proposition 2.2.3, X an object in $\mathcal{C}$ and $f : X \to X$ a morphism.

1. The **left** and **right trace** of f are defined as

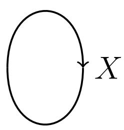
$$\mathrm{tr}_L(f) = \mathrm{ev}_X^R \circ (1_{X^*} \otimes f) \circ \mathrm{coev}_X^L \qquad \mathrm{tr}_R(f) = \mathrm{ev}_X^L \circ (f \otimes 1_{X^*}) \circ \mathrm{coev}_X^R$$

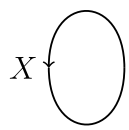
$\mathrm{tr}_L(f) =$ 

$\mathrm{tr}_R(f) =$ 

2. The **left** and **right dimension** of X are defined as

$$\dim_L(X) = \mathrm{tr}_L(1_X) = \mathrm{ev}_X^R \circ \mathrm{coev}_X^L \qquad \dim_R(X) = \mathrm{tr}_R(1_X) = \mathrm{ev}_X^L \circ \mathrm{coev}_X^R$$

$\dim_L(X) =$ 

$\dim_R(X) =$ 

3. The category $\mathcal{C}$ is called **spherical** if $\mathrm{tr}_L(f) = \mathrm{tr}_R(f)$ for all endomorphisms f in $\mathcal{C}$.

The left and right traces in a pivotal category have many properties that are familiar from the traces in $\mathrm{Vect}_{\mathbb{F}}^{fd}$ such as cyclic invariance and compatibility with duality. They are also compatible with tensor products, provided the morphisms satisfy a mild addition assumption. In particular, this implies that the left and right dimensions of objects in $\mathcal{C}$ behave in a way that is very similar to the dimensions of vector spaces.

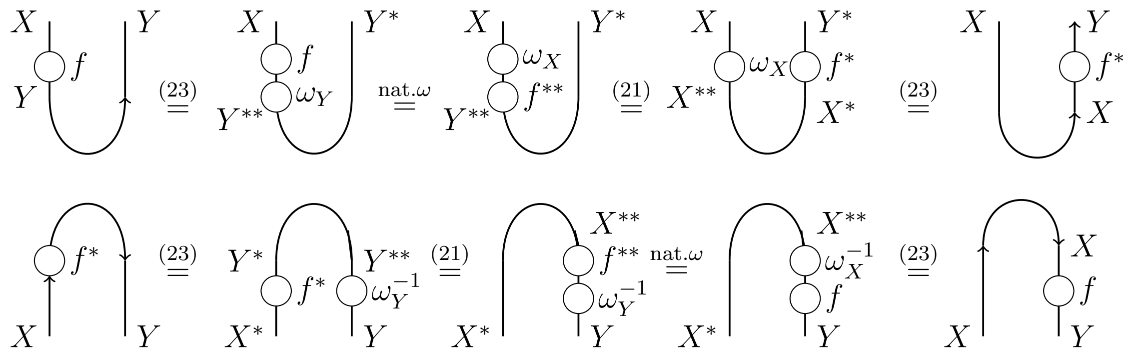
Lemma 2.2.5: Let $\mathcal{C}$ be a pivotal category. The traces in $\mathcal{C}$ have the following properties:

1. **cyclic invariance:** $\mathrm{tr}_{L,R}(g \circ f) = \mathrm{tr}_{L,R}(f \circ g)$ for all morphisms $f : X \to Y$, $g : Y \to X$.

2. **duality:** $\mathrm{tr}_{L,R}(f) = \mathrm{tr}_{R,L}(f^*)$ for all endomorphisms $f : X \to X$.

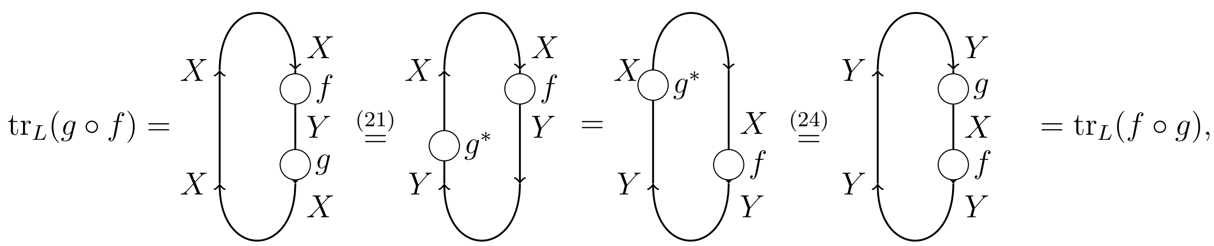
3. **compatibility with tensor products:**
If $r_Y \circ (1_Y \otimes h) \circ r_Y^{-1} = l_Y \circ (h \otimes 1_Y) \circ l_Y^{-1}$ for all endomorphisms $h : e \to e$ and an object Y, then $\mathrm{tr}_{L,R}(f \otimes g) = \mathrm{tr}_{L,R}(g) \cdot \mathrm{tr}_{L,R}(f)$ for all objects X and endomorphisms $f : X \to X$, $g : Y \to Y$.

Proof:
We prove these identities graphically for the left traces. The proofs for the right traces are analogous. The definition of the left evaluation and coevaluation and the pivot implies:

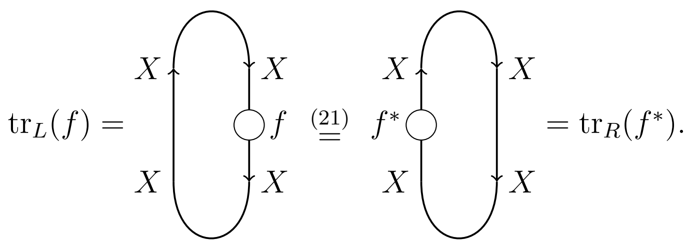


(24)

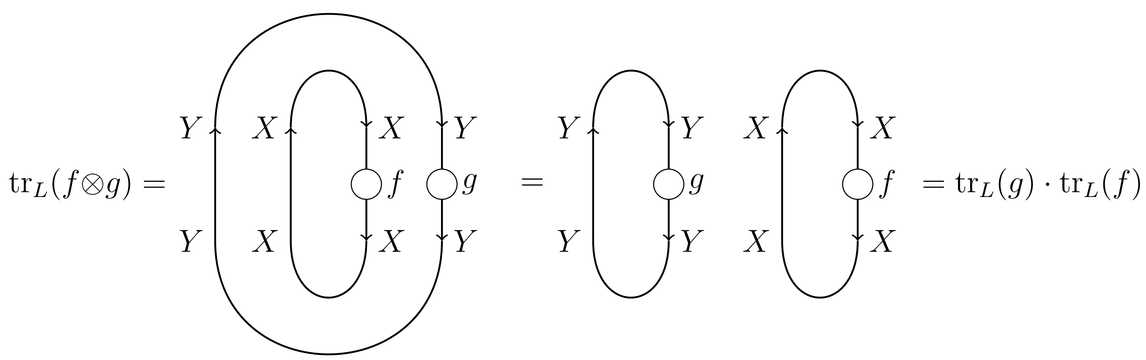
By combining these identities with the corresponding identities for the right evaluation and coevaluation in (21), we obtain



and for all endomorphisms $f : X \to X$



The condition $r_Y \circ (1_Y \otimes h) = l_Y \circ (h \otimes 1_Y)$ implies that we can move $\mathrm{tr}_L(f) : e \to e$ to the right of the line labelled with Y in the picture for $\mathrm{tr}_L(f \otimes g)$. This yields



where $\cdot$ is the multiplication in commutative monoid $\mathrm{Hom}_{\mathcal{C}}(e, e)$ from Corollary 1.2.2. $\square$

Corollary 2.2.6: Let $\mathcal{C}$ be a pivotal category. Then:

(i) $X \cong Y$ implies $\dim_{L,R}(X) = \dim_{L,R}(Y)$,

(ii) $\dim_{L,R}(X) = \dim_{R,L}(X^*)$ for all objects X,

(iii) If $r_Y \circ (1_Y \otimes h) \circ r_Y^{-1} = l_Y \circ (h \otimes 1_Y) \circ l_Y^{-1}$ for all endomorphisms $h : e \to e$ and an object Y, then $\dim_{L,R}(X \otimes Y) = \dim_{L,R}(X) \cdot \dim_{L,R}(Y)$ for all objects X, Y,

(iv) $\dim_L(e) = \dim_R(e) = 1_e$.

Proof:
(i) If $X \cong Y$, then there is an isomorphism $f : X \to Y$. With the cyclic invariance of the trace one obtains

$$\dim_L(X) = \mathrm{tr}_L(1_X) = \mathrm{tr}_L(f^{-1} \circ f) = \mathrm{tr}_L(f \circ f^{-1}) = \mathrm{tr}_L(1_Y) = \dim_L(Y).$$

(ii) Follows directly from Lemma 2.2.5, 2. by setting $f = 1_X$ and using the identity $1_{X^*} = 1_X^*$, which follows from the fact that $* : \mathcal{C} \to \mathcal{C}^{op}$ is a functor. Similarly, (iii) is obtained from Lemma 2.2.5, 3. by setting $f = 1_X$ and $g = 1_Y$.

(vi) By Lemma 1.2.1 we have $r_e = l_e$, and the naturality of the unit constraints implies that $l_e \circ (1_e \otimes h) \circ l_e^{-1} = h = r_e \circ (h \otimes 1_e) \circ r_e^{-1}$ for all enomorphisms $h : e \to e$. With the cyclic invariance of the trace from Lemma 2.2.5, 1. this yields for all endomorphisms $h : e \to e$

$$\mathrm{tr}_{L,R}(h) \cdot \dim_{L,R}(e) = \mathrm{tr}_{L,R}(h \otimes 1_e) = \mathrm{tr}_{L,R}(r_e^{-1} \circ h \circ r_e) = \mathrm{tr}_{L,R}(h).$$

and hence $\dim_{L,R}(e) = 1_e$, the multiplicative unit in the commutative monoid $\mathrm{Hom}_{\mathcal{C}}(e, e)$. $\square$

3 Braided monoidal categories

3.1 Braided monoidal categories

In this section, we consider monoidal categories $\mathcal{C}$ with additional structure, namely a natural isomorphism $c : \otimes \to \otimes^{op}$, where $\otimes^{op} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is the opposite tensor product from Example 1.1.4, 8. It assigns to a pair (U, V) of objects in $\mathcal{C}$ the object $U \otimes^{op} V = V \otimes U$ and to a pair of morphisms $f : U \to U'$, $g : V \to V'$ the morphism $f \otimes^{op} g = g \otimes f : V \otimes U \to V' \otimes U'$. The component morphisms $c_{U,V} : U \otimes V \to V \otimes U$ of this natural isomorphism generalise the flip maps $\tau_{U,V} : U \otimes V \to V \otimes U$, $u \otimes v \mapsto v \otimes u$ in the category $\mathrm{Vect}_{\mathbb{F}}$.

Just as the flip maps $\tau_{U,V}$ in $\mathrm{Vect}_{\mathbb{F}}$, these component morphisms need to satisfy a compatibility condition with the tensor product in $\mathcal{C}$, namely that flipping an object with a tensor product of two other objects is the same as flipping it first with one and then with the other. A natural isomorphism $c : \otimes \to \otimes^{op}$ that satisfies this condition is called a *braiding*. Unlike the flip map τ, a braiding does not need to be involutive or symmetric, i. e. to satisfy $c_{V,U} \circ c_{U,V} = \mathrm{id}_{U \otimes V}$.

Definition 3.1.1: Let $(\mathcal{C}, \otimes, e, a, l, r)$ be a monoidal category.

1. A **braiding** for $\mathcal{C}$ is a natural isomorphism $c : \otimes \to \otimes^{op}$ that satisfies the **hexagon axioms**: the following diagrams commute for all objects U, V, W in $\mathcal{C}$

$$\begin{tikzcd}
(U\otimes V)\otimes W \arrow[r, "c_{U,V}\otimes 1_W"] \arrow[d, "a_{U,V.W}"'] & (V\otimes U)\otimes W \arrow[r, "a_{V,U,W}"] & V\otimes(U\otimes W) \arrow[d, "1_V\otimes c_{U,W}"] \\
U\otimes(V\otimes W) \arrow[r, "c_{U,V\otimes W}"'] & (V\otimes W)\otimes U \arrow[r, "a_{V,W,U}"'] & V\otimes(W\otimes U)
\end{tikzcd}$$

$$\begin{tikzcd}
U\otimes(V\otimes W) \arrow[r, "1_U\otimes c_{V,W}"] \arrow[d, "a^{-1}_{U,V,W}"'] & U\otimes(W\otimes V) \arrow[r, "a^{-1}_{U,W,V}"] & (U\otimes W)\otimes V \arrow[d, "c_{U,W}\otimes 1_V"] \\
(U\otimes V)\otimes W \arrow[r, "c_{U\otimes V,W}"'] & W\otimes(U\otimes V) \arrow[r, "a^{-1}_{W,U,V}"] & (W\otimes U)\otimes V
\end{tikzcd}.$$

2. A braiding is called **symmetric** if $c_{W,V} = c^{-1}_{V,W}$ for all objects V, W in $\mathcal{C}$.

3. A monoidal category with a braiding is called a **braided monoidal category**. If the braiding is symmetric, it is called a **symmetric monoidal category**.

Remark 3.1.2:

1. If $(\mathcal{C}, \otimes, e)$ is a strict monoidal category, then the hexagon axioms reduce to the equations

$$c_{U\otimes V,W} = (c_{U,W}\otimes 1_V) \circ (1_U \otimes c_{V,W}) \qquad c_{U,V\otimes W} = (1_V \otimes c_{U,W}) \circ (c_{U,V} \otimes 1_W).$$

2. If $c : \otimes \to \otimes^{op}$ is a braiding for $(\mathcal{C}, \otimes, e, a, l, r)$, then $c' : \otimes \to \otimes^{op}$ with component morphisms $c'_{U,V} = c^{-1}_{V,U} : U \otimes V \to V \otimes U$ is a braiding as well (Exercise). It is called the **opposite braiding** and shows that a braiding is a choice of *structure*, not a *property*.

3. For all objects V in a braided monoidal category $(\mathcal{C}, \otimes, e, a, l, r, c)$ one has

$$c_{V,e} = l^{-1}_V \circ r_V = c^{-1}_{e,V}.$$

This is obtained from the diagram

$$\begin{tikzcd}
(V\otimes e)\otimes W \arrow[r, "a_{V,e,W}"] \arrow[dd, "c_{V,e}\otimes 1_W"'] \arrow[dr, "r_V\otimes 1_W"'] & V\otimes(e\otimes W) \arrow[r, "c_{V,e\otimes W}"] \arrow[d, "1_V\otimes l_W"] & (e\otimes W)\otimes V \arrow[d, "l_W\otimes 1_V"'] \arrow[dd, bend left, "a_{e,W,V}"] \\
 & V\otimes W \arrow[r, "c_{V,W}"] & W\otimes V \\
(e\otimes V)\otimes W \arrow[r, "a_{e,V,W}"'] \arrow[ur, "l_V\otimes 1_W"] & e\otimes(V\otimes W) \arrow[r, "1_e\otimes c_{V,W}"'] \arrow[u, "l_{V\otimes W}"'] & e\otimes(W\otimes V), \arrow[u, "l_{W\otimes V}"]
\end{tikzcd}$$

in which the upper rectangle commutes by the naturality of c, the lower rectangle by naturality of l, the triangle on the upper left by the triangle axiom, the triangles on the lower left and on the right by Lemma 1.2.1, and the outer hexagon by the first hexagon axiom. As all arrows are labelled by isomorphisms, this implies that the middle triangle on the left commutes as well and hence $(l_V \circ c_{V,e})\otimes 1_W = r_V\otimes 1_W$ for all objects V, W. Setting $W = e$ and applying the same argument as in the proof of Lemma 1.2.1 one then obtains $c_{V,e} = l_V^{-1} \circ r_V$. The proof of the second identity is analogous.

4. For all objects U, V, W in in a braided tensor category $(\mathcal{C}, \otimes, e, a, l, r, c)$ the **dodecagon diagram** commutes:

$$\begin{tikzcd}
(V\otimes U)\otimes W \arrow[d, "a_{V,U,W}"'] & (U\otimes V)\otimes W \arrow[l, "c_{U,V}\otimes 1_W"'] \arrow[r, "a_{U,V,W}"] & U\otimes(V\otimes W) \arrow[d, "1_U\otimes c_{V,W}"] \arrow[dddll, "c_{U,V\otimes W}"'] \\
V\otimes(U\otimes W) \arrow[d, "1_V\otimes c_{U,W}"'] & & U\otimes(W\otimes V) \arrow[d, "a^{-1}_{U,W,V}"] \arrow[dddll, "c_{U,W\otimes V}"] \\
V\otimes(W\otimes U) \arrow[d, "a^{-1}_{V,W,U}"'] & & (U\otimes W)\otimes V \arrow[d, "c_{U,W}\otimes 1_V"] \\
(V\otimes W)\otimes U \arrow[d, "c_{V,W}\otimes 1_U"'] & & (W\otimes U)\otimes V \arrow[d, "a_{W,U,V}"] \\
(W\otimes V)\otimes U \arrow[r, "a_{W,V,U}"'] & W\otimes(V\otimes U) & W\otimes(U\otimes V) \arrow[l, "1_W\otimes c_{U,V}"]
\end{tikzcd}$$

This follows because the two hexagons commute by the hexagon axioms and the parallelogram by naturality of the braiding.

If $(\mathcal{C}, \otimes, e, a, l, r, c)$ is strict, it reduces to the **Yang-Baxter equation**

$$(c_{V,W}\otimes 1_U) \circ (1_V\otimes c_{U,W}) \circ (c_{U,V}\otimes 1_W) = (1_W\otimes c_{U,V}) \circ (c_{U,W}\otimes 1_V) \circ (1_U\otimes c_{V,W}).$$

Example 3.1.3:

1. The category $\mathrm{Vect}_{\mathbb{F}}$ is a symmetric monoidal category with the braiding given by the flip map $c_{U,V} : U\otimes V \to V\otimes U$, $u\otimes v \mapsto v\otimes u$. More generally, for any commutative ring k, the category (k-Mod, $\otimes_k$, k) is a symmetric monoidal category with $c_{U,V} : U\otimes_k V \to V\otimes_k U$, $u\otimes v \mapsto v\otimes u$.

2. For any group G, the category $\mathbb{F}[G]-$Mod is a symmetric monoidal category with the braiding $c_{U,V} : U\otimes V \to V\otimes U$, $u\otimes v \mapsto v\otimes u$ from $\mathrm{Vect}_{\mathbb{F}}$. As $g \rhd (u\otimes v) = (g \rhd u)\otimes(g \rhd v)$, the maps $c_{U,V}$ become $\mathbb{F}[G]$-linear.

3. The categories Set and Top with, respectively, the cartesian product of sets and the product of topological spaces are symmetric monoidal categories with the braiding $c_{X,Y}: X \times Y \to Y \times X$, $(x, y) \mapsto (y, x)$.

4. More generally, any monoidal category $\mathcal{C}$ whose tensor product is given by a categorical product or coproduct in $\mathcal{C}$ is a symmetric monoidal category.

5. Let $\mathbb{F}$ be a field of characteristic $\mathrm{char}\,\mathbb{F} \neq 2$. Then the category $\mathrm{Vect}^{\mathbb{F}}_{\mathbb{Z}/2\mathbb{Z}}$ is a symmetric monoidal category with $c_{U,V}: U \otimes V \to V \otimes U$, $u \otimes v \mapsto (-1)^{\deg(u)\deg(v)} v \otimes u$, where $\deg(u) = 0$ if $u \in U_0$ and $\deg(u) = 1$ if $u \in U_1$.

6. The category $\mathrm{Ch}_{R\text{-Mod}}$ of chain complexes and chain maps from Example 1.1.4, 6. is a symmetric monoidal category.

Example 3.1.4: Let G be a group.

- A **crossed** G**-set** is a triple $(X, \rhd, \mu)$ of a set X, a left action $\rhd: X \times G \to X$ and a map $\mu: X \to G$ that satisfy $\mu(g \rhd x) = g \cdot \mu(x) \cdot g^{-1}$ for all $x \in X$ and $g \in G$.
- A **morphism of crossed** G**-sets** from $(X, \rhd_X, \mu_X)$ to $(Y, \rhd_Y, \mu_Y)$ is a map $f: X \to Y$ with $f(g \rhd_X x) = g \rhd_Y f(x)$ for all $x \in X$ and $g \in G$ and $\mu_Y \circ f = \mu_X$.
- The **tensor product** of crossed G-sets $(X, \rhd_X, \mu_X)$ and $(Y, \rhd_Y, \mu_Y)$ is the crossed G-set $(X \times Y, \rhd, \mu)$ with $g \rhd (x, y) = (g \rhd_X x, g \rhd_Y y)$ and $\mu(x, y) = \mu_X(x) \cdot \mu_Y(y)$. The tensor product of morphisms $f: X \to Y$ and $h: U \to V$ of crossed G-sets is the morphism $h \times f: U \times X \to V \times Y$.

Crossed G-sets form a monoidal category $X(G)$, where the tensor unit is the singleton $\{\bullet\}$ with the trivial group action and the map $\mu: \{\bullet\} \to G$, $\bullet \mapsto e$. It is braided, but not symmetric, with the component morphisms of the braiding given by

$$c_{X,Y}: X \times Y \to Y \times X, \quad (x, y) \mapsto (y, \mu_Y(y)^{-1} \rhd_X)$$

and the ones of the opposite braiding by

$$c^{op}_{X,Y} = c^{-1}_{Y,X}: x \times Y \to Y \times X, \quad (x, y) \mapsto (\mu_X(x) \rhd_Y y, x).$$

Example 3.1.5:

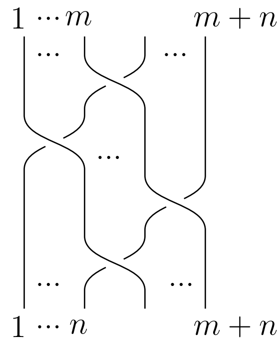
1. The braid category $\mathcal{B}$ is a strict braided monoidal category.
2. The permutation category $\mathcal{S}$ is a strict symmetric monoidal category.

Proof:
We take as the component morphisms of the braiding in $\mathcal{B}$ the morphisms

$$c_{m,n} = (\sigma_n \circ ... \circ \sigma_2 \circ \sigma_1) \circ (\sigma_{n+1} \circ ... \circ \sigma_3 \circ \sigma_2) \circ ... \circ (\sigma_{n+m-1} \circ ... \circ \sigma_{m+1} \circ \sigma_m) \qquad (25)$$

that braid the first m strands over the last n strands.

$c_{m,n} =$

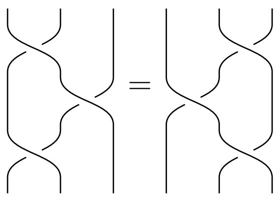


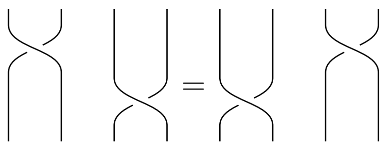
(26)

Then the hexagon axioms follow directly from the definition of the braiding. To prove naturality of the braiding, it is sufficient to show that

$$c_{m,n} \circ (\sigma_i \otimes \sigma_j) = (\sigma_j \otimes \sigma_i) \circ c_{m,n}$$

for all $i \in \{1, ..., m-1\}$ and $j \in \{1, ..., n-1\}$. This follows by repeatedly applying the relations





$\sigma_i \circ \sigma_{i+1} \circ \sigma_i = \sigma_{i+1} \circ \sigma_i \circ \sigma_{i+1}$
for all $i \in \{1, ..n+m-2\}$

$\sigma_i \circ \sigma_j = \sigma_j \circ \sigma_i$
for all $i \in \{1, ..n+m-2\}, |i-j| > 1$.

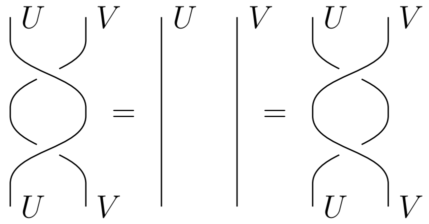
The permutation category $\mathcal{S}$ is described by the same relations as the braid category $\mathcal{B}$, plus the additional relations $\sigma_i^2 = \mathrm{id}$. It follows that it is braided with the isomorphisms $c_{m,n} : S_n \otimes S_m \to S_m \otimes S_n$ given again by (25). The additional relations $\sigma_i^2 = \mathrm{id}$ imply that overcrossings can be changed into undercrossings and hence $c_{m,n} = c_{n,m}^{-1}$ for all $n, m \in \mathbb{N}_0$. This shows that $\mathcal{S}$ is a symmetric monoidal category. □

The braiding in a braided monoidal category has a diagrammatic interpretation. As suggested by Example 3.1.5 and the name *braiding*, the component morphisms of the natural isomorphism $c : \otimes \to \otimes^{op}$ in a braided monoidal category are represented diagrammatically by undercrossings and overcrossings of the lines that represent the objects

$c_{U,V} : U \otimes V \to V \otimes U \quad c_{U,V}^{-1} : V \otimes U \to U \otimes V \quad c_{V,U} : V \otimes U \to U \otimes V \quad c_{V,U}^{-1} : U \otimes V \to V \otimes U.$

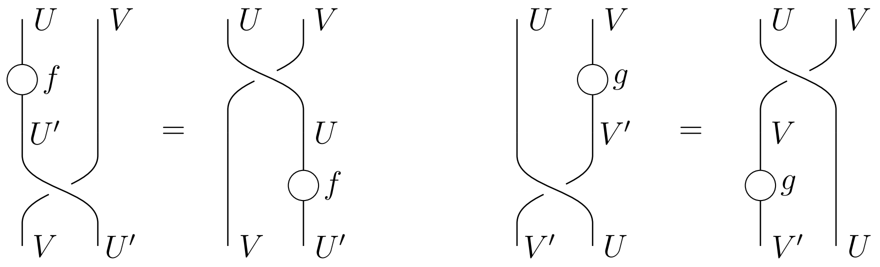
The identities in Remark 3.1.2, 3. ensure that it is still consistent to omit the tensor unit from the graphical calculus, since they imply that the braiding of the tensor unit with any other object is given by the left and right unit constraints: $c_{e,V} = l_V^{-1} \circ r_V = c_{V,e}^{-1}$ and the left and right unit constraints are not represented.

The conditions $c_{U,V}^{-1} \circ c_{U,V} = 1_{U\otimes V} = c_{V,U} \circ c_{V,U}^{-1}$ read



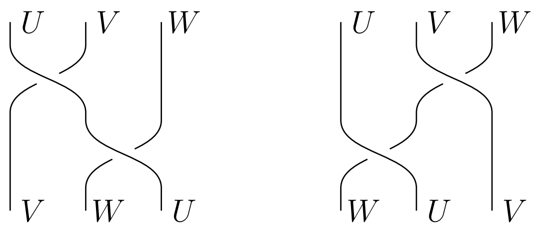
(27)

The naturality of the braiding implies that morphisms can slide above or below a crossing:



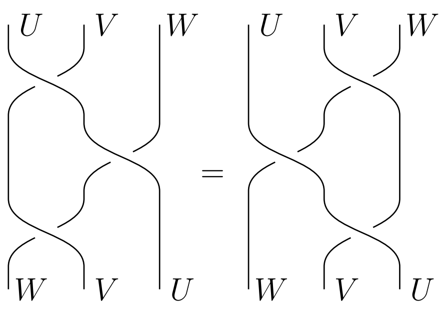
(28)

The hexagon axioms state that the two possible interpretations of the following diagrams, as (i) the composite of two braidings and (ii) a braiding of the object for one strand with the tensor product of the objects for the other two, coincide:



(29)

The dodecagon identity states that the following two diagrams represent the same morphism:



(30)

A *symmetric* braiding is represented by the analogous diagrams, but in this case one need not distinguish overcrossings and undercrossings and may represent them as simple crossings.

That the diagrams for a braided monoidal category resemble the diagrams for the braid category $\mathcal{B}$ from Definition 1.1.10 is not a coincidence. The braid category plays a special role among the braided monoidal categories that becomes apparent once one considers monoidal functors that respect the braiding. Such a functor is called a *braided monoidal functor*, and monoidal natural transformations between such functors are called *braided natural transformations*.

Definition 3.1.6: Let $\mathcal{C}$, $\mathcal{D}$ be braided monoidal categories.

1. A monoidal functor $(F, \phi^e, \phi^\otimes) : \mathcal{C} \to \mathcal{D}$ is called a a **braided monoidal functor** from $\mathcal{C}$ to $\mathcal{D}$ if for all objects V, W in $\mathcal{C}$ the following diagram commutes

$$\begin{tikzcd} F(V)\otimes F(W) \arrow[r, "c^{\mathcal{D}}_{F(V),F(W)}"] \arrow[d, "\phi^\otimes_{V,W}"'] & F(W)\otimes F(V) \arrow[d, "\phi^\otimes_{W,V}"] \\ F(V\otimes W) \arrow[r, "F(c^{\mathcal{C}}_{V,W})"] & F(W\otimes V). \end{tikzcd}$$

   If $\mathcal{C}$ and $\mathcal{D}$ are symmetric tensor categories, then a braided monoidal functor $F : \mathcal{C} \to \mathcal{D}$ is also called a **symmetric monoidal functor**.

2. A **braided natural transformation** is a monoidal natural transformation between braided monoidal functors. A **braided natural isomorphism** is a braided natural transformation that is a natural isomorphism.

Example 3.1.7:

1. The forgetful functors $V : \mathrm{Top} \to \mathrm{Set}$, $\mathrm{Vect}_{\mathbb{F}} \to \mathrm{Set}$ from Example 1.1.14 and the forgetful functors $V : k-\mathrm{Mod} \to \mathrm{Ab}$ for a commutative ring k and $V : \mathbb{F}[G]{-}\mathrm{Mod} \to \mathrm{Vect}_{\mathbb{F}}$ for a group G are symmetric monoidal functors.

2. The functor $F : \mathrm{Set} \to \mathrm{Vect}_{\mathbb{F}}$ from from Example 1.1.14 that assigns to a set X the free vector space $\langle X \rangle_{\mathbb{F}}$ generated by X and to a map $f : X \to Y$ the induced map $F(f) : \langle X \rangle_{\mathbb{F}} \to \langle Y \rangle_{\mathbb{F}}$ is a symmetric monoidal functor.

3. The family $(\Pi_n)_{n\in\mathbb{N}_0}$ of group homomorphisms $\Pi_n : B_n \to S_n$, $\sigma_i \mapsto \sigma_i$ introduced after Definition 1.1.9 define a strict braided monoidal functor $F : \mathcal{B} \to \mathcal{S}$ with $F(n) = n$ for all $n \in \mathbb{N}_0$ and $F(f) = \Pi_n(f)$ for all morphisms $f \in \mathrm{Hom}_{\mathcal{B}}(n, n) = B_n$.

4. One can show (Exercise 19) that for any braided monoidal category $(\mathcal{C}, \otimes, c)$ there is a *strict* braided monoidal category $\mathcal{C}'$ and a braided monoidal equivalence $F : \mathcal{C}' \to \mathcal{C}$. This can be viewed as a braided version of MacLanes strictification theorem.

The braid category $\mathcal{B}$ captures the essence of a braided monoidal category. All morphisms in $\mathcal{B}$ are obtained by composing braidings, their inverses and identity morphisms, via the tensor product and the composition of morphisms. This makes it especially simple to construct braided monoidal functors $F : \mathcal{B} \to \mathcal{C}$ into a braided monoidal category $\mathcal{C}$.

Proposition 3.1.8: (**universality of the braid category**)
Let $(\mathcal{C}, \otimes, c)$ be a strict braided monoidal category. Then for any object V of $\mathcal{C}$ there is a unique strict braided monoidal functor $F_V : \mathcal{B} \to \mathcal{C}$ with $F_V(1) = V$.

Proof:
1. Any strict monoidal functor $F : \mathcal{B} \to \mathcal{C}$ satisfies $F(0) = e$ for the tensor units and for $n \in \mathbb{N}$, one has $F(n) = F(1 + ... + 1) = F(1)\otimes ...\otimes F(1)$. This shows that F is determined uniquely on the objects by $F(1) =: V$.

The condition that F is a strict *braided* monoidal functor implies that the image of the morphism $c_{1,1} = \sigma_1 : 2 \to 2$ is given by $F(c_{1,1}) = c_{V,V} : V\otimes V \to V\otimes V$ and $F(c^{-1}_{1,1}) = F(c_{1,1})^{-1} = c^{-1}_{V,V}$.

As the generating morphisms in $\mathcal{B}$ are given by $\sigma_i^{\pm 1} = 1_{i-1} \otimes c_{1,1}^{\pm 1} \otimes 1_{n-i-1} : n \to n$ for all $n \in \mathbb{N}$ and $i \in \{1, ..., n-1\}$ and F is monoidal, we have $F(\sigma_i^{\pm 1}) = 1_{V^{\otimes(i-1)}} \otimes c_{V,V}^{-1} \otimes 1_{V^{\otimes(n-i-1)}}$. As any morphism in $\mathcal{B}$ is a composite of the morphisms σ_i, this determines F on the morphisms.

2. Conversely, for any object V in $\mathcal{C}$, we construct a functor $F_V : \mathcal{B} \to \mathcal{C}$ by setting $F(0) = e$, $F(n) = V^{\otimes n}$ for $n \in \mathbb{N}$, $F_V(\sigma_i^{\pm 1}) = 1_{V^{\otimes(i-1)}} \otimes c_{V,V}^{\pm 1} \otimes 1_{V^{\otimes(n-i-1)}}$. To show that this defines a functor $F_V : \mathcal{B} \to \mathcal{C}$, one needs to check that F_V respects the defining relations of the braid category: The functoriality of the tensor product in $\mathcal{C}$ implies $F_V(\sigma_i) \circ F_V(\sigma_j) = F_V(\sigma_j) \circ F_V(\sigma_i)$ for all $i, j \in \{1, ..., n-1\}$ with $|i-j| > 1$ and the dodecagon identity in $\mathcal{C}$ implies $F_V(\sigma_i) \circ F_V(\sigma_{i+1}) \circ F_V(\sigma_i) = F_V(\sigma_{i+1}) \circ F_V(\sigma_i) \circ F_V(\sigma_{i+1})$. Thus, F_V respects the relations in B_n and is a functor, which is strict monoidal by definition.

3. It remains to show that F_V is braided, that is $F_V(c_{m,n}) = c_{F_V(m), F_V(n)}$ for all $m, n \in \mathbb{N}_0$. This follows from the definition of $c_{m,n}$ in (26) and the definition of F_V together with the hexagon axiom for the braiding in $\mathcal{C}$. $\square$

If we do not suppose that the category $\mathcal{C}$ is braided, the procedure in the proof of Proposition 3.1.8 still allows us to construct functors $F : \mathcal{B} \to \mathcal{C}$ as long as there is an object V in $\mathcal{C}$ with an isomorphism $\sigma : V \otimes V \to V \otimes V$ that satisfies the dodecagon identity. We can then just consider multiple tensor products of the object V with itself and take the morphism σ as the image of the morphism $c_{1,1} : 2 \to 2$. This does not even require that $\mathcal{C}$ is strict.

Definition 3.1.9: Let $(\mathcal{C}, \otimes, e, a, l, r)$ be a monoidal category.

1. A **Yang-Baxter operator** in $\mathcal{C}$ is an object V in $\mathcal{C}$ together with an isomorphism $\sigma : V \otimes V \to V \otimes V$ such that the **dodecagon diagram** commutes

$$\begin{tikzcd}
(V\otimes V)\otimes V \arrow[d, "a_{V,V,V}"'] & (V\otimes V)\otimes V \arrow[l, "\sigma\otimes 1_V"'] \arrow[r, "a_{V,V,V}"] & V\otimes(V\otimes V) \arrow[d, "1_V\otimes\sigma"] \\
V\otimes(V\otimes V) \arrow[d, "1_V\otimes\sigma"'] & & V\otimes(V\otimes V) \arrow[d, "a^{-1}_{V,V,V}"] \\
V\otimes(V\otimes V) \arrow[d, "a^{-1}_{V,V,V}"'] & & (V\otimes V)\otimes V \arrow[d, "\sigma\otimes 1_V"] \\
(V\otimes V)\otimes V \arrow[d, "\sigma\otimes 1_V"'] & & (V\otimes V)\otimes V \arrow[d, "a_{V,V,V}"] \\
(V\otimes V)\otimes V \arrow[r, "a_{V,V,V}"'] & V\otimes(V\otimes V) & V\otimes(V\otimes V) \arrow[l, "1_V\otimes\sigma"]
\end{tikzcd}$$

2. A **morphism of Yang-Baxter operators** in $\mathcal{C}$ from (V, σ) to (W, τ) is a morphism $f : V \to W$ with $\tau \circ (f \otimes f) = (f \otimes f) \circ \sigma$.

Yang-Baxter operators in $\mathcal{C}$ and morphisms of Yang-Baxter operators in $\mathcal{C}$ form a category $\mathcal{YB}(\mathcal{C})$. A Yang-Baxter operator in $\mathrm{Vect}_{\mathbb{F}}$ is also called a **braided vector space**.

Example 3.1.10:

1. If $\mathcal{C}$ is a braided monoidal category with braidings $c_{U,V} : U \otimes V \to V \otimes U$, then $(V, c_{V,V})$ is a Yang-Baxter operator for any object V of $\mathcal{C}$.

2. If $F : \mathcal{C} \to \mathcal{D}$ is a monoidal functor, then for any Yang-Baxter operator (V, σ) in $\mathcal{C}$, the pair $(F(V), \sigma')$ with $\sigma' = \phi^{\otimes\,-1}_{V,V} \circ F(\sigma) \circ \phi^{\otimes}_{V,V}$ is a Yang-Baxter operator in $\mathcal{D}$.

3. Let $q, \lambda \in \mathbb{F}^\times$ with $q \neq 1$ and V a vector space over $\mathbb{F}$ with an ordered basis $(v_1, ..., v_n)$. Then V becomes a braided vector space with

$$\sigma : V \otimes V \to V \otimes V, \quad v_i \otimes v_j \mapsto \begin{cases} \lambda\, v_j \otimes v_i & i < j \\ \lambda q\, v_i \otimes v_i & i = j \\ \lambda\, v_j \otimes v_i + \lambda(q - q^{-1}) v_i \otimes v_j & i > j. \end{cases}$$

Corollary 3.1.11: Let (V, σ) be a Yang-Baxter operator in $\mathcal{C}$. Then there is a monoidal functor $F : \mathcal{B} \to \mathcal{C}$, unique up to coherence data in $\mathcal{C}$, with $F(1) = V$ and $F(c_{1,1}) = \sigma$.

Proof:
The proof is analogous to the one of Proposition 3.1.8. We set $F(0) = e$ and take for $F(n)$ an n-fold tensor product of V with itself with a chosen bracketing. This defines F on the objects. On the morphisms, we define $F(c_{1,1}) = \sigma$, and for the generators $\sigma_i \in B_n$ we set $F(\sigma_i^{\pm 1}) = 1_{V^{\otimes(i-1)}} \otimes \sigma^{\pm 1} \otimes 1_{V^{\otimes(n-i-1)}}$, up to bracketings and associators that are determined uniquely by the bracketing on the objects.

The functoriality of the tensor product in $\mathcal{C}$ then implies $F(\sigma_i) \circ F(\sigma_j) = F(\sigma_j) \circ F(\sigma_i)$ for all $i, j \in \{1, ..., n-1\}$ with $|i - j| > 1$ and the dodecagon identity for σ implies $F(\sigma_i) \circ F(\sigma_{i+1}) \circ F(\sigma_i) = F(\sigma_{i+1}) \circ F(\sigma_i) \circ F(\sigma_{i+1})$. We thus obtain a functor $F : \mathcal{B} \to \mathcal{C}$. The functor F is monoidal with $\phi^e = 1_e$ and the natural isomorphism $\phi^\otimes$ given by the associators in $\mathcal{C}$. It is unique up to the choice of the bracketings of multiple tensor products of V. □

Remark 3.1.12: One can show (Exercise 20) that the category $\mathcal{YB}$ of Yang-Baxter operators and morphisms of Yang-Baxter operators in a strict monoidal category $\mathcal{C}$ is equivalent to the category $\mathrm{Fun}_\otimes(\mathcal{B}, \mathcal{C})$ of strict monoidal functors $F : \mathcal{B} \to \mathcal{C}$ and monoidal natural transformations between them.

Corollary 3.1.13: Let (V, σ) be a braided vector space. Then the maps

$$\rho_n : B_n \to \mathrm{Aut}_{\mathbb{F}}(V^{\otimes n}), \quad \sigma_i \mapsto \mathrm{id}_{V^{\otimes(i-1)}} \otimes \sigma \otimes \mathrm{id}_{V^{\otimes(n-i-1)}}$$

define a family of representations of the braid groups B_n on $V^{\otimes n}$.

Proof:
We define ρ on the generators of B_n by setting $\rho(\sigma_i^{\pm 1}) = \mathrm{id}_{V^{\otimes(i-1)}} \otimes \sigma^{\pm} \otimes \mathrm{id}_{V^{\otimes(n-i-1)}}$ for all $i \in \{1, ..., n-1\}$. As (V, σ) is a Yang-Baxter operator in $\mathrm{Vect}_{\mathbb{F}}$, the functoriality of the tensor product and the dodecagon identity allow one to extend ρ to a group homomorphism $\rho : B_n \to \mathrm{Aut}_{\mathbb{F}}(V^{\otimes n})$. □

3.2 The centre construction

The examples of braided monoidal categories in the previous sections might give the impression that braided monoidal categories are rather special and that non-trivial examples are hard to find. This is in fact not the case. In this section, we show that there is a canonical construction that assigns to any monoidal category $\mathcal{C}$ a braided monoidal category $\mathcal{Z}(\mathcal{C})$.

It is called the *centre construction*, because it can be viewed as a categorical version of the centre of a ring. The centre of a ring R is the subring $Z(R) = \{r \in R \mid r \cdot s = s \cdot r \,\forall s \in R\}$. If we replace the multiplication of the ring by the tensor product of a monoidal category and translate the definition naively, we would consider the full subcategory whose objects V satisfy $X \otimes V = V \otimes X$ for all objects X in $\mathcal{C}$. However, this is not very useful, since it is too strict. There are very few such objects in typical monoidal categories such as $\mathrm{Vect}_{\mathbb{F}}$. This makes it more natural to impose $X \otimes V \cong V \otimes X$ instead.

One should also view the isomorphisms $X \otimes V \cong V \otimes X$ for different objects X as *structures* associated with the object V and not as a property of V. This forces us to consider the interaction of these isomorphisms with tensor products and with morphisms in $\mathcal{C}$ and to impose consistency conditions.

Theorem 3.2.1: Let $(\mathcal{C}, \otimes, e)$ be a monoidal category. Then there is a braided monoidal category $\mathcal{Z}(\mathcal{C})$, the **centre** of $\mathcal{C}$, defined as follows:

- The objects of $\mathcal{Z}(\mathcal{C})$ are pairs $(V, c_{-,V})$ of an object V in $\mathcal{C}$ and a family $c_{-,V}$ of isomorphisms $c_{X,V} : X \otimes V \to V \otimes X$ defined for all objects X in $\mathcal{C}$ such that:

  (i) $c_{X,V}$ is natural in X: For all morphisms $g : X \to Y$ the following diagram commutes

$$\begin{tikzcd} X\otimes V \arrow[r, "c_{X,V}"] \arrow[d, "g\otimes 1_V"'] & V\otimes X \arrow[d, "1_V\otimes g"] \\ Y\otimes V \arrow[r, "c_{Y,V}"'] & V\otimes Y. \end{tikzcd} \tag{31}$$

  (ii) for all $X, Y, V \in \mathrm{Ob}\,\mathcal{C}$ the following diagram commutes

$$\begin{tikzcd} & (X\otimes Y)\otimes V \arrow[r, "c_{X\otimes Y,V}"] \arrow[dl, "a_{X,Y,V}"'] & V\otimes(X\otimes Y) & \\ X\otimes(Y\otimes V) \arrow[dr, "1_X\otimes c_{Y,V}"'] & & & (V\otimes X)\otimes Y \arrow[ul, "a_{V,X,Y}"'] \\ & X\otimes(V\otimes Y) \arrow[r, "a^{-1}_{X,V,Y}"'] & (X\otimes V)\otimes Y \arrow[ur, "c_{X,V}\otimes 1_Y"'] & \end{tikzcd} \tag{32}$$

- A morphism $f : (V, c_{-,V}) \to (W, c_{-,W})$ is a morphism $f : V \to W$ in $\mathcal{C}$ such that for all objects X in $\mathcal{C}$ the following diagram commutes

$$\begin{tikzcd} X\otimes V \arrow[r, "c_{X,V}"] \arrow[d, "1_X\otimes f"'] & V\otimes X \arrow[d, "f\otimes 1_X"] \\ X\otimes W \arrow[r, "c_{X,W}"'] & W\otimes X. \end{tikzcd} \tag{33}$$

The identity morphisms are the identity morphisms of $\mathcal{C}$, and the composition of morphisms is the composition of morphisms in $\mathcal{C}$.

- The tensor product in $\mathcal{Z}(\mathcal{C})$ is given by $(V, c_{-,V})\otimes(W, c_{-,W}) = (V\otimes W, c_{-,V\otimes W})$ with $c_{X,V\otimes W}$ defined by

$$\begin{tikzcd}
& X\otimes(V\otimes W) \arrow[rr, "c_{X,V\otimes W}"] \arrow[dl, "a^{-1}_{X,V,W}"'] & & (V\otimes W)\otimes X & \\
(X\otimes V)\otimes W \arrow[dr, "c_{X,V}\otimes 1_Y"'] & & & & V\otimes(W\otimes X) \arrow[ul, "a^{-1}_{V,W,X}"'] \\
& (V\otimes X)\otimes W \arrow[rr, "a_{V,X,W}"'] & & V\otimes(X\otimes W) \arrow[ur, "1_V\otimes c_{X,W}"'] &
\end{tikzcd} \tag{34}$$

- The braiding of $\mathcal{Z}(\mathcal{C})$ is given by $c_{V,W} : (V, c_{-,V})\otimes(W, c_{-,W}) \to (W, c_{-,W})\otimes(V, c_{-,V})$.

Proof:
1. We show that $\mathcal{Z}(\mathcal{C})$ is a category. For this, it is sufficient to note that diagram (33) commutes for $f = 1_V$ and that for morphisms $f : (U, c_{-,U}) \to (V, c_{-,V})$ and $g : (V, c_{-,V}) \to (W, c_{-,W})$ in $\mathcal{Z}(\mathcal{C})$, the morphism $g \circ f : U \to W$ again makes the diagram (33) commute for all objects X in $\mathcal{C}$. This follows directly by stacking the diagrams (33) for f and g vertically.

2. We show that $\mathcal{Z}(\mathcal{C})$ is monoidal. To show that the tensor product is well-defined on the objects, we prove that the morphisms $c_{X,V\otimes W}$ from (34) are natural in the first argument and make the diagram (32) commute. The naturality follows from the naturality of the associator and of the morphisms $c_{X,V}$ and $c_{X,W}$. We show that diagram (32) commutes for $c_{-,V\otimes W}$ for the case where $\mathcal{C}$ is strict. In this case we have the diagram

$$\begin{tikzcd}
X\otimes Y\otimes V\otimes W \arrow[rrrr, "c_{X\otimes Y,V\otimes W}"] \arrow[d, "1\otimes c_{Y,V}\otimes 1"'] \arrow[drr, "c_{X\otimes Y,V}\otimes 1"] & & & & V\otimes W\otimes X\otimes Y \\
X\otimes V\otimes Y\otimes W \arrow[rr, "c_{X,V}\otimes 1\otimes 1"] \arrow[drr, "1\otimes 1\otimes c_{Y,W}"'] & & V\otimes X\otimes Y\otimes W \arrow[rr, "1\otimes 1\otimes c_{Y,W}"] \arrow[urr, "1\otimes c_{X\otimes Y,W}"] & & V\otimes X\otimes W\otimes Y \arrow[u, "1\otimes c_{X,W}\otimes 1"'] \\
& & X\otimes V\otimes W\otimes Y, \arrow[urr, "c_{X,V}\otimes 1\otimes 1"'] & &
\end{tikzcd}$$

in which the top triangle commutes by (34), the triangles on the left and right by (32) and the quadrilateral at the bottom by the functoriality of the tensor product. As the paths from the left and right top entry to the middle entry at the bottom are the morphisms $1_X\otimes c_{Y,V\otimes W}$ and $c_{X,V\otimes W}\otimes 1_Y$ by (34), this shows that diagram (32) commutes for $c_{-,V\otimes W}$ if $\mathcal{C}$ is strict. The general case is obtained by inserting associators and subdividing the diagram into pentagon axioms, diagrams with two associators and one braiding, which commute by the naturality of the associators, and diagrams (31) and (34).

To show that the tensor product is well-defined on morphisms, we need to show that for all morphisms $f : V \to V'$ and $g : W \to W'$ for which diagram (33) commutes, diagram (33) also commutes for $f\otimes g : V\otimes W \to V'\otimes W'$ with $c_{-,V\otimes W}$ and $c_{-,V'\otimes W'}$ given by (34). This follows from (34), the naturality of the associator and the commuting diagrams (33) for f, g. This shows that the tensor product of $\mathcal{Z}(\mathcal{C})$ is well-defined and defines a functor $\otimes : \mathcal{Z}(\mathcal{C}) \times \mathcal{Z}(\mathcal{C}) \to \mathcal{Z}(\mathcal{C})$.

The tensor unit is the pair $(e, c_{-,e})$ with $c_{X,e} = l_X^{-1} \circ r_X : X\otimes e \to e\otimes X$, the unit constraints are the unit constraints of $\mathcal{C}$, and the associator is the associator of $\mathcal{C}$. To show that this gives

$\mathcal{Z}(\mathcal{C})$ the structure of a monoidal category, we need to show that these morphisms satisfy (33) and hence are morphisms in $\mathcal{Z}(\mathcal{C})$. The pentagon and the triangle axiom then follow from the pentagon and triangle axiom in $\mathcal{C}$.

That the left unit constraints of $\mathcal{C}$ satisfy (33) follows from the commuting diagram

$$\begin{tikzcd}
(e\otimes X)\otimes V \arrow[r, "a_{e,X,V}"] \arrow[dr, "l_X\otimes 1"'] & e\otimes(X\otimes V) \arrow[r, "1\otimes c_{X,V}"] \arrow[d, "l_{X\otimes V}"] & e\otimes(V\otimes X) \arrow[r, "a^{-1}_{e,V,X}"] \arrow[d, "l_{V\otimes X}"] & (e\otimes V)\otimes X \arrow[dl, "l_V\otimes 1"] \\
(X\otimes e)\otimes V \arrow[u, "c_{X,e}"] \arrow[r, "r_X\otimes 1"] & X\otimes V \arrow[r, "c_{X,V}"] & V\otimes X & \\
X\otimes(e\otimes V) \arrow[u, "a^{-1}_{X,e,V}"] \arrow[ur, "1\otimes l_V"'] & & &
\end{tikzcd}$$

in which the path from the lower left to the upper right along the left side and top of the diagram represents $c_{X,e\otimes V}$, the rectangle in the middle commutes by naturality of l, the two triangles adjacent to it by Lemma 1.2.1, the triangle on the lower left by the triangle axiom for $\mathcal{C}$ and the triangle above it by definition of $c_{-,e}$. This shows that the morphism $l_V : e\otimes V \to V$ is a morphism in $\mathcal{Z}(\mathcal{C})$, and the proof for the morphism $r_V : V\otimes e \to V$ is analogous.

The proof that the associators $a_{U,V,W} : (U\otimes V)\otimes W \to U\otimes(V\otimes W)$ are morphisms in $\mathcal{Z}(\mathcal{C})$ proceeds by expressing the morphisms $c_{X,(U\otimes V)\otimes W}$ and $c_{X,U\otimes(V\otimes W)}$ as a composite of associators and the morphisms $c_{X,U}$, $c_{X,V}$, $c_{X,W}$ via (34) and inserting this into diagram (33) for $f = a_{U,V,W}$. By introducing additional arrows labelled with associators, one can subdivide this diagram into pentagons, which commute by the pentagon axiom for $\mathcal{C}$, and squares that involve two associators and two morphisms $c_{X,Y}$ for $Y = U, V, W$, which commute by naturality of the associators (Exercise). This proves that $\mathcal{Z}(\mathcal{C})$ is a monoidal category.

3. We show that $\mathcal{Z}(\mathcal{C})$ is braided. For this, it is sufficient to prove that the morphisms $c_{V,W} : (V, c_{-,V})\otimes(W, c_{-,W}) \to (W, c_{-,W})\otimes(V, c_{-,V})$ are morphisms in $\mathcal{Z}(\mathcal{C})$. The second hexagon axiom in Definition 3.1.1 then follows directly from condition (32) and the first hexagon axiom from the definition of $c_{X,V\otimes W}$ in (34).

To see that $c_{U,V}$ is a morphism in $\mathcal{Z}(\mathcal{C})$, we consider the commuting diagram

$$\begin{tikzcd}
(X\otimes U)\otimes V \arrow[r, "c_{X,U}\otimes 1_V"] \arrow[dddrr, "c_{X\otimes U,V}"] & (U\otimes X)\otimes V \arrow[r, "a_{U,X,V}"] \arrow[dddrr, "c_{U\otimes X,V}"] & U\otimes(X\otimes V) \arrow[r, "1_U\otimes c_{X,V}"] & U\otimes(V\otimes X) \arrow[d, "a^{-1}_{U,V,X}"] \\
X\otimes(U\otimes V) \arrow[u, "a^{-1}_{X,U,V}"] \arrow[d, "1_X\otimes c_{U,V}"'] & & & (U\otimes V)\otimes X \arrow[d, "c_{U,V}\otimes 1_X"] \\
X\otimes(V\otimes U) \arrow[d, "a^{-1}_{X,V,U}"'] & & & (V\otimes U)\otimes X \\
(X\otimes V)\otimes U \arrow[r, "c_{X,V}\otimes 1_U"'] & (V\otimes X)\otimes U \arrow[r, "a_{V,X,U}"'] & V\otimes(X\otimes U) \arrow[r, "1_V\otimes c_{X,U}"'] & V\otimes(U\otimes X), \arrow[u, "a^{-1}_{V,U,X}"']
\end{tikzcd}$$

where the hexagons on the left and right commute by (32) and the parallelogram in the middle by naturality of $c_{-,V}$ in the first argument.

By subdividing this commuting diagram as

$$\begin{tikzcd}
(X\otimes U)\otimes V \arrow[r, "c_{X,U}\otimes 1_V"] & (U\otimes X)\otimes V \arrow[r, "a_{U,X,V}"] & U\otimes(X\otimes V) \arrow[r, "1_U\otimes c_{X,V}"] & U\otimes(V\otimes X) \arrow[d, "a^{-1}_{U,V,X}"] \\
X\otimes(U\otimes V) \arrow[u, "a^{-1}_{X,U,V}"] \arrow[rrr, "c_{X,U\otimes V}"] \arrow[d, "1_X\otimes c_{U,V}"'] & & & (U\otimes V)\otimes X \arrow[d, "c_{U,V}\otimes 1_X"] \\
X\otimes(V\otimes U) \arrow[rrr, "c_{X,V\otimes U}"] \arrow[d, "a^{-1}_{X,V,U}"'] & & & (V\otimes U)\otimes X \\
(X\otimes V)\otimes U \arrow[r, "c_{X,V}\otimes 1_U"'] & (V\otimes X)\otimes U \arrow[r, "a_{V,X,U}"'] & V\otimes(X\otimes U) \arrow[r, "1_V\otimes c_{X,U}"'] & V\otimes(U\otimes X) \arrow[u, "a^{-1}_{V,U,X}"']
\end{tikzcd}$$

in which the middle rectangle is (33) for $c_{U,V}$ and in which the top and bottom rectangle commute by definition of $c_{X,U\otimes V}$ and $c_{X,V\otimes U}$ in (34), one finds that diagram (33) for $c_{U,V}$ commutes and $c_{U,V}$ is a morphism in $\mathcal{Z}(\mathcal{C})$. □

Theorem 3.2.1 associates to each monoidal category a braided monoidal category. It is the categorical counterpart to passing from a ring R to its centre $Z(R)$. However, the relation between a monoidal category $\mathcal{C}$ and its centre $\mathcal{Z}(\mathcal{C})$ is more complicated. For instance, any commutative ring coincides with its centre, but this is not true for a braided monoidal category $\mathcal{C}$ and its centre $\mathcal{Z}(\mathcal{C})$. It is obvious that in any braided monoidal category $\mathcal{C}$, one can construct objects in $\mathcal{Z}(\mathcal{C})$ as pairs $(V, c_{-,V})$, where $c_{-,V}$ is given by the braiding in $\mathcal{C}$. However, there is also the opposite braiding from Remark 3.1.2, 2. which defines additional objects in $\mathcal{Z}(\mathcal{C})$ and there may be even more objects obtained by modifying the braidings.

This raises the question if the centre of a monoidal category can be characterised more abstractly via a characteristic property or a universality condition, and if a braided monoidal category can at least be embedded into its centre. One also wonders under which conditions a functor $F:\mathcal{C}\to\mathcal{D}$ from a braided monoidal category $\mathcal{C}$ can be lifted to a functor $F:\mathcal{C}\to\mathcal{Z}(\mathcal{D})$. For this, we consider the forgetful functor $\Pi:\mathcal{Z}(\mathcal{C})\to\mathcal{C}$ with $\Pi(V,c_{-,V})=V$ and $\Pi(f)=f$ for all objects V and morphisms $f:V\to W$ in $\mathcal{Z}(\mathcal{C})$. It forgets the additional data that defines the braiding of $\mathcal{Z}(\mathcal{C})$ and is monoidal by definition of $\mathcal{Z}(\mathcal{C})$.

Proposition 3.2.2: Let $\mathcal{C}$ be a braided monoidal category and $F:\mathcal{C}\to\mathcal{D}$ a monoidal functor that is essentially surjective and full. Then there is a unique braided monoidal functor $\mathcal{Z}(F):\mathcal{C}\to\mathcal{Z}(\mathcal{D})$ with $\Pi\mathcal{Z}(F)=F$.

Proof:
1. Uniqueness of $\mathcal{Z}(F)$:
If $G:\mathcal{C}\to\mathcal{Z}(\mathcal{D})$ is a monoidal functor with $\Pi G=F$, then $G(V)=(F(V),c_{-,F(V)})$ for all objects V in $\mathcal{C}$ and $G(f)=F(f)$ for all morphisms f in $\mathcal{C}$. If additionally G is braided, then $c_{F(X),F(V)}=\phi^{\otimes -1}_{V,X}\circ F(c_{X,V})\circ\phi^{\otimes}_{X,V}$ for all objects X and V in $\mathcal{C}$, where $c_{X,V}$ denotes the braiding in $\mathcal{C}$. Naturality of $c_{-,F(V)}$ in the first argument then implies that the following diagram commutes for all objects V in $\mathcal{C}$ and morphisms $g:D\to F(X)$ in $\mathcal{D}$

$$\begin{tikzcd}
D\otimes F(V) \arrow[r, "g\otimes 1"] \arrow[d, "c_{D,F(V)}"'] & F(X)\otimes F(V) \arrow[r, "\phi^{\otimes}_{X,V}"] \arrow[d, "c_{F(X),F(V)}"] & F(X\otimes V) \arrow[d, "F(c_{X,V})"] \\
F(V)\otimes D \arrow[r, "1\otimes g"'] & F(V)\otimes F(X) \arrow[r, "\phi^{\otimes -1}_{V,X}"'] & F(V\otimes X).
\end{tikzcd} \tag{35}$$

As F is essentially surjective, for every object D in $\mathcal{D}$, there is an object X in $\mathcal{C}$ and an isomorphism $g : D \to F(X)$, and hence $c_{D,F(V)}$ is defined uniquely by (35).

2. Existence of $\mathcal{Z}(F)$:
Using the essential surjectivity of F, we choose an isomorphism $g : D \to F(X)$ for each object D in $\mathcal{D}$ and define $\mathcal{Z}(F)$ by $\mathcal{Z}(F)(f) = F(f)$ on the morphisms and by $\mathcal{Z}(F)(V) = (F(V), c_{-,F(V)})$ on the objects, where $c_{D,F(V)}$ is given by (35).

2.(a) To show that $(F(V), c_{-,F(V)})$ is an object in $\mathcal{Z}(\mathcal{D})$, we check that (i) the morphisms $c_{D,F(V)}$ are natural in the first argument and (ii) they make the diagram (32) commute.

(i) The naturality in the first argument follows, because for every morphism $f : D \to D'$ in $\mathcal{D}$ and isomorphisms $g_D : D \to F(X)$ and $g_{D'} : D' \to F(X')$, there is a morphism $f' : X \to X'$ with $F(f') = g_{D'} \circ f \circ g_D^{-1}$ by fullness of F. Naturality of $\phi^\otimes$ and of c yields

$$\begin{tikzcd}
D\otimes F(V) \arrow[r, "g_D\otimes 1"'] \arrow[d, "f\otimes 1"] \arrow[rrrrr, bend left, "c_{D,F(V)}"] & F(X)\otimes F(V) \arrow[r, "\phi^\otimes_{X,V}"'] \arrow[d, "F(f')\otimes 1"] & F(X\otimes V) \arrow[r, "F(c_{X,V})"'] \arrow[d, "F(f'\otimes 1)"] & F(V\otimes X) \arrow[r, "\phi^{\otimes -1}_{V,X}"'] \arrow[d, "F(1\otimes f')"] & F(V)\otimes F(X) \arrow[r, "1\otimes g_D^{-1}"'] \arrow[d, "1\otimes F(f')"] & F(V)\otimes D \arrow[d, "1\otimes f"] \\
D'\otimes F(V) \arrow[r, "g_{D'}\otimes 1"] \arrow[rrrrr, bend right, "c_{D',F(V)}"'] & F(X')\otimes F(V) \arrow[r, "\phi^\otimes_{X',V}"] & F(X'\otimes V) \arrow[r, "F(c_{X',V})"] & F(V\otimes X') \arrow[r, "\phi^{\otimes -1}_{V,X'}"] & F(V)\otimes F(X') \arrow[r, "1\otimes g_{D'}^{-1}"] & F(V)\otimes D'.
\end{tikzcd}$$

(ii) To show that the diagram (32) commutes for the morphisms $c_{-,F(V)}$, we first consider the case, where the first argument is the tensor product of two objects in the image of F. We subdivide diagram (32) into a diagram that is the image for the second hexagon axiom of $\mathcal{C}$ under F, the defining diagrams for $c_{-,F(V)}$ and several diagrams that encode the naturality of the braiding in $\mathcal{C}$ and the monoidal structure of F. With the shorthand notation $X' := F(X)$ for all objects X in $\mathcal{C}$, we then obtain the commuting diagram

$$\begin{tikzcd}
X'\otimes(Y'\otimes V') \arrow[rr, "a_{X',Y',V'}"] \arrow[dddddd, "1\otimes c_{Y',V'}"] \arrow[dddr, "1\otimes\phi^\otimes_{Y,V}"] & & (X'\otimes Y')\otimes V' \arrow[r, "c_{X'\otimes Y',V'}"] \arrow[d, "\phi^\otimes_{X,Y}\otimes 1"'] & V'\otimes(X'\otimes Y') \arrow[d, "1\otimes\phi^\otimes_{X,Y}"] & & (V'\otimes X')\otimes Y' \arrow[ll, "a_{V',X',Y'}"'] \arrow[dddl, "\phi^\otimes_{V,X}\otimes 1"'] \\
& & (X\otimes Y)'\otimes V' \arrow[r, "c_{(X\otimes Y)',V'}"] \arrow[d, "\phi^\otimes_{X\otimes Y,V}"'] & V'\otimes(X\otimes Y)' \arrow[d, "\phi^\otimes_{V,X\otimes Y}"] & & \\
& & ((X\otimes Y)\otimes V)' \arrow[r, "F(c_{X\otimes Y,V})"] \arrow[d, "F(a^{-1}_{X,Y,V})"'] & (V\otimes(X\otimes Y))' & & \\
& X'\otimes(Y\otimes V)' \arrow[r, "\phi^\otimes_{X,Y\otimes V}"'] \arrow[d, "1\otimes F(c_{Y,V})"'] & (X\otimes(Y\otimes V))' \arrow[d, "F(1\otimes c_{Y,V})"'] & ((V\otimes X)\otimes Y)' \arrow[u, "F(a_{V,X,Y})"'] & (V\otimes X)'\otimes Y' \arrow[l, "\phi^\otimes_{V\otimes X,Y}"'] & \\
& X'\otimes(V\otimes Y)' \arrow[r, "\phi^\otimes_{X,V\otimes Y}"'] & (X\otimes(V\otimes Y))' \arrow[r, "F(a^{-1}_{X,V,Y})"'] & ((X\otimes V)\otimes Y)' \arrow[u, "F(c_{X,V}\otimes 1)"] \arrow[r, "\phi^{\otimes -1}_{X\otimes V,Y}"'] & (X\otimes V)'\otimes Y' \arrow[u, "F(c_{X,V})\otimes 1"'] \arrow[dr, "\phi^{\otimes -1}_{X,V}\otimes 1"] & \\
& & & & & \\
X'\otimes(V'\otimes Y') \arrow[uur, "1\otimes\phi^\otimes_{V,Y}"] \arrow[rrrrr, "a^{-1}_{X',V',Y'}"'] & & & & & (X'\otimes V')\otimes Y'. \arrow[uuuuuu, "c_{X',V'}\otimes 1"']
\end{tikzcd}$$

in which the upper middle rectangle commutes by naturality of $c_{-,F(V)}$ in the first argument, the rectangle below it by definition of $c_{-,F(V)}$, and the small middle hexagon below it by the second hexagon axiom for the braiding in $\mathcal{C}$. The hexagons at the upper left, the upper right and at the bottom of the diagram commute because F is monoidal, the quadrilaterals at the left and right by definition of $c_{-,F(V)}$ and the two lower middle rectangles by naturality of $\phi^{\otimes}$.

This shows that (32) commutes if the objects X, Y in (32) are in the image of F. To prove this for all objects in $\mathcal{D}$, choose isomorphisms $g_D : D \to F(X)$ and $g_E : E \to F(Y)$. The diagram

$$\begin{tikzcd}
(D\otimes E)\otimes F(V) \arrow[rrr, "c_{D\otimes E,F(V)}"] \arrow[dr, "(g_D\otimes g_E)\otimes 1"] \arrow[dd, "a_{D,E,F(V)}"'] & & & F(V)\otimes(D\otimes E) \arrow[dl, "1\otimes(g_D\otimes g_E)"'] \\
& (F(X)\otimes F(Y))\otimes F(V) \arrow[r, "c_{F(X)\otimes F(Y),V}"] \arrow[d, "a_{F(X),F(Y),F(V)}"] & F(V)\otimes(F(X)\otimes F(Y)) & \\
D\otimes(E\otimes F(V)) \arrow[r, "g_D\otimes(g_E\otimes 1)"] \arrow[dd, "1\otimes c_{E,F(V)}"'] & F(X)\otimes(F(Y)\otimes F(V)) \arrow[d, "1\otimes c_{F(Y),F(V)}"] & (F(V)\otimes F(X))\otimes F(Y) \arrow[u, "a_{F(V),F(X),F(Y)}"] & (F(V)\otimes D)\otimes E \arrow[l, "(1\otimes g_D)\otimes g_E"'] \arrow[uuu, "a_{F(V),D,E}"'] \\
& F(X)\otimes(F(V)\otimes F(Y)) \arrow[r, "a^{-1}_{F(X),F(V),F(Y)}"'] & (F(X)\otimes F(V))\otimes F(Y) \arrow[u, "c_{F(X),F(V)}\otimes 1"] & \\
D\otimes(F(V)\otimes E) \arrow[rrr, "a^{-1}_{D,F(V),E}"'] \arrow[ur, "g_D\otimes(1\otimes g_E)"] & & & (D\otimes F(V))\otimes E, \arrow[ul, "(g_D\otimes 1)\otimes g_E"'] \arrow[uu, "c_{D,F(V)}\otimes 1"']
\end{tikzcd}$$

commutes. Its inner hexagon commutes, because it involves only objects in the image of F, and the outer diagrams commute by naturality of $c_{-,F(V)}$ in the first argument, the properties of the tensor product and the naturality of the associator in $\mathcal{D}$. This shows that $\mathcal{Z}(F)$ is well-defined on the objects of $\mathcal{C}$.

2.(b) To show that $\mathcal{Z}(F)$ is well-defined on the morphisms of $\mathcal{C}$, we need to check that for each morphism $f : V \to W$ in $\mathcal{C}$ the morphism $F(f) : F(V) \to F(W)$ in $\mathcal{D}$ makes the diagram (33) commute. This follows, because this diagram can be subdivided as

$$\begin{tikzcd}
F(X)\otimes F(V) \arrow[rrr, "c_{F(X),F(V)}"] \arrow[dr, "\phi^{\otimes}_{X,V}"] \arrow[ddd, "1\otimes F(f)"'] & & & F(V)\otimes F(X) \arrow[dl, "\phi^{\otimes}_{V,X}"'] \arrow[ddd, "F(f)\otimes 1"] \\
& F(X\otimes V) \arrow[r, "F(c_{X,V})"] \arrow[d, "F(1\otimes f)"] & F(V\otimes X) \arrow[d, "F(f\otimes 1)"] & \\
& F(X\otimes W) \arrow[r, "F(c_{X,W})"] \arrow[dl, "\phi^{\otimes -1}_{X,W}"'] & F(W\otimes X) \arrow[dr, "\phi^{\otimes -1}_{W,X}"] & \\
F(X)\otimes F(W) \arrow[rrr, "c_{F(X),F(W)}"'] & & & F(W)\otimes F(X),
\end{tikzcd}$$

where the top and bottom quadrilateral commute by definition of $c_{-,F(V)}$ and $c_{-,F(W)}$, the left and right quadrilateral by naturality of $\phi^{\otimes}$ and the middle rectangle by naturality of c.

As $\mathcal{Z}(F)$ coincides with F on the morphisms, we have shown that $\mathcal{Z}(F) : \mathcal{C} \to \mathcal{Z}(\mathcal{D})$ is a functor, and by definition we have $\Pi\mathcal{Z}(F) = F$.

2.(c) To show that $\mathcal{Z}(F)$ is monoidal, note that the only natural candidates for its coherence data are the isomorphism $\phi^e : F(e) \to e$ and the natural isomorphism $\phi^{\otimes} : \otimes(F \times F) \to F\otimes$ for

F. We thus need to show that (i) $\phi^e : F(e) \to e$ is a morphism in $\mathcal{Z}(\mathcal{D})$, (ii) the isomorphisms $\phi^{\otimes}_{V,W}$ are morphisms in $\mathcal{Z}(\mathcal{D})$ and (iii) that diagram (34) commutes for $c_{-,F(V)}$.

(i) That ϕ^e is a morphism in $\mathcal{Z}(\mathcal{C})$ follows from the diagram

$$\begin{tikzcd}
F(X)\otimes F(e) \arrow[rrrr, "c_{F(X),F(e)}"] \arrow[dr, "\phi^{\otimes}_{X,e}"] \arrow[dd, "1\otimes\phi^e"'] & & & & F(e)\otimes F(X) \arrow[dd, "\phi^e\otimes 1"] \\
& F(X\otimes e) \arrow[r, "F(r_X)"] & F(X) \arrow[r, "F(l_X^{-1})"] \arrow[drr, "l^{-1}_{F(X)}"'] & F(e\otimes X) \arrow[ur, "\phi^{\otimes -1}_{e,X}"'] & \\
F(X)\otimes e \arrow[urr, "r_{F(X)}"] \arrow[rrrr, "c_{F(X),e}"'] & & & & e\otimes F(X),
\end{tikzcd}$$

in which the top quadrilateral commutes by definition of $c_{-,F(e)}$ and by Remark 3.1.2, 3. for the braiding in $\mathcal{C}$. The bottom triangle commutes by Remark 3.1.2, 3. for the braiding in $\mathcal{Z}(\mathcal{D})$, and the quadrilaterals at the left and right because F is monoidal.

(ii) To show that the isomorphisms $\phi^{\otimes}_{V,W} : F(V)\otimes F(W) \to F(V\otimes W)$ are morphisms in $\mathcal{Z}(\mathcal{D})$, we need to show that they make diagram (33) commute. For this, we first consider the case, where the first argument of $c_{-,F(V)\otimes F(W)}$ and of $c_{-,F(V\otimes W)}$ is an object $F(X)$ in the image of F and subdivide the diagram

$$\begin{tikzcd}
F(X)\otimes(F(V)\otimes F(W)) \arrow[r, "c_{F(X),F(V)\otimes F(W)}"] \arrow[d, "1\otimes\phi^{\otimes}_{V,W}"] & (F(V)\otimes F(W))\otimes F(X) \arrow[d, "\phi^{\otimes}_{V,W}\otimes 1"] \\
F(X)\otimes F(V\otimes W) \arrow[r, "c_{F(X),F(V\otimes W)}"'] & F(V\otimes W)\otimes F(X)
\end{tikzcd}$$

into the subdiagram (34) that relates $c_{F(X),F(V)\otimes F(W)}$ to $c_{F(X),F(V)}$ and $c_{F(X),F(W)}$ attached to the top horizontal arrow, the subdiagrams given by (35) with $g_D = 1$ that relate $c_{F(X),F(V)}$, $c_{F(X),F(W)}$ and $c_{F(X),F(V\otimes W)}$ to $F(c_{X,V})$, $F(c_{X,W})$ and $F(c_{X,V\otimes W})$, the subdiagram that relates $F(c_{X,V\otimes W})$ to $F(c_{X,V})$ and $F(c_{X,W})$ and additional diagrams that encode the naturality of $\phi^{\otimes}$ and the compatibility condition of $\phi^{\otimes}$ with the associators in $\mathcal{C}$ and $\mathcal{D}$. The claim for general objects D in $\mathcal{C}$ then follows as in 2.(ii) by choosing an isomorphism $g_D : D \to F(X)$, which exists by essential surjectivity of F.

(iii) That diagram (34) commutes for $c_{-,F(V)}$ follows by splitting it into subdiagrams as in 2.(ii), only that now one uses the first hexagon axiom for $\mathcal{C}$ instead of the second one.

2.(d) That $\mathcal{Z}(F)$ is braided then follows directly from the definition of $c_{-,F(V)}$ in (35). □

By applying Proposition 3.2.2 to the identity functor $F = \mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ on a braided monoidal category $\mathcal{C}$, we obtain a functor that embeds a *braided* monoidal category into its centre and is a right inverse of the forgetful functor $\Pi : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$.

Corollary 3.2.3: For any braided monoidal category $\mathcal{C}$, there is a unique braided monoidal functor $Z : \mathcal{C} \to \mathcal{Z}(\mathcal{C})$ with $\Pi Z = \mathrm{id}_{\mathcal{C}}$.

We now consider the interaction of the centre construction with monoidal equivalences. Any ring isomorphism $\phi : R \to S$ induces a ring isomorphism $\phi' : Z(R) \to Z(S)$, that is an isomorphism of commutative rings. Replacing rings by monoidal categories, the ring isomorphism by a monoidal equivalence and the centres by categorical centres leads to the question if a monoidal equivalence $F : \mathcal{C} \to \mathcal{D}$ induces a braided monoidal equivalence $F' : \mathcal{Z}(\mathcal{C}) \to \mathcal{Z}(\mathcal{D})$. Indeed, we can lift F to a braided monoidal functor $F' : \mathcal{Z}(\mathcal{C}) \to \mathcal{Z}(\mathcal{D})$ with $F\Pi = \Pi F'$. As $F\Pi : \mathcal{Z}(\mathcal{C}) \to \mathcal{D}$ is essentially surjective and full, Proposition 3.2.2 guarantees the existence and uniqueness of F'. We show that F' is an equivalence of categories.

Proposition 3.2.4: Let $F : \mathcal{C} \to \mathcal{D}$ be a monoidal equivalence. Then there is a unique braided equivalence $F' : \mathcal{Z}(\mathcal{C}) \to \mathcal{Z}(\mathcal{D})$ with $F\Pi = \Pi F'$.

Proof:
1. Uniqueness: Let $F' : \mathcal{Z}(\mathcal{C}) \to \mathcal{Z}(\mathcal{D})$ be a braided equivalence with $F\Pi = \Pi F'$. Then the identity $F\Pi = \Pi F'$ implies $F'(f) = F(f)$ for all morphisms f in $\mathcal{Z}(\mathcal{C})$ and $F'(V, c_{-,V}) = (F(V), c_{-,F(V)})$. As F' is braided, one has $c_{F(U),F(V)} = \phi^{\otimes -1}_{V,U} \circ F(c_{U,V}) \circ \phi^{\otimes}_{U,V}$ for all objects U, V in $\mathcal{C}$, and the naturality of $c_{-,F(V)}$ in the first argument implies that diagram

$$\begin{tikzcd}
D\otimes F(V) \arrow[r, "g\otimes 1"] \arrow[d, "c_{D,F(V)}"'] & F(X)\otimes F(V) \arrow[r, "\phi^{\otimes}_{X,V}"] \arrow[d, "c_{F(X),F(V)}"] & F(X\otimes V) \arrow[d, "F(c_{X,V})"] \\
F(V)\otimes D \arrow[r, "1\otimes g"'] & F(V)\otimes F(X) \arrow[r, "\phi^{\otimes -1}_{V,X}"'] & F(V\otimes X).
\end{tikzcd} \tag{36}$$

commutes for all morphisms $g : D \to F(X)$. As F is essentially surjective, for each object D is $\mathcal{D}$, there is an isomorphism $g : D \to F(X)$ for some object X in $\mathcal{C}$, and (36) determines $c_{-,F(V)}$ and hence F' uniquely.

2. Existence: We define $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{Z}(\mathcal{D})$ by $F'(f) = F(f)$ for all morphisms f in $\mathcal{Z}(\mathcal{C})$. To define F' on the objects, we choose for each object D in $\mathcal{D}$ an isomorphism $g = g_D : D \to F(X)$, which exists by essential surjectivity of F and set $F'((V, c_{-,V})) = (F(V), c_{-,F(V)})$, where $c_{-,V}$ is defined by (36) for $g = g_D$.

2.(a) That F' is well-defined on the objects and morphisms of $\mathcal{Z}(\mathcal{C})$ follows as in the proof of Propositon 3.2.2. That F' becomes monoidal when equipped with the coherence data of F also follows analogously to the proof of Propositon 3.2.2. Diagram (36) with $g = 1_{F(X)}$ then shows that F' is braided.

2.(b) We show that F' is an equivalence of categories. As F is an equivalence of categories and $F'(f) = F(f)$ for all morphisms, it follows directly that F' is faithful. To show that F' is full, let $h : (F(V), c_{-,F(V)}) \to (F(W), c_{-,F(W)})$ be a morphism in $\mathcal{Z}(\mathcal{D})$. As F is fully faithful, there is a unique morphism $h' : V \to W$ with $F(h') = h$. It remains to show hat h is a morphism in $\mathcal{Z}(\mathcal{C})$ from $(V, c_{-,V})$ to $(W, c_{-,W})$, i. e. that diagram (33) commutes for all objects X in $\mathcal{C}$. As F is fully faithful, it is sufficient to consider its image under F

$$\begin{tikzcd}
F(X\otimes V) \arrow[r, "\phi^{\otimes -1}_{X,V}"] \arrow[d, "F(1\otimes h')"] & F(X)\otimes F(V) \arrow[r, "c_{F(X),F(V)}"] \arrow[d, "1\otimes F(h')=", "1\otimes h"'] & F(V)\otimes F(X) \arrow[r, "\phi^{\otimes}_{V,X}"] \arrow[d, "h\otimes 1"] & F(V\otimes X) \arrow[d, "F(h'\otimes 1)"] \\
F(X\otimes W) \arrow[r, "\phi^{\otimes -1}_{X,W}"'] & F(X)\otimes F(W) \arrow[r, "c_{F(X),F(W)}"'] & F(W)\otimes F(X) \arrow[r, "\phi^{\otimes}_{W,X}"'] & F(W\otimes X),
\end{tikzcd}$$

in which the outer rectangles commute by naturality of $\phi^{\otimes}$ and the middle one because h is a morphism in $\mathcal{Z}(\mathcal{D})$. This shows that h' is a morphism in $\mathcal{Z}(\mathcal{C})$ with $F'(h') = f$ and F' is full.

To see that F' is essentially surjective, let $(D, c_{-,D})$ be an object in $\mathcal{Z}(\mathcal{D})$. As F is essentially surjective, there is an isomorphism $g_D : D \to F(X)$ for some object X in $\mathcal{C}$. As F is fully faithful, for every object X' in $\mathcal{C}$ there is a unique morphism $c_{X',X} : X'\otimes X \to X\otimes X'$ with

$$\begin{aligned} F(c_{X,X'}) : F(X'\otimes X) &\xrightarrow{\phi^{\otimes -1}_{X',X}} F(X')\otimes F(X) \xrightarrow{1\otimes g_D^{-1}} F(X')\otimes D \xrightarrow{c_{F(X'),D}} D\otimes F(X') \\ &\xrightarrow{g_D\otimes 1} F(X)\otimes F(X') \xrightarrow{\phi^{\otimes}_{X,X'}} F(X\otimes X'). \end{aligned} \tag{37}$$

To show that $F'(X, c_{-,X}) \cong (D, c_{-,D})$ in $\mathcal{Z}(\mathcal{C})$, we need to show that (i) the morphisms $c_{X',X}$ are natural in the first argument and (ii) satisfy condition (32) and that (iii) g_D is an isomorphism in $\mathcal{Z}(\mathcal{D})$ from $(D, c_{-,D})$ to $F'(X, c_{-,X})$. As F is fully faithful, claim (i) follows from the diagram

$$\begin{tikzcd}
F(X'\otimes X) \arrow[r, "\phi^{\otimes -1}_{X',X}"] \arrow[d, "F(f\otimes 1)"] & F(X')\otimes F(X) \arrow[r, "1\otimes g_D^{-1}"] \arrow[d, "F(f)\otimes 1"] & F(X')\otimes D \arrow[r, "c_{F(X'),D}"] \arrow[d, "F(f)\otimes 1"] & D\otimes F(X') \arrow[r, "g_D\otimes 1"] \arrow[d, "1\otimes F(f)"] & F(X)\otimes F(X') \arrow[r, "\phi^{\otimes}_{X,X'}"] \arrow[d, "1\otimes F(f)"] & F(X\otimes X') \arrow[d, "F(1\otimes f)"] \\
F(X''\otimes X) \arrow[r, "\phi^{\otimes -1}_{X'',X}"] & F(X'')\otimes F(X) \arrow[r, "1\otimes g_D^{-1}"] & F(X'')\otimes D \arrow[r, "c_{F(X''),D}"] & D\otimes F(X'') \arrow[r, "g_D\otimes 1"] & F(X)\otimes F(X'') \arrow[r, "\phi^{\otimes}_{X,X''}"] & F(X\otimes X''),
\end{tikzcd}$$

which commutes by naturality of $\phi^{\otimes}$ and c. Claim (ii) follows from the second hexagon identity for $c_{-,D}$, the condition that F is monoidal and (37). Claim (iii) follows directly from (35) and (37). This shows that $(X, c_{-,X})$ is an object of $\mathcal{Z}(\mathcal{C})$ and $(D, c_{-,D}) \cong F'(X, c_{-,X})$ in $\mathcal{Z}(\mathcal{D})$. Hence, F' is essentially surjective and hence a braided equivalence. □

The centre construction is quite abstract, and it is often difficult to describe the resulting braided monoidal categories explicitly and concretely. In particular, it is often complicated to write down explicitly the class of objects of the centre. However, this is often not necesary, since one is only interested in the centre up to braided equivalence. The point is not to show that a braided monoidal category *is equal* to the centre of a monoidal category, only that it is *braided equivalent* to it. In Section 6.3 we will see an alternative, but less general construction that gives a more concrete description of the centre of certain monoidal categories.

3.3 Ribbon categories

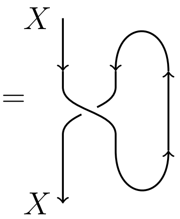
We now investigate the interaction of braidings with duals in a braided monoidal category $\mathcal{C}$. For this, we suppose that $\mathcal{C}$ is pivotal and that the left duals are constructed from the right duals and the pivot as in Proposition 2.2.3. This allows us to use the simplified graphical calculus for pivotal categories from Section 2.2.

The naturality of the braiding then implies that evaluations and coevaluations can be moved below and above overcrossings and undercrossings in the diagrams, which allows one to undo many braidings. However, there is a basic diagram that combines a braidings with an evaluation and a coevaluation that cannot be undone by pulling evaluations and coevaluations under crossings. It describes the *twist morphism* in $\mathcal{C}$ and merits further investigation.

Definition 3.3.1: Let $\mathcal{C}$ be a braided pivotal category.

1. For any object X in $\mathcal{C}$ the **twist** on X is the morphism

$$\theta_X = r_X \circ (1_X\otimes \mathrm{ev}^L_X) \circ a_{X,X,X^*} \circ (c_{X,X}\otimes 1_{X^*}) \circ a^{-1}_{X,X,X^*} \circ (1_X\otimes \mathrm{coev}^R_X) \circ r^{-1}_X \tag{38}$$

$\theta_X =$ 

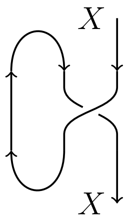
2. The category $\mathcal{C}$ is called a **ribbon category** if all twists are self-dual: $\theta_X^* = \theta_{X^*} : X^* \to X^*$ for all objects X in $\mathcal{C}$.

Clearly, the twist morphism θ_X depicted in Definition 3.3.1 exists in four variants, which are obtained by reflecting its diagram on a vertical line and by exchanging over- and undercrossings in the twist and its reflection. The former is its inverse θ_X^{-1}, while the latter two are the dual $\theta_{X^*}^*$ and its inverse. They coincide with the twist and its inverse if and only if $\mathcal{C}$ is ribbon. This is a consequence of the following lemma, which also shows that the twist defines a natural isomorphism $\theta : \mathrm{id}_\mathcal{C} \to \mathrm{id}_\mathcal{C}$, but that this natural isomorphism is in general *not monoidal*.

Lemma 3.3.2: Let $\mathcal{C}$ be a braided pivotal category.

1. The twist is invertible with inverse

   $\theta_X^{-1} = l_X \circ (\mathrm{ev}_X^R \otimes 1_X) \circ a^{-1}_{X^*,X,X} \circ (1_{X^*} \otimes c^{-1}_{X,X}) \circ a_{X^*,X,X} \circ (\mathrm{coev}_X^L \otimes 1_X) \circ l_X^{-1}$

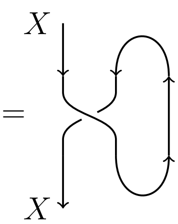
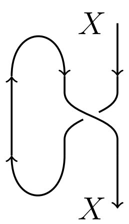
   $\theta_X^{-1} =$  (39)

2. The twist satisfies $\theta_e = 1_e$ and $\theta_{X \otimes Y} = c_{Y,X} \circ c_{X,Y} \circ (\theta_X \otimes \theta_Y) = (\theta_X \otimes \theta_Y) \circ c_{Y,X} \circ c_{X,Y}$.

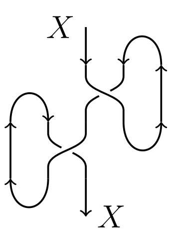
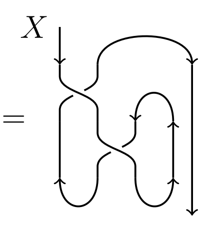
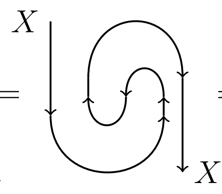
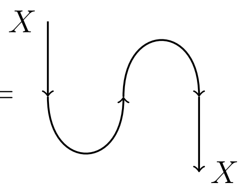
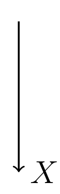
3. The twist is natural: $f \circ \theta_X = \theta_Y \circ f$ for all morphisms $f : X \to Y$.

4. $\mathcal{C}$ is ribbon if and only if for all objects X one has

   $\theta_X = \theta'_X := l_X \circ (\mathrm{ev}_X^R \otimes 1_X) \circ a^{-1}_{X^*,X,X} \circ (1_{X^*} \otimes c_{X,X}) \circ a_{X^*,X,X} \circ (\mathrm{coev}_X^L \otimes 1_X) \circ l_X^{-1}$

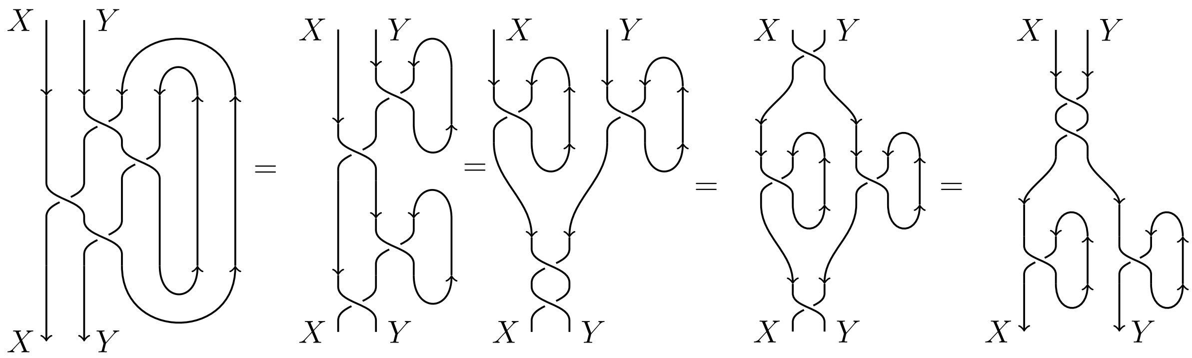
   $\theta_X =$  $=$  $= \theta'_X$ (40)

Proof:
1. We prove graphically that $\theta_X^{-1} \circ \theta_X = 1_X$:

 $=$  $=$  $=$  $=$ 

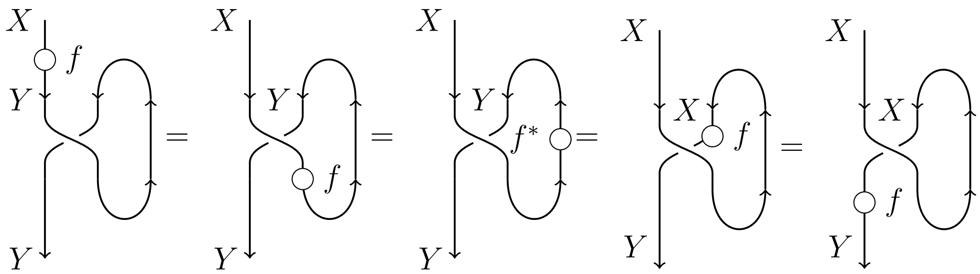
where we used the naturality of the braiding with respect to the twist in the first step, the naturality of the braiding with respect to $\cup$ in the second step and then the snake identity. The graphical proof that $\theta_X \circ \theta_X^{-1} = 1$ is analogous (Exercise).

2. That $\theta_e = 1_e$ follows directly from the identities $c_{e,e} = 1_{e\otimes e}$ and $\mathrm{tr}_L(1_e) = \mathrm{tr}_R(1_e) = 1$. The identities for $\theta_{X\otimes Y}$ can be proved graphically. From the diagram for the twist $\theta_{X\otimes Y}$ we obtain

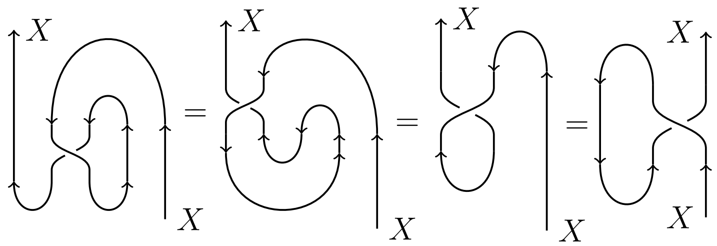


where we used the naturality of the braiding with respect to θ_Y and the dodecagon identity (30) in the first step and then twice the naturality of the braiding with respect to the twists.

3. The naturality of the twist follows from the naturality of the braiding and the identities (21) and (24) for the left and right evaluation and coevaluation:



4. To prove 4. we use the definition of the dual morphism θ_X^* in (20) and compute

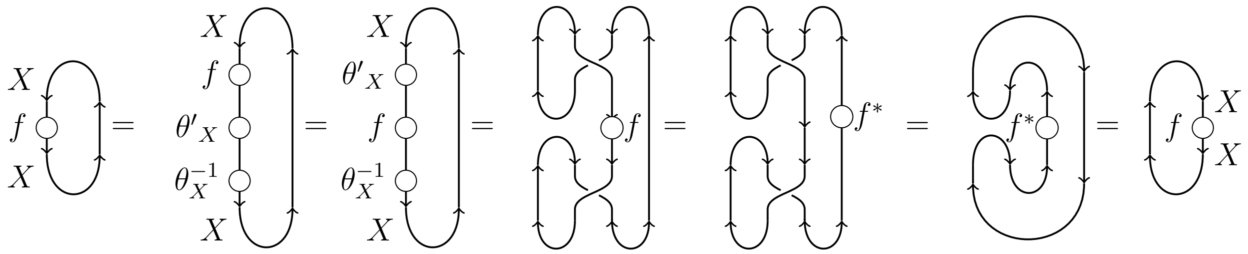
$\theta_X^* =$  $= {\theta'}_{X^*}$

where we used identity (27) and the naturality of the braiding with respect to $\cup$ in the first step, the snake identity in the second step, then again (27) and the naturality of the braiding with respect to $\cap$ in the third step. Hence we have $\theta_X^* = \theta_{X^*}$ if and only if $\theta_X = {\theta'}_X$. □

The fact that a pivotal braided monoidal category $\mathcal{C}$ is ribbon has consequences for the left and right traces of morphisms $f : X \to X$ and for its quantum dimensions. It allows one to transform a left trace into a right trace by pulling one side of the diagram for the trace behind or in front of the other, creating two twists and passing it to the other side. It follows that the left and right traces of any morphism in a ribbon category agree.

Corollary 3.3.3: Every ribbon category is spherical.

Proof:
We give a diagrammatic proof as follows



where we used the identity $\theta_X = \theta'_X$ in the first step, the naturality of θ' in the second step, the definition of θ_X and θ'_X in the third step, the naturality of the braiding with respect to f and the pivotality of $\mathcal{C}$ in the fourth step, the inverse and the naturality of the braiding with respect to the the evaluations and coevaluations and f in the fifth step and the snake identity and the pivotality in the last step. $\square$

Example 3.3.4: The categories $\mathrm{Vect}_{\mathbb{F}}^{fd}$ and $\mathbb{F}[G]-\mathrm{Mod}^{fd}$ for a group G are ribbon.

We will see more non-trivial examples of ribbon categories in Exercise 22 and 24 and in the following sections, where we also discuss their relations to knot and ribbon invariants.

4 Applications

The graphical calculus for monoidal categories is not just a convenient tool for computations but encodes connections between algebra, geometry and topology. A first example is knot theory. This was a hot topic in 19th century mathematics and then did not advance much further, until it had a sudden revival in the 1990s. The source of this revival was the discovery that representations of certain algebras and certain topological gauge theories from mathematical physics give rise to knot invariants. This was subsequently formalised and encoded in the language of ribbon categories.

4.1 Knots, link and ribbon invariants

The goal of knot theory is to establish simple and manageable criteria that tell one if

(i) two knots can be transformed into each other without cutting them,

(ii) a knot can be unknotted without cutting it.

Similar questions are considered for generalised knots with several connected components such as the olympic rings, which are called *links*. An efficient way to address these questions are *knot* or *link invariants*. These are maps from the set of knots or links into a commutative monoid that take the same value on all knots or links that can be transformed into each other without cutting them. To introduce them, we require some background on knots an links and their representations by diagrams.

Definition 4.1.1:

1. An **oriented link** is a smooth embedding $L : \amalg_n S^1 \to \mathbb{R}^3$ for some $n \in \mathbb{N}$. An **oriented knot** is an oriented link $L : S^1 \to \mathbb{R}^3$.
2. A **link** is an equivalence class of oriented links with respect to the equivalence relation defined by individual orientation reversals on each copy of S^1.
3. Two oriented links L, L' are called **equivalent** or **ambient isotopic** if there is a smooth map $F : [0,1] \times \mathbb{R}^3 \to \mathbb{R}^3$ such that $F_t = F(t, -) : \mathbb{R}^3 \to \mathbb{R}^3$ is a diffeomorphism for all $t \in [0,1]$, $F_0 = \mathrm{id}$ and $F_1 \circ L = L'$, an **ambient isotopy** from L to L'.
4. Two links are called **ambient isotopic** if they are ambient isotopic up to individual orientation reversal on each copy of S^1.

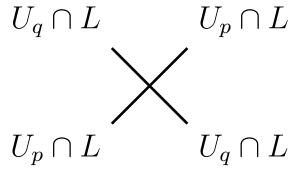
Here, $\amalg_n S^1$ denotes the direct sum of n copies of S^1, that is, the n-fold disjoint union with the sum topology and the induced smooth manifold structure. The orientation reversal on S^1 is given by the map $o : S^1 \to S^1$, $z \mapsto z^{-1}$. An embedding of a smooth manifold M into a smooth manifold N is an injective smooth map $f : M \to N$ that is a homeomorphism onto its image and such that $T_m f : T_m M \to T_{f(m)} N$ is injective for all $m \in M$. The notion of ambient isotopy encodes the intuitive picture of deforming a link smoothly, without passing it through itself. By an abuse of notation, we often do not distinguish links or oriented links from their images and write $L \subset \mathbb{R}^3$ instead of $L(\amalg_n S^1) \subset \mathbb{R}^3$.

Links in $\mathbb{R}^3$ can be described by link diagrams. A link diagram is obtained by projecting a link $L \subset \mathbb{R}^3$ with the map $P : \mathbb{R}^3 \to \mathbb{R}^2$, $(x,y,z) \mapsto (x,y)$. By applying an ambient isotopy to the link L, one can always achieve that the projection $P(L) \subset L$ is **generic**, i. e. satisfies:

(i) $|P^{-1}(x) \cap L| < 3$ for all $x \in \mathbb{R}^2$,

(ii) there are only finitely many points $x \in \mathbb{R}^2$ with $|P^{-1}(x) \cap L| = 2$,

(iii) if $P^{-1}(x) \cap L = \{p, q\}$, then there are neighbourhoods $U_p, U_q \subset \mathbb{R}^3$ and an orientation preserving diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ that maps $P(U_p \cup U_q)$ to the following diagram



This means that each point where the projection of the link overlaps with itself is a transversal crossing of exactly two strands and that there are only finitely many such crossings.

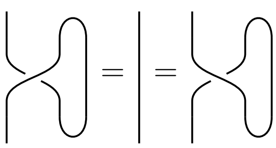
To reconstruct a link from its projection, at least up to equivalence, one requires the information which strand lies above he other in each crossing point. This information is indicated in diagrams by drawing the crossing as an **overcrossing** or **undercrossing**, where the strand with greater z coordinate at the crossing point crosses over the one with smaller z-coordinate. The same diagrams are used for oriented links, with the orientation of the link indicated by arrows on each connected component. A knot diagram without crossing points is called a **unknot**.

Definition 4.1.2: Let $L \subset \mathbb{R}^3$ be a link.

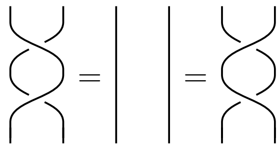
1. A **link diagram** for L is a generic link projection of L together with the information which of the two points in $P^{-1}(x) \cap L = \{p, q\}$ has the greater z-coordinate for each $x \in \mathbb{R}^2$ with $|P^{-1}(x) \cap L| = 2$.
2. An **oriented link diagram** for L is a link diagram for L together with a choice of orientation on each connected component of L.

It was shown by Reidemeister that (oriented) link diagrams capture all information about the equivalence classes of (oriented) links. The proof uses just basic topology and basic facts about smooth manifolds and can be found for instance in [Mn, Mu].

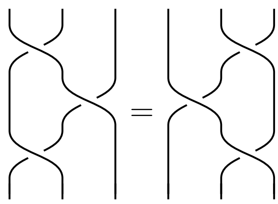
Theorem 4.1.3: Two (oriented) links $L, L' \subset \mathbb{R}^3$ are equivalent if and only if their (oriented) link diagrams D_L and $D_{L'}$ are related by a finite sequence of orientation preserving diffeomorphisms $f : \mathbb{R}^2 \to \mathbb{R}^2$ and the three **Reidemeister moves**:



Reidemeister move RM1



Reidemeister move RM2



Reidemeister move RM3.

In this case, the link diagrams D_L, D'_L are called **equivalent**.

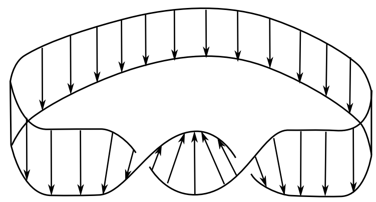
The Reidemeister moves are local moves that change only the depicted region in the link diagram and leave the rest of the link diagram invariant. They are defined analogously for oriented links, where each connected component receives an orientation. They encode the fact that an ambient isotopy $F : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}^3$ from a link L to L' need not induce diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ with $f(D) = D'$. Instead, it may give rise to additional crossing points in the projections (RM1 and RM2) or move crossing points past other crossing points (RM3).

Besides the notion of a link, there is also the related concept of a *framed link* or *ribbon*, which can also be oriented. It can be viewed as a link that is thickened to a strip or ribbon. The information needed to define the thickening is contained in the framing.

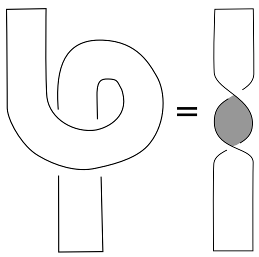
Definition 4.1.4:

1. A **framed link** or **ribbon** is a link $L : \amalg_n S^1 \to \mathbb{R}^3$ together with a vector field X along L that is nowhere tangent to L: a smooth map $X : \amalg_n S^1 \to \mathbb{R}^3$ with $X(z) \notin T_{L(z)}L \subset \mathbb{R}^3$ for all $z \in \amalg_n S^1$.
2. Two framed links (L, X) and (L', X') are called **equivalent** or **ambient isotopic** if there is an ambient isotopy $F : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}^3$ from L to L' with $X'(z) = T_{L(z)}F_1(X(z))$ for all $z \in \amalg_n S^1$.

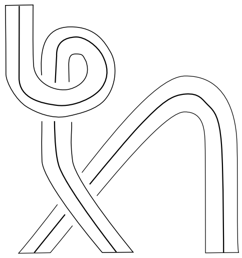
Given a link $L \subset \mathbb{R}^3$ and a vector field X along L that is nowhere tangent to L, we can thicken the link to a ribbon, that is twisted around itself only by multiples of 2π. Note that this excludes Möbius strips.



If one is only interested in equivalence classes of framed links, one can forget about the vector field and define a framed link as a link with an assignment of an integer $z \in \mathbb{Z}$ to each connected component that indicates how many times the connected component is twisted around itself. With the relation

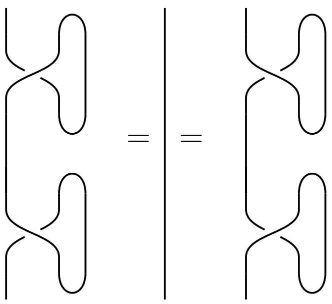


one can transform any projection of an associated ribbon into a **blackboard framed** ribbon projection that only involves twists of the type on the left but not the twist on the right. This corresponds to colouring the ribbon in $\mathbb{R}^3$ in two colours, black and white, and projecting in such a way that the white side is up in all parts of the projection. Blackboard framed links can be characterised by the same diagrams as links, where the link diagram represents the projection of a line in the middle of a ribbon, the **core**.

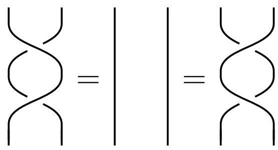


The only difference is that link diagrams that are related by the the Reidemeister move RM1 no longer describe projections of equivalent ribbons. Instead, one has a modified Reidemeister move RM1′.

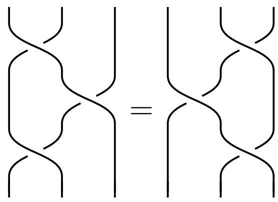
Theorem 4.1.5: Two framed (oriented) links $L, L' \subset \mathbb{R}^3$ are equivalent if and only if their link diagrams $D_L, D_{L'}$ are related by a finite sequence of orientation preserving diffeomorphisms $f : \mathbb{R}^2 \to \mathbb{R}^2$ and of the three **Reidemeister moves**



Reidemeister move RM1’



Reidemeister move RM2



Reidemeister move RM3.

In this case, the diagrams $D_L, D_{L'}$ are called **equivalent**.

The central question of knot theory is to decide from two given link diagrams $D_L, D_{L'}$ if the associated links $L, L' \in \mathbb{R}^3$ or framed links in $L, L' \subset \mathbb{R}^3$ are equivalent. By Theorem 4.1.3 and 4.1.5 this is the case if and only if the associated link diagrams are related by finite sequences of orientation preserving diffeomorphisms $f : \mathbb{R}^2 \to \mathbb{R}^2$ and the three Reidemeister moves RM1, RM2, RM3 or the three Reidemeister moves RM1’, RM2, RM3, respectively.

However, it is not practical to address this question by the Reidemeister moves alone. Instead, one considers **link invariants** or **ribbon invariants**, which are functions from the set of links or ribbons into a commutative monoid R that are constant on the equivalence classes of links or framed links. A good link or ribbon invariant should

- be easy to compute from a link or ribbon diagram,
- distinguish as many nonequivalent links or ribbons in $\mathbb{R}^3$ as possible.

By Theorems 4.1.3 and 4.1.5, link or ribbon invariants can also be defined in terms of diagrams.

Definition 4.1.6: Let R be a commutative monoid and $\mathcal{D}$ the set of (oriented) link diagrams.

1. An **(oriented) link invariant** is a map $I : \mathcal{D} \to R$ that is invariant under orientation preserving diffeomorphisms $f : \mathbb{R}^2 \to \mathbb{R}^2$ and the Reidemeister moves RM1, RM2, RM3.
2. An **(oriented) ribbon invariant** is a map $I : \mathcal{D} \to R$ that is invariant under orientation preserving diffeomorphisms $f : \mathbb{R}^2 \to \mathbb{R}^2$ and the Reidemeister moves RM1’, RM2, RM3.

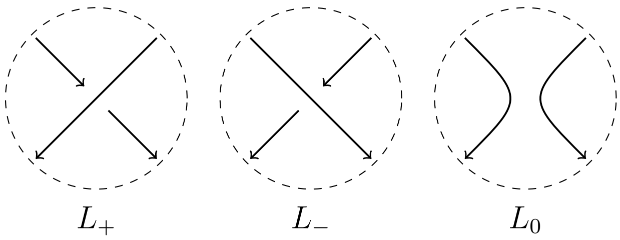
An obvious but not very useful invariant of a link or ribbon is the number of connected components. It is indeed easy to read off from a diagram, but does not distinguish enough links to be useful. Two famous and important link invariants that can distinguish many links are the *HOMFLY polynomial* and the *Jones-Kauffman polynomial* of a link. For more background on these link invariants and the proofs of the following theorems see [Ka] and [Mn].

Theorem 4.1.7: There is a unique invariant of oriented links with values in $\mathbb{Z}[x, x^{-1}, y, y^{-1}]$, the **HOMFLY polynomial** H, that satisfies the following conditions:

(i) It takes the value 1 on the **unknot**: $H(O) = 1$.

(ii) If the diagrams of oriented links $L, L' \in \mathbb{R}^3$ are related by the three Reidemeister moves RM1-RM3 and orientation preserving diffeomorphisms $f : \mathbb{R}^2 \to \mathbb{R}^2$, then $H(L) = H(L')$.

(iii) If the diagrams of the oriented links L_+, L_-, L_0 are **skein related**, i. e. locally related by



while the rest of their diagrams coincide, then

$$x \cdot H(L_+) - x^{-1} \cdot H(L_-) = y \cdot H(L_0). \tag{41}$$

Proof:
That the HOMFLY polynomial is determined uniquely by these conditions follows because every link can be transformed into an unknot by applying the skein relation and the Reidemeister moves RM1-RM3.

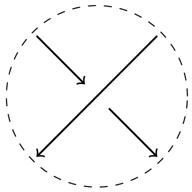
To show that the HOMFLY polynomial exists, it is sufficient to prove that the skein relation is compatible with the Reidemeister relations RM1-RM3. This follows by applying the skein relation to the diagrams on the left and right in the Reidemeister relations RM2 and RM3 and show that this does not give rise to any contradictions (Exercise). □

Another important link invariant is the *Jones-Kauffman invariant*. It is obtained from an invariant of framed links by rescaling it with the *writhe*, which describes how often each connected component of the framed link twists around itself.

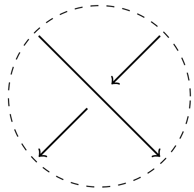
Definition 4.1.8: The **writhe** $w(K)$ of a framed knot K is the sum over all crossing points p in the knot diagram for K over the signs of the crossing

$$w(K) = \sum_{p \in K \cap K} \mathrm{sgn}(p),$$

where K is given an arbitrary orientation and the sign $\mathrm{sgn}(p)$ of a crossing point p is



lower strand crosses from right to left
$\mathrm{sgn}(p) = 1$



lower strand crosses from left to right
$\mathrm{sgn}(p) = -1$

The **writhe** of a link is the sum of the writhes of its connected components.

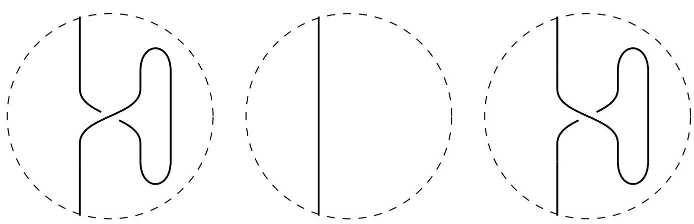
Note that in the definition of the writhe, only the crossings of each connected component of the link with itself are taken into account, but not crossings involving two different connected components. This implies that the writhe does not depend on the orientation of a framed link. Reversing the orientation of a connected component $K \subset L$ reverses the orientation of both strands in each crossing point and hence does not change the sign of the crossing.

It also follows directly from the definition that the writhe is invariant under the Reidemeister moves RM2 and RM3, since the Reidemeister move RM2 for one connected component of a link creates two additional crossings with opposite signs and the Reidemeister move RM3 does not change the number or signs of crossings.

It is also invariant under the Reidemeister move RM1', which creates or removes two crossings with opposite signs, but not under the Reidemeister move RM1.The latter creates or removes a crossing point with sign 1 or -1 and hence changes the writhe by ± 1. It follows that the writhe is an *invariant of framed links*, but not a link invariant.

Theorem 4.1.9:

1. There is a unique invariant P of framed links with values in $\mathbb{Z}[z, z^{-1}, a, a^{-1}]$, the **Kauffman polynomial** K, that satisfies the following conditions:
   (i) It takes the value 1 on the **unknot**: $K(O) = 1$.
   (ii) If the diagrams of two links $L, L' \in \mathbb{R}^3$ are related by the Reidemeister moves RM2, RM3 and orientation preserving diffeomorphisms $f : \mathbb{R}^2 \to \mathbb{R}^2$, then $K(L) = K(L')$.
   (iii) If the diagrams of links L_{++}, L_{00}, L_{--} are locally related by

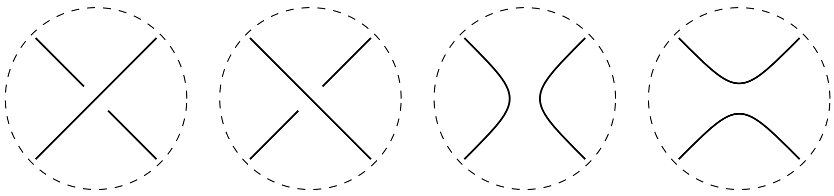
   

   L_{++} $\quad$ L_{00} $\quad$ L_{--}

   while the rest of their diagrams coincide, then

   $$a^{-1} \cdot K(L_{++}) = K(L_{00}) = a \cdot K(L_{--}).$$

   (iv) If the diagrams of the links L_+, L_-, L_0, L_∞ are locally related by



L_+ L_- L_0 L_∞

while the rest of their diagrams coincide, then

$$K(L_+) + K(L_-) = z \cdot K(L_0) + z \cdot K(L_\infty).$$

2. The **rescaled Kauffman polynomial** P given by $P(L) = a^{-w(L)}K(L)$, where $w(L)$ is the writhe of L, is a link invariant.

Proof:
It is clear that the Kauffman polynomial and the rescaled Kauffman polynomial are determined uniquely by conditions (i)-(iv), since any link L can be transformed into an unknot by removing twists as in (iii) and transforming overcrossings into undercrossings and vice versa as in (iv).

To show that the Kauffman polynomial and the rescaled Kauffman polynomial are well-defined and invariants of framed links, it is sufficient to show that the relations in (iii) and in (iv) are compatible with the Reidemeister relations RM1', RM2 and RM3. For the Reidemeister relation RM1', this follows directly from (iii). For the Reidemeister relations RM2 and RM3, it follows by applying the relations in (iii) and (iv) to the diagrams on the left and right in the Reidemeister relations RM2 and RM3 and to show that the resulting polynomials are indeed equal. (Exercise). That the rescaled Kauffman polynomial is a link invariant follows directly from its definition and from condition (iii). □

Many other famous link invariants can be viewed as special cases or rescalings of the HOMFLY polynomial or the rescaled Kauffman polynomial. Examples are the following:

Definition 4.1.10:

1. The **Alexander polynomial** of an oriented link L is the polynomial in $\mathbb{Z}[t]$ given by

$$A(L)(t) = H(L)(1, t^{1/2} - t^{-1/2})$$

2. The **Jones polynomial** of an oriented link L is the polynomial in $\mathbb{Z}[t^{1/2}, t^{-1/2}]$ given by

$$J(L)(t) = H(L)(t^{-1}, t^{1/2} - t^{-1/2}) = -K(L)(-t^{-1/4} - t^{1/4}, t^{-3/4})$$

3. The **Jones-Conway polynomial** of an oriented link is the polynomial in $\mathbb{Z}[q, q^{-1}]$ given by $C(O) = (q^n - q^{-n})/(q - q^{-1})$ for the unknot and the skein relation

$$q^n C(L_+) - q^{-n} C(L_-) = (q - q^{-1}) C(L_0),$$

where L_+, L_-, L_0 are as in Theorem 4.1.7 and $n \in \mathbb{N}$, $n > 1$.

These are less powerful link invariants than the HOMFLY and the rescaled Kauffman polynomial. Even the HOMFLY and the Kauffman polynomial are not perfect, as there are non-equivalent knots with the same HOMFLY and Kauffman polynomial. However, the goal is not to construct a perfect link invariant, but to construct managable link invariants *systematically*. This can be achieved by considering a suitable rigid monoidal category

4.2 Link and ribbon invariants via the tangle and ribbon category

It is clear that (framed) links are closely connected to the diagrammatic calculus for ribbon categories. The over- and undercrossings resemble the diagrammatic representation of the braiding in a monoidal category, the maxima and minima in a (framed) link diagram the diagrams for the evaluation and coevaluation in a pivotal category. These can be viewed as the elementary building blocks of a link diagram.

These elementary building blocks satisfy additional relations. The Reidemeister move RM2 in Theorem 4.1.5 corresponds to diagram (27) for the inverse of the braiding morphism, Reidemeister move RM 3 to diagram (30) for the dodecagon identity and the modified Reidemeister move RM1' is the diagrammatic expression of the statement that a pivotal braided monoidal category is ribbon from Lemma 3.3.2.

This suggests to describe (framed) links via a very basic ribbon category that only contains the minimal number of objects and morphisms required. On the levels of objects that should be a basic object, its dual and all finite tensor products of this object and its dual. On the level of morphisms, these are evaluation and coevaluation morphisms for this basic object, a braiding of the basic object with itself and its inverse as well as all possible tensor products and composites of these morphisms.

Just as in the case of the braid category, it seems sensible to define this ribbon category as a strict monoidal category and to present it in terms of generators and relations. This means that every object can be expressed as a tensor product of certain generating objects and every morphism as a composite of the generating morphisms via the composition and the tensor product. All relations between morphisms are obtained via the composition of morphisms and the tensor product from the generating relations and the monoidal structure. This can be formalised in a similar way to group presentations - the only difference is that there are two compositions and the morphisms need not be invertible and the procedure becomes quite technical. For details, see for instance [Ka, XII.1].

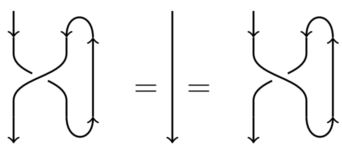
To account for links and framed links, we introduce two such categories that differ only in one relation. The tangle category $\mathcal{T}$ describes oriented link diagrams, and the category $\mathcal{R}$ of ribbon tangles that plays an analogous role for ribbons.

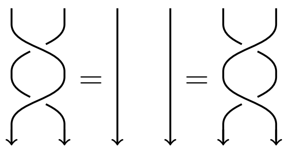
Definition 4.2.1: The **tangle category** $\mathcal{T}$ is the strict monoidal category with finite sequences $(\epsilon_1, ..., \epsilon_n)$ in $\mathbb{Z}/2\mathbb{Z}$ as objects, six generating morphisms $\cup, \cup', \cap, \cap', X, X^{-1}$

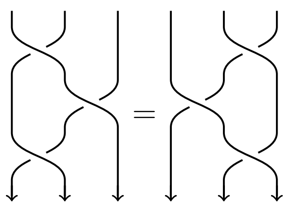
$\cup$: $-\otimes+ \to \emptyset$ $\quad$ $\cap$: $\emptyset \to +\otimes-$ $\quad$ $\cup'$: $+\otimes- \to \emptyset$ $\quad$ $\cap'$: $\emptyset \to -\otimes+$ $\quad$ X : $+\otimes+ \to +\otimes+$ $\quad$ X^{-1} : $+\otimes+ \to +\otimes+$

and the following generating relations:

- 1. **RM1:** $\quad$ 2. **RM2** $\quad$ 3. **RM3**

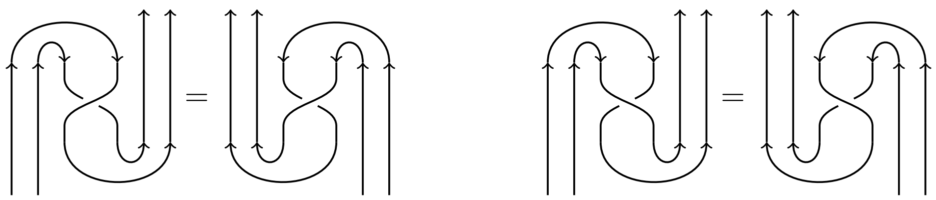




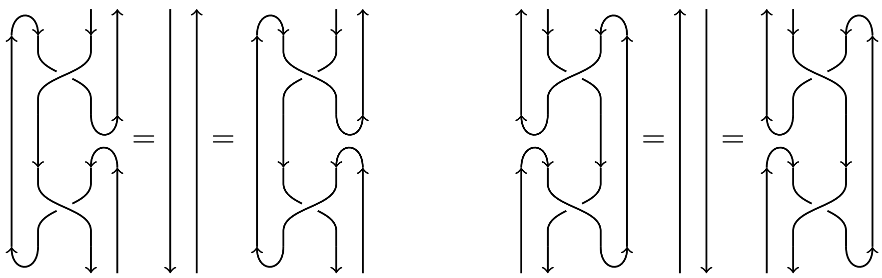


- 4. **Snake identities:**

- 5. **Snaked braiding:**



- 6. **Modified RM2:**



The identity morphisms $1_+ : + \to +$ and $1_- : - \to -$ are denoted

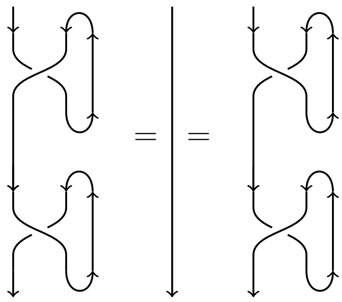
$$1_+ = \Big\downarrow \qquad\qquad 1_- = \Big\uparrow$$

- The composition of morphisms is the vertical composition of diagrams, whenever the sequences at the bottom of the top diagram and at the top of the bottom diagram match.
- The monoidal structure is given by the concatenation of finite sequences in $\mathbb{Z}/2\mathbb{Z}$ and the horizontal composition of diagrams.

A morphism $f : (\epsilon_1, ..., \epsilon_m) \to (\epsilon'_1, ..., \epsilon'_n)$ in $\mathcal{T}$ is called an (m, n)**-tangle**.

The tangle category exhibits as relations the Reidemeister moves RM1 to RM3, as well as the snake identities and other relations between diagrams that are related by diffeomorphisms. In particular, each $(0, 0)$-tangle is an oriented link diagram, up to Reidemeister moves and diffeomorphisms of the diagram. This makes this category suitable for the description of oriented link invariants. There is an analogous strict monoidal category for the description of oriented ribbons. It has the same generators and relations, just that the Reidemeister move RM1 is replaced by the Reidemeister move RM1'.

Definition 4.2.2: The **category $\mathcal{R}$ of ribbon tangles** is the strict monoidal category with finite sequences $(\epsilon_1, ..., \epsilon_n)$ in $\mathbb{Z}/2\mathbb{Z}$ as objects, the same generating morphisms and relations as in Definition 4.2.1, except that relation RM1 is replaced by

RM1'  (42)

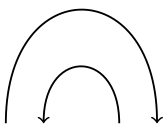
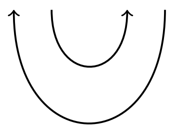
A morphism $f : (\epsilon_1, ..., \epsilon_m) \to (\epsilon'_1, ..., \epsilon'_n)$ in $\mathcal{R}$ is called an (m, n)**-ribbon tangle**.

As suggested by their names and the diagrammatic representation, the tangle category $\mathcal{T}$ and the ribbon category $\mathcal{R}$ are indeed ribbon categories, with the evaluations and coevaluations given by $\cup, \cup;, \cap, \cap'$ and the braidings constructed from the morphism X and its inverse.

Proposition 4.2.3: The categories $\mathcal{T}$ and $\mathcal{R}$ are strict ribbon categories.

Proof:
1. By definition, the categories $\mathcal{T}$ and $\mathcal{R}$ are strict monoidal categories. It also follows directly from the snake identities that they are left and right rigid with $(\epsilon_1, ..., \epsilon_n)^* = {}^*(\epsilon_1, ..., \epsilon_n) = (-\epsilon_n, ..., -\epsilon_1)$ and evaluation and coevaluation maps given as composites of the morphisms $\cap, \cup, \cap', \cup'$ as indicated in this example:

$\mathrm{coev}^R_{(-,+)} = \mathrm{coev}^L_{(-,+)}$ $\mathrm{ev}^R_{(-,+)} = \mathrm{ev}^L_{(-,+)}$

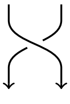
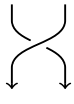
 

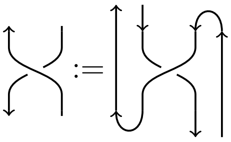
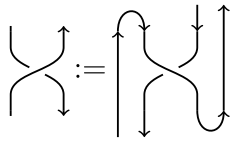
The snake identities and the snaked braiding relations imply that they are pivotal with the identity morphisms as the pivot.

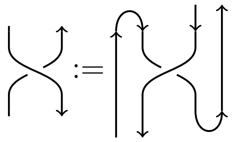
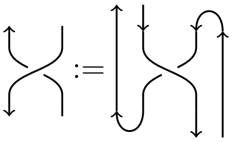
2. We show that $\mathcal{T}$ and $\mathcal{R}$ are braided. For this, note that a braiding in $\mathcal{T}$ and $\mathcal{R}$ is defined uniquely by its component morphisms $c_{\epsilon,\epsilon'} : (\epsilon, \epsilon') \to (\epsilon', \epsilon)$ for $\epsilon, \epsilon' \in \{\pm\}$ since every object $(\epsilon_1, ..., \epsilon_n) = \epsilon_1 \otimes ... \otimes \epsilon_n$ is a multiple tensor product of the objects $\pm$, and the braiding in a strict monoidal category satisfies

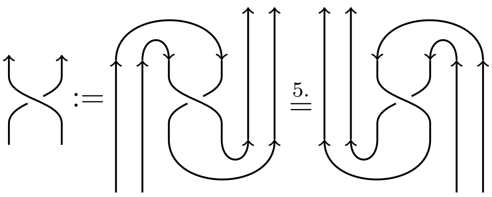
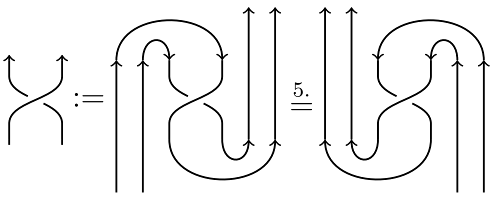
$$c_{e,U} = c_{U,e} = 1_U \qquad c_{U\otimes V,W} = (c_{U,W} \otimes 1_V) \circ (1_U \otimes c_{V,W}) \qquad c_{U,V\otimes W} = (1_V \otimes c_{U,W}) \circ (c_{U,V} \otimes 1_W)$$

for all objects U, V, W by the hexagon identity, see Remark 3.1.2. We define

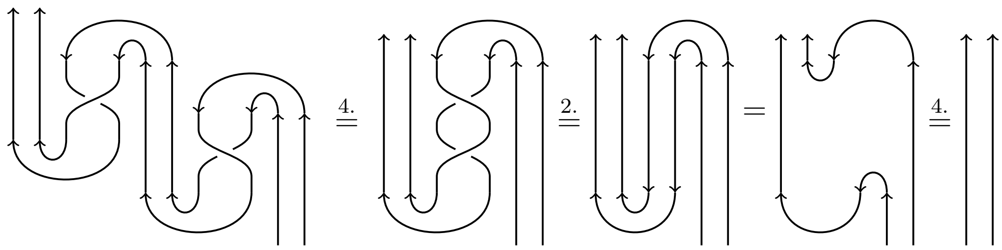
$c_{+,+} := X =$  $c^{-1}_{+,+} := X^{-1} =$ 

$c_{-,+} :=$  $c^{-1}_{-,+} :=$ 

$c_{+,-} :=$  $c^{-1}_{+,-} :=$ 

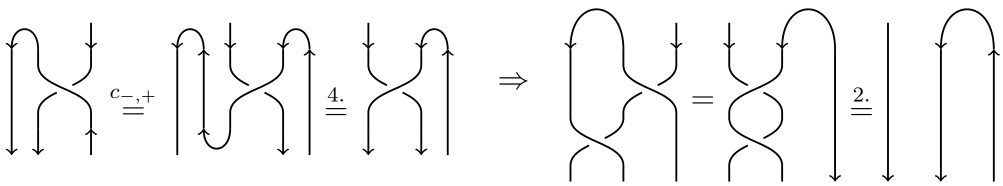
$c_{-,-} :=$  $c^{-1}_{-,-} :=$ 

We verify that $c^{-1}_{\epsilon,\epsilon'}$ is indeed the inverse of $c_{\epsilon,\epsilon'}$. The identities $c^{\mp 1}_{+,+} \circ c^{\pm 1}_{+,+} = 1_{+\otimes +}$ follow directly from the defining relation RM2, and the identities $c^{\mp 1}_{+,-} \circ c^{\pm 1}_{+,-} = 1_{\pm\otimes\mp}$ and $c^{\mp 1}_{-,+} \circ c^{\pm 1}_{-,+} = 1_{\mp\otimes\pm}$ follow from the modified RM2 relation. For the identity $c_{-,-} \circ c^{-1}_{-,-} = 1_{-\otimes -}$ we compute

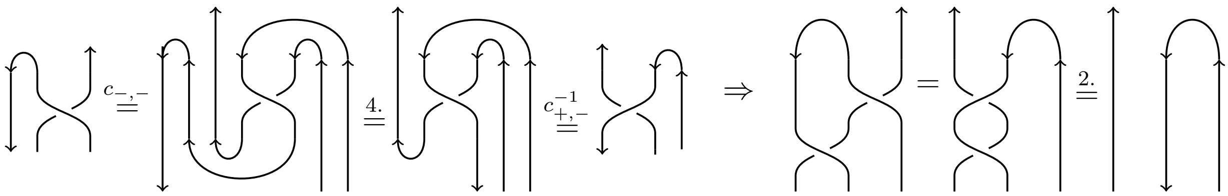


and the identity $c^{-1}_{-,-} \circ c_{-,-} = 1_{-\otimes -}$ follows analogously. This shows that $c_{\epsilon,\epsilon'} : \epsilon\otimes\epsilon' \to \epsilon'\otimes\epsilon$ for $\epsilon, \epsilon' \in \{\pm\}$ are isomorphisms.

It remains to prove the naturality of the braiding. As the morphisms $\cap$, $\cap$, $\cup'$, $\cap'$ and $X^{\pm 1}$ generate $\mathcal{T}, \mathcal{R}$ and the braidings for tensor products of objects are given as composites of the braidings $c_{\epsilon,\epsilon'}$ for $\epsilon, \epsilon' \in \{\pm 1\}$, it is sufficient to prove naturality for $\cap$, $\cap$, $\cup'$, $\cap'$ and X. To prove the naturality for $\cap$ we note that the definition of $c_{-,+}$ and the snake identity imply

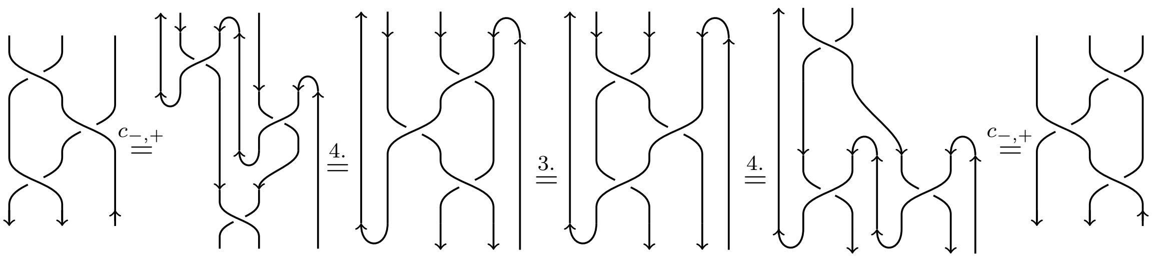


This proves the naturality for $\cap$ for a line that is oriented downwards and crosses under the strands $\cap$. The corresponding identity for a line that is oriented upwards and crosses under the strands of $\cap$ follows in a similar way from the definition of $c_{-,-}$ and $c^{-1}_{+,-}$, which imply



The corresponding identities where the line crosses over the strands of $\cap$ and the identities for $\cap'$, $\cup$ and $\cup'$ follow analogously. This proves the naturality of the braiding with respect to the morphisms $\cap, \cup, \cap', \cup'$. It remains to prove the naturality of the braiding with respect to the morphism X. For a line that is oriented downwards and crosses over or under the strands of X

this follows directly from the RM3 relation. For a line that is oriented upwards it follows from the definition of $c_{-,+}$, the snake identity and the RM3 relation:



3. That the category $\mathcal{R}$ is ribbon follows directly from the modified Reidemeister 1 relation RM1' in (42), which is equivalent to the condition $\theta_+ = \theta'_+$ from Lemma 3.3.2, 4. and hence to $\theta^*_+ = \theta_-$. By taking another dual one obtains $\theta_+ = \theta^{**}_+ = \theta^*_-$. That the tangle category $\mathcal{T}$ is ribbon as well follows directly, since the relation RM1 implies the relation RM1'. □

In fact, the category $\mathcal{R}$ is not just a ribbon category, but a minimal ribbon category in the same way as the braid category $\mathcal{B}$ is a minimal braided monoidal category. It contains the minimum amount of objects and morphisms that are required for a ribbon category, an object $+$, its dual $-$ and tensor products of these objects, the evaluations and coevaluations for $+$ and the braidings, as well as composites and tensor products of these morphisms. As a consequence, the category $\mathcal{R}$ has a similar universality property for ribbon categories as the universality of the braid category for braided monoidal categories from Proposition 3.1.8.

Proposition 4.2.4: (universality of the ribbon category)
Let $\mathcal{C}$ be a ribbon category and $V \in \mathrm{Ob}\,\mathcal{C}$. Then there is a braided monoidal functor $F_V : \mathcal{R} \to \mathcal{C}$, unique up to natural isomorphisms composed of associators and unit constraints in $\mathcal{C}$, with

$$F_V(+) = V \qquad F_V(-) = V^* \qquad F_V(\cup) = \mathrm{ev}^R_V : V^* \otimes V \to e \qquad F_V(\cup') = \mathrm{ev}^L_V : V \otimes V^* \to e.$$

Proof:
We prove the claim for the case where $\mathcal{C}$ is a strict braided monoidal category and $F_V : \mathcal{R} \to \mathcal{C}$ is a strict braided monoidal functor.

If $F_V : \mathcal{R} \to \mathcal{C}$ is a strict braided monoidal functor with $F_V(+) = V$ and $F_V(-) = V^*$, then one has $F_V(\emptyset) = e$ and $F(\epsilon_1, ..., \epsilon_n) = F_V(\epsilon_1) \otimes ... \otimes F_V(\epsilon_n)$ for all $n \in \mathbb{N}$ and $\epsilon_1, ..., \epsilon_n \in \{\pm\}$ and hence F_V is determined uniquely on the objects by $F_V(+) = V$ and $F_V(-) = V^*$. By assumption, one has $F_V(\cup) = \mathrm{ev}^R_V$, $F_V(\cup') = \mathrm{ev}^L_V$. The snake identity then enforces $F_V(\cap) = \mathrm{coev}^R_V$, , $F_V(\cap') = \mathrm{coev}^L_V$.

As F_V is braided, one has $F_V(X^{\pm 1}) = F_V(c^{\pm 1}_{+,+}) = c^{\pm 1}_{V,V} : V \otimes V \to V \otimes V$, and as the morphisms $\cup, \cap, \cup', \cap', X^{\pm 1}$ generate $\mathcal{R}$, this defines F_V uniquely on the morphisms.

To show that these assignments define a braided monoidal functor $F_V : \mathcal{R} \to \mathcal{C}$, we need to show that the morphisms $F_V(\cup)$, $F_V(\cap)$, $F_V(\cup')$, $F_V(\cap')$ and $F_V(X^{\pm 1})$ satisfy analogues of the defining relations from Definition 4.2.2.

• The Reidemeister relation RM1' follows directly from the condition $\theta_V = \theta'_V$ in Lemma 3.3.2 and the associated diagrams (38), (39) and (40) for the component morphisms of the twist θ and its dual.

• The Reidemeister relations RM2 and RM3 follow directly from the fact that $\mathcal{C}$ is braided and $F_V(X) = c_{V,V}$.

• The snake identities for $F_V(\cup)$, $F_V(\cap)$, $F_V(\cup')$ and $F_V(\cap')$ follow from the fact that $\mathcal{C}$ is pivotal and hence left and right rigid, see the corresponding diagrams after Example 2.1.3.

• The snaked braiding identities in Definition 4.2.2 follow from the identities (21) and (24), applied to the braiding in $\mathcal{C}$ and the snake identities in $\mathcal{C}$.

• The modified RM2 relations follow again from the identities (21) and (24), applied to the braiding, and the snake identities.

This shows that the assignments are compatible with the relations in $\mathcal{T}$ and define a monoidal functor $F_V : \mathcal{R} \to \mathcal{C}$. This functor is braided, since it sends X to $c_{V,V}$. As the braiding in $\mathcal{R}$ is determined uniquely by the morphism X, it follows that F_V is braided. □

The main motivation to consider braided monoidal functors $F : \mathcal{T} \to \mathcal{C}$ or $F : \mathcal{R} \to \mathcal{C}$ is that every *monoidal* functor $F : \mathcal{T} \to \mathcal{C}$ or $F : \mathcal{R} \to \mathcal{C}$ defines oriented link and ribbon invariants. If we view oriented link or ribbon diagrams as $(0,0)$-tangles or $(0,0)$-ribbon tangles, then any monoidal functor $F : \mathcal{T} \to \mathcal{C}$ or $F : \mathcal{R} \to \mathcal{C}$ into a monoidal category $\mathcal{C}$ assigns to a link or ribbon diagram an endomorphism of the unit object in $\mathcal{C}$. This assignment is invariant under Reidemeister moves and certain diffeomorphisms of the plane encoded in the other relations of $\mathcal{R}$ and $\mathcal{T}$ and hence depends only on the link or ribbon up to ambient isotopy.

Theorem 4.2.5: Let $\mathcal{C}$ be a monoidal category. Every monoidal functor $F : \mathcal{T} \to \mathcal{C}$ defines an oriented link invariant and every monoidal functor $F : \mathcal{R} \to \mathcal{C}$ an oriented ribbon invariant.

Sketch of Proof:
The oriented link or ribbon invariant is obtained by projecting a link or ribbon L to a link or ribbon diagram D_L. By applying orientation preserving diffeomorphisms $f : \mathbb{R}^2 \to \mathbb{R}^2$, one can transform the projection to a diagram representing a $(0,0)$-tangle in $\mathcal{T}$ or a $(0,0)$-ribbon tangle in $\mathcal{R}$.

It remains to show that the relations in $\mathcal{T}$ and $\mathcal{R}$ ensure that the resulting morphisms in $\mathcal{T}$ or $\mathcal{R}$ do not depend on the projection and agree for all oriented links or ribbons that are ambient isotopic. For diagrams related by Reidemeister moves, this follows directly from the Reidemeister relations in $\mathcal{T}$ and $\mathcal{R}$. For diagrams related by orientation preserving diffeomorphisms $f : \mathbb{R}^2 \to \mathbb{R}^2$, this is more difficult to show. For a proof, see [Ka, Chapters X.5 and XII] and [T, Chapters I.2-I.4] and the references therein.

Applying the functor F to this $(0,0)$-tangle or ribbon tangle yields an morphism $F(D) : e \to e$ that depends only on the ambient isotopy class of the link or the ribbon. □

Corollary 4.2.6: Every object in a ribbon category $\mathcal{C}$, together with a choice of left and right evaluation, defines an oriented ribbon invariant.

In particular, Corollary 6.4.15 can be applied to symmetric pivotal categories such as the categories $\mathrm{Vect}_{\mathbb{F}}^{fd}$ or $\mathbb{F}[G]{-}\mathrm{Mod}^{fd}$ for a group G, which are ribbon categories by Example 6.4.15. However, the resulting ribbon invariants are not very interesting, because they cannot distinguish ribbons that are obtained from each other by changing overcrossings into undercrossings or vice versa. By exchanging over- and undercrossings and applying the three Reidemeister moves, one can transform any ribbon into a disjoint union of unknots, possibly with a number of twists. As pairs of twists on one connected component cancel, the resulting ribbon invariants can at most measure the number of connected components of the ribbon and determine if the writhe of each connected component is odd or even.

Example 4.2.7: The following defines a braided monoidal functor $F : \mathcal{T} \to \mathrm{Vect}_{\mathbb{F}}^{fd}$

- $F(+) = V$ for some finite-dimensional vector space V over $\mathbb{F}$ and $F(-) = V^*$,
- $F(\cup) = \mathrm{ev}_V^R : V^* \otimes V \to \mathbb{F}$, $F(\cap) = \mathrm{coev}_V^R : \mathbb{F} \to V \otimes V^*$,
- $F(\cup') = \mathrm{ev}_V^L : V \otimes V^* \to \mathbb{F}$, $F(\cap') = \mathrm{coev}_V^L : \mathbb{F} \to V^* \otimes V$.
- $F(X) = F(X^{-1}) = \tau : V \otimes V \to V \otimes V$, $v \otimes w \mapsto w \otimes v$.

The associated ribbon invariant is a link invariant and assigns to a link with n connected components the number $(\dim_{\mathbb{F}} V)^n$.

To obtain more interesting link or ribbon invariants, one needs to consider non-symmetric ribbon categories, which we will construct systematically in the following sections. Another option is to consider objects with additional structures in monoidal categories that give rise to monoidal functors as in Theorem 4.2.5.

This is similar to the relation between Proposition 3.1.8, which states that every object in a strict braided monoidal category $\mathcal{C}$ defines a *braided monoidal* functor $F : \mathcal{B} \to \mathcal{C}$, and Corollary 3.1.11, which states that any Yang-Baxter operator in a monoidal category $\mathcal{C}$ defines a *monoidal* functor $F : \mathcal{B} \to \mathcal{C}$. Clearly, if $\mathcal{C}$ is braided, any object in $\mathcal{C}$ has a canonical Yang-Baxter operator structure, but this is not required to construct a *monoidal* functor $F : \mathcal{B} \to \mathcal{C}$.

Similarly, by Proposition 4.2.4 every object in a ribbon category $\mathcal{C}$ defines a monoidal functor $F : \mathcal{R} \to \mathcal{C}$ as in Theorem 4.2.5, but we may also consider objects with additional structure in more general monoidal categories to construct such functors. One possibility to obtain such functors is to consider braided vector spaces as in Definition 3.1.9 with additional structure.

Example 4.2.8: Suppose that $q \in \mathbb{C}^\times$ is not a root of unity and (V, σ) the braided vector space from Example 3.1.10 with ordered basis $(v_1, ..., v_n)$ and $\lambda = q^{-n}$ and

$$\sigma : V \otimes V \to V \otimes V, \quad v_i \otimes v_j \mapsto \begin{cases} q^{-n} v_j \otimes v_i & i < j \\ q^{1-n} v_i \otimes v_i & i = j \\ q^{-n} v_j \otimes v_i + q^{-n}(q - q^{-1}) v_i \otimes v_j & i > j. \end{cases}$$

Then there is a monoidal functor $F : \mathcal{T} \to \mathrm{Vect}_{\mathbb{F}}$ with $F(+) = V$, $F(-) = V^*$ and

$$\begin{aligned} &F(X) = \sigma : V \otimes V \to V \otimes V \\ &F(\cup) : V^* \otimes V \to \mathbb{F}, \ \alpha^j \otimes v_i \mapsto \delta_{ij} \\ &F(\cup') : V \otimes V^* \to \mathbb{F}, \ v_i \otimes \alpha^j \mapsto q^{1+n-2i} \delta_{ij}. \end{aligned}$$

where $(\alpha^1, ..., \alpha^n)$ is the dual basis of V^*. It satisfies the additional relation

$$q^n F(X) - q^{-n} F(X^{-1}) = (q - q^{-1}) \mathrm{id}_{V \otimes V}$$

and assigns to a link L the evaluation of its Jones-Conway polynomial from Definition 4.1.10:

$$F(L) = \mathrm{ev}_{q^{-1}} C(L).$$

Proof:
1. To show that this defines a monoidal functor $F : \mathcal{T} \to \mathrm{Vect}_{\mathbb{F}}$, we verify that the images of the morphisms $\cup, \cap, \cup', \cap'$ and $X^{\pm 1}$ under F satisfy analogues of the relations in Definition 4.2.1. By Example 3.1.10 (V, σ) is a braided vector space, and this implies that relations RM2 and RM3 are satisfied. The snake relations fix the images of the morphisms $\cap, \cap'$ as

$$F(\cap) : \mathbb{F} \to V \otimes V^*,\ 1 \mapsto \Sigma_{i=1}^n v_i \otimes \alpha^i \qquad F(\cap') : \mathbb{F} \to V^* \otimes V, 1 \mapsto \ \Sigma_{i=1}^n q^{2i-1-n} \alpha^i \otimes v_i$$

The analogues of the relations RM1, the snaked braiding relations and the modified RM2 relations follow by direct, but lengthy computations. This shows that we obtain a functor $F : \mathcal{T} \to \mathrm{Vect}_{\mathbb{F}}$.

2. The relation $q^n F(X) - q^{-n} F(X^{-1}) = (q - q^{-1}) \mathrm{id}_{V \otimes V}$ follows by a direct computation from the definition of σ. Up to exchanging $q \mapsto q^{-1}$, this is precisely the relation for the Jones-Conway polynomial from Definition 4.1.10. We also obtain the expression from Definition 4.1.10 for the value on the unknot

$$F(O) = F(\cup')F(\cap) = F(\cup)F(\cap') = \Sigma_{i=1}^n q^{\pm(n+1-2i)} = q^{\pm(n+1)} \Sigma_{i=1}^n q^{\mp 2i} = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

Thus, the evaluation of F yields the evaluation of the Jones-Conway polynomial in q^{-1}. $\square$

This example illustrates the power and usefulness of the diagrammatic calculus for ribbon categories and shows that it has nice applications in geometry and topology. However, one might wonder about the origin of the map $\sigma : V \otimes V \to V \otimes V$ in Example 4.2.8 and, more generally, how to construct braided vector spaces or ribbon categories systematically. We will develop methods to do this in the following sections, where we construct ribbon categories as representation categories of certain algebras with additional structure.

4.3 Topological quantum field theories

The concept of a topological quantum field theory was developed by Atiyah in [At], originally to describe quantum field theories on manifolds. The basic idea is to assign to each oriented closed $(n-1)$-manifold S a vector space $Z(S)$. To the manifold $\bar{S}$ with the opposite orientation, one assigns the dual vector space $Z(S)^*$ and to each oriented, compact n-manifold M with boundary $\partial M = \bar{S} \amalg S'$ a linear map $Z(M) : Z(S) \to Z(S')$. This assignment is required to be compatible with disjoint unions of manifolds and with gluing.

For this, one constructs a category $\mathcal{C}$ with $(n-1)$-manifolds as objects and n-manifolds with boundary as morphisms between them. Compatibility with gluing amounts to the statement that $Z : \mathcal{C} \to \mathrm{Vect}_{\mathbb{F}}$ is a functor. The disjoint union of manifolds defines a tensor product in the category $\mathcal{C}$. The compatibility of Z with disjoint unions then states that the category $\mathcal{C}$ is a symmetric monoidal category and $Z : \mathcal{C} \to \mathrm{Vect}_{\mathbb{F}}$ a symmetric monoidal functor.

Definition 4.3.1: The **cobordism category** $\mathrm{Cob}_{n,n-1}$ with $n \in \mathbb{N}$ is the symmetric monoidal category given as follows:

- Objects of $\mathrm{Cob}_{n,n-1}$ are oriented closed smooth $(n-1)$-manifolds.
- Morphisms in $\mathrm{Cob}_{n,n-1}$ are equivalence classes of cobordisms:

- A **cobordism** from a closed oriented smooth $(n-1)$-manifold S to a closed oriented smooth $(n-1)$-manifold S' is a pair (M, ϕ) of a smooth compact oriented n-manifold M with boundary ∂M and an orientation preserving diffeomorphism $\phi : \bar{S} \amalg S' \to \partial M$, where $\bar{S}$ is S with the reversed orientation and $\amalg$ the disjoint union of manifolds.
- Two cobordisms $(M, \phi), (M', \phi') : S \to S'$ are called **equivalent** if there is an orientation preserving diffeomorphism $\psi : M \to M'$ such that the following diagram commutes

$$\begin{tikzcd} M \arrow[rr, "\psi"] & & M' \\ & \bar{S} \amalg S' \arrow[ul, "\phi"] \arrow[ur, "\phi'"'] & \end{tikzcd}.$$

- The identity morphism 1_S is the equivalence class of the cobordism $([0,1] \times S, \phi_S)$, where $\phi_S : \bar{S} \amalg S \to \{0,1\} \times S$ with $\phi_S(x) = (1, x)$ and $\phi_S(y) = (0, y)$ for all $x \in \bar{S}$, $y \in S$.

- The composite of morphisms $[(M, \rho)] : S \to S'$ and $[(N, \sigma)] : S' \to S''$ is the equivalence class of the cobordism (P, τ), where $P = M \#_{S'} N$ is the n-manifold obtained by gluing M and N along S' with the gluing maps given by $\rho|_{S'} : S' \to \partial M$ and $\sigma|_{\bar{S}'} : \bar{S}' \to \partial N$. The diffeomorphism $\tau : \bar{S} \amalg S'' \to \partial P$ is induced by $\rho|_{\bar{S}} : \bar{S} \to M$, $\sigma|_{S''} : S'' \to N$ and the canonical surjection $\pi : M \amalg N \to P$. The smooth structure on P is constructed with a choice of collars around S', but the the equivalence class of the resulting cobordism does not depend on this choices.

- The tensor product of cobordisms is given by the disjoint union of manifolds and the tensor unit is the empty manifold $\emptyset$, viewed as an oriented smooth $(n-1)$-manifold[1].

Remark 4.3.2:

1. There are other versions of topological quantum field theories based on topological or piecewise linear manifolds with boundary. For $n \leq 3$ the associated cobordism categories are equivalent. For $n \geq 4$ the smooth framework is the most common and well-developed.
2. Orientation reversal defines a functor $* : \mathrm{Cob}_{n,n-1} \to \mathrm{Cob}_{n,n-1}^{op}$ with $** = \mathrm{id}_{\mathrm{Cob}_{n,n-1}}$. It assigns to a smooth oriented $(n-1)$-manifold S the manifold $\bar{S}$ with the opposite orientation and to the equivalence class of a cobordism $(M, \phi) : S \to S'$ the equivalence class of the cobordism $(\bar{M}, \phi) : \bar{S}' \to \bar{S}$, where $\bar{M}$ is equipped with the opposite orientation. Thus, the symmetric monoidal category $\mathrm{Cob}_{n,n-1}$ is pivotal.

With the notion of the cobordism category it is simple to define a topological quantum field theory. Although one usually considers topological quantum field theories with values in the category $\mathrm{Vect}_{\mathbb{F}}^{fd}$, the notion can be generalised to any symmetric monoidal category.

Definition 4.3.3: Let $\mathcal{C}$ be a symmetric monoidal category.

1. An oriented n-dimensional **topological quantum field theory** with values in $\mathcal{C}$ is a symmetric monoidal functor $Z : \mathrm{Cob}_{n,n-1} \to \mathcal{C}$.

2. Two oriented topological n-dimensional quantum field theories $Z, Z' : \mathrm{Cob}_{n,n-1} \to \mathcal{C}$ are called **equivalent** if there is a monoidal natural isomorphism $\phi : Z \to Z'$.

[1]Note that the empty set $\emptyset$ is by definition an n-dimensional smooth oriented manifold for all $n \in \mathbb{N}_0$.

The goals in the investigation of topological quantum field theories are the construction of interesting examples, the use of topological quantum field theories to gain information on manifolds and the classification of topological quantum field theories.

For this we need describe the category $\mathrm{Cob}_{n,n-1}$ as explicitly and concretely as possible, namely to present it in terms of generators and relations, as for the tangle category and the ribbon category. Such presentations are obtained with techniques from Morse theory. In principle, these techniques allow one to obtain a presentation of $\mathrm{Cob}_{n,n-1}$ for any $n \in \mathbb{N}$, but the description in terms of generators and relations becomes more complicated with growing dimension.

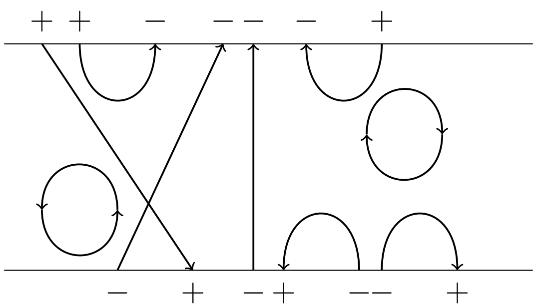
We therefore focus on $n = 1$ and $n = 2$. For general background on Morse theory and Cobordisms, see for instance [H, Chapter 6,7], for a brief summary and the application to $n = 1$ and $n = 2$, see [Kock, Chapter 1].

Example 4.3.4: The cobordism category $\mathrm{Cob}_{1,0}$

The symmetric monoidal category $\mathrm{Cob}_{1,0}$ has

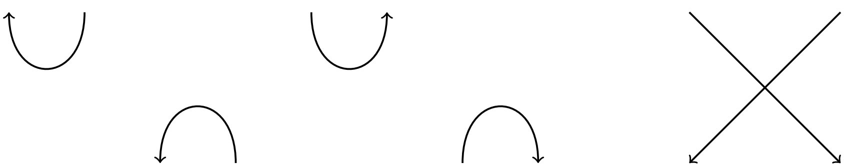
- as objects finiteb disjoint unions of oriented points, given by finite sequences in $\mathbb{Z}/2\mathbb{Z}$,
- as morphisms finite unions of oriented circles and of oriented lines such that the orientations of lines match the orientations of the objects.

This can be depicted by a diagram in the plane as follows:



A morphism $f : (+,+,-,-,-,-,+) \to (-,+,-,+,-,+)$ in $\mathrm{Cob}_{1,0}$.

The category $\mathrm{Cob}_{1,0}$ is generated by the the morphisms



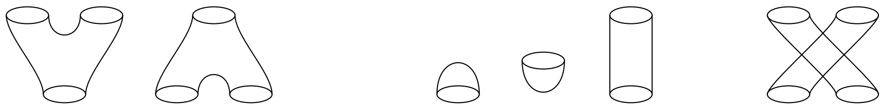
Its defining relations are analogous to the defining relations of the tangle category 4.2.1 with the additional relation that the braiding coincides with its inverse.

Note that the relations in $\mathrm{Cob}_{1,0}$ imply that any cobordism from $\emptyset$ to $\emptyset$ is a disjoint union of circles. As $\mathrm{Cob}_{1,0}$ has the relations of the tangle category with a symmetric braiding, where over- and undercrossings agree, any link is equal to a disjoint union of unknots. Due to the naturality conditions, circles can be moved freely in the diagrams.

Example 4.3.5: The cobordism category $\mathrm{Cob}_{2,1}$

• **objects:**
The cobordism category $\mathrm{Cob}_{2,1}$ has as objects finite unions of oriented circles.

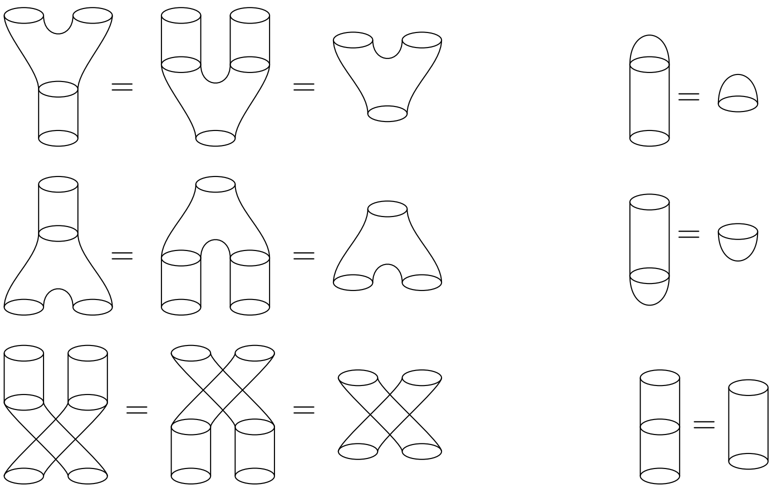
• **generating morphisms:**
The cobordism category $\mathrm{Cob}_{2,1}$ is generated by the following six morphisms. Each of them arises in two versions with opposite orientation, and the orientation of the boundary circles is induced by the orientations of the surfaces:



trinion **opposite trinion** **cap** **cup** **cylinder** **exchanging cylinder**

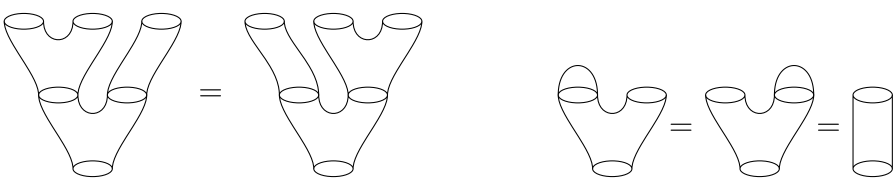
• **relations:**
The generators are subject to the following defining relations:

(a) **identity relations**

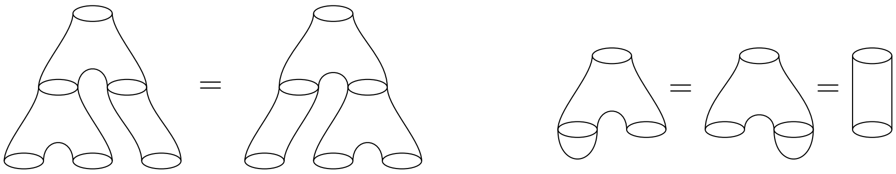


These relations state that the identity morphism on a finite union of oriented circles is the finite union of cylinders over these circles.

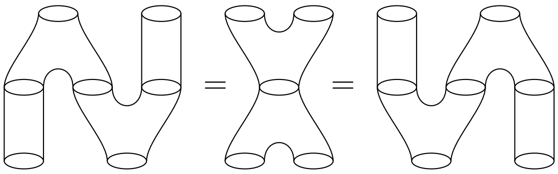
(b) **associativity and unitality**



(c) **coassociativity and counitality**

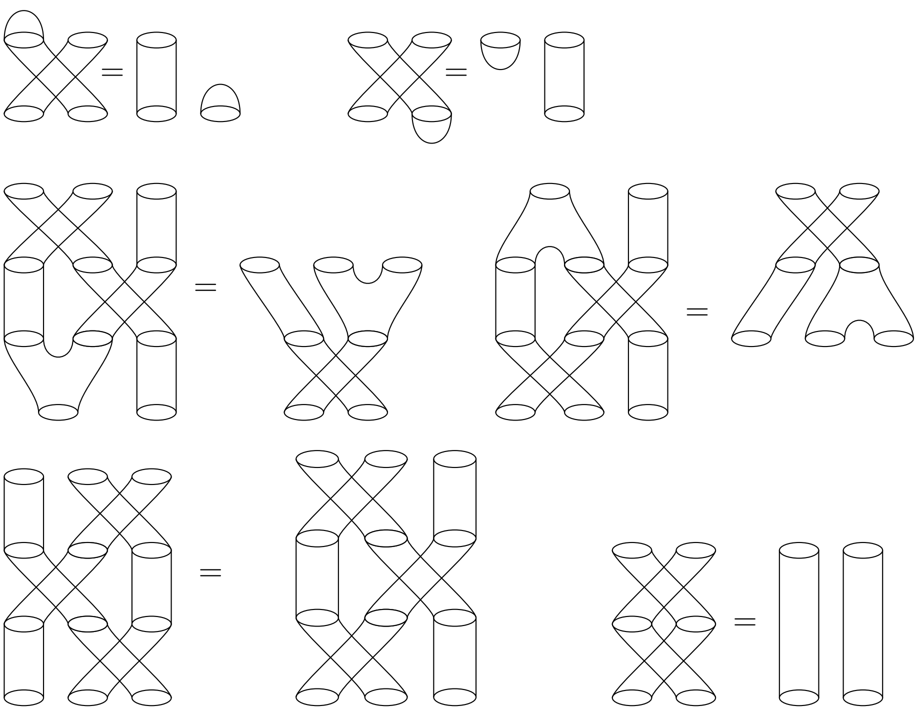


(d) **Frobenius relation**



(e) **commutativity and cocommutativity**

(f) **relations for the exchanging cylinder**



The first four relations state that the exchanging cylinder is natural with respect to cap, cup and trinions. The last two are a symmetric version of the Reidemeister relations RM2 and RM3.

We will now use the presentation of the cobordism category $\mathrm{Cob}_{2,1}$ to classify topological quantum field theories $Z : \mathrm{Cob}_{2,1} \to \mathrm{Vect}_{\mathbb{F}}$. Note that as *monoidal* functors TQFTs send duals to duals by Exercise 14 and .duals are unique up to unique isomorphisms by Proposition 2.1.2. Thus any TQFT $Z : \mathrm{Cob}_{2,1} \to \mathrm{Vect}_{\mathbb{F}}^{fd}$ is determined on the objects by the image of the oriented circle $V := Z(O_+)$, and its value on the circke with the opposite orientation is given by the dual $Z(O_-) = V^*$. On the morphisms it is determined by the images of cup, cap, and the trinions. The condition that Z is monoidal implies that the empty manifold is sent to the field $\mathbb{F}$, the tensor unit. The condition that Z is a functor implies that the cylinders are sent to identity morphisms, and the condition that Z is symmetric monoidal implies that the exchanging cylinder is sent to the flip map $\tau : V\otimes V \to V\otimes V$, $v\otimes v' \mapsto v'\otimes v$.

The image of the cup and cap define linear maps $\epsilon = F(\cup) : V \to \mathbb{F}$ and $\eta = F(\cap) : V \to \mathbb{F}$. The images of the trinions define linear maps $m : V\otimes V \to V$ and $\Delta : V \to V\otimes V$. The associativity and unitality relation state that (V, m, η) is an algebra over $\mathbb{F}$, and the commutativity relation states that it is commutative. The linear maps $\Delta : V \to V\otimes V$ and $\epsilon : V \to \mathbb{F}$ define additional structure on this algebra. It turns out that this additional structure is that of a *Frobenius algebra*. The concept of a Frobenius algebra can be defined in two equivalent ways.

Definition 4.3.6: A **Frobenius algebra** over $\mathbb{F}$ is an algebra A over $\mathbb{F}$ together with a linear map $\kappa : A\otimes A \to \mathbb{F}$, $a\otimes b \mapsto \kappa(a\otimes b)$, the **Frobenius form**, such that

1. $\kappa((a \cdot b)\otimes c) = \kappa(a\otimes(b \cdot c))$ for all $a, b, c \in A$,
2. κ is **non-degenerate**: the map $\phi_\kappa : A \to A^*$, $b \mapsto \kappa(-\otimes b)$ is a linear isomorphism.

It is called **symmetric** if $\kappa(a\otimes b) = \kappa(b\otimes a)$ for all $a, b \in A$.

A **morphism of Frobenius algebras** $f : (A, \kappa) \to (A, \kappa')$ is an algebra homomorphism $f : A \to A'$ with $\kappa'(f(a)\otimes f(b)) = \kappa(a\otimes b)$ for all $a, b \in A$.

Definition 4.3.7: A **(Δ, ϵ)-Frobenius algebra** over $\mathbb{F}$ is an algebra A over $\mathbb{F}$ together with linear maps $\Delta : A \to A\otimes A$ and $\epsilon : A \to \mathbb{F}$ such that the following conditions are satisfied:

1. coassociativity and counitality:
   $(\Delta\otimes\mathrm{id}) \circ \Delta = (\mathrm{id}\otimes\Delta) \circ \Delta$ and $l_A \circ (\epsilon\otimes\mathrm{id}) \circ \Delta = \mathrm{id} = r_A \circ (\mathrm{id}\otimes\epsilon) \circ \Delta$,
2. Δ is a morphism of $A\otimes A^{op}$-modules:
   $\Delta(ab) = (a\otimes 1) \cdot \Delta(b) = \Delta(a) \cdot (1\otimes b)$ for all $a, b \in A$.

A **morphism of (Δ, ϵ)-Frobenius algebras** $f : (A, \Delta, \epsilon) \to (A', \Delta', \epsilon')$ is an algebra homomorphism $f : A \to A'$ with $(f\otimes f) \circ \Delta = \Delta' \circ f$ and $\epsilon' \circ f = \epsilon$.

To see that these two definitions are indeed equivalent, one needs to construct the linear maps $\Delta : A \to A\otimes A$ and $\epsilon : A \to \mathbb{F}$ from the algebra structure and the Frobenius form and to construct a Frobenius form from the data of a (Δ, ϵ)-Frobenius algebra. This is left as an exercise (Exercise 29).

Lemma 4.3.8: Every Frobenius algebra has a canonical (Δ, ϵ)-Frobenius algebra structure, and every (Δ, ϵ)-Frobenius algebra a canonical Frobenius algebra structure.

Remark 4.3.9:

1. Every Frobenius form satisfies $\kappa(a\otimes b) = \kappa(ab\otimes 1) = \kappa(1\otimes ab)$ and hence is given by a linear form $\lambda : A \to \mathbb{F}$ with $\kappa(a\otimes b) = \lambda(a \cdot b)$ for all $a, b \in A$.
2. Every commutative Frobenius algebra is symmetric: if A is commutative, then 1. implies $\kappa(a\otimes b) = \kappa(b\otimes a)$ for all $a, b \in A$.

3. Every Frobenius algebra is finite-dimensional, since the non-degeneracy condition implies $A^* \cong A$ and hence $\dim_{\mathbb{F}} A < \infty$.

Example 4.3.10:

1. Any invertible matrix $A \in \mathrm{Mat}(n \times n, \mathbb{F})$ defines a Frobenius form κ_A on $\mathrm{Mat}(n \times n, \mathbb{F})$
$$\kappa_A : \mathrm{Mat}(n \times n, \mathbb{F}) \otimes \mathrm{Mat}(n \times n, \mathbb{F}) \to \mathbb{F}, \quad M \otimes N \mapsto \mathrm{Tr}(M \cdot N \cdot A).$$
It is symmetric if and only if $A = \lambda 1_n$ for some $\lambda \in \mathbb{F}$.

2. Let G be a finite group. Then $\kappa : \mathbb{F}[G] \otimes \mathbb{F}[G] \to \mathbb{F}$, $\kappa(g \otimes h) = \delta_e(g \cdot h)$ is a Frobenius form on $\mathbb{F}[G]$ and $\kappa' : \mathrm{Fun}(G, \mathbb{F}) \otimes \mathrm{Fun}(G, \mathbb{F}) \to \mathbb{F}$, $\kappa'(f \otimes h) = \Sigma_{g \in G} f(g) h(g)$ is a Frobenius form on $\mathrm{Fun}(G, \mathbb{F})$. Both Frobenius algebras are symmetric.

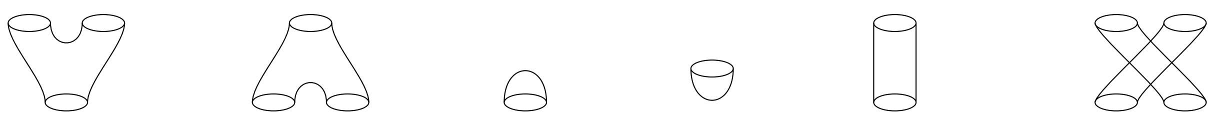
3. If (A, κ) is a Frobenius algebra and $a \in A$ is invertible, then
$$\kappa_a : A \otimes A \to \mathbb{F}, \quad b \otimes c \mapsto \kappa(b \otimes ca) \qquad \kappa'_a : A \otimes A \to \mathbb{F}, \quad b \otimes c \mapsto \kappa(ab \otimes c)$$
are Frobenius forms on A as well. One says they are obtained by **twisting** with a.

4. The tensor product $A \otimes B$ of two Frobenius algebras (A, κ_A) and (B, κ_B) over $\mathbb{F}$ has a natural Frobenius algebra structure with the Frobenius form
$$\kappa : (A \otimes B) \otimes (A \otimes B) \to \mathbb{F}, \quad (a \otimes b) \otimes (a' \otimes b') \mapsto \kappa_A(a \otimes a') \kappa_B(b \otimes b').$$

Given the concept of a Frobenius algebra and some relevant examples, we can now classify TQFTs $Z : \mathrm{Cob}_{2,1} \to \mathrm{Vect}_{\mathbb{F}}$. For this, note that the images of the cap and one trinion define a commutative algebra structure on the image of the circle. The image of the cap and the other trinion have to satisfy the coassociativity and counitality condition (c) in Example 4.3.5, which coincides with the first condition in Example 4.3.7. Finally, the Frobenius condition condition (d) in Example 4.3.5 is precisely the second condition in Definition 4.3.7.

Theorem 4.3.11: Equivalence classes of 2-dimensional oriented topological quantum field theories $Z : \mathrm{Cob}_{2,1} \to \mathrm{Vect}^{fd}_{\mathbb{F}}$ are in bijection with isomorphism classes of commutative Frobenius algebras over $\mathbb{F}$.

Proof:
A monoidal functor $Z : \mathrm{Cob}_{2,1} \to \mathrm{Vect}^{fd}_{\mathbb{F}}$ is determined uniquely (up to rebracketing and left and right unit constraints) by its value on the positively oriented circle and on the six generating morphisms. If Z assigns to the positively oriented circle a vector space $Z(O) = V$, then it assigns to the circle with the opposite orientation the dual vector space $Z(\bar{O}) = V^*$, to an n-fold union of circles the vector space $Z(O \amalg \ldots \amalg O) = V \otimes \ldots \otimes V$ and to the empty set the underlying field $Z(\emptyset) = \mathbb{F}$. This implies that Z associates to the six generating morphisms linear maps



$m : V \otimes V \to V$ $\quad \Delta : V \to V \otimes V$ $\quad \eta : \mathbb{F} \to V$ $\quad \epsilon : V \to \mathbb{F}$ $\quad \mathrm{id}_V : V \to V$ $\quad \tau : V \otimes V \to V \otimes V$

where we took already into account the identity relations for cylinders and suppose that all circles match the oprientation of the cobordism. In order to define a functor $Z : \mathrm{Cob}_{2,1} \to \mathrm{Vect}^{fd}_{\mathbb{F}}$

the linear maps m, η, Δ, ϵ, τ must satisfy certain relations that correspond to the defining relations of $\mathrm{Cob}_{2,1}$.

(b) **associativity and unitality:** they state that (V, m, η) is an algebra:

$$m \circ (m \otimes \mathrm{id}) = m \circ (\mathrm{id} \otimes m) \qquad m \circ (\eta \otimes \mathrm{id}_V) = \mathrm{id}_V = m \circ (\mathrm{id}_V \otimes \eta).$$

(c) **coassociativity and counitality:**
coincide with the coassociativity and counitality condition from Definition 4.3.7

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta \qquad l_V \circ (\epsilon \otimes \mathrm{id}_V) \circ \Delta = \mathrm{id}_V = r_V \circ (\mathrm{id}_V \otimes \epsilon) \circ \Delta.$$

(d) **Frobenius relation:** coincide with condition that Δ is $A \otimes A^{op}$-linear from Definition 4.3.7

$$(\mathrm{id}_V \otimes m) \circ (\Delta \otimes \mathrm{id}_V) = \Delta \circ m = (m \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes \Delta)$$

Together (b), (c), (d) state that $(V, m, \eta, \Delta, \epsilon)$ is a (Δ, ϵ)-Frobenius algebra and hence a Frobenius algebra by Lemma 4.3.8.

(f) **relations for the exchanging cylinder:** The last two relations state that the linear map $\tau : V \otimes V \to V \otimes V$ is an involution and defines a functor $\mathcal{S} \to \mathrm{Vect}_{\mathbb{F}}^{fd}$. The remaining ones are

$$\begin{aligned} &\tau \circ (\eta \otimes \mathrm{id}_V) = \mathrm{id}_V \otimes \eta, && (m \otimes \mathrm{id}) \circ (\mathrm{id}_V \otimes \tau) \circ (\tau \otimes \mathrm{id}_V) = \tau \circ (\mathrm{id}_V \otimes m), \\ &(\mathrm{id}_V \otimes \epsilon) \circ \tau = \epsilon \otimes \mathrm{id}_V && (\tau \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes \tau) \circ \Delta = (\mathrm{id}_V \otimes \Delta) \circ \tau. \end{aligned}$$

We conclude that τ is the flip map $\tau : V \otimes V \to V \otimes V$, $v \otimes v' \mapsto v' \otimes v$.

(e) **commutativity and cocommutativity relations:** They state that the (Δ, ϵ)-Frobenius algebra $(V, m, \eta, \Delta, \epsilon)$ is commutative and cocommutative:

$$m \circ \tau = m \qquad \tau \circ \Delta = \Delta.$$

One can show that a (Δ, ϵ)-Frobenius algebra is commutative and cocommutative if and only if the associated Frobenius algebra is commutative. This shows that oriented topological quantum field theories $Z : \mathrm{Cob}_{2,1} \to \mathrm{Vect}_{\mathbb{F}}$ correspond to commutative Frobenius algebras.

Due to the conditions in Definition 1.1.11, 2. a monoidal natural isomorphism $\phi : Z \to Z'$ between two topological quantum field theories Z and Z' is specified uniquely by the linear map $\phi_O : V = Z(O) \to Z'(O) = V'$. The naturality of ϕ implies that the map ϕ_O is an algebra and coalgebra isomorphism, which is the case if and only if ϕ is an algebra isomorphism that preserves the Frobenius form. Conversely, every algebra isomorphism $\phi : V = Z(O) \to Z'(O) = V'$ defines a monoidal natural isomorphism $\phi : Z \to Z'$. $\square$

5 Bialgebras and Hopf algebras

5.1 Bialgebras

In this section we focus on monoidal categories and rigid monoidal categories that arise from representations of algebras over a field $\mathbb{F}$. As already noted in Example 1.1.14, the category $\mathbb{F}[G]-\mathrm{Mod}$ of modules over a group algebra $\mathbb{F}[G]$ is a monoidal category with the tensor product, tensor unit, associators and unit constraints from $\mathrm{Vect}_{\mathbb{F}}$. By Example 2.1.3 the full subcategory $\mathbb{F}[G]-\mathrm{Mod}^{fd}$ of finite-dimensional modules is a rigid monoidal category when equipped with the left and right evaluations and coevaluations of $\mathrm{Vect}_{\mathbb{F}}^{fd}$.

These statements do not hold for the category $A-\mathrm{Mod}$ of modules over a general algebra A over $\mathbb{F}$. In this case there is no canonical A-module structure on the field $\mathbb{F}$, the tensor product of two A-modules over $\mathbb{F}$ does not inherit an A-module structure, and there is no canonical A-module structure on the dual vector space V^* of an A-module V.

We start by investigating which additional structure on A is needed to ensure that the category $A-\mathrm{Mod}$ is a monoidal category when equipped with the the tensor product over $\mathbb{F}$. Clearly, this requires an A-module structure on the tensor unit $\mathbb{F}$ in $\mathrm{Vect}_{\mathbb{F}}$ and an A-module structure on the tensor product over $\mathbb{F}$ of any two A-modules.

An A-module structure $\rho_{\mathbb{F}}: A \to \mathrm{End}_{\mathbb{F}}(\mathbb{F})$ is given by an algebra homomorphism $\epsilon : A \to \mathbb{F}$ with $\rho_{\mathbb{F}}(a)\lambda = \epsilon(a)\lambda$. If we require A-module structures on all tensor products of A-modules over $\mathbb{F}$, we require in particular an A-module structure $\rho_A : A \to \mathrm{End}_{\mathbb{F}}(A\otimes A)$, where A is viewed as an A-module over itself with the left multiplication. This defines an algebra homomorphism $\Delta : A \to A\otimes A$, $a \mapsto \rho(a)(1\otimes 1)$. Given such an algebra homomorphism and two A-modules V, W, we obtain an A-module structure $\rho_{V\otimes W} = (\rho_V\otimes\rho_W)\circ\Delta : A \to \mathrm{End}_{\mathbb{F}}(V\otimes W)$ on $V\otimes W$.

To ensure that $A-\mathrm{Mod}$ inherits the monoidal structure from $\mathrm{Vect}_{\mathbb{F}}$, we need to impose that the associativity isomorphisms $a_{U,V,W} : (U\otimes V)\otimes W \to U\otimes(V\otimes W)$ and the left and right unit isomorphisms $r_V : V\otimes\mathbb{F} \to V$ and $l_V : \mathbb{F}\otimes V \to V$ in $\mathrm{Vect}_{\mathbb{F}}$ are homomorphisms of A-modules. This imposes additional conditions on the algebra homomorphisms ϵ and Δ.

The A-module structures on the tensor products $U\otimes(V\otimes W)$ and on $U\otimes(V\otimes W)$ are given by

$$\rho_{(U\otimes V)\otimes W} = ((\rho_U\otimes\rho_V)\otimes\rho_W)\circ(\Delta\otimes\mathrm{id}_A)\circ\Delta \qquad \rho_{U\otimes(V\otimes W)} = ((\rho_U\otimes\rho_V)\otimes\rho_W)\circ(\mathrm{id}_A\otimes\Delta)\circ\Delta.$$

From these expressions, it follows that the associator $a_{U,V,W}$ is an isomorphism of A-modules if

$$a_{A,A,A}\circ(\Delta\otimes\mathrm{id}_A)\circ\Delta = (\mathrm{id}_A\otimes\Delta)\circ\Delta. \tag{43}$$

By setting $U = V = W = A$ and applying the endomorphisms $\rho_{(A\otimes A)\otimes A}(a)$ to $(1\otimes 1)\otimes 1$ and $\rho_{A\otimes(A\otimes A)}(a)$ to $1\otimes(1\otimes 1)$, we find that this condition is not just sufficient, but necessary. Similarly, the representations of A on $V\otimes\mathbb{F}$ and $\mathbb{F}\otimes V$ are given by

$$\rho_{V\otimes\mathbb{F}} = (\rho_V\otimes\rho_{\mathbb{F}})\circ\Delta \qquad \rho_{\mathbb{F}\otimes V} = (\rho_{\mathbb{F}}\otimes\rho_V)\circ\Delta,$$

and $r_V : V\otimes\mathbb{F} \to V$ and $l_V : \mathbb{F}\otimes V \to V$. The condition

$$l_A\circ(\epsilon\otimes\mathrm{id}_A)\circ\Delta = r_A\circ(\mathrm{id}_A\otimes\epsilon)\circ\Delta = \mathrm{id}_A. \tag{44}$$

ensures that r_V and l_V are isomorphisms of A-modules. By setting $V = A$ and applying them to the elements $1\otimes 1$, one finds that these conditions are also necessary.

A vector space A over $\mathbb{F}$ together with linear maps $\epsilon : A \to \mathbb{F}$ and $\Delta : A \to A\otimes A$ subject to (43) and (44) is called a *coalgebra*. If we also require that the linear maps $\epsilon : A \to \mathbb{F}$ and $\Delta : A \to A\otimes A$ are *algebra homomorphisms*, we obtain the concept of a *bialgebra*.

Definition 5.1.1:

1. A **coalgebra** over a field $\mathbb{F}$ is a triple (C, Δ, ϵ) of an $\mathbb{F}$-vector space C and linear maps $\Delta : C \to C\otimes C$, $\epsilon : C \to \mathbb{F}$, the **comultiplication** and the **counit**, such that the following diagrams commute

$$\begin{tikzcd}
 & (C\otimes C)\otimes C \arrow[dl, "a_{C,C,C}"'] & & C\otimes C \arrow[ll, "\Delta\otimes\mathrm{id}"'] \\
C\otimes(C\otimes C) & & C\otimes C \arrow[ll, "\mathrm{id}\otimes\Delta"] & C \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{F}\otimes C & C \arrow[l, "c\mapsto 1\otimes c"', "\cong"] \arrow[r, "c\mapsto c\otimes 1", "\cong"'] \arrow[d, "\Delta"] & C\otimes\mathbb{F} \\
 & C\otimes C \arrow[ul, "\epsilon\otimes\mathrm{id}"] \arrow[ur, "\mathrm{id}\otimes\epsilon"'] &
\end{tikzcd}$$

coassociativity **counitality**

A coalgebra (C, Δ, ϵ) is called **cocommutative** if $\Delta^{op} := \tau \circ \Delta = \Delta$, where $\tau : C\otimes C \to C\otimes C$, $c\otimes c' \mapsto c'\otimes c$ is the **flip map**.

2. A **homomorphism of coalgebras** or **coalgebra map** from $(C, \Delta_C, \epsilon_C)$ to $(D, \Delta_D, \epsilon_D)$ is a linear map $\phi : C \to D$ for which the following diagrams commute

$$\begin{tikzcd}
C\otimes C \arrow[d, "\phi\otimes\phi"'] & C \arrow[l, "\Delta_C"'] \arrow[d, "\phi"] \\
D\otimes D & D \arrow[l, "\Delta_D"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{F} \arrow[d, "\mathrm{id}"'] & C \arrow[l, "\epsilon_C"'] \arrow[d, "\phi"] \\
\mathbb{F} & D \arrow[l, "\epsilon_D"]
\end{tikzcd}$$

Note that the comultiplication Δ is a *structure* on C, whereas the counit is a *property*. One can show that for each pair (C, Δ) there is at most one linear map $\epsilon : C \to \mathbb{F}$ that satisfies the counitality condition (Exercise 34).

Note also that the commuting diagrams in Definition 5.1.1 are obtained from the corresponding diagrams for algebras and algebra homomorphisms in Definition A.3 by reversing all arrows labelled by m or η and labelling them with Δ and ϵ instead. In this sense, the concepts of an algebra and a coalgebra are dual to each other, which motivates the name *coalgebra*.

Remark 5.1.2: For a coalgebra (C, Δ, ϵ) we use the symbolic notation $\Delta(c) = \Sigma_{(c)}\, c_{(1)}\otimes c_{(2)}$, which stands for a finite sum $\Delta(c) = \Sigma_{i=1}^n c_i\otimes c_i'$ with $c_i, c_i' \in C$. It is called **Sweedler notation**.

It is symbolic since the properties of the tensor product imply that the elements $c_{(1)}$ and $c_{(2)}$ are not defined uniquely. However, this ambiguity causes no problems as long as all maps composed with Δ are $\mathbb{F}$-linear. The coassociativity of Δ then implies for all $c \in C$

$$(\Delta\otimes\mathrm{id}) \circ \Delta(c) = \Sigma_{(c)} c_{(1)(1)}\otimes c_{(1)(2)}\otimes c_{(2)} = \Sigma_{(c)} c_{(1)}\otimes c_{(2)(1)}\otimes c_{(2)(2)} = (\mathrm{id}\otimes\Delta) \circ \Delta(c).$$

This allows us to renumber the factors in the tensor product as

$$\Sigma_{(c)} c_{(1)}\otimes c_{(2)}\otimes c_{(3)} = \Sigma_{(c)} c_{(1)(1)}\otimes c_{(1)(2)}\otimes c_{(2)} = \Sigma_{(c)} c_{(1)}\otimes c_{(2)(1)}\otimes c_{(2)(2)}$$

and similarly for higher composites of Δ.

Example 5.1.3:

1. For any coalgebra (C, Δ, ϵ), the opposite comultiplication $\Delta^{op} = \tau \circ \Delta : C \to C \otimes C$ defines another coalgebra structure on C with counit ϵ. The coalgebra $(C, \Delta^{op}, \epsilon)$ is called the **opposite coalgebra** and denoted C^{cop}.

2. For any pair of coalgebras $(C, \Delta_C, \epsilon_C)$ and $(D, \Delta_D, \epsilon_D)$ the vector space $C \otimes D$ has a canonical coalgebra structure given by

$$\begin{aligned} &\Delta_{C\otimes D} = \tau_{23} \circ (\Delta_C \otimes \Delta_D) : C\otimes D \to (C\otimes D)\otimes(C\otimes D) && c\otimes d \mapsto \Sigma_{(c),(d)}\, c_{(1)}\otimes d_{(1)}\otimes c_{(2)}\otimes d_{(2)} \\ &\epsilon_{C\otimes D} = \epsilon_C \otimes \epsilon_D : C\otimes D \to \mathbb{F} && c\otimes d \mapsto \epsilon_C(c)\epsilon_D(d), \end{aligned}$$

   with $\tau_{23} : C\otimes C\otimes D\otimes D \to C\otimes D\otimes C\otimes D$, $c\otimes c'\otimes d\otimes d' \mapsto c\otimes d\otimes c'\otimes d'$. This coalgebra structure on $C\otimes D$ is called the **tensor product** of the coalgebras C, D.

3. If $(A, m, \cdot)$ is a finite-dimensional algebra over $\mathbb{F}$, then the dual vector space A^* has a coalgebra structure (A^*, m^*, η^*), where $m^* : A^* \to (A\otimes A)^* = A^*\otimes A^*$ and $\eta^* : A^* \to \mathbb{F}$ are the duals of the multiplication and unit map given by

$$m^*(\alpha)(a\otimes b) = \alpha(ab) \qquad \eta^*(\alpha) = \alpha(1) \qquad \forall \alpha \in A^*, a, b \in A.$$

   If A is infinite-dimensional, then the dual of the multiplication is a linear map $m^* : A^* \to (A\otimes A)^*$. However, in this case we can have $A^*\otimes A^* \subsetneq (A\otimes A)^*$, and then m^* does not define a coalgebra structure on A^*. However, we obtain a coalgebra structure on the **finite dual** $A^\circ = \{\alpha \in A^* \mid m^*(\alpha) \in A^*\otimes A^*\}$ (Exercise).

4. The dual statement of 3. holds also in the infinite-dimensional case. If (C, Δ, ϵ) is a coalgebra over $\mathbb{F}$, then the $(C^*, \Delta^*|_{C^*\otimes C^*}, \epsilon^*)$ is an algebra over $\mathbb{F}$.

5. We consider the algebra $\mathrm{Mat}(n\times n, \mathbb{F})$ with the basis given by the elementary matrices E_{ij} that have the entry 1 in the ith row and jth column and zero elsewhere. The dual basis of $\mathrm{Mat}(n\times n, \mathbb{F})^*$ is given by the **matrix elements** $M_{ij} : \mathrm{Mat}(n\times n, \mathbb{F}) \to \mathbb{F}$, $M \mapsto m_{ij}$. The comultiplication and counit of $\mathrm{Mat}(n \times n, \mathbb{F})^*$ are given by

$$\Delta(M_{ij}) = \Sigma_{k=1}^n M_{ik}\otimes M_{kj} \qquad \epsilon(M_{ij}) = \delta_{ij}.$$

6. By Definition 4.3.7 every (Δ, ϵ)-Frobenius algebra and hence by Lemma 4.3.8 every Frobenius algebra over $\mathbb{F}$ is a coalgebra over $\mathbb{F}$.

As we can view a coalgebra as the dual of an algebra, we can also introduce subcoalgebras and left, right and two-sided coideals by dualising the concepts of subalgebras, left, right and two-sided ideals. In particular, we can take the quotient of a coalgebra by a two-sided coideal and obtain another coalgebra.

Definition 5.1.4: Let (C, Δ, ϵ) be a coalgebra.

1. A **subcoalgebra** of C is a linear subspace $I \subset C$ with $\Delta(I) \subset I\otimes I$.

2. A **left coideal** in C is a linear subspace $I \subset C$ with $\Delta(I) \subset C\otimes I$, a **right coideal** is a linear subspace $I \subset C$ with $\Delta(I) \subset I\otimes C$ and a **coideal** is a linear subspace $I \subset C$ with $\Delta(I) \subset I\otimes C + C\otimes I$ and $\epsilon(I) = 0$.

Proposition 5.1.5: If C is a coalgebra and $I \subset C$ a coideal, then the quotient space C/I inherits a canonical coalgebra structure with the following **characteristic property**:

The canonical surjection $\pi : C \to C/I$ is a coalgebra map. For any coalgebra map $\phi : C \to D$ with $\ker(\phi) \subset I$ there is a unique coalgebra map $\tilde{\phi} : C/I \to D$ with $\tilde{\phi} \circ \pi = \phi$.

Proof:
As $I \subset C$ is a coideal, the map $\Delta' : C/I \to C/I \otimes C/I$, $c + I \mapsto (\pi \otimes \pi)\Delta(c)$ is well defined and satisfies $\Delta' \circ \pi = (\pi \otimes \pi) \circ \Delta$. Its coassociativity follows directly from the coassociativity of Δ and the surjectivity of π

$$\begin{aligned}(\Delta' \otimes \mathrm{id}) \circ \Delta' \circ \pi &= (\Delta' \otimes \mathrm{id}) \circ (\pi \otimes \pi) \circ \Delta = (\pi \otimes \pi \otimes \pi) \circ (\Delta \otimes \mathrm{id}) \circ \Delta \\ &= (\pi \otimes \pi \otimes \pi) \circ (\mathrm{id} \otimes \Delta) \circ \Delta = (\mathrm{id} \otimes \Delta') \circ (\pi \otimes \pi) \circ \Delta = (\mathrm{id} \otimes \Delta') \circ \Delta' \circ \pi.\end{aligned}$$

As I is a coideal, we have $I \subset \ker(\epsilon)$ by definition and obtain a linear map $\epsilon' : C/I \to \mathbb{F}$ with $\epsilon' \circ \pi = \epsilon$. The counitality of ϵ' then follows from the counitality of ϵ and the surjectivity of π:

$$\begin{aligned}(\epsilon' \otimes \mathrm{id}) \circ \Delta' \circ \pi &= (\epsilon' \otimes \mathrm{id}) \circ (\pi \otimes \pi) \circ \Delta = (\mathrm{id} \otimes \pi) \circ (\epsilon \otimes \mathrm{id}) \circ \Delta = 1_{\mathbb{F}} \otimes \pi \\ (\mathrm{id} \otimes \epsilon') \circ \Delta' \circ \pi &= (\mathrm{id} \otimes \epsilon') \circ (\pi \otimes \pi) \circ \Delta = (\mathrm{id} \otimes \pi) \circ (\mathrm{id} \otimes \epsilon) \circ \Delta = 1_{\mathbb{F}} \otimes \pi.\end{aligned}$$

The canonical surjection is a coalgebra map by definition of the coalgebra structure on C/I. The rest of the characteristic property follows directly from the characteristic property of the quotient spaces and the definition of the coalgebra structure. □

In a similar manner, we can dualise the concept of a *module* over an *algebra* to obtain the notion of a *comodule* over a *coalgebra*. One also can define subcomodules, quotients of comodules by subcomodules and related structures. All of them are obtained by taking the defining commuting diagrams for modules over algebras and reversing all arrows labelled by m, η and $\rhd$.

Definition 5.1.6: Let (C, Δ, ϵ) be a coalgebra over $\mathbb{F}$.

1. A **left comodule** over C is a pair (V, δ) of a vector space V over $\mathbb{F}$ and a linear map $\delta : V \to C \otimes V$ such that the following diagrams commute

$$\begin{tikzcd} & C\otimes(C\otimes V) & C\otimes V \arrow[l, "\mathrm{id}\otimes\delta"'] \\ (C\otimes C)\otimes V \arrow[ur, "a_{C,C,V}"] & C\otimes V \arrow[l, "\Delta\otimes\mathrm{id}"'] & V \arrow[l, "\delta"'] \arrow[u, "\delta"'] \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "v\mapsto 1\otimes v", "\cong"'] \arrow[d, "\delta"'] & \mathbb{F}\otimes V \\ C\otimes V. \arrow[ur, "\epsilon\otimes\mathrm{id}"'] & \end{tikzcd}$$

2. A **homomorphism of left comodules** or an C-**colinear map** from (V, δ_V) to (W, δ_W) is an $\mathbb{F}$-linear map $\phi : V \to W$ for which the following diagram commutes

$$\begin{tikzcd} C\otimes V \arrow[d, "\mathrm{id}\otimes\phi"'] & V \arrow[l, "\delta_V"'] \arrow[d, "\phi"] \\ C\otimes W & W. \arrow[l, "\delta_W"'] \end{tikzcd}$$

Analogously, one defines right comodules over C as left comodules over C^{cop} and (C, C)-bicomodules as left comodules over $C \otimes C^{cop}$.

One often uses a variant of Sweedler notation and denotes the map $\delta : V \to V \otimes C$ for a *right* C-comodule V by $\delta(v) = \Sigma_{(v)} v_{(0)} \otimes v_{(1)}$, which stands for as a finite sum over elements of $C \otimes V$. By definition of a *right* comodule, one then has

$$(\delta \otimes \mathrm{id}_C) \circ \delta(v) = \Sigma_{(v)} v_{(0)(0)} \otimes v_{(0)(1)} \otimes v_{(1)} =: \Sigma_{(v)} v_{(0)} \otimes v_{(1)} \otimes v_{(2)} := \Sigma_{(v)} v_{(0)} \otimes v_{(1)(1)} \otimes v_{(1)(2)} = (\mathrm{id} \otimes \Delta) \circ \delta(v).$$

Analogously, for a *left* C-comodule V with $\delta : V \to C \otimes V$ one writes $\delta(v) = \Sigma_{(v)} v_{(1)} \otimes v_{(0)}$ and

$$(\mathrm{id}_C \otimes \delta) \circ \delta(v) = \Sigma_{(v)} v_{(1)} \otimes v_{(0)(1)} \otimes v_{(0)(0)} =: \Sigma_{(v)} v_{(2)} \otimes v_{(1)} \otimes v_{(0)} := \Sigma_{(v)} v_{(1)(1)} \otimes v_{(1)(2)} \otimes v_{(0)} = (\Delta \otimes \mathrm{id}) \circ \delta(v).$$

Example 5.1.7:

1. Every coalgebra (C, Δ, ϵ) is a left and right comodule over itself with the comultiplication $\delta = \Delta : C \to C \otimes C$. This gives C the structure of a (C, C)-bicomodule.
2. If V is a left comodule over a coalgebra (C, Δ, ϵ) with $\delta : V \to C \otimes V$, $v \mapsto \Sigma_{(v)} v_{(1)} \otimes v_{(0)}$, then it is a right module over the algebra $(C^*, \Delta^*|_{C^* \otimes C^*}, \epsilon^*)$ with $\triangleleft : V \otimes C^* \to V$, $v \otimes \alpha \mapsto \Sigma_{(v)} \alpha(v_{(1)})\, v_{(0)}$. However, not every right module over $(C^*, \Delta^*, \epsilon^*)$ arises from a comodule over C, if C is infinite-dimensional. The modules that arise in this way are called **rational modules**.
3. If (C, Δ, ϵ) is a coalgebra and $I \subset C$ a linear subspace, then the comultiplication of C induces a left (right) comodule structure on the quotient C/I if and only if I is a left (right) coideal in C (Exercise 34).

The statements in Proposition 5.1.5 and in Example 5.1.7, 3. justify the definition of left coideal, right coideal and coideal in Definition 5.1.4, which seem slightly odd at first sight. They are defined in such a way that quotients of coalgebras by coideals have properties analogous to quotients of algebras by ideals: quotients of (co)algebras by (co)ideals yield (co)algebras, and quotients by left or right (co)ideals yield left or right (co)modules.

If we require that a coalgebra over $\mathbb{F}$ also has an algebra structure and that its comultiplication and counit are algebra homomorphisms, we obtain the notion of a bialgebra. Note that the condition that comultiplication and the counit are algebra homomorphisms is equivalent to imposing that the multiplication and unit are coalgebra homomorphisms. Hence, the coalgebra structure and the algebra structure enter the definition of a bialgebra on an equal footing.

Definition 5.1.8:

1. A **bialgebra** over a field $\mathbb{F}$ is a pentuple $(B, m, \eta, \Delta, \epsilon)$ such that (B, m, η) is an algebra over $\mathbb{F}$, (B, Δ, ϵ) is a coalgebra over $\mathbb{F}$ and $\Delta : B \to B \otimes B$ and $\epsilon : B \to \mathbb{F}$ are algebra homomorphisms.
2. A **bialgebra homomorphism** from $(B, m, \eta, \Delta, \epsilon)$ to $(B', m', \eta', \Delta', \epsilon')$ is a linear map $\phi : B \to B'$ that is a homomorphism of algebras and a homomorphism of coalgebras:

$$m' \circ (\phi \otimes \phi) = \phi \circ m \qquad \phi \circ \eta = \eta' \qquad \Delta' \circ \phi = (\phi \otimes \phi) \circ \Delta \qquad \epsilon' \circ \phi = \epsilon.$$

Example 5.1.9:

1. For any bialgebra $(B, m, \eta, \Delta, \epsilon)$, reversing the multiplication or the comultiplication yields another bialgebra structure on B. The three new bialgebras obtained in this way are $B^{op} := (B, m^{op}, \eta, \Delta, \epsilon)$, $B^{cop} := (B, m, \eta, \Delta^{op}, \epsilon)$ and $B^{op,cop} := (B, m^{op}, \eta, \Delta^{op}, \epsilon)$.

2. For any two bialgebras B, C over $\mathbb{F}$, the vector space $B \otimes C$ becomes a bialgebra when equipped with the tensor product algebra and coalgebra structures. This is called the **tensor product bialgebra** and denoted $B \otimes C$.

3. For any finite-dimensional bialgebra $(B, m, \eta, \Delta, \epsilon)$, the dual vector space has a canonical bialgebra structure given by $(B^*, \Delta^*, \epsilon^*, m^*, \eta^*)$. If B is infinite-dimensional, the **finite dual** $B^\circ = \{b \in B \,|\, m^*(b) \in B^* \otimes B^*\}$ is a bialgebra with the restriction of the maps $m^* : B^* \to B^* \otimes B^*$, $\eta^* : B^* \to \mathbb{F}$, $\Delta^* : B^* \otimes B^* \to B^*$ and $\epsilon^* : \mathbb{F} \to B^*$ (Exercise).

4. A **subbialgebra** of a bialgebra B is a linear subspace $U \subset B$ that is a subalgebra and a subcoalgebra of B.

Theorem 5.1.10: Let $(B, m, \eta, \Delta, \epsilon)$ be a bialgebra over $\mathbb{F}$. Then $B-\mathrm{Mod}$ has the structure of a monoidal category that makes the forgetful functor $F : B-\mathrm{Mod} \to \mathrm{Vect}_{\mathbb{F}}$ monoidal.

Proof:
We define the B-module structure on $\mathbb{F}$ and the B-module structure on the tensor product of two A-modules V, W by

$$\rhd_{\mathbb{F}} : B \otimes \mathbb{F} \to \mathbb{F}, \quad b \otimes \lambda \mapsto \epsilon(b)\lambda$$
$$\rhd_{V \otimes W} : B \otimes (V \otimes W) \to V \otimes W,\ b \otimes (v \otimes w) \mapsto \Sigma_{(b)} (b_{(1)} \rhd_V v) \otimes (b_{(2)} \rhd_W w).$$

The fact that Δ and ϵ are algebra homomorphism ensures that these are indeed B-module structures. We define the functor $\otimes : B-\mathrm{Mod} \times B-\mathrm{Mod} \to B-\mathrm{Mod}$ on the objects by $(V, \rhd_V) \otimes (W, \rhd_W) = (V \otimes W, \rhd_{V \otimes W})$ and as the usual tensor product of $\mathbb{F}$-linear maps on the morphisms. A direct computation shows that for any two B-linear maps $f : V \to V'$ and $g : W \to W'$, the map $f \otimes g : V \otimes W \to V' \otimes W'$ is again B-linear:

$$\begin{aligned} b \rhd_{V' \otimes W'} (f \otimes g)(v \otimes w) &= \Sigma_{(b)} (b_{(1)} \rhd_{V'} f(v)) \otimes (b_{(2)} \rhd_{W'} g(w)) \\ &= \Sigma_{(b)} (f(b_{(1)} \rhd_V v)) \otimes (g(b_{(2)} \rhd_{W'} w)) = (f \otimes g)(b \rhd_{V \otimes W} (v \otimes w)). \end{aligned}$$

As the identity map on any B-module and the composite of two B-linear maps are again B-linear, this defines a functor $\otimes : B-\mathrm{Mod} \times B-\mathrm{Mod} \to B-\mathrm{Mod}$.

The associator is given by its component morphisms $a_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$, $(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w)$, the tensor unit by $e = (\mathbb{F}, \rhd_{\mathbb{F}})$ and the left and right unit constraints by their component morphisms $l_V : \mathbb{F} \otimes V \to V$, $\lambda \otimes v \mapsto \lambda v$ and $r_V : V \otimes \mathbb{F} \to V$, $v \otimes \lambda \mapsto \lambda v$.

That the associator and the unit constraints are morphisms of B-modules follows from the coassociativity and counitality of Δ and ϵ, as shown at the beginning of this section. That they satisfy the pentagon and triangle axiom follows from the pentagon and triangle axiom in $\mathrm{Vect}_{\mathbb{F}}$. This shows that $B-\mathrm{Mod}$ is a monoidal category.

That the forgetful functor $F : B-\mathrm{Mod} \to \mathrm{Vect}_{\mathbb{F}}$ is monoidal follows directly from the definition of the tensor product, the associator and the unit constraints in $B-\mathrm{Mod}$. $\square$

5.2 Hopf algebras

We now investigate which additional properties a bialgebra needs so that the monoidal category $B-\mathrm{Mod}^{fd}$ of *finite-dimensional* B-modules becomes a right rigid, left rigid or rigid monoidal category, when equipped with the coevaluations and evaluations in $\mathrm{Vect}_{\mathbb{F}}$. The evaluation and coevaluation maps for a finite-dimensional vector space V over $\mathbb{F}$ were introduced in Example 2.1.3 and are given by

$$\begin{aligned}
&\mathrm{coev}_V^R : \mathbb{F} \to V \otimes V^*, \quad \lambda \mapsto \lambda \Sigma_{i=1}^n v_i \otimes \alpha^i && \mathrm{ev}_V^R : V^* \otimes V \to \mathbb{F}, \quad \alpha \otimes v \mapsto \alpha(v)\\
&\mathrm{coev}_V^L : \mathbb{F} \to V^* \otimes V, \quad \lambda \mapsto \lambda \Sigma_{i=1}^n \alpha^i \otimes v_i && \mathrm{ev}_V^L : V \otimes V^* \to \mathbb{F}, \quad v \otimes \alpha \mapsto \alpha(v),
\end{aligned}$$

where $(v_1, ..., v_n)$ is a basis of V and $(\alpha^1, ..., \alpha^n)$ the dual basis of V^*.

We thus require an B-module structure on the dual vector space V^* of each finite-dimensional B-module V, such that the right evaluation and coevaluation, the left evaluation and coevaluation or all of them are homomorphisms of B-modules.

For this, note that the B-module structure on V induces a B-*right* module structure on V^* that is given by $\alpha \lhd b = \alpha \circ (b \rhd -)$ for all $\alpha \in V^*$ and $b \in B$. To turn this into a B-left module structure, we need an anti-algebra homomorphism $S : B \to B$ and then define

$$\rhd : B \otimes V^* \to V^*, \quad b \otimes \alpha \mapsto \alpha \circ (S(b) \rhd -).$$

The condition that the right evaluation and coevaluation are B-linear read in Sweedler notation

$$\begin{aligned}
&\mathrm{ev}_V^R(b \rhd (\alpha \otimes v)) = \alpha((\Sigma_{(b)} S(b_{(1)}) \cdot b_{(2)}) \rhd v) = \epsilon(b)\alpha(v) = b \rhd \mathrm{ev}_V^R(\alpha \otimes v) \qquad (45)\\
&b \rhd \mathrm{coev}_V^R(1) = \Sigma_{i=1}^n ((\Sigma_{(b)} b_{(1)} \cdot S(b_{(2)})) \rhd v_i) \otimes \alpha^i = \epsilon(b) \Sigma_{i=1}^n v_i \otimes \alpha^i = \mathrm{coev}_V^R(b \rhd 1),
\end{aligned}$$

and the corresponding conditions for the left evaluation and coevaluation read

$$\begin{aligned}
&\mathrm{ev}_V^L(b \rhd (v \otimes \alpha)) = \alpha((\Sigma_{(b)} S(b_{(2)}) b_{(1)}) \rhd v) = \epsilon(b)\alpha(v) = b \rhd \mathrm{ev}_V^L(v \otimes \alpha) \qquad (46)\\
&b \rhd \mathrm{coev}_V^L(1) = \Sigma_{i=1}^n (\alpha^i \otimes (\Sigma_{(b)} b_{(2)} S(b_{(1)})) \rhd v_i)) = \epsilon(b) \Sigma_{i=1}^n \alpha^i \otimes v^i = \mathrm{coev}_V^L(b \rhd 1),
\end{aligned}$$

where we used the identity $\Sigma_{i=1}^n \phi(v_i) \otimes \alpha^i = \Sigma_{i=1}^n v_i \otimes \phi^*(\alpha^i)$ for any linear map $\phi : V \to V$ and its dual $\phi^* : V^* \to V^*$, $\alpha \mapsto \alpha \circ \phi$. They conditions for the right evaluation and coevaluation are satisfied for all finite-dimensional B-modules V if

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \epsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta \qquad (47)$$

and the ones for the left evaluation and coevaluation if

$$m^{op} \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \epsilon = m^{op} \circ (\mathrm{id} \otimes S) \circ \Delta. \qquad (48)$$

If B is finite dimensional, we can set $V = B$, $v = 1$ in (45) and (46) and work with a basis of B that contains the unit of B as the first basis element: $v_1 = 1$. In this case, it follows that (47) and (48) are not only sufficient, but also necessary conditions.

We will see later that the conditions (47) and (48) are in general not equivalent, and that either of them implies that $S : B \to B$ is an anti-algebra morphism. Imposing condition (47) for the right duals leads to the following definition.

Definition 5.2.1: A bialgebra $(H, m, \eta, \Delta, \epsilon)$ is called a **Hopf algebra** if there is a linear map $S : H \to H$, called the **antipode**, with

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = m \circ (\mathrm{id} \otimes S) \circ \Delta = \eta \circ \epsilon.$$

Remark 5.2.2:

1. In Sweedler notation, the axioms for a Hopf algebra read:

$$\begin{array}{ll} \Sigma_{(a)} a_{(1)(1)} \otimes a_{(1)(2)} \otimes a_{(2)} = \Sigma_{(a)} a_{(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)} & \Sigma_{(a)} \epsilon(a_{(1)}) a_{(2)} = \Sigma_{(a)} a_{(1)} \epsilon(a_{(2)}) = a \\ \Sigma_{(ab)} (ab)_{(1)} \otimes (ab)_{(2)} = \Sigma_{(a)} \Sigma_{(b)} a_{(1)} b_{(1)} \otimes a_{(2)} b_{(2)} & \epsilon(ab) = \epsilon(a)\epsilon(b) \\ \Sigma_{(a)} S(a_{(1)}) a_{(2)} = \Sigma_{(a)} a_{(1)} S(a_{(2)}) = \epsilon(a)\, 1 & \forall a, b \in H. \end{array}$$

2. As indicated by the wording of Definition 5.2.1, the existence of the antipode is a *property* of a given bialgebra $(H, m, \eta, \Delta, \epsilon)$ and not a choice of structure. We will see in Corollary 5.2.5 that there is at most one antipode for a given bialgebra structure on H.
3. Although the antipode replaces the inverse for representations of a finite group, it does not need to be an involution. In fact, it is not even guaranteed to be invertible. A Hopf algebra with $S^2 = \mathrm{id}$ is called **involutive**. We will show in Theorem 6.2.9 that for every *finite-dimensional* Hopf algebra H the antipode is invertible.

The defining condition on the antipode in Definition 5.2.1 is motivated from the representation theoretical viewpoint by right rigidity of $H{-}\mathrm{Mod}^{fd}$. Nevertheless, it is possible to understand it conceptually without representations. This is achieved with the *convolution product* of a bialgebra H, which is defined more generally for a pair of an algebra A and a coalgebra C.

Lemma 5.2.3: Let (A, m, η) be an algebra and (C, Δ, ϵ) a coalgebra over $\mathbb{F}$.

1. The **convolution product** defines an algebra structure on $\mathrm{Hom}_{\mathbb{F}}(C, A)$

$$* : \mathrm{Hom}_{\mathbb{F}}(C, A) \otimes \mathrm{Hom}_{\mathbb{F}}(C, A) \to \mathrm{Hom}_{\mathbb{F}}(C, A), \quad f \otimes g \mapsto f * g = m \circ (f \otimes g) \circ \Delta.$$

2. The convolution invertible elements in $\mathrm{Hom}_{\mathbb{F}}(C, A)$ form a group with unit $\eta \circ \epsilon$.

Proof:
That $*$ is $\mathbb{F}$-linear follows from the $\mathbb{F}$-linearity of Δ, m and the properties of the tensor product. The associativity of $*$ follows from the associativity of m and the coassociativity of Δ

$$\begin{aligned} (f * g) * h &= m \circ ((f * g) \otimes h) \circ \Delta = m \circ (m \otimes \mathrm{id}) \circ (f \otimes g \otimes h) \circ (\Delta \otimes \mathrm{id}) \circ \Delta \\ &= m \circ (\mathrm{id} \otimes m) \circ (f \otimes g \otimes h) \circ (\mathrm{id} \otimes \Delta) \circ \Delta = m \circ (f \otimes (g * h)) \circ \Delta = f * (g * h). \end{aligned}$$

That $\eta \circ \epsilon : C \to A$ is a unit for $*$ follows because η is the unit of A and ϵ the counit of C

$$(\eta \circ \epsilon) * f = m \circ ((\eta \circ \epsilon) \otimes f) \circ \Delta = m \circ (\mathrm{id} \otimes f) \circ (\eta \otimes \mathrm{id}) \circ (\epsilon \otimes \mathrm{id}) \circ \Delta = m \circ (1_A \otimes f) = f$$

$$f * (\eta \circ \epsilon) = m \circ (f \otimes (\eta \circ \epsilon)) \circ \Delta = m \circ (f \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \eta) \circ (\mathrm{id} \otimes \epsilon) \circ \Delta = m \circ (f \otimes 1_A) = f.$$

This shows that the vector space $\mathrm{Hom}_{\mathbb{F}}(C, A)$ with the convolution product is an associative algebra over $\mathbb{F}$. The uniqueness of two-sided inverses and the statement that the elements with a two-sided inverse form a group holds for any monoid. $\square$

If $(B, m, \eta, \Delta, \epsilon)$ is a bialgebra, we can choose $(C, \Delta, \epsilon) = (B, \Delta, \epsilon)$ and $(A, m, \eta) = (B, m, \eta)$ and consider the convolution product on $\mathrm{End}_{\mathbb{F}}(B)$. It is then natural to ask if the element $\mathrm{id}_B \in \mathrm{End}_{\mathbb{F}}(B)$ is convolution invertible. We find that this is the case if and only if B is a Hopf algebra, and then the convolution inverse is the antipode of B. This gives an additional motivation for Definition 5.2.1 and allows us to investigate the properties of the antipode.

Proposition 5.2.4: Let $(B, m, \eta, \Delta, \epsilon)$ and $(B', m', \eta', \Delta', \epsilon')$ bialgebras over $\mathbb{F}$.

1. The identity map $\mathrm{id}_B : B \to B$ is convolution invertible if and only if $(B, m, \eta, \Delta, \epsilon)$ is a Hopf algebra, and in this case the convolution inverse of id_B is the antipode $S : B \to B$.
2. If $f \in \mathrm{Hom}_{\mathbb{F}}(B, B)$ is a convolution invertible algebra homomorphism, then its convolution inverse $f^{-1} : B \to B$ is an anti-algebra homomorphism: $m^{op} \circ (f^{-1} \otimes f^{-1}) = f^{-1} \circ m$.
3. If $f \in \mathrm{Hom}_{\mathbb{F}}(B, B)$ is a convolution invertible coalgebra homomorphism, then its convolution inverse $f^{-1} : B \to B$ is an anti-coalgebra homomorphism: $\Delta \circ f^{-1} = (f^{-1} \otimes f^{-1}) \circ \Delta^{op}$.
4. If $\phi : B \to B'$ is an algebra homomorphism, then $L_\phi : \mathrm{Hom}_{\mathbb{F}}(B, B) \to \mathrm{Hom}_{\mathbb{F}}(B, B')$, $f \mapsto \phi \circ f$ is an algebra homomorphism with respect to the convolution products.
5. If $\psi : B \to B'$ is a coalgebra homomorphism, then $R_\psi : \mathrm{Hom}_{\mathbb{F}}(B', B') \to \mathrm{Hom}_{\mathbb{F}}(B, B')$, $g \mapsto g \circ \psi$ is an algebra homomorphism with respect to the convolution products.

Proof:
1. By definition, the identity map id_B is convolution invertible if and only if there is a linear map $f : B \to B$ with $f * \mathrm{id}_B = m \circ (f \otimes \mathrm{id}_B) \circ \Delta = \eta \circ \epsilon = m \circ (\mathrm{id}_B \otimes f) \circ \Delta = \mathrm{id}_B * f$. This is the defining condition for the antipode in Definition 5.2.1.

2. For any algebra homomorphism $f : B \to B$ with convolution inverse $f^{-1} : B \to B$ we have

$$f^{-1} \circ \eta = m \circ (f^{-1} \circ \eta \otimes f \circ \eta) = m \circ (f^{-1} \otimes f) \circ \Delta \circ \eta = (f^{-1} * f) \circ \eta = \eta \circ \epsilon \circ \eta = \eta,$$

where we used the identities $m \circ (\mathrm{id} \otimes \eta) = \mathrm{id}$ and $f \circ \eta = \eta$ for an algebra homomorphism f in the first step, then the identity $\Delta \circ \eta = \eta \otimes \eta$, the definition of the convolution product and the identity $\epsilon \circ \eta = \mathrm{id}_{\mathbb{F}}$.

To show that $m^{op} \circ (f^{-1} \otimes f^{-1}) = f^{-1} \circ m$, we consider the convolution algebra $\mathrm{Hom}_{\mathbb{F}}(B \otimes B, B)$, where $B \otimes B$ is equipped with the tensor product coalgebra structure from Example 5.1.3, 2. We show that $f^{-1} \circ m$ and $m^{op} \circ (f^{-1} \otimes f^{-1})$ are both convolution inverses of $m \circ (f \otimes f) = f \circ m : B \otimes B \to B$. The uniqueness of the convolution inverse then implies $m^{op} \circ (f^{-1} \otimes f^{-1}) = f^{-1} \circ m$.

To show that both, $f^{-1} \circ m$ and $m^{op} \circ (f^{-1} \otimes f^{-1})$, are convolution inverses of $f \circ m$, we compute

$$\begin{aligned}(f^{-1} \circ m) * (f \circ m) &= m \circ (f^{-1} \circ m \otimes f \circ m) \circ \Delta_{B \otimes B} = m \circ (f^{-1} \otimes f) \circ \Delta \circ m \\ &= (f^{-1} * f) \circ m = \eta \circ \epsilon \circ m = \eta \circ (\epsilon \otimes \epsilon) = \eta \circ \epsilon_{B \otimes B},\end{aligned}$$

and an analogous computation shows that $(f \circ m) * (f^{-1} \circ m) = \eta \circ \epsilon_{B \otimes B}$.

To show the corresponding identity for $m^{op} \circ (f^{-1} \otimes f^{-1})$, we note that in Sweedler notation we have $(f * g)(b) = \Sigma_{(b)} f(b_{(1)}) g(b_{(2)})$ and $\Delta_{B \otimes B}(b \otimes c) = \Sigma_{(b)(c)} b_{(1)} \otimes c_{(1)} \otimes b_{(2)} \otimes c_{(2)}$. This yields

$$\begin{aligned}&(m^{op} \circ (f^{-1} \otimes f^{-1})) * (f \circ m)(b \otimes c) \\ &= \Sigma_{(b)(c)} (m^{op} \circ (f^{-1} \otimes f^{-1}))(b_{(1)} \otimes c_{(1)}) \cdot (f \circ m)(b_{(2)} \otimes c_{(2)}) = \Sigma_{(b)(c)} f^{-1}(c_{(1)}) \cdot f^{-1}(b_{(1)}) \cdot f(b_{(2)} c_{(2)}) \\ &= \Sigma_{(b)(c)} f^{-1}(c_{(1)}) \cdot f^{-1}(b_{(1)}) \cdot f(b_{(2)}) \cdot f(c_{(2)}) = \Sigma_{(b)(c)} f^{-1}(c_{(1)}) \cdot (f^{-1} * f)(b) \cdot f(c_{(2)}) \\ &= \epsilon(b) \Sigma_{(c)} f^{-1}(c_{(1)}) \cdot f(c_{(2)}) = \epsilon(b)(f^{-1} * f)(c) = \epsilon(b)\epsilon(c) 1_B = \eta \circ \epsilon_{B \otimes B}(b \otimes c),\end{aligned}$$

and a similar computation proves $(f \circ m) * (f^{-1} \circ m) = (f * m) \circ (m^{op} \circ (f^{-1} \circ f^{-1}) = \eta \circ \epsilon_{B \otimes B}$.

3. The proof for 3. is analogous to 2. One considers the convolution algebra $\mathrm{Hom}_{\mathbb{F}}(B, B \otimes B)$, where $B \otimes B$ is the tensor product of the algebra B with itself from Example A.5, 5. and proves that $\Delta \circ f^{-1}$ and $(f^{-1} \otimes f^{-1}) \circ \Delta^{op}$ are convolution inverses of $\Delta \circ f = (f \otimes f) \circ \Delta : B \to B \otimes B$ (Exercise).

4. and 5. For any algebra homomorphism $\phi : B \to B'$, coalgebra homomorphism $\psi : B \to B'$ and linear maps $f, g \in \mathrm{End}_{\mathbb{F}}(B)$, $h, k \in \mathrm{End}_{\mathbb{F}}(B')$ one has

$$\phi \circ (f * g) = \phi \circ m \circ (f \otimes g) \circ \Delta = m' \circ (\phi \otimes \phi) \circ (f \otimes g) \circ \Delta = (\phi \circ f) * (\phi \circ g)$$
$$(h * k) \circ \psi = m' \circ (h \otimes k) \circ \Delta' \circ \psi = m' \circ (h \otimes k) \circ (\psi \otimes \psi) \circ \Delta = (h \circ \psi) * (k \circ \psi).$$

As any algebra homomorphism $\phi : B \to B'$ satisfies $L_\phi(\eta \circ \epsilon) = \phi \circ \eta \circ \epsilon = \eta' \circ \epsilon$ and any coalgebra homomorphism $\psi : B \to B'$ satisfies $R_\psi(\eta' \circ \epsilon') = \eta' \circ \epsilon' \circ \psi = \eta' \circ \epsilon$, we estabished that $L_\phi : \mathrm{End}_{\mathbb{F}}(B) \to \mathrm{Hom}_{\mathbb{F}}(B, B')$ and $R_\phi : \mathrm{End}_{\mathbb{F}}(B') \to \mathrm{Hom}_F(B, B')$ are algebra homomorphisms. □

Corollary 5.2.5: (**Properties of the antipode 1**)

1. If a bialgebra $(B, m, \eta, \Delta, \epsilon)$ is a Hopf algebra, then its antipode $S : B \to B$ is unique.
2. The antipode of a Hopf algebra is an anti-algebra and anti-coalgebra homomorphism

$$m^{op} \circ (S \otimes S) = S \circ m \qquad S \circ \eta = \eta \qquad \Delta \circ S = (S \otimes S) \circ \Delta^{op} \qquad \epsilon \circ S = \epsilon.$$

3. If $(B, m, \eta, \Delta, \epsilon, S)$ and $(B', m', \eta', \Delta', \epsilon', S')$ are Hopf algebras, then any bialgebra homomorphism $\phi : B \to B'$ satisfies $S' \circ \phi = \phi \circ S$.

Proof:
1. The first claim follows from the uniqueness of the convolution inverse in Lemma 5.2.3 and because the antipode is the convolution inverse of the identity map by Proposition 5.2.4, 1.

2. The second claim follows from Proposition 5.2.4, 2. and 3., since id_H is an algebra and coalgebra homomorphism.

3. The last claim follows directly from Proposition 5.2.4, 4. and 5. since for any bialgebra homomorphism $\phi : B \to B'$, one has

$$(\phi \circ S) * \phi = L_\phi(S * \mathrm{id}_B) = L_\phi(\eta \circ \epsilon) = \eta' \circ \epsilon = L_\phi(\eta \circ \epsilon) = L_\phi(\mathrm{id}_B * S) = \phi * (\phi \circ S)$$
$$(S' \circ \phi) * \phi = R_\phi(S' * \mathrm{id}_{B'}) = R_\phi(\eta' \circ \epsilon') = \eta' \circ \epsilon = R_\phi(\eta' \circ \epsilon') = R_\phi(\mathrm{id}_{B'} * S') = \phi * (S' \circ \phi).$$

This shows that both, $S' \circ \phi$ and $\phi \circ S$ are convolution inverses of $\phi : B \to B'$ in $\mathrm{Hom}_{\mathbb{F}}(B, B')$. The uniqueness of the convolution inverse then implies $S' \circ \phi = \phi \circ S$. □

Before considering examples, it remains to clarify the dependence of our definitions on the choices involved. Our definition of a Hopf algebra took as the defining condition for the antipode equation (47), which ensures that the *right* evaluation and coevaluation maps are homomorphisms of representations. Note that this is also the defining condition for the antipode of the bialgebra $B^{op,cop} = (m^{op}, \eta, \Delta^{op}, \epsilon)$, since equation (47) is invariant under simultaneously replacing m and Δ by m^{op} and Δ^{op}. In contrast, the corresponding condition (48) for the *left* evaluation and coevaluation is the defining condition on the antipode of the bialgebras $B^{op} = (m^{op}, \eta, \Delta, \epsilon)$ and $B^{cop} = (m, \eta, \Delta^{op}, \epsilon)$.

It turns out that for a Hopf algebra with a bijective antipode $S : H \to H$, switching the conditions (47) and (48) amounts to replacing its antipode by its inverse. Imposing both conditions amounts to the requirement that the antipode is an involution.

Lemma 5.2.6: (**Properties of the antipode 2**)
Let $(H, m, \eta, \Delta, \epsilon, S)$ be a Hopf algebra.

1. If S is invertible, then $m^{op} \circ (S^{-1} \otimes \mathrm{id}) \circ \Delta = m^{op} \circ (\mathrm{id} \otimes S^{-1}) \circ \Delta = \eta \circ \epsilon$.
2. $S^2 = \mathrm{id}_H$ if and only if $m^{op} \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \epsilon = m^{op} \circ (\mathrm{id} \otimes S) \circ \Delta$.
3. If H is commutative or cocommutative, then $S^2 = \mathrm{id}_H$.

Proof:
1. If $S^{-1} : H \to H$ is the inverse of the antipode $S : H \to H$, one has

$$\begin{aligned} S \circ m^{op} \circ (S^{-1} \otimes \mathrm{id}) \circ \Delta &= m \circ (S \otimes S) \circ (S^{-1} \otimes \mathrm{id}) \circ \Delta = m \circ (\mathrm{id} \otimes S) \circ \Delta = \eta \circ \epsilon \\ S \circ m^{op} \circ (\mathrm{id} \otimes S^{-1}) \circ \Delta &= m \circ (S \otimes S) \circ (\mathrm{id} \otimes S^{-1}) \circ \Delta = m \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \epsilon. \end{aligned}$$

As $S \circ \eta \circ \epsilon = \eta \circ \epsilon = S^{-1} \circ \eta \circ \epsilon$, applying S^{-1} to both sides of these equations proves 1.

2. If $S^2 = \mathrm{id}_H$, then $S = S^{-1}$ and from 1. one obtains $m^{op} \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \epsilon = m^{op} \circ (\mathrm{id} \otimes S) \circ \Delta$. To prove the other implication, one computes with the convolution product in $\mathrm{End}_{\mathbb{F}}(H)$

$$\begin{aligned} S * S^2 &= m \circ (S \otimes S^2) \circ \Delta = S \circ m^{op} \circ (\mathrm{id} \otimes S) \circ \Delta = m \circ (\mathrm{id} \otimes S) \circ \Delta^{op} \circ S \qquad (49) \\ S^2 * S &= m \circ (S^2 \otimes S) \circ \Delta = S \circ m^{op} \circ (S \otimes \mathrm{id}) \circ \Delta = m \circ (S \otimes \mathrm{id}) \circ \Delta^{op} \circ S. \end{aligned}$$

If $m^{op} \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \epsilon = m^{op} \circ (\mathrm{id} \otimes S) \circ \Delta$, this implies $S * S^2 = S^2 * S = S \circ \eta \circ \epsilon = \eta \circ \epsilon$ and by uniqueness of the convolution inverse $S^2 = \mathrm{id}_H$. Claim 3. also follows directly from (49) since $m^{op} = m$ or $\Delta^{op} = \Delta$ imply $S * S^2 = S^2 * S = \mathrm{id}_H$ in (49) and hence $S^2 = \mathrm{id}_H$. □

This corollary shows that for any Hopf algebra $(H, m, \eta, \Delta, \epsilon, S)$, reversing the multiplication *and* the comultiplication yields another Hopf algebra structure on H, namely the Hopf algebra $H^{op,cop} = (H, m^{op}, \eta, \Delta^{op}, \epsilon, S)$. If S is *invertible*, then reversing the multiplication *or* the comultiplication and taking the inverse of the antipode yields new Hopf algebra structures $H^{op} = (H, m^{op}, \eta, \Delta, \epsilon, S^{-1})$ and $H^{cop} = (H, m, \eta, \Delta^{op}, \epsilon, S^{-1})$ on H.

By combining this result with the investigation of the right and left evaluation and coevaluation maps at the beginning of this subsection, one obtains sufficient conditions that ensure that the representation category $H\text{-}\mathrm{Mod}^{fd}$ of finite-dimensional modules over a bialgebra H is rigid.

Corollary 5.2.7: Let H be a Hopf algebra over $\mathbb{F}$.

1. The monoidal categories $H\text{-}\mathrm{Mod}^{fd}$ and $H^{op,cop}\text{-}\mathrm{Mod}^{fd}$ are right rigid.
2. The monoidal categories $H^{op}\text{-}\mathrm{Mod}^{fd}$ and $H^{cop}\text{-}\mathrm{Mod}^{fd}$ are left rigid.
3. If the antipode of H is bijective, then $H\text{-}\mathrm{Mod}^{fd}$ is rigid.
4. If the antipode of H is an involution, then right and left duals in $H\text{-}\mathrm{Mod}^{fd}$ coincide.

Proof:
By Theorem 5.1.10 the representation category $H-\mathrm{Mod}$ is a monoidal category, and so is its full subcategory $H-\mathrm{Mod}^{fd}$ of finite-dimensional H-modules. At the beginning of this subsection, we established that the right evaluation and coevaluation of $\mathrm{Vect}_{\mathbb{F}}^{fd}$ become H-linear if H is equipped with a linear map $S: H \to H$ that satisfies (47) and the left evaluation and coevaluation become H-linear if $S: H \to H$ satisfies (48). The former is the case for H and $H^{op,cop}$, the latter for H^{op} and H^{cop}. The last two claims then follow directly from Lemma 5.2.6, 1. and 2. □

With these results on the properties of the antipode, we can now consider our first examples, which are rather trivial but structurally important, because they care used in many constructions. More interesting and advanced examples will be considered in the next subsection.

Example 5.2.8:

1. For any two Hopf algebras H, K over $\mathbb{F}$, the tensor product bialgebra $H \otimes K$ is a Hopf algebra with antipode $S = S_H \otimes S_K$. This is called the **tensor product Hopf algebra** and denoted $H \otimes K$.
2. For any *finite-dimensional* Hopf algebra $(H, m, \eta, \Delta, \epsilon, S)$, the dual bialgebra $(H^*, \Delta^*, \epsilon^*, m^*, \eta^*)$ is a Hopf algebra with antipode S^*. For any Hopf algebra $(H, m, \eta, \Delta, \epsilon, S)$, the finite dual $(H^\circ, \Delta^*|_{H^\circ \otimes H^\circ}, \epsilon^*, m^*|_{H^\circ}, \eta^*|_{H^\circ})$ from Example 5.1.9, 3. is a Hopf algebra with antipode $S^*|_{H^\circ}$ (Exercise).
3. A **Hopf subalgebra** of a Hopf algebra H is a subbialgebra $U \subset H$ with $S(U) \subset U$.

Example 5.2.9: Let G be a group and $\mathbb{F}$ a field.
The group algebra $\mathbb{F}[G]$ is a cocommutative Hopf algebra with the algebra structure from Example A.10, comultiplication $\Delta: \mathbb{F}[G] \to \mathbb{F}[G] \otimes \mathbb{F}[G]$, $g \mapsto g \otimes g$, counit $\epsilon: \mathbb{F}[G] \to \mathbb{F}$, $g \mapsto 1$ and antipode $S: \mathbb{F}[G] \to \mathbb{F}[G]$, $g \mapsto g^{-1}$.
Proof:
As the elements of $\mathbb{F}[G]$ are finite linear combinations $\Sigma_{g\in G} \lambda_g g$ with $\lambda_g \in \mathbb{F}$, it is sufficient to verify that the axioms hold for the basis elements. This follows by a direct computation

$$(\Delta \otimes \mathrm{id}) \circ \Delta(g) = \Delta(g) \otimes g = g \otimes g \otimes g = g \otimes \Delta(g) = (\mathrm{id} \otimes \Delta) \circ \Delta(g)$$
$$(\epsilon \otimes \mathrm{id}) \circ \Delta(g) = \epsilon(g) \otimes g = 1 \otimes g \qquad (\mathrm{id} \otimes \epsilon) \circ \Delta(g) = g \otimes \epsilon(g) = g \otimes 1$$
$$\Delta(g \cdot h) = (gh) \otimes (gh) = (g \otimes g) \cdot (h \otimes h) = \Delta(g) \cdot \Delta(h)$$
$$\epsilon(g \cdot h) = 1 = 1 \cdot 1 = \epsilon(g) \cdot \epsilon(h)$$
$$m \circ (S \otimes \mathrm{id}) \circ \Delta(g) = m(g^{-1} \otimes g) = g^{-1} g = 1 = \eta(\epsilon(g)) = g g^{-1} = m(g \otimes g^{-1}) = m \circ (\mathrm{id} \otimes S) \circ \Delta(g).$$

□

Example 5.2.10: Let G be a finite group and $\mathbb{F}$ a field.
The dual vector space $\mathbb{F}[G]^*$ is the vector space $\mathrm{Map}(G, \mathbb{F})$ of maps $f: G \to \mathbb{F}$ with the pointwise addition and scalar multiplication. In terms of the basis elements $\delta_g: G \to \mathbb{F}$ with $\delta_g(g) = 1$ and $\delta_g(h) = 0$ for $g \neq h$, the dual Hopf algebra $(\mathrm{Map}(G, \mathbb{F}), \Delta^*, \epsilon^*, m^*, \eta^*, S^*)$ is

$$\Delta^*(\delta_g \otimes \delta_h) = \delta_g \cdot \delta_h = \delta_g(h)\, \delta_h \qquad \epsilon^*(\lambda) = \lambda \Sigma_{g\in G} \delta_g$$
$$m^*(\delta_g) = \Sigma_{h \in G} \delta_h \otimes \delta_{h^{-1}g} \qquad \eta^*(\delta_g) = \delta_g(e) \qquad S'(\delta_g) = S^*(\delta_g) = \delta_{g^{-1}}$$

This Hopf algebra is commutative, and its algebra structure is given by the pointwise multiplication of functions $f: G \to \mathbb{F}$.

Proof:
This follows by a direct computation from the definition of the dual Hopf algebra structure. We have for all $g, h, u, v \in G$

$$\begin{aligned}
&\Delta^*(\delta_g \otimes \delta_h)(u) = (\delta_g \otimes \delta_h)(\Delta(u)) = (\delta_g \otimes \delta_h)(u \otimes u) = \delta_g(u)\delta_h(u) = \delta_g(h)\,\delta_h(u)\\
&\epsilon^*(\lambda)(u) = \lambda\epsilon(u) = \lambda = \Sigma_{g\in G}\lambda\delta_g(u)\\
&m^*(\delta_g)(u \otimes v) = \delta_g(u \cdot v) = \Sigma_{h\in G}\delta_h(u)\delta_g(hv) = \Sigma_{h\in G}\delta_h(u)\delta_{h^{-1}g}(v)\\
&\eta^*(\delta_g) = \delta_g(e)\\
&S^*(\delta_g)(u) = \delta_g(S(u)) = \delta_g(u^{-1}) = \delta_{g^{-1}}(u).
\end{aligned}$$

This implies for all $u \in G$ and maps $f_1, f_2 : G \to \mathbb{F}$

$$\begin{aligned}
(f_1 \cdot f_2)(u) &= \Delta^*(f_1 \otimes f_2)(u) = \Sigma_{g,h\in G} f_1(g) f_2(h) \Delta^*(\delta_g \otimes \delta_h)(u) = \Sigma_{g,h\in G} f_1(g) f_2(h)\delta_g(u)\delta_h(u)\\
&= f_1(u) f_2(u).
\end{aligned}$$

□

5.3 Examples

In this section, we consider examples of Hopf algebras, which show that the concept goes beyond group algebras and other familiar constructions. In particular, we construct parameter dependent examples that are non-commutative and non-cocommutative and can be viewed as deformations of other, more basic Hopf algebras. We start with two most basic examples, namely the tensor algebra of a vector space (cf. Example A.6) and the universal enveloping algebra of a Lie algebra (cf. Example A.9).

Example 5.3.1: The tensor algebra $T(V)$ of a vector space V over $\mathbb{F}$ is a cocommutative Hopf algebra over $\mathbb{F}$ with the algebra structure from Example A.6 and the comultiplication, counit and antipode given by

$$\begin{aligned}
\Delta(v_1 \otimes \ldots \otimes v_n) &= \Sigma_{p=0}^n \Sigma_{\sigma\in \mathrm{Sh}(p,n-p)} (v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(p)}) \otimes (v_{\sigma(p+1)} \otimes \ldots \otimes v_{\sigma(n)})\\
\epsilon(v_1 \otimes \ldots \otimes v_n) &= 0 \qquad S(v_1 \otimes ... \otimes v_n) = (-1)^n v_n \otimes ... \otimes v_1,
\end{aligned}$$

where $\mathrm{Sh}(p, q)$ is the set of (p, q)-**shuffle permutations**

$$\sigma \in S_{p+q} \quad \text{with} \quad \sigma(1) < \sigma(2) < ... < \sigma(p) \text{ and } \sigma(p+1) < \sigma(p+2) < ... < \sigma(p+q),$$

and we set $v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(p)} = 1$ for $p = 0$ and $v_{\sigma(p+1)} \otimes \ldots \otimes v_{\sigma(n)} = 1$ for $p = n$.

Proof:
By the universal property of the tensor algebra, see Example A.6, the linear maps

$$\Delta' : V \to T(V) \otimes T(V), \;\; v \mapsto 1 \otimes v + v \otimes 1 \qquad \epsilon' : V \to \mathbb{F}, \;\; v \mapsto 0 \qquad S : V \to T(V)^{op}, \;\; v \mapsto -v$$

induce algebra homomorphisms $\Delta : T(V) \to T(V) \otimes T(V)$, $\epsilon : T(V) \to \mathbb{F}$, $S : T(V) \to T(V)^{op}$ with $\Delta \circ \iota_V = \Delta'$, $\epsilon \circ \iota_V = \epsilon'$ and $S \circ \iota_V = S'$. To show that Δ and ϵ are coassociative and counital and S is an antipode it is sufficient to prove that

$$\begin{aligned}
&(\Delta \otimes \mathrm{id}) \circ \Delta \circ \iota_V = (\mathrm{id} \otimes \Delta) \circ \Delta \circ \iota_V \qquad (50)\\
&l_{T(V)} \circ (\epsilon \otimes \mathrm{id}) \circ \Delta \circ \iota_V = \iota_V = r_{T(V)} \circ (\mathrm{id} \otimes \epsilon) \circ \Delta \circ \iota_V\\
&m \circ (S \otimes \mathrm{id}) \circ \Delta \circ \iota_V = \eta \circ \epsilon \circ \iota_V = m \circ (\mathrm{id} \otimes S) \circ \Delta \circ \iota_V
\end{aligned}$$

by the universal property of the tensor algebra. As Δ, ϵ are algebra homomorphisms and S is an anti-algebra homomorphism, they satisfy $\Delta(1_\mathbb{F}) = 1_\mathbb{F}\otimes 1_\mathbb{F}$, $\epsilon(1_\mathbb{F}) = 1$ and $S(1_\mathbb{F}) = 1_\mathbb{F}$ for the unit $1_\mathbb{F}$ of $T(V)$. The identities (50) follow by a direct computation from the expressions for Δ', ϵ', S'.

The formulas for the comultiplication, counit and antipode follow by induction over n. If they hold for all products of vectors in V of length $\leq n$, then we have by definition of Δ, ϵ and S

$$\begin{aligned}
&\epsilon(v_1\otimes ...\otimes v_{n+1}) = \epsilon(v_1\otimes ...\otimes v_n) \cdot \epsilon(v_{n+1}) = 0\\
&S(v_1\otimes ...\otimes v_{n+1}) = S(v_{n+1})\otimes S(v_1\otimes ...\otimes v_n) = (-1)^{n+1} v_{n+1}\otimes v_n\otimes ...\otimes v_1\\
&\Delta(v_1\otimes ...\otimes v_{n+1}) = \Delta(v_1\otimes ...\otimes v_n) \cdot \Delta(v_{n+1})\\
&= \left(\Sigma_{p=0}^n \Sigma_{\sigma\in \mathrm{Sh}(p,n-p)} \left(v_{\sigma(1)}\otimes ...\otimes v_{\sigma(p)}\right)\otimes\left(v_{\sigma(p+1)}\otimes ...\otimes v_{\sigma(n)}\right)\right) \cdot (v_{n+1}\otimes 1 + 1\otimes v_{n+1})\\
&= \Sigma_{p=0}^n \Sigma_{\sigma\in \mathrm{Sh}(p,n-p)} \left(v_{\sigma(1)}\otimes ...\otimes v_{\sigma(p)}\otimes v_{n+1}\right)\otimes\left(v_{\sigma(p+1)}\otimes ...\otimes v_{\sigma(n)}\right)\\
&+ \Sigma_{p=0}^n \Sigma_{\sigma\in \mathrm{Sh}(p,n-p)} \left(v_{\sigma(1)}\otimes ...\otimes v_{\sigma(p)}\right)\otimes\left(v_{\sigma(p+1)}\otimes ...\otimes v_{\sigma(n)}\otimes v_{n+1}\right)\\
&= \Sigma_{p=0}^{n+1} \Sigma_{\sigma\in \mathrm{Sh}(p,n+1-p)} \left(v_{\sigma(1)}\otimes ...\otimes v_{\sigma(p)}\right)\otimes\left(v_{\sigma(p+1)}\otimes ...\otimes v_{\sigma(n+1)}\right).
\end{aligned}$$

In the last step we used that every shuffle permutation $\sigma \in \mathrm{Sh}(p, n-p)$ defines a shuffle permutation $\sigma' \in \mathrm{Sh}(p+1, n-p)$ and a shuffle permutation $\sigma'' \in \mathrm{Sh}(p, n+1-p)$

$$\sigma'(i) = \begin{cases} \sigma(i) & 1 \leq i \leq p \\ n+1 & i = p+1 \\ \sigma(i-1) & p+2 \leq i \leq n+1 \end{cases} \qquad \sigma''(i) = \begin{cases} \sigma(i) & 1 \leq i \leq n \\ n+1 & i = n+1. \end{cases}$$

Conversely, for every shuffle permutation $\pi \in \mathrm{Sh}(p, n+1-p)$, one has either $\pi(p) = n+1$ or $\pi(n+1) = n+1$. In the first case, one has $p > 0$ and $\pi = \sigma'$ for a shuffle permutation $\sigma \in \mathrm{Sh}(p-1, n+1-p)$ and in the second $\pi = \sigma''$ for a shuffle permutation $\sigma \in \mathrm{Sh}(p, n-p)$. The cocommutativity of $T(V)$ follows directly from the formula for Δ'. □

Example 5.3.2: The universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is a cocommutative Hopf algebra with the algebra structure from Example A.9 and the comultiplication, counit and antipode given by $\Delta(x) = x\otimes 1 + 1\otimes x$, $\epsilon(x) = 0$ and $S(x) = -x$ for all $x \in \mathfrak{g}$.

Proof:
The linear maps

$$\Delta' : \mathfrak{g} \to U(\mathfrak{g})\otimes U(\mathfrak{g}),\ x \mapsto x\otimes 1 + 1\otimes x \qquad \epsilon' : \mathfrak{g} \to \mathbb{F},\ x \mapsto 0 \qquad S' : \mathfrak{g} \to U(\mathfrak{g})^{op},\ x \mapsto -x$$

are Lie algebra homomorphisms, since one has for all $x, y \in \mathfrak{g}$

$$\begin{aligned}
&[\Delta'(x), \Delta'(y)] = \Delta'(x) \cdot \Delta'(y) - \Delta'(y) \cdot \Delta'(x) = [x, y]\otimes 1 + 1\otimes [x, y] = \Delta'([x, y])\\
&[\epsilon'(x), \epsilon'(y)]) = \epsilon'(x)\epsilon'(y) - \epsilon'(y)\epsilon'(x) = 0 = \epsilon'([x, y])\\
&[S'(x), S'(y)] = S'(y)S'(x) - S'(x)S'(y) = y \cdot x - x \cdot y = -[x, y] = S'([x, y]).
\end{aligned}$$

By the universal property of $U(\mathfrak{g})$, they induce algebra homomorphisms $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g})\otimes U(\mathfrak{g})$ $\epsilon : U(\mathfrak{g}) \to \mathbb{F}$ and $S : U(\mathfrak{g}) \to U(\mathfrak{g})^{op}$ with $\Delta \circ \iota_\mathfrak{g} = \Delta'$, $\epsilon \circ \iota_\mathfrak{g} = \epsilon'$ and $S \circ \iota_\mathfrak{g} = S'$. To prove the coassociativity and counitality and that S is an antipode, it is sufficient to show that

$$\begin{aligned}
&(\Delta\otimes\mathrm{id}) \circ \Delta \circ \iota_\mathfrak{g} = (\mathrm{id}\otimes\Delta) \circ \Delta \circ \iota_\mathfrak{g} \qquad (51)\\
&l_{U(\mathfrak{g})} \circ (\epsilon\otimes\mathrm{id}) \circ \Delta \circ \iota_\mathfrak{g} = \iota_\mathfrak{g} = r_{U(\mathfrak{g})} \circ (\mathrm{id}\otimes\epsilon) \circ \Delta \circ \iota_\mathfrak{g}\\
&m \circ (S\otimes\mathrm{id}) \circ \Delta \circ \iota_\mathfrak{g} = \eta \circ \epsilon \circ \iota_\mathfrak{g} = m \circ (\mathrm{id}\otimes S) \circ \Delta \circ \iota_\mathfrak{g}.
\end{aligned}$$

The claim then follows from the universal property of $U(\mathfrak{g})$. The identities (51) follow by a direct computation that yields the same formulas as in the proof of Example 5.3.1. That this bialgebra is cocommutative follows from the fact that $\Delta \circ \iota_{\mathfrak{g}}(x) = 1\otimes x + x\otimes 1 = \Delta^{op} \circ \iota_{\mathfrak{g}}(x)$ for all $x \in \mathfrak{g}$. With the universal property of $U(\mathfrak{g})$, this implies $\Delta = \Delta^{op} : U(\mathfrak{g}) \to U(\mathfrak{g})\otimes U(\mathfrak{g})$. □

All examples of Hopf algebras treated so far are cocommutative, and hence their (finite) duals are commutative. To construct more interesting examples that are neither commutative nor cocommutative, we consider certain polynomials in $\mathbb{Z}[q]$, the so-called *q-factorials* and *q-binomials*. Their name is due to the fact that they exhibit relations that resemble the relations between factorials of natural numbers and binomial coefficients.

Definition 5.3.3: Let $\mathbb{Z}[q]$ the ring of polynomials with coefficients in $\mathbb{Z}$ and $\mathbb{Z}(q)$ the associated fraction field of rational functions. We define:

- the q-**natural** $(n)_q = 1 + q + ... + q^{n-1} = \frac{q^n-1}{q-1}$ for all $n \in \mathbb{N}$,
- the q-**factorial** $(0)!_q = 1$ and $(n)!_q = (n)_q(n-1)_q \cdots (1)_q = \frac{(q^n-1)(q^{n-1}-1)\cdots(q-1)}{(q-1)^n}$ for $n \in \mathbb{N}$,
- the q-**binomial** or **Gauß polynomial** $\binom{n}{k}_q = \frac{(n)!_q}{(n-k)!_q\,(k)!_q}$ for $k, n \in \mathbb{N}_0$ with $0 \leq k \leq n$.

Lemma 5.3.4:

1. The q-naturals, the q-factorials and the q-binomials are elements of $\mathbb{Z}[q]$.
2. For all $k, n \in \mathbb{N}_0$ with $0 \leq k \leq n$ the q-binomials satisfy the identity
$$\binom{n}{k}_q = \binom{n}{n-k}_q$$
3. For all $k, n \in \mathbb{N}_0$ with $0 \leq k < n$ the q-binomials satisfy the q-**Pascal identity**
$$\binom{n+1}{k+1}_q = \binom{n}{k}_q + q^{k+1}\binom{n}{k+1}_q = \binom{n}{k+1}_q + q^{n-k}\binom{n}{k}_q$$
4. If A is an algebra over $\mathbb{Z}(q)$ and $x, y \in A$ with $xy = q\,yx$ one has the q-**binomial formula**
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k}_q y^k x^{n-k},$$

Proof:
That $(n)_q$ and $(n)!_q$ are polynomials in q follows directly from their definition. That this also holds for the q-binomials follows by induction from 3. and from the fact that they are equal to 1 for $k = 0$ or $k = n$. The second claim follows directly from the definition of the q-binomial, and the third follows by a direct computation:

$$\binom{n}{k}_q + q^{k+1}\binom{n}{k+1}_q = \frac{(q^n-1)\cdots(q^{k+1}-1)}{(q^{n-k}-1)\cdots(q-1)} + q^{k+1}\frac{(q^n-1)\cdots(q^{k+2}-1)}{(q^{n-k-1}-1)\cdots(q-1)}$$
$$= \frac{(q^n-1)\cdots(q^{k+2}-1)}{(q^{n-k}-1)\cdots(q-1)} \cdot \left(q^{k+1} - 1 + q^{k+1}(q^{n-k}-1)\right) = \frac{(q^{n+1}-1)\cdots(q^{k+2}-1)}{(q^{n-k}-1)\cdots(q-1)} = \binom{n+1}{k+1}_q$$
$$\binom{n}{k+1}_q + q^{n-k}\binom{n}{k}_q = \frac{(q^n-1)\cdots(q^{k+2}-1)}{(q^{n-k-1}-1)\cdots(q-1)} + q^{n-k}\frac{(q^n-1)\cdots(q^{k+1}-1)}{(q^{n-k}-1)\cdots(q-1)}$$
$$= \frac{(q^n-1)\cdots(q^{k+2}-1)}{(q^{n-k}-1)\cdots(q-1)} \cdot \left(q^{n-k} - 1 + q^{n-k}(q^{k+1}-1)\right) = \frac{(q^{n+1}-1)\cdots(q^{k+2}-1)}{(q^{n-k}-1)\cdots(q-1)} = \binom{n+1}{k+1}_q.$$

4. To prove the last claim, we show by induction over n that for all $x, y \in A$ and $n \in \mathbb{N}_0$

$$(x+y)^{n+1} - \sum_{k=0}^{n+1} \binom{n+1}{k}_q y^k x^{n+1-k} \in (xy - qyx) =: I. \tag{52}$$

For $n = 0$ this holds by definition. Suppose it holds for $n \in \mathbb{N}_0$ and $x, y \in A$. With the identity

$$xy^k - q^k y^k x = \Sigma_{l=0}^{k-1} q^l y^l (xy - qyx) y^{k-l-1}, \tag{53}$$

which follows by induction over k for all $x, y \in A$ and with 3. we compute

$$\begin{aligned}
&(x+y)^{n+1} - \sum_{k=0}^{n+1} \binom{n+1}{k}_q y^k x^{n+1-k} \\
&\overset{\text{ind}}{=} (x+y) \sum_{k=0}^{n} \binom{n}{k}_q y^k x^{n-k} - \sum_{k=0}^{n+1} \binom{n+1}{k}_q y^k x^{n+1-k} \bmod I \\
&\overset{3.}{=} \sum_{k=0}^{n-1} \binom{n}{k}_q y^{k+1} x^{n-k} + \sum_{k=1}^{n} \binom{n}{k}_q x y^k x^{n-k} - \sum_{k=1}^{n} \binom{n}{k-1}_q y^k x^{n+1-k} - \sum_{k=1}^{n} \binom{n}{k}_q q^k y^k x^{n+1-k} \bmod I \\
&= \sum_{k=1}^{n} \binom{n}{k}_q (x y^k x^{n-k} - q^k y^k x^{n+1-k}) \bmod I \overset{(53)}{=} \sum_{k=1}^{n} \sum_{l=0}^{k-1} \binom{n}{k}_q q^l y^l (xy - qyx) y^{k-l-1} x^{n-k} = 0 \bmod I,
\end{aligned}$$

where we write $a = b \bmod I$ for $a - b \in I = (xy - qyx)$. If $xy = q\,yx$, then the ideal $(xy - qyx) \subset A$ is trivial and the claim follows. □

To evaluate the q-naturals, q-factorials and q-binomials, we recall that for every integral domain R, we have a unital ring homomorphism $\phi : \mathbb{Z} \to R$, $z \mapsto z = z1$. By the universal property of $\mathbb{Z}[q]$ it induces a unique unital ring homomorphism $\phi' : \mathbb{Z}[q] \to R[q]$ with $\phi'(1) = 1$. By composing it with the evaluation homomorphism $\mathrm{ev}_r : R[q] \to R$ for an element $r \in R$, we obtain a unital ring homomorphism $\mathrm{ev}'_r : \mathbb{Z}[q] \to R$, $\Sigma_{n\in\mathbb{N}_0} a_n q^n \mapsto \Sigma_{n\in\mathbb{N}_0} a_n r^n$.

Definition 5.3.5: Let R be an integral domain. The **evaluation** of the q-naturals, q-binomials and q-factorials at an element $r \in R$ is

$$(n)'_r = \mathrm{ev}'_{\mathrm{r}}(n)_q \qquad (n)!'_r = \mathrm{ev}'_{\mathrm{r}}(n)!_q \qquad \binom{n}{k}'_r = \mathrm{ev}'_r \binom{n}{k}_q.$$

Clearly, there are two cases of special interest. The first is $R = \mathbb{Z}$ and $r = 1$, where the evaluations of q-factorials and q-binomials in $r = 1$ coincide with the usual factorials and binomials. This justifies the names q-naturals, q-binomials and q-factorials.

The second is the case, where $r \in R$ is a primitive nth root of unity: $r^n = 1$ and $r^k \neq 1$ for $1 \leq k < n$. In this case, one has $(n)'_r = 0$ and $(k)'_r \neq 0$ for all $k < n$ since the roots of the polynomial $(m)_q$ are precisely the non-trivial mth roots of unity. This implies that the evaluations of all q-factorials $(m)!_q$ with $m \geq n$ vanish, since they contain a factor $(n)_q$. The same holds for the evaluations of all q-binomials with entries n and $0 < k < n$, since $(n)'_r = 0$ and $(k)'_r \neq 0$ for all $k < n$ implies

$$\binom{n}{k}'_r = \mathrm{ev}'_r \frac{(n)!_q}{(n-k)!_q (k)!_q} = \frac{(n)'_q \cdots (n-k+1)'_q}{(k)'_q (k-1)'_q \cdots (1)'_q} = 0.$$

We now use the q-naturals, q-factorials and q-binomials to construct an example of a Hopf algebra that is neither commutative nor cocommutative. The simplest way to proceed is to present its algebra structure in terms of generators and relations, see Definition A.7. Clearly, the minimum number of linearly independent generators that can give rise to a non-commutative and non-cocommutative bialgebra is two. The simplest relations that can be imposed on such an algebra without making it commutative or trivial are quadratic relations in the two generators, i. e. relations of the form $xy - qyx$ for some $q \in \mathbb{F}$ and generators x, y. This defines an infinite-dimensional non-commutative and non-cocommutative bialgebra. If we impose additional relations of the form $x^n = 0$ and $y^n = 1$ to make the bialgebra finite-dimensional, we have to take q as a primitive nth root of unity and obtain *Taft's example*.

Example 5.3.6: (Taft's example)
Let $\mathbb{F}$ be a field of characteristic zero, $n \in \mathbb{N}$ and $q \in \mathbb{F}$ a primitive nth root of unity. Let T_q be the algebra over $\mathbb{F}$ with generators x, y and relations

$$xy - q\,yx = 0, \qquad x^n = 0, \qquad y^n - 1 = 0. \tag{54}$$

Then T_q is a Hopf algebra with the comultiplication, counit and antipode given by

$$\Delta(x) = 1 \otimes x + x \otimes y,\ \Delta(y) = y \otimes y,\ \epsilon(x) = 0,\ \epsilon(y) = 1,\ S(x) = -xy^{n-1},\ S(y) = y^{n-1}. \tag{55}$$

Proof:
Let V be the free $\mathbb{F}$-vector space generated by $B = \{x, y\}$. Then the algebra T_q is given as $T_q = T(V)/I$, where $I = (xy - q\,yx, x^n, y^n - 1)$ is the two-sided ideal in $T(V)$ generated by the relations (54) and $\cdot$ denotes the multiplication of the tensor algebra $T(V)$.

1. We show that (55) defines a bialgebra structure on $T(V)$:

By the universal property of the tensor algebra, the linear maps $\Delta' : V \to T(V) \otimes T(V)$ and $\epsilon' : V \to \mathbb{F}$ that are determined by their values on B from (55) induce algebra homomorphisms $\Delta'' : T(V) \to T(V) \otimes T(V)$, $\epsilon'' : T(V) \to \mathbb{F}$ with $\Delta'' \circ \iota_V = \Delta'$ and $\epsilon'' \circ \iota_V = \epsilon'$.

To prove that these algebra homomorphisms satisfy the coassociativity and the counitality condition, it is sufficient to show that

$$(\Delta' \otimes \mathrm{id}) \circ \Delta'(z) = (\mathrm{id} \otimes \Delta') \circ \Delta'(z) \qquad\qquad (\epsilon' \otimes \mathrm{id}) \circ \Delta'(z) = 1 \otimes z = (\mathrm{id} \otimes \epsilon') \circ \Delta'(z)$$

for $z \in B$. This follows by a direct computation from (55) and shows that Δ'' and ϵ'' define a bialgebra structure on the tensor algebra $T(V)$.

2. We show that it induces a bialgebra structure on $T_q = T(V)/I$:

By Proposition 5.1.5 it is sufficient to show that I is a coideal in $T(V)$. As I is the two-sided ideal generated by the relations in (54) and Δ'' and ϵ'' are algebra homomorphisms, it is sufficient to show that $\Delta''(r) \in I \otimes T(V) + T(V) \otimes I$ and $\epsilon''(r) = 0$ for each relation r in (54). The latter follows directly from the definition of ϵ''. For the former, we compute

$$\begin{aligned}
\Delta''(xy - qyx) &= \Delta''(x) \cdot \Delta''(y) - q\Delta''(y) \cdot \Delta''(x) \\
&= y \otimes (xy) + (xy) \otimes y^2 - q\, y \otimes (yx) - q(yx) \otimes y^2 \\
&= y \otimes (xy - q\,yx) + (xy - qyx) \otimes y^2 \in T(V) \otimes I + I \otimes T(V).
\end{aligned}$$

$$\begin{aligned}
\Delta''(y^n - 1) &= \Delta''(y)^n - 1 \otimes 1 = (y \otimes y)^n - (1 \otimes 1) = y^n \otimes y^n - 1 \otimes 1 \\
&= y^n \otimes (y^n - 1) + (y^n - 1) \otimes 1 \in T(V) \otimes I + I \otimes T(V).
\end{aligned}$$

To prove this for $\Delta''(x^n) = (1\otimes x + x\otimes y)^n$, note that (52) from the proof of Lemma 5.3.4 implies

$$(1\otimes x + x\otimes y)^n - \sum_{k=0}^{n} \binom{n}{k}_q' (x\otimes y)^k \cdot (1\otimes x)^{n-k} \in J$$

where the binomial coefficient is evaluated in $q \in \mathbb{F}$ and J is the two-sided idea generated by the element $(1\otimes x) \cdot (x\otimes y) - q\,(x\otimes y)(1\otimes x) = x\otimes(xy) - q\,x\otimes(yx) = x\otimes(xy - qyx) \in T(V)\otimes I$. As $T(V)\otimes I$ is a two-sided ideal in $T(V)\otimes T(V)$, we have $J \subset T(V)\otimes I$. As q is a primitive nth root of unity, the evaluations of the binomial coefficients for $0 < k < n$ vanish. This yields

$$\Delta''(x^n) - 1\otimes x^n - x^n\otimes y^n \in T(V)\otimes I \quad \Rightarrow \quad \Delta''(x^n) \in T(V)\otimes I + I\otimes T(V).$$

Hence, we have shown that Δ'' and ϵ'' induce algebra homomorphisms $\Delta : T_q \to T_q\otimes T_q$ and $\epsilon : T_q \to \mathbb{F}$ with $(\pi\otimes\pi)\circ\Delta'' = \Delta\circ\pi$ and $\epsilon'' = \epsilon\circ\pi$, where $\pi : T(V) \to T(V)/I$ is the canonical surjection, and that this defines a bialgebra structure on $T_q = T(V)/I$.

3. We show that T_q is a Hopf Algebra:

For this we consider the linear map $S' : V \to T(V)^{op}$ defined by (55). By the universal property of the tensor algebra it induces an algebra morphism $S'' : T(V) \to T(V)^{op}$ with $S''\circ\iota_V = S'$. By composing it with the canonical surjection $\pi : T(V) \to T_q$, we obtain an algebra morphism $\pi\circ S'' : T(V) \to T_q^{op}$. To show that this induces an algebra morphism $S : T_q \to T_q$, we have to show that $\pi\circ S''(r) = 0$ for all relations r in (54). With (53) we compute

$$\begin{aligned}
\pi\circ S''(xy - q\,yx) &= \pi(S''(y))\pi(S''(x)) - q\,\pi(S''(x))\pi(S''(y)) = -y^{n-1}xy^{n-1} + q\,xy^{n-1}\cdot y^{n-1}\\
&= -y^{n-1}\,xy^{n-1} + q^n y^{n-1}xy^{n-1} = (q^n - 1)y^{n-1}xy^{n-1} = 0,\\
\pi\circ S''(x^n) &= \pi(S''(x)^n) = (-xy^{n-1})^n = (-1)^n(xy^{n-1})\cdots(xy^{n-1}) = (-1)^n q^{(n-1)(1+\ldots+n)} y^{n(n-1)} x^n = 0,\\
\pi\circ S''(y^n - 1) &= \pi(S''(y)^n) - 1 = y^{n(n-1)} - 1 = 1^{n-1} - 1 = 0.
\end{aligned}$$

To show that the algebra morphism $S : T_q \to T_q^{op}$ satisfies the defining condition on the antipode, it is sufficient to verify

$$m\circ(S\otimes\mathrm{id})\circ\Delta(z) = \epsilon(z) = m\circ(\mathrm{id}\otimes S)\circ\Delta(z)$$

for $z \in \{x, y\}$, which follows by a direct computation. This shows that T_q is a Hopf algebra. $\square$

Remark 5.3.7:

1. Taft's example for $q = -1$, $n = 2$ is also known as **Sweedler's example**.

2. The elements $x^i y^j \in T_q$ with $0 \leq i, j \leq n-1$ form a basis of the vector space T_q. This follows because every mixed monomial in x and y can be transformed into one of them by applying the relations, and the elements $x^i y^j$ are linearly independent. The elements $y^i x^j \in T_q$ with $0 \leq i, j \leq n-1$ form as basis as well. In particular, one has $\dim_{\mathbb{F}} T_q = n^2$.

3. It follows from the proof of Example 5.3.6 that the algebra with generators x, y and the relation $xy - qyx$ is also a bialgebra for any $q \in \mathbb{F}$, because the ideal $I' = (xy - qyx)$ is a coideal in $T(V)$. This infinite-dimensional bialgebra is sometimes called the **quantum plane**. It is not cocommutative, commutative if and only if $q = 1$, and the elements $x^i y^j$ for $i, j \in \mathbb{N}_0$ form a basis of this bialgebra.

4. The antipode in Taft's example satisfies $S^2(y) = y$ and $S^2(x) = yxy^{-1}$. This shows that S^2 and hence S are invertible, but we do not have $S^2 = \mathrm{id}$.

Our next example of a non-cocommutative and non-commutative bialgebra and Hopf algebra are the so-called q-deformed matrix algebras $\mathrm{M}_q(2,\mathbb{F})$ and $\mathrm{SL}_q(2,\mathbb{F})$. They are again presented in terms of generators and relations and their coalgebra structure will be interpreted later as a generalisation and deformation of the coalgebra $\mathrm{Mat}(2\times 2,\mathbb{F})^*$ from Example 5.1.3, 5.

Example 5.3.8: Let $\mathbb{F}$ be a field and $q\in\mathbb{F}\setminus\{0\}$.

1. The **matrix algebra** $\mathrm{M}_q(2,\mathbb{F})$ is the algebra over $\mathbb{F}$ with generators a,b,c,d and relations
$$ba=q\,ab,\quad db=q\,bd,\quad ca=q\,ac,\quad dc=q\,cd,\quad bc=cb,\quad da-ad=(q-q^{-1})\,bc. \tag{56}$$
It has a bialgebra structure with comultiplication and counit given by
$$\begin{array}{llll}\Delta(a)=a\otimes a+b\otimes c, & \Delta(b)=a\otimes b+b\otimes d, & \Delta(c)=c\otimes a+d\otimes c, & \Delta(d)=c\otimes b+d\otimes d\\ \epsilon(a)=1 & \epsilon(b)=0 & \epsilon(c)=0 & \epsilon(d)=1.\end{array} \tag{57}$$

2. The **q-determinant** $\det_q=ad-q^{-1}bc$ is central in $\mathrm{M}_q(2,\mathbb{F})$ with
$$\Delta(\det_q)=\det_q\otimes\det_q \qquad \epsilon(\det_q)=1.$$

3. The bialgebra structure of $\mathrm{M}_q(2,\mathbb{F})$ induces a Hopf algebra structure on the algebra $\mathrm{SL}_q(2,\mathbb{F})=\mathrm{M}_q(2,\mathbb{F})/(\det_q-1)$ with the antipode given by
$$S(a)=d,\quad S(b)=-qb,\quad S(c)=-q^{-1}c,\quad S(d)=a.$$

Proof:
1. The proof is similar to the one of Example 5.3.6. The algebra $\mathrm{M}_q(2,\mathbb{F})$ is given as the quotient $\mathrm{M}_q(2,\mathbb{F})=T(V)/I$, where V is the free vector space with basis $\{a,b,c,d\}$ and $I\subset T(V)$ the two-sided ideal generated by the six relations in (56). By the universal property of the tensor algebra, the maps $\Delta':V\to T(V)\otimes T(V)$ and $\epsilon':V\to\mathbb{F}$ specified by (57) induce algebra homomorphisms $\Delta'':T(V)\to T(V)\otimes T(V)$ and $\epsilon'':T(V)\to\mathbb{F}$. To show that Δ'' and ϵ'' are coassociative and counital, it is again sufficient to show that for $x\in\{a,b,c,d\}$
$$\begin{aligned}&(\Delta''\otimes\mathrm{id})\circ\Delta''(x)=(\mathrm{id}\otimes\Delta'')\circ\Delta''(x)\\ &l_{T(V)}\circ(\epsilon''\otimes\mathrm{id})\circ\Delta''(x)=r_{T(V)}\circ(\mathrm{id}\otimes\epsilon'')\circ\Delta''(x).\end{aligned}$$
This follows by a direct computation from (57). To show that this induces a bialgebra structure on $\mathrm{M}_q(2,\mathbb{F})$ it is sufficient to prove that I is a two-sided coideal in $T(V)$, i. e. that we have $\Delta''(r)\in I\otimes T(V)+T(V)\otimes I$ and $\epsilon''(r)=0$ for each relation r. For the latter, note that $\epsilon''(xy)=0$ if $x,y\in\{a,b,c,d\}$ with $\{x,y\}\cap\{b,c\}\neq\emptyset$. This proves that $\epsilon''(r)=0$ for the first five relations. For the last relation, we have
$$\epsilon''(da-ad)=\epsilon''(d)\epsilon''(a)-\epsilon''(a)\epsilon''(d)=1-1=0=(q-q^{-1})\epsilon''(b)\epsilon''(c)=(q-q^{-1})\epsilon''(bc).$$
The identities $\Delta''(r)\in I\otimes T(V)+T(V)\otimes I$ follow from a direct computation, which we perform for the first relation, since the other computations are similar
$$\begin{aligned}&\Delta''(ba-qab)=(a\otimes b+b\otimes d)\cdot(a\otimes a+b\otimes c)-q(a\otimes a+b\otimes c)\cdot(a\otimes b+b\otimes d)\\ &=a^2\otimes(ba-q\,ab)+b^2\otimes(dc-q\,cd)+(ba-q\,ab)\otimes da+q\,ab\otimes(da-ad+(q^{-1}-q)bc)+ab\otimes(cb-bc).\end{aligned}$$
2. That the element $\det_q$ is central in $\mathrm{M}_q(2,\mathbb{F})$ follows by a direct computation from the relations in $\mathrm{M}_q(2,\mathbb{F})$, and so do the formulas coproduct and the counit of the q-determinant.

3. As we have $\Delta(\det_q -1) = \det_q \otimes \det_q -1\otimes 1 = \det_q \otimes(\det_q -1)+(\det_q -1)\otimes 1$, $\epsilon(\det_q -1) = 0$, the two-sided ideal $(\det_q -1)$ in $\mathrm{M}_q(2,\mathbb{F})$ is a coideal in $\mathrm{M}_q(2,\mathbb{F})$. This implies that the quotient $\mathrm{M}_q(2,\mathbb{F})/(\det_q -1)$ inherits a bialgebra structure from $\mathrm{M}_q(2,\mathbb{F})$.

To show that this bialgebra is a Hopf algebra, one verifies with the expressions for S that $m\circ(S\otimes\mathrm{id})\circ\Delta(x) = m\circ(\mathrm{id}\otimes S)\circ\Delta(x) = \epsilon(x)$ for all $x\in\{a,b,c,d\}$ in $\mathrm{SL}_q(2,\mathbb{F})$. This shows that S is an antipode for the bialgebra $\mathrm{SL}_q(2,\mathbb{F})$ and $\mathrm{SL}_q(2,\mathbb{F})$ is a Hopf algebra. □

To understand the names $\mathrm{M}_q(2,\mathbb{F})$ and $\mathrm{SL}_q(2,\mathbb{F})$ for these algebras, we note that for $q=1$ the relations of the matrix algebra $M_q(2,\mathbb{F})$ in (56) imply that $M_1(2,\mathbb{F})$ is a *commutative* algebra with four generators and the coalgebra structure given by (57). If we interpret the generators a,b,c,d as linear maps $a,b,c,d\in\mathrm{Mat}(2\times 2,\mathbb{F})^*$ given by

$$a:\begin{pmatrix} a' & b'\\ c' & d'\end{pmatrix}\mapsto a',\quad b:\begin{pmatrix} a' & b'\\ c' & d'\end{pmatrix}\mapsto b',\quad c:\begin{pmatrix} a' & b'\\ c' & d'\end{pmatrix}\mapsto c',\quad d:\begin{pmatrix} a' & b'\\ c' & d'\end{pmatrix}\mapsto d',$$

we see that $M_1(2,\mathbb{F})$ is isomorphic to the algebra of functions $f:\mathrm{Mat}(2\times 2,\mathbb{F})\to\mathbb{F}$ that are polynomials in the entries a',b',c',d', with the pointwise addition, scalar multiplication and multiplication.

Moreover, the coalgebra structure defined by (57) coincides with the one from Example 5.1.3, 5. Hence, we extended the coalgebra structure on $\mathrm{Mat}(2\times 2,\mathbb{F})$ to the commutative algebra $\mathrm{M}_1(2,\mathbb{F})$ and obtained a bialgebra structure on $\mathrm{M}_1(2,\mathbb{F})$. We can therefore interpret q as a deformation parameter that changes the algebra structure of the commutative bialgebra $\mathrm{M}_1(2,\mathbb{F})$ to a non-commutative one given by (56). This justifies the name $M_q(2,\mathbb{F})$.

Note also that for $q=1$, we have $\det_1 = ad-bc$ and hence can interpret $\det_1\in\mathrm{M}_1(2,\mathbb{F})$ as the determinant $\det:\mathrm{Mat}(2\times 2,\mathbb{F})\to\mathbb{F}$. The algebra $\mathrm{SL}_1(2,\mathbb{F}) = \mathrm{M}_1(2,\mathbb{F})/(\det_1 -1)$ is obtained from $\mathrm{M}_1(2,\mathbb{F})$ by identifying those polynomial functions that agree on the subset $\mathrm{SL}(2,\mathbb{F}) = \{M\in\mathrm{Mat}(2\times 2,\mathbb{F})\,|\,\det(M)=1\}$. Hence, we can interpret $\mathrm{SL}_1(2,\mathbb{F})$ as the bialgebra of functions $f:\mathrm{SL}(2,\mathbb{F})\to\mathbb{F}$ that are polynomials in the matrix entries, with the pointwise addition, scalar multiplication and multiplication.

The antipode of $\mathrm{SL}_1(2,\mathbb{F})$ is given by $S(a)=d$, $S(b)=-b$, $S(c)=-c$ and $S(d)=a$, and we can interpret it as a map that sends the matrix elements of a matrix in $\mathrm{SL}(2,\mathbb{F})$ to the matrix elements of the inverse matrix. The algebra $\mathrm{SL}_q(2,\mathbb{F})$ for general q can then be viewed a deformation of this algebra, in which the multiplication becomes non-commutative, and the matrix elements of the inverse matrix are replaced by their image under the antipode.

Our last important example of a q-deformation are the so-called *q-deformed universal enveloping algebras*. The simplest non-trivial one is the q-deformed universal enveloping algebra $U_q(\mathfrak{sl}(2))$, which is related to the Lie algebra $\mathfrak{sl}(2)$ of traceless (2×2)-matrices with the Lie bracket given by the matrix commutator. We first give its bialgebra structure in the simplest presentation and then discuss its relation to the Lie algebra $\mathfrak{sl}(2)$ and its universal enveloping algebra $U(\mathfrak{sl}(2))$.

Example 5.3.9: Let $\mathbb{F}$ be a field and $q\in\mathbb{F}\setminus\{0,1,-1\}$.
The **q-deformed universal enveloping algebra** $U_q(\mathfrak{sl}_2)$ is the algebra over $\mathbb{F}$ with generators E,F,K,K^{-1} and relations

$$K^{\pm 1}K^{\mp 1}=1,\quad KEK^{-1}=q^2E,\quad KFK^{-1}=q^{-2}F,\quad [E,F]=\frac{K-K^{-1}}{q-q^{-1}}.\qquad (58)$$

A Hopf algebra structure on $U_q(\mathfrak{sl}_2)$ is given by

$$\begin{aligned}
&\Delta(K^{\pm 1}) = K^{\pm 1}\otimes K^{\pm 1}, && \Delta(E) = 1\otimes E + E\otimes K, && \Delta(F) = F\otimes 1 + K^{-1}\otimes F\\
&\epsilon(K^{\pm 1}) = 1, && \epsilon(E) = 0, && \epsilon(F) = 0,\\
&S(K^{\pm 1}) = K^{\mp 1} && S(E) = -EK^{-1} && S(F) = -KF.
\end{aligned} \tag{59}$$

Proof:
The proof is analogous to the one for the previous two examples and is left as an exercise. □

Remark 5.3.10: One can show that the set $B = \{E^i F^j K^k \,|\, i, j \in \mathbb{N}_0, k \in \mathbb{N}\}$ is a basis of $U_q(\mathfrak{sl}_2)$ and that the Hopf algebra $\mathrm{SL}_q(2,\mathbb{F})$ from Example 5.3.8 is the finite dual of $U_q(\mathfrak{sl}_2)$ and vice versa. The duality is given by the unique linear map $\langle\ ,\ \rangle : \mathrm{SL}_q(2)\otimes U_q(\mathfrak{sl}_2) \to \mathbb{F}$ with

$$\langle a, K^{\pm 1}\rangle = q^{\mp 1} \qquad \langle d, K^{\pm 1}\rangle = q^{\pm 1} \qquad \langle b, E\rangle = 1 \qquad \langle c, F\rangle = 1$$

and $\langle x, U\rangle = 0$ for all other combinations of $x \in \{a, b, c, d\}$ and $U \in \{K^{\pm 1}, E, F\}$. The proofs of these statements, which are are lengthy and technical, are given in [Ka].

We will now relate the bialgebra $U_q(\mathfrak{sl}_2)$ to the universal enveloping algebra of the Lie algebra $\mathfrak{sl}_2$ of traceless (2×2)-matrices. However, for this the presentation of $U_q(\mathfrak{sl}_2)$ in Example 5.3.9 is unsuitable, since it is ill-defined for $q = 1$. It turns out that this is not a problem with its Hopf algebra structure but with its presentation in terms of generators and relations.

We show that there is a bialgebra $U'_q(\mathfrak{sl}_2)$ defined for all $q \in \mathbb{F}\setminus\{0\}$ which is isomorphic to $U_q(\mathfrak{sl}_2)$ for $q \neq \pm 1$ and closely related to the universal enveloping algebra $U(\mathfrak{sl}_2)$ for $q = 1$. The price one has to pay is a higher number of generators and relations.

Proposition 5.3.11: Let $q \in \mathbb{F}\setminus\{0\}$.

For $q \neq \pm 1$ the Hopf algebra $U_q(\mathfrak{sl}_2)$ is isomorphic to the Hopf algebra $U'_q(\mathfrak{sl}_2)$ over $\mathbb{F}$ with generators e, f, k, k^{-1}, l and relations

$$\begin{aligned}
&kk^{-1} = k^{-1}k = 1, && kek^{-1} = q^2 e, && kfk^{-1} = q^{-2} f, && [e, f] = l,\\
&(q - q^{-1})l = k - k^{-1}, && [l, e] = q(ek + k^{-1}e), && [l, f] = -q^{-1}(fk + k^{-1}f) &&
\end{aligned} \tag{60}$$

and the Hopf algebra structure

$$\begin{aligned}
&\Delta'(k^{\pm 1}) = k^{\pm 1}\otimes k^{\pm 1}, && \Delta'(e) = 1\otimes e + e\otimes k, && \Delta'(f) = f\otimes 1 + k^{-1}\otimes f, && \Delta'(l) = l\otimes k + k^{-1}\otimes l\\
&\epsilon'(k^{\pm 1}) = 1, && \epsilon'(e) = 0, && \epsilon'(f) = 0, && \epsilon'(l) = 0,\\
&S'(k^{\pm 1}) = k^{\mp 1}, && S'(e) = -ek^{-1}, && S'(f) = -kf, && S'(l) = -l.
\end{aligned} \tag{61}$$

For $q = 1$, the element k is central in $U'_q(\mathfrak{sl}_2)$ with $k^2 = 1$, and $U'_1(\mathfrak{sl}_2)/(k-1)$ is isomorphic to $U(\mathfrak{sl}_2)$ as a bialgebra.

Proof:
1. Let V be the free vector space generated by $E, F, K^{\pm 1}$ and V' be the free vector space generated by $e, f, k^{\pm 1}, l$. Let $I \subset T(V)$ and $I' \subset T(V')$ be the two-sided ideals generated by

the relations (58) and (60), respectively. To show that $U_q(\mathfrak{sl}_2)$ and $U'_q(\mathfrak{sl}_2)$ are isomorphic, we consider for $q\neq\pm 1$ the linear maps

$$\begin{array}{lll} \phi: V\to T(V') & \text{with} & \phi(E)=e,\ \phi(F)=f,\ \phi(K^{\pm 1})=k^{\pm 1}\\ \psi: V'\to T(V) & \text{with} & \psi(e)=E,\ \psi(f)=F,\ \psi(k^{\pm 1})=K^{\pm 1},\ \psi(l)=[E,F]. \end{array}$$

By the universal property of the tensor algebra, there are unique algebra homomorphisms $\phi': T(V)\to T(V')$ and $\psi': T(V')\to T(V)$ with $\phi'\circ\iota_V=\phi$ and $\psi'\circ\iota_{V'}=\psi$. To prove that the latter descend to algebra homomorphisms between $U_q(\mathfrak{sl}_2)$ and $U'_q(\mathfrak{sl}_2)$, we have to show that $\phi'(r)\in I'$ and $\psi'(r')\in I$ for each relation r of $U_q(\mathfrak{sl}_2)$ and r' of $U'_q(\mathfrak{sl}_2)$. For the first four relations of $U_q(\mathfrak{sl}_2)$ and the first five relations of $U'_q(\mathfrak{sl}_2)$, this is obvious. For the 5th relation of $U_q(\mathfrak{sl}_2)$ and the 6th relation of $U'_q(\mathfrak{sl}_2)$, we have

$$\begin{aligned} &\phi'([E,F]-(q-q^{-1})^{-1}(K-K^{-1}))=[e,f]-(q-q^{-1})^{-1}(k-k^{-1})=l-l=0 \bmod I'\\ &\psi'((q-q^{-1})l-k+k^{-1})=(q-q^{-1})[E,F]-K+K^{-1}=0 \bmod I, \end{aligned}$$

and for the last two relations in I', we obtain

$$\begin{aligned} &\psi'([l,e]-q(ek+k^{-1}e))=[[E,F],E]-q(EK+K^{-1}E)\\ &=(q-q^{-1})^{-1}[K-K^{-1},E]-q(EK+K^{-1}E) \text{ mod } I\\ &=(q^2-1)(q-q^{-1})^{-1}(EK+K^{-1}E)-q(EK+K^{-1}E) \text{ mod } I=0 \text{ mod } I\\ &\psi'([l,f]+q^{-1}(fk+k^{-1}f))=[[E,F],F]+q^{-1}(FK+K^{-1}F)\\ &=(q-q^{-1})^{-1}[K-K^{-1},F]+q^{-1}(FK+K^{-1}F) \text{ mod } I\\ &=(q^{-2}-1)(q-q^{-1})^{-1}(FK+K^{-1}F)+q^{-1}(FK+K^{-1}F) \text{ mod } I=0 \text{ mod } I, \end{aligned}$$

where we use the shorthand notation $a=b \bmod I$ for $a-b\in I$. This shows that ϕ' and ψ' induce algebra homomorphisms $\phi: U_q(\mathfrak{sl}_2)\to U'_q(\mathfrak{sl}_2)$ and $\psi: U'_q(\mathfrak{sl}(2))\to U_q(\mathfrak{sl}_2)$.

That they are mutually inverse algebra isomorphisms follows with the universal properties of the tensor algebra and the characteristic property of the quotient and the identities $\psi\circ\phi(X)=X$ for $X\in\{E,F,K^{\pm 1}\}$ and $\phi\circ\psi(x)=x$ for $x\in\{e,f,k^{\pm 1},l\}$, which follow by a direct computation. By setting $\Delta'=(\phi\otimes\phi)\circ\Delta\circ\psi$, $\epsilon'=\epsilon\circ\phi$ and $S'=\phi\circ S\circ\psi$, we obtain the Hopf algebra structure from (61), and ϕ becomes an isomorphism of Hopf algebras with inverse ψ.

2. The algebra $U'_q(\mathfrak{sl}_2)$ is defined for $q=1$. In this case its relations (60) reduce to

$$k^2=1,\qquad [k,e]=0,\qquad [k,f]=0,\qquad [e,f]=l,\qquad [l,e]=2ek,\qquad [l,f]=-2fk,$$

and its Hopf algebra structure is given by $\epsilon(k)=1$, $\epsilon(e)=\epsilon(f)=\epsilon(l)=0$, and

$$\begin{array}{llll} \Delta(k)=k\otimes k, & \Delta(e)=1\otimes e+e\otimes k, & \Delta(f)=f\otimes 1+k\otimes f, & \Delta(l)=l\otimes k+k\otimes l\\ S(k)=k^{-1}, & S(e)=-e, & S(f)=-f, & S(l)=-l. \end{array}$$

As k is central in $U'_1(\mathfrak{sl}(2))$ with $k^2=1$ and $\Delta(k)=k\otimes k$, the quotient $U'_1(\mathfrak{sl}(2))/(k-1)$ inherits a bialgebra structure from $U'_1(\mathfrak{sl}_2)$. Its algebra structure is given by

$$[e,f]=l,\quad [l,e]=2e,\quad [l,f]=-2f, \tag{62}$$

and its Hopf algebra structure by

$$\epsilon(X)=0,\quad \Delta(X)=X\otimes 1+1\otimes X,\quad S(X)=-X\qquad \forall X\in\{e,f,l\}. \tag{63}$$

If we choose as a basis of $\mathfrak{sl}_2 = \{M \in \mathrm{Mat}(2\times 2, \mathbb{F}) \,|\, \mathrm{tr}(M) = 0\}$ the matrices

$$l = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{64}$$

then the Lie bracket of $\mathfrak{sl}_2$ is given by (62), and the bialgebra structure of the universal enveloping algebra $U(\mathfrak{sl}_2)$ from Example 5.3.2 by (62) and (63). This shows that the Hopf algebras $U'_1(\mathfrak{sl}_2)/(k-1)$ and $U(\mathfrak{sl}_2)$ are isomorphic. □

Proposition 5.3.11 motivates the name q-deformed universal enveloping algebra and the notation $U_q(\mathfrak{sl}_2)$, since it relates $U_q(\mathfrak{sl}_2)$ for $q = 1$ to the universal enveloping algebra $U(\mathfrak{sl}_2)$.

Besides $q = 1$, there are other values of q, for which the q-deformed universal enveloping algebra $U_q(\mathfrak{sl}_2)$ has special structure, namely the case where q is a root of unity. In this case, one can take a quotient of $U_q(\mathfrak{sl}_2)$ by a two-sided ideal to obtain a finite-dimensional Hopf algebra. This finite-dimensional Hopf algebra is often called the **q-deformed universal enveloping algebra $U_q(\mathfrak{sl}_2)$ at a root of unity**, but the name is slightly misleading since it is a quotient of $U_q(\mathfrak{sl}_2)$. The proof of the following proposition is left as an exercise.

Proposition 5.3.12: Let $\mathbb{F}$ be a field, $q \in \mathbb{F} \setminus \{1, -1\}$ a primitive dth root of unity and $r := d$ if d is odd and $r := d/2$ if d is even.

1. The elements $K^{\pm r}$, E^r, F^r are central in $U_q(\mathfrak{sl}_2)$.
2. $U_q^r(\mathfrak{sl}_2) = U_q(\mathfrak{sl}_2)/(F^r, E^r, K^r - 1)$ inherits a Hopf algebra structure from $U_q(\mathfrak{sl}_2)$.
3. $U_q^r(\mathfrak{sl}_2)$ is finite-dimensional and spanned by $\{E^i F^j K^k \,|\, i, j, k = 0, 1, ..., r-1\}$.

Clearly, the q-deformed universal enveloping algebra $U_q(\mathfrak{sl}_2)$, its counterpart $U'_q(\mathfrak{sl}_2)$ and its quotient $U_q^r(\mathfrak{sl}_2)$ at a root of unity have a complicated mathematical structure, and it is not obvious at all how to generalise this construction to other Lie algebras in a systematic way.

Nevertheless, they are part of a general construction that is possible for all complex, simple Lie algebras and can be generalised to affine Kac-Moody algebras. These are the so-called *Drinfeld-Jimbo deformations* of of universal enveloping algebras. For complex simple Lie-algebras of type A, D, E, they take a particularly simple form.

Remark 5.3.13: (**Drinfeld-Jimbo deformations**)

Let $\mathfrak{g}$ be a complex, simple Lie algebra and $B = \{H_i, E_i, F_i \,|\, i = 1, ..., r\}$, the **Chevalley basis** of $\mathfrak{g}$, in which the Lie bracket takes the form

$$\begin{aligned}
&[H_i, H_j] = 0, \quad [H_i, E_j] = a_{ij} E_j, \quad [H_i, F_j] = -a_{ij} F_j, \quad [E_i, F_j] = \delta_{ij} H_i,\\
&(\mathrm{ad}_{E_i})^{1-a_{ij}} E_j = 0, \qquad\qquad\qquad (\mathrm{ad}_{F_i})^{1-a_{ij}} F_j = 0,
\end{aligned}$$

where ad_X is the linear map $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$, $Y \mapsto [X, Y]$ and $A = (a_{ij}) \in \mathrm{Mat}(r \times r, \mathbb{Z})$ the **Cartan matrix** of $\mathfrak{g}$.

If $\mathfrak{g}$ is a complex simple Lie algebra of type A, D or E, its Cartan matrix is positive definite and symmetric with $a_{ii} = 2$ for $i \in \{1, ..., r\}$ and $a_{ij} \in \{0, -1\}$ for $i \neq j$. In this case, the q-deformed universal enveloping algebra $U_q(\mathfrak{g})$ is has generators $\{K_i, E_i, F_i \,|\, i = 1, ..., r\}$ and relations

$$
\begin{array}{lll}
K_i^{\pm 1} K_i^{\pm 1} = 1, \quad [K_i, K_j] = 0, & & \\
[E_i, F_j] = \delta_{ij}(q - q^{-1})^{-1}(K_i - K_i^{-1}) & K_i E_j K_i^{-1} = q^{a_{ij}} E_j, & K_i F_j K_i^{-1} = q^{-a_{ij}} F_j, \\
[E_i, E_j] = 0 & [F_i, F_j] = 0 & \text{if } a_{ij} = 0, \\
E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = 0 & F_i^2 F_j - (q + q^{-1}) F_i F_j F_i + F_j F_i^2 = 0 & \text{if } a_{ij} = -1.
\end{array}
$$

Its Hopf algebra structure is given by

$$
\begin{array}{lll}
\Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1} & \Delta(E_i) = 1 \otimes E_i + E_i \otimes K_i & \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i \\
\epsilon(K_i^{\pm 1}) = 1 & \epsilon(E_i) = 0 & \epsilon(F_i) = 0 \\
S(K_i^{\pm 1}) = K_i^{\mp 1} & S(E_i) = -E_i K_i^{-1} & S(F_i) = -K_i F_i.
\end{array}
$$

There is also a presentation of $U_q(\mathfrak{g})$ similar to the one in Proposition 5.3.11 that is well-defined at $q = 1$ and relates the Hopf algebra $U_q(\mathfrak{g})$ to the universal enveloping algebra $U(\mathfrak{g})$. If q is a root of unity, then there is a finite-dimensional quotient $U_q^r(\mathfrak{g})$, which inherits a Hopf algebra structure from $U_q(\mathfrak{g})$ and generalises the Hopf algebra $U_q^r(\mathfrak{sl}_2)$ from Proposition 5.3.12.

6 Structure and Properties of Hopf algebras

6.1 Grouplike and primitive elements

Before investigating the representation theory of bialgebras and Hopf algebras in more depth, we focus on their structure. Many features of Hopf algebras can be understood by focusing on two types of special elements. The first are the *grouplike* elements, which behave in a similar way to the elements $g \in G$ in the group algebra $\mathbb{F}[G]$. The second are *primitive elements*, which generalise the elements $v \in V$ in the tensor algebra $T(V)$ and the elements $x \in \mathfrak{g}$ in the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra. Such elements play an important role in the classification of Hopf algebras, especially in the cocommutative and finite-dimensional case. Although these classification results are rather involved and cannot be proven here, understanding the properties of grouplike and primitive elements is helpful to develop an intuition.

Definition 6.1.1: Let H be a Hopf algebra.

1. An element $g \in H$ is called **grouplike** if $g \neq 0$ and $\Delta(g) = g \otimes g$. The set of grouplike elements of H is denoted $\mathrm{Gr}(H)$.
2. An element $h \in H$ is called **primitive** if $\Delta(h) = 1 \otimes h + h \otimes 1$. The set of primitive elements of H is denoted $\mathrm{Pr}(H)$.

Example 6.1.2:

1. The elements $g \in G$ are grouplike elements of the group algebra $\mathbb{F}[G]$. The element y in Taft's example and the elements $K^{\pm 1}$ in $U_q(\mathfrak{sl}_2)$ are grouplike.
2. The elements $v \in V$ are primitive elements of the tensor algebra $T(V)$ and the elements $x \in \mathfrak{g}$ are primitive elements of the universal enveloping algebra $U(\mathfrak{g})$.

3. Let H^* be the (finite) dual of a Hopf algebra H.

   Then grouplike elements of H^* are the algebra homomorphisms $\alpha : H \to \mathbb{F}$, also called **characters** of H. This follows, because an element $0 \neq \alpha \in H^*$ is grouplike if and only if for all $h, k \in H$ $\alpha(h \cdot k) = \Delta(\alpha)(h \otimes k) = (\alpha \otimes \alpha)(h \otimes k) = \alpha(h)\alpha(k)$. As $\alpha \neq 0$, this implies $\alpha(1) = \epsilon(\alpha) = 1$.

   An element $\beta \in H^*$ is primitive if and only if for all $h, k \in H$ it satisfies the condition $\beta(h \cdot k) = \Delta(\beta)(h \otimes k) = (1 \otimes \beta + \beta \otimes 1)(h \otimes k) = \epsilon(h)\beta(k) + \epsilon(k)\beta(h)$. A linear map $\beta : H \to \mathbb{F}$ that satisfies this condition is called a **derivation** on H.

The reason for the name *grouplike element* is not only that grouplike elements mimic the behaviour of elements $g \in G$ in the group algebra $\mathbb{F}[G]$, but one can show that they form indeed a group. *Primitive elements* could in principle be called *Lie algebra-like* since they form a Lie algebra with the commutator. One can show that both, grouplike and primitive elements generate Hopf subalgebras of H.

Proposition 6.1.3: Let H be a Hopf algebra.

1. Every grouplike element $g \in H$ satisfies $\epsilon(g) = 1$ and $S(g) = g^{-1}$.
2. The set $\mathrm{Gr}(H) \subset H$ is a group and $\mathrm{span}_{\mathbb{F}}\mathrm{Gr}(H) \subset H$ is a Hopf subalgebra.
3. Every primitive element $h \in H$ satisfies $\epsilon(h) = 0$ and $S(h) = -h$.
4. The set $\mathrm{Pr}(H) \subset H$ is a Lie subalgebra of the Lie algebra H with the commutator, and the subalgebra of H generated by $\mathrm{Pr}(H)$ is a Hopf subalgebra.
5. If $g \in H$ is grouplike and $h \in H$ primitive, then ghg^{-1} is primitive.

Proof:
1. If $g \in H$ is grouplike, then $\Delta(g) = g \otimes g$ and $g \neq 0$. The counitality condition then implies $1 \otimes g = (\epsilon \otimes \mathrm{id}) \circ \Delta(g) = (\epsilon \otimes \mathrm{id})(g \otimes g) = \epsilon(g) \otimes g = 1 \otimes \epsilon(g)g$. As $g \neq 0$ it follows that $\epsilon(g) = 1$. Similarly, we have $\eta \circ \epsilon(g) = 1 = m \circ (S \otimes \mathrm{id}) \circ \Delta(g) = S(g) \cdot g = m \circ (\mathrm{id} \otimes S) \circ \Delta(g) = g \cdot S(g)$. This shows that $S(g)$ is an inverse of g.

2. As Δ is an algebra homomorphism, we have $\Delta(1) = 1 \otimes 1$ and $1 \in \mathrm{Gr}(H)$. If $g, h \in \mathrm{Gr}(H)$, then $\Delta(gh) = \Delta(g) \cdot \Delta(h) = (g \otimes g)(h \otimes h) = gh \otimes gh$ and hence $gh \in \mathrm{Gr}(H)$. As S is an anti-algebra homomorphism, we obtain $\Delta(g^{-1}) = \Delta(S(g)) = (S \otimes S)(\Delta(g)) = S(g) \otimes S(g) = g^{-1} \otimes g^{-1}$ and hence $gh, g^{-1} \in \mathrm{Gr}(H)$. This shows that $\mathrm{Gr}(H)$ is a group. By definition of a grouplike element, one has $\Delta(g) = g \otimes g$ and hence $\mathrm{span}_{\mathbb{F}}\mathrm{Gr}(H) \subset H$ is a Hopf subalgebra.

3. As $1 \otimes h = (\epsilon \otimes \mathrm{id}) \circ \Delta(h) = (\epsilon \otimes \mathrm{id}) \circ (1 \otimes h + h \otimes 1) = \epsilon(1) \otimes h + \epsilon(h) \otimes 1 = 1 \otimes h + \epsilon(h) \otimes 1$ for each primitive $h \in H$, we have $\epsilon(h) = 0$. Similarly, for each primitive element $h \in H$ we have $m \circ (S \otimes \mathrm{id}) \circ \Delta(h) = (m \circ S)(1 \otimes h + h \otimes 1) = S(1) \cdot h + S(h) \cdot 1 = h + S(h) = \eta \circ \epsilon(h) = 0$ and hence $S(h) = -h$.

4. It follows from the definition of a primitive element that $\mathrm{Pr}(H) \subset H$ is a linear subspace. If $h, k \in \mathrm{Pr}(H)$, then their commutator $[h, k] = h \cdot k - k \cdot h$ satisfies

$$\begin{aligned}\Delta([h,k]) &= [\Delta(h), \Delta(k)] = (1 \otimes h + h \otimes 1) \cdot (1 \otimes k + k \otimes 1) - (1 \otimes k + k \otimes 1) \cdot (1 \otimes h + h \otimes 1) \\ &= 1 \otimes hk + k \otimes h + h \otimes k + hk \otimes 1 - (1 \otimes kh + h \otimes k + k \otimes h + kh \otimes 1) = 1 \otimes [h,k] + [h,k] \otimes 1.\end{aligned}$$

This shows that $[h, k] \in \mathrm{Pr}(H)$ and hence $\mathrm{Pr}(H) \subset H$ is a Lie subalgebra of the Lie algebra H with the commutator.

As $\Delta(h) = 1\otimes h + h\otimes 1$ and $S(h) = -h$ for every primitive element $h \in H$ and the maps $\Delta : H \to H\otimes H$ and $S : H \to H^{op}$ are algebra homomorphisms, it follows that the subalgebra of H generated by the primitive elements is a Hopf subalgebra of H.

5. If $g \in H$ is grouplike and $h \in H$ primitive, then

$$\Delta(ghg^{-1}) = \Delta(g)\Delta(h)\Delta(g^{-1}) = (g\otimes g)(1\otimes h + h\otimes 1)(g^{-1}\otimes g^{-1}) = 1\otimes ghg^{-1} + ghg^{-1}\otimes 1.$$

□

Proposition 6.1.3 suggests that every Hopf algebra H should contains a Hopf subalgebra K that is a semidirect product $K = \mathbb{F}[\mathrm{Gr}(H)] \ltimes A$ of the group algebra of $\mathrm{Gr}(H)$ and the Hopf subalgebra $A \subset H$ generated by the primitive elements. In other words, $K \cong \mathbb{F}[G]\otimes A$ as a vector space with the multiplication law $(a\otimes g)\cdot(b\otimes h) = a(gbg^{-1})\otimes gh$ for all $a, b \in A$ and $g, h \in \mathrm{Gr}(H)$. To show that this is indeed the case, we need to prove that different grouplike elements of H are linearly independent, i. e. that $\mathrm{span}_{\mathbb{F}}\mathrm{Gr}(H) \cong \mathbb{F}[\mathrm{Gr}(H)]$ and that $\mathrm{Gr}(H) \cap A = \{1_H\}$.

Proposition 6.1.4: Let H be a Hopf algebra over $\mathbb{F}$.

1. The set $\mathrm{Gr}(H)$ of grouplike elements is linearly independent.
2. If H is generated as an algebra by primitive elements, then $\mathrm{Gr}(H) = \{1\}$.

Proof:
1. We show by induction over n that $\Sigma_{i=1}^n \lambda_i g_i = 0$ with $\lambda_i \in \mathbb{F}$ and distinct $g_i \in \mathrm{Gr}(H)$ implies $\lambda_1 = ... = \lambda_n = 0$. For $n = 1$, this follows from $g \neq 0$ for all $g \in \mathrm{Gr}(H)$. Suppose the claim holds for all linear combinations with at most n nontrivial coefficients, and let $\Sigma_{i=1}^{n+1}\lambda_i g_i = 0$ with pairwise distinct $g_i \in \mathrm{Gr}(H)$.

Let $\iota : H \to H^{**}$, $h \mapsto h'$ be the canonical injection defined by $h'(\alpha) = \alpha(h)$ for all $\alpha \in H^*$. Then the elements $g_i' \in H^{**}$ satisfy $g_i'(\alpha \cdot \beta) = (\alpha \cdot \beta)(g_i) = (\alpha\otimes\beta)\Delta(g_i) = \alpha(g_i)\beta(g_i) = g_i'(\alpha) \cdot g_i'(\beta)$ for all $\alpha, \beta \in H^*$ and $g_i'(1) = \epsilon(g_i) = 1_{\mathbb{F}}$. As $g_{n+1} \notin \{g_1, ..., g_n\}$ there is an element $\alpha \in H^*$ with $g_{n+1}'(\alpha) = \alpha(g_{n+1}) = 1$ and $g_i'(\alpha) = \alpha(g_i) \neq 1$ for all $i \in \{1, ..., n\}$ by Exercise 37 (a). This implies for all $\beta \in H^*$

$$0 = \Sigma_{i=1}^{n+1}\lambda_i g_i'(\beta) - \Sigma_{i=1}^{n+1}\lambda_i g_i'(\alpha \cdot \beta) = \Sigma_{i=1}^{n+1}\lambda_i(1 - g_i'(\alpha))g_i'(\beta) = \beta(\Sigma_{i=1}^{n}\lambda_i(1 - \alpha(g_i))g_i)$$

and hence $\Sigma_{i=1}^n\lambda_i(1 - \alpha(g_i))g_i = 0$. With the induction hypothesis and $\alpha(g_i) \neq 1$ one obtains $\lambda_1 = ... = \lambda_n = 0$, and this implies $\lambda_{n+1} = 0$ since $g_{n+1} \neq 0$.

2. Let $H_0 = \mathbb{F}1_H$, $X \subset H$ a set of primitive generators and H_n the linear subspace of H spanned by all elements of the form $x_{i_1}^{m_1} \cdots x_{i_k}^{m_k}$ with $x_{i_j} \in X$ and $m_1 + ... + m_k \leq n$. Then we have $H = \cup_{n=0}^{\infty} H_n$, $H_n \subset H_m$ for all $m \geq n$, and $H_n \cdot H_m \subset H_{n+m}$. It follows by induction with the Pascal identity for the binomial coefficients that for any primitive element $h \in H$, one has

$$\Delta(h^n) = \sum_{k=0}^{n} \binom{n}{k} h^k \otimes h^{n-k}. \tag{65}$$

This implies

$$\Delta(x_{i_1}^{m_1} \cdots x_{i_k}^{m_k}) = \Delta(x_{i_1})^{m_1} \cdots \Delta(x_{i_k})^{m_k} = \sum_{l_1=0}^{m_1} \cdots \sum_{l_k=0}^{m_k} \binom{m_1}{l_1} \cdots \binom{m_k}{l_k} x_{i_1}^{l_1} \cdots x_{i_k}^{l_k} \otimes x_{i_1}^{m_1-l_1} \cdots x_{i_k}^{m_k-l_k}$$

and hence $\Delta(H_n) \subset \Sigma_{k=0}^n H_k \otimes H_{n-k}$. If $g \in H$ is grouplike with $m = \min\{n \in \mathbb{N}_0 \,|\, g \in H_n\} \geq 1$, then there is an $\alpha \in H^*$ with $\alpha(g) = 1$ and $\alpha(1_H) = 0$, and this implies

$$1_{\mathbb{F}} \otimes g = (\alpha \otimes \mathrm{id})(g \otimes g) = (\alpha \otimes \mathrm{id}) \circ \Delta(g) \in (\alpha \otimes \mathrm{id})\left(\Sigma_{k=0}^m H_k \otimes H_{m-k}\right) \subset 1_{\mathbb{F}} \otimes H_{m-1}$$

where we used in the last step that $\alpha(H_0) = \alpha(\mathbb{F}1_H) = \{0\}$. As $g \neq 0$, it follows that $g \in H_{m-1}$, which contradicts the minimality of m. Hence $\mathrm{Gr}(H) \subset H_0$, and the only grouplike element in H_0 is 1_H. □

Corollary 6.1.5: Let G be a group and $\mathbb{F}$ a field. Then $\mathrm{Pr}(\mathbb{F}[G]) = \{0\}$ and $\mathrm{Gr}(\mathbb{F}[G]) = G$.

Proof:
If $x = \Sigma_{g \in G} \lambda_g\, g$ is primitive, then $\Delta(x) = \Sigma_{g \in G} \lambda_g\, g \otimes g = \Sigma_{g \in G} \lambda_g (1 \otimes g + g \otimes 1)$. As the set $\{g \otimes h \,|\, g, h \in G\}$ is a basis of $\mathbb{F}[G] \otimes \mathbb{F}[G]$, this implies $\lambda_g = 0$ for all $g \in G$ and $x = 0$. Clearly, every element $g \in G$ is grouplike. If there was a grouplike element $y \in \mathbb{F}[G] \setminus G$, then the set $G \cup \{y\} \supsetneq G$ would be linearly independent by Proposition 6.1.4, a contradiction to the fact that $G \subset \mathbb{F}[G]$ is a basis of $\mathbb{F}[G]$. □

This corollary confirms the expectation that the only grouplike elements in a group algebra $\mathbb{F}[G]$ are the group elements $g \in G$ and that the group algebra contains no non-trivial primitive elements. Similarly, Proposition 6.1.4 implies that the only grouplike element in the tensor algebra $T(V)$ and in a universal enveloping algebra $U(\mathfrak{g})$ is the unit element, since both Hopf algebras are generated by primitive elements. In analogy to the statement about the grouplike elements in a group algebra $\mathbb{F}[G]$, one would expect that the primitive elements in a universal enveloping algebra $U(\mathfrak{g})$ are precisely the elements of the Lie algebra $\mathfrak{g} \subset U(\mathfrak{g})$. However, this is only true for Lie algebras over fields of characteristic zero.

Proposition 6.1.6: Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{F}$ and $U(\mathfrak{g})$ its universal enveloping algebra.

1. If $\mathrm{char}(\mathbb{F}) = 0$ then $\mathrm{Pr}(U(\mathfrak{g})) = \mathfrak{g}$.
2. If $\mathrm{char}(\mathbb{F}) = p$ then $\mathrm{Pr}(U(\mathfrak{g})) = \mathrm{span}_{\mathbb{F}}\{x^{p^l} \,|\, x \in \mathfrak{g}, l \in \mathbb{N}_0\}$.

Proof:
Every element $x \in \mathfrak{g} \subset U(\mathfrak{g})$ is primitive, and hence $\mathfrak{g} \subset \mathrm{Pr}(U(\mathfrak{g}))$. If $B = (x_1, ..., x_n)$ is an ordered basis of $\mathfrak{g}$, then the Poincaré-Birkhoff-Witt basis $B = \{x_1^{m_1} \cdots x_n^{m_n} \,|\, m_1, ..., m_n \in \mathbb{N}_0\}$ is a basis of $U(\mathfrak{g})$, and hence every element $x \in U(\mathfrak{g})$ can be expressed as a linear combination

$$x = \sum_{m_1=0}^{K} \cdots \sum_{m_n=0}^{K} \lambda_{m_1 \ldots m_n} x_1^{m_1} \cdots x_n^{m_n}$$

with $\lambda_{m_1 \ldots m_n} \in \mathbb{F}$ for some $K \in \mathbb{N}$. Equation (65) from the proof of Proposition 6.1.4 implies

$$\Delta(x) = \sum_{m_1=0}^{K} \cdots \sum_{m_n=0}^{K} \sum_{k_1=0}^{m_1} \cdots \sum_{k_n=0}^{m_n} \lambda_{m_1 \ldots m_n} \binom{m_1}{k_1} \cdots \binom{m_n}{k_n} x_1^{k_1} \cdots x_n^{k_n} \otimes x_1^{m_1-k_1} \cdots x_n^{m_n-k_n}.$$

As the set $\{x_1^{k_1}\cdots x_n^{k_n}\,|\,k_1,...,k_n\in\{0,...,K\}\}$ is linearly independent by the Poincaré-Birkhoff-Witt Theorem, this shows that x cannot be primitive, unless it is of the form $x=\Sigma_{i=1}^n\mu_i x_i^{m_i}$ for some $\mu_i\in\mathbb{F}$. In this case, one has

$$\Delta(x)=\sum_{i=1}^n\sum_{k=0}^{m_i}\mu_i\binom{m_i}{k}x_i^k\otimes x_i^{m_i-k}.$$

If $\mathrm{char}(\mathbb{F})=0$, all binomial coefficients in this formula are non-zero, and this shows that x can only be primitive if $m_i\in\{0,1\}$ for all $i\in\{1,...,n\}$ with $\mu_i\neq 0$, which implies $x\in\mathfrak{g}$. If $\mathrm{char}(\mathbb{F})=p$, then all binomial coefficients for $i\in\{1,...,n\}$ with $\mu_i\neq 0$ and $k=1<m_i$ must vanish in order for x to be primitive. This is the case if and only if $m_i=p^{l_i}$ for some $l_i\in\mathbb{N}$ and all $i\in\{1,...,n\}$ with $\mu_i\neq 0$. Conversely, if $m_i=p^{l_i}$ with $l_i\in\mathbb{N}_0$, then all binomial coefficients for $k\notin\{0,m_i\}$ vanish, since they are divisible by p, and this shows that x is a linear combination of elements y^{p^l} with $y\in\mathfrak{g}$ and $l\in\mathbb{N}_0$. □

As the restrictions of the comultiplication of a Hopf algebra H to the Hopf subalgebras $\mathrm{span}_{\mathbb{F}}\mathrm{Gr}(H)$ and to the Hopf subalgebra generated by the set $\mathrm{Pr}(H)$ are cocommutative, one cannot hope in general that every Hopf algebra can be decomposed into Hopf subalgebras spanned by grouplike or generated by primitive elements, since this would imply that H is cocommutative. However, one can show that this is indeed possible for every *cocommutative Hopf algebra* over an *algebraically closed field of characteristic zero*. This is known as the *Cartier-Kostant-Milnor-Moore Theorem*. Parts of the proof are given in [Mo, Chapter 5].

Theorem 6.1.7: (**Cartier-Kostant-Milnor-Moore Theorem**)
If H is a cocommutative Hopf algebra over an algebraically closed field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F})=0$, then H is isomorphic to the semidirect product $U(\mathrm{Pr}(H))\rtimes\mathbb{F}[\mathrm{Gr}(H)]$, i. e. to the vector space $U(\mathrm{Pr}(H))\otimes\mathbb{F}[\mathrm{Gr}(H)]$ with the product

$$(x\otimes g)\cdot(y\otimes h)=(x+gyg^{-1})\otimes gh\qquad\forall g,h\in\mathrm{Gr}(H),x,y\in\mathrm{Pr}(H).$$

Corollary 6.1.8: Every finite-dimensional cocommutative Hopf algebra over an algebraically closed field of characteristic zero is isomorphic to the group algebra of a finite group.

6.2 Integrals and the antipode

The fact that representations of bialgebras and Hopf algebras behave quite similarly to group representations allows one to generalise concepts from the representation theory of groups to Hopf algebras. In particular, there is a notion of invariants for each module over a bialgebra and a dual notion of coinvariants for a comodule.

For a module M over a group algebra $\mathbb{F}[G]$, an element $m\in M$ is called an *invariant*, if $g\rhd m=m$ for all group elements $g\in G$. Using the fact that the group elements $g\in G$ form a basis of $\mathbb{F}[G]$ and the expression for the counit of $\mathbb{F}[G]$ from Example 5.2.9, we can reformulate this condition as $h\rhd m=\epsilon(h)m$ for all $h\in H$. The latter can be formulated and imposed for any bialgebra B over $\mathbb{F}$. Replacing the module over a bialgebra by a comodule, we can then formulate a dual condition, namely $\delta(m)=1\otimes m$. This leads to the notion of a coinvariant.

Definition 6.2.1: Let B be a bialgebra over $\mathbb{F}$.

1. An element $m \in M$ of a B-module $(M, \rhd)$ is called an **invariant**, if $b \rhd m = \epsilon(b)\, m$ for all $b \in B$. The submodule of invariants of M is denoted M^B.
2. An element $m \in M$ of a B-comodule (M, δ) is called a **coinvariant** of M if $\delta(m) = 1_B \otimes m$. The subcomodule of invariants in M is denoted M^{coB}.

The notions of invariants and coinvariants capture many familiar concepts such as invariant functions on a set with a group action, the centre of a Hopf algebra H and H-linear maps between H-modules.

Example 6.2.2:

1. Let $\rhd : G \times X \to X$ be an action of a group G on a set X and denote by $\rhd : \mathbb{F}[G] \otimes \mathrm{Map}(X, \mathbb{F}) \to \mathrm{Map}(X, \mathbb{F})$ be the associated $\mathbb{F}[G]$-module structure on $\mathrm{Map}(X, \mathbb{F})$ with $(g \rhd f)(x) = f(g^{-1} \rhd x)$ for all $g \in G$ and $x \in X$. Then the invariants are given by
$$\mathrm{Map}(X, \mathbb{F})^{\mathbb{F}[G]} = \{f : X \to \mathbb{F} \mid f(g \rhd x) = f(x)\, \forall g \in G\}.$$
2. Let H be a Hopf algebra acting on itself via the adjoint action $\rhd_{ad} : H \otimes H \to H$ with $h \rhd_{ad} k = \Sigma_{(h)}\, h_{(1)} \cdot k \cdot S(h_{(2)})$. Then the submodule of invariants is the centre of H
$$H^{\rhd_{ad}} = Z(H) = \{k \in H \mid h \cdot k = k \cdot h\, \forall h \in H\}.$$
If $k \in Z(H)$, then $h \rhd_{ad} k = \Sigma_{(h)} h_{(1)} \cdot k \cdot S(h_{(2)}) = (\Sigma_{(h)} h_{(1)} S(h_{(2)})) \cdot k = \epsilon(h)\, k$ for all $h \in H$. Conversely, if $k \in H^{\rhd_{ad}}$ then for all $h \in H$
$$\begin{aligned} h \cdot k &= (\Sigma_{(h)} h_{(1)} \epsilon(h_{(2)})) \cdot k = \Sigma_{(h)} h_{(1)} k S(h_{(2)(1)}) h_{(2)(2)} = \Sigma_{(h)} (h_{(1)(1)} k S(h_{(1)(2)})) h_{(2)} \\ &= \Sigma_{(h)} (h_{(1)} \rhd_{ad} k) h_{(2)} = \Sigma_{(h)} \epsilon(h_{(1)})\, k h_{(2)} = k \cdot \big(\Sigma_{(h)} \epsilon(h_{(1)}) h_{(2)}\big) = k \cdot h. \end{aligned}$$
3. Let H be a Hopf algebra and M, N modules over H. Then the H-linear maps $f : M \to N$ are the invariants of the H-left module structure
$$\rhd : H \otimes \mathrm{Hom}_{\mathbb{F}}(M, N) \to \mathrm{Hom}_{\mathbb{F}}(M, N), \qquad (h \rhd f)(m) = \Sigma_{(h)}\, h_{(1)} \rhd_N f(S(h_{(2)}) \rhd_M m).$$
That $\rhd$ defines an H-module structure on $\mathrm{Hom}_{\mathbb{F}}(M, N)$ follows by a direct computation (Exercise). For each H-linear map $f : M \to N$ we have
$$(h \rhd f)(m) = \Sigma_{(h)} h_{(1)} \rhd_N f(S(h_{(2)}) \rhd_M m) = \Sigma_{(h)}\, (h_{(1)} S(h_{(2)})) \rhd_N f(m) = \epsilon(h) f(m)$$
and hence $f \in \mathrm{Hom}_{\mathbb{F}}(M, N)^H$. Conversely, if $f \in \mathrm{Hom}_{\mathbb{F}}(M, N)^H$, then for all $m \in M$
$$\begin{aligned} h \rhd_N f(m) &= \Sigma_{(h)} h_{(1)} \rhd_N f((S(h_{(2)}) h_{(3)}) \rhd_M m) = \Sigma_{(h)} (h_{(1)} \rhd f)(h_{(2)} \rhd_M m) \\ &= \Sigma_{(h)} \epsilon(h_{(1)})\, f(h_{(2)} \rhd_M h) = \Sigma_{(h)} f((\Sigma_{(h)} \epsilon(h_{(1)}) h_{(2)}) \rhd_M m) = f(h \rhd_M m). \end{aligned}$$

Just as for any ring, every module M over a bialgebra B can be related to the action of B on itself by left multiplication. This is achieved by the B-linear maps $\rhd_m : B \to M$, $b \mapsto b \rhd m$ for elements $m \in M$. Any B-linear map $f : M \to N$ sends invariants to invariants, since one has $b \rhd f(m) = f(b \rhd m) = \epsilon(b) f(m)$ for all $m \in M^B$. We can thus describe the invariants of an B-module M in terms of the invariants for the left action of B on itself by left multiplication.

Analogously, one can describe invariants of H-right modules in terms of the invariants of B with the right action by right multiplication.

The invariants of a bialgebra action on itself by left or right multiplication play a special role and are called *integrals*. They can be viewed as the counterparts of left or right invariant integrals on a Lie group. For the same reason, an integral in a bialgebra B that is both left and right invariant is sometimes called a *Haar integral* in B. Just as Haar integrals over compact Lie groups can be used to construct invariant functions on Lie groups, integrals in a bialgebra that satisfy certain normalisation conditions define projectors on the invariants of B-modules.

Definition 6.2.3: Let B be a bialgebra over $\mathbb{F}$.

1. A **left (right) integral** in B is an invariant for the left (right) regular action of B on itself: an element $\ell \in B$ with $b \cdot \ell = \epsilon(b)\,\ell$ (with $\ell \cdot b = \epsilon(b)\,\ell$) for all $b \in B$.
2. The linear subspaces of left and right integrals in B are denoted $I_L(B)$ and $I_R(B)$. If $I_L(B) = I_R(B)$, then the bialgebra B is called **unimodular**.
3. A left or right integral $\ell \in B$ is called **normalised** if $\epsilon(\ell) = 1$. A (normalised) element $\ell \in I_L(B) \cap I_R(B)$ is called a **(normalised) Haar integral**.

Example 6.2.4:

1. If G is a finite group, then $\mathbb{F}[G]$ and $\mathbb{F}[G]^*$ are unimodular with
$$I_L(\mathbb{F}[G]) = I_R(\mathbb{F}[G]) = \mathrm{span}_{\mathbb{F}}\,\{\Sigma_{g\in G}\, g\} \qquad I_L(\mathbb{F}[G]^*) = I_R(\mathbb{F}[G]^*) = \mathrm{span}_{\mathbb{F}}\,\{\delta_e\}\,.$$
The integral δ_e is normalised. The integral $\Sigma_{g\in G} g$ can be normalised iff $\mathrm{char}(\mathbb{F}) \not| \, |G|$.
2. The Taft algebra from Example 5.3.6. is not unimodular. One has (Exercise 50)
$$I_L(H) = \mathrm{span}_{\mathbb{F}}\{\Sigma_{j=0}^{n-1} y^j x^{n-1}\} \qquad I_R(H) = \mathrm{span}_{\mathbb{F}}\{\Sigma_{j=0}^{n-1} q^{-j} y^j x^{n-1}\}.$$
3. The q-deformed universal enveloping algebra $U_q(\mathfrak{sl}_2)$ from Example 5.3.9 has no non-trivial left or right integrals: $I_L(U_q(\mathfrak{sl}_2)) = I_R(U_q(\mathfrak{sl}_2)) = \{0\}$. The q-deformed universal enveloping algebra $U_q^r(\mathfrak{sl}_2)$ at a root of unity from Proposition 5.3.12 is unimodular with
$$I_L(U^r(\mathfrak{sl}_2)) = I_R(U^r(\mathfrak{sl}_2)) = \mathrm{span}_{\mathbb{F}}\{\Sigma_{j=0}^{r-1} K^j E^{r-1} F^{r-1}\}$$

The usefulness of integrals comes from the fact that given a normalised left or right integral for a bialgebra, we can construct a projector on the invariants of any left or right module over B. This allows one to determine the invariants of any module over B explicitly and systematically.

Lemma 6.2.5: Let B be a bialgebra and $\ell \in B$ a normalised left integral. Then for any B-module $(M, \rhd)$, the linear map $P : M \to M$, $m \mapsto \ell \rhd m$ is a projector on M^B.

Proof:
This follows directly from the properties of the normalised left integral $\ell \in B$. One has
$$b \rhd P(m) = b \rhd (\ell \rhd m) = (b\ell) \rhd m = (\epsilon(b)\ell) \rhd m = \epsilon(b)\,(\ell \rhd m) = \epsilon(b)\,P(m)$$
$$(P \circ P)(m) = \ell \rhd (\ell \rhd m) = (\ell^2) \rhd m = (\epsilon(\ell)\,\ell) \rhd m = \ell \rhd m = P(m)$$

for all $m \in M$ and $b \in B$, and this shows that P is a projector with $\mathrm{im}(P) \subset M^B$. Conversely, if $m \in M^B$, one has $P(m) = \ell \rhd m = \epsilon(\ell)\, m = m$ and hence $m \in P(M)$. □

This lemma and Example 6.2.4 make it worthwhile to investigate left and right integrals in bialgebras and Hopf algebras. The first point to address is their existence and uniqueness. For finite-dimensional Hopf algebras, the existence and uniqueness of integrals can be established using the concept of a *Hopf module*.

A Hopf module can be seen as a structure that generalises a bialgebra in the same way as a module generalises an algebra and a comodule a coalgebra. It has both, a module and a comodule structure, and these structures must satisfy a compatibility condition that generalises the compatibility condition between the multiplication and comultiplication in a bialgebra. Hopf modules exist in several variants, depending on if one works with a right or left module and comodule structure. For us, it is sufficient to consider right Hopf modules.

Definition 6.2.6: Let B be a bialgebra over $\mathbb{F}$.

1. A **right Hopf module** over B is an B-right module and B-right comodule
$$\lhd : V \otimes B \to V, \;\; b \otimes v \mapsto v \lhd b \qquad \delta : V \to V \otimes B, \;\; v \mapsto \Sigma_{(v)} v_{(0)} \otimes v_{(1)},$$
such that δ is a morphism of B-right modules:
$$\delta(v \lhd b) = \delta(v) \lhd b = \Sigma_{(v)(b)}\, (v_{(0)} \lhd b_{(1)}) \otimes (v_{(1)} \cdot b_{(2)})$$

2. A **homomorphism of right Hopf modules** from $(V, \rhd_V, \delta_V)$ to $(W, \rhd_W, \delta_W)$ is a linear map $\phi : V \to W$ that is a homomorphism of right b-modules and right b-comodules:
$$\phi \circ \lhd_V = \lhd_W \circ (\phi \otimes \mathrm{id}_B) \qquad (\phi \otimes \mathrm{id}_B) \circ \delta_V = \delta_W \circ \phi.$$

Example 6.2.7:

1. Every bialgebra B is a right Hopf module over itself with its right action by right multiplication $\lhd_R : B \otimes B \to B$, $k \lhd b = k \cdot b$ and the comultiplication $\delta = \Delta : B \to B \otimes B$.

2. For every right module $(M, \lhd)$ over a bialgebra B, the vector space $M \otimes B$ is a Hopf module with
$$\begin{aligned} &\lhd : (M \otimes B) \otimes B \to M \otimes B, && (m \otimes k) \lhd b = \Sigma_{(b)}\, (m \lhd b_{(1)}) \otimes (k \cdot b_{(2)}) \\ &\delta = (\mathrm{id}_M \otimes \Delta) : M \otimes B \to (M \otimes B) \otimes B, && \delta(m \otimes k) = \Sigma_{(k)} m \otimes k_{(1)} \otimes k_{(2)}. \end{aligned}$$

3. In particular, for every vector space V over $\mathbb{F}$, the vector space $V \otimes B$ is a Hopf module over B with the **trivial Hopf module structure**
$$\begin{aligned} &\lhd = \mathrm{id} \otimes m : (V \otimes B) \otimes B \to V \otimes B, && (v \otimes k) \lhd b = v \otimes kb \\ &\delta = (\mathrm{id} \otimes \Delta) : V \otimes B \to (V \otimes B) \otimes B, && \delta(v \otimes k) = \Sigma_{(k)} v \otimes k_{(1)} \otimes k_{(2)}. \end{aligned}$$

The fact that these examples are largely trivial is not due to the choice of examples. In fact, the distinguishing property of Hopf modules over a Hopf algebra is that they *factorise* into their subspace of coinvariants and the underlying Hopf algebra H. Every Hopf module is isomorphic as a Hopf module to the tensor product of its coinvariants with H, equipped with the trivial Hopf module structure from Example 6.2.7, 3.

Theorem 6.2.8: (**Fundamental theorem of Hopf modules**)
Let H be a Hopf algebra over $\mathbb{F}$ and $(M, \lhd, \delta)$ a right Hopf module over H.

1. An H-invariant projector on $M^{coH} \subset M$ is given by
$$P = \lhd \circ (\mathrm{id} \otimes S) \circ \delta : M \to M, \quad m \mapsto \Sigma_{(m)} m_{(0)} \lhd S(m_{(1)}).$$
2. If $M^{coH} \otimes H$ is equipped with the trivial Hopf module structure from Example 6.2.7, 3, then the action map defines an isomorphism of Hopf modules
$$\phi = \lhd : M^{coH} \otimes H \to M, \quad m \otimes h \mapsto m \lhd h.$$

Proof:
We denote by $\lhd : M \otimes H \to M$ the H-right module structure and by $\delta : M \to M \otimes H$ the H-right comodule structure on M. Then we have in Sweedler notation $\delta(m) = \Sigma_{(m)} m_{(0)} \otimes m_{(1)}$ and the Hopf module conditions on δ read

$$\begin{aligned}
&(\delta \otimes \mathrm{id}) \circ \delta(m) = \Sigma_{(m)} m_{(0)} \otimes m_{(1)} \otimes m_{(2)} = (\mathrm{id} \otimes \Delta) \circ \delta(m) \\
&(\mathrm{id} \otimes \epsilon) \circ \delta(m) = \Sigma_{(m)} m_{(0)} \epsilon(m_{(1)}) = m \\
&\delta(m \lhd h) = \delta(m) \lhd h = \Sigma_{(m)(h)} m_{(0)} \lhd h_{(1)} \otimes m_{(1)} h_{(2)}.
\end{aligned}$$

The trivial Hopf module structure on $M^{coH} \otimes H$ is given by

$$(m \otimes k) \lhd' h = m \otimes kh \qquad \delta'(m \otimes k) = \Sigma_{(k)} m \otimes k_{(1)} \otimes k_{(2)}.$$

1. We show that P is an H-invariant projector on M^{coH}: for all $m \in M$ and $h \in H$ we have

$$\begin{aligned}
\delta(P(m)) &= \delta\left(\Sigma_{(m)} m_{(0)} \lhd S(m_{(1)})\right) = \Sigma_{(m)} \delta(m_{(0)}) \lhd S(m_{(1)}) \\
&= \Sigma_{(m)} (m_{(0)} \otimes m_{(1)}) \lhd S(m_{(2)}) = \Sigma_{(m)} [m_{(0)} \lhd S(m_{(2)})_{(1)}] \otimes [m_{(1)} S(m_{(2)})_{(2)}] \\
&= \Sigma_{(m)} [m_{(0)} \lhd S(m_{(2)(2)})] \otimes [m_{(1)} S(m_{(2)(1)})] = \Sigma_{(m)} [m_{(0)} \lhd S(m_{(3)})] \otimes [m_{(1)} S(m_{(2)})] \\
&= \epsilon(m_{(1)}) [\Sigma_{(m)} m_{(0)} \lhd S(m_{(2)})] \otimes 1 = \Sigma_{(m)} [m_{(0)} \lhd S(m_{(1)})] \otimes 1 = P(m) \otimes 1 \\
P(m \lhd h) &= \lhd \circ (\mathrm{id} \otimes S) \circ \delta(m \lhd h) = \lhd \circ (\mathrm{id} \otimes S)(\delta(m) \lhd h) \\
&= \lhd \circ (\mathrm{id} \otimes S)(\Sigma_{(m)(h)} m_{(0)} \lhd h_{(1)} \otimes m_{(1)} h_{(2)}) \\
&= \Sigma_{(m)(h)} \, (m_{(0)} \lhd h_{(1)}) \lhd S(m_{(1)} h_{(2)}) = \Sigma_{(m)(h)} (m_{(0)} \lhd h_{(1)}) \lhd (S(h_{(2)}) S(m_{(1)})) \\
&= \Sigma_{(m)(h)} (m_{(0)} \lhd (h_{(1)} S(h_{(2)}))) \lhd S(m_{(1)})) = \epsilon(h) \Sigma_{(m)} (m_{(0)} \lhd 1) \lhd S(m_{(1)}) \\
&= \epsilon(h) \, P(m).
\end{aligned}$$

For all coinvariants $m \in M^{coH}$ this yields

$$P(m) = \lhd \circ (\mathrm{id} \otimes S) \circ \delta(m) = \lhd \circ (\mathrm{id} \otimes S)(m \otimes 1) = m \lhd S(1) = m \lhd 1 = m.$$

This implies $(P \circ P)(m) = P(m)$ for all $m \in M$ and shows that P is a projector on M^{coH}.

2. We show that the map $\phi : M^{coH} \otimes H \to M$ is a homomorphism of Hopf modules.
For all $m \in M^{coH}$ and $h, k \in H$, we have

$$\begin{aligned}
\phi(m \otimes k) \lhd h &= (m \lhd k) \lhd h = m \lhd (kh) = \phi(m \otimes kh) = \phi((m \otimes k) \lhd' h) \\
(\delta \circ \phi)(m \otimes k) &= \delta(m \lhd k) = \delta(m) \lhd k = (m \otimes 1) \lhd k = \Sigma_{(k)} (m \lhd k_{(1)}) \otimes k_{(2)} \\
&= (\phi \otimes \mathrm{id}) \left(\Sigma_{(k)} m \otimes k_{(1)} \otimes k_{(2)}\right) = ((\phi \otimes \mathrm{id}) \circ \delta')(m \otimes k).
\end{aligned}$$

3. We show that the linear map

$$\chi = (P \otimes \mathrm{id}) \circ \delta : M \to M^{coH} \otimes H, \quad m \mapsto \Sigma_{(m)} [m_{(0)} \lhd S(m_{(1)})] \otimes m_{(2)}$$

is inverse to ϕ. Using the fact that P is a projector on M^{coH} that is invariant under the action of H on M, we obtain for all $m \in M$, $n \in M^{coH}$ and $h \in H$

$$\begin{aligned}
&\phi \circ \chi(m) = \phi(\Sigma_{(m)} [m_{(0)} \lhd S(m_{(1)})] \otimes m_{(2)}) = \Sigma_{(m)} (m_{(0)} \lhd S(m_{(1)})) \lhd m_{(2)} \\
&= \Sigma_{(m)} m_{(0)} \lhd (S(m_{(1)}) m_{(2)}) = \Sigma_{(m)} m_{(0)} \lhd (\epsilon(m_{(1)}) 1) = \Sigma_{(m)} (\epsilon(m_{(1)}) m_{(0)}) \lhd 1 = m
\end{aligned}$$

$$\begin{aligned}
&\chi \circ \phi(n \otimes h) = \chi(n \lhd h) = (P \otimes \mathrm{id}) \delta(n \lhd h) = (P \otimes \mathrm{id})(\delta(n) \lhd h) = (P \otimes \mathrm{id})((n \otimes 1) \lhd h) \\
&= \Sigma_{(h)} P(n \lhd h_{(1)}) \otimes h_{(2)} = \Sigma_{(h)} \epsilon(h_{(1)}) P(n) \otimes h_{(2)} = P(n) \otimes (\Sigma_{(h)} \epsilon(h_{(1)}) h_{(2)}) = n \otimes h.
\end{aligned}$$

This proves that $\chi = \phi^{-1}$ and that $\phi : M^{coH} \otimes H \to M$ is an isomorphism of Hopf modules. $\square$

Note that the projector P on the subspace $M^{coH} \subset M$ can be viewed as the Hopf module counterpart of the defining condition $m \circ (\mathrm{id} \otimes S) \circ \Delta = \eta \circ \epsilon$ of the antipode of a Hopf algebra and the isomorphism χ as the counterpart of the identity $(m \circ (S \otimes \mathrm{id}) \circ \Delta \otimes \mathrm{id}) \circ \Delta = (\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id}$ in a Hopf algebra H.

With the fundamental theorem of Hopf modules, we can settle the question of existence and uniqueness of left and right integrals for finite-dimensional Hopf algebras H. This is achieved by defining an H-Hopf module structure on H^* that has the right integrals of H^* as its coinvariants.

Theorem 6.2.9: Let H be a finite-dimensional Hopf algebra, $(x_1, ..., x_n)$ an ordered basis of H and $(\alpha^1, ..., \alpha^n)$ the dual basis of H^*.

1. There is an H-right Hopf module structure on H^* with $H^{*coH} = I_L(H^*)$ given by

$$\begin{aligned}
&\lhd : H^* \otimes H \to H^*, \qquad && \alpha \otimes h \mapsto \Sigma_{(\alpha)} \alpha_{(2)}(S(h))\, \alpha_{(1)} \\
&\delta : H^* \to H^* \otimes H, && \alpha \mapsto \Sigma_{i=1}^n \alpha^i \alpha \otimes x_i.
\end{aligned} \tag{66}$$

2. For any $0 \neq \lambda \in I_L(H^*)$, the **Frobenius map** is an isomorphism of right H-Hopf modules:

$$\phi_\lambda : H \to H^*, \quad h \mapsto \Sigma_{(\lambda)} \lambda_{(2)}(S(h))\, \lambda_{(1)}.$$

3. One has $\dim_{\mathbb{F}} I_L(H) = \dim_{\mathbb{F}} I_R(H) = 1$.
4. The antipode of H is bijective with $S^{\pm 1}(I_L(H)) = I_R(H)$ and $S^{\pm 1}(I_R(H)) = I_L(H)$.

Proof:
1. That (66) defines a right H-Hopf module structure on H^* follows by a direct, but lengthy computation. We show first that $\lhd$ is an H-right module structure on H^*:

$$\begin{aligned}
(\alpha \lhd h) \lhd k &= \Sigma_{(\alpha)} \alpha_{(2)}(S(h))\, \alpha_{(1)} \lhd k = \Sigma_{(\alpha)} \alpha_{(3)}(S(h)) \alpha_{(2)}(S(k))\, \alpha_{(1)} \\
&= \Sigma_{(\alpha)} \alpha_{(2)}(S(k)S(h))\, \alpha_{(1)} = \Sigma_{(\alpha)} \alpha_{(2)}(S(hk))\, \alpha_{(1)} = \alpha \lhd (hk) \\
\alpha \lhd 1 &= \Sigma_{(\alpha)} \alpha_{(2)}(S(1))\, \alpha_{(1)} = \Sigma_{(\alpha)} \alpha_{(2)}(1)\, \alpha_{(1)} = \Sigma_{(\alpha)} \epsilon(\alpha_{(2)}) \alpha_{(1)} = \alpha.
\end{aligned}$$

To show that δ is an H-right comodule structure on H^*, we use the auxiliary identities

$$\Sigma_{i=1}^n \alpha^i \otimes \Delta(x_i) = \Sigma_{i,j=1}^n \alpha^i \alpha^j \otimes x_i \otimes x_j \qquad \Sigma_{i=1}^n \Delta(\alpha^i) \otimes x_i = \Sigma_{i,j=1}^n \alpha^i \otimes \alpha^j \otimes x_i x_j \tag{67}$$

which follow by evaluating both sides of the equation on elements $\beta, \gamma \in H^*$ or $x, y \in H$

$$\begin{aligned}
(\mathrm{id}\otimes\beta\otimes\gamma)(\Sigma_{i=1}^n \alpha^i \otimes \Delta(x_i)) &= \Sigma_{i=1}^n (\beta\gamma)(x_i)\,\alpha^i = \beta\gamma = \Sigma_{i=1}^n \alpha^i\alpha^j \beta(x_i)\gamma(x_j)\\
&= (\mathrm{id}\otimes\beta\otimes\gamma)(\Sigma_{i,j=1}^n \alpha^i\alpha^j \otimes x_i \otimes x_j)\\
(x\otimes y\otimes \mathrm{id})(\Sigma_{i=1}^n \Delta(\alpha^i)\otimes x_i) &= \Sigma_{i=1}^n (xy)(\alpha^i)\otimes x_i = xy = \Sigma_{i=1}^n x(\alpha^i) y(\alpha^j) x_i x_j\\
&= (x\otimes y\otimes \mathrm{id})(\Sigma_{i=1}^n \alpha^i \otimes \alpha^j \otimes x_i x_j).
\end{aligned}$$

Applying these identities we obtain

$$\begin{aligned}
(\delta\otimes\mathrm{id})\circ\delta(\alpha) &= \Sigma_{i=1}^n \delta(\alpha^i\alpha)\otimes x_i = \Sigma_{i,j=1}^n \alpha^j\alpha^i\alpha \otimes x_j \otimes x_i \overset{(67)}{=} \Sigma_{i=1}^n \alpha^i\alpha\otimes\Delta(x_i) = (\mathrm{id}\otimes\Delta)\circ\delta(\alpha)\\
(\mathrm{id}\otimes\epsilon)\delta(\alpha) &= \Sigma_{i=1}^n \alpha^i\alpha\epsilon(x_i) = \alpha.
\end{aligned}$$

To show that δ is H-linear, we compute

$$\begin{aligned}
\delta(\alpha)\lhd h &= \Sigma_{i=1}^n (\alpha^i\alpha\otimes x_i)\lhd h = \Sigma_{i=1}^n \Sigma_{(h)} [(\alpha^i\alpha)\lhd h_{(1)}]\otimes x_i h_{(2)}\\
&= \Sigma_{i=1}^n \Sigma_{(h),(\alpha),(\alpha^i)} (\alpha^i_{(2)}\alpha_{(2)})(S(h_{(1)}))\ \alpha^i_{(1)}\alpha_{(1)}\otimes x_i h_{(2)}\\
&= \Sigma_{i=1}^n \Sigma_{(h),(\alpha),(\alpha^i)} \alpha^i_{(2)}(S(h_{(2)}))\alpha_{(2)}(S(h_{(1)}))\ \alpha^i_{(1)}\alpha_{(1)}\otimes x_i h_{(3)}\\
&\overset{(67)}{=} \Sigma_{i=1}^n \Sigma_{(h),(\alpha)} \alpha_{(2)}(S(h_{(1)}))\ \alpha^i\alpha_{(1)}\otimes x_i S(h_{(2)})h_{(3)}\\
&= \Sigma_{i=1}^n \Sigma_{(\alpha)} \alpha_{(2)}(S(h))\ \alpha^i\alpha_{(1)}\otimes x_i = \Sigma_{(\alpha)} \alpha_{(2)}(S(h))\ \delta(\alpha_{(1)}) = \delta(\alpha\lhd h).
\end{aligned}$$

This shows that (66) defines a right H-Hopf module structure on H^*.

Its coinvariants are the elements $\alpha \in H^*$ with $\Sigma_{i=1}^n \alpha^i\alpha\otimes x_i = \delta(\alpha) = \alpha\otimes 1$. This condition implies $\beta\alpha = (\mathrm{id}\otimes\beta)(\Sigma_{i=1}^n \alpha^i\alpha\otimes x_i) = (\mathrm{id}\otimes\beta)\delta(\alpha) = (\mathrm{id}\otimes\beta)(\alpha\otimes 1) = \epsilon(\beta)\alpha$ for all $\beta \in H^*$, and hence one has $H^{*coH} \subset I_L(H^*)$. Conversely, for each left integral $\alpha \in I_L(H^*)$ one has $\delta(\alpha) = \Sigma_{i=1}^n \alpha^i\alpha\otimes x_i = \Sigma_{i=1}^n \epsilon(\alpha^i)\alpha\otimes x_i = \alpha\otimes 1$, and hence $I_L(H^*) = H^{*coH}$.

2. and 3.: Theorem 6.2.8 implies with 1. that $\lhd : I_L(H^*)\otimes H \to H^*$, $\lambda\otimes h \mapsto \lambda \lhd h$ is an isomorphism of Hopf modules. As H is finite-dimensional, we obtain

$$\dim_{\mathbb{F}} I_L(H^*)\cdot \dim_{\mathbb{F}} H^* = \dim_{\mathbb{F}}(I_L(H^*)\otimes H) = \dim_{\mathbb{F}}(H) \quad\Rightarrow\quad \dim_{\mathbb{F}} I_L(H^*) = 1.$$

It follows that the Frobenius map $\phi_\lambda = \lambda \lhd - : H \to H^*$ is an isomorphism of Hopf modules for all $0 \neq \lambda \in I_L(H^*)$. Claim 3. for $I_L(H)$ then follows, because the finite-dimensionality of H implies $H^{**} \cong H$, and claim 3. for $I_R(H)$ from 4.

4. If $h \in \ker(S)$, then $\phi_\lambda(h) = 0$ and hence $\ker(S) \subset \ker(\phi_\lambda) = \{0\}$. This shows that the antipode is injective, and because H is finite-dimensional, it follows that S is bijective. As $S^{\pm 1} : H \to H^{op,cop}$ is a Hopf algebra homomorphism, one has for $\ell \in I_L(H)$, $\ell' \in I_R(H)$, $h \in H$

$$\begin{aligned}
S^{\pm1}(\ell)\cdot h &= S^{\pm1}(S^{\mp1}(h)\cdot\ell) = \epsilon(S^{\mp1}(h))S^{\pm1}(\ell) = \epsilon(h)S^{\pm1}(\ell) &&\Rightarrow\quad S^{\pm1}(\ell)\in I_R(H)\\
h\cdot S^{\pm1}(\ell') &= S^{\pm1}(\ell'\cdot S^{\mp1}(h)) = \epsilon(S^{\mp1}(h))\,S^{\pm1}(\ell') = \epsilon(h)S^{\pm1}(\ell') &&\Rightarrow\quad S^{\pm1}(\ell')\in I_L(H).
\end{aligned}$$

This shows that $S^{\pm1}(I_L(H)) = I_R(H)$ and $S^{\pm1}(I_R(H)) = I_L(H)$. □

Corollary 6.2.10: Let H be a finite-dimensional Hopf algebra and $0 \neq \lambda \in I_L(H^*)$ a non-trivial left integral. Then H is a Frobenius algebra with Frobenius form

$$\kappa : H\otimes H \to \mathbb{F}, \quad \kappa(h\otimes k) = \lambda(h\cdot k).$$

Proof:
The Frobenius condition follows directly from the associativity of the product in H:

$$\kappa((h\cdot k)\otimes l)=\lambda((h\cdot k)\cdot l)=\lambda(h\cdot(k\cdot l))=\kappa(h\otimes(k\cdot l))\qquad \forall h,k,l\in H.$$

That κ is non-degenerate follows, because by Theorem 6.2.9 the antipode $S: H\to H$ and the Frobenius map $\phi_\lambda: H\to H^*$ are linear isomorphisms and

$$\kappa(h\otimes k)=\lambda(h\cdot k)=\Sigma_{(\lambda)}\lambda_{(1)}(h)\lambda_{(2)}(k)=\phi_\lambda(S^{-1}(k))(h).$$

Hence $\kappa(h\otimes k)=0$ for all $h\in H$ implies $\phi_\lambda(S^{-1}(k))=0$ and $k=0$. As H is finite-dimensional, this shows that $\phi_\kappa: H\to H^*$, $h\mapsto\kappa(-\otimes h)$ is a linear isomorphism. $\square$

Theorem 6.2.9 clarifies the existence and uniqueness of left and right integrals for finite-dimensional Hopf algebras H. In fact, one can show that the finite-dimensionality of H is not only a sufficient but also a necessary condition for the existence of non-trivial left and right integrals: if a Hopf algebra H has a left or right integral integral $\ell\neq 0$, then it follows that H is finite-dimensional. We will not prove this statement here. A proof is given in [R, Prop. 10.2.1].

Having established the existence and uniqueness of integrals for a finite-dimensional Hopf algebra H, we may ask what is the role of a *normalised* integral. In Lemma 6.2.5, we established that a normalised left integral in H defines a projector on the invariants of any H-module M. It is clear that the existence of a normalised integral on H is equivalent to the existence of an integral $\ell\in H$ with $\epsilon(\ell)\neq 0$, since any such integral can be normalised by rescaling it. It turns out that the existence of such an integral is closely relate to the representation theoretical properties of H. More specifically, it encodes the semisimplicity of H, and one obtains a generalisation of Maschke's theorem for representations of finite groups.

Theorem 6.2.11: (**Maschke's Theorem for Hopf algebras**)
Let H be a finite-dimensional Hopf algebra over $\mathbb{F}$. The the following are equivalent:

(i) H is semisimple.
(ii) There is a left integral $\ell\in H$ with $\epsilon(\ell)\neq 0$.
(iii) There is a right integral $\ell\in H$ with $\epsilon(\ell)\neq 0$.

Proof:
We prove the claim for left integrals. The claim for right integrals then follows because $S(I_L(H))=I_R(H)$ and $\epsilon\circ S=\epsilon$.

(i)$\Rightarrow$ (ii): The linear map $\epsilon: H\to\mathbb{F}$ is a module homomorphism with respect to the action of H on itself by left multiplication and the trivial H-module structure on $\mathbb{F}$, since we have $\epsilon(h\rhd_L k)=\epsilon(hk)=\epsilon(h)\epsilon(k)=h\rhd\epsilon(k)$ for all $h,k\in H$. Hence $\ker(\epsilon)\subset H$ is a submodule, or, equivalently, a left ideal in H. As H is semisimple, there is a left ideal $I\subset H$ with $H=\ker(\epsilon)\oplus I$. As we have $(h-\epsilon(h)\,1)\cdot k\in\ker(\epsilon)$ and $h\cdot i\in I$ for all $h,k\in H$ and $i\in I$, we obtain

$$\underbrace{h\cdot i}_{\in I}=\underbrace{(h-\epsilon(h)\,1)i}_{\in\ker(\epsilon)}+\underbrace{\epsilon(h)i}_{\in I}=\underbrace{\epsilon(h)i}_{\in I},$$

and this implies $I\subset I_L(H)$. As $\epsilon(1)=1$ implies $\dim_{\mathbb{F}}(\ker(\epsilon))<\dim_{\mathbb{F}}H$ and $I\cap\ker(\epsilon)=\{0\}$, we have $1\leq\dim_{\mathbb{F}}I$, and there is a left integral $\ell\in I\subset I_L(H)$ with $\epsilon(\ell)\neq 0$.

(ii) $\Rightarrow$ (i): Suppose there is an integral $\ell \in I_L(H)$ with $\epsilon(\ell) \neq 0$. Then by multiplying with an element $\lambda \in \mathbb{F}$, one can achieve $\epsilon(\ell) = 1$. Let M be a module over H, $U \subset M$ a submodule and choose a linear map $\phi : M \to U$ with $\phi|_U = \mathrm{id}_U$. If we equip the vector space $\mathrm{Hom}_{\mathbb{F}}(M, U)$ with the H-module structure from Example 6.2.2, 3. then the linear map

$$\pi = \ell \rhd \phi : M \to U, \quad m \mapsto (\ell \rhd \phi)(m) = \Sigma_\ell\, \ell_{(1)} \rhd \phi(S(\ell_{(2)}) \rhd m)$$

is an invariant of the module $\mathrm{Hom}_{\mathbb{F}}(M, U)$ by Lemma 6.2.5 and hence H-linear by Example 6.2.2, 3. It follows that $\ker(\pi) \subset M$ is a submodule of M. Moreover, one has for all $u \in U$

$$\pi(u) = (\ell \rhd P)(u) = \Sigma_{(\ell)}\, \ell_{(1)} \rhd \phi(S(\ell_{(2)}) \rhd u) = \ell_{(1)} \rhd (S(\ell_{(2)}) \rhd u) = \epsilon(\ell)\, u = u.$$

This implies $\ker(\pi) \cap U = \{0\}$ and since every element $m \in M$ can be written as $m = m - \pi(m) + \pi(m)$ with $m - \pi(m) \in \ker(\pi)$ and $\pi(m) \in U$ it follows that $M = U \oplus \ker(\pi)$. Hence, every submodule of M has a complement and H is semisimple. $\square$

Corollary 6.2.12: (**Maschke's Theorem for finite groups**)
Let G be a finite group. Then the group algebra $\mathbb{F}[G]$ is semisimple if and only if $\mathrm{char}(\mathbb{F}) \not| |G|$.

Proof:
By Example 6.2.4, we have $I_L(\mathbb{F}[G]) = \mathrm{span}_{\mathbb{F}} \{\Sigma_{g\in G}\, g\}$. As $\epsilon\left(\Sigma_{g\in G}\, g\right) = \Sigma_{g\in G}\, \epsilon(g) = |G|$, it follows that $\epsilon(I_L(\mathbb{F}[G])) \neq \{0\}$ if and only if $\mathrm{char}(\mathbb{F}) \nmid |G|$, and by Theorem 6.2.11, this is equivalent to the semisimplicity of $\mathbb{F}[G]$. $\square$

We will now relate the existence of normalised left and right integrals and hence the semisimplicity of a Hopf algebra H to the properties of its antipode. This requires some technical results on the properties of left and right integrals and the relation between them.

The first result is a characterisation of the unimodularity of a finite-dimensional Hopf algebra H in terms of a special element $\alpha \in H^*$. This is obtained by realising that the linear subspace of left integrals is invariant under right multiplication with H. As it is one-dimensional, it follows that multiplying a left integral ℓ on the right by an element $h \in H$ yields a scalar multiple $\alpha(h)\ell$ of the left integral ℓ. The assignment $h \mapsto \alpha(h)$ then defines a distinguished element $\alpha \in H^*$ that determines if H is unimodular.

Proposition 6.2.13: Let H be a finite-dimensional Hopf algebra over $\mathbb{F}$. Then:

1. There is a unique $\alpha \in H^*$, the the **modular element** of H, with $\ell \cdot h = \alpha(h)\ell$ for all $h \in H$ and $\ell \in I_L(H)$.
2. The element $\alpha \in H^*$ is grouplike.
3. One has $h \cdot \ell' = \alpha^{-1}(h)\ell'$ for all $h \in H$ and $\ell' \in I_R(H)$.
4. The Hopf algebra H is unimodular if and only if $\alpha = \epsilon$.

Proof:
1. For every left integral $\ell \in I_L(H)$ and $h, k \in H$, one has $k(\ell h) = (k\ell)h = \epsilon(k)\, \ell h$ for all $k \in H$ and hence $\ell h \in I_L(H)$. As $\dim_{\mathbb{F}} I_L(H) = 1$ by Theorem 6.2.9 and the multiplication is linear, this implies $\ell h = \alpha(h)\ell$ for some element $\alpha \in H^*$. If $\ell \neq 0$ and $\ell' \in I_L(H)$ is another left integral, then $\ell' = \mu\ell$ for some $\mu \in \mathbb{F}$, and we have $\ell' h = \mu\, \ell h = \mu\alpha(h)\, \ell = \alpha(h)\, \ell'$ for all $h \in H$. This shows that the identity $\ell \cdot h = \alpha(h)\, \ell$ holds for all $\ell \in I_L(H)$.

2. We have for all $h, k \in H$

$$\alpha(hk)\,\ell = \ell hk = (\ell h)k = (\alpha(h)\ell)k = \alpha(h)\,\ell k = \alpha(h)\alpha(k)\ell \qquad \ell = \ell 1_H = \ell = \alpha(1_H)\,\ell.$$

This shows that $\alpha : H \to \mathbb{F}$ is an algebra homomorphism and hence a grouplike element of H^* by Example 6.1.2.

3. As $I_R(H) = S^{\pm 1}(I_L(H))$ by Theorem 6.2.9 and $S^{\pm 1}(\alpha) = \alpha \circ S^{\pm 1} = \alpha^{-1}$ for any $\alpha \in \mathrm{Gr}(H^*)$, we obtain for all right integrals $\ell' \in I_R(H)$

$$h \cdot \ell' = S^{-1}(S(h)) \cdot S^{-1}(S(\ell')) = S^{-1}(S(\ell') \cdot S(h)) = \alpha(S(h))\, S^{-1}(S(\ell')) = \alpha^{-1}(h)\,\ell'.$$

4. The Hopf algebra H is unimodular if and only if $I_L(H) = I_R(H)$. This is the case if and only if $\ell h = \alpha(h)\ell = \epsilon(h)\ell$ for all $h \in H$ and left integrals $\ell \in I_L(H)$, which is equivalent to $\alpha = \epsilon$ since $\dim_{\mathbb{F}} I_L(H) = 1$. □

Corollary 6.2.14: Every finite-dimensional semisimple Hopf algebra is unimodular.

Proof:
If H is finite-dimensional and semisimple, there is a left integral $\ell \in I_L(H)$ with $\epsilon(\ell) = 1$ by Theorem 6.2.11. Then the modular element $\alpha \in \mathrm{Gr}(H^*)$ with $\ell \cdot h = \alpha(h)\,\ell$ for all $h \in H$ from Proposition 6.2.13 satisfies

$$\alpha(h)\,\ell = \alpha(h)\,\epsilon(\ell)\ell = \alpha(h)\,\ell^2 = (\ell h)\ell = \ell(h\ell) = \epsilon(h)\,\ell^2 = \epsilon(h)\epsilon(\ell)\,\ell = \epsilon(h)\,\ell.$$

As $\ell \neq 0$, this implies $\alpha(h) = \epsilon(h)$ for all $h \in H$, and by Proposition 6.2.13 H is unimodular. □

To relate the existence of normalised left and right integrals and hence the semisimplicity of H to the properties of the antipode, we require some additional results on the properties of the integrals and need to relate the integrals of H and H^*.

Lemma 6.2.15: Let H be a finite-dimensional Hopf algebra over $\mathbb{F}$.

1. For each right integral $0 \neq \lambda \in I_R(H^*)$ there is a left integral $\ell \in I_L(H)$ with $\lambda(\ell) = 1$.
2. Under these assumptions one has for each element $h \in H$

$$h = \Sigma_{(\ell)}\, \lambda(h\ell_{(1)})S(\ell_{(2)}). \tag{68}$$

Proof:
1. Let $0 \neq \lambda \in I_R(H^*)$ be a non-trivial left integral. Then by Theorem 6.2.9 one has $0 \neq S^{-1}(\lambda) \in I_L(H^*)$ and the Frobenius map $\phi_{S^{-1}(\lambda)} : H \to H^*$, $h \mapsto \Sigma_{(\lambda)}\lambda_{(1)}(h)S^{-1}(\lambda_{(2)})$ is a linear isomorphism. This implies $\phi_{S^{-1}(\lambda)}(\ell) \neq 0$ for each non-trivial left integral $0 \neq \ell \in I_L(H)$, and hence there is a $k \in H$ with

$$0 \neq \phi_{S^{-1}(\lambda)}(\ell)S(k) = \Sigma_{(\lambda)}\lambda_{(1)}(\ell)\,\lambda_{(2)}(k) = \lambda(\ell \cdot k) = \alpha(k)\lambda(\ell).$$

This implies $\lambda(\ell) \neq 0$, and by multiplying ℓ with a suitable element of $\mathbb{F}^\times$, we obtain $\lambda(\ell) = 1$.

2. Let now $\ell \in I_L(H)$, $\lambda \in I_R(H^*)$ left and right integrals with $\lambda(\ell) = 1$. Then we have

$$\Sigma_{(\ell)}\, h\ell_{(1)} \otimes S(\ell_{(2)}) = \Sigma_{(\ell)}\, \ell_{(1)} \otimes S(\ell_{(2)})h \qquad \forall h \in H. \tag{69}$$

This follows by a direct computation using the fact that $\ell \in I_L(H)$ is a left integral, the counitality and antipode condition:

$$\begin{aligned}&\Sigma_{(\ell)}\, h\ell_{(1)}\otimes S(\ell_{(2)}) = \Sigma_{(\ell)(h)} h_{(1)}\ell_{(1)}\otimes S(\epsilon(h_{(2)})\ell_{(2)}) = \Sigma_{(\ell)(h)}\, h_{(1)}\ell_{(1)}\otimes S(S^{-1}(h_{(3)})h_{(2)}\ell_{(2)})\\ &= \Sigma_{(\ell)(h)}\, h_{(1)}\ell_{(1)}\otimes S(h_{(2)}\ell_{(2)})h_{(3)} = \Sigma_{(h)}(\mathrm{id}\otimes S)(\Delta(h_{(1)}\ell))\cdot(1\otimes h_{(2)})\\ &= \Sigma_{(h)}\epsilon(h_{(1)})(\mathrm{id}\otimes S)\circ\Delta(\ell)\cdot(1\otimes h_{(2)}) = (\mathrm{id}\otimes S)\circ\Delta(\ell)\cdot(1\otimes h) = \Sigma_{(\ell)}\ell_{(1)}\otimes S(\ell_{(2)})h.\end{aligned}$$

As $\lambda \in I_R(H^*)$ is a right integral, we have $(\lambda\otimes\mathrm{id})\circ\Delta(h) = \Sigma_{(h)}\,\lambda(h_{(1)})h_{(2)} = \lambda(h)$ for all $h \in H$. Combining this with the condition $\lambda(\ell) = 1$ and identity (69) we obtain

$$\Sigma_{(\ell)}\,\lambda(h\ell_{(1)})S(\ell_{(2)}) \overset{(69)}{=} \Sigma_{(\ell)}\,\lambda(\ell_{(1)})S(\ell_{(2)})h = \lambda(\ell)\,S(1)\cdot h = h.$$

$\square$

We will now show that the semisimplicity of a finite-dimensional Hopf algebra H is related to the square of its antipode. This is achieved via formula (68) in Lemma 6.2.15. The fact that every element of H can be expressed in terms of a left integral $\ell \in I_L(H)$ and right integral $\lambda \in I_R(H^*)$ allows one to express the traces of a linear maps $\phi : H \to H$ in terms of integrals and to relate the existence of normalised left or right integrals to the square of the antipode.

Proposition 6.2.16: Let H be a finite-dimensional Hopf algebra over $\mathbb{F}$, $\ell \in I_L(H)$ a left integral and $\lambda \in I_R(H^*)$ a right integral with $\lambda(\ell) = 1$, as in Lemma 6.2.15. Then:

1. For all linear maps $\phi : H \to H$ one has $\mathrm{Tr}(\phi) = \Sigma_{(\ell)}\lambda\big(\phi(S(\ell_{(2)}))\cdot\ell_{(1)}\big)$,
2. The square of the antipode satisfies $\mathrm{Tr}(S^2) = \epsilon(\ell)\,\lambda(1)$.

Proof:
We choose an ordered basis $(x_1, ..., x_n)$ of H and the dual basis $(\alpha^1, ..., \alpha^n)$ of H^*. Then we have $\mathrm{Tr}(\phi) = \Sigma_{i=1}^n \alpha^i(\phi(x_i))$ for all $\phi \in \mathrm{End}_{\mathbb{F}}(H)$. As $\phi(x_i) = \Sigma_{(\ell)}\;\lambda(\phi(x_i)\ell_{(1)})\,S(\ell_{(2)})$ for all $i = 1, ..., n$ by formula (68) in Lemma 6.2.15, we obtain

$$\begin{aligned}\mathrm{Tr}(\phi) &= \Sigma_{i=1}^n\alpha^i(\phi(x_i)) = \Sigma_{i=1}^n\;\alpha^i\big(\Sigma_{(\ell)}\;\lambda(\phi(x_i)\ell_{(1)})S(\ell_{(2)})\big) = \Sigma_{i=1}^n\Sigma_{(\ell)}\;\alpha^i(S(\ell_{(2)}))\lambda(\phi(x_i)\ell_{(1)})\\ &= \Sigma_{(\ell)}\lambda\big(\phi(\Sigma_{i=1}^n\alpha^i(S(\ell_{(2)}))x_i)\cdot\ell_{(1)}\big) = \Sigma_{(\ell)}\lambda\big(\phi(S(\ell_{(2)}))\cdot\ell_{(1)}\big).\end{aligned}$$

Inserting $\phi = S^{-2}$ into this equation yields

$$\mathrm{Tr}(S^{-2}) = \Sigma_{(\ell)}\;\lambda\big(S^{-1}(\ell_{(2)})\ell_{(1)}\big) = \lambda\big(\Sigma_{(\ell)}\,S^{-1}(\ell_{(2)})\ell_{(1)}\,\big) = \lambda(\epsilon(\ell)) = \epsilon(\ell)\lambda(1).$$

Replacing H with H^{cop} exchanges the antipode and its inverse and yields $\mathrm{Tr}(S^2) = 1$. $\square$

Corollary 6.2.17: Let H be a finite-dimensional Hopf algebra over $\mathbb{F}$.

1. H and H^* are semisimple if and only if $\mathrm{Tr}(S^2) \neq 0$.
2. If $S^2 = \mathrm{id}_H$ and $\mathrm{char}(\mathbb{F}) \not|\, \dim_{\mathbb{F}} H$, then H and H^* are semisimple.

Proof:
1. By Lemma 6.2.15 there are integrals $\ell \in I_L(H)$ and $\lambda \in I_R(H^*)$ with $\lambda(\ell) = 1$, and by Proposition 6.2.16 we have $\mathrm{Tr}(S^2) = \epsilon(\ell)\lambda(1)$. Hence, we have $\mathrm{Tr}(S^2) \neq 0$ if and only if $\epsilon(\ell), \lambda(1) \neq 0$. As $\dim_{\mathbb{F}} I_L(H) = \dim_{\mathbb{F}} I_R(H^*) = 1$, this is equivalent to the existence of a

left integral $\ell \in I_L(H)$ and a right integral $\lambda \in I_R(H^*)$ with $\epsilon(\ell) \neq 0$ and $\lambda(1) \neq 0$. By Theorem 6.2.11 the first condition is equivalent to the semisimplicity of H and the second to the semisimplicity of H^*.

2. If $S^2 = \mathrm{id}_H$, then $\mathrm{Tr}(S^2) = \mathrm{Tr}(\mathrm{id}_H) = \dim_{\mathbb{F}} H$. If $\mathrm{char}(\mathbb{F}) \not| \dim_{\mathbb{F}}(H)$, this implies $\mathrm{Tr}(S^2) \neq 0$, and 1. implies that H and H^* are semisimple. $\square$

Corollary 6.2.17 relates the the semisimplicity of a finite-dimensional Hopf algebra H and its dual to the square of the antipode. In fact, for fields of characteristic zero there is a stronger result that shows that the condition $S^2 = \mathrm{id}_H$ is not just sufficient, but also necessary for the semisimplicity of H and H^*. This is the theorem of Larson and Radford, for a proof see the original article [LR].

Theorem 6.2.18: (Larson-Radford Theorem)
Let H be a finite-dimensional Hopf algebra over a field $\mathbb{F}$ of characteristic zero. Then the following are equivalent:

(i) H is semisimple.

(ii) H^* is semisimple.

(iii) $S^2 = \mathrm{id}_H$.

The Larson-Radford Theorem is useful to establish that a given Hopf algebra over a field of characteristic zero is not semisimple. Instead of a time-consuming search for indecomposable modules that are not simple, one can simply compute the square of the antipode.

Example 6.2.19: Let $\mathbb{F}$ be a field of characteristic zero.

1. The q-deformed universal enveloping algebra $U_q^r(\mathfrak{sl}_2)$ at a primitive nth root of unity from Proposition 5.3.12 is unimodular, but it is not semisimple. This follows because the square of its antipode is given by

$$S^2(K) = K, \qquad S^2(E) = KEK^{-1} = q^2E \qquad S^2(F) = -KFK^{-1} = q^{-2}F$$

Hence, the antipode of $U_q^r(\mathfrak{sl}_2)$ has order $2r > 2$ with $r = n/2$ for n even and $r = n$ for n odd and $U_q^r(\mathfrak{sl}_2)$ cannot be semisimple by Theorem 6.2.18.

2. The Taft algebra T_q from Example 5.3.6 is not semisimple, since one has

$$S^2(x) = S(-xy^{-1}) = -S(xy^{-1}) = -S(y)^{-1}S(x) = -y(-xy^{-1}) = yxy^{-1} = q^{-1}x \neq x.$$

The theorem by Larson and Radford also relates the semisimplicity of a Hopf algebra H to the properties of the Frobenius algebra from Corollary 6.2.10. By combining Theorem 6.2.18 with the formulas from Lemma 6.2.15, one finds that this Frobenius algebra is symmetric, if H is semisimple. This implies in particular that the coproduct of an integral in a finite-dimensional semisimple Hopf algebra is always symmetric.

Corollary 6.2.20: Let H be a finite-dimensional semisimple Hopf algebra over a field $\mathbb{F}$ of characteristic zero. Then:

1. The Frobenius algebra from Corollary 6.2.10 is symmetric.
2. For all integrals $\ell \in H$ the element $\Delta^{(n)}(\ell) = (\Delta \otimes \mathrm{id}_{H^{\otimes(n-1)}}) \circ \cdots \circ (\Delta \otimes \mathrm{id}_H) \circ \Delta(\ell)$ is invariant under cyclic permutations for all $n \in \mathbb{N}$.

Proof:
1. Let H be a finite-dimensional semisimple Hopf algebra over $\mathbb{F}$. The by Theorem 6.2.18 one has $S^2 = \mathrm{id}_H$, the dual H^* is semisimple as well and H and H^* are unimodular by Corollary 6.2.14. By Lemma 6.2.15 there is a right integral $\lambda \in I_R(H^*)$ and a left integral $\ell \in I_L(H)$ with $\lambda(\ell) = 1$. As H is unimodular, we have $S(\ell) \in I_L(H)$ and hence $S(\ell) = \mu\ell$ with $\mu \in \mathbb{F}^\times$. The identity $\epsilon(S(\ell)) = \mu\epsilon(\ell) = \epsilon(\ell)$ and $\epsilon(\ell) \neq 0$ imply $\mu = 1$ and $S(\ell) = \ell$.

Using the identity $S^2 = \mathrm{id}_H$, one can then derive a counterpart of formula (69) where h is on the right in the first and on the left in the second argument

$$\Sigma_{(\ell)}\, h\ell_{(1)} \otimes S(\ell_{(2)}) = \Sigma_{(\ell)}\, \ell_{(1)} \otimes S(\ell_{(2)})h \qquad\qquad \Sigma_{(\ell)}\, \ell_{(1)}h \otimes S(\ell_{(2)}) = \ell_{(1)} \otimes hS(\ell_{(2)}). \tag{70}$$

As $\lambda \in I_L(H^*) = I_R(H^*)$ with $\lambda(\ell) = 1$, we have for all $h \in H$

$$(\lambda \otimes \mathrm{id})\Delta(h) = \Sigma_{(h)}\lambda(h_{(1)})h_{(2)} = \lambda(h)1 = \Sigma_{(h)}\lambda(h_{(2)})h_{(1)} = (\mathrm{id} \otimes \lambda)\Delta(h),$$

and applying these identities to (70) yields a generalisation of (68)

$$h = \Sigma_{(\ell)}\, \lambda(h\ell_{(1)})\, S(\ell_{(2)}) = \Sigma_{(\ell)}\, \lambda(S(\ell_{(2)})h)\, \ell_{(1)} = \Sigma_{(\ell)}\lambda(h\ell_{(2)})\, S(\ell_{(1)}) = \Sigma_{(\ell)}\lambda(S(\ell_{(1)})h)\, \ell_{(2)}. \tag{71}$$

This yields for all $h, k \in H$

$$\begin{aligned}\lambda(k \cdot h) &\overset{(71)}{=} \lambda\big(\big(\Sigma_{(\ell)}\lambda(S(\ell_{(1)})k)\, \ell_{(2)}\big) \cdot h\big) = \Sigma_{(\ell)}\, \lambda(S(\ell_{(1)})k)\, \lambda(\ell_{(2)}h) \overset{(70)}{=} \Sigma_{(\ell)}\, \lambda(hS(\ell_{(1)})k)\, \lambda(\ell_{(2)})\\ &\overset{(70)}{=} \Sigma_{(\ell)}\lambda(hS(\ell_{(1)}))\, \lambda(k\ell_{(2)}) = \lambda\big(h \cdot \big(\Sigma_{(\ell)}\lambda(k\ell_{(2)})\, S(\ell_{(1)})\big)\big) \overset{(71)}{=} \lambda(h \cdot k).\end{aligned}$$

2. Clearly, this holds if $\ell = 0$. We prove it for non-trivial integrals $0 \neq \ell \in H$. As the associated Frobenius form κ on H^* is symmetric by 1, one has

$$(\alpha \otimes \beta)(\Delta(\ell)) = (\alpha \cdot \beta)(\ell) = \kappa(\alpha \otimes \beta) = \kappa(\beta \otimes \alpha) = (\beta \cdot \alpha)(\ell) = (\alpha \otimes \beta)(\Delta^{op}(\ell))$$

for all $\alpha, \beta \in H^*$ and hence $\Delta(\ell) = \Delta^{op}(\ell)$. This yields for all $k \in \mathbb{N}_0$

$$\begin{aligned}\Sigma_{(\ell)}\ell_{(1)} \otimes \ell_{(2)} \otimes ... \otimes \ell_{(k+1)} &= \Sigma_{(\ell)}\Delta^{(k-1)}(\ell_{(1)}) \otimes \ell_{(2)} = (\Delta^{(k-1)}) \otimes \mathrm{id}) \circ \Delta(\ell)\\ &= (\Delta^{(k-1)}) \otimes \mathrm{id}) \circ \Delta^{op}(\ell) = \Sigma_{(\ell)}\Delta^{(k-1)}(\ell_{(2)}) \otimes \ell_{(1)} = \Sigma_{(\ell)}\ell_{(2)} \otimes ... \otimes \ell_{(k+1)} \otimes \ell_{(1)}.\end{aligned}$$

□

6.3 Quasitriangular Hopf algebras

Bialgebras were motivated by the requirement that their representation categories are monoidal (Theorem 5.1.10) and Hopf algebras by the requirement that their categories of finite-dimensional representations are right rigid monoidal categories (Corollary 5.2.7). In this section we investigate the additional structure on a bialgebra or Hopf algebra B that ensures that its representation category $B-\mathrm{Mod}$ is a *braided* monoidal category.

This additional structure must relate the representation of B on the tensor product of two representation spaces to the one on the opposite tensor product. More specifically, for all B-modules V, W the B-modules $V \otimes W$ and $W \otimes V$ must be *isomorphic*, these isomorphisms are natural in both arguments and compatible with tensor products. We will show that this is equivalent to the following conditions on the bialgebra B.

Definition 6.3.1:

1. A **quasitriangular bialgebra** is is a pair (B, R) of a bialgebra B and an an invertible element $R = R_{(1)} \otimes R_{(2)} \in B \otimes B$, the **universal R-matrix**, that satisfies
$$\Delta^{op}(b) = R \cdot \Delta(b) \cdot R^{-1} \qquad (\Delta \otimes \mathrm{id})(R) = R_{13} \cdot R_{23} \qquad (\mathrm{id} \otimes \Delta)(R) = R_{13} \cdot R_{12},$$
where $R_{12} = R_{(1)} \otimes R_{(2)} \otimes 1$, $R_{13} = R_{(1)} \otimes 1 \otimes R_{(2)}$ and $R_{23} = 1 \otimes R_{(1)} \otimes R_{(2)} \in B \otimes B \otimes B$.
2. A **homomorphism** of quasitriangular bialgebras from (B, R) to (B', R') is a bialgebra homomorphis $\phi : B \to B'$ with $R' = (\phi \otimes \phi)(R)$.
3. A quasitriangular bialgebra is called **triangular** if its R-matrix satisfies $\tau \circ R = R^{-1}$, where $\tau : B \otimes B \to B \otimes B$, $b \otimes b' \mapsto b' \otimes b$ denotes the flip map.
4. A **(quasi)triangular** Hopf algebra is a (quasi)triangular bialgebra that is a Hopf algebra.

Note that the notation $R = R_{(1)} \otimes R_{(2)}$ is symbolic. It stands for a finite sum $R = \Sigma_{i=1}^n b_i \otimes b_i'$ with $b_i, b_i' \in B$. To distinguish it from the Sweedler notation for a coproduct $\Delta(b) = \Sigma_{(b)} b_{(1)} \otimes b_{(2)}$, we do not use a summation sign in this case.

It is plausible that a quasitriangular structure on a bialgebra is related to a braiding on its representation category $B-\mathrm{Mod}$, since the representation of B on the tensor product $V \otimes W$ is given by the coproduct $(\rho_V \otimes \rho_W) \circ \Delta : B \to \mathrm{End}_{\mathbb{F}}(V \otimes W)$, while its representation on $W \otimes V$ is given by $(\rho_W \otimes \rho_V) \circ \Delta = \tau \circ (\rho_V \otimes \rho_W) \circ \Delta^{op} : B \to \mathrm{End}_{\mathbb{F}}(W \otimes V)$. The braiding of the representations on V and W should thus be given by the action of the universal R-matrix.

Theorem 6.3.2: Let B be a bialgebra. Then (symmetric) braidings on the representation category $B-\mathrm{Mod}$ correspond bijectively to quasitriangular (triangular) structures on B.

Proof:
1. Suppose that (B, R) is a quasitriangular bialgebra and denote by $R_{21} = \tau(R)$ the flipped R-matrix. We define for B-modules $(V, \rhd_V)$ and $(W, \rhd_W)$ a linear map $c_{V,W} : V \otimes W \to W \otimes V$
$$c_{V,W}(v \otimes w) = \tau(R \rhd_{V \otimes W} v \otimes w) = R_{21} \rhd_{W \otimes V} \tau(v \otimes w) \qquad \forall v \in V, w \in W. \tag{72}$$

Then $c_{V,W}$ has inverse $c^{-1}_{V,W}(w\otimes v) = \tau(R^{-1}_{21} \rhd_{W\otimes V} w\otimes v) = R^{-1} \rhd_{V\otimes W} \tau(w\otimes v)$ and is B-linear:

$$\begin{aligned} &b \rhd_{W\otimes V} c_{V,W}(v\otimes w) = \tau((\Delta^{op}(b)\cdot R) \rhd_{V\otimes W} v\otimes w) = \tau((R\cdot\Delta(b)) \rhd_{V\otimes W} v\otimes w) \\ &= \tau(R \rhd_{V\otimes W} (b \rhd_{V\otimes W} v\otimes w)) = c_{V,W}(b \rhd_{V\otimes W} (v\otimes w)). \end{aligned}$$

To show that this defines a natural isomorphism $c : \otimes \to \otimes^{op}$ in B-Mod, we consider B-linear maps $f : V \to V'$ and $g : W \to W'$ and compute

$$\begin{aligned} (c_{V',W'} \circ (f\otimes g))(v\otimes w) &= \tau(R \rhd_{V',W'} f(v)\otimes g(w)) = (\tau \circ (f\otimes g))(R \rhd_{V\otimes W} v\otimes w) \\ &= ((g\otimes f)\circ\tau)(R \rhd_{V\otimes W} v\otimes w) = ((g\otimes f)\circ c_{V,W})(v\otimes w) \end{aligned}$$

where we used first the definition of $c_{V',W'}$, then the B-linearity of f and g and then the definition of $c_{V,W}$. This proves the naturality of c.

For the proof of the hexagon relations note that in Sweedler notation the conditions on the universal R-matrix read

$$\begin{aligned} (\Delta\otimes\mathrm{id})(R) = R_{13}R_{23} \quad &\Leftrightarrow \quad \Sigma_{(R_{(1)})} R_{(1)(1)}\otimes R_{(1)(2)}\otimes R_{(2)} = R'_{(1)}\otimes R_{(1)}\otimes R'_{(2)}R_{(2)} \\ (\mathrm{id}\otimes\Delta)(R) = R_{13}R_{12} \quad &\Leftrightarrow \quad \Sigma_{(R_{(2)})} R_{(1)}\otimes R_{(2)(1)}\otimes R_{(2)(2)} = R'_{(1)}R_{(1)}\otimes R_{(2)}\otimes R'_{(2)}. \end{aligned}$$

With this, we obtain the two hexagon relations

$$\begin{aligned} &(1_V\otimes c_{U,W})\circ a_{V,U,W}\circ(c_{U,V}\otimes 1_W)((u\otimes v)\otimes w) = (1_V\otimes c_{U,W})(R_{(2)} \rhd_V v)\otimes((R_{(1)} \rhd_U u)\otimes w) \\ &= (R_{(2)} \rhd_V v)\otimes((R'_{(2)} \rhd_W w)\otimes((R'_{(1)}R_{(1)}) \rhd_U u)) \\ &= \Sigma_{(R_{(2)})} (R_{(2)(1)} \rhd_V v)\otimes((R_{(2)(2)} \rhd_W w)\otimes(R_{(1)} \rhd_U u)) \\ &= a_{V,W,U}((R_{(2)} \rhd_{V\otimes W} (v\otimes w))\otimes(R_{(1)} \rhd_U u)) = a_{V,W,U}\circ c_{U,V\otimes W}\circ a_{U,V,W}((u\otimes v)\otimes w) \end{aligned}$$

$$\begin{aligned} &(c_{U,W}\otimes 1_V)\circ a^{-1}_{U,W,V}\circ(1_U\otimes c_{V,W})(u\otimes(v\otimes w)) = (c_{U,W}\otimes 1_V)((u\otimes(R_{(2)} \rhd_W w))\otimes(R_{(1)} \rhd_V v)) \\ &= ((R'_{(2)}R_{(2)}) \rhd_W w)\otimes(R'_{(1)} \rhd_U u))\otimes(R_{(1)} \rhd_V v) \\ &= \Sigma_{(R_{(1)})} ((R_{(2)} \rhd_W w)\otimes(R_{(1)(1)} \rhd_U u))\otimes(R_{(1)(2)} \rhd_V v) \\ &= a^{-1}_{W,U,V}((R_{(2)} \rhd_W w)\otimes(R_{(1)} \rhd_{U\otimes V} (u\otimes v)) = a^{-1}_{W,U,V}\circ c_{U\otimes V,W}\circ a^{-1}_{U,V,W}(u\otimes(v\otimes w)). \end{aligned}$$

This shows that the B-module isomorphisms $c_{V,W} : V\otimes W \to W\otimes V$ define a braiding.

2. Let now B be a bialgebra and $c : \otimes \to \otimes^{op}$ a braiding for B-Mod. We define

$$R := \tau\circ c_{B,B}(1\otimes 1) \in B\otimes B \quad \Rightarrow \quad R^{-1} := c^{-1}_{B,B}(1\otimes 1) \in B\otimes B. \tag{73}$$

To show that R^{-1} is inverse to R, we use the B-linearity of the braiding and compute

$$\begin{aligned} &R\cdot R^{-1} = R \rhd c^{-1}_{B,B}(1\otimes 1) = c^{-1}_{B,B}(R_{21} \rhd (1\otimes 1)) = c^{-1}_{B,B}(R_{21}\cdot(1\otimes 1)) = c^{-1}_{B,B}(c_{B,B}(1\otimes 1)) = 1\otimes 1 \\ &R^{-1}\cdot R = R^{-1} \rhd \tau(c_{B,B}(1\otimes 1)) = \tau(c_{B,B}(R^{-1} \rhd (1\otimes 1))) = \tau\circ c_{B,B}(c^{-1}_{B,B}(1\otimes 1)) = \tau(1\otimes 1) = 1\otimes 1. \end{aligned}$$

To prove that R satisfies the conditions on the universal R-matrix, note that for all B-modules V, W and $v\in V$, $w\in W$, the map $\phi_{v,w} : B\otimes B \to V\otimes W$, $a\otimes b \mapsto (a \rhd_V v)\otimes(b \rhd_W w)$ is B-linear. The naturality of the braiding then implies $c_{V,W}\circ\phi_{v,w} = \phi_{w,v}\circ c_{B,B}$ and for all $v\in V$, $w\in W$

$$R \rhd (v\otimes w) = \phi_{v,w}(R) = \tau(\phi_{w,v}\circ c_{B,B}(1\otimes 1)) = \tau(c_{V,W}\circ\phi_{v,w}(1\otimes 1)) = \tau(c_{V,W}(v\otimes w)). \tag{74}$$

With the B-linearity of the braiding we then obtain

$$\Delta^{op}(b) \cdot R = \tau(\Delta(b) \cdot c_{B,B}(1\otimes 1)) \overset{**}{=} \tau(b \rhd_{B\otimes B} c_{B,B}(1\otimes 1)) \overset{*}{=} \tau(c_{B,B}(b \rhd_{B\otimes B} 1\otimes 1)) \overset{**}{=} \tau(c_{B,B}(\Delta(b)))$$
$$\overset{(74)}{=} R \rhd_{B\otimes B} \Delta(b) \overset{**}{=} R \cdot \Delta(b),$$

where we used first the definition of R, in * the B-linearity of $c_{B,B}$ and in ** the B-module structure on $B\otimes B$. This shows that $\Delta^{op} = R \cdot \Delta \cdot R^{-1}$.

With the hexagon relations we compute

$$\begin{aligned}
(\mathrm{id}_B\otimes\Delta)(R) &= R \rhd_{B\otimes(B\otimes B)} (1\otimes(1\otimes 1)) \overset{(74)}{=} \tau(c_{B,B\otimes B}(1\otimes(1\otimes 1))) \\
&\overset{\text{hex}}{=} \tau((a^{-1}_{B,B,B} \circ (\mathrm{id}_B\otimes c_{B,B}) \circ a_{B,B,B} \circ (c_{B,B}\otimes \mathrm{id}_B) \circ a^{-1}_{B,B,B})(1\otimes(1\otimes 1)) \\
&\overset{\text{def } R}{=} R_{13}R_{12} \rhd_{(B\otimes B)\otimes B} ((1\otimes 1)\otimes 1) = R_{13}R_{12}
\end{aligned}$$

$$\begin{aligned}
(\Delta\otimes\mathrm{id}_B)(R) &= R \rhd_{(B\otimes B)\otimes B} ((1\otimes 1)\otimes 1) \overset{(74)}{=} \tau(c_{B\otimes B,B}((1\otimes 1)\otimes 1)) \\
&\overset{\text{hex}}{=} \tau(a_{B,B,B} \circ (c_{B,B}\otimes\mathrm{id}_B) \circ a^{-1}_{B,B,B} \circ (\mathrm{id}_B\otimes c_{B,B}) \circ a_{B,B,B})((1\otimes 1)\otimes 1) \\
&\overset{\text{def } R}{=} R_{13}R_{23} \rhd_{B\otimes(B\otimes B)} (1\otimes(1\otimes 1)) = R_{13}R_{23}.
\end{aligned}$$

3. That the representation category B-Mod is symmetric if and only if (B, R) is triangular follows directly from the expressions for the braiding in terms of the universal R-matrix:

$$c_{V,W}(v\otimes w) = \tau(R \rhd_{V\otimes W} v\otimes w) \qquad c^{-1}_{W,V}(v\otimes w) = \tau(R^{-1}_{21} \rhd_{V\otimes W} v\otimes w).$$

If $R = R^{-1}_{21}$, this implies $c_{V,W} = c^{-1}_{W,V}$ for all B-modules V, W. Conversely, if $c_{V,W} = c^{-1}_{W,V}$ for all B-modules V, W, then this holds in particular for $V = W = B$ and we obtain with (73)

$$R \overset{(73)}{=} \tau \circ c_{B,B}(1\otimes 1) = \tau \circ c^{-1}_{B,B}(1\otimes 1) \overset{(73)}{=} \tau(R^{-1}) = R^{-1}_{21}.$$

4. It remains to show that the construction of a quasitriangular structure for B from a braiding on $B-$Mod and the construction of a braiding on $B-$Mod from a quasitriangular structure on B are inverse to each other. If (B, R) is a quasitriangular bialgebra, c the braiding defined by (72) and R' the universal R-matrix defined by c via (73), we have

$$R' \overset{(73)}{=} \tau(c_{B,B}(1\otimes 1)) \overset{(72)}{=} \tau(\tau(R \rhd_{B\otimes B} 1\otimes 1)) = \tau \circ \tau(R) = R.$$

Conversely, if c is a braiding on B-Mod, R the associated universal R-matrix defined by (73) and c' is the braiding defined by R via (72), then we have

$$c'_{V,W}(v\otimes w) \overset{(72)}{=} \tau(R \rhd_{V\otimes W} (v\otimes w)) \overset{(74)}{=} \tau(\tau(c_{V,W}(v\otimes w))) = c_{V,W}(v\otimes w)$$

for all B-modules V, W and all $v \in V$, $w \in W$. This shows that the two constructions are inverse to each other. □

Example 6.3.3:

1. Every *cocommutative* bialgebra is quasitriangular with universal R-matrix $R = 1\otimes 1$. This includes group algebras of finite groups, tensor algebras of vector spaces and universal enveloping algebras of Lie algebras.

2. A *commutative* bialgebra B is quasitriangular if and only if it is cocommutative, since in this case, one has $\Delta^{op}(b) = R \cdot \Delta(b) \cdot R^{-1}$ for all $b \in B$ if and only if $\Delta = \Delta^{op}$. This shows that the algebra of functions $\mathbb{F}[G]^* = \mathrm{Map}_{\mathbb{F}}(G)$ on a non-abelian finite group G cannot be quasitriangular - it is commutative, but not cocommutative.

3. The group algebra $\mathbb{C}[\mathbb{Z}/n\mathbb{Z}]$ is quasitriangular with universal R-matrix

$$R = \tfrac{1}{n}\Sigma_{j,k=0}^{n-1} e^{2\pi \mathrm{i}\, jk/n}\, \bar{\jmath} \otimes \bar{k}.$$

4. For $n = 2$ and $\mathrm{char}(\mathbb{F}) \neq 2$, the Taft algebra from Example 5.3.6 is presented with generators x, y and relations $xy = -yx$, $y^2 = 1$, $x^2 = 0$, and Hopf algebra structure

$$\Delta(y) = y\otimes y, \quad \Delta(x) = 1\otimes x + y\otimes x, \quad \epsilon(y) = 1, \quad \epsilon(x) = 0, \quad S(y) = y, \quad S(x) = -xy.$$

   This Hopf algebra is quasitriangular with universal R-matrices

$$R_\alpha = \tfrac{1}{2}\,(1\otimes 1 + 1\otimes y + y\otimes 1 - y\otimes y) + \tfrac{\alpha}{2}\,(x\otimes x - x\otimes xy + xy\otimes xy + xy\otimes x) \quad \text{for } \alpha \in \mathbb{F}.$$

5. Let $q \in \mathbb{F}$ be a primitive rth root of unity with $r > 1$ odd. Then the Hopf algebra $U_q^r(\mathfrak{sl}_2)$ from Proposition 5.3.12 is quasitriangular with universal R-matrix

$$R = \frac{1}{r} \sum_{i,j,k=0}^{r-1} \frac{(q-q^{-1})^k}{(k)!_{q^2}} q^{k(k-1)+2k(i-j)-2ij}\, E^k K^i \otimes F^k K^j. \tag{75}$$

The examples show that quasitriangularity is a *structure*, not a property, just like a braiding in a monoidal category. In fact, every universal R-matrix that is not triangular yields another universal R-matrix. This is a consequence of the following proposition that contains the bialgebra counterparts of the statement that the braiding with the tensor unit is trivial, of the dodecagon identity and of the opposite braiding from Remark 3.1.2.

Proposition 6.3.4: Let (B, R) be a quasitriangular bialgebra.

1. Then the universal R-matrix satisfies

$$(\epsilon\otimes\mathrm{id})(R) = (\mathrm{id}\otimes\epsilon)(R) = 1 \qquad\qquad R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$

2. (B, R_{21}^{-1}) with $R_{21}^{-1} = \tau(R^{-1})$ is a quasitriangular bialgebra.

The equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ is called **quantum Yang-Baxter equation** (**QYBE**).

Proof:
1. With the defining condition on the universal R-matrix we compute

$$\begin{aligned} R &= (\mathrm{id}\otimes\mathrm{id})(R) = (\epsilon\otimes\mathrm{id}\otimes\mathrm{id}) \circ (\Delta\otimes\mathrm{id})(R) = (\epsilon\otimes\mathrm{id}\otimes\mathrm{id})(R_{13} \cdot R_{23}) \\ &= (\epsilon\otimes\mathrm{id}\otimes\mathrm{id})(R_{13}) \cdot (\epsilon\otimes\mathrm{id}\otimes\mathrm{id})(R_{23}) = \epsilon(1)\,(\epsilon\otimes\mathrm{id}\otimes\mathrm{id})(R_{13}) \cdot R = (1\otimes(\epsilon\otimes\mathrm{id})(R)) \cdot R \end{aligned}$$

As R is invertible, right multiplication of this equation with R^{-1} yields $1\otimes(\epsilon\otimes\mathrm{id})(R) = 1\otimes 1$, and applying $\epsilon\otimes\mathrm{id}$ to this equation we obtain $(\epsilon\otimes\mathrm{id})(R) = (\epsilon\otimes\mathrm{id})(1\otimes 1) = \epsilon(1)1 = 1$. The proof of the identity $(\mathrm{id}\otimes\epsilon)(R) = 1$ is analogous. The QYBE follows directly from the defining properties of the universal R-matrix:

$$R_{12}R_{13}R_{23} = R_{12} \cdot (\Delta\otimes\mathrm{id})(R) = (\Delta^{op}\otimes\mathrm{id})(R) \cdot R_{12} = (\tau\otimes\mathrm{id})(R_{13}R_{23}) \cdot R_{12} = R_{23}R_{13}R_{12}.$$

2. Applying the flip map $\tau : B\otimes B \to B\otimes B$, $b\otimes c \mapsto c\otimes b$ to the defining conditions in Definition 6.3.1 yields for all $b \in B$

$$\Delta(b) = \tau \circ \Delta^{op}(b) = \tau(R \cdot \Delta(b) \cdot R^{-1}) = R_{21} \cdot \Delta^{op}(b) \cdot R_{21}^{-1}$$
$$(\mathrm{id}\otimes\Delta)(R_{21}^{-1}) = (\tau\otimes\mathrm{id}) \circ (\mathrm{id}\otimes\tau) \circ (\Delta\otimes\mathrm{id})(R^{-1}) = (\tau\otimes\mathrm{id}) \circ (\mathrm{id}\otimes\tau)(R_{23}^{-1}R_{13}^{-1}) = (R_{21}^{-1})_{13}(R_{21}^{-1})_{12}$$
$$(\Delta\otimes\mathrm{id})(R_{21}^{-1}) = (\mathrm{id}\otimes\tau) \circ (\tau\otimes\mathrm{id}) \circ (\mathrm{id}\otimes\Delta)(R^{-1}) = (\mathrm{id}\otimes\tau) \circ (\tau\otimes\mathrm{id})(R_{12}^{-1}R_{13}^{-1}) = (R_{21}^{-1})_{13}(R_{21}^{-1})_{23}.$$

Conjugating the first equation with R_{21}^{-1} shows that R_{21}^{-1} is another universal R-matrix for B. □

If (H, R) is not just a quasitriangular bialgebra, but a quasitriangular *Hopf* algebra, then the quasitriangularity has important consequences for the antipode. It implies that the antipode is invertible and that its square is given by conjugation with a distinguished element of H, the *Drinfeld element*. Moreover, there is a distinguished grouplike $g \in H$ element such that the fourth power of the antipode is given by conjugation with g.

Theorem 6.3.5: Let (H, R) be a quasitriangular Hopf algebra. Then:

1. The antipode of H is invertible.
2. The universal R-matrix satisfies
$$(S\otimes\mathrm{id})(R) = (\mathrm{id}\otimes S^{-1})(R) = R^{-1} \qquad (S\otimes S)(R) = (S^{-1}\otimes S^{-1})(R) = R.$$
3. The **Drinfeld element** $u = S(R_{(2)})R_{(1)}$ is invertible with inverse $u^{-1} = R_{(2)}S^2(R_{(1)})$ and coproduct $\Delta(u) = (u\otimes u) \cdot (R_{21}R)^{-1}$.
4. The element $g = uS(u)^{-1}$ is grouplike.
5. The antipode satisfies $S^2(h) = uhu^{-1}$ and $S^4(h) = ghg^{-1}$ for all $h \in H$.

Proof:
2. To prove the identities $(S\otimes\mathrm{id})(R) = R^{-1}$ and $(S\otimes S)(R) = R$, we compute

$$\begin{aligned}&(m\otimes\mathrm{id}) \circ (S\otimes\mathrm{id}\otimes\mathrm{id}) \circ (\Delta\otimes\mathrm{id})(R) = (1_H\epsilon\otimes\mathrm{id})(R) = 1\otimes 1\\ =&(m\otimes\mathrm{id}) \circ (S\otimes\mathrm{id}\otimes\mathrm{id})(R_{13}R_{23}) = (S\otimes\mathrm{id})(R) \cdot R.\end{aligned}$$

Right multiplication by R^{-1} then yields $(S\otimes\mathrm{id})(R) = R^{-1}$. As $\tau \circ R^{-1} = R_{21}^{-1}$ is another universal R-matrix for H by Proposition 6.3.4, we also obtain $(\mathrm{id}\otimes S)(R^{-1}) = R$ and

$$(S\otimes S)(R) = (\mathrm{id}\otimes S) \circ (S\otimes\mathrm{id})(R) = (\mathrm{id}\otimes S)(R^{-1}) = R. \tag{76}$$

The identities $(\mathrm{id}\otimes S^{-1})(R) = R^{-1}$ and $(S^{-1}\otimes S^{-1})(R) = R$ then follow by applying $S^{-1}\otimes S^{-1}$ to these two equations, once it is established that S is invertible.

3.(a) We show that u is invertible with inverse $u^{-1} = R_{(2)}S^2(R_{(1)})$ with the auxiliary identity

$$S^2(h)u = hu \qquad \forall h \in H. \tag{77}$$

To prove (77) note that the condition $\Delta^{op}(h) \cdot R = R \cdot \Delta(h)$ reads in Sweedler notation $\Sigma_{(h)}h_{(2)}R_{(1)}\otimes h_{(1)}R_{(2)} = \Sigma_{(h)}R_{(1)}h_{(1)}\otimes R_{(2)}h_{(2)}$. With the condition on the antipode this gives

$$\begin{aligned}S^2(h)u &= S^2(h)S(R_{(2)})R_{(1)} = \Sigma_{(h)}S^2(h_{(3)})S(R_{(2)})S(h_{(1)})h_{(2)}R_{(1)} = \Sigma_{(h)}S^2(h_{(3)})S(h_{(1)}R_{(2)})h_{(2)}R_{(1)}\\ &= \Sigma_{(h)}S^2(h_{(3)})S(R_{(2)}h_{(2)})R_{(1)}h_{(1)} = \Sigma_{(h)}S^2(h_{(3)})S(h_{(2)})S(R_{(2)})R_{(1)}h_{(1)} = \Sigma_{(h)}S^2(h_{(3)})S(h_{(2)})uh_{(1)}\\ &= \Sigma_{(h)}S(h_{(2)}S(h_{(3)}))uh_{(1)} = \Sigma_{(h)}\epsilon(h_{(2)})uh_{(1)} = uh,\end{aligned}$$

where we used first the definition of u, then the identity $\Sigma_{(h)}S(h_{(1)})h_{(2)}\otimes h_{(3)}=1\otimes h$, then the fact that S is an anti-algebra homomorphism, then the identity $\Delta^{op}(h)\cdot R=R\cdot\Delta(h)$, then that S is an anti-algebra homomorphism, the definition of u and $\Sigma_{(h)}h_{(1)}\otimes h_{(2)}S(h_{(3)})=h\otimes 1$.

To show that u is invertible, we compute with (77) and the identity $(S\otimes S)(R)=R$

$$u\cdot R_{(2)}S^2(R_{(1)})\overset{(77)}{=}S^2(R_{(2)})uS^2(R_{(1)})\overset{(76)}{=}R_{(2)}uR_{(1)}\overset{(77)}{=}R_{(2)}S^2(R_{(1)})\cdot u. \tag{78}$$

Using this equation together with the definition of u, the identities $(S\otimes S)(R)=R$ and $(S\otimes\mathrm{id})(R)=R^{-1}$, that S is an anti-algebra homomorphism and (77) we then obtain

$$\begin{aligned}&u\cdot R_{(2)}S^2(R_{(1)})\overset{(78)}{=}R_{(2)}S^2(R_{(1)})\cdot u\overset{(77)}{=}R_{(2)}uR_{(1)}=R_{(2)}S(R'_{(2)})R'_{(1)}R_{(1)}\\&\overset{(76)}{=}S(R_{(2)})S(R'_{(2)})R'_{(1)}S(R_{(1)})=S(R'_{(2)}R_{(2)})R'_{(1)}S(R_{(1)})\\&=m^{op}\circ(\mathrm{id}\otimes S)(R\cdot(S\otimes\mathrm{id})(R))=m^{op}\circ(\mathrm{id}\otimes S)(R\cdot R^{-1})=m^{op}(1\otimes 1)=1.\end{aligned}$$

2. Identity (77) and the invertibility of u from 2.(a) imply that $S^2(h)=uhu^{-1}$ for all $h\in H$, and this implies that S^2 and hence S are invertible. This concludes the proof of 1. and 2. and proves the first identity in 5.

3.(b) To prove the identity $\Delta(u)=(u\otimes u)\cdot(R_{21}R)^{-1}$ we use the identities $(\Delta\otimes\mathrm{id})(R)=R_{13}R_{23}$ and $(\mathrm{id}\otimes\Delta)(R)=R_{13}R_{12}$ to compute

$$\begin{aligned}&\Delta(u)=\Delta(S(R_{(2)})R_{(1)})=\Delta(S(R_{(2)}))\cdot\Delta(R_{(1)})=\Delta(S(R_{(2)}R'_{(2)}))\cdot(R_{(1)}\otimes R'_{(1)})\\&=(S\otimes S)(\Delta^{op}(R_{(2)})\cdot\Delta^{op}(R'_{(2)}))\cdot(R_{(1)}\otimes R'_{(1)})=(S\otimes S)(R_{(2)}R'_{(2)}\otimes\tilde R_{(2)}\tilde R'_{(2)})\cdot(R_{(1)}\tilde R_{(1)}\otimes R'_{(1)}\tilde R'_{(1)})\\&=S(R'_{(2)})S(R_{(2)})R_{(1)}\tilde R_{(1)}\otimes S(\tilde R'_{(2)})S(\tilde R_{(2)})R'_{(1)}\tilde R'_{(1)}=S(R'_{(2)})u\tilde R_{(1)}\otimes S(\tilde R'_{(2)})S(\tilde R_{(2)})R'_{(1)}\tilde R'_{(1)})\\&\overset{(77)}{=}uS^{-1}(R'_{(2)})\tilde R_{(1)}\otimes S(\tilde R'_{(2)})S(\tilde R_{(2)})R'_{(1)}\tilde R'_{(1)})\overset{(76)}{=}uR'_{(2)}\tilde R_{(1)}\otimes S(\tilde R'_{(2)})S(\tilde R_{(2)})S(R'_{(1)})\tilde R'_{(1)})\end{aligned}$$

To simplify this expression, we consider the QYBE and multiply it with from the left and from the right with R_{12}^{-1}, which yields $R_{13}R_{23}R_{12}^{-1}=R_{12}^{-1}R_{23}R_{13}$. In Sweedler notation, this reads

$$R_{(1)}S(R'_{(1)})\otimes\tilde R_{(1)}R'_{(2)}\otimes R_{(2)}\tilde R_{(2)}=S(R_{(1)})R'_{(1)}\otimes R_{(2)}\tilde R_{(1)}\otimes\tilde R_{(2)}R'_{(2)}.$$

Applying the map $(m\otimes\mathrm{id})\circ\tau_{12}\circ\tau_{23}\circ(\mathrm{id}\otimes\mathrm{id}\otimes S)$ to both sides of this equation yields

$$S(\tilde R_{(2)})S(R_{(2)})R_{(1)}S(R'_{(1)})\otimes\tilde R_{(1)}R'_{(2)}=S(R'_{(2)})S(\tilde R_{(2)})S(R_{(1)})R'_{(1)}\otimes R_{(2)}\tilde R_{(1)},$$

and by inserting this equation into the last term in the expression for $\Delta(u)$, we obtain

$$\begin{aligned}\Delta(u)&=u\tilde R_{(1)}R'_{(2)}\otimes S(\tilde R_{(2)})S(\tilde R'_{(2)})\tilde R'_{(1)}S(R'_{(1)})=u\tilde R_{(1)}R'_{(2)}\otimes S(\tilde R_{(2)})uS(R'_{(1)})\\&\overset{(77)}{=}u\tilde R_{(1)}R'_{(2)}\otimes uS^{-1}(\tilde R_{(2)})S(R'_{(1)})\overset{(76)}{=}uS(\tilde R_{(1)})R'_{(2)}\otimes u\tilde R_{(2)}S(R'_{(1)})\\&=(u\otimes u)\cdot(R^{-1}R_{21}^{-1})=(u\otimes u)\cdot(R_{21}R)^{-1}.\end{aligned}$$

5. To prove that $S^4(h)=ghg^{-1}$ with $g=uS(u)^{-1}$, we consider the quasitriangular Hopf algebra $(H^{op,cop},R)$. Then $S(u)=R_{(1)}S(R_{(2)})$ takes the role of $u=S(R_{(2)})R_{(1)}$ for $(H^{op,cop},R)$. This implies $S^2(h)=S(u)^{-1}hS(u)$ and $S^4(h)=uS^2(h)u^{-1}=uS(u)^{-1}hS(u)u^{-1}=ghg^{-1}$ for all $h\in H$ and concludes the proof of 5.

4. The identities $S^2(h)=S(u)^{-1}hS(u)$ and $(S\otimes S)(R)=R$ imply

$$(S(u)^{-1}\otimes S(u)^{-1})(R_{21}R)=(S^2\otimes S^2)(R_{21}R)(S(u)^{-1}\otimes S(u)^{-1})=(R_{21}R)\cdot(S(u)^{-1}\otimes S(u)^{-1}),$$

and by combining these equations with the expression for the coproduct of u we obtain

$$\begin{aligned}\Delta(g) &= \Delta(uS(u)^{-1}) = \Delta(u)\cdot\Delta(S(u))^{-1} = (u\otimes u)(R_{21}R)^{-1}\cdot(S(u)^{-1}\otimes S(u)^{-1})\cdot(R_{21}R)\\ &= (u\otimes u)(R_{21}R)^{-1}\cdot(R_{21}R)\cdot(S(u)^{-1}\otimes S(u)^{-1}) = uS(u)^{-1}\otimes uS(u)^{-1} = g\otimes g.\end{aligned}$$

As $\epsilon(g)=\epsilon(u)\epsilon(u^{-1})=1\neq 0$, it follows that g is grouplike. □

In Section 3.2 we showed that there is a canonical construction, the *centre construction* that associates to each monoidal category a braided monoidal category. For monoidal categories that arise as the representation categories of finite-dimensional Hopf algebras, this construction has a Hopf algebraic counterpart. This is the *Drinfeld double* construction. It assigns to each finite-dimensional Hopf algebra H over $\mathbb{F}$ a quasitriangular Hopf algebra $D(H)$ that contains both, H and the dual H^{*cop} as Hopf subalgebras.

Theorem 6.3.6: For every finite-dimensional Hopf algebra H there is a unique quasitriangular Hopf algebra structure on the vector space $H^*\otimes H$ such that the inclusion maps

$$\iota_H : H\to H^*\otimes H,\ h\mapsto 1\otimes h \qquad\qquad \iota_{H^*} : H^{*cop}\to H^*\otimes H,\ \alpha\mapsto\alpha\otimes 1$$

are homomorphisms of Hopf algebras. It is called the **Drinfeld double** or **quantum double** $D(H)$ and given by

$$\begin{aligned}&(\alpha\otimes h)\cdot(\beta\otimes k) = \Sigma_{(h),(\beta)}\,\beta_{(3)}(h_{(1)})\,\beta_{(1)}(S^{-1}(h_{(3)}))\,\alpha\beta_{(2)}\otimes h_{(2)}k && 1 = 1_{H^*}\otimes 1_H\\ &\Delta(\alpha\otimes h) = \Sigma_{(h),(\alpha)}\,\alpha_{(2)}\otimes h_{(1)}\otimes\alpha_{(1)}\otimes h_{(2)} && \epsilon(\alpha\otimes h)=\epsilon_{H^*}(\alpha)\epsilon_H(h)\\ &S(\alpha\otimes h) = (1\otimes S(h))\cdot(S^{-1}(\alpha)\otimes 1)\end{aligned}\tag{79}$$

A universal R-matrix for $D(H)$ is given by $R=\Sigma_{i=1}^n 1\otimes x_i\otimes\alpha^i\otimes 1$, where $(x_1,...,x_n)$ is an ordered basis of H with dual basis $(\alpha^1,...,\alpha^n)$.

Proof:
1. We show that (79) defines a Hopf algebra structure on $H^*\otimes H$.

The coassociativity and counitality follow directly from the coassociativity and counitality for H and H^{*cop}. That Δ and ϵ are algebra homomorphisms follows, because this holds in H and H^* and from the identity $\alpha\otimes h=(\alpha\otimes 1)\cdot(1\otimes h)$ for all $h\in H$ and $\alpha\in H^*$. The same holds for the condition on the antipode. As S^{-1} is an antipode for H^{*cop} by Lemma 5.2.6, we have

$$\begin{aligned}&m\circ(S\otimes\mathrm{id})\circ\Delta(\alpha\otimes h) = \Sigma_{(h)(\alpha)}\,(1\otimes S(h_{(1)}))\cdot(S^{-1}(\alpha_{(2)})\otimes 1)\cdot(\alpha_{(1)}\otimes 1)\cdot(1\otimes h_{(2)})\\ &= (1\otimes S(h_{(1)}))\cdot(S(\alpha_{(1)})\alpha_{(2)}\otimes 1)\cdot(1\otimes h_{(2)}) = \epsilon(\alpha)\Sigma_{(h)}\,1\otimes S(h_{(1)})h_{(2)} = \epsilon(\alpha)\epsilon(h)\,1_{H^*}\otimes 1_H\end{aligned}$$

and the identity $m\circ(S\otimes\mathrm{id})\circ\Delta(\alpha\otimes h)=\eta\circ\epsilon$ follows analogously. That $1_{H^*}\otimes 1_H$ is a unit for the multiplication follows directly from the formulas

$$\begin{aligned}(\alpha\otimes h)\cdot(1_{H^*}\otimes 1_H) &= \Sigma_{(h)}\,1_{H^*}(h_{(1)})\,1_{H^*}(S^{-1}(h_{(3)}))\,\alpha 1_{H^*}\otimes h_{(2)}1_H\\ &= \Sigma_{(h)}\,\epsilon(h_{(1)})\,\epsilon(S^{-1}(h_{(3)}))\,\alpha\epsilon\otimes h_{(2)} = \alpha\otimes h\\ (1_{H^*}\otimes 1_H)\cdot(\beta\otimes k) &= \Sigma_{(\beta)}\,\beta_{(3)}(1_H)\,\beta_{(1)}(S^{-1}(1_H))\,1_{H^*}\beta_{(2)}\otimes 1_H k\\ &= \Sigma_{(\beta)}\,\epsilon(\beta_{(3)})\epsilon(\beta_{(1)})\beta_{(2)}\otimes k = \beta\otimes k.\end{aligned}$$

It remains to prove the associativity of the multiplication, which follows by a direct computation:

$$\begin{aligned}
&\big((\alpha\otimes h)\cdot(\beta\otimes k)\big)\cdot(\gamma\otimes l) = \Sigma_{(h),(\beta)}\,\beta_{(3)}(h_{(1)})\,\beta_{(1)}(S^{-1}(h_{(3)}))\;(\alpha\beta_{(2)}\otimes h_{(2)}k)\cdot(\gamma\otimes l)\\
&= \Sigma_{(h),(k),(\beta),(\gamma)}\;\beta_{(3)}(h_{(1)})\,\beta_{(1)}(S^{-1}(h_{(3)}))\gamma_{(3)}(h_{(2)}k)_{(1)})\gamma_{(1)}(S^{-1}((h_{(2)}k)_{(3)}))\;\alpha\beta_{(2)}\gamma_{(2)}\otimes(h_{(2)}k)_{(2)}l\\
&= \Sigma_{(h),(k),(\beta),(\gamma)}\;\beta_{(3)}(h_{(1)})\,\beta_{(1)}(S^{-1}(h_{(5)}))\gamma_{(3)}(h_{(2)}k_{(1)})\gamma_{(1)}(S^{-1}(h_{(4)}k_{(3)}))\;\alpha\beta_{(2)}\gamma_{(2)}\otimes h_{(3)}k_{(2)}l
\end{aligned}$$

$$\begin{aligned}
&(\alpha\otimes h)\cdot\big((\beta\otimes k)\;\cdot(\gamma\otimes l)\big) = \Sigma_{(k),(\gamma)}\,\gamma_{(3)}(k_{(1)})\,\gamma_{(1)}(S^{-1}(k_{(3)}))\;(\alpha\otimes h)\cdot(\beta\gamma_{(2)}\otimes k_{(2)}l)\\
&= \Sigma_{(k),(h),(\beta),(\gamma)}\,\gamma_{(3)}(k_{(1)})\,\gamma_{(1)}(S^{-1}(k_{(3)}))(\beta\gamma_{(2)})_{(3)}(h_{(1)})(\beta\gamma_{(2)})_{(1)}(S(h_{(3)}))\;\alpha(\beta\gamma_{(2)})_{(2)}\otimes h_{(2)}k_{(2)}l\\
&= \Sigma_{(k),(h),(\beta),(\gamma)}\,\gamma_{(5)}(k_{(1)})\,\gamma_{(1)}(S^{-1}(k_{(3)}))(\beta_{(3)}\gamma_{(4)})(h_{(1)})(\beta_{(1)}\gamma_{(2)})(S^{-1}(h_{(3)}))\;\alpha\beta_{(2)}\gamma_{(3)}\otimes h_{(2)}k_{(2)}l\\
&= \Sigma_{(k),(h),(\beta),(\gamma)}\,\beta_{(3)}(h_{(1)})\gamma_{(3)}(h_{(2)}k_{(1)})\,\gamma_{(1)}(S^{-1}(h_{(4)}k_{(3)}))\beta_{(1)}(S^{-1}(h_{(5)}))\;\alpha\beta_{(2)}\gamma_{(2)}\otimes h_{(3)}k_{(2)}l.
\end{aligned}$$

2. That the inclusion maps $\iota_H : H \to H^*\otimes H$ and $\iota_{H^*} : H^{*cop} \to H^*\otimes H$ are homomorphisms of Hopf algebras follows directly from the expressions for (co)multiplication, (co)unit and antipode.

3. To show that $R = \Sigma_{i=1}^n 1\otimes x_i\otimes\alpha^i\otimes 1$ is a universal R-matrix for $D(H)$ we use the auxiliary identities (67) from the proof of Theorem 6.2.9

$$\Sigma_{i=1}^n \Delta(x_i)\otimes\alpha^i = \Sigma_{i,j=1}^n x_i\otimes x_j\otimes\alpha^i\alpha^j \qquad \Sigma_{i=1}^n x_i\otimes\Delta(\alpha^i) = \Sigma_{i,j=1}^n x_ix_j\otimes\alpha^i\otimes\alpha^j.$$

With the identities (67) we compute

$$\begin{aligned}
(\Delta\otimes\mathrm{id})(R) &= \Sigma_{i=1}^n\Delta(1\otimes x_i)\otimes\alpha^i\otimes 1 = \Sigma_{i=1}^n 1\otimes x_{i(1)}\otimes 1\otimes x_{i(2)}\otimes\alpha^i\otimes 1 = \Sigma_{i,j=1}^n 1\otimes x_i\otimes 1\otimes x_j\otimes\alpha^i\alpha^j\otimes 1\\
&= (\Sigma_{i=1}^n 1\otimes x_i\otimes 1\otimes 1\otimes\alpha^i\otimes 1)\cdot(\Sigma_{j=1}^n 1\otimes 1\otimes 1\otimes x_j\otimes\alpha^j\otimes 1) = R_{13}\cdot R_{23}\\
(\mathrm{id}\otimes\Delta)(R) &= \Sigma_{i=1}^n 1\otimes x_i\otimes\Delta(\alpha^i\otimes 1) = \Sigma_{i=1}^n 1\otimes x_i\otimes\alpha^i_{(2)}\otimes 1\otimes\alpha^i_{(1)}\otimes 1 = \Sigma_{i,j=1}^n 1\otimes x_ix_j\otimes\alpha^j\otimes 1\otimes\alpha^i\otimes 1\\
&= (\Sigma_{i=1}^n 1\otimes x_i\otimes 1\otimes 1\otimes\alpha^i\otimes 1)\cdot(\Sigma_{j=1}^n 1\otimes x_j\otimes\alpha^j\otimes 1\otimes 1\otimes 1) = R_{13}\cdot R_{12},
\end{aligned}$$

and show that R is invertible:

$$\begin{aligned}
R\cdot(S\otimes\mathrm{id})(R) &= (\Sigma_{i=1}^n 1\otimes x_i\otimes\alpha^i\otimes 1)\cdot(\Sigma_{i=1}^n 1\otimes S(x_j)\otimes\alpha^j\otimes 1) = \Sigma_{i,j=1}^n 1\otimes x_iS(x_j)\otimes\alpha^i\alpha^j\otimes 1\\
&= \Sigma_{i=1}^n 1\otimes x_{i(1)}S(x_{i(2)})\otimes\alpha^i\otimes 1 = \Sigma_{i=1}^n\epsilon(x_i)1\otimes 1\otimes\alpha^i\otimes 1 = 1\otimes 1\otimes 1\otimes 1.
\end{aligned}$$

The condition $R\cdot\Delta = \Delta^{op}\cdot R$ then follows again from (67) by a direct computation

$$\begin{aligned}
&R\cdot\Delta(\alpha\otimes h) = \Sigma_{(h),(\alpha)}\Sigma_{i=1}^n(1\otimes x_i\otimes\alpha^i\otimes 1)\cdot(\alpha_{(2)}\otimes h_{(1)}\otimes\alpha_{(1)}\otimes h_{(2)})\\
&= \Sigma_{(h),(\alpha)}\Sigma_{i=1}^n\alpha_{(2)(3)}(x_{i(1)})\alpha_{(2)(1)}(S^{-1}(x_{i(3)}))\alpha_{(2)(2)}\otimes x_{i(2)}h_{(1)}\otimes\alpha^i\alpha_{(1)}\otimes h_{(2)}\\
&= \Sigma_{(h),(\alpha)}\Sigma_{i,j,k=1}^n\alpha_{(4)}(x_i)\alpha_{(2)}(S^{-1}(x_k))\alpha_{(3)}\otimes x_jh_{(1)}\otimes\alpha^i\alpha^j\alpha^k\alpha_{(1)}\otimes h_{(2)}\\
&= \Sigma_{(h),(\alpha)}\Sigma_{j=1}^n\alpha_{(3)}\otimes x_jh_{(1)}\otimes\alpha_{(4)}\alpha^jS^{-1}(\alpha_{(2)})\alpha_{(1)}\otimes h_{(2)}\\
&= \Sigma_{(h),(\alpha)}\Sigma_{j=1}^n\epsilon(\alpha_{(1)})\alpha_{(2)}\otimes x_jh_{(1)}\otimes\alpha_{(3)}\alpha^j\otimes h_{(2)} = \Sigma_{(h),(\alpha)}\Sigma_{j=1}^n\alpha_{(1)}\otimes x_jh_{(1)}\otimes\alpha_{(2)}\alpha^j\otimes h_{(2)}
\end{aligned}$$

$$\begin{aligned}
&\Delta^{op}(\alpha\otimes h)\cdot R = \Sigma_{(h),(\alpha)}\Sigma_{i=1}^n(\alpha_{(1)}\otimes h_{(2)}\otimes\alpha_{(2)}\otimes h_{(1)})\cdot(1\otimes x_i\otimes\alpha^i\otimes 1)\\
&= \Sigma_{(h),(\alpha)}\Sigma_{i=1}^n\alpha^i_{(3)}(h_{(1)(1)})\alpha^i_{(1)}(S^{-1}(h_{(1)(3)}))\;\alpha_{(1)}\otimes h_{(2)}x_i\otimes\alpha_{(2)}\alpha^i_{(2)}\otimes h_{(1)(2)}\\
&= \Sigma_{(h),(\alpha)}\Sigma_{i,j,k=1}^n\alpha^k(h_{(1)})\alpha^i(S^{-1}(h_{(3)}))\;\alpha_{(1)}\otimes h_{(4)}x_ix_jx_k\otimes\alpha_{(2)}\alpha^j\otimes h_{(2)}\\
&= \Sigma_{(h),(\alpha)}\Sigma_{i,j,k=1}^n\alpha_{(1)}\otimes h_{(4)}S^{-1}(h_{(3)})x_jh_{(1)}\otimes\alpha_{(2)}\alpha^j\otimes h_{(2)}\\
&= \Sigma_{(h),(\alpha)}\Sigma_{i,j,k=1}^n\epsilon(h_{(3)})\alpha_{(1)}\otimes x_jh_{(1)}\otimes\alpha_{(2)}\alpha^j\otimes h_{(2)} = \Sigma_{(h),(\alpha)}\Sigma_{i,j,k=1}^n\alpha_{(1)}\otimes x_jh_{(1)}\otimes\alpha_{(2)}\alpha^j\otimes h_{(2)}.\quad\square
\end{aligned}$$

By definition, the algebra structure on $D(H)$ is given by the relations $(\alpha\otimes 1)(1\otimes h) = \alpha\otimes h$, by the relations $(1\otimes h)(1\otimes k) = 1\otimes hk$ and $(\alpha\otimes 1)(\beta\otimes 1) = \alpha\beta\otimes 1$, which encode the inclusions of the algebras H and H^{*cop} and by the **crossed relations**

$$(1\otimes h)\cdot(\alpha\otimes 1) = \Sigma_{(h),(\alpha)}\;\alpha_{(3)}(h_{(1)})\alpha_{(1)}(S^{-1}(h_{(3)}))\;(\alpha_{(2)}\otimes 1)\cdot(1\otimes h_{(2)}).$$

In proofs it is often simpler to work with these relations than with the full multiplication law of $D(H)$, since this reduces the complexity of the computations.

Example 6.3.7: Let G be a finite group. Then the Drinfeld double $D(\mathbb{F}[G])$ is the vector space $\mathrm{Map}(G,\mathbb{F})\otimes\mathbb{F}[G]$ with the quasitriangular Hopf algebra structure

$$\begin{aligned}
&(\delta_u\otimes g)\cdot(\delta_v\otimes h)=\delta_u(gvg^{-1})\delta_u\otimes gh && 1=1\otimes e=\Sigma_{g\in G}\,\delta_g\otimes e\\
&\Delta(\delta_u\otimes g)=\Sigma_{xy=u}\,\delta_y\otimes g\otimes\delta_x\otimes g && \epsilon(\delta_u\otimes g)=\delta_u(e)\\
&S(\delta_u\otimes g)=\delta_{g^{-1}u^{-1}g}\otimes g^{-1} && R=\Sigma_{g\in G}1\otimes g\otimes\delta_g\otimes e
\end{aligned}$$

The formulas for the Hopf algebra structure of the Drinfeld double $D(H)$ describe its Hopf algebra structure in terms of the Hopf algebras H and H^*. Over a field of characteristic zero, this allows one to directly draw conclusions about its integrals and semisimplicity.

Corollary 6.3.8: For a finite-dimensional Hopf algebra H over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F})=0$, the following are equivalent:

(i) H is semisimple.
(ii) H^* is semisimple.
(iii) $D(H)$ is semisimple.

If one of these conditions is satisfied, then the normalised Haar integral for $D(H)$ is given by $\lambda\otimes\ell$, where $\ell\in H$ and $\lambda\in H^*$ are the normalised Haar integrals of H and H^*.

Proof:
1. The equivalence of (i) and (ii) follows from Theorem 6.2.18 by Larson and Radford, which also implies that (i) and (ii) are equivalent to $S_H^2=\mathrm{id}_H$ and $S_{H^*}^2=\mathrm{id}_{H^*}$. As one has from (79) for all $\alpha\in H^*$ and $h\in H$

$$\begin{aligned}
S^2_{D(H)}(\alpha\otimes h)&=S^2_{D(H)}((\alpha\otimes 1)(1\otimes h))=S^2_{D(H)}(\alpha\otimes 1)S^2_{D(H)}(1\otimes h)=(S^{-2}_{H^*}(\alpha)\otimes 1)(1\otimes S^2_H(h))\\
&=(S^{-2}_{H^*}(\alpha)\otimes S^2_H(h)),
\end{aligned}$$

if follows that (i) and (ii) are equivalent to the condition $S^2_{D(H)}=\mathrm{id}$, which by Theorem 6.2.18 is equivalent to (iii).

2. With formulas (79) one computes

$$\begin{aligned}
(\lambda\otimes\ell)\cdot(\alpha\otimes h)&=\Sigma_{(\ell),(\alpha)}\alpha_{(3)}(\ell_{(1)})\alpha_{(1)}(S^{-1}(\ell_{(3)}))\;\lambda\alpha_{(2)}\otimes\ell_{(2)}h\\
&=\Sigma_{(\ell),(\alpha)}\alpha_{(3)}(\ell_{(1)})\alpha_{(1)}(S^{-1}(\ell_{(3)}))\epsilon(\alpha_{(2)})\;\lambda\otimes\ell_{(2)}h\\
&=\Sigma_{(\ell),(\alpha)}\alpha_{(2)}(\ell_{(1)})\alpha_{(1)}(S^{-1}(\ell_{(3)}))\;\lambda\otimes\ell_{(2)}h=\Sigma_{(\ell)}\alpha(S^{-1}(\ell_{(3)})\ell_{(1)})\;\lambda\otimes\ell_{(2)}h\\
&=\Sigma_{(\ell)}\alpha(S(\ell_{(3)})\ell_{(1)})\;\lambda\otimes\ell_{(2)}h\overset{*}{=}\Sigma_{(\ell)}\alpha(S(\ell_{(2)})\ell_{(3)})\;\lambda\otimes\ell_{(1)}h\\
&=\Sigma_{(\ell)}\epsilon(\ell_{(2)})\epsilon(\alpha)\lambda\otimes\ell_{(1)}h=\epsilon(\alpha)\;\lambda\otimes\ell h=\epsilon(\alpha)\epsilon(h)\;\lambda\otimes\ell,
\end{aligned}$$

where we used that $\lambda\in I_R(H^*)$ to pass to the second line, the identity $S^2=\mathrm{id}$ to pass to the fourth line and in * that $\Delta^{(2)}(\ell)$ is invariant under cyclic permutations by Corollary 6.2.20. This shows that $\lambda\otimes\ell$ is a left integral in $D(H)$. As $D(H)$ is semisimple, it is a right integral as well by Corollary 6.2.14. That it is normalised follows directly from (79) $\epsilon_{D(H)}(\lambda\otimes\ell)=\epsilon(\lambda)\epsilon(\ell)=1$.
$\square$

Given the centre construction from Section 3.2 and the Drinfeld double construction from Theorem 6.3.6, one has two constructions that associate to a finite-dimensional Hopf algebra H

a braided monoidal category. One may either consider the centre of its representation category $H-\mathrm{Mod}$ or the representation category $D(H)-\mathrm{Mod}$ of its Drinfeld double. In fact, these braided monoidal categories are braided equivalent. This gives a more concrete description of the centre construction and allows one to realise it as a representation category.

Theorem 6.3.9: Let H be a finite-dimensional Hopf algebra over $\mathbb{F}$ and $D(H)$ its Drinfeld double. Then the representation category $D(H)-\mathrm{Mod}$ is braided equivalent to the centre $\mathcal{Z}(H-\mathrm{Mod})$, and $D(H)-\mathrm{Mod}^{fd}$ is braided equivalent to $\mathcal{Z}(H-\mathrm{Mod}^{fd})$.

Proof:
1. We consider the functor $G : D(H)-\mathrm{Mod} \to \mathcal{Z}(H-\mathrm{Mod})$ that assigns to a $D(H)$-module $(M, \rhd)$ the H-module $(M, \rhd')$ with $h \rhd' m = (1 \otimes h) \rhd m$ and the family of linear maps

$$c_{N,M} : N \otimes M \to M \otimes N, \quad n \otimes m \mapsto \tau \circ (R \rhd n \otimes m) = \Sigma_{i=1}^n [(\alpha^i \otimes 1) \rhd m] \otimes [x_i \rhd_N n]$$

for each H-module $(N, \rhd_N)$. To a $D(H)$-linear map $f : M \to M'$ it assigns the H-linear map $G(f) = f : M \to M'$.

The H-linearity of the maps $c_{N,M}$ follows from the identity $\Delta^{op}(1 \otimes h) R = R \Delta(1 \otimes h)$, and they are natural in the first argument because for every H-linear map $f : N \to N'$ one has

$$\begin{aligned}(\mathrm{id} \otimes f) \circ c_{N,M}(n \otimes m) &= (\mathrm{id} \otimes f)\big(\Sigma_{i=1}^n [(\alpha^i \otimes 1) \rhd m] \otimes [x_i \rhd n]\big) = \Sigma_{i=1}^n [(\alpha^i \otimes 1) \rhd m] \otimes f(x_i \rhd n) \\ &= \Sigma_{i=1}^n [(\alpha^i \otimes 1) \rhd m] \otimes [x_i \rhd f(n)] = c_{N,M} \circ (f \otimes \mathrm{id})(n \otimes m).\end{aligned}$$

They also satisfy condition (32) due to the identity $(\Delta \otimes \mathrm{id})(R) = R_{13} R_{23}$. This shows that $(M, c_{-,M})$ is indeed an element of $\mathcal{Z}(H-\mathrm{Mod})$. That $G(f) = f : M \to M'$ is a morphism in $\mathcal{Z}(H-\mathrm{Mod})$ for each $D(H)$-linear map $f : M \to M'$ follows again directly from the definition:

$$\begin{aligned}(f \otimes \mathrm{id}) \circ c_{N,M}(n \otimes m) &= \Sigma_{i=1}^n f((\alpha^i \otimes 1) \rhd' m) \otimes [x_i \rhd n] \\ &= \Sigma_{i=1}^n [(\alpha^i \otimes 1) \rhd' f(m)] \otimes [x_i \rhd n] = c_{N,M'} \circ (\mathrm{id} \otimes f)(n \otimes m).\end{aligned}$$

This shows that we have a functor $G : D(H)-\mathrm{Mod} \to \mathcal{Z}(H-\mathrm{Mod})$. The functor G is monoidal due to the identity $(\mathrm{id} \otimes \Delta)(R) = R_{13} R_{12}$ and braided, since for all $D(H)$-modules M, N the braiding of $G(M)$ and $G(N)$ in $\mathcal{Z}(H-\mathrm{Mod})$ is

$$c_{G(N),G(M)}(n \otimes m) = \tau \circ (R \rhd n \otimes m) = \Sigma_{i=1}^n [(\alpha^i \otimes 1) \rhd m] \otimes [(1 \otimes x_i) \rhd n] = G(c^{D(H)}_{N,M})(n \otimes m).$$

2. We consider the functor $F : \mathcal{Z}(H-\mathrm{Mod}) \to D(H)-\mathrm{Mod}$ that assigns to an object $(V, c_{-,V})$ in $\mathcal{Z}(H-\mathrm{Mod})$ the $D(H)$-module $(V, \rhd')$ with

$$(\alpha \otimes h) \rhd' v = (\mathrm{id} \otimes \alpha) \circ c_{H,V}(1 \otimes (h \rhd v))$$

and to a morphism $f : V \to W$ in $\mathcal{Z}(H-\mathrm{Mod})$ the $D(H)$-linear map $F(f) = f : V \to W$.

To show that F is well-defined, we need to verify that $\rhd' : D(H) \otimes V \to V$ is indeed a $D(H)$-module structure on V and that $f : V \to W$ is $D(H)$-linear. For the former, we note that $(\alpha \otimes h) \rhd' v = (\alpha \otimes 1) \rhd' ((1 \otimes h) \rhd' v)$ and compute

$$\begin{aligned}&(\alpha \otimes 1) \rhd' ((\beta \otimes 1) \rhd' v) = (\mathrm{id} \otimes \alpha) c_{H,V}(1 \otimes (\mathrm{id} \otimes \beta) c_{H,V}(1 \otimes v)) \\ &= (\mathrm{id} \otimes \alpha \otimes \beta)(c_{H,V} \otimes \mathrm{id})(\mathrm{id} \otimes c_{H,V})(1 \otimes 1 \otimes v) \\ &= (\mathrm{id} \otimes \alpha \otimes \beta) c_{H \otimes H,V}(1 \otimes 1 \otimes v) = (\mathrm{id} \otimes \alpha \otimes \beta) c_{H \otimes H,V}(\Delta \otimes \mathrm{id})(1 \otimes v) \\ &= (\mathrm{id} \otimes \alpha \otimes \beta)(\mathrm{id} \otimes \Delta) \circ c_{H,V}(1 \otimes v) = (\mathrm{id} \otimes (\alpha\beta)) c_{H,V}(1 \otimes v) = (\alpha\beta \otimes 1) \rhd' v,\end{aligned}$$

where we used first the definition of $\rhd'$, then identity (32) to pass to the third line, then the fact that $\Delta : H \to H \otimes H$ is H-linear to pass to the fourth line and then the duality between H^* and H.

To prove the corresponding identity for elements $1\otimes h \in 1\otimes H \subset D(H)$, $\alpha\otimes 1 \in H^*\otimes 1 \subset D(H)$, we consider the maps $L_h : H \to H$, $k \mapsto hk$ and $R_h : H \to H$, $k \mapsto kh$ and compute

$$
\begin{aligned}
&(1\otimes h) \rhd' ((\alpha\otimes 1) \rhd' v) = h \rhd (\mathrm{id}\otimes\alpha)c_{H,V}(1\otimes v) = (\mathrm{id}\otimes\alpha)((h \rhd -)\otimes \mathrm{id})c_{H,V}(1\otimes v)\\
&= \Sigma_{(h)}(\mathrm{id}\otimes\alpha)((h_{(1)} \rhd -)\otimes L_{S^{-1}(h_{(3)})h_{(2)}})c_{H,V}(1\otimes v)\\
&= \Sigma_{(h)}(\mathrm{id}\otimes\alpha)(\mathrm{id}\otimes L_{S^{-1}(h_{(3)})})((h_{(1)} \rhd -)\otimes L_{h_{(2)}})c_{H,V}(1\otimes v))\\
&= \Sigma_{(h)}(\mathrm{id}\otimes\alpha)(\mathrm{id}\otimes L_{S^{-1}(h_{(2)})})(h_{(1)} \rhd c_{H,V}(1\otimes v)) = \Sigma_{(h)}(\mathrm{id}\otimes\alpha)(\mathrm{id}\otimes L_{S^{-1}(h_{(2)})})c_{H,V}(h_{(1)} \rhd (1\otimes v))\\
&= \Sigma_{(h)}(\mathrm{id}\otimes\alpha)(\mathrm{id}\otimes L_{S^{-1}(h_{(3)})})c_{H,V}(h_{(1)}\otimes h_{(2)} \rhd v)\\
&= \Sigma_{(h)}(\mathrm{id}\otimes\alpha)(\mathrm{id}\otimes L_{S^{-1}(h_{(3)})})(\mathrm{id}\otimes R_{h_{(1)}})c_{H,V}(1\otimes h_{(2)} \rhd v)\\
&= \Sigma_{(h),(\alpha)}\alpha_{(3)}(h_{(1)})S^{-1}(\alpha_{(1)})(h_{(3)})\ (\mathrm{id}\otimes\alpha_{(2)})c_{H,V}(1\otimes h_{(2)} \rhd v) = ((1\otimes h)\cdot(\alpha\otimes 1)) \rhd' v,
\end{aligned}
$$

where we first used the definition of $\rhd'$, then the identities $m^{op} \circ (\mathrm{id}\otimes S^{-1}) \circ \Delta = \eta \circ \epsilon$ and $L_e = \mathrm{id}_H$ to pass to the second line, then the identity $L_{hh'} = L_h L_{h'}$ to pass to the third line, the definition of the H-module structure on $V\otimes H$ to pass to the fourth line and then the H-linearity of $c_{H,V}$, the definition of the H-module structure on $H\otimes V$ to pass to the fifth line, then the fact that $R_h : H \to H$ is H-linear and the naturality of $c_{H,V}$ in H, then the duality between H and H^* and finally the multiplication law of $D(H)$. The corresponding condition for the action of two elements $1\otimes h, 1\otimes k \in 1\otimes H \subset D(H)$ follows directly from the definition of the H-module. This shows that $\rhd'$ is a $D(H)$-module structure on V.

To see that $f : V \to W$ is $D(H)$-linear for all morphisms $f : V \to W$ in $\mathcal{Z}(\mathcal{C})$, we compute

$$
\begin{aligned}
&f((\alpha\otimes h) \rhd' v) = (f\otimes\alpha)c_{H,V}(1\otimes(h \rhd v)) = (\mathrm{id}\otimes\alpha)(f\otimes\mathrm{id})c_{H,V}(1\otimes(h \rhd v))\\
&= (\mathrm{id}\otimes\alpha)c_{H,W}(1\otimes f(h \rhd v)) = (\mathrm{id}\otimes\alpha)c_{H,W}(1\otimes(h \rhd f(v))) = (\alpha\otimes h) \rhd' f(v),
\end{aligned}
$$

where we first used the definition of $\rhd'$ and the properties of the tensor product, then the fact that f is a morphism in $\mathcal{Z}(H-\mathrm{Mod})$ to pass to the second line, then the H-linearity of f and then again the definition of $\rhd'$. This shows that the assignments define a functor $F : \mathcal{Z}(H-\mathrm{Mod}) \to D(H)-\mathrm{Mod}$.

We show that F is a braided monoidal functor. To show that F is monoidal, we have to show that the $D(H)$-module structure on $V\otimes V'$ for the tensor product of two objects $(V, c_{-,V})$, $(V', c_{-,V'})$ is the one of the tensor product of the $D(H)$-modules V and V' and that the $D(H)$-module structure on the tensor unit $(\mathbb{F}, c_{-,\mathbb{F}})$ in $\mathcal{Z}(H-\mathrm{Mod})$ is the trivial $D(H)$-module structure. The second statement follows directly from the definition of $\rhd'$ and the fact that $c_{M,\mathbb{F}} = l_M^{-1} \circ r_M$. To prove the first statement, note that the two H-module structures on $V\otimes V'$ agree by definition. To show that the H^*-module structures agree, note that the H-linearity of $R_h : H \to H$, $k \mapsto kh$ and the naturality of $c_{H,V}$ in the first argument imply $c_{H,V}(h\otimes v) = c_{H,V} \circ (R_h\otimes\mathrm{id})(1\otimes v) = (\mathrm{id}\otimes R_h)c_{H,V}(1\otimes v)$ for all $v \in V$, $h \in H$. Consequently, one has

$$(\mathrm{id}\otimes\alpha)c_{H,V}(h\otimes v) = (\mathrm{id}\otimes\alpha)(\mathrm{id}\otimes R_h)c_{H,V}(1\otimes v) = (\mathrm{id}\otimes\Delta(\alpha))(c_{H,V}(1\otimes v)\otimes h).$$

Using the definition of the tensor product in $\mathcal{Z}(H-\mathrm{Mod})$ from (34), one obtains

$$
\begin{aligned}
&(\alpha\otimes 1) \rhd' (v\otimes v') = (\mathrm{id}\otimes\alpha)c_{H,V\otimes V'}(1\otimes v\otimes v') = (\mathrm{id}\otimes\alpha)(\mathrm{id}\otimes c_{H,V'})(c_{H,V}\otimes\mathrm{id})(1\otimes v\otimes v')\\
&= \Sigma_{(\alpha)}(\mathrm{id}\otimes\alpha_{(2)}\otimes\mathrm{id}\otimes\alpha_{(1)})(c_{H,V}\otimes c_{H,V'})(1\otimes v\otimes 1\otimes v')\\
&= \Sigma_{(\alpha)}((\alpha_{(2)}\otimes 1) \rhd' v)\otimes((\alpha_{(1)}\otimes 1) \rhd' v') = \Delta(\alpha\otimes 1)(\rhd\otimes\rhd)(v\otimes v').
\end{aligned}
$$

This shows that F is monoidal. To see that F is braided, we note that for any H-module V and $v \in V$ the H-linear map $f_v : H \to V$, $h \mapsto h \rhd v$ is given in terms of the basis $(x_1, ..., x_n)$ of H and the dual basis $(\alpha^1, ..., \alpha^n)$ of H^* by $f_v(h) = \Sigma_{i=1}^n \alpha^i(h) x_i \rhd v$. With the formula for the braiding in $D(H)-\mathrm{Mod}$ we then compute for objects $(V, c_{-,V})$ and $(W, c_{-,W})$ in $\mathcal{Z}(H-\mathrm{Mod})$

$$
\begin{aligned}
&c^{D(H)}_{V,W}(v \otimes w) = \Sigma_{i=1}^n [(\alpha^i \otimes 1) \rhd' w] \otimes [(1 \otimes x_i) \rhd' v] \\
&= \Sigma_{i=1}^n [(\mathrm{id} \otimes \alpha^i) c_{H,W}(1 \otimes w)] \otimes [x_i \rhd v] = (\mathrm{id} \otimes f_v) c_{H,W}(1 \otimes v) = c_{V,W}(f_v \otimes \mathrm{id})(1 \otimes w) = c_{V,W}(v \otimes w),
\end{aligned}
$$

where we used the formula for the braiding of $D(H)-\mathrm{Mod}$, then the identity for the H-linear map f_v, then the naturality of $c_{H,V}$ in the first argument and then the definition of f_v. This shows that F is braided.

3. We show that F and G form a braided equivalence between $\mathcal{Z}(H-\mathrm{Mod})$ and $D(H)-\mathrm{Mod}$. For each object $(V, c_{-,V})$ in $\mathcal{Z}(H-\mathrm{Mod})$, we have $GF(V, c_{-,V}) = (V, c'_{-,V})$ with

$$
\begin{aligned}
&c'_{M,V}(m \otimes v) = \Sigma_{i=1}^n [(\alpha^i \otimes 1) \rhd' v] \otimes [x_i \rhd_M m] = \Sigma_{i=1}^n [(\mathrm{id} \otimes \alpha^i) c_{H,V}(1 \otimes v)] \otimes [x_i \rhd_M m] \\
&= (\mathrm{id} \otimes f_m) c_{H,V}(1 \otimes v) = c_{M,V}(f_m \otimes \mathrm{id})(1 \otimes v) = c_{M,V}(m \otimes v)
\end{aligned}
$$

for each H-module $(M, \rhd_M)$. This shows that $GF(V, c_{-,V}) = (V, c_{-,V})$ for each object $(V, c_{-,V})$ in $\mathcal{Z}(H-\mathrm{Mod})$. As F and G are the identity on the morphisms, we have $FG = \mathrm{id}_{\mathcal{Z}(H-\mathrm{Mod})}$.

For each $D(H)$-module $(M, \rhd_M)$, we have $FG(M) = M$ with the $D(H)$-module structure

$$
\begin{aligned}
&(\alpha \otimes h) \rhd m = (\mathrm{id} \otimes \alpha) c^{D(H)}_{H,M}(1 \otimes ((1 \otimes h) \rhd_M m)) = \Sigma_{i=1}^n \alpha(x_i)\ (\alpha^i \otimes 1) \rhd_M ((1 \otimes h) \rhd_M m) \\
&= (\alpha \otimes 1) \rhd_M ((1 \otimes h) \rhd_M m) = (\alpha \otimes h) \rhd_M m
\end{aligned}
$$

and hence $FG(M, \rhd_M) = (M, \rhd_M)$. As F and G are the identity on the morphisms, this implies $FG = \mathrm{id}_{D(H)-\mathrm{Mod}}$. We thus have shown that F and G form a braided monoidal equivalence between $\mathcal{Z}(H-\mathrm{Mod})$ and $D(H)-\mathrm{Mod}$.

4. The claim for the categories $\mathcal{Z}(H-\mathrm{Mod}^{fd})$ and $D(H)-\mathrm{Mod}^{fd}$ follows directly, since the functors F and G send finite-dimensional H-modules to finite-dimensional H-modules. $\square$

6.4 Pivotal and ribbon Hopf algebras

Given a quasitriangular Hopf algebra (H, R), we can deduce from Theorem 6.3.2 that its representation category $H-\mathrm{Mod}$ and its full subcategory $H-\mathrm{Mod}^{fd}$ of finite-dimensional representations are braided. By Theorem 6.3.5 the antipode of H is invertible, and this implies with Corollary 5.2.7, 3. that $H-\mathrm{Mod}^{fd}$ is rigid.

We may thus ask under which additional assumptions on H the category $H-\mathrm{Mod}^{fd}$ of finite-dimensional H-modules is *pivotal* or *ribbon*. The first question does not have anything to do with quasitriangularity, except that quasitriangularity guarantees rigidity, which is required for pivotality. It turns out that $H-\mathrm{Mod}^{fd}$ is *pivotal* if and only if the square of the antipode is given by conjugation with a grouplike element.

Definition 6.4.10: A **pivotal Hopf algebra** is a Hopf algebra H over $\mathbb{F}$ together with an element $g \in \mathrm{Gr}(H)$, the **pivot**, such that $S^2(h) = g \cdot h \cdot g^{-1}$ for all $h \in H$.

Example 6.4.11:

1. Every involutive Hopf algebra H is pivotal with pivot $1 \in H$.
2. Every *triangular* Hopf algebra is pivotal with the Drinfeld element $u = S(R_{(2)})R_{(1)}$ as the pivot, since triangularity implies $R_{21}R = 1$ and u grouplike by Theorem 6.3.5, 4.

   Note that this does not hold for *quasitriangular* Hopf algebras. Although their antipode satisfies $S^2(h) = uhu^{-1}$ for all $h \in H$, the element $u \in H$ need not be grouplike.
3. In general, the pivot of a pivotal Hopf algebra is not unique. If $g \in \mathrm{Gr}(H)$ is a pivot for H and $v \in \mathrm{Gr}(H)$ a central grouplike element, then gv is another pivot for H.

Proposition 6.4.12:

1. If (H, g) is a pivotal Hopf algebra, then the representation category $H{-}\mathrm{Mod}^{fd}$ is pivotal with $\omega_V : V \to V^{**}$, $v \mapsto \mathrm{can}_V(g \rhd v)$ for all finite-dimensional H-modules (V, ρ_V).

2. If H is a finite-dimensional Hopf algebra and $H{-}\mathrm{Mod}^{fd}$ is pivotal with pivot $\omega : \mathrm{id}_{\mathcal{C}} \Rightarrow **$, then H is a pivotal Hopf algebra with pivot $g = \mathrm{can}_H^{-1} \circ \omega_H(1)$.

Proof:
1. Let (H, g) be a pivotal Hopf algebra. Then $S : H \to H$ is invertible with $S^{-1}(h) = g^{-1}S(h)g$ and hence $H{-}\mathrm{Mod}^{fd}$ is rigid by Corollary 5.2.7, 3.

Denote for each finite-dimensional H-module V by $\mathrm{can}_V : V \to V^{**}$ the canonical linear isomorphism with $\mathrm{can}_V(v)(\alpha) = \alpha(v)$ for all $v \in V$ and $\alpha \in V^*$. Define $\omega_V : V \to V^{**}$, $v \mapsto \mathrm{can}_V(g \rhd v)$. Then ω_V is invertible with inverse $\omega_V^{-1} : V^{**} \to V$, $x \mapsto g^{-1} \rhd \mathrm{can}_V^{-1}(x)$. The functor $** : H{-}\mathrm{Mod}^{fd} \to H{-}\mathrm{Mod}^{fd}$ assigns to an H-module $(V, \rhd)$ the H-module $(V^{**}, \rhd^{**})$ with $(h \rhd^{**} \mathrm{can}_V(v))(\alpha) = \alpha(S^2(h) \rhd v) = \mathrm{can}_V(S^2(h) \rhd v)(\alpha)$ for $v \in V$, $\alpha \in V^*$. This implies

$$h \rhd^{**} \omega_V(v) = h \rhd^{**} \mathrm{can}_V(g \rhd v) = \mathrm{can}_V((S^2(h)g) \rhd v) = \mathrm{can}_V((gh) \rhd v) = \omega_V(h \rhd v)$$

for all $h \in H$ and $v \in V$ and shows that the maps ω_V are H-linear. The naturality of ω follows from the naturality of can, which implies for all H-linear maps $f : V \to W$ and $v \in V$

$$f^{**} \circ \omega_V(v) = f^{**} \circ \mathrm{can}_V(g \rhd v) = \mathrm{can}_W \circ f \circ (g \rhd v) = \mathrm{can}_W(g \rhd f(v)) = \omega_W \circ f(v)$$

That ω is a *monoidal* natural isomorphism follows from the fact that g is grouplike

$$\omega_{V \otimes V}(v \otimes v') = \mathrm{can}_{V \otimes V}(g \rhd (v \otimes v')) = (\mathrm{can}_V \otimes \mathrm{can}_V)((g \rhd v) \otimes (g \rhd v')) = \omega_V(v) \otimes \omega_V(v')$$

This shows that $H{-}\mathrm{Mod}^{fd}$ is pivotal with pivot ω.

2. Let H be a finite-dimensional Hopf algebra such that $H{-}\mathrm{Mod}^{fd}$ is pivotal and consider the element $g = \mathrm{can}_H^{-1}(\omega_H(1)) \in H$. Then $g \neq 0$ since can_H is a linear isomorphism and ω_H is invertible, which implies $\omega_H(1) \neq 0$.

For all $h \in H$, the linear map $\phi_h : H \to H$, $k \mapsto kh$ is H-linear, and by the naturality of ω, one has $\omega_H \circ \phi_h = \phi_h^{**} \circ \omega_H$. Using the identity $h \rhd^{**} \mathrm{can}_H(k) = \mathrm{can}_H(S^2(h)k)$ from 1. and setting $k = g$, we then obtain

$$\begin{aligned}&\mathrm{can}_H(S^2(h)g) = h \rhd^{**} \mathrm{can}_H(g) = h \rhd^{**} \omega_H(1) = \omega_H(h \rhd 1)\\ &= \omega_H(h) = \omega_H \circ \phi_h(1) = \phi_h^{**} \circ \omega_H(1) = \phi_h^{**} \circ \mathrm{can}_H(g) = \mathrm{can}_H \circ \phi_h(g) = \mathrm{can}_H(gh),\end{aligned}$$

where we used that $\omega_H : H \to H^{**}$ is H-linear in the first line and then the condition $\omega_H \circ \phi_h = \phi_h^{**} \circ \omega_H$ together with the definition of g. As $\mathrm{can}_H : H \to H^{**}$ is a linear isomorphism, this implies $S^2(h)g = gh$ for all $h \in H$.

To show that g is grouplike, we note that $\Delta : H \to H \otimes H$ is a module homomorphism, which implies $\omega_{H\otimes H} \circ \Delta = \Delta^{**} \circ \omega_H$, and that ω is *monoidal*, which implies $\omega_{H\otimes H} = \omega_H \otimes \omega_H$. Identifying the vector spaces $(H\otimes H)^{**} \cong H^{**} \otimes H^{**}$, we then obtain

$$\begin{aligned}&\mathrm{can}_H(g)\otimes\mathrm{can}_H(g) = \omega_H(1)\otimes\omega_H(1) = \omega_{H\otimes H}(1\otimes 1) = \omega_{H\otimes H} \circ \Delta(1)\\ &= \Delta^{**} \circ \omega_H(1) = \Delta^{**} \circ \mathrm{can}_H(g) = \mathrm{can}_{H\otimes H} \circ \Delta(g) = (\mathrm{can}_H \otimes \mathrm{can}_H) \circ \Delta(g)\end{aligned}$$

As can_H is an isomorphism, this shows that $\Delta(g) = g\otimes g$ and (H, g) is a pivotal Hopf algebra. □

We have thus clarified under which additional assumptions on a Hopf algebra H its representation category $H-\mathrm{Mod}^{fd}$ is braided or pivotal. The former corresponds to quasitriangularity of H and he latter to pivotality. We now assume that H is equipped with both, a quasitriangular and a pivotal structure, and determine under which additional conditions the category $H-\mathrm{Mod}^{fd}$ is ribbon.

Definition 6.4.13: A **ribbon Hopf algebra** is a quasitriangular Hopf algebra (H, R) together with an invertible central element $\nu \in H$, the **ribbon element**, such that

$$uS(u) = \nu^2 \qquad\qquad \Delta(\nu) = (\nu\otimes\nu) \cdot (R_{21}R)^{-1}.$$

Remark 6.4.14:

1. A ribbon element is unique only up to right multiplication with a central grouplike element $g \in H$ satisfying $g^2 = 1$.
2. One can show that any ribbon element satisfies $\epsilon(\nu) = 1$ and $S(\nu) = \nu$ (Exercise).

Example 6.4.15:

1. If H is quasitriangular and involutive, then H is ribbon with ribbon element u.

   The identity $S^2 = \mathrm{id}_H$ implies $u = S(R_{(2)})R_{(1)} = R_{(2)}S(R_{(1)})$, $u^{-1} = R_{(2)}R_{(1)}$ and $S(u) = S(R_{(1)})R_{(2)} = R_{(1)}S(R_{(2)})$. It follows that

   $$S(u)u^{-1} = R_{(1)}S(R_{(2)})R'_{(2)}R'_{(1)} = 1 = R'_{(2)}R'_{(1)}S(R_{(1)})R_{(2)} = u^{-1}S(u)$$

   This shows that $u = S(u)$ and $uS(u) = u^2$. we also have $\Delta(u) = (u\otimes u)(R_{21}R)^{-1}$ by Theorem 6.3.5, 3. and $uhu^{-1} = S^2(h) = h$ for all $h \in H$ by Theorem 6.3.5, 5., which shows that u is central.

2. The Drinfeld double of any finite-dimensional semisimple Hopf algebra H over a field of characteristic zero is a ribbon Hopf algebra with ribbon element $\nu = u = S(R_{(2)})R_{(1)}$.

   By corollary 6.3.8, the Drinfeld double $D(H)$ of a finite-dimensional semisimple Hopf algebra H is semisimple as well, and its antipode satisfies $S^2 = \mathrm{id}_H$. Hence, $D(H)$ is a ribbon Hopf algebra by 1.

3. Let $q \in \mathbb{F}$ be a primitive rth root of unity with $r > 1$ odd. Then the Hopf algebra $U_q^r(\mathfrak{sl}_2)$ from Proposition 5.3.12 is ribbon with ribbon element $\nu = K^{-1}u = uK^{-1}$ where $u = S(R_{(2)})R_{(1)}$ for the R-matrix (75).

   That ν is central follows from the identity $S^2(h) = KhK^{-1} = uhu^{-1}$ for all $h \in U_q^r(\mathfrak{sl}_2)$. The identity $\Delta(\nu) = (\nu\otimes\nu)(R_{21}R)^{-1}$ follows from the identity $\Delta(K^{\pm 1}) = K^{\pm 1}\otimes K^{\pm 1}$ and the corresponding identity for u in Theorem 6.3.5.

Proposition 6.4.16:

1. If (H, R, ν) is a ribbon Hopf algebra, then $H-\mathrm{Mod}^{fd}$ is a ribbon category.
2. If (H, R, g) is a finite-dimensional quasitriangular pivotal Hopf algebra and $\nu \in H$ such that the linear maps $\theta_V : V \to V$, $v \mapsto \nu^{-1} \rhd v$ define a twist on $H-\mathrm{Mod}^{fd}$, then $\nu = g^{-1}u$ and ν is a ribbon element.

Proof:
1. If $\nu \in H$ is invertible and central with $uS(u) = \nu^2$ and $\Delta(\nu) = (\nu\otimes\nu)(R_{21}R)^{-1}$, then its inverse satisfies $\Delta(\nu^{-1}) = (\nu^{-1}\otimes\nu^{-1})(R_{21}R)$. It follows that the element $g := u\nu^{-1}$ is grouplike, since we obtain with the coproduct of u from Theorem 6.3.5 and the fact that ν is central

$$\Delta(g) = \Delta(u) \cdot \Delta(\nu^{-1}) = (u\otimes u)(R_{21}R)^{-1} \cdot (\nu^{-1}\otimes\nu^{-1})(R_{21}R) = u\nu^{-1}\otimes u\nu^{-1} = g\otimes g.$$

Moreover, since ν is central, we have $ghg^{-1} = (u\nu^{-1})h(\nu u^{-1}) = uhu^{-1} = S^2(h)$ for all $h \in H$. This shows that $g = u\nu^{-1}$ is a pivot for H and hence $H-\mathrm{Mod}^{fd}$ a braided pivotal category by Theorem 6.3.2 and Proposition 6.4.12.

It remains to check that the twist in $H-\mathrm{Mod}^{fd}$ satisfies the condition in Lemma 3.3.2, 4. Let V be a finite-dimensional H-module with basis $(x_1, ..., x_n)$ and dual basis $(\alpha^1, ..., \alpha^n)$ of V^*. With formulas (17) and (23) for the evaluation and coevaluation maps, formula (72) for the braiding and the formulas from Theorem 6.3.5 we compute

$$\begin{aligned}\theta_V(v) &= \Sigma_{i=1}^n \alpha^i((gR_{(1)}) \rhd v)\, R_{(2)} \rhd x_i = (R_{(2)}gR_{(1)}) \rhd v = (R_{(2)}S^2(R_{(1)})g) \rhd v = (u^{-1}g) \rhd v \\ &= \nu^{-1} \rhd v \\ \theta'_V(v) &= \Sigma_{i=1}^n \alpha^i(R_{(2)}) \rhd v)(R_{(1)}g^{-1}) \rhd x_i = (R_{(1)}g^{-1}R_{(2)}) \rhd v = (g^{-1}S^2(R_{(1)})R_{(2)}) \rhd v \\ &= (\nu u^{-1}S(u^{-1})) \rhd v = (\nu(S(u)u)^{-1}) \rhd v = (\nu(uS(u))^{-1}) \rhd v = (\nu\nu^{-2}) \rhd v = \nu^{-1} \rhd v.\end{aligned}$$

This shows that the condition from Lemma 3.3.2 is satisfied and $H-\mathrm{Mod}^{fd}$ is a ribbon category.

2. Let H be a finite-dimensional quasitriangular pivotal Hopf algebra with pivot $g \in H$ and $\nu \in H$ such that $\theta_V : V \to V$, $v \mapsto \nu^{-1} \rhd v$ define a twist for $H-\mathrm{Mod}^{fd}$. Then by Lemma 3.3.2, 4. and the computation above we have

$$u^{-1}g = \theta_H(1) = \nu^{-1} = \theta'_H(1) = g^{-1}S(u)^{-1}$$

and this implies $\nu^{-2} = (u^{-1}g)(g^{-1}S(u)^{-1}) = (S(u)u)^{-1} = (uS(u))^{-1}$ and $\nu^2 = uS(u)$. By Lemma 3.3.2, 2, we also have

$$\Delta(\nu^{-1}) = \Delta(\nu)^{-1} = \theta_{H\otimes H}(1) = c_{H,H} \circ c_{H,H} \circ (\theta_H\otimes\theta_H)(1) = (R_{21}R)(\nu^{-1}\otimes\nu^{-1}),$$

which implies $\Delta(\nu) = (\nu\otimes\nu)(R_{21}R)^{-1}$. As $\theta_H : H \to H$ is a H-module homomorphism, we have $\nu^{-1}h = \theta_H(h) = \theta_H(h\rhd 1) = h\rhd\theta_H(1) = h\nu^{-1}$ for all $h \in H$ and hence ν is central in H. $\square$

7 Application: Kitaev models

7.1 Kitaev lattice models

Kitaev models were first introduced in 2003 by Alexei Kitaev [Ki] to obtain a realistic model for a quantum computer that is protected against errors by topological effects. The original model, the *toric code*, was based on the group algebra $\mathbb{C}[\mathbb{Z}/2\mathbb{Z}]$. The model was then generalised to the group algebra $\mathbb{C}[G]$ for a finite group G in [BMD] and to finite-dimensional semisimple Hopf algebras in [BMCA]. These models became very prominent and are a topic of current research in condensed matter physics and topological quantum computing. They are also interesting from the mathematical perspective since they are related to topological quantum field theories.

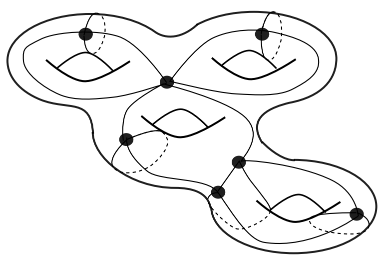
The ingredients of the Kitaev model are

(i) a finite-dimensional semisimple Hopf algebra H over a field $\mathbb{F}$ of characteristic zero,
(ii) an oriented surface Σ: a connected, compact oriented 2d topological manifold Σ,
(iii) a finite directed graph Γ embedded into Σ such that $\Sigma \setminus \Gamma$ is a disjoint union of discs.

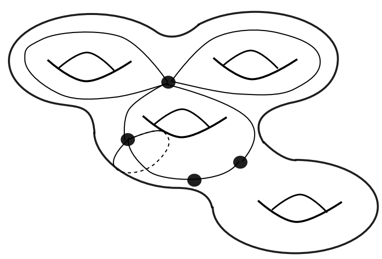
Recall that compact oriented surfaces are classified up to homeomorphisms by their genus g, the number of handles, and their fundamental group is presented as

$$\pi_1(\Sigma) = \langle a_1, b_1, ..., a_g, b_g \mid [b_g, a_g] \cdots [b_1, a_1] = 1 \rangle,$$

where a_i, b_i are the a and b-cycles of the ith handle. The condition (iii) that $\Sigma \setminus \Gamma$ is a disjoint union of discs ensures that the graph Γ is sufficiently refined to resolve the topology of the surface Σ: one can realise a set of generators of the fundamental group $\pi_1(\Sigma)$ as paths in Γ.



an embedded graph satisfying (iii)



an embedded graph that does not satisfy (iii).

We denote by E and V the sets of edges and vertices of Γ and use the same letters for their cardinalities. We also require the notion of a *face*. A **face** of Γ is defined as a connected component of $\Sigma \setminus \Gamma$. It is represented by closed paths in Γ that start and end at a vertex $v \in V$, turn maximally right at each vertex and traverse each edge at most once in each direction. More precisely, it is an equivalence class of such paths under *cyclic permutations*, which change the starting vertex of the path. The set of faces of Γ and its cardinality are denoted F.

By placing a marking at a vertex $v \in V$ between two incident edges at v, one obtains an ordering of the edges at v by counting them counterclockwise from the marking. Similarly, placing a marking at one of the vertices in a face yields an ordering of the edges in the face by counting them counterclockwise from the marking. A vertex together with a marking is called

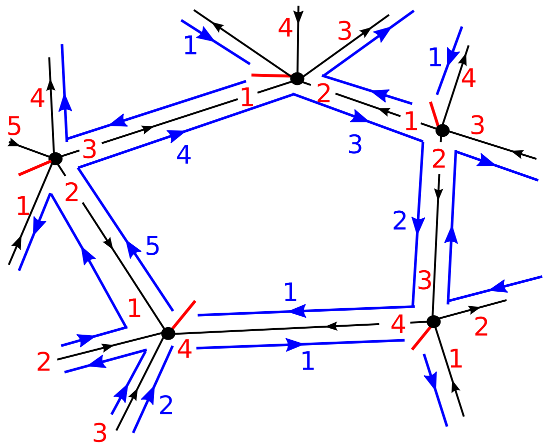


Figure 1: Embedded graph Γ with markings (red) and the induced ordering of edges at the vertices and faces of Γ.

a **marked vertex** and a face with a marking a **marked face**. In the following, we assume that each face and each vertex is equipped with a marking, as in Figure 1. A pair (v, f) of a marked vertex v and a marked face f that share a marking is called a **site**.

With these definitions, we can define the Kitaev model associated with the triple (Σ, Γ, H) that satisfies conditions (i) to (iii).

Definition 7.1.1: The **Kitaev model** for (Σ, Γ, H) consists of the following data:

1. The **extended space**: the vector space $H^{\otimes E}$.
2. The **edge operators:**

   The edge operators for a triple (e, h, α) of an edge $e \in E$ and elements $h \in H$, $\alpha \in H^*$ are the linear maps $L^h_{e\pm}, T^\alpha_{e\pm} : H^{\otimes E} \to H^{\otimes E}$ that act as the identity on all copies of H in $H^{\otimes E}$ that are not associated with e and on the copy for e according to

$$
\begin{aligned}
&L^h_{e+} : ...\otimes k^e \otimes ... \mapsto ...\otimes hk^e \otimes ... && L^h_{e-} : ...\otimes k^e \otimes ... \mapsto ...\otimes k^e S(h) \otimes ... \\
&T^\alpha_{e+} : ...\otimes k^e \otimes ... \mapsto \Sigma_{(k^e)} \alpha(k^e_{(2)})\, ...\otimes k^e_{(1)} \otimes ... && T^\alpha_{e-} : ...\otimes k^e \otimes ... \mapsto \Sigma_{(k^e)} \alpha(S(k^e_{(1)}))\, ...\otimes k^e_{(2)} \otimes ...
\end{aligned}
\tag{80}
$$

3. The **vertex and face operators:**

   - The **vertex operator** for a pair (v, h) of a vertex $v \in V$ and $h \in H$ is the linear map

$$A^h_v = \Sigma_{(h)}\, L^{h_{(1)}}_{e_1,\epsilon_1} \circ L^{h_{(2)}}_{e_2,\epsilon_2} \circ ... \circ L^{h_{(n)}}_{e_n,\epsilon_n} : H^{\otimes E} \to H^{\otimes E}$$

   where $e_1, ..., e_n$ are the incident edge ends at v, numbered counterclockwise from the marking at v, $\epsilon_i = +$ if e_i is incoming at v and and $\epsilon_i = -$ if e_i is outgoing from v.

   - The **face operator** for a pair (f, α) of a face $f \in F$ and $\alpha \in H^*$ is the linear map

$$B^\alpha_f = \Sigma_{(\alpha)}\, T^{\alpha_{(1)}}_{e_1,\epsilon_1} \circ T^{\alpha_{(2)}}_{e_2,\epsilon_2} \circ ... \circ T^{\alpha_{(n)}}_{e_n,\epsilon_n} : H^{\otimes E} \to H^{\otimes E},$$

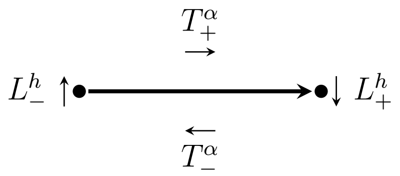
where $e_1, ..., e_n$ are the edges traversed by f, numbered counterclockwise from the marking, $\epsilon_i = +$ if e_i is traversed in its orientation and $\epsilon_i = -$ if e_i is traversed against its orientation.

4. The **protected space** or **ground state**:

   The protected space of a Kitaev model is the linear subspace

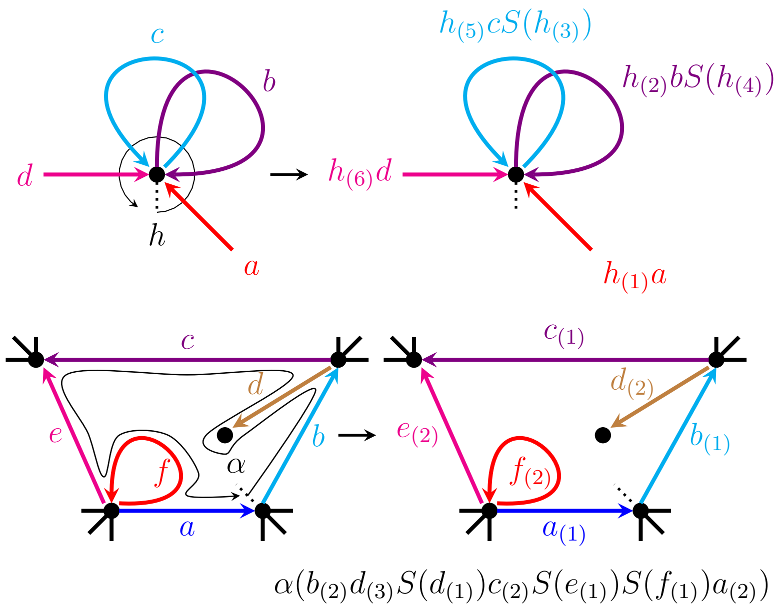
$$H_{inv}^{\otimes E} = \{x \in H^{\otimes E} \,|\, A_v^h x = \epsilon(h)\, x,\; B_f^\alpha(x) = \epsilon(\alpha)\, x \quad \forall h \in H, \alpha \in H^*, v \in V, f \in F\}.$$

We can visualise the definition of vertex, edge and face operators by assigning the edge operators to the two ends and the two sides of an edge as follows



The vertex A_v^h and face operators B_f^α are obtained by appplying the comultiplication to h and α to create as many copies as there are edge ends in the path around v and edge sides in the face, assigning these copies to the edge ends and edge sides in the order in which they are traversed by these paths and then applying $L_\pm$ and $T_\pm$ depending on the relative orientation. If we visualise the factors of H in the extended space by assigning Hopf algebra elements to the edges of a graph, the vertex and face operator are then given as follows.

Example 7.1.2:



The main reason why Kitaev models are interesting from the mathematics perspective is that their protected space $H_{inv}^{\otimes E}$ does not depend on the choice of the graph Γ or its embedding into Σ but only on the homeomorphism class of the surface Σ. It is a topological invariant of Σ, which was related to Turaev-Viro topological quantum field theories in [BK].

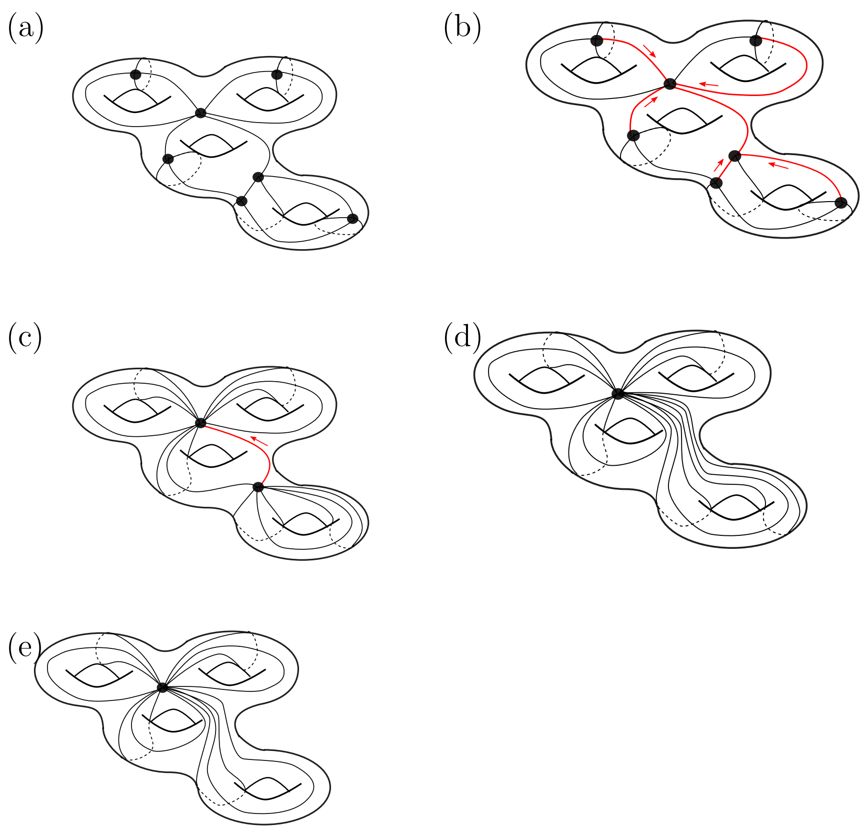


Figure 2: Contracting the edges of a maximal tree in a graph: (b)→(c), (c)→(d) and removing loops: (d)→(e).

Theorem 7.1.3 ([BMCA])**:** The protected space of a Kitaev model is a **topological invariant**: Its dimension depends only on the homeomorphism class of the surface Σ an not on the embedded graph Γ.

Sketch of Proof: The proof is performed by selecting a maximal tree $T \subset \Gamma$ as in Figure 2 (b). This is a subgraph $T \subset \Gamma$ with no non-trivial closed paths or, equivalently, with trivial fundamental group that contains each vertex of Γ. One then contracts all edges in the tree towards a chosen vertex as in Figure 2 (b),(c),(d). One can show that these edge contractions induce isomorphisms between the protected spaces of the associated Kitaev models. By contracting all edges in the tree T one obtains a graph Γ' as in Figure 2 (d) whose ground state is isomorphic to the one of Γ and which contains only a single vertex.

By removing those loops of Γ' that can be removed without violating the condition that $\Sigma \setminus \Gamma$ is a disjoint union of discs as in Figure 2 (d),(e) one obtains another graph Γ'' with a single vertex, a single face and $2g$ edges, where g is the genus of Σ. As removing loops induces isomorphisms between the protected spaces, its protected space is isomorphic to the one of Γ,

After performing some further graph transformations which again induce isomorphisms between the protected spaces of the associated Kitaev models, one obtains a standard graph Γ''' which depends only on the genus of the surface Σ and such that the protected space of the associated Kitaev model is isomorphic to the one for Γ. This shows that the protected space of the Kitaev model for (Σ, Γ, H) depends only on H and the genus of Σ. □

7.2 Algebraic structures in Kitaev lattice models

We now investigate the algebraic structure of the Kitaev model and and show that it is an application of the concepts introduced in the last chapter. For this, we fix a finite-dimensional semisimple Hopf algebra H over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) = 0$ and a directed graph Γ with vertex set V, edge set E and face set F that is embedded into an oriented surface Σ such that $\Sigma \setminus \Gamma$ is a disjoint union of discs.

We consider the Kitaev lattice model associated with (Σ, Γ, H) from Definition 7.1.1 and start by noting that the edge operators $L^h_{e\pm}$ and $T^\alpha_{e\pm}$ associated to each edge e of Γ are related to two standard right H-Hopf module structures on H.

Lemma 7.2.1: Let H be a finite-dimensional semisimple Hopf algebra over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) = 0$. Then:

1. The H-right module and H-right comodule structures

$$\begin{array}{ll} \lhd : H\otimes H \to H,\ k \lhd h = kh & \delta = \Delta : H \to H\otimes H,\ h \mapsto \Sigma_{(h)} h_{(1)} \otimes h_{(2)} \\ \lhd' : H\otimes H \to H,\ k \lhd h = S(h)k & \delta' = (\mathrm{id}\otimes S)\Delta^{op} : H \to H\otimes H,\ h \mapsto \Sigma_{(h)} h_{(2)} \otimes S(h_{(1)}) \end{array}$$

   define H-right Hopf module structures $(H, \lhd, \delta)$ and $(H, \lhd', \delta')$ on H.
2. The edge operators from (80) are given by

$$L^h_+ k = k \lhd' S(h) \qquad L^h_- k = k \lhd S(h) \qquad T^\alpha_+ k = (\mathrm{id}\otimes\alpha)\delta(k) \qquad T^\alpha_- k = (\mathrm{id}\otimes\alpha)\delta'(k),$$

   where we omit the copies of $H^{\otimes E}$ on which they act as the identity.

Proof:
1. That the H-right actions and H-right coactions are indeed right actions and coactions follows by a direct computation. The same holds for the claim that they define Hopf module structures on H. Using the properties of the antipode and the identity $S^2 = \mathrm{id}_H$, which follows from the semisimplicity by the Larson-Radford theorem, we obtain

$$\begin{aligned} &\delta(k \lhd h) = \delta(kh) = \Sigma_{(h),(k)} k_{(1)} h_{(1)} \otimes k_{(2)} h_{(2)} = \Sigma_{(h),(k)} (k_{(1)} \lhd h_{(1)}) \otimes k_{(2)} h_{(2)} = \delta(k) \lhd h \\ &\delta'(k \lhd' h) = \delta'(S(h)k) = \Sigma_{(h),(k)} S(h_{(1)}) k_{(2)} \otimes S(k_{(1)}) h_{(2)} = \delta'(k) \lhd h. \end{aligned}$$

2. The second claim follows directly from the definition of the H-module and comodule structures and the definition of the edge operators in (80). □

For a better understanding of the edge operators, we need to describe their interaction in terms of commutation relations and to identify the subalgebra of $\mathrm{End}_{\mathbb{F}}(H^{\otimes E})$ that is generated by these operators. The latter can be achieved most easily via the Haar integrals of the Hopf algebra H and its dual H^*.

Proposition 7.2.2 ([BMCA]):

1. The edge operators associated with distinct edges commute.
2. The edge operators for an edge e satisfy the commutation relations

$$\begin{array}{ll} L^h_{e\pm} L^k_{e\pm} = L^{hk}_{e\pm} & L^h_{e\pm} L^k_{e\mp} = L^k_{e\mp} L^h_{e\pm} \\ T^\alpha_{e\pm} T^\beta_{e\pm} = T^{\alpha\beta}_{e\pm} & T^\alpha_{e\pm} T^\beta_{e\mp} = T^\beta_{e\mp} T^\alpha_{e\pm} \\ T^\alpha_{e\pm} L^h_{e\pm} = \Sigma_{(h),(\alpha)} \alpha_{(1)}(h_{(2)})\, L^{h_{(1)}}_{e\pm} T^{\alpha_{(2)}}_{e\pm} & T^\alpha_{e\pm} L^h_{e\mp} = \Sigma_{(h),(\alpha)} \alpha_{(2)}(S(h_{(1)}))\, L^{h_{(2)}}_{e\mp} T^{\alpha_{(1)}}_{e\pm}. \end{array} \tag{81}$$

3. Let $S_e : H^{\otimes E} \to H^{\otimes E}$ be the application of the antipode to the copy of H for e. Then:
$$L^h_{e\pm} \circ S_e = S_e \circ L^h_{e\mp} \qquad\qquad T^\alpha_{e\pm} \circ S_e = S_e \circ T^\alpha_{e\mp}.$$
4. The edge operators L^h_{e+}, T^α_{e+} for edges e generate the algebra $\mathrm{End}_{\mathbb{F}}(H^{\otimes E}) \cong \mathrm{End}_{\mathbb{F}}(H)^{\otimes E}$.

Proof:
1. The first claim follows from the fact that the edge operators $L^h_{e\pm}$, $T^\alpha_{e\pm}$ act trivially on all copies of H that are associated with edges $f \neq e$.

2. and 3. The second and third claim follow by a direct computation using the formulas (80) for the edge operators $L^h_{e\pm}$, $T^\alpha_{e\pm}$ and the identity $S^2 = \mathrm{id}_H$ which holds because H is semisimple.

4. To show that the edge operators L^h_{e+}, T^α_{e+} generate the algebra $\mathrm{End}_{\mathbb{F}}(H^{\otimes E}) \cong \mathrm{End}_{\mathbb{F}}(H)^{\otimes E}$, it is sufficient to show that for each edge e, the associated edge operators L^h_{e+}, T^α_{e+} generate the algebra $\mathrm{End}_{\mathbb{F}}(H)$. For this, we consider an ordered basis $(x_1, ..., x_n)$ of H with the dual basis $(\alpha^1, ..., \alpha^n)$ of H^*. Then the linear maps $\phi_{ij} : H \to H$, $x_k \mapsto \delta_{ik} x_j$ form a basis of $\mathrm{End}_{\mathbb{F}}(H)$.

We show that they can be expressed as a product of the edge operators L^h_{e+}, T^α_{e+}. For this, let $\ell \in H$ and $\lambda \in H^*$ be two-sided integrals of H and H^* with $\lambda(\ell) = 1$. Thewy exist by Lemma 6.2.15, because H is semisimple by assumption, the semisimplicity of H implies the semisimplicity of H^* by the Larson-Radford Theorem 6.2.18 and finite-dimensional semisimple Hopf algebras are unimodular by Corollary 6.2.14. Then we have
$$\begin{aligned} L^{x_j}_{e+} T^\lambda_{e+} L^\ell_{e+} T^{\alpha^i}_{e+}(x_k) &= \Sigma_{(k)} \alpha^i(x_{k(2)})\, L^{x_j}_{e+} T^\lambda_{e+}(\ell x_{k(1)}) = \Sigma_{(k)} \alpha^i(x_{k(2)}) \epsilon(x_{k(1)})\, L^{x_j}_{e+} T^\lambda_{e+}(\ell) \\ &= \alpha^i(x_k) L^{x_j}_{e+} T^\lambda_{e+}(\ell) = \Sigma_{(\ell)} \delta_{ik} \lambda(\ell_{(2)}) x_j \ell_{(1)} = \delta_{ik} \lambda(\ell) x_j = \delta_{ik} x_j = \phi_{ij}(x_k) \end{aligned}$$
for all $i, j, k \in \{1, .., n\}$ and this shows that $L^{x_j}_{e+} T^\lambda_{e+} L^\ell_{e+} T^{\alpha^i}_{e+} = \phi_{ij}$. □

We now consider the vertex and face operators in the Kitaev model. From the definition of the model it is apparent that they play the role of a symmetry algebra that acts on the extended space and defines the protected space by the condition that their action on an element of $H^{\otimes E}_{inv}$ is trivial, that is, given by the counits of H and H^*. It turns out that these are the invariants of associated module and comodule structures on $H^{\otimes E}$.

Moreover, for every pair (v, f) of a vertex and a face that share a marking, it is natural to combine the vertex and face operators A^h_v and B^α_f and to investigate their commutation relations. It turns out that for every such pair the vertex and face operators form a representation of the Drinfeld double $D(H)$. They have the interpretation of excited states in Kitaev models.

Proposition 7.2.3 ([BMCA]):

1. For every vertex $v \in V$ and face $f \in F$ the associated vertex and face operators define representations
$$\rho_v : H \to \mathrm{End}_{\mathbb{F}}(H^{\otimes E}), \quad h \mapsto A^h_v \qquad \rho_f : H^* \to \mathrm{End}_{\mathbb{F}}(H^{\otimes E}), \quad \alpha \mapsto B^\alpha_f.$$
This induces representations
$$\rho_V : H^{\otimes V} \to \mathrm{End}_{\mathbb{F}}(H^{\otimes E}) \qquad \rho_F : H^{*\otimes F} \to \mathrm{End}_{\mathbb{F}}(H^{\otimes E}).$$

2. If a vertex v and face f share a marking, then the associated vertex and face operator define a representation of $D(H)$ on $H^{\otimes E}$

$$\rho_{(v,f)}: D(H) \to \mathrm{End}_{\mathbb{F}}(H^{\otimes E}), \quad \alpha \otimes h \mapsto B_f^\alpha \circ A_v^h.$$

3. Let $\lambda \in H^*$ and $\ell \in H$ be normalised Haar integrals. Then all vertex and face operators A_v^ℓ and B_f^λ are independent of the choice of markings and commute.
4. A projector on $H_{inv}^{\otimes E}$ is given by

$$P = \Pi_{v\in V}\Pi_{f\in F}\, B_f^\lambda A_v^\ell : H^{\otimes E} \to H^{\otimes E}$$

Proof:
1. To prove the first claim, note that for each edge e the maps $\rho_{e\pm}: H \to \mathrm{End}_{\mathbb{F}}(H^{\otimes E})$, $h \mapsto L^h_{e\pm}$ are representations of H and the maps $\rho'_{e\pm}: H^* \to \mathrm{End}_{\mathbb{F}}(H^{\otimes E})$, $\alpha \mapsto T^\alpha_{e\pm}$ are representations of H^* by the first and third relation in (81). Also by (81) the operator $L^h_{e\pm}$ commute with $L^k_{e\mp}$ and with $L^k_{e'\pm}$, $L^k_{e'\mp}$ for all $e' \neq e \in E$ and likewise for the operators $T^\alpha_{f\pm}$. As the vertex and face operators are composites of these commuting operators constructed via the coproducts of H and H^* the first claim follows.

As a direct consequence of the commutation relations in (81) we have $A^h_v \circ A^k_{v'} = A^k_{v'} \circ A^h_v$ and $B^\alpha_f \circ B^\beta_{f'} = B^\beta_{f'} \circ B^\alpha_f$ for all $v \neq v' \in V$ and $f \neq f' \in F$. This proves the second claim.

2. For simplicity, we suppose that there are no loops at v and that the edges at v are only traversed once by f. The proof for the general case without this assumptions is similar, but more complicated and requires case distinctions.

Suppose that $e_1, ..., e_n$ are the edges at v, numbered counterclockwise from the marking as in Figure 1. By Proposition 7.2.2, 3. applying the antipode to the copy of H for an edge e exchanges the operators L^h_{e+} and L^h_{e-} which corresponds to a reversal of the edge orientation. We may thus assume that all edges $e_1, ..., e_n$ are incoming. Then the vertex operator for v is

$$A^h_v = L^{h_{(1)}}_{e_1+} \circ X(h_{(2)}) \circ L^{h_{(3)}}_{e_n+} \tag{82}$$

with a linear map $X: H \to \mathrm{End}_{\mathbb{F}}(H^{\otimes E})$, $h \mapsto X(h)$ such that $X(h)$ commutes with $L^k_{e_1\pm}$ and $L^k_{e_n\pm}$ for all $k \in H$ and with $T^\alpha_{e_1\pm}$ and $T^\alpha_{e_n\pm}$ for all $\alpha \in H^*$. This follows because the vertex operator A^v_h contains no contributions from edge operators of e_1 and e_n apart from the ones at the beginning and the end.

As the face f shares a marking with v and turns maximally right at each vertex, the edge e_n is the first edge in f and traversed with its orientation. The edge e_1 is the last edge in f and traversed against its orientation, as shown in Figure 1. The face operator for f has the form

$$B^\alpha_f = \Sigma_{(\alpha)}\, T^{\alpha_{(1)}}_{e_n+} \circ Y(\alpha_{(2)}) \circ T^{\alpha_{(3)}}_{e_1-} \tag{83}$$

with a linear map $Y: H^* \to \mathrm{End}_{\mathbb{F}}(H^{\otimes E})$, $\alpha \mapsto Y(\alpha)$. Because e_1 and e_n are only traversed once by f, the face operator contains no further contributions from edge operators for e_1 and e_n, and this implies that the linear endomorphism $Y(\alpha)$ commutes with $T^\beta_{e_1\pm}$, $T^\beta_{e_n\pm}$ and with $L^h_{e_1\pm}$, $L^h_{e_n\pm}$ for all $\beta \in H^*$ and $h \in H$.

If f does not traverse any edges incident at v except e_1, e_n then this also implies that $Y(\alpha)$ commutes with $X(h)$ for all $h \in H$.

If f traverses any edge e_i with $i \in \{2, .., n-1\}$ in the direction of its orientation, it also traverses the edge e_{i+1} against its orientation and $i < n-1$, since f turns maximally right at each vertex and traverses each edge only once. Conversely, if f traverses the edge e_{i+1} against its orientation, then it also traverses e_i with its orientation. This is pictured in Figure 1. The contribution for the edges e_i and e_{i+1} to the face operator is then of the form $\Sigma_{(\alpha)}\, T^{\alpha_{(1)}}_{e_{i+1}-} \circ T^{\alpha_{(2)}}_{e_i+}$ with $i \in \{2, ..., n-2\}$. This commutes with $L^h_{e_k+}$ for all $k \notin \{i, i+1\}$. To show that $L(\alpha)$ commutes with $X(h)$ for all $h \in H$, we use the commutation relations (81) of the edge operators and the identity $S^2 = \mathrm{id}_H$, which follows with the semisimplicity of H from Theorem 6.2.18. This implies

$$
\begin{aligned}
&\big(\Sigma_{(\alpha)}\, T^{\alpha_{(1)}}_{e_{i+1}-} T^{\alpha_{(2)}}_{e_i+}\big)\, (\Sigma_{(h)}\, L^{h_{(1)}}_{e_i+} L^{h_{(2)}}_{e_{i+1}+})\\
&\overset{(81)}{=} \Sigma_{(\alpha)(h)}\, \alpha_{(2)(1)}(h_{(1)(2)})\alpha_{(1)(2)}(S(h_{(2)(1)}))\, L^{h_{(1)(1)}}_{e_i+} L^{h_{(2)(2)}}_{e_{i+1}+} T^{\alpha_{(1)(1)}}_{e_{i+1}-} T^{\alpha_{(2)(2)}}_{e_i+}\\
&= \Sigma_{(\alpha)(h)}\, \alpha_{(3)}(h_{(2)})\alpha_{(2)}(S(h_{(3)}))\, L^{h_{(1)}}_{e_i+} L^{h_{(4)}}_{e_{i+1}+} T^{\alpha_{(1)}}_{e_{i+1}-} T^{\alpha_{(4)}}_{e_i+}\\
&= \Sigma_{(\alpha)(h)}\, \alpha_{(2)}(S(h_{(3)})h_{(2)})\, L^{h_{(1)}}_{e_i+} L^{h_{(4)}}_{e_{i+1}+} T^{\alpha_{(1)}}_{e_{i+1}-} T^{\alpha_{(3)}}_{e_i+}\\
&\overset{S^2=\mathrm{id}}{=} \Sigma_{(\alpha)(h)}\, \epsilon(\alpha_{(2)})\epsilon(h_{(2)})\, L^{h_{(1)}}_{e_i+} L^{h_{(3)}}_{e_{i+1}+} T^{\alpha_{(1)}}_{e_{i+1}-} T^{\alpha_{(3)}}_{e_i+} = (\Sigma_{(h)}\, L^{h_{(1)}}_{e_i+} L^{h_{(2)}}_{e_{i+1}+})(\Sigma_{(\alpha)}\, T^{\alpha_{(1)}}_{e_{i+1}-} T^{\alpha_{(2)}}_{e_i+}),
\end{aligned}
$$

and hence the elements $Y(\alpha)$ and $X(h)$ commute for all $h \in H$, $\alpha \in H^*$. With this, we compute

$$
\begin{aligned}
&\rho_{(v,f)}(\alpha\otimes h)\rho_{(v,f)}(\beta\otimes k) = B^\alpha_f L^h_v B^\beta_f A^k_v\\
&= \Sigma_{\substack{(h)(k)\\(\alpha)(\beta)}}\, [T^{\alpha_{(1)}}_{e_n+} Y(\alpha_{(2)}) T^{\alpha_{(3)}}_{e_1-}] \cdot [L^{h_{(1)}}_{e_1+} X(h_{(2)}) L^{h_{(3)}}_{e_n+}] \cdot [T^{\beta_{(1)}}_{e_n+} Y(\beta_{(2)}) T^{\beta_{(3)}}_{e_1-}] \cdot [L^{k_{(1)}}_{e_1+} X(k_{(2)}) L^{k_{(3)}}_{e_n+}]\\
&= \Sigma_{\substack{(h)(h)\\(\alpha)(\beta)}}\, \beta_{(1)(1)}(S(h_{(3)(2)}))\beta_{(3)(2)}(h_{(1)(1)})\\
&\qquad T^{\alpha_{(1)}}_{e_n+} T^{\beta_{(1)(2)}}_{e_n+} Y(\alpha_{(2)}) Y(\beta_{(2)}) T^{\alpha_{(3)}}_{e_1-} T^{\beta_{(3)(1)}}_{e_1-} L^{h_{(1)(1)}}_{e_1+} L^{k_{(1)}}_{e_1+} X(h_{(2)}) X(k_{(2)}) L^{h_{(3)(1)}}_{e_n+} L^{k_{(3)}}_{e_n+}\\
&= \Sigma_{\substack{(h)(k)\\(\alpha)(\beta)}}\, \beta_{(1)}(S(h_{(5)}))\beta_5(h_{(1)})\, T^{\alpha_{(1)}\beta_{(2)}}_{e_n+} Y(\alpha_{(2)}\beta_{(3)}) T^{\alpha_{(3)}\beta_{(4)}}_{e_1-} L^{h_{(2)}k_{(1)}}_{e_1+} X(h_{(3)}k_{(2)}) L^{h_{(4)}k_{(3)}}_{e_n+}\\
&= \Sigma_{(\beta)(h)}\, \beta_{(1)}(S(h_{(3)}))\beta_3(h_{(1)})\, \rho_{(v,f)}(\alpha\beta_{(2)}\otimes h_{(2)}k) \overset{S^2=\mathrm{id}}{=} \rho_{(v,f)}((\alpha, h) \cdot_{D(H)} (\beta\otimes k)),
\end{aligned}
$$

where we used equations (82) and (83) to pass to the second line, then the fact that $X(h)$ and $Y(\alpha)$ commute with each other and with $L^h_{e_1\pm}$, $L^h_{e_n\pm}$, $T^\alpha_{e_1\pm}$ and $T^\alpha_{e_n\pm}$ and the commutation relations (81) to pass to the third line. We then combined adjacent edge operators using again (81) and then simplified the resulting expressions using the properties of the Hopf algebra. In the last step, we used the identity $S^2 = \mathrm{id}_H$ and the formula (79) for the multiplication of the Drinfeld double $D(H)$. As we also have $\rho(1_{H^*}\otimes 1_H) = B^{1_{H^*}}_f A^{1_H}_v = \mathrm{id}_{H^{\otimes E}}$, this proves that $\rho_{(v,f)}$ is a representation of $D(H)$.

3. As H is finite-dimensional semisimple, by Theorem 6.2.18, the dual Hopf algebra H^* is finite-dimensional semisimple as well. By Theorem 6.2.11 and Corollary 6.2.14 there are normalised Haar integrals $\ell \in H$ and $\lambda \in H^*$, and by Corollary 6.2.20 the elements $\Delta^{(n-1)}(\ell)$ and $\Delta^{(m-1)}(\lambda)$ are invariant under cyclic permutations of the tensor factors for all $m, n \in \mathbb{N}$.

As different choices of the markings at a vertex v are related by cyclic permutations of the tensor factors of $\Delta^{(n-1)}(\ell)$ in A^ℓ_v, it follows that A^ℓ_v does not depend on the choice of the marking at v. Similarly, different choices of a marking of a face f correspond to cyclic permutations of the tensor factors of $\Delta^{(m-1)}(\lambda)$ in B^λ_f, and the cyclic invariance of $\Delta^{(m-1)}(\lambda)$ implies that B^λ_f does not depend on the choice of the marking at f.

As vertex operators for different vertices and face operators for different faces commute by 1, it remains to show that any vertex operator A^ℓ_v commutes with all face operators B^λ_f. If f is a face that is not incident at v, this follows because there is no edge that is both, incident at v and traversed by f. Thus, A^h_v and B^α_f act on different copies of H in the tensor product $H^{\otimes E}$

and hence commute. If f is a face incident at v, we can choose a common marking for v and f and obtain a representation of $D(H)$ as in 2. The claim then follows by a direct computation

$$\begin{aligned}
&A_v^\ell \circ B_f^\lambda = \rho_{(v,f)}(1\otimes\ell) \circ \rho_{(v,f)}(\lambda\otimes 1) = \rho_{(v,f)}(\Sigma_{(\ell)(\lambda)}\, \lambda_{(3)}(\ell_{(1)})\, \lambda_{(1)}(S(\ell_{(3)}))\lambda_{(2)}\otimes\ell_{(2)}\,)\\
&= \rho_{(v,f)}(\Sigma_{(\ell)(\lambda)}\, \lambda_{(1)}(\ell_{(2)})\, \lambda_{(2)}(S(\ell_{(1)}))\lambda_{(3)}\otimes\ell_{(3)}\,) = \rho_{(v,f)}(\Sigma_{(\ell)(\lambda)}\, \lambda_{(1)}(\ell_{(2)}S(\ell_{(1)}))\lambda_{(2)}\otimes\ell_{(3)}\,)\\
&= \rho_{(v,f)}(\Sigma_{(\ell)(\lambda)}\, \epsilon(\lambda_{(1)})\epsilon(\ell_{(1)})\lambda_{(2)}\otimes\ell_{(2)}\,) = \rho_{(v,f)}(\lambda\otimes\ell) = \rho_{(v,f)}(\lambda\otimes 1) \circ \rho_{(v,f)}(1\otimes\ell) = B_f^\lambda \circ A_v^\ell,
\end{aligned}$$

where we used first the fact that $\rho_{(v,f)}$ is an algebra homomorphism and the multiplication law of $D(H)$, then the cyclic invariance of $\Delta^{(2)}(\ell)$ and $\Delta^{(2)}(\lambda)$ to pass to the second line, then the duality between multiplication in H and comultiplication in H^*, the antipode condition to pass to the third line, the counit condition and again the fact that $\rho_{(v,f)}$ is an algebra homomorphism and the multiplication law of $D(H)$.

4. By Corollary 6.3.8 the normalised Haar integrals for H and H^* define projectors on the invariants of any H-module or H^* module. Thus, for any vertex v and face f, the operators A_v^ℓ and B_f^α are projectors on the invariants on the associated representation ρ_v and ρ_f of H and H^*. By 3. all such projectors commutes and thus define a projector on the linear subspace of H that consists of the elements that are invariant under all vertex and face operators. □

8 Reconstruction theory

8.1 $\mathbb{F}$-linear abelian categories

In the last section we constructed monoidal categories as the categories $B-\mathrm{Mod}$ of B-modules and B-linear maps for a bialgebra B over $\mathbb{F}$ and their full subcategories $B-\mathrm{Mod}^{fd}$ of finite-dimensional B-modules. It is clear that not every monoidal category is of this form or even monoidally equivalent to one of these categories. The categories $B-\mathrm{Mod}$ and $B-\mathrm{Mod}^{fd}$ have additional structure and properties that are carried over by any equivalence of categories. For instance, they are abelian, and their Hom sets are vector spaces over $\mathbb{F}$.

Many monoidal categories such as the simplex category from Example 1.1.7 or the braid category from Example 3.1.5 do not have these additional structures and properties and hence cannot be equivalent to representation categories of bialgebras. To understand which monoidal categories arise as representation categories of bialgebras, we need to systematically determine the additional structures and properties of these representation categories and then restrict attention to monoidal categories with these structures and properties.

As this problem becomes very difficult if one admits infinite-dimensional B-modules, we focus on the representation categories $B-\mathrm{Mod}^{fd}$ of *finite-dimensional modules* over a bialgebra B and also impose that B is *finite-dimensional* in the following.

Proposition 8.1.1: For any bialgebra B over $\mathbb{F}$ the categories $B-\mathrm{Mod}$ and $B-\mathrm{Mod}^{fd}$ have the following structures and properties:

1. They are abelian.
2. The functor $\otimes : B-\mathrm{Mod}^{(fd)} \times B-\mathrm{Mod}^{(fd)} \to B-\mathrm{Mod}^{(fd)}$ is biexact.
3. Their Hom sets are vector spaces over $\mathbb{F}$, and the composition of morphisms is $\mathbb{F}$-bilinear.
4. The endomorphisms of the unit object satisfy $\mathrm{End}_B(e) = \mathrm{End}_{\mathbb{F}}(\mathbb{F}) \cong \mathbb{F}$.
5. The maps $\otimes : \mathrm{Hom}_B(M, M') \times \mathrm{Hom}_B(N, N') \to \mathrm{Hom}_B(M\otimes N, M'\otimes N')$ are $\mathbb{F}$-bilinear for all B-modules M, M', N, N'.

The category $B-\mathrm{Mod}^{fd}$ has the following additional properties:

6. The Hom sets are finite-dimensional $\mathbb{F}$-vector spaces.
7. Every object is a B-module of finite length.

Proof:
The first claim holds for the category $B-\mathrm{Mod}$ of modules over any unital ring. The full subcategory $B-\mathrm{Mod}^{fd}$ is additive, because direct sums of finite-dimensional B-modules are again finite-dimensional. It is abelian, because the kernel, cokernel, image and coimage of a B-linear map between finite-dimensional modules are again finite-dimensional.

Property 2. follows because the functors $M\otimes -, -\otimes M : \mathrm{Vect}_{\mathbb{F}} \to \mathrm{Vect}_{\mathbb{F}}$ are exact for each vector space M over $\mathbb{F}$, since every $\mathbb{F}$-vector space is a free $\mathbb{F}$-module and hence projective. As the kernel $\iota : \ker(f) \to M$, the image $\iota' : \mathrm{im}(f) \to N$, the cokernel $\pi : N \to N/\mathrm{im}(f)$ and the coimage $\pi' : M \to M/\ker(f)$ in $\mathrm{Vect}_{\mathbb{F}}$ for a B-linear map $f : M \to N$ are B-linear, it follows that they are also the kernels, images, cokernels and coimages in $B-\mathrm{Mod}$. This implies that the functors $M\otimes -, -\otimes M : B-\mathrm{Mod} \to B-\mathrm{Mod}$ for a B-module M are exact as well.

Properties 3., 4., 5. and 6. follow directly, since the B-linear maps $f : M \to N$ form a vector space over $\mathbb{F}$ with the pointwise addition and scalar multiplication, the composition of B-linear maps is $\mathbb{F}$-bilinear, and tensoring $\mathbb{F}$-bilinear maps is $\mathbb{F}$-bilinear by the properties of the tensor product. One has $\dim_{\mathbb{F}} \mathrm{Hom}_{B-\mathrm{Mod}}(M,N) \leq \dim_{\mathbb{F}} \mathrm{Hom}_{\mathbb{F}}(M,N) = \dim_{\mathbb{F}} M \dim_{\mathbb{F}} N$ for finite-dimensional B-modules M, N, which proves 6. Property 4. follows, since one has $\mathrm{Hom}_{B-\mathrm{Mod}}(e) = \mathrm{Hom}_{B-\mathrm{Mod}}(\mathbb{F}) = \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}) \cong \mathbb{F}$. Property 7. is just Lemma A.24. $\square$

Any monoidal category that is monoidally equivalent to a category $B-\mathrm{Mod}$ or $B-\mathrm{Mod}^{fd}$ must have structures and properties analogous to the ones in Proposition 8.1.1. We thus need to formulate counterparts of these structures and properties for more general monoidal categories. For items 1. and 2. in Proposition 8.1.1, this requires no additional work. Items 3., 4., 5. and 6. in Proposition 8.1.1 require an $\mathbb{F}$-linear structure on the Hom sets, which can be imposed on any additive category and is encoded in the concept of an $\mathbb{F}$-linear category.

Definition 8.1.2: Let $\mathbb{F}$ be a field.

1. An additive category $\mathcal{A}$ is called $\mathbb{F}$-**linear**, if $\mathrm{Hom}_{\mathcal{A}}(A, A')$ is a vector space over $\mathbb{F}$ for all objects A, A' in $\mathcal{A}$ and the composition of morphisms is $\mathbb{F}$-linear.
2. A functor $F : \mathcal{A} \to \mathcal{B}$ between $\mathbb{F}$-linear categories $\mathcal{A}, \mathcal{B}$ is called $\mathbb{F}$-**linear**, if it is additive and the maps $\mathrm{Hom}_{\mathcal{A}}(A, A') \to \mathrm{Hom}_{\mathcal{B}}(F(A), F(A'))$ are $\mathbb{F}$-linear.

An $\mathbb{F}$-linear functor $F : \mathcal{A} \to \mathcal{B}$ that is an equivalence of categories is called an $\mathbb{F}$-**linear equivalence**, and the categories $\mathcal{A}$ and $\mathcal{B}$ are called $\mathbb{F}$-**linearly equivalent**.

With this definition of $\mathbb{F}$-linear categories and $\mathbb{F}$-linear functors, one can implement conditions 3. to 6. from Proposition 8.1.1. To formulate condition 7. in Proposition 8.1.1, we need a notion of a subobject in an abelian category that replaces the notion of a submodule in $B-\mathrm{Mod}$. We also need a concept of a simple objects and a generalisation of Jordan-Hölder series.

The proper notion of a subobject in an abelian category $\mathcal{A}$ is given by monomorphisms, and, dually, there is a notion of quotient objects given by epimorphisms. Simple, semisimple and indecomposable objects then can be defined as for modules over rings, by replacing submodules with subobjects and direct sums with coproducts.

Definition 8.1.3: Let $\mathcal{A}$ be an abelian category and A an object in $\mathcal{A}$. A **subobject** of A is a monomorphism $\iota : X \to A$. A **quotient object** of A is an epimorphism $\pi : A \to Y$.

Definition 8.1.4: Let $\mathcal{A}$ be an abelian category. A non-zero object A in $\mathcal{A}$ is called

- **simple**, if the source objects of its subobjects are all isomorphic to 0 or to A,
- **semisimple**, if it is a coproduct of simple objects,
- **indecomposable**, if it is not a coproduct of at least two non-zero subobjects.

The category $\mathcal{A}$ is called **semisimple**, if every object in $\mathcal{A}$ is semisimple.

With analogous arguments as for modules, one can derive a version of Schur's lemma for abelian categories (Exercise 56). The definition of a Jordan-Hölder series also generalises directly from modules to abelian categories. The only difference is that submodules are replaced by subobjects and quotient modules by cokernels. There is also a Jordan-Hölder Theorem for abelian categories that ensures that any two Jordan-Hölder series for a given object have the same lengths and isomorphic subfactors.

Definition 8.1.5: An object A in an abelian category $\mathcal{A}$ has **finite length**, if there is a **Jordan-Hölder series** of **length** $n \in \mathbb{N}_0$ for A, a sequence

$$0 = A_0 \xrightarrow{\iota_0} A_1 \xrightarrow{\iota_1} A_2 \xrightarrow{\iota_2} \ldots \xrightarrow{\iota_{n-2}} A_{n-1} \xrightarrow{\iota_{n-1}} A_n = A \tag{84}$$

of monomorphisms $\iota_i : A_i \to A_{i+1}$ such that $A_{i+1}/A_i := \mathrm{coker}(\iota_i)$ is simple for all $i = 0, ..., n-1$.

The **multiplicity** $[A : A']$ of a simple object A' in the Jordan-Hölder series (84) is the number of i with $\mathrm{coker}(\iota_i) \cong A'$.

Theorem 8.1.6: (**Jordan-Hölder theorem**)
Let A be an object of finite length in an abelian category $\mathcal{A}$. Then any sequence of monomorphisms as in (84) can be extended to a Jordan-Hölder series of A. All Jordan-Hölder series of A have the same lengths and the same multiplicities for each simple object A' in $\mathcal{A}$.

Properties 6. and 7. in Proposition 8.1.1 for the category $B{-}\mathrm{Mod}^{fd}$ both follow from 3. to 5. and the fact that the modules are finite-dimensional. In a general $\mathbb{F}$-linear abelian category, there is no notion of dimension for objects, and these conditions are imposed separately.

Definition 8.1.7: An $\mathbb{F}$-linear abelian category $\mathcal{A}$ is called **locally finite** if

(i) $\mathrm{Hom}_{\mathcal{A}}(A, B)$ is a finite-dimensional $\mathbb{F}$-vector space for all objects A, B in $\mathcal{A}$,
(ii) every object in $\mathcal{A}$ has finite length.

The conditions in Definition 8.1.7 are almost sufficient to ensure that an $\mathbb{F}$-linear abelian category $\mathcal{A}$ is equivalent to the category $A{-}\mathrm{Mod}^{fd}$ for a finite-dimensional algebra A over $\mathbb{F}$. In fact this requires just two supplementary conditions:

(i) there are only finitely many isomorphism classes of simple objects in $\mathcal{A}$,
(ii) every simple object in $\mathcal{A}$ has a **projective cover**: there is a projective object P_A and an epimorphism $\pi : P_A \to A$ with the following property: for each epimorphism $p : P \to A$ from a projective object P there is an epimorphism $p' : P \to P_A$ with $\pi \circ p' = p$.

An $\mathbb{F}$-linear abelian category $\mathcal{A}$ that satisfies these two condition is called *finite*.

The first condition generalises the well-known fact that any finite-dimensional algebra has only finitely many isomorphism classes of simple modules. Understanding the second condition requires more background on projective objects and projective covers. For this reason, we will not prove that any finite $\mathbb{F}$-linear abelian category is equivalent to the category $A{-}\mathrm{Mod}^{fd}$ of a finite-dimensional algebra over $\mathbb{F}$, but take equivalence to the representation category of a finite-dimensional algebra as the definition. A sketch of proof is given in [EGNO, Def 1.8.6 ff].

Definition 8.1.8: An $\mathbb{F}$-linear abelian category $\mathcal{A}$ is called **finite** if it is $\mathbb{F}$-linearly equivalent to the category $A{-}\mathrm{Mod}^{fd}$ of finite-dimensional modules over a finite-dimensional $\mathbb{F}$-algebra A.

Proposition 8.1.1 implies that any monoidal category $(\mathcal{C}, \otimes)$ that is monoidally equivalent to a representation category $B{-}\mathrm{Mod}^{fd}$ for a bialgebra B over $\mathbb{F}$ must have the structure of a locally finite abelian $\mathbb{F}$-linear category with a biexact and $\mathbb{F}$-bilinear tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and with $\mathrm{End}_{\mathcal{C}}(e) \cong \mathbb{F}$. If we require that B is finite-dimensional, then $\mathcal{C}$ must be finite.

This gives a strong motivation to understand $\mathbb{F}$-linear additive functors between finite $\mathbb{F}$-linear abelian categories and the natural transformations between them. Due to their equivalence with the representation category $A{-}\mathrm{Mod}^{fd}$ for a finite-dimensional algebra A over $\mathbb{F}$, such functors can be characterised in terms of bimodules. More specifically, it turns out that any *right exact* $\mathbb{F}$-linear functor $F : A{-}\mathrm{Mod}^{fd} \to B{-}\mathrm{Mod}^{fd}$ for finite-dimensional $\mathbb{F}$-algebras A, B is obtained, up to equivalence, by tensoring over A with an (B,A)-bimodule. Natural transformations between two such functors are then given by (B,A)-bimodule morphisms.

Definition 8.1.9: Let A, B finite-dimensional algebras over $\mathbb{F}$. An additive $\mathbb{F}$-linear functor $F : A{-}\mathrm{Mod}^{fd} \to B{-}\mathrm{Mod}^{fd}$ is called $\otimes$-**representable** if there is a (B,A)-bimodule V such that F is naturally isomorphic to $V \otimes_A - : A{-}\mathrm{Mod}^{fd} \to B{-}\mathrm{Mod}^{fd}$.

Proposition 8.1.10: An $\mathbb{F}$-linear functor $G : A{-}\mathrm{Mod}^{fd} \to B{-}\mathrm{Mod}^{fd}$ is $\otimes$-representable if and only if it is right exact.

Proof:
If G is naturally isomorphic to a functor $V \otimes_A - : A{-}\mathrm{Mod}^{fd} \to B{-}\mathrm{Mod}^{fd}$ for a (B,A)-bimodule V, then it is right exact, since $V \otimes_A -$ is right exact, see [Me, Cor 3.1.16]. We show that any right exact functor $G : A{-}\mathrm{Mod}^{fd} \to B{-}\mathrm{Mod}^{fd}$ is $\otimes$-representable.

1. We consider the finite-dimensional B-module $V = G(A)$ and for each $a \in A$ the A-linear map $r_A : A \to A$, $a' \mapsto a'a$. Its image $G(r_a) : V \to V$ is B-linear by definition of G, and the map $\lhd : V \times A \to V$, $v \lhd a = G(r_a)v$ equips V with the structure of a (B,A)-bimodule:

$$(v \lhd a) \lhd a' = G(r_{a'})G(r_a)v = G(r_{a'} \circ r_a)v = G(r_{aa'})v = v \lhd (aa')$$
$$v \lhd 1_A = G(r_{1_A})v = G(\mathrm{id}_A)v = v \qquad b \rhd (v \lhd a) = b \rhd G(r_a)v = G(r_a)(b \rhd v) = (b \rhd v) \lhd a.$$

2. We construct a natural isomorphism $\phi : G \to V \otimes_A -$:

• As G is additive, one has for each free A-module $F = \oplus_I A$ an A-linear isomorphism

$$\phi_F : G(F) = G(\oplus_I A) \cong \oplus_I G(A) \cong \oplus_I (G(A) \otimes_A A) \cong G(A) \otimes_A (\oplus_I A) = V \otimes_A F \qquad (85)$$

that satisfies $\phi_{F'} \circ f = (V \otimes_A f) \circ \phi_F$ for all A-linear maps $f : F \to F'$ between free A-modules.

• To construct isomorphisms $\phi_M : G(M) \to V \otimes_A M$ for general finite-dimensional A-modules M, we choose for each finite-dimensional A-module M an exact sequence

$$F_\bullet = F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$$

in $A{-}\mathrm{Mod}^{fd}$ with free A-modules F_0, F_1. For instance, we may choose

$$F_0 = \oplus_B A, \qquad d_0 : F_0 \to M, (a_m)_{m \in B} \mapsto \Sigma_{m \in B} a_m \rhd m$$
$$F_1 = \oplus_{B'} A, \qquad d_1 : F_1 \to F_0, (a_m)_{m \in B'} \mapsto \Sigma_{m \in B'} a_m \rhd m$$

for vector space bases B of M and B' of $\ker(d_0)$.

The exactness of $F_\bullet$ is equivalent to the statement that d_0 is a cokernel of d_1. As G is right exact, it follows that $G(d_0) : G(F_0) \to G(M)$ is a cokernel of $G(d_1) : G(F_1) \to G(F_0)$, and consequently $G(d_0) \circ \phi_{F_0}^{-1} : V \otimes_A F_0 \to G(M)$ is a cokernel of $V \otimes_A d_1 : V \otimes_A F_1 \to V \otimes_A F_0$. As $V \otimes_A d_0 : V \otimes_A F_0 \to V \otimes_A M$ is another cokernel of $V \otimes_A d_1$ due to right exactness of $V \otimes_A -$, we have a unique isomorphism $\phi_M : G(M) \to V\otimes_A M$ such that the following diagram commutes

$$\begin{tikzcd}
G(F_1) \arrow[r, "G(d_1)"] \arrow[d, "\cong"', "\phi_{F_1}"] & G(F_0) \arrow[r, "G(d_0)"] \arrow[d, "\cong"', "\phi_{F_0}"] & G(M) \arrow[r] \arrow[d, "\cong"', "\phi_M"] & 0 \\
V\otimes_A F_1 \arrow[r, "V\otimes_A d_1"'] & V\otimes_A F_0 \arrow[r, "V\otimes_A d_0"'] & V\otimes_A M \arrow[r] & 0.
\end{tikzcd} \tag{86}$$

The morphism ϕ_M does not depend on the choice of F_0 and F_1. If $F'_\bullet = F'_1 \xrightarrow{d'_1} F'_0 \xrightarrow{d'_0} M \to 0$ is another exact sequence in $A-\mathrm{Mod}^{fd}$ with free A-modules F'_0, F'_1, then we can construct A-linear maps $f_0 : F_0 \to F'_0$ and $f'_1 : F_1 \to F'_1$ such that the following diagram commutes

$$\begin{tikzcd}
F_1 \arrow[r, "d_1"] \arrow[d, "f_1"] & F_0 \arrow[r, "d_0"] \arrow[d, "f_0"] & M \arrow[r] \arrow[d, "\mathrm{id}_M"] & 0 \\
F'_1 \arrow[r, "d'_1"] & F'_0 \arrow[r, "d'_0"] & M \arrow[r] & 0.
\end{tikzcd}$$

The isomorphisms ϕ_F in (86) satisfy $\phi_{F'_n} \circ f_n = \phi_{F_n}$ for $n = 0, 1$. This follows due to the exactness and projectivity, see for instance the proof of Theorem 4.1.6 in [Me]. By applying G to this diagram and combining the resulting diagrams with diagram (86) for $F_\bullet$ and $F'_\bullet$, one finds that that the morphisms ϕ_M for $F_\bullet$ and ϕ'_M for $F'_\bullet$ in (86) agree.

The isomorphisms $\phi_M : G(M) \to V\otimes_A M$ define a natural isomorphism $\phi : G \to V\otimes_A -$. Given finite-dimensional A-modules M, M' and exact sequences

$$F_\bullet = F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0 \qquad\qquad F'_\bullet = F'_1 \xrightarrow{d'_1} F'_0 \xrightarrow{d'_0} M' \to 0,$$

we can extend any A-linear map $f : M \to M'$ to a chain map $f_\bullet : F_\bullet \to F'_\bullet$ with $f_{-1} = f$ by [Me, Thm 4.1.6]. Applying the functors G and $V\otimes_A -$ yields chain maps $G(f_\bullet) : G(F_\bullet) \to G(F'_\bullet)$ and $V\otimes_A f_\bullet : V\otimes_A F_\bullet \to V\otimes_A F'_\bullet$ that link the diagrams (86) for M and M'. This implies $\phi_N \circ G(f) = (V\otimes_A f) \circ \phi_M$ in degree -1. □

Proposition 8.1.11: Let A, B be finite-dimensional algebras over $\mathbb{F}$ and V, W finite-dimensional (B, A)-bimodules. Then natural transformations $\nu : V\otimes_A - \to W\otimes_A -$ are in bijection with elements of $\mathrm{End}_{B\otimes A^{op}}(V, W)$.

Proof:
We identify $V\otimes_A A$ with V via the $B\otimes A^{op}$-linear isomorphism $r_V : V\otimes_A A \to V$, $v\otimes a \mapsto v \lhd a$ with inverse $r_V^{-1} : V \to V\otimes_A A$, $v \mapsto v\otimes 1$.

1. To a natural transformation $\nu : V\otimes_A - \to W\otimes_A -$ we assign the $B\otimes A^{op}$-linear map

$$f_\nu = r_W \circ \nu_A \circ r_V^{-1} : V \to V, \quad v \mapsto r_W \circ \nu_A(v\otimes 1).$$

It is B-linear, because ν_A is B-linear by definition and A^{op}-linear by naturality of ν: by considering the A-linear maps $R_a : A \to A$, $a' \mapsto a'a$ we obtain for all $a \in A$ and $v \in V$

$$\begin{aligned} f_\nu(v \lhd a) &= r_W \circ \nu_A(v \lhd a\otimes 1) = r_W \circ \nu_A(v\otimes a) = r_W \circ \nu_A \circ (\mathrm{id}_V\otimes R_a)(v\otimes 1) \\ &= r_W \circ (\mathrm{id}_V\otimes R_a) \circ \nu_A(1\otimes v) = (r_W \circ \nu_A(1\otimes v)) \lhd a = f_\nu(v) \lhd a. \end{aligned}$$

2. To a $B\otimes A^{op}$-linear map $f : V \to W$, we assign the natural transformation ν^f with component morphisms $\nu^f_M = (f\otimes\mathrm{id}_M) : V\otimes_A M \to W\otimes_A M$, $v\otimes m \mapsto f(v)\otimes m$. The $B\otimes A^{op}$-linearity of f ensures that the maps ν^f_M are well-defined and B-linear:

$$\nu^f_M((v \lhd a)\otimes m) = f(v \lhd a)\otimes m = (f(v) \lhd a)\otimes m = f(v)\otimes(a \rhd m) = \nu^f_M(v\otimes(a \rhd m))$$
$$\nu^f_M((b \rhd v)\otimes m) = f(b \rhd v)\otimes m = (b \rhd f(v))\otimes m = b \rhd \nu^f_M(v\otimes m).$$

They define a natural transformation, because we have for each A-linear map $g : M \to N$

$$\nu^f_N \circ (\mathrm{id}_V\otimes g) = (f\otimes\mathrm{id}_N) \circ (\mathrm{id}_V\otimes g) = (\mathrm{id}_V\otimes g) \circ (f\otimes\mathrm{id}_N) = (\mathrm{id}_V\otimes g) \circ \nu^f_M.$$

3. To show that the assignments $\phi : f \to \nu_f$ and $\psi : \nu \to f_\nu$ are mutually inverse bijections, we compute for each each $B\otimes A^{op}$-linear map $f : V \to W$ and $v \in V$

$$f_{\nu_f}(v) = r_W \circ \nu_f(v\otimes 1) = r_W(f(v)\otimes 1) = f(v)$$

and hence $f_{\nu_f} = f$. By considering for each A-module M and $m \in M$ the A-linear maps $\rhd_m : A \to M$, $a \mapsto a \rhd m$, we obtain for each natural transformation $\nu : V\otimes_A - \to V\otimes_A -$

$$\begin{aligned}\nu^{f_\nu}_M(v\otimes m) &= f_\nu(v)\otimes m = (\mathrm{id}_V\otimes\rhd_m)(f_\nu(v)\otimes 1) = (\mathrm{id}_V\otimes\rhd_m)(r_W(\nu_A(v\otimes 1))\otimes 1)\\ &=(\mathrm{id}_V\otimes\rhd_m) \circ \nu_A(v\otimes 1) = \nu_M \circ (\mathrm{id}_V\otimes\rhd_m)(v\otimes 1) = \nu_M(v\otimes m),\end{aligned}$$

for all $v \in V$, $m \in M$ and finite-dimensional A-modules M. To pass to the second line we used the identity $(u \lhd a)\otimes \rhd_m (1) = u\otimes \rhd_m (a)$ for all $a \in A$, $m \in M$ that follows from the A-linearity of $\rhd_m$ and the properties of the tensor product $\otimes_A$. This shows that $\nu^{f_\nu} = \nu$. □

8.2 Fiber functors and reconstruction

In the last section, we determined which monoidal categories have a chance to be monoidally equivalent to the representation category $B-\mathrm{Mod}^{fd}$ of finite dimensional-modules over a bialgebra B over $\mathbb{F}$. From Proposition 8.1.1 we obtained that any such category must have the structure of a locally finite abelian $\mathbb{F}$-linear category with a biexact and $\mathbb{F}$-bilinear tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and with $\mathrm{End}_{\mathcal{C}}(e) \cong \mathbb{F}$.

We can also impose additional requirements on B that yield additional conditions on $\mathcal{C}$:

- If B is a finite-dimensional bialgebra, then $\mathcal{C}$ must not only be locally finite, but finite.
- If B is a Hopf algebra, then $\mathcal{C}$ must also be right rigid by Corollary 5.2.7.
- If B is a finite-dimensional Hopf algebra, then $\mathcal{C}$ must be rigid by Corollary 5.2.7.
- If B is a finite-dimensional semisimple Hopf algebra, then $\mathcal{C}$ must be rigid and semisimple by Proposition A.19.

Definition 8.2.1: Let $\mathcal{C}$ be a locally finite, $\mathbb{F}$-linear abelian monoidal category with $\mathbb{F}$-bilinear tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and $\mathrm{End}_{\mathbb{F}}(e) \cong \mathbb{F}$. Then $\mathcal{C}$ is called

- a **ring category** if $\otimes$ is biexact,
- a **tensor category** if $\mathcal{C}$ is rigid,
- a **fusion category** if $\mathcal{C}$ is rigid, finite and semisimple.

Remark 8.2.2: Every tensor category is a ring category.

This follows, because in a rigid monoidal category $\mathcal{C}$ the functor $C\otimes- : \mathcal{C} \to \mathcal{C}$ and the functor $-\otimes C : \mathcal{C} \to \mathcal{C}$ have both, left and right adjoints for each object C in $\mathcal{C}$ by Proposition 2.1.4. If $\mathcal{C}$ is abelian, this implies that the functors $C\otimes-$ and $-\otimes C$ are exact for all objects C in $\mathcal{C}$ by Lemma B.36 and hence $\otimes$ is biexact.

The conditions in Definition 8.2.1 necessary for a category $\mathcal{C}$ to be monoidally equivalent to a representation category for a bialgebra or Hopf algebra, but not yet fully sufficient. They do not take into account that the monoidal structure in a category $B-\text{Mod}^{fd}$ is given by the monoidal structure in $\text{Vect}_{\mathbb{F}}^{fd}$. This is encoded in the forgetful functor $V : B-\text{Mod}^{fd} \to \text{Vect}_{\mathbb{F}}^{fd}$, which is monoidal, $\mathbb{F}$-linear, faithful and exact. Any monoidal equivalence $\phi : \mathcal{C} \to B-\text{Mod}^{fd}$ can be composed with this forgetful functor, and if $\mathcal{C}$ is equipped with the $\mathbb{F}$-linear abelian structure induced by ϕ, then this yields an $\mathbb{F}$-linear exact faithful monoidal functor $F = V\phi : \mathcal{C} \to \text{Vect}_{\mathbb{F}}^{fd}$.

Hence, any monoidal category $\mathcal{C}$ that is monoidally equivalent to a category $B-\text{Mod}^{fd}$ must be equipped with an an $\mathbb{F}$-linear exact faithful monoidal functor $F : \mathcal{C} \to \text{Vect}_{\mathbb{F}}^{fd}$. Such a functor is called a *fiber functor*.

Definition 8.2.3: Let $\mathcal{C}$ be a ring category. A **fiber functor** for $\mathcal{C}$ is an exact faithful $\mathbb{F}$-linear monoidal functor $F : \mathcal{C} \to \text{Vect}_{\mathbb{F}}^{fd}$.

Example 8.2.4: For any bialgebra B over $\mathbb{F}$, the forgetful functor $V : B-\text{Mod}^{fd} \to \text{Vect}_{\mathbb{F}}^{fd}$ is a fiber functor.

We will now show that every *finite* ring category $\mathcal{C}$ with a fiber functor $F : \mathcal{C} \to \text{Vect}_{\mathbb{F}}^{fd}$ defines a *finite-dimensional* bialgebra B over $\mathbb{F}$. This bialgebra is obtained as the algebra $\text{End}(F)$ of natural transformations $\nu : F \to F$ with the composition of natural transformations as multiplication. Its coalgebra structure is induced by the coherence data of the fiber functor, the isomorphism $\phi^e : \mathbb{F} \to F(e)$ and the natural isomorphism $\phi^{\otimes} : \otimes(F \times F) \to F\otimes$.

Theorem 8.2.5: Let $\mathcal{C}$ be a finite ring category, $F : \mathcal{C} \to \text{Vect}_{\mathbb{F}}^{fd}$ be a fiber functor and $\text{End}(F)$ the vector space of natural transformations $\eta : F \to F$.

1. Then $\text{End}(F)$ has the structure of a finite-dimensional bialgebra over $\mathbb{F}$.
2. If $\mathcal{C}$ is right rigid, then $\text{End}(F)$ is a finite-dimensional Hopf algebra over $\mathbb{F}$.

Proof:
1. As a finite $\mathbb{F}$-linear abelian category, the category $\mathcal{C}$ is equivalent to the category $A-\text{Mod}^{fd}$ for a finite-dimensional algebra A over $\mathbb{F}$ by Definition 8.1.8. By Exercise 9, this defines a monoidal structure on $A-\text{Mod}^{fd}$ such that the equivalence becomes monoidal. As an equivalence of abelian categories, it is also fully faithful and exact. By composing it with the fiber functor, we obtain a fiber functor $F : A-\text{Mod}^{fd} \to \text{Vect}^{fd}$. We may thus assume that $\mathcal{C} = A-\text{Mod}^{fd}$ for a finite-dimensional algebra A over $\mathbb{F}$.

Proposition 8.1.10 with $B = \mathbb{F}$ implies that there is a finite-dimensional A-right module V such that F is naturally isomorphic to $V\otimes_A- : A-\text{Mod}^{fd} \to \text{Vect}_{\mathbb{F}}^{fd}$. Proposition 8.1.11 shows that $\text{End}(F) \cong \text{End}_{A^{op}}(V) \subset \text{End}_{\mathbb{F}}(V)$ is a finite-dimensional vector space over $\mathbb{F}$. Proposition 8.1.11 also shows that the composition of natural transformations in $\text{End}(F)$ corresponds to the composition in $\text{End}_{A^{op}}(V)$ and the identity natural transformation id_F to the unit element $\text{id}_V \in \text{End}_{A^{op}}(V)$. Thus, $\text{End}(F) \cong \text{End}_{A^{op}}(V)$ is a finite-dimensional algebra over $\mathbb{F}$.

2. The counit and the coproduct on $\mathrm{End}(F)\cong \mathrm{End}_{A^{op}}(V)$ are defined by the monoidal structure of F, the isomorphism $\phi^e:\mathbb{F}\to F(e)$ and the natural isomorphism $\phi^\otimes:\otimes_{\mathbb{F}}(F\times F)\to F\otimes$, respectively. The counit $\epsilon:\mathrm{End}(F)\to\mathbb{F}$, $\nu\mapsto\epsilon(\nu)$ is defined by the commuting diagram

$$\begin{tikzcd}
\mathbb{F} \arrow[r, "\epsilon(\nu)"] \arrow[d, "\phi^e"', "\cong"] & \mathbb{F} \arrow[d, "\phi^e"', "\cong"] \\
F(e) \arrow[r, "\nu_e"'] & F(e).
\end{tikzcd} \tag{87}$$

The comultiplication $\Delta:\mathrm{End}(F)\to\mathrm{End}(F)\otimes\mathrm{End}(F)$ is defined by the commuting diagram

$$\begin{tikzcd}
F(X)\otimes F(Y) \arrow[r, "\Delta(\nu)_{X,Y}"] \arrow[d, "\phi^\otimes_{X,Y}"', "\cong"] & F(X)\otimes F(Y) \arrow[d, "\phi^\otimes_{X,Y}"', "\cong"] \\
F(X\otimes Y) \arrow[r, "\nu_{X\otimes Y}"'] & F(X\otimes Y).
\end{tikzcd} \tag{88}$$

Due to the naturality of $\phi^\otimes$ and ν, the $\mathbb{F}$-linear maps $\Delta(\nu)_{X,Y}:F(X)\otimes F(Y)\to F(X)\otimes F(Y)$ define a natural transformation $\Delta'(\nu):(V\otimes V)_{A\otimes A}-\to(V\otimes V)_{A\otimes A}-$, where $V\otimes V$ carries the canonical $A\otimes A$-right module structure $(v\otimes v')\lhd(a\otimes a')=(v\lhd a)\otimes(v'\lhd a')$.

The canonical isomorphism $\mathrm{End}_{\mathbb{F}}(V\otimes V)\cong\mathrm{End}_{\mathbb{F}}(V)\otimes\mathrm{End}_{\mathbb{F}}(V)$ induces an $\mathbb{F}$-linear isomorphism $\mathrm{End}_{(A\otimes A)^{op}}(V\otimes V)\cong\mathrm{End}_{A^{op}}(V)\otimes\mathrm{End}_{A^{op}}(V)$. This yields an $\mathbb{F}$-linear isomorphism

$$\chi:\mathrm{End}((V\otimes V)_{A\otimes A}-)\cong\mathrm{End}_{(A\otimes A)^{op}}(V\otimes V)\cong\mathrm{End}_{A^{op}}(V)\otimes\mathrm{End}_{A^{op}}(V)\cong\mathrm{End}(F)\otimes\mathrm{End}(F)$$

that is an algebra isomorphism with respect to the composition of natural transformations. This defines an $\mathbb{F}$-linear map $\Delta:\mathrm{End}(F)\to\mathrm{End}(F)\otimes\mathrm{End}(F)$, $\nu\mapsto\chi(\Delta'(\nu))$.

3. That $\epsilon:\mathrm{End}(F)\to\mathbb{F}$ is an algebra homomorphism follows by composing the diagrams (87) for natural transformations ν,ν' horizontally. Similarly, composing the diagrams (88) horizontally yields $\Delta'(\nu')\circ\Delta'(\nu)=\Delta'(\nu'\circ\nu)$ and hence Δ' is an algebra homomorphism. As χ is an algebra isomorphism χ, it follows that Δ is an algebra homomorphism as well.

4. We verify that Δ and ϵ satisfy the coassociativity and counit axioms. The coassociativity of Δ follows from the commuting diagram

$$\begin{tikzcd}
(F(X)\otimes F(Y))\otimes F(Z) \arrow[r, "(\Delta\otimes\mathrm{id})\circ\Delta(\nu)_{X,Y,Z}"] \arrow[d, "\phi^\otimes_{X,Y}\otimes 1_{F(Z)}"', "\cong"] \arrow[ddddd, bend right, "a_{F(X),F(Y),F(Z)}"', "\cong"] & (F(X)\otimes F(Y))\otimes F(Z) \arrow[d, "\phi^\otimes_{X,Y}\otimes 1_{F(Z)}"', "\cong"] \arrow[ddddd, bend left, "a_{F(X),F(Y),F(Z)}", "\cong"'] \\
F(X\otimes Y)\otimes F(Z) \arrow[r, "\Delta(\nu)_{X\otimes Y,Z}"] \arrow[d, "\phi^\otimes_{X\otimes Y,Z}"', "\cong"] & F(X\otimes Y)\otimes F(Z) \arrow[d, "\phi^\otimes_{X\otimes Y,Z}"', "\cong"] \\
F((X\otimes Y)\otimes Z) \arrow[r, "\nu_{(X\otimes Y)\otimes Z}"] \arrow[d, "F(a_{X,Y,Z})"', "\cong"] & F((X\otimes Y)\otimes Z) \arrow[d, "F(a_{X,Y,Z})"', "\cong"] \\
F(X\otimes(Y\otimes Z)) \arrow[r, "\nu_{X\otimes(Y\otimes Z)}"] \arrow[d, "\phi^{\otimes-1}_{X,Y\otimes Z}"', "\cong"] & F(X\otimes(Y\otimes Z)) \arrow[d, "\phi^{\otimes-1}_{X,Y\otimes Z}"', "\cong"] \\
F(X)\otimes F(Y\otimes Z) \arrow[r, "\Delta'(\nu)_{X,Y\otimes Z}"] \arrow[d, "1_{F(x)}\otimes\phi^{\otimes-1}_{Y\otimes Z}"', "\cong"] & F(X)\otimes F(Y\otimes Z) \arrow[d, "1_{F(X)}\otimes\phi^{\otimes-1}_{Y\otimes Z}"', "\cong"] \\
F(X)\otimes(F(Y)\otimes F(Z)) \arrow[r, "(\mathrm{id}\otimes\Delta)\circ\Delta(\nu)_{X,Y,Z}"'] & F(X)\otimes(F(Y)\otimes F(Z)),
\end{tikzcd}$$

in which the left and right diagrams commute, because F is a monoidal functor, the top two and bottom two rectangles in the middle by definition of $\Delta(\nu)$ and the rectangle in the middle by naturality of ν. The counitality follows from the commuting diagram

$$
\begin{tikzcd}
\mathbb{F}\otimes F(Y) \arrow[r, "(\epsilon\otimes\mathrm{id})\circ\Delta(\nu)_{e,Y}"] \arrow[d, "\phi^e\otimes 1_{F(Y)}"', "\cong"] \arrow[ddd, bend right=60, "l_{F(Y)}"', "\cong"] & \mathbb{F}\otimes F(Y) \arrow[d, "\phi^e\otimes 1_{F(Y)}"', "\cong"] \arrow[ddd, bend left=60, "l_{F(Y)}", "\cong"'] \\
F(e)\otimes F(Y) \arrow[r, "\Delta(\nu)_Y"] \arrow[d, "\phi^{\otimes}_{e,Y}"', "\cong"] & F(e)\otimes F(Y) \arrow[d, "\phi^{\otimes}_{e,Y}"', "\cong"] \\
F(e\otimes Y) \arrow[r, "\nu_{e\otimes Y}"] \arrow[d, "F(l_Y)"', "\cong"] & F(e\otimes Y) \arrow[d, "F(l_Y)"', "\cong"] \\
F(Y) \arrow[r, "\nu_Y"] & F(Y)
\end{tikzcd}
$$

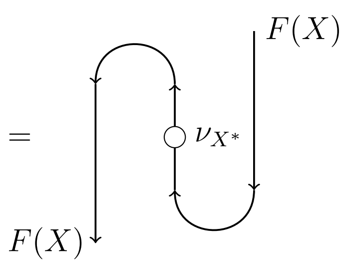
and its counterpart for the right unit constraint, in which the left and right diagram commute because F is monoidal, the top rectangle by definition of the counit, the middle rectangle by definition of the comultiplication and the bottom rectangle by naturality of ν.

This shows that $(\mathrm{End}(F), \circ, \mathrm{id}_F, \Delta, \epsilon)$ is a bialgebra.

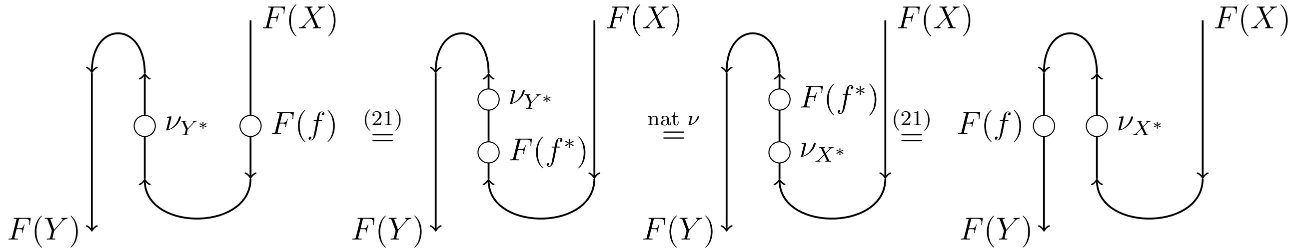
5. Suppose now that $\mathcal{C} = A-\mathrm{Mod}$ is right rigid. Then we define the antipode of $\mathrm{End}(F)$ by

$$
\begin{tikzcd}
F(X) \arrow[r, "S(\nu)_X"] \arrow[d, "l^{-1}_{F(X)}"', "\cong"] & F(X) \\
\mathbb{F}\otimes F(X) \arrow[d, "\phi^e\otimes 1_{F(X)}"', "\cong"] & F(X)\otimes\mathbb{F} \arrow[u, "\cong", "r_{F(X)}"'] \\
F(e)\otimes F(X) \arrow[d, "F(\mathrm{coev}^R_X)\otimes 1_{F(X)}"'] & F(X)\otimes F(e) \arrow[u, "\cong", "1_{F(X)}\otimes\phi^{e-1}"'] \\
F(X\otimes X^*)\otimes F(X) \arrow[d, "\phi^{\otimes -1}_{X,X^*}\otimes 1_{F(X)}"', "\cong"] & F(X)\otimes F(X^*\otimes X) \arrow[u, "1_{F(X)}\otimes F(\mathrm{ev}^R_X)"'] \\
(F(X)\otimes F(X^*))\otimes F(X) \arrow[d, "a_{F(X),F(X^*),F(X)}"'] & F(X)\otimes(F(X^*)\otimes F(X)) \arrow[u, "\cong", "1_{F(X)}\otimes\phi^{\otimes}_{X^*,X}"'] \\
F(X)\otimes(F(X^*)\otimes F(X)) \arrow[ur, "1_{F(X)}\otimes(\nu_{X^*}\otimes 1_{F(X)})"'] &
\end{tikzcd}
\tag{89}
$$

Diagrammatically, this morphism is given by

$S(\nu)_X = \nu^*_{X^*} =$ 

All morphisms in diagram (89) except ev^R_X and coev^R_X are natural in X. For the latter, the diagrammatic identity (21) implies for all morphisms $f : X \to Y$



and thus $S(\nu)_Y \circ F(f) = \nu^*_{Y^*} \circ F(f) = F(f) \circ \nu^*_{X^*} = F(f) \circ S(\nu)_X$. This shows that the morphisms $S(\nu)_X : F(X) \to F(X)$ define a natural transformation $S(\nu) : F \to F$. As all morphisms in this diagram are $\mathbb{F}$-linear, this defines an $\mathbb{F}$-linear map $S : \mathrm{End}(F) \to \mathrm{End}(F)$.

It remains to show that S satisfies the axiom on the antipode from Definition 5.2.1. For this, note first that for all natural transformations $\tau, \rho : F \to F$ the following diagram commutes

$$\begin{tikzcd}
F(X) \arrow[r, "\rho_X"] \arrow[d, "l^{-1}_{F(X)}"', "\cong"] \arrow[rr, bend left, "m\circ(S\otimes\mathrm{id})(\tau\otimes\rho)_X"] & F(X) \arrow[r, "S(\tau)_X"] \arrow[d, "l^{-1}_{F(X)}"', "\cong"] & F(X) \\
\mathbb{F}\otimes F(X) \arrow[r, "1\otimes\rho_X"] \arrow[d, "\phi^e\otimes 1"', "\cong"] & \mathbb{F}\otimes F(X) \arrow[d, "\phi^e\otimes 1"', "\cong"] & F(X)\otimes\mathbb{F} \arrow[u, "\cong", "r_{F(X)}"'] \\
F(e)\otimes F(X) \arrow[r, "1\otimes\rho_X"] \arrow[d, "F(\mathrm{coev}^R_X)\otimes 1"'] & F(e)\otimes F(X) \arrow[d, "F(\mathrm{coev}^R_X)\otimes 1"'] & F(X)\otimes F(e) \arrow[u, "\cong", "1\otimes\phi^{e-1}"'] \\
F(X\otimes X^*)\otimes F(X) \arrow[r, "1\otimes\rho_X"] \arrow[d, "\phi^{\otimes -1}_{X,X^*}\otimes 1"', "\cong"] & F(X\otimes X^*)\otimes F(X) \arrow[d, "\phi^{\otimes -1}_{X,X^*}\otimes 1"', "\cong"] & F(X)\otimes F(X^*\otimes X) \arrow[u, "1\otimes F(\mathrm{ev}^R_X)"'] \\
(F(X)\otimes F(X^*))\otimes F(X) \arrow[r, "(1\otimes 1)\otimes\rho_X"] \arrow[d, "a"] & (F(X)\otimes F(X^*))\otimes F(X) \arrow[d, "a"] & \\
F(X)\otimes(F(X^*)\otimes F(X)) \arrow[r, "1\otimes(1\otimes\rho_X)"'] \arrow[rr, bend right, "1\otimes(\tau_{X^*}\otimes\rho_X)"'] & F(X)\otimes(F(X^*)\otimes F(X)) \arrow[r, "1\otimes(\tau_{X^*}\otimes 1)"'] & F(X)\otimes(F(X^*)\otimes F(X)) \arrow[uu, "\cong", "1\otimes\phi^{\otimes}_{X^*,X}"']
\end{tikzcd}$$

This follows, because the rectangle on the lower left commutes by naturality of the associator, the other rectangles on the left and the curved triangle at the bottom by the properties of the tensor product, the diagram on the right by definition of the antipode and the curved triangle on top by definition of the multiplication $m : \mathrm{End}(F)\otimes\mathrm{End}(F) \to \mathrm{End}(F)$.

By setting $\tau\otimes\rho = \Delta(\nu)$ for a natural transformation $\nu : F \to F$ in the previous diagram, we find that the component morphisms of the natural transformation $m \circ (S\otimes\mathrm{id}) \circ \Delta(\nu) : F \to F$

are given by the outer rectangle in the diagram

$$\begin{tikzcd}
F(X) \arrow[r, "1_{F(X)}"] \arrow[d, "\cong"', "l^{-1}_{F(X)}"] \arrow[rr, bend left, "m\circ(S\otimes\mathrm{id})\circ\Delta(\nu)_X"] & F(X) \arrow[r, "\epsilon(\nu)1_{F(X)}"] & F(X) \\
\mathbb{F}\otimes F(X) \arrow[d, "\cong"', "\phi^e\otimes 1"] & F(X)\otimes\mathbb{F} \arrow[u, "\cong", "r_{F(X)}"'] \arrow[r, "1\otimes\epsilon(\nu)"] & F(X)\otimes\mathbb{F} \arrow[u, "\cong", "r_{F(X)}"'] \\
F(e)\otimes F(X) \arrow[d, "F(\mathrm{coev}^R_X)\otimes 1"] & F(X)\otimes F(e) \arrow[u, "\cong", "1\otimes\phi^{e-1}"'] \arrow[r, "1\otimes\nu_e"] & F(X)\otimes F(e) \arrow[u, "\cong", "1\otimes\phi^{e-1}"'] \\
F(X\otimes X^*)\otimes F(X) \arrow[d, "\cong"', "\phi^{\otimes -1}_{X,X^*}\otimes 1"] & F(X)\otimes F(X^*\otimes X) \arrow[u, "1\otimes F(\mathrm{ev}^R_X)"'] \arrow[r, "1\otimes\nu_{X^*\otimes X}"] & F(X)\otimes F(X\otimes X^*) \arrow[u, "1\otimes F(\mathrm{ev}^R_X)"'] \\
(F(X)\otimes F(X^*))\otimes F(X) \arrow[r, "\cong", "a"'] & F(X)\otimes(F(X^*)\otimes F(X)) \arrow[u, "\cong", "1\otimes\phi^{\otimes}_{X^*,X}"'] \arrow[r, "1\otimes\Delta'(\nu)_{X^*,X}"'] & F(X)\otimes(F(X^*)\otimes F(X)). \arrow[u, "\cong", "1\otimes\phi^{\otimes}_{X^*,X}"']
\end{tikzcd}$$

The top rectangle on the right commutes by definition of the tensor product, the second rectangle on the right by definition (87) of the counit, the third rectangle on the right by naturality of ν and the rectangle at the bottom right by definition (88) of the comultiplication. The diagram on the left commutes by Exercise 14, since $F(X^*)$ is a right dual of $F(X)$, the lower three arrows on the left compose to $\mathrm{coev}_{F(X)}\otimes 1_{F(X)}$ and the lower three arrows on the right to $1_{F(X)}\otimes\mathrm{ev}_{F(X)}$. The diagram on the left thus commutes by the snake identity (14). This shows that $m\circ(S\otimes\mathrm{id})\circ\Delta(\nu)=\epsilon(\nu)\mathrm{id}_F$. The proof of the identity $m\circ(\mathrm{id}\otimes S)\circ\Delta(\nu)=\epsilon(\nu)\mathrm{id}$ is analogous. □

Before investigating the bialgebra $\mathrm{End}(F)$ further, we consider the simplest example. Suppose that the finite ring category $\mathcal{C}$ is the representation category $B-\mathrm{Mod}^{fd}$ for a finite-dimensional bialgebra B over $\mathbb{F}$ and $F: B-\mathrm{Mod}^{fd}\to\mathrm{Vect}^{fd}_{\mathbb{F}}$ is the forgetful functor.

In this case, there is a distinguished B-module, namely B as a module over itself. It is related to all other B-modules via B-linear maps: every element $m\in M$ of a B-module $(M,\rhd)$, determines a B-linear map $\rhd_m: B\to M$, $b\mapsto b\rhd m$. This allows one to characterise any natural transformation $\nu: V\to V$ by its component morphism ν_B. As right multiplication with elements of b also defines a B-linear map $R_b: B\to B$, $b\mapsto b'$, this component morphism is determined uniquely by the element $\nu_B(1)$.

Example 8.2.6: Let B be a finite-dimensional bialgebra over $\mathbb{F}$ and $V: B-\mathrm{Mod}^{fd}\to\mathrm{Vect}^{fd}_{\mathbb{F}}$ the forgetful functor. Then the map $\phi_B:\mathrm{End}(V)\to B$, $\nu\mapsto\nu_B(1)$ is a bialgebra isomorphism. If B is a Hopf algebra, it is an isomorphism of Hopf algebras.

Proof:
1. The $\mathbb{F}$-linear map ϕ is injective, since any natural transformation $\nu: V\to V$ satisfies

$$\nu_M(m)=\nu_M\circ\rhd_m(1)=V(\rhd_m)\circ\nu_B(1)=\nu_B(1)\rhd m$$

for all B-modules M and $m\in M$, where $\rhd_m: B\to M$, $b\mapsto b\rhd m$. It is surjective since any element $b\in B$ defines a natural transformation $\nu^b: V\to V$ with component morphisms $\nu^b_M: M\to M$, $m\mapsto b\rhd m$. Its naturality follows, because for any B-linear map $f: M\to N$

and any $m \in M$ one has $\nu^b_N \circ f(m) = b \rhd f(m) = f(b \rhd m) = V(f) \circ \nu^b_M(m)$, and it satisfies $\phi(\nu^b) = \nu_B(1) = b \rhd 1 = b$. As ϕ is $\mathbb{F}$-linear, it is an $\mathbb{F}$-linear isomorphism.

2. The map ϕ is an algebra homomorphism, since $\phi(\mathrm{id}_V) = \mathrm{id}_B(1) = 1$ and

$$\phi(\nu \circ \rho) = \nu_B \circ \rho_B(1) = \nu_B(\rho_B(1)) = \nu_B(1 \cdot \rho_B(1)) = \nu_B(1) \cdot \rho_B(1) = \phi(\nu) \cdot \phi(\rho),$$

where we used the B-linearity of $R_{\rho_B(1)} : B \to B$, $b \mapsto b\rho_B(1)$ and the naturality of ν_B.

By definition (87) of the counit in $\mathrm{End}(V)$ we have $\epsilon(\nu)1_{\mathbb{F}} = \nu_{\mathbb{F}}$ and hence

$$\epsilon(\nu) = \nu_{\mathbb{F}}(1_{\mathbb{F}}) = \nu_B(1) \rhd 1_{\mathbb{F}} = \epsilon(\nu_B(1)) = \epsilon \circ \phi(\nu).$$

Similarly, we have $\Delta(\nu)_{M,N} = \nu_{M\otimes N}$ by (88) for all finite-dimensional B-modules M, N and

$$(\phi \otimes \phi) \circ \Delta(\nu) = \nu_{B\otimes B}(1\otimes 1) = \nu_B(1) \rhd (1\otimes 1) = \Delta(\nu_B(1)) = \Delta \circ \phi(\nu).$$

This shows that ϕ is an isomorphism of bialgebras. As any bialgebra isomorphism between Hopf algebras is an isomorphism of Hopf algebras, the second claim follows. $\square$

In Theorem 5.1.10 and Proposition 8.1.1 we established that the representation category $B{-}\mathrm{Mod}^{fd}$ for a finite-dimensional bialgebra B over $\mathbb{F}$ is a finite ring category, equipped with a the forgetful functor V as a fibre functor. Theorem 8.2.5 shows that any finite ring category together with a fibre functor $F : \mathcal{C} \to \mathrm{Vect}^{fd}_{\mathbb{F}}$ defines a finite-dimensional bialgebra over $\mathbb{F}$. In Example 8.2.6 we saw that for the pair $(B{-}\mathrm{Mod}^{fd}, V)$ this bialgebra is isomorphic to B.

We will now show that this correspondence defines in fact an equivalence of categories between

- the category $\mathrm{Bialg}_{\mathbb{F}}$ with finite-dimensional bialgebras over $\mathbb{F}$ as objects and bialgebra homomorphisms as morphisms,
- the category $\mathrm{Fib}_{\mathbb{F}}$ whose objects are pairs $(\mathcal{C}, F)$ of a finite ring category $\mathcal{C}$ over $\mathbb{F}$ and a fiber functor $F : \mathcal{C} \to \mathrm{Vect}^{fd}_{\mathbb{F}}$. Morphisms from $(\mathcal{C}, F)$ to $(\mathcal{C}', F')$ are $\mathbb{F}$-linear functors $H : \mathcal{C}' \to \mathcal{C}$ with $FH = F'$.

Theorem 5.1.10 and Proposition 8.1.1 define a functor $\mathrm{Mod} : \mathrm{Bialg}_{\mathbb{F}} \to \mathrm{Fib}_{\mathbb{F}}$ that assigns

- to a finite-dimensional bialgebra B the pair $(B{-}\mathrm{Mod}^{fd}, V^B)$ with the forgetful functor $V^B : B{-}\mathrm{Mod}^{fd} \to \mathrm{Vect}^{fd}_{\mathbb{F}}$,
- to a bialgebra morphism $f : B \to B'$ the pullback functor $f^* : B'{-}\mathrm{Mod}^{fd} \to B{-}\mathrm{Mod}^{fd}$ with $V^B f^* = V^{B'}$ that sends a B'-module $(M, \rhd')$ to the B-module $(M, \rhd)$ with $b \rhd m = f(b) \rhd' m$ and a B'-linear map $f : M \to N$ to the B-linear map $f : M \to N$.

Theorem 8.2.5 defines a functor $\mathrm{Rec} : \mathrm{Fib}_{\mathbb{F}} \to \mathrm{Bialg}_{\mathbb{F}}$ that assigns

- to a pair $(\mathcal{C}, F)$ the bialgebra $\mathrm{End}(F)$,
- to a morphism $H : (\mathcal{C}, F) \to (\mathcal{C}', F')$, given by a functor $H : \mathcal{C}' \to \mathcal{C}$ with $FH = F'$, the bialgebra morphism $\mathrm{Rec}(H) : \mathrm{End}(F) \to \mathrm{End}(F')$, $\nu \mapsto \nu H$.

With these definitions, we can formulate the correspondence between finite-dimensional bialgebras and $\mathbb{F}$-linear abelian monoidal categories from Theorem 5.1.10, Proposition 8.1.1 and Theorem 8.2.5 as an equivalence of categories.

We can also consider the full subcategory $\mathrm{Hopf}_{\mathbb{F}} \subset \mathrm{Bialg}_{\mathbb{F}}$ with finite-dimensional Hopf algebras over $\mathbb{F}$ as objects and the full subcategory $\mathrm{Fibrg}_{\mathbb{F}} \subset \mathrm{Fibt}_{\mathbb{F}}$, whose objects are pairs $(\mathcal{C}, F)$ of a finite tensor category $\mathcal{C}$ and a fiber functor F. That $\mathrm{Fibt}_{\mathbb{F}}$ is indeed a full subcategory of $\mathrm{Fib}_{\mathbb{F}}$ follows, because every monoidal functor sends duals to duals by Exercise 14.

Corollary 8.2.7:

1. The functor $\mathrm{Rec} : \mathrm{Fib}_{\mathbb{F}} \to \mathrm{Bialg}_{\mathbb{F}}$ is an equivalence of categories.
2. It induces an equivalence of categories $\mathrm{Rec} : \mathrm{Fibt}_{\mathbb{F}} \to \mathrm{Hopf}_{\mathbb{F}}$.

Proof. 1. The bialgebra isomorphisms $\phi_B : \mathrm{End}(V^B) \to B$, $\nu \mapsto \nu_B(1)$ from Example 8.2.6 define a natural isomorphism $\phi : \mathrm{RecMod} \to \mathrm{id}_{\mathrm{Bialg}_{\mathbb{F}}}$. The naturality follows, because the morphism $\mathrm{RecMod}(f) : \mathrm{End}(V^B) \to \mathrm{End}(V^{B'})$ for a bialgebra homomorphism $f : B \to B'$ sends a natural transformation $\nu : V^B \to V^B$ to the natural transformation $\nu' = \nu f^* : V^{B'} \to V^{B'}$ with $\nu'_{B'}(1) = f(\nu_B(1))$ and hence

$$\phi_{B'} \circ \mathrm{RecMod}(f)(\nu) = \phi_{B'}(\nu' f^*) = (\nu f^*)_{B'}(1) = f(\nu_B(1)) = f \circ \phi_B(\nu).$$

for all bialgebra morphisms $f : B \to B'$ and $\nu \in \mathrm{End}(F)$.

This implies that Rec is essentially surjective, and that for all bialgebras B, B' the induced maps $\mathrm{RecMod} : \mathrm{Hom}_{\mathrm{Bialg}_{\mathbb{F}}}(B, B') \to \mathrm{Hom}_{\mathrm{Bialg}_{\mathbb{F}}}(B, B')$ are bijective. It follows that the maps

$$\mathrm{Rec} : \mathrm{Hom}_{\mathrm{Fib}_{\mathbb{F}}}((\mathcal{C}, F), (\mathcal{C}, F')) \to \mathrm{Hom}_{\mathrm{Bialg}_{\mathbb{F}}}(\mathrm{End}(F), \mathrm{End}(F')), \quad H \mapsto \mathrm{Rec}(H) \tag{90}$$

are surjective, i. e. that Rec is full. As the maps

$$\mathrm{Mod} : \mathrm{Hom}_{\mathrm{Bialg}}(B, B') \to \mathrm{Hom}_{\mathrm{Fib}}((B\!-\!\mathrm{Mod}, V^B), (B'\!-\!\mathrm{Mod}, V^{B'})), \quad f \mapsto f^*$$

are surjective by Exercise 57, it follows that the maps Rec from (90) are also injective and hence Rec is faithful. We have shown that Rec is essentially surjective and fully faithful and hence an equivalence of categories.

2. By Corollary 5.2.7 the functor Mod sends a finite-dimensional Hopf algebra, whose antipode is invertible by Theorem 6.2.9, to a finite tensor category. By Theorem 8.2.5 the functor Rec sends a finite tensor category with a fiber functor to a finite-dimensional Hopf algebra. As $\mathrm{Hopf}_{\mathbb{F}} \subset \mathrm{Bialg}_{\mathbb{F}}$ and $\mathrm{Fibt}_{\mathbb{F}} \subset \mathrm{Fib}_{\mathbb{F}}$ are full subcategories, it follows that Rec induces an equivalence of categories $\mathrm{Rec} : \mathrm{Fibt}_{\mathbb{F}} \to \mathrm{Hopf}_{\mathbb{F}}$. □

This shows that essentially all finite ring categories with fiber functors arise from representations of finite-dimensional bialgebras and essentially all finite tensor categories with fiber functors from representations of Hopf algebras. There is also a generalisation of these reconstruction results to the infinite-dimensional case. This generalisation states that bialgebras over a field $\mathbb{F}$ correspond to ring categories with fiber functors, Hopf algebras over $\mathbb{F}$ to right rigid ring categories with fiber functors and Hopf algebras over $\mathbb{F}$ with invertible antipodes to tensor categories. These correspondences can also be formulated as equivalences of categories as in Corollary 8.2.7. However, the proof of these statements requires more advanced methods, such as Deligne's tensor product and coends. For a reference see, [EGNO, Chapter 5.4].

9 Exercises

9.1 Exercises for Chapter 1

Exercise 1: The **tensor product** of two vector spaces V, W over a field $\mathbb{F}$ is the quotient

$$V \otimes W := \langle V \times W \rangle / U,$$

where $\langle V \times W \rangle$ is the free vector space generated by the set $V \times W$ and $U \subset \langle V \times W \rangle$ is the linear span of all elements of the form

$$\begin{aligned} &(\lambda v, w) - \lambda(v, w), && (v + v', w) - (v, w) - (v', w), \\ &(v, \lambda w) - \lambda(v, w), && (v, w + w') - (v, w) - (v, w') \end{aligned}$$

for $\lambda \in \mathbb{F}$, $v, v' \in V$, $w, w' \in W$. We denote by $\tau : V \times W \to V \otimes W$, $(v, w) \mapsto (v, w) + U$ the canonical projection and write $v \otimes w := \tau(v, w) = (v, w) + U$

(a) Show that the tensor product over a field $\mathbb{F}$ has the following universal property:
The map τ is $\mathbb{F}$-bilinear. For every $\mathbb{F}$-bilinear map $\phi : V \times W \to X$ into an $\mathbb{F}$-vector space X, there is a unique linear map $\phi' : V \otimes W \to X$ with $\phi' \circ \tau = \phi$

$$\begin{tikzcd} V \times W \arrow[r, "\phi"] \arrow[d, "\tau"'] & X \\ V \otimes W \arrow[ur, dashed, "\exists ! \phi'"'] & \end{tikzcd}$$

(b) Show that for any basis B of V and any basis C of W, the set $B \times C$ is a basis of $V \otimes W$.

(c) Show that for all $\mathbb{F}$-linear maps $f : V \to V'$ and $g : W \to W'$, there is a unique $\mathbb{F}$-linear map $f \otimes g : V \otimes W \to V' \otimes W'$ such that the following diagram commutes

$$\begin{tikzcd} V \times W \arrow[r, "f \times g"] \arrow[d, "\tau"] & V' \otimes W' \arrow[d, "\tau'"] \\ V \otimes W \arrow[r, dashed, "\exists ! f \otimes g"'] & V' \otimes W'. \end{tikzcd}$$

(d) Show that this defines a functor $F : \mathrm{Vect}_{\mathbb{F}} \times \mathrm{Vect}_{\mathbb{F}} \to \mathrm{Vect}_{\mathbb{F}}$, where $\mathrm{Vect}_{\mathbb{F}}$ is the category of vector spaces and linear maps over $\mathbb{F}$.

(e) Show that this gives $\mathrm{Vect}_{\mathbb{F}}$ the structure of a monoidal category.

Exercise 2:

- A **representation** of a group G on a vector space V over $\mathbb{F}$ is a group homomorphism $\rho : G \to \mathrm{Aut}_{\mathbb{F}}(V)$.
- A **morphism of representations** from $\rho : G \to \mathrm{Aut}_{\mathbb{F}}(V)$ to $\rho' : G \to \mathrm{Aut}_{\mathbb{F}}(V')$ is an $\mathbb{F}$-linear map $f : V \to V'$ with $\rho'(g)f(v) = f(\rho(g)v)$ for all $g \in G$ and $v \in V$.

The category $\mathbb{F}[G] - \mathrm{Mod}$ has as objects representations of G and morphisms of representations as morphisms. Show that it has the structure of a monoidal category with the tensor product, tensor unit, associator and unit constraints from $\mathrm{Vect}_{\mathbb{F}}$.

Exercise 3: Show that the following definition of a monoidal category is equivalent to Definition 1.1.1:

A monoidal category is a pentuple $(\mathcal{C}, \otimes, a, e, \iota)$ of a category $\mathcal{C}$, a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a natural isomorphism $a : \otimes(\otimes \times \mathrm{id}) \to \otimes(\mathrm{id} \times \otimes)$, an object e and an isomorphism $\iota : e \otimes e \to e$ such that:

(i) the natural isomorphism a satisfies the pentagon axiom,
(ii) the functors $e \otimes - : \mathcal{C} \to \mathcal{C}$ and $- \otimes e : \mathcal{C} \to \mathcal{C}$ are equivalences of categories.

Exercise 4: Let G be a group and $\omega : G \times G \times G \to \mathbb{F}^\times$ a 3-cocycle, i. e. a map that satisfies

$$\omega(gh, k, l)\omega(g, h, kl) = \omega(g, h, k)\omega(g, hk, l)\omega(h, k, l) \qquad \forall g, h, k, l \in G.$$

Let Vect_G^ω be the category of G-**graded vector spaces** over $\mathbb{F}$ with

- vector spaces V over $\mathbb{F}$ with a decomposition $V = \oplus_{g \in G} V_g$ as objects,
- $\mathbb{F}$-linear maps $f : V \to W$ with $f(V_g) \subset W_g$ as morphisms.

Show that Vect_G^ω has the structure of a monoidal category with the tensor product

$$(V \otimes W)_g = \oplus_{xy=g} V_x \otimes_{\mathbb{F}} W_y,$$

and the associator given by the linear maps

$$a_{U_g, V_h, W_k} : (U_g \otimes_{\mathbb{F}} V_h) \otimes_{\mathbb{F}} W_k \to U_g \otimes_{\mathbb{F}} (V_h \otimes_{\mathbb{F}} W_k),\ (u \otimes v) \otimes w \mapsto \omega(g, h, k) u \otimes (v \otimes w).$$

Exercise 5: A **crossed module** is a quadruple $(A, B, \rhd, \delta)$ of groups A, B, a group action $\rhd : B \times A \to A$ by automorphisms and a group homomorphism $\delta : A \to B$ such that

$$\delta(b \rhd a) = b\delta(a)b^{-1} \qquad \delta(a) \rhd a' = aa'a^{-1} \qquad \forall a, a' \in A, b \in B.$$

(a) Show that every crossed module defines a category $\mathcal{C}$ with
  - $\mathrm{Ob}\,\mathcal{C} = B$,
  - $\mathrm{Hom}_{\mathcal{C}}(b, b') = \{(a, b) \in A \times B \mid \delta(a)b = b'\}$,
  - the composition of morphisms given by $(a', \delta(a)b) \circ (a, b) = (a'a, b)$.

(b) Show that $\mathcal{C}$ has the structure of a strict monoidal category.

Exercise 6: Show that the alternative definitions of a monoidal functor and a monoidal natural transformation from Remark 1.1.12 are equivalent to the one from Definition 1.1.11.

Proceed as follows:

(a) Show that the assumptions of Remark 1.1.12 imply that there is a unique isomorphism $\phi^e : e_{\mathcal{D}} \to F(e_{\mathcal{C}})$ such that the following diagram commutes:

$$\begin{tikzcd}
e_{\mathcal{D}} \otimes F(e_{\mathcal{C}}) \arrow[r, "l_{F(e_{\mathcal{C}})}"] \arrow[d, "\phi^e \otimes 1"] & F(e_{\mathcal{C}}) \arrow[d, "F(l_{e_{\mathcal{C}}}^{-1})"] \\
F(e_{\mathcal{C}}) \otimes F(e_{\mathcal{C}}) \arrow[r, "\phi^{\otimes}_{e_{\mathcal{C}}, e_{\mathcal{C}}}"'] & F(e_{\mathcal{C}} \otimes e_{\mathcal{C}}).
\end{tikzcd}$$

(b) Show that this isomorphism satisfies the compatibility conditions with the unit constraints from Definition 1.1.11.

(c) Show that any natural transformation that satisfies the assumptions in Remark 1.1.12 satisfies the compatibility condition with the unit constraints from Definition 1.1.11.

Exercise 7: Let G_1, G_2 be groups and $\omega_i : G_i \times G_i \times G_i \to \mathbb{F}^\times$ 3-cocycles.

(a) Show that monoidal functors from $\mathrm{Vect}_{G_1}^{\omega_1}$ to $\mathrm{Vect}_{G_2}^{\omega_2}$ correspond to pairs (f, μ) of a group homomorphism $f : G_1 \to G_2$ and a map $\mu : G_1 \times G_1 \to \mathbb{F}^\times$ satisfying for all $g, h, k \in G_1$

$$\omega_1(g,h,k) = \omega_2(f(g), f(h), f(k))d\mu(g,h,k)$$
$$d\mu(g,h,k) = \mu(h,k)\mu^{-1}(gh,k)\mu(g,hk)\mu^{-1}(g,h).$$

(b) Let $(f,\mu), (f',\mu') : \mathrm{Vect}_{G_1}^{\omega_1} \to \mathrm{Vect}_{G_2}^{\omega_2}$ be monoidal functors. Show that monoidal natural transformations between (f,μ) and (f',μ') exist if and only if $f = f'$, are always monoidal natural isomorphisms and are given by maps $\eta : G_1 \to \mathbb{F}^\times$ with

$$\mu(g,h) = \mu'(g,h)d\eta(g,h) \qquad d\eta(g,h) = \eta(g)\eta(h)\eta(gh)^{-1} \qquad \forall g, h \in G_1.$$

Hint: Consider the objects $\delta^h = \oplus_{g \in G}\delta^h_g$ with $\delta^h_g = 0$ for $g \neq h$ and $\delta^h_h = \mathbb{F}$ in $\mathrm{Vect}_{G_1}^{\omega_1}$. Work with the alternative definitions from Remark 1.1.12 and Exercise 6.

Exercise 8: Let $(G_n)_{n\in\mathbb{N}_0}$ and $(H_n)_{n\in\mathbb{N}_0}$, families of groups, $(\rho_{n,m})_{n,m\in\mathbb{N}_0}$ and $(\tau_{n,m})_{n,m\in\mathbb{N}_0}$ families of group homomorphisms $\rho_{n,m} : G_m \times G_n \to G_{n+m}$ and $\tau_{m,n} : H_m \times H_n \to H_{n+m}$ that satisfy the conditions from Example 1.1.8. Denote by $\mathcal{G}$, $\mathcal{H}$ the associated strict monoidal categories from Example 1.1.8.

(a) Characterise essentially surjective strict monoidal functors $F : \mathcal{G} \to \mathcal{H}$ by families $(\mu_n)_{n\in\mathbb{N}_0}$ of group homomorphisms $\mu_n : G_n \to H_n$.

(b) Let $\mathbb{F}$ be a field. Consider the groups $H_n = \mathrm{GL}(n,\mathbb{F})$ of invertible $n \times n$-matrices with entries in $\mathbb{F}$ and

$$\tau_{m,n} : \mathrm{GL}(m,\mathbb{F}) \times \mathrm{GL}(n,\mathbb{F}) \to \mathrm{GL}(n+m,\mathbb{F}), \quad (A,B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Determine all essentially surjective strict monoidal functors $F : \mathcal{B} \to \mathcal{H}$ and $F : \mathcal{S} \to \mathcal{H}$.

Hint: For (b) note that any generator $\sigma_i \in B_n = \mathrm{Hom}_{\mathcal{B}}(n,n)$ can be expressed as a tensor product of identity morphisms and the generator $\sigma_1 \in B_2 = \mathrm{Hom}_{\mathcal{B}}(2,2)$ and similarly for the elementary transpositions $\sigma_i = (i, i+1) \in S_n$.

Exercise 9: Let $(\mathcal{C}, \otimes, e, a, l, r)$ be a monoidal category. Show that any equivalence of categories $F : \mathcal{C} \to \mathcal{D}$ defines a monoidal structure on $\mathcal{D}$ such that F is monoidal.

Hint: You can use without proof that any equivalence of categories can be made into an adjoint equivalence.

Exercise 10: Let $F : \mathcal{C} \to \mathcal{D}$ be a monoidal functor that is an equivalence of categories. Show that there is a monoidal functor $G : \mathcal{D} \to \mathcal{C}$ such that $FG : \mathcal{D} \to \mathcal{D}$ and $GF : \mathcal{C} \to \mathcal{C}$ are isomorphic to the identity functors by monoidal natural transformations.

Exercise 11: (**Mac Lane's proof of the strictification theorem**)

Let $(\mathcal{C}, \otimes, e, a, l, r)$ be a monoidal category. We consider finite lists $\ell = (C_1, .., C_n)$ of objects in $\mathcal{C}$, including the empty list $()$, and denote by $*$ the concatenation of lists:

$$(C_1, ..., C_n) * (C_{n+1}, ..., C_{n+m}) = (C_1, ..., C_n, C_{n+1}, ..., C_{n+m}) \qquad () * \ell = \ell * () = \ell.$$

We assign to each list ℓ an object $F(\ell)$ in $\mathcal{C}$ defined inductively by $F(()) = e$, $F((C)) = C$ and $F(\ell * (C)) = F(\ell) \otimes C$ for all $C \in \mathrm{Ob}\,\mathcal{C}$ and non-empty lists ℓ.

(a) Show that this defines a category $\mathcal{D}$ with finite lists of objects in $\mathcal{C}$ as objects and an equivalence of categories $F : \mathcal{D} \to \mathcal{C}$ that is the identity on the Hom-sets.

(b) Define isomorphisms $\phi_{\ell,\ell'} : F(\ell) \otimes F(\ell') \to F(\ell * \ell')$ by setting for all non-empty lists ℓ, ℓ' and $C \in \mathrm{Ob}\,\mathcal{C}$

$$\phi_{(),()} = r_e = l_e,\ \phi_{(),\ell} = l_{F(\ell)}, ; \phi_{\ell,()} = r_{F(\ell)},\ \phi_{\ell,(C)} = 1_{F(\ell)\otimes C},\ \phi_{\ell,\ell'*(C)} = (\phi_{\ell,\ell'} \otimes 1_C) \circ a^{-1}_{F(\ell),F(\ell'),C}.$$

Show that $\phi_{\ell,m*n} \circ (1_{F(\ell)} \otimes \phi_{m,n}) \circ a_{F(\ell),F(m),F(n)} = \phi_{\ell*m,n} \circ (\phi_{\ell,m} \otimes 1_{F(n)})$ for all lists ℓ, m, n.

(c) Show that $\mathcal{D}$ has the structure of a strict monoidal category with the tensor product given by $\ell \otimes \ell' = \ell * \ell'$ on the objects and on morphisms $f : \ell \to \ell'$ and $g : m \to m'$ by

$$\begin{tikzcd} F(\ell * m) \arrow[r, "F(f*g)"] & F(\ell' * m') \\ F(\ell) \otimes F(m) \arrow[u, "\phi_{\ell,m}"] \arrow[r, "F(f) \otimes F(g)"'] & F(\ell') \otimes F(m'). \arrow[u, "\phi_{\ell',m'}"'] \end{tikzcd}$$

Show that (F, ϕ) is a monoidal equivalence.

9.2 Exercises for Chapter 2

Exercise 12: Let $\mathcal{C} = \mathrm{Fun}(\mathcal{D}, \mathcal{D})$ the strict monoidal category of endofunctors $F : \mathcal{D} \to \mathcal{D}$ and natural transformations between them. Show that an object $F : \mathcal{D} \to \mathcal{D}$ is left (right) dualisable, if and only if it has a right (left) adjoint.

Exercise 13: Let G be a finite group and $\omega : G \times G \times G \to \mathbb{F}^\times$ a 3-cocycle. Show that the monoidal category $\mathrm{Vect}_G^{\omega\, fd}$ of *finite-dimensional* G-graded vector spaces is rigid. Proceed as follows:

(a) Show that the 3-cocycle ω can be made into a normalised cocycle $\omega' : G \times G \times G \to \mathbb{F}^\times$ with $\omega'(g, 1, h) = 1$ for all $g, h \in G$ by setting

$$\omega'(g, h, k) = \omega(g, h, k) d\mu(g, h, k) \qquad d\mu(g, h, k) = \mu(h, k)\mu(gh, k)^{-1}\mu(g, hk)\mu(g, h)^{-1}$$

for a suitable map $\mu : G \times G \to \mathbb{F}^\times$.

(b) Use (a) to assume without restriction of generality that ω is normalised ($\omega(g, 1, h) = 1$ for all $g, h \in G$) and show that $\mathrm{Vect}_G^{\omega\, fd}$ is rigid.

Exercise 14: Let $F, F' : \mathcal{C} \to \mathcal{D}$ be monoidal functors and $\eta : F \to F'$ a monoidal natural transformation.

(a) Show that $F(X)$ is a right (left) dualisable object in $\mathcal{D}$ for every right (left) dualisable object X in $\mathcal{C}$.

(b) Show that for every right dualisable object X in $\mathcal{C}$

$$\mathrm{ev}^R_{F'(X)} \circ (\eta_{X^*} \otimes \eta_X) = \mathrm{ev}^R_{F(X)} \qquad (\eta_X \otimes \eta_{X^*}) \circ \mathrm{coev}^R_{F(X)} = \mathrm{coev}^R_{F'(X)}$$

and for every left dualisable object X in $\mathcal{C}$

$$\mathrm{ev}^L_{F'(X)} \circ (\eta_X \otimes \eta_{^*X}) = \mathrm{ev}^L_{F(X)} \qquad (\eta_{^*X} \otimes \eta_X) \circ \mathrm{coev}^L_{F(X)} = \mathrm{coev}^L_{F'(X)}$$

Exercise 15: Let $\mathcal{C}, \mathcal{D}$ be rigid monoidal categories and $F, F' : \mathcal{C} \to \mathcal{D}$ monoidal functors. For a rigid monoidal category $\mathcal{C}$ we denote by $*^L_{\mathcal{C}}, *^R_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}^{op,op}$ the functors from Proposition 2.1.5 induced by its left and right duals.

(a) Show that for any rigid monoidal category $\mathcal{C}$, the functor $*^L_{\mathcal{C}} *^R_{\mathcal{C}}$ is naturally isomorphic to $\mathrm{id}_{\mathcal{C}}$.

(b) Show that for any monoidal functor $F : \mathcal{C} \to \mathcal{D}$, the functors $F *^R_{\mathcal{C}} : \mathcal{C} \to \mathcal{D}^{op,op}$ and $*^R_{\mathcal{D}} F : \mathcal{C} \to \mathcal{D}^{op,op}$ are naturally isomorphic.

(c) Show that any monoidal natural transformation $\eta : F \to F'$ is a monoidal isomorphism.

Hint: Use (a) and (b) to define the inverse in (c). Investigate how η interacts with the natural isomorphism in (b).

9.3 Exercises for Chapter 3

Exercise 16: Let $(\mathcal{C}, \otimes, e, a, l, r)$ be a monoidal category. Show that if $c : \otimes \to \otimes^{op}$ is a braiding, then $c' : \otimes \to \otimes^{op}$ with $c'_{U,V} = c^{-1}_{V,U} : U \otimes V \to V \otimes U$ is also a braiding.

Exercise 17: Show that a braiding in a monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, e, a, l, r)$ defines a monoidal equivalence $F : \mathcal{C} \to \mathcal{C}^{\otimes op}$, where $\mathcal{C}^{\otimes op} = (\mathcal{C}, \otimes^{op}, e, a^{-1}, r, l)$ is the monoidal category with the opposite tensor product.

Exercise 18: Consider the family $(\mathrm{GL}(n, \mathbb{F}))_{n \in \mathbb{N}_0}$ of groups $\mathrm{GL}(n, \mathbb{F})$ of invertible $n \times n$-matrices with entries in $\mathbb{F}$ and the family $(\rho_{m,n})_{m,n \in \mathbb{N}_0}$ of group homomorphisms

$$\rho_{m,n} : \mathrm{GL}(m, \mathbb{F}) \times \mathrm{GL}(n, \mathbb{F}) \to \mathrm{GL}(m+n, \mathbb{F}), \quad (A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Show that the associated monoidal category from Example 1.1.8 is symmetric.

Exercise 19: Show that for any braided monoidal category $\mathcal{C}$ there is a strict braided monoidal category $\mathcal{D}$ and a braided equivalence $F : \mathcal{D} \to \mathcal{C}$.

Hint: Use Exercise 11 and equip the category $\mathcal{D}$ constructed there with a suitable braiding.

Exercise 20: Let $\mathcal{B}$ be the braid category and $\mathcal{C}$ a strict monoidal category. Denote by $\mathcal{YB}(\mathcal{C})$ the category of Yang-Baxter operators and morphisms of Yang-Baxter operators in $\mathcal{C}$.

(a) Show that the category $\mathcal{YB}(\mathcal{C})$ is equivalent to the category $\mathrm{Fun}_{\otimes}(\mathcal{B}, \mathcal{C})$ of strict monoidal functors $F : \mathcal{B} \to \mathcal{C}$ and monoidal natural transformations between them.

(b) Show that if $\mathcal{C}$ is *braided*, then the category $\mathrm{Fun}_{br}(\mathcal{B}, \mathcal{C})$ of strict braided monoidal functors $F : \mathcal{B} \to \mathcal{C}$ and monoidal natural transformations between them is equivalent to $\mathcal{C}$.

Exercise 21: Let G be a group. Show that the category $X(G)$ of crossed G-sets from Example 3.1.4 with the tensor product and braiding given in Example 3.1.4 is a braided monoidal category.

Exercise 22: Let G be a finite group, $\mathbb{F}$ a field and Vect_G^{fd} the category of finite-dimensional G-graded vector spaces for the trivial 3-cocycle $\omega : G \times G \times G \to \mathbb{F}$, $(g, h, k) \mapsto 1$.

Denote by $\delta^g = \oplus_{h \in G} \delta_h^g$ the G-graded vector spaces with $\delta_h^g = \mathbb{F}$ for $h = g$ and $\delta_h^g = 0$ else.

(a) Consider the linear maps $c_{U,V} : U \otimes V \to V \otimes U$ with $c_{U,V}(u \otimes v) = \gamma(g, h) v \otimes u$ for $u \in U_g$, $v \in V_h$ and a map $\gamma : G \times G \to \mathbb{F}^\times$. Determine the conditions on γ under which the morphisms $c_{U,V}$ define a braiding on Vect_G^{fd}.

(b) Show that under the conditions from (a) the category Vect_G^{fd} is ribbon.

(c) Give a non-trivial example of a group G and a map $\gamma : G \times G \to \mathbb{F}^\times$ that satisfies the conditions in (a).

(d) Suppose that $f : \mathbb{F} \to \mathbb{F}$ is a morphism in Vect_G^{fd} that is composed via $\otimes$ and $\circ$ of the left and right coevaluation maps for δ_g and δ_g^* and braidings $c_{U,V}^{\pm 1}$ with $U, V \in \{\delta_g, \delta_g^*\}$. Give a simple formula for f in terms of $\gamma(g, g)$ and the geometrical properties of its diagram.

Hint: Recall from Exercise 13 that Vect_G^{fd} is pivotal with a trivial pivot, with $V^* = \bigoplus_{h \in G} V_{h^{-1}}^*$ and with evaluation maps $\mathrm{ev}^R(\alpha \otimes v) = \mathrm{ev}^L(v \otimes \alpha) = \delta_{gh} \alpha(v)$ for $v \in V_g$ and $\alpha \in V_{h^{-1}}^*$.

Exercise 23: Let G be a finite group, $\mathbb{F}$ a field and $\mathcal{C}$ the category with

- crossed G-sets $(X, \rhd, \mu)$ as objects,
- $\mathbb{F}[G]$-linear maps $f : \langle X \rangle_{\mathbb{F}} \to \langle Y \rangle_{\mathbb{F}}$ with $f(x) \in \mathrm{span}_{\mathbb{F}}\{y \in Y \mid \mu_Y(y) = \mu_X(x)\} \subset \langle Y \rangle_{\mathbb{F}}$ for all $x \in X$ as morphisms $f : (X, \rhd_X, \mu_X) \to (Y, \rhd_Y, \mu_Y)$.

Denote by $\mathcal{C}^{fin}$ the full subcategory of $\mathcal{C}$ with *finite* crossed G-sets as objects. Show:

(a) The category $\mathcal{C}$ is a braided monoidal category with the monoidal structure and braiding induced by the ones of crossed G-sets.

(b) There is a fully faithful braided monoidal functor $F : \mathcal{C} \to \mathcal{Z}(\mathbb{F}[G]\text{-Mod})$.

(c) The category $\mathcal{C}^{fin}$ is pivotal.

Exercise 24: Let G be a finite abelian group and $\mathbb{F}$ a field with $\mathrm{char}(\mathbb{F}) = 0$. We consider the pivotal braided monoidal category $\mathcal{C}^{fin}$ from Exercise 23 with

- finite crossed G-sets $(X, \rhd_X, \mu_X)$ as objects,

- $\mathbb{F}[G]$-linear maps $f : \langle X\rangle_{\mathbb{F}} \to \langle Y\rangle_{\mathbb{F}}$ with $f(x) \in \mathrm{span}_{\mathbb{F}}\{y \in Y \mid \mu_Y(y) = \mu_X(x)\} \subset \langle Y\rangle_{\mathbb{F}}$ for all $x \in X$ as morphisms $f : (X, \rhd_X, \mu_X) \to (Y, \rhd_Y, \mu_Y)$,
- monoidal structure and braiding induced by the ones of crossed G-sets: $(X, \rhd_X, \mu_X)\otimes(Y, \rhd_Y, \mu_Y) = (X \times Y, \rhd_{X\times Y}, \mu_{X\times Y})$ with

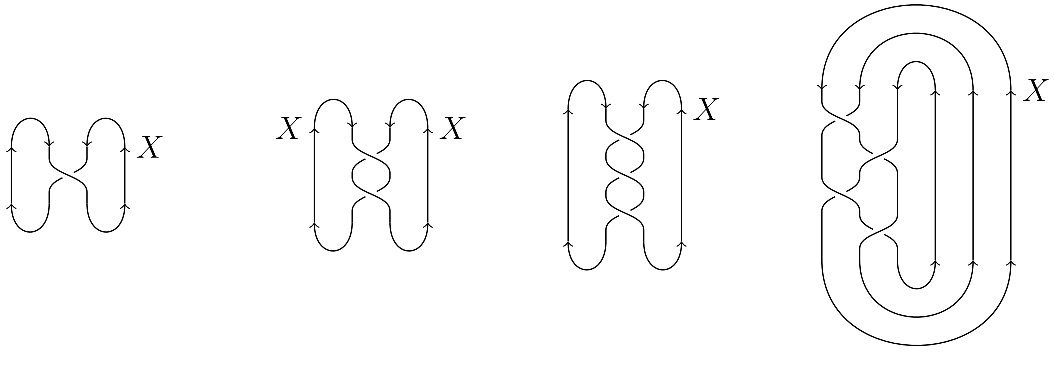
$$g \rhd_{X\times Y} (x, y) = (g \rhd_X x, g \rhd_Y y), \quad \mu_{X\times Y}(x, y) = \mu_X(x)\mu_Y(y)$$
$$c_{X,Y} : \langle X \times Y\rangle_{\mathbb{F}} \to \langle Y \times X\rangle_{\mathbb{F}},\ (x, y) \mapsto (y, \mu_Y(y)^{-1} \rhd_X x),$$

- right duals given by $(X, \rhd, \mu)^* = (X, \rhd, \iota \circ \mu)$, where $\iota : G \to G$, $g \mapsto g^{-1}$, and

$$\mathrm{ev}^R_X : \langle X \times X\rangle_{\mathbb{F}} \to \mathbb{F}, (x, x') \mapsto \delta_x(x') \qquad \mathrm{coev}^R_X : \mathbb{F} \to \langle X \times X\rangle_{\mathbb{F}}, \lambda \mapsto \lambda\Sigma_{x\in X}(x, x)$$

(a) Compute the twists for each finite crossed G-set X and show that $\mathcal{C}^{fin}$ is ribbon.

(b) Consider the crossed G-set $X = (G, \rhd, \mu)$ with $g \rhd h = ghg^{-1}$ and $\mu(g) = g$ for all $g, h \in G$ and compute the linear maps $\mathbb{F} \to \mathbb{F}$ given by the following diagrams:



Hopf link trefoil knot figure eight knot

(c) Look up the knot groups of the Hopf link, the trefoil knot and the figure eight knot and interpret the morphisms in (b).

Exercise 25: Let $q \in \mathbb{C} \setminus \{0\}$ be not a root of unity. Let V be the complex vector space with ordered basis $(v_1, ..., v_n)$ and $(\alpha^1, ..., \alpha^n)$ the dual basis of V^*. Equip V with the braided vector space structure

$$\sigma : V\otimes V \to V\otimes V, \quad v_i\otimes v_j \mapsto \begin{cases} q^{-n} v_j\otimes v_i & i < j \\ q^{1-n} v_i\otimes v_i & i = j \\ q^{-n} v_j\otimes v_i + q^{-n}(q - q^{-1}) v_i\otimes v_j & i > j. \end{cases}$$

(a) Show that V becomes a right- and left dualisable object in $\mathrm{Vect}_{\mathbb{C}}$ with the following evaluation maps and suitable right and left coevaluations

$$\mathrm{ev}^R_V : V^*\otimes V \to \mathbb{C}, \quad \alpha^j\otimes v_i \mapsto \delta_{ij} \qquad \mathrm{ev}^L_V : V\otimes V^* \to \mathbb{C}, \quad v_i\otimes\alpha^j \mapsto q^{1+n-2i}\delta_{ij}.$$

(b) Compute the twist θ_V and the twist θ'_V and show that the ribbon condition is satisfied.

9.4 Exercises for Chapter 4

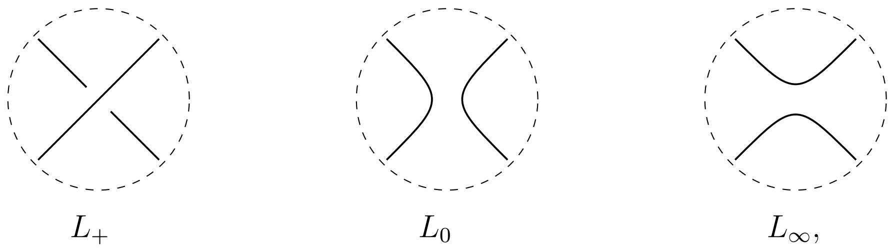
Exercise 26: Denote by $\mathcal{D}$ the set of link diagrams.
A **Kauffman bracket** is a map $\langle\ \rangle : \mathcal{D} \to \mathbb{Z}[A, B, d]$ defined by the following conditions:

(i) $\langle D\rangle = \langle D'\rangle$ if D and D' are related by an orientation preserving diffeomorphism,

(ii) $\langle O\rangle = 1$ for the unknot O,

(iii) $\langle D \amalg O\rangle = d\langle D\rangle$, where $D \amalg O$ denotes the disjoint union of D with the unknot O,

(iv) $\langle L_+\rangle = A\langle L_\infty\rangle + B\langle L_0\rangle$, if L_+, L_-, L_0 agree outside of a circle and are given inside by



(a) Determine the relations R between A, B, d that need to be quotiented out to ensure that the induced map $\langle\ \rangle' : \mathcal{D} \to \mathbb{Z}[A, B, d]/(R)$ is a ribbon invariant. In this, exclude relations of the form $d = \text{const} \in \mathbb{Z}$.

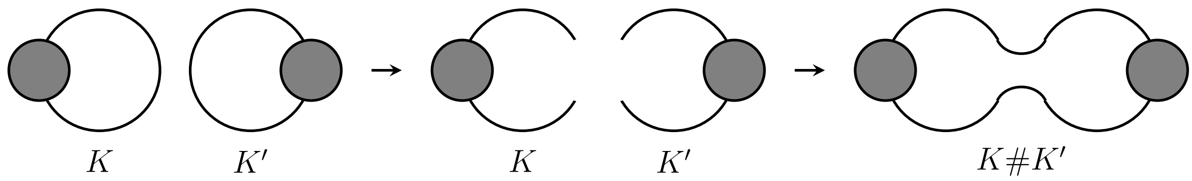
(b) Show that a suitable rescaling of $\langle\ \rangle'$ yields a link invariant.

(c) A **state** of a link diagram D is a link diagram s obtained by replacing each crossing L_+ in D either with L_0 or with L_∞. Show that the Kauffman bracket of D is given by

$$\langle D\rangle = \sum_{s \in S_D} A^{s_A} B^{s_B} d^{|s|-1},$$

where S_D is the set of states of D, s_A and s_B are the number of crossings in D replaced by L_∞ and by L_0 in s, respectively, and $|s|$ is the number of connected components of s.

(d) The connected sum $K\#K'$ of two knots K, K' is the knot obtained by cutting K, K' and joining the open ends of K to the ones of K' such that no additional crossings are created



Relate the Kauffman bracket $\langle K\#K'\rangle$ to the Kauffman brackets $\langle K\rangle$ and $\langle K'\rangle$.

Exercise 27:

(a) Compute the HOMFLY polynomial for the Hopf link and the trefoil knot.

(b) Show that the HOMFLY polynomial of an oriented knot is invariant under orientation reversal of the knot.

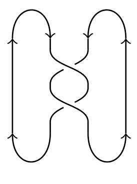
(c) Give an example of two oriented links that define the same link but do not have the same HOMFLY polynomial.

(d) The **Conway polynomial** of a link is its HOMFLY polynomial evaluated in $x = 1$. Show that the Conway polynomial of a knot is an even polynomial in y.

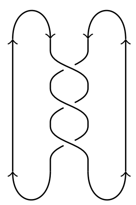
(e) Show that the Conway polynomial of a link with two connected components is an odd polynomial in y and give a simple formula for its lowest non-trivial coefficient.

Remark: A polynomial $p \in \mathbb{Z}[y]$ is called even if it is of the form $p = \Sigma_{k=0}^{\infty} a_k x^{2k}$ and odd if it is of the form $p = \Sigma_{k=0}^{\infty} a_k x^{2k+1}$, where $a_k \in \mathbb{Z}$.

Hint: In (e) consider the linking number, which is given by as one half the sum of the signs of the crossings of two different components.



Hopf link



trefoil knot

Exercise 28: Let (A, κ) be a Frobenius algebra. Show that there is a unique algebra automorphism $\rho : A \to A$, the **Nakayama automorphism**, such that $\kappa(a \otimes b) = \kappa(\rho(b) \otimes a)$ for all $a, b \in A$.

Exercise 29: Show that every Frobenius algebra (A, κ) has the structure of a (Δ, ϵ)-Frobenius algebra and vice versa.

Hint: Consider the dual map $m^* : A^* \to A^* \otimes A^*$ with $m^*(\alpha)(a \otimes b) = \alpha(a \cdot b)$ for all $\alpha \in A^*$, $a, b \in A$ and the linear isomorphism $\phi_\kappa : A \to A^*$, $a \mapsto \kappa(- \otimes a)$. Define

$$\Delta = (\phi_\kappa^{-1} \otimes \phi_\kappa^{-1}) \circ m^{op*} \circ \phi_\kappa : A \to A \otimes A \qquad \epsilon : A \to \mathbb{F}, a \mapsto \kappa(a \otimes 1) = \kappa(1 \otimes a).$$

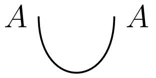
Exercise 30:

(a) Generalise the concepts of a Frobenius algebra and a (Δ, ϵ)-Frobenius algebra to general monoidal categories $\mathcal{C}$ and describe their defining properties by diagrams.

(b) Show with a *diagrammatical proof* that every (Δ, ϵ)-Frobenius algebra has the structure of a Frobenius algebra.

(c) Show with a *diagrammatical proof* that every Frobenius algebra has the structure of a (Δ, ϵ)-Frobenius algebra.

Hint: In (a) the non-degeneracy of the Frobenius form κ implies that there are diagrams



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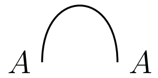
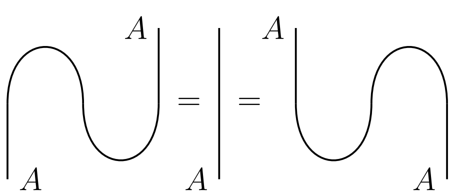


such that



Exercise 31: Use the presentation of the cobordism category $\mathrm{Cob}_{1,0}$ to classify all 1d oriented topological quantum field theories $Z : \mathrm{Cob}_{1,0} \to \mathrm{Vect}_{\mathbb{F}}^{fd}$ with $Z* = *Z$, up to monoidal natural isomorphisms.

Exercise 32: Let G be a finite group, $\mathbb{F}$ a field of characteristic zero and $\lambda \in \mathbb{F} \setminus \{0\}$. We consider the commutative Frobenius algebra $\mathrm{Map}(G, \mathbb{F})$ of maps $f : G \to \mathbb{F}$ with the pointwise addition, product and scalar multiplication and the Frobenius form

$$\kappa(f \otimes h) = \lambda \sum_{g \in G} f(g)h(g)$$

and the associated 2d topological quantum field theory $Z : \mathrm{Cob}_{2,1} \to \mathrm{Vect}_{\mathbb{F}}^{fd}$.

(a) Determine the corresponding (Δ, ϵ)-Frobenius structure on $\mathrm{Map}(G, \mathbb{F})$.

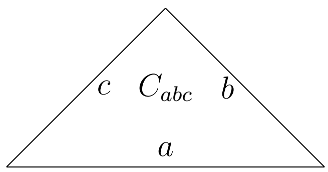
(b) Compute $Z(S)$ for the case where S is an oriented surface of genus $g \in \mathbb{N}_0$.

(c) Compute $Z(S)$ for the case where S is the disjoint union of $n \in \mathbb{N}$ tori.

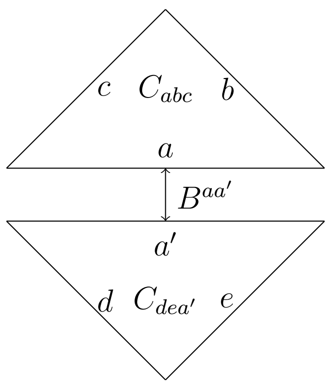
Exercise 33: (**Fukuma-Hosono-Kawai model**)

Let Σ be a triangulated oriented surface, i. e. an oriented surface with a semisimplicial complex structure, I a finite set and $\mathbb{F}$ a field. We associate to Σ a number $Z(\Sigma) \in \mathbb{F}$ defined as follows:

1. Assign to each edge in an oriented triangle t an element of I, and to the oriented triangle $t = (abc)$ formed by edges labelled with $a, b, c \in I$ a number $C_{abc} \in \mathbb{F}$, the **triangle constant**, satisfying $C_{abc} = C_{cab} = C_{bca}$.



2. Each edge in the surface Σ occurs in two adjacent triangles t, t' and carries two labels $a, a' \in I$. Assign to such an edge a number $B^{aa'} \in \mathbb{F}$, the **gluing constant**, such that the matrix $B = (B^{aa'})_{a,a' \in I}$ is symmetric and invertible.

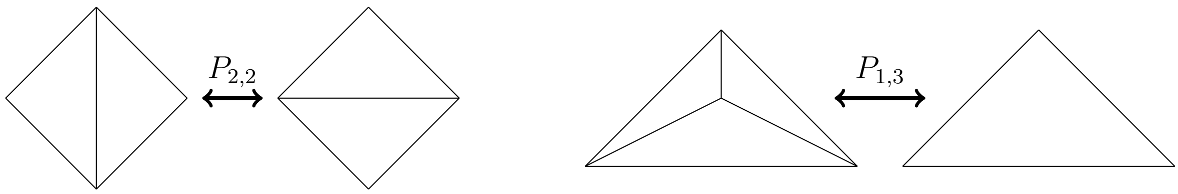


3. Assign to the triangulated surface Σ the number

$$Z(\Sigma) = R^{-V} \sum_{f: E \to I \times I} \prod_{e \in E} B^{ee'} \prod_{t=(efg) \in T} C_{efg}$$

where $R \in \mathbb{F}^{\times}$, V is the number of vertices, E the set of edges and T the set of triangles, the sum runs over all labelings $f : E \to I \times I$ and the products are taken over all labelled edges and labelled triangles.

One can show that two triangulated surfaces are homeomorphic if and only if the simplicial complex structures are related by a finite sequence of the two **Pachner moves**



Hence, if $Z(\Sigma)$ is invariant under $P_{2,2}$ and $P_{1,3}$, then $Z(\Sigma)$ depends only on the homeomorphism class of the oriented surface Σ and is a topological invariant.

Show that $Z(\Sigma)$ is a non-trivial topological invariant if and only if the constants C_{abc} and $B^{aa'}$ define a symmetric Frobenius algebra A over $\mathbb{F}$ with $m \circ \Delta = R\,\mathrm{id}$. Proceed as follows:

(a) Derive sufficient and necessary conditions on the constants C_{abc}, B^{ab}, B_{cd} for invariance of $Z(\Sigma)$ under the two Pachner moves.

(b) Consider the free vector space $A = \langle I \rangle_{\mathbb{F}}$ with basis I, define a linear map $\kappa : A \otimes A \to \mathbb{F}$ and a multiplication map $m : A \otimes A \to A$ by

$$a \cdot b = m(a \otimes b) = \sum_{c,d \in I} C_{abc} B^{cd} d \qquad \kappa(c \otimes d) = B_{cd} \qquad \forall a, b, c, d \in I,$$

where $B_{aa'}$ are the coefficients of the inverse matrix $B^{-1} = (B_{aa'})_{a,a' \in I}$. Show that the conditions from (a) imply:

(i) the multiplication is associative,

(ii) κ is a symmetric Frobenius form on A,

(iii) if A has a unit element, it is given by $1 = R^{-1} \sum_{a,b,c,d \in I} C_{abc} B^{ab} B^{cd} d$.

(c) Show that the Frobenius algebra from (b) satisfies the additional condition $m \circ \Delta = R\,\mathrm{id}$.

(d) Show that any symmetric Frobenius algebra with $m \circ \Delta = R\,\mathrm{id}$ for some $R \in \mathbb{F}^\times$ gives rise to a finite set I, triangle constants C_{abc} and gluing constants B^{ab} such that $Z(\Sigma)$ is a topological invariant.

9.5 Exercises for Chapter 5

Exercise 34: Let (C, Δ, ϵ) and (C', Δ', ϵ') be coalgebras over $\mathbb{F}$.

(a) The counit is unique: If $\epsilon'' : C \to \mathbb{F}$ is a linear map that satisfies for all $c \in C$ $r_C \circ (\mathrm{id}_C \otimes \epsilon'') \circ \Delta(c) = l_C \circ (\epsilon'' \otimes \mathrm{id}_C) \circ \Delta(c) = c$, then $\epsilon'' = \epsilon$.

(b) For every coalgebra homomorphism $\phi : C \to C'$, the kernel $\ker(\phi) \subset C$ is a coideal in C and the image $\mathrm{im}(\phi') \subset C'$ is a subcoalgebra of C'.

(c) If $\pi : C \to C/I$, $c \mapsto c + I$ the canonical surjection for a linear subspace $I \subset C$, then $\delta : C/I \to C \otimes (C/I)$, $c + I \mapsto (\mathrm{id} \otimes \pi) \circ \Delta(c)$ defines a C-left comodule structure on C/I if and only if I is a left coideal.

(d) If $I \subset C$ is a left coideal, a right coideal or a subcoalgebra of C, then $\epsilon(I) = \{0\}$ if and only if $I = \{0\}$.

Exercise 35: Show that the tensor product coalgebra $C \otimes C'$ for two coalgebras (C, Δ, ϵ) and (C', Δ', ϵ') has the following universal property:

The projection maps $\pi : C \otimes C' \to C$, $c \otimes c' \mapsto \epsilon'(c')\, c$ and $\pi' : C \otimes C' \to C'$, $c \otimes c' \mapsto \epsilon(c)\, c'$ are coalgebra homomorphisms. For every *cocommutative* coalgebra D and every pair of coalgebra homomorphisms $f : D \to C$, $f' : D \to C'$ there is a unique coalgebra homomorphism $\tilde{f} : D \to C \otimes C'$ with $\pi \circ \tilde{f} = f$ and $\pi' \circ \tilde{f} = f'$.

Exercise 36: Let V be a vector space over $\mathbb{F}$. The rank of an element $v \in V \otimes V$ is

$$\mathrm{rk}(v) = \min\{n \in \mathbb{N} \mid v = \Sigma_{i=1}^n v_i \otimes v_i' \text{ for some } v_i, v_i' \in V\}.$$

(a) Show that for $v = \Sigma_{i,j=1}^n M_{ij} v_i \otimes v_j'$ with $M_{ij} \in \mathbb{F}$, linearly independent $v_1, ..., v_n \in V$ and linearly independent $v_1', ..., v_n' \in V$ one has $\mathrm{rk}(v) = \mathrm{rk}(M)$, where $M = (M_{ij})$.

(b) Suppose that V is infinite-dimensional with a countable basis $B = \{v_i \mid i \in \mathbb{N}\}$. Show that $V^* \otimes V^* \subsetneq (V \otimes V)^*$ by considering the element $\delta : V \otimes V \to \mathbb{F}$, $v_i \otimes v_j \mapsto \delta_{ij}$ and its restrictions $\delta_n = \delta|_{V_n \otimes V_n} : V_n \otimes V_n \to \mathbb{F}$ to the subspaces $V_n = \mathrm{span}_{\mathbb{F}}\{v_1, ..., v_n\}$.

Exercise 37: Let (A, m, η) be an algebra over $\mathbb{F}$, $m^* : A^* \to (A \otimes A)^*$ and $\eta^* : A^* \to \mathbb{F}$ the dual maps of $m : A \otimes A \to A$ and $\eta : \mathbb{F} \to A$ and $A^\circ = \{\alpha \in A^* \mid m^*(\alpha) \in A^* \otimes A^*\}$.

Prove that $(A^\circ, m^*|_{A^\circ}, \eta^*|_{A^\circ})$ is a coalgebra over $\mathbb{F}$:

(a) Prove first the following fact from linear algebra:

Let V be a vector space over $\mathbb{F}$ and $\alpha^1, ..., \alpha^n \in V^*$ linearly independent. Then for all $\lambda_1, ..., \lambda_n \in \mathbb{F}$, there is a vector $v \in V$ with $\alpha^i(v) = \lambda_i$ for $i = 1, ..., n$.

(b) Use (a) and the coassociativity of $m^* : A^* \to (A \otimes A)^*$ to prove that for every element $\alpha \in A^\circ$, one has $m^*(\alpha) \in A^\circ \otimes A^\circ$.

Exercise 38: Let H be a Hopf Algebra and $a, b, c, d \in H$. Simplify the expressions for the following elements of H:

$$x = \Sigma_{(a),(b)} \; \left[S(a_{(1)}) b_{(1)} S(b_{(4)}) a_{(4)}\right] \otimes \left[a_{(3)} S(b_{(2)}) S(S^{-2}(a_{(2)}) S^{-1}(b_{(3)}))\right]$$
$$y = \Sigma_{(a),(b)} \; \epsilon(b_{(2)}) \; \left[S^2(b_{(3)}) S(a_{(1)} b_{(1)})_{(1)} a_{(2)}\right] \otimes \left[S(a_{(1)} b_{(1)})_{(2)}\right]$$
$$z = \Sigma_{(a),(b),(c),(d)} \; \left[S^{-1}(d_{(2)}) S(a_{(1)} S(b_{(2)}) S(a_{(3)}) c_{(2)} S^{-1}(d_{(1)})) a_{(2)} S(b_{(1)}) S(c_{(1)})\right] \otimes \Delta(S(a_{(4)}))$$

Exercise 39: Consider the vector space $\mathbb{F}[x]$ of polynomials with coefficients in $\mathbb{F}$ with the multiplication $m : \mathbb{F}[x] \otimes \mathbb{F}[x] \to \mathbb{F}[x]$ and comultiplication $\Delta : \mathbb{F}[x] \to \mathbb{F}[x] \otimes \mathbb{F}[x]$ given by

$$m(x^n \otimes x^m) = \binom{n+m}{n} x^{m+n} \qquad \Delta(x^m) = \sum_{n=0}^m x^n \otimes x^{m-n} \qquad \forall n, m \in \mathbb{N}_0.$$

Show that these maps define a Hopf algebra structure on $\mathbb{F}[x]$.

Exercise 40: Let B, B' be finite-dimensional bialgebras over $\mathbb{F}$. Show the following:

(a) The algebra $\mathrm{Hom}_{\mathbb{F}}(B, B')$ with the convolution product has a canonical bialgebra structure. If B, B' are Hopf algebras, it is a Hopf algebra structure.

(b) The bialgebra from (a) is isomorphic to the tensor product bialgebra structure on $B^* \otimes B'$.

Exercise 41: Let V be a vector space over $\mathbb{F}$ and $T(V)$ the tensor algebra over V with the Hopf algebra structure from Example 5.3.1. Show that the Hopf algebra structure of $T(V)$ induces a Hopf algebra structure on the symmetric algebra $S(V) = T(V)/(x \otimes y - y \otimes x)$.

Exercise 42: Prove the q-**Chu-Vandermonde formula** for the q-binomials:

$$\binom{m+n}{p}_q = \sum_{k=0}^{p} q^{(m-k)(p-k)} \binom{m}{k}_q \binom{n}{p-k}_q \qquad \forall 0 \leq p \leq n, m.$$

Exercise 43: Let q be a primitive nth root of unity and T_q Taft's Hopf algebra.

(a) Show that T_q is isomorphic as an algebra to a semidirect product $\mathbb{F}[\mathbb{Z}/n\mathbb{Z}] \ltimes \mathbb{F}[x]/(x^n)$, i. e. the vector space $\mathbb{F}[\mathbb{Z}/n\mathbb{Z}] \otimes \mathbb{F}[x]/(x^n)$ with the multiplication

$$(\bar{k} \otimes p) \cdot (\bar{m} \otimes q) = (\overline{k+m}) \otimes (p \cdot \rho(\bar{k}) q)$$

for a group homomorphism $\rho : \mathbb{Z}/n\mathbb{Z} \to \mathrm{Aut}(\mathbb{F}[x]/(x^n))$ into the group of unital algebra automorphisms of $\mathbb{F}[x]/(x^n)$. Determine the group homomorphism ρ.

(b) Compute the dual bialgebra structure on T_q^*.

(c) Show that $\alpha, \beta \in T_q^*$ with $\alpha(y^j x^k) = q^j \delta_{k,0}$ and $\beta(y^j x^k) = \delta_{k,1}$ for $j, k \in \{0, 1, ..., n-1\}$ satisfy the multiplicative relations of T_q.

(d) Show that the elements $\alpha^j \beta^k$ for $j, k \in \{0, 1, ..., n-1\}$ form a basis of the vector space T_q^*.

(e) Show that the Hopf algebra T_q is self-dual by constructing a Hopf algebra isomorphism $\phi : T_q \to T_q^*$.

Exercise 44: Let $\mathbb{F}$ be a field and $q \in \mathbb{F} \setminus \{0, 1, -1\}$.
The q-**deformed universal enveloping algebra** $U_q(\mathfrak{sl}_2)$ is the algebra over $\mathbb{F}$ with generators E, F, K, K^{-1} and relations

$$K^{\pm 1} K^{\mp 1} = 1, \quad KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}. \tag{91}$$

Show that $U_q(\mathfrak{sl}_2)$ is a Hopf algebra with the comultiplication, counit and antipode

$$\begin{aligned}
&\Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1}, && \Delta(E) = 1 \otimes E + E \otimes K, && \Delta(F) = F \otimes 1 + K^{-1} \otimes F \\
&\epsilon(K^{\pm 1}) = 1, && \epsilon(E) = 0, && \epsilon(F) = 0 \\
&S(K^{\pm 1}) = K^{\mp 1} && S(E) = -EK^{-1} && S(F) = -KF.
\end{aligned} \tag{92}$$

Exercise 45: Let $\mathbb{F}$ be a field, $q \in \mathbb{F} \setminus \{0, 1, -1\}$ and $U_q(\mathfrak{sl}_2)$ the q-deformed universal enveloping algebra from Example 5.3.9. Prove the following:

(a) The **quantum Casimir** element

$$C_q = EF + \frac{q^{-1}K + qK^{-1}}{(q-q^{-1})^2} = FE + \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2}$$

is in the centre of $U_q(\mathfrak{sl}_2)$: $C_q \cdot X = X \cdot C_q$ for all $X \in U_q(\mathfrak{sl}_2)$.

(b) The antipode of $U_q(\mathfrak{sl}_2)$ is invertible.

(c) For all elements $X \in U_q(\mathfrak{sl}_2)$ one has $S^2(X) = KXK^{-1}$.

(d) There is a unique Hopf algebra isomorphism $\phi : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2)^{cop}$ with

$$\phi(E) = F \qquad \phi(F) = E \qquad \phi(K) = K^{-1}.$$

It is called the **Cartan automorphism** of $U_q(\mathfrak{sl}_2)$.

Exercise 46: Suppose that $q \in \mathbb{C}^\times$ is not a root of unity and V a complex $U_q(\mathfrak{sl}_2)$-module.

- A **weight** of V is an eigenvalue of the linear map $\phi : V \to V$, $v \mapsto K \rhd v$.
- A **highest weight vector** of weight λ is an eigenvector $0 \neq v \in V^\lambda = \ker(\phi - \lambda \mathrm{id}_V)$ with $E \rhd v = 0$.
- V is called a **highest weight module** of weight λ, if it is generated as a $U_q(\mathfrak{sl}_2)$-module by a highest weight vector of weight λ.

Prove the following:

(a) $E \rhd V^\lambda \subset V^{q^2\lambda}$, $F \rhd V^\lambda \subset V^{q^{-2}\lambda}$ and for any highest weight vector $v \in V^\lambda$

$$E \rhd (F^{k+1} \rhd v) = \frac{\lambda(k+1)_{q^{-2}} - \lambda^{-1}(k+1)_{q^2}}{q - q^{-1}} F^k \rhd v \qquad \forall k \in \mathbb{N}_0.$$

(b) If V is an n-dimensional highest weight module with highest weight vector v, the set $B = \{v, F \rhd v, ..., F^{n-1} \rhd v\}$ is a vector space basis of V, $\lambda = \pm q^{n-1}$ and $\phi : V \to V$ is diagonalisable with eigenvalues $\pm q^{n-1}, \pm q^{n-3}, ..., \pm q^{3-n}, \pm q^{1-n}$.

(c) Every finite-dimensional $U_q(\mathfrak{sl}_2)$-module has a highest weight vector and every simple finite-dimensional $U_q(\mathfrak{sl}_2)$-module is a highest weight module.

9.6 Exercises for Chapter 6

Exercise 47: Let $\mathbb{F}$ be a field of prime characteristic char$(\mathbb{F}) = p$. A **restricted Lie algebra** over $\mathbb{F}$ is a Lie algebra $(\mathfrak{g}, [\,,\,])$ over $\mathbb{F}$ together with a map $\phi : \mathfrak{g} \to \mathfrak{g}$, $x \mapsto x^{[p]}$ that satisfies

$$(\lambda x)^{[p]} = \lambda^p x^{[p]}, \quad \mathrm{ad}_{x^{[p]}} = \mathrm{ad}_x^p = \mathrm{ad}_x \circ ... \circ ad_x, \quad (x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{k=1}^{p-1} \frac{\sigma_k(x,y)}{k}$$

where $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$, $y \mapsto [x, y]$ is the adjoint action and $\sigma_k(x, y)$ is given by

$$\mathrm{ad}^{p-1}_{\lambda x + y}(x) = \Sigma_{k=1}^{p-1} \lambda^{k-1} \sigma_k(x, y)$$

(a) Show that any algebra A over $\mathbb{F}$ is a restricted Lie algebra with the commutator as the Lie bracket and the map $\phi : A \to A$, $a \mapsto a^p$.

(b) Let $(\mathfrak{g}, [\,,\,], \phi)$ be a restricted Lie algebra and $\mathcal{U} = U(\mathfrak{g})/(x^p - x^{[p]})$. Denote by $\pi : U(\mathfrak{g}) \to \mathcal{U}$ the canonical surjection, by $\iota_{\mathfrak{g}} : \mathfrak{g} \to U(\mathfrak{g})$ the canonical inclusion and set $\tau = \pi \circ \iota_{\mathfrak{g}} : \mathfrak{g} \to \mathcal{U}$. Show that $\mathcal{U}$ is a Hopf algebra with the comultiplication, counit and antipode given by

$$\Delta(\tau(x)) = 1 \otimes \tau(x) + \tau(x) \otimes 1 \qquad \epsilon(\tau(x)) = 0 \qquad S(\tau(x)) = -\tau(x) \quad \forall x \in \mathfrak{g}.$$

(c) Show that $\mathcal{U}$ is a finite-dimensional cocommutative Hopf algebra, but that it is *not* isomorphic to the group algebra of a finite group.

Exercise 48: Let B be a bialgebra over $\mathbb{F}$.

- An **algebra object** in $B-$Mod or a B-**module algebra** is an algebra A over $\mathbb{F}$ with a B-module structure such that $m : A \otimes A \to A$ and $\eta : \mathbb{F} \to A$ are B-linear.
- A **module object** over an algebra object A in $B-$Mod is an A-module M whose action map $\rhd : A \otimes M \to M$ is B-linear.

Show the following:

(a) An algebra A over $\mathbb{F}$ with a B-module structure $\rhd : B \otimes A \to A$ is an algebra object in $B-$Mod if and only if for all $b \in B$ and $a, a' \in A$, one has

$$b \rhd (a \cdot a') = \Sigma_{(b)} (b_{(1)} \rhd a) \cdot (b_{(2)} \rhd a') \qquad b \rhd 1_A = \epsilon(b)\, 1_A.$$

(b) For any algebra object A in $B-$Mod, the submodule A^B of invariants is a subalgebra of A.

(c) Any algebra object A in $B-$Mod defines an algebra structure on the vector space $A \otimes B$ by

$$(a \otimes b) \cdot (a' \otimes b') = \Sigma_{(b)} a(b_{(1)} \rhd a') \otimes b_{(2)} b'.$$

It is called the **smash product** of A and B and denoted $A\#B$.

(d) Module objects over an algebra object A in $B-$Mod are in bijection with modules over the smash product $A\#B$.

(e) Taft's algebra T_q and the q-deformed universal enveloping algebra $U_q(\mathfrak{sl}_2)$ are isomorphic as algebras to smash products $T_q \cong A\#\mathrm{span}_{\mathbb{F}}\mathrm{Gr}(T_q)$ and $U_q(\mathfrak{sl}_2) \cong A'\#\mathrm{span}_{\mathbb{F}}\mathrm{Gr}(U_q(\mathfrak{sl}_2))$ for suitable subalgebras $A \subset T_q$ and $A' \subset U_q(\mathfrak{sl}_2)$.

Exercise 49: Determine the submodule of invariants for the following Hopf algebras H and H-module structures $\rhd : H \otimes M \to M$

(a) $H = M = \mathbb{F}[G]$ for a finite group G and $h \rhd g = hgh^{-1}$.

(b) H finite-dimensional, $M = H^*$ and $h \rhd \alpha = \Sigma_{(\alpha)} \alpha_{(2)}(h)\, \alpha_{(1)}$.

(c) H finite-dimensinal, $M = H^*$ and $h \rhd \alpha = \Sigma_{(\alpha),(h)} \alpha_{(3)}(h_{(1)}) \alpha_{(1)}(S(h_{(2)}))\, \alpha_{(2)}$.

Exercise 50: Let $q \in \mathbb{F}$ be a primitive nth root of unity. Show that the space of left and right integrals for Taft's Hopf Algebra T_q are given by

$$I_L(H) = \mathrm{span}_{\mathbb{F}}\{\Sigma_{j=0}^{n-1} y^j x^{n-1}\} \qquad I_R(H) = \mathrm{span}_{\mathbb{F}}\{\Sigma_{j=0}^{n-1} q^{-j} y^j x^{n-1}\}.$$

Exercise 51: Let H be a finite-dimensional Hopf algebra. Show the following:

(a) If there is a left integral $\ell \in H$ with $\epsilon(\ell) \neq 0$, then H is unimodular.

(b) The converse of this statement is false.

Exercise 52: An algebra $(A, m, 1_A)$ over $\mathbb{F}$ is called **separable** if there is an $A\otimes A^{op}$-linear map $\phi : A \to A\otimes A^{op}$ with $m \circ \phi = \mathrm{id}_A$, where the $A\otimes A^{op}$-module structure on A is given by $(a\otimes b) \rhd c = acb$ for $a, b, c \in A$.

(a) Show that A is separable if and only if there is an element $e \in A\otimes A^{op}$, the **separability idempotent**, with $e^2 = e$, $m(e) = 1$ and $(a\otimes 1) \cdot e = e \cdot (1\otimes a)$ for all $a \in A$.

(b) Let H be a finite-dimensional semisimple Hopf algebra. Show that H is separable by constructing a separability idempotent for H.

Hint: In (b) use the normalised Haar integral.

Exercise 53: Let $\mathbb{F}$ be a field with $\mathrm{char}(\mathbb{F}) \neq 2$. The algebra H_8 is the algebra over $\mathbb{F}$ with generators x, y, z and relations

$$xy = yx \qquad zx = yz, \qquad zy = xz, \qquad x^2 = y^2 = 1, \qquad z^2 = \tfrac{1}{2}(1 + x + y - xy).$$

Assume without proof that $\dim_{\mathbb{F}} H_8 = 8$ and that

$$\Delta(x) = x\otimes x \qquad \Delta(y) = y\otimes y \qquad \Delta(z) = \tfrac{1}{2}(z\otimes z + yz\otimes z + z\otimes xz - yz\otimes xz)$$

defines a Hopf algebra structure on H_8.

(a) Determine a basis of H_8 and the counit and antipode of the Hopf algebra structure.

(b) Determine the group $\mathrm{Gr}(H_8)$ and the Lie algebra $\mathrm{Pr}(H_8)$.

(c) Show that H_8 is semisimple and determine the left and right integrals of H_8 and H_8^*.

Remark: H_8 is an important example, because it is the lowest-dimensional semisimple Hopf algebra that is not a group algebra of a finite group

Exercise 54: Let G be a finite group and $\mathbb{F}[G]$ its group algebra over $\mathbb{F}$. Verify that the Drinfeld double $D(\mathbb{F}[G])$ is given by

$$\begin{aligned}
&(\delta_u\otimes g) \cdot (\delta_v\otimes h) = \delta_u(gvg^{-1})\delta_u\otimes gh && 1 = 1\otimes e = \Sigma_{g\in G}\, \delta_g\otimes e\\
&\Delta(\delta_u\otimes g) = \Sigma_{xy=u}\, \delta_y\otimes g\otimes\delta_x\otimes g && \epsilon(\delta_u\otimes g) = \delta_u(e)\\
&S(\delta_u\otimes g) = \delta_{g^{-1}u^{-1}g}\otimes g^{-1}
\end{aligned}$$

and that $R = \Sigma_{g\in G} 1\otimes g\otimes\delta_g\otimes e$ is a universal R-matrix for $D(\mathbb{F}[G])$.

Exercise 55: Let G be a finite group. The Drinfeld double $D(\mathbb{F}[G])$ is the vector space $\mathrm{Map}(G, \mathbb{F})\otimes\mathbb{F}[G]$ with the Hopf algebra structure

$$\begin{aligned}
&(\delta_u\otimes g) \cdot (\delta_v\otimes h) = \delta_u(gvg^{-1})\delta_u\otimes gh && 1 = 1\otimes e = \Sigma_{g\in G}\, \delta_g\otimes e\\
&\Delta(\delta_u\otimes g) = \Sigma_{xy=u}\, \delta_y\otimes g\otimes\delta_x\otimes g && \epsilon(\delta_u\otimes g) = \delta_u(e)\\
&S(\delta_u\otimes g) = \delta_{g^{-1}u^{-1}g}\otimes g^{-1}.
\end{aligned}$$

(a) Show that modules over $D(\mathbb{F}[G])$ are in bijection with modules $(V, \triangleright)$ over $\mathbb{F}[G]$ with a decomposition $V = \oplus_{g\in G} V_g$ such that $h \triangleright V_g \subset V_{hgh^{-1}}$ for all $g, h \in G$. Show that $D(\mathbb{F}[G])$-linear maps $f : V \to W$ are in bijection with $\mathbb{F}[G]$-linear maps $f : V \to W$ that satisfy $f(V_g) \subset W_g$ for all $g \in G$.

(b) Determine the direct sum decomposition of the tensor product of two $D(\mathbb{F}[G])$-modules V, W and of the tensor unit in $D(\mathbb{F}[G])$-Mod.

(c) Let $\tau_{V,W} : V\otimes W \to W\otimes V$, $v\otimes w \mapsto w\otimes v$ and $R = \Sigma_{g\in G} 1\otimes g\otimes\delta_g\otimes e \in D(\mathbb{F}[G])\otimes D(\mathbb{F}[G])$. Show that the maps $c_{V,W} = \tau_{V,W} \circ (R \triangleright -) : V\otimes W \to W\otimes V$ define a braiding on $D(\mathbb{F}[G])$-Mod by computing their action on the vector spaces in the decomposition.

9.7 Exercises for Chapter 8

Exercise 56: Prove **Schur's lemma** for abelian categories:

1. If A, A' are simple objects in an abelian category $\mathcal{A}$, then every non-zero morphism $f : A \to A'$ is an isomorphism.
2. If $\mathcal{A}$ is a locally finite $\mathbb{F}$-linear abelian category and $\mathbb{F}$ algebraically closed, then $\mathrm{End}_{\mathcal{A}}(A) \cong \mathbb{F}$ for every simple object A.

Exercise 57: Let B, B' be finite-dimensional bialgebras over $\mathbb{F}$ and $V^B : B-\mathrm{Mod} \to \mathrm{Vect}_{\mathbb{F}}$, $V^{B'} : B'-\mathrm{Mod} \to \mathrm{Vect}_{\mathbb{F}}$ the forgetful functors. Show that every functor $H : B'-\mathrm{Mod} \to B-\mathrm{Mod}$ with $V^B H = V^{B'}$ is a pullback functor: $H = \phi^*$ for a bialgebra homomorphism $\phi : B \to B'$.

A Modules over algebras

A.1 Algebras

In this subsection, we recall some basic concepts, definitions and constructions and discuss important examples of algebras. In the following we will always take *algebra* to mean *associative unital algebra*, and all algebra homomorphisms are assumed to be *unital* as well. For the tensor product of vector spaces V and W over $\mathbb{F}$, we use the notation $V\otimes W = V\otimes_{\mathbb{F}} W$.

Definition A.1:

1. An **algebra** A over a field $\mathbb{F}$ is a vector space A over $\mathbb{F}$ together with a multiplication map $\cdot : A\times A\to A$ such that $(A,+,\cdot)$ is a unital ring and $a\cdot(\lambda a') = (\lambda a)\cdot a' = \lambda(a\cdot a')$ for all $a, a'\in A$ and $\lambda\in\mathbb{F}$. An algebra A is called **commutative** if $a\cdot a' = a'\cdot a$ for all $a, a'\in A$.

2. An **algebra homomorphism** from an algebra A to an algebra B over $\mathbb{F}$ is an $\mathbb{F}$-linear map $\phi : A\to B$ that is also a unital ring homomorphism, i. e. satisfies $\phi(1_A) = 1_B$ and
$$\phi(a+a') = \phi(a)+\phi(a'),\quad \phi(\lambda a) = \lambda\phi(a),\quad \phi(a\cdot a') = \phi(a)\cdot\phi(a')\qquad \forall a, a'\in A, \lambda\in\mathbb{F}.$$

As an algebra can be viewed as a unital ring with a compatible vector space structure, the concepts of a unital subring, of a left right or two-sided ideal and of a quotient by an ideal have direct analogues for algebras. In particular, a left, right or two-sided ideal in an *algebra* A is simply a left, right or two-sided ideal in the *ring* A. That such an ideal is also a linear subspace of A follows because $\lambda a = (\lambda 1)\cdot a = a\cdot(\lambda 1)\in I$ for all $a\in I$ and $\lambda\in\mathbb{F}$. Consequently, the quotient A/I by a two sided ideal $I\subset A$ is not only a ring, but also inherits a vector space structure and hence the structure of an algebra.

Definition A.2: Let $\mathbb{F}$ be a field and A an algebra over $\mathbb{F}$.

1. A **subalgebra** of A is a subset $B\subset A$ that is an algebra with the restriction of the addition, scalar multiplication and multiplication, i. e. a subset $B\subset A$ with $1_A\in B$, $b+b'\in B$, $\lambda b\in B$, and $b\cdot b'\in B$ for all $b, b'\in B$ and and $\lambda\in\mathbb{F}$.

2. The **quotient algebra** of A by a two-sided ideal $I\subset A$ is the quotient vector space A/I with the multiplication map $\cdot : A/I\times A/I\to A/I$, $(a+I, a'+I)\mapsto aa'+I$.

There is an equivalent definition of an algebra that is formulated purely in terms of vector spaces and linear maps. For this, note that we can view the unit $1\in A$ as a linear map $\eta : \mathbb{F}\to A$, $\lambda\mapsto\lambda 1$. Similarly, we can interpret the multiplication as an $\mathbb{F}$-linear map $m : A\otimes A\to A$ instead of a map $\cdot : A\times A\to A$ that is compatible with scalar multiplication and satisfies the distributive laws. This follows because the distributive laws and the compatibility condition on scalar multiplication and algebra multiplication are equivalent to the statement that the map $\cdot$ is $\mathbb{F}$-bilinear. By the universal property of the tensor product, it therefore induces a unique linear map $m : A\otimes A\to A$ with $m(a\otimes b) = a\cdot b$. The remaining conditions are the associativity of the multiplication map m and the condition that 1 is a unit, which can be stated as follows.

Definition A.3:

1. An **algebra** (A, m, η) over a field $\mathbb{F}$ is a vector space A over $\mathbb{F}$ together with linear maps $m : A\otimes A \to A$ and $\eta : \mathbb{F} \to A$, the **multiplication** and the **unit**, such that the following two diagrams commute

$$\begin{tikzcd}
A\otimes(A\otimes A) \arrow[r, "\cong"] \arrow[dr, "\mathrm{id}\otimes m"'] & (A\otimes A)\otimes A \arrow[r, "m\otimes\mathrm{id}"] & A\otimes A \arrow[d, "m"] \\
& A\otimes A \arrow[r, "m"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{F}\otimes A \arrow[r, "\lambda\otimes a\mapsto\lambda a", "\cong"'] \arrow[dr, "\eta\otimes\mathrm{id}"'] & A & A\otimes\mathbb{F} \arrow[l, "a\otimes\lambda\mapsto\lambda a"', "\cong"] \arrow[dl, "\mathrm{id}\otimes\eta"] \\
& A\otimes A. \arrow[u, "m"] &
\end{tikzcd}$$

associativity **unitality**

An algebra A is called **commutative** if $m^{op} := m \circ \tau = m$, where $\tau : A\otimes A \to A\otimes A$, $a\otimes b \mapsto b\otimes a$ is called the **flip map**.

2. An **algebra homomorphism** from an $\mathbb{F}$-algebra A to an $\mathbb{F}$-algebra B is a linear map $\phi : A \to B$ such that the following two diagrams commute

$$\begin{tikzcd}
A\otimes A \arrow[r, "m_A"] \arrow[d, "\phi\otimes\phi"'] & A \arrow[d, "\phi"] \\
B\otimes B \arrow[r, "m_B"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{F} \arrow[r, "\eta_A"] \arrow[d, "\mathrm{id}"'] & A \arrow[d, "\phi"] \\
\mathbb{F} \arrow[r, "\eta_B"'] & B.
\end{tikzcd}$$

Remark A.4:

1. Definition A.3 can be generalised by replacing the field $\mathbb{F}$ with a commutative unital ring k, $\mathbb{F}$-vector spaces by k-modules, tensor products of vector spaces by tensor products of k-modules and $\mathbb{F}$-linear maps by k-linear maps. This defines an **algebra over a commutative unital ring**.

2. Note that the multiplication from Definition A.3 is a *structure* - there may be many associative multiplication maps on a vector space A. The existence of a unit map η that satisfies the conditions in Definition A.3 is a *property* of m. As two-sided units in monoids are unique, there is at most one unit for m.

Example A.5:

1. Every field $\mathbb{F}$ is an algebra over itself. If $\mathbb{F} \subset \mathbb{G}$ is a field extension, then $\mathbb{G}$ is an algebra over $\mathbb{F}$.
2. For every field $\mathbb{F}$, the $(n \times n)$-matrices with entries in $\mathbb{F}$ form an algebra $\mathrm{Mat}(n \times n, \mathbb{F})$ with the matrix addition, scalar multiplication and matrix multiplication. The diagonal matrices, the upper triangular matrices and the lower triangular matrices form subalgebras of $\mathrm{Mat}(n \times n, \mathbb{F})$.

3. For any $\mathbb{F}$-vector space V, the linear endomorphisms of V form an algebra $\mathrm{End}_{\mathbb{F}}(V)$ with the pointwise addition and scalar multiplication and composition.

4. For any algebra A, the vector space A with the opposite multiplication $m^{op} : A\otimes A \to A$, $a\otimes b \mapsto b \cdot a$ is an algebra. It is called the **opposite algebra** and denoted A^{op}.

5. For two $\mathbb{F}$-algebras A and B, the vector space $A\otimes B$ has a canonical algebra structure with multiplication and unit

$$\begin{aligned} m_{A\otimes B} &: (A\otimes B)\otimes(A\otimes B) \to A\otimes B & \qquad (a\otimes b)\otimes(a'\otimes b') &\mapsto (aa')\otimes(bb') \\ \eta_{A\otimes B} &: \mathbb{F} \to A\otimes B, & \lambda &\mapsto \lambda(1\otimes 1). \end{aligned}$$

   This algebra is called the **tensor product** of the algebras A and B and denoted $A\otimes B$.

6. The maps $f : \mathbb{N}_0 \to \mathbb{F}$ form an associative algebra over $\mathbb{F}$ with

$$(f+g)(n) = f(n) + g(n) \qquad (\lambda f)(n) = \lambda\, f(n) \qquad (f\cdot g)(n) = \Sigma_{k=0}^n f(n-k)g(k).$$

   This is called the algebra of **formal power series** with coefficients in $\mathbb{F}$ and denoted $\mathbb{F}[[x]]$. The name is due to the following. If we describe a power series $\Sigma_{n\in\mathbb{N}_0} a_n x^n$ by its coefficient function $f : \mathbb{N}_0 \to \mathbb{F}$, $n \mapsto a_n$, then the formulas above give the familiar addition, scalar multiplication and multiplication law for power series

$$\begin{aligned} &\Sigma_{n\in\mathbb{N}_0} a_n x^n + \Sigma_{n\in\mathbb{N}_0} b_n x^n = \Sigma_{n\in\mathbb{N}_0} (a_n + b_n) x^n \\ &\lambda \Sigma_{n\in\mathbb{N}_0} a_n x^n = \Sigma_{n\in\mathbb{N}_0} \lambda a_n x^n \\ &(\Sigma_{n\in\mathbb{N}_0} a_n x^n)\cdot(\Sigma_{n\in\mathbb{N}_0} b_n x^n) = \Sigma_{n\in\mathbb{N}_0} (\Sigma_{k=0}^n a_{n-k} b_k) x^k. \end{aligned}$$

7. The polynomials with coefficients in $\mathbb{F}$ form a subalgebra

$$\mathbb{F}[x] = \{f : \mathbb{N}_0 \to \mathbb{F} \,|\, f(n) = 0 \text{ for almost all } n \in \mathbb{N}_0\} \subset \mathbb{F}[[x]].$$

8. For any set M and any $\mathbb{F}$-algebra A algebra, the maps $f : M \to A$ form an algebra over $\mathbb{F}$ with the pointwise addition, scalar multiplication and multiplication.

An important example of an algebra that will be used extensively in the following is the tensor algebra of a vector space V over $\mathbb{F}$. As a vector space, it is the direct sum $T(V) = \oplus_{n=0}^\infty V^{\otimes n}$, where $V^{\otimes 0} := \mathbb{F}$ and $V^{\otimes n} := V\otimes ...\otimes V$ is the n-fold tensor product of V with itself for $n \in \mathbb{N}$. Its algebra structure is given by the concatenation, and the unit is the element $1 = 1_\mathbb{F} \in \mathbb{F}$. The symmetric and the exterior algebra of V are two further examples of algebras associated with a vector space V. They are obtained by taking quotients of $T(V)$ by two-sided ideals.

Example A.6: Let V be a vector space over $\mathbb{F}$.

1. The **tensor algebra** of V is the vector space $T(V) = \oplus_{n=0}^\infty V^{\otimes n}$ with the multiplication

$$(v_1\otimes ...\otimes v_m)\cdot(w_1\otimes ...\otimes w_n) = v_1\otimes ...\otimes v_m \otimes w_1 \otimes ...\otimes w_n$$

   for all $v_1, ..., v_m, w_1, ..., w_n \in V$ and $n, m \in \mathbb{N}_0$, where $v_1\otimes ...\otimes v_n := 1_\mathbb{F}$ for $n = 0$. It is an algebra over $\mathbb{F}$ with unit $1_\mathbb{F} \in V^0$. The injective $\mathbb{F}$-linear map $\iota_V : V \to T(V)$, $v \mapsto v$ is called the **inclusion map**.

2. If B is a basis of V, then $B_\otimes = \{b_1\otimes ...\otimes b_n \,|\, n \in \mathbb{N}_0, b_i \in B\}$ is a basis of $T(V)$.

3. The tensor algebra is $\mathbb{N}$-**graded**: it is given as the direct sum $T(V) = \oplus_{n=0}^\infty V^{\otimes n}$ of the linear subspaces $V^{\otimes n}$, and one has $V^{\otimes m}\cdot V^{\otimes n} \subset V^{\otimes(n+m)}$ for all $m, n \in \mathbb{N}_0$.

4. The tensor algebra has the following **universal property**:

   For every $\mathbb{F}$-linear map $\phi : V \to A$ into an $\mathbb{F}$-algebra A, there is a unique algebra homomorphism $\tilde{\phi} : T(V) \to A$ such that the following diagram commutes

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, hook, "\iota_V"'] & A \\ T(V). \arrow[ur, dashed, "\exists!\tilde{\phi}"'] & \end{tikzcd}$$

5. The **symmetric algebra** of V is the quotient algebra of $T(V)$

$$S(V) = T(V)/(v\otimes w - w\otimes v)$$

   by the two-sided ideal $(v\otimes w - w\otimes v)$ generated by the elements $v\otimes w - w\otimes v$ for $v, w \in V$.

6. The **exterior algebra** of V or **alternating algebra** of V is the quotient algebra of $T(V)$

$$\Lambda V = T(V)/(v\otimes w + w\otimes v)$$

   by the two-sided ideal $(v\otimes w + w\otimes v)$ generated by the elements $v\otimes w + w\otimes v$ for $v, w \in V$.

7. The tensor algebra of the vector space $\mathbb{F}$ is the algebra of polynomials in $\mathbb{F}$: $T(\mathbb{F}) = \mathbb{F}[x]$.

Tensor algebras play a similar role for algebras as free groups for groups and free modules for modules. They allow one to describe an algebra in terms of *generators* and *relations*.

The universal property of the tensor algebra implies that any algebra A over $\mathbb{F}$ is isomorphic to a quotient of a tensor algebra by a suitable two-sided ideal. This follows, because the identity map $\mathrm{id}_A : A \to A$ induces a surjective algebra homomorphism $\phi : T(A) \to A$. Its kernel is a two-sided ideal in $T(A)$ and $T(A)/\ker(\phi) \cong A$ as an algebra.

In practice, it is inconvenient to describe an algebra as a quotient of its own tensor algebra. Rather, one considers a set B and the associated free vector space $\langle B\rangle$, such that the induced algebra homomorphism $\phi : T(\langle B\rangle) \to A$ is surjective. One then chooses a subset $U \subset T(\langle B\rangle)$ such that the two-sided ideal $(U) \subset T(\langle B\rangle)$ generated by U is $(U) = \ker(\phi)$. One then has $A \cong T(\langle B\rangle)/(U)$ and calls (B, U) a *presentation* of A.

Definition A.7: Let A be an algebra over $\mathbb{F}$. A **presentation** of A is a pair (B, U) of a set B and a subset $U \subset T(\langle B\rangle)$ such that A is isomorphic to $T(\langle B\rangle)/(U)$. The elements of B are called **generators** and the elements of U **relations**. One often lists the relations $u \in U$ as equations $u = 0$ for $u \in U$.

As the set $B_\otimes$ from Example A.6 is a basis of $T(\langle B\rangle)$ every element of A is a linear combination of equivalence classes of tensor products of elements in B, where 1_A is viewed as the empty tensor product. This yields a generating set of A, but not a basis, unless $U = \emptyset$.

The sets B and U are in general not unique, and one usually presents an algebra A with as few generators and relations as possible. In particular, one requires that no proper subset $U' \subsetneq U$ generates $\ker(\phi)$. Even if these additional conditions are imposed, an algebra may have many different presentations that are not related in an obvious way. In general, it is very difficult to decide if two algebras presented in terms of generators and relations are isomorphic, and there are no algorithms that solve this problem in general.

In some textbooks, presentations of algebras are defined in terms of the **free algebra** generated by a set B and relations in this free algebra. This is equivalent to our definition, since the tensor algebra of a vector space V with basis B is canonically isomorphic to the *free algebra* generated by the set B - both are characterised by the same universal property. For a detailed discussion, see [Ka, Chapter I.2 and Chapter II.5].

Another important example in the following is the *universal enveloping algebra* of a Lie algebra, which is obtained as a quotient of its tensor algebra. It has a universal property that relates Lie algebra homomorphisms and modules over Lie algebras to algebra homomorphisms and modules over algebras.

Definition A.8: Let $\mathbb{F}$ be a field.

1. A **Lie algebra** over $\mathbb{F}$ is an $\mathbb{F}$-vector space $\mathfrak{g}$ together with an alternating $\mathbb{F}$-bilinear map $[\,,\,]: \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$, $(x,y)\mapsto[x,y]$, the **Lie bracket**, that satisfies the **Jacobi identity**
$$[x,[y,z]]+[z,[x,y]]+[y,[z,x]]=0 \qquad \forall x,y,z\in\mathfrak{g}.$$
2. A **Lie algebra homomorphism** from $\mathfrak{g}$ to $\mathfrak{h}$ is an $\mathbb{F}$-linear map $\phi:\mathfrak{g}\to\mathfrak{h}$ with
$$[\,,\,]_{\mathfrak{h}}\circ(\phi\otimes\phi)=\phi\circ[\,,\,]_{\mathfrak{g}}.$$

Every associative (not necessarily unital) algebra A has a canonical Lie algebra structure with the **commutator** $[\,,\,]: A\otimes A\to A$, $a\otimes b\mapsto[a,b]=a\cdot b-b\cdot a$ as the Lie bracket, whose Jacobi identity follows from the associativity of A. If we speak about the Lie algebra structure of an associative algebra or Lie algebra homomorphisms into an associative algebra A, we assume that A is equipped with this Lie bracket.

Example A.9: Let $\mathfrak{g}$ be a Lie algebra.

1. The **universal enveloping algebra** of $\mathfrak{g}$ is the quotient algebra $U(\mathfrak{g})=T(\mathfrak{g})/I$, where $I=(x\otimes y-y\otimes x-[x,y])$ is the two-sided ideal generated by the elements $x\otimes y-y\otimes x-[x,y]$ for $x,y\in\mathfrak{g}$.
2. The universal enveloping algebra has the following **universal property**:

   The **inclusion maps** $\iota_{\mathfrak{g}}:\mathfrak{g}\to U(\mathfrak{g})$, $x\mapsto x+I$ are Lie algebra homomorphisms. For any Lie algebra homomorphism $\phi:\mathfrak{g}\to A$ into an algebra A, there is a unique algebra homomorphism $\tilde\phi: U(\mathfrak{g})\to A$ such that the following diagram commutes
$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\phi"] \arrow[d, hook, "\iota_{\mathfrak{g}}"'] & A \\ U(\mathfrak{g}). \arrow[ur, dashed, "\exists!\tilde\phi"'] & \end{tikzcd}$$
3. If $B=(b_i)_{i\in I}$ is an ordered basis of $\mathfrak{g}$, then the Lie bracket of $\mathfrak{g}$ is given by
$$[b_i,b_j]=\Sigma_{k\in I}f_{ij}{}^k b_k$$
   with $f_{ij}{}^k=-f_{ji}{}^k\in\mathbb{F}$, $f_{ij}^k=0$ for almost all $k\in I$. The universal enveloping algebra $U(\mathfrak{g})$ is presented with generators b_i, $i\in I$, and relations
$$b_i\otimes b_j-b_j\otimes b_i=[b_i,b_j]=\Sigma_{k\in I}f_{ij}{}^k b_k.$$

The set $B' = \{\iota_{\mathfrak{g}}(b_{i_1}) \cdots \iota_{\mathfrak{g}}(b_{i_n}) \mid n \in \mathbb{N}_0, i \in I, i_1 \leq i_2 \leq ... \leq i_n\}$ is a basis of $U(\mathfrak{g})$, the **Poincaré-Birkhoff-Witt basis** of $U(\mathfrak{g})$.

4. The universal enveloping algebra is a **filtered algebra**:

   It is the union $U(\mathfrak{g}) = \cup_{n=0}^{\infty} U^n(\mathfrak{g})$ of subspaces $U^n(\mathfrak{g}) = \oplus_{k=0}^{n} V^{\otimes k}/I$, which satisfy $U^0(\mathfrak{g}) \subset U^1(\mathfrak{g}) \subset ...$ and $U^m(\mathfrak{g}) \cdot U^n(\mathfrak{g}) \subset U^{n+m}(\mathfrak{g})$ for all $n, m \in \mathbb{N}_0$.

The universal property of the universal enveloping algebra follows from the universal property of the tensor algebra and the fact that for any Lie algebra morphism $\phi : \mathfrak{g} \to A$, the induced algebra homomorphism $\phi' : T(V) \to A$ satisfies $\phi'(x \otimes y - y \otimes x - [x, y]) = 0$ for all $x, y \in \mathfrak{g}$. The proof of the Poincaré-Birkhoff-Witt Theorem, which states that the Poincaré-Birkhoff-Witt basis is a basis of $U(\mathfrak{g})$, and the proof that $U(\mathfrak{g})$ is filtered are more cumbersome and proceed by induction. These proofs and more details on universal enveloping algebras can be found in [Di] and [Se, Chapter II].

Other important examples of algebras are *group algebras*. The group algebra of a group G is simply its group ring $R[G]$, in the case where the ring $R = \mathbb{F}$ is a field. In this case, the group ring becomes an algebra over $\mathbb{F}$ with the pointwise multiplication by $\mathbb{F}$ as scalar multiplication.

Example A.10: (The **group algebra** $\mathbb{F}[G]$)
Let G be a group and $\mathbb{F}$ a field. The free $\mathbb{F}$-vector space generated by G

$$\langle G \rangle_{\mathbb{F}} = \{f : G \to \mathbb{F} \mid f(g) = 0 \text{ for almost all } g \in G\}$$

with the pointwise addition and scalar multiplication and the **convolution product**:

$$(f_1 + f_2)(g) = f_1(g) + f_2(g) \qquad (\lambda f)(g) = \lambda f(g) \qquad (f_1 \cdot f_2)(g) = \Sigma_{h \in G} f_1(gh^{-1}) \cdot f_2(h)$$

is an associative unital $\mathbb{F}$-algebra, called the group **group algebra** of G and denoted $\mathbb{F}[G]$. The maps $\delta_g : G \to \mathbb{F}$ with $\delta_g(g) = 1$ and $\delta_g(h) = 0$ for $g \neq h$ form a basis of $\mathbb{F}[G]$.

Remark A.11:

1. In terms of the maps $\delta_g : G \to \mathbb{F}$ the multiplication of $\mathbb{F}[G]$ takes the form $\delta_g \cdot \delta_h = \delta_{gh}$ for all $g, h \in G$. We therefore write g for δ_g and denote elements of $\mathbb{F}[G]$ by $f = \Sigma_{g \in G} \lambda_g \, g$ with $\lambda_g \in \mathbb{F}$ for all $g \in G$. The algebra structure of $\mathbb{F}[G]$ is then given by

$$\begin{aligned}
&(\Sigma_{g \in G} \lambda_g \, g) + (\Sigma_{h \in G} \mu_h \, h) = \Sigma_{g \in G} (\lambda_g + \mu_g) \, g \\
&\lambda \, (\Sigma_{g \in G} \lambda_g \, g) = \Sigma_{g \in G} (\lambda \lambda_g) \, g \\
&(\Sigma_{g \in G} \lambda_g \, g) \cdot (\Sigma_{h \in G} \mu_h \, h) = \Sigma_{g \in G} \left(\Sigma_{h \in G} \lambda_{gh^{-1}} \mu_h\right) \, g.
\end{aligned}$$

2. A group homomorphism $\rho : G \to H$ induces an algebra homomorphism $\phi_\rho : \mathbb{F}[G] \to \mathbb{F}[H]$, $\Sigma_{g \in G} \lambda_g \, g \mapsto \Sigma_{g \in G} \lambda_g \, \rho(g)$, but not every algebra homomorphism $\phi : \mathbb{F}[G] \to \mathbb{F}[H]$ arises from a group homomorphism. Similarly, for every subgroup $U \subset G$, the linear subspace $\mathrm{span}_{\mathbb{F}}(U) \cong \mathbb{F}[U] \subset \mathbb{F}[G]$ is a subalgebra, but not all subalgebras of $\mathbb{F}[G]$ arise this way.

A.2 Basic representation theory of algebras

In this section, we discuss basic properties and examples of modules over algebras. As an algebra A over $\mathbb{F}$ is a unital ring with a compatible vector space structure over $\mathbb{F}$, a module over an algebra is simply defined as a module over the underlying ring. In particular, this ensures that all known constructions for modules over rings such as submodules, quotients, direct sums, products and tensor products are defines and all known results about modules over rings can be applied to modules over algebras.

The only difference to modules over general rings is that modules over an algebra A are vector spaces over $\mathbb{F}$ and all module homomorphisms between them are $\mathbb{F}$-linear maps. We can thus apply concepts from linear algebra to modules over an algebra A and module homomorphisms.

Definition A.12: Let $\mathbb{F}$ be a field and A an algebra over $\mathbb{F}$.

1. A **left module** over A or a **representation** of A is an abelian group $(V,+)$ together with a map $\rhd : A \times V \to V$, $(a,v) \mapsto a \rhd v$ that satisfies for all $a,b \in A$, $v,v' \in V$

   $$a \rhd (v+v') = a \rhd v + a \rhd v', \quad (a+b) \rhd v = a \rhd v + b \rhd v, \quad (a \cdot b) \rhd v = a \rhd (b \rhd v), \quad 1 \rhd v = v.$$

2. A **homomorphism of representations**, an **A-linear map** or a **homomorphism of A-left modules** from $(V, \rhd_V)$ to $(W, \rhd_W)$ is a group homomorphism $\phi : (V,+) \to (W,+)$ with $\phi(a \rhd_V v) = a \rhd_W \phi(v)$ for all $a \in A$ and $v \in V$.

Definition A.13: Let $\mathbb{F}$ be a field and A an algebra over $\mathbb{F}$.

1. The **dimension** of an A-module $(V, \rhd)$ is the dimension $\dim_{\mathbb{F}} V$ of the vector space V.

2. For an A-linear map $f : V \to W$ between finite-dimensional A-modules V, W, the rank, defect, determinant, trace of f are defined as the rank, defect, determinant, trace of the $\mathbb{F}$-linear map f.

Note that the left modules over A form a category A-Mod. The objects of A-Mod are left modules over A, and the morphisms of A-Mod are homomorphisms of A-left modules. There are analogous concepts of right modules over A and of (A,A)-bimodules. The former are equivalent to left modules over the algebra A^{op} and the latter to left modules over the algebra $A \otimes A^{op}$. In the following we use the term *module* over A as a synonym of *left module* over A.

Remark A.14:

1. A representation of a $\mathbb{F}$-algebra A can be defined equivalently as a pair (V, ρ) of an $\mathbb{F}$-vector space V and an algebra homomorphism $\rho : A \to \mathrm{End}_{\mathbb{F}}(V)$.

   This holds because every A-module $(V, \rhd)$ has a canonical $\mathbb{F}$-vector space structure with the scalar multiplication $\lambda v := (\lambda 1) \rhd v$, and the map $\rho : A \to \mathrm{End}_{\mathbb{F}}(V)$ with $\rho(a)v := a \rhd v$ is an algebra homomorphism. Conversely, each algebra homomorphism $\rho : A \to \mathrm{End}_{\mathbb{F}}(V)$ determines an A-left module structure on V given by $a \rhd v := \rho(a)v$.

2. A representation of a group G over $\mathbb{F}$ can be defined equivalently as
   - a module V over the group algebra $\mathbb{F}[G]$,

- a pair (V, ρ) of a vector space V and a group homomorphism $\rho : G \to \mathrm{Aut}_{\mathbb{F}}(V)$.

This follows because an $\mathbb{F}[G]$-module structure on V defines an algebra homomorphism $\rho' : \mathbb{F}[G] \to \mathrm{End}_{\mathbb{F}}(V)$ by 1. As every algebra homomorphism $\rho' : \mathbb{F}[G] \to \mathrm{End}_{\mathbb{F}}(V)$ induces a group homomorphism $\rho : G \to \mathrm{End}_{\mathbb{F}}(V)$, $g \mapsto \rho'(\delta_g)$ and vice versa, this corresponds to the choice group homomorphism $\rho : G \to \mathrm{End}_{\mathbb{F}}(V)$. As $\rho(g^{-1}) = \rho(g)^{-1}$ for all $g \in G$, one has $\rho(g) \in \mathrm{Aut}_{\mathbb{F}}(V)$ for all $g \in G$.

3. Equivalently, a representation of an $\mathbb{F}$-algebra A is a pair $(V, \rhd)$ of an $\mathbb{F}$-vector space V and an $\mathbb{F}$-linear map $\rhd : A \otimes V \to V$ such that the following diagrams commute

$$\begin{tikzcd}
(A\otimes A)\otimes V \arrow[r, "\cong"] \arrow[dr, "m\otimes \mathrm{id}"'] & A\otimes(A\otimes V) \arrow[r, "\mathrm{id}\otimes\rhd"] & A\otimes V \arrow[d, "\rhd"] \\
& A\otimes V \arrow[r, "\rhd"'] & V
\end{tikzcd}
\qquad
\begin{tikzcd}
V & \mathbb{F}\otimes V \arrow[l, "\lambda\otimes v\mapsto \lambda v"', "\cong"] \arrow[dl, "\eta\otimes\mathrm{id}"] \\
A\otimes V. \arrow[u, "\rhd"] &
\end{tikzcd}$$

A homomorphism of representations can then be defined as a $\mathbb{F}$-linear map $\phi : V \to W$ such that the following diagram commutes

$$\begin{tikzcd}
A\otimes V \arrow[r, "\rhd_V"] \arrow[d, "\mathrm{id}\otimes\phi"'] & V \arrow[d, "\phi"] \\
A\otimes W \arrow[r, "\rhd_W"] & W.
\end{tikzcd}$$

Example A.15:

1. Any group G can be represented on any $\mathbb{F}$-vector space V by the **trivial representation** $\rho : G \to \mathrm{Aut}_{\mathbb{F}}(V)$, $g \mapsto \mathrm{id}_V$.

2. Any $\mathbb{F}$-vector space V carries representations of $\mathrm{Aut}_{\mathbb{F}}(V)$ and $\mathrm{End}_{\mathbb{F}}(V)$.

3. A representation of the group $\mathbb{Z}/2\mathbb{Z}$ on an $\mathbb{F}$-vector space V corresponds to the choice of an involution on V, i. e. an $\mathbb{F}$-linear map $I : V \to V$ with $I \circ I = \mathrm{id}_V$. If $\mathrm{char}(\mathbb{F}) \neq 2$, this amounts to a decomposition $V = V_+ \oplus V_-$, where $V_\pm = \ker(I \mp \mathrm{id}_V)$.

4. A representations of the group $\mathbb{Z}$ on an $\mathbb{F}$-vector space V corresponds to the choice of an automorphism $\phi \in \mathrm{Aut}_{\mathbb{F}}(V)$. This holds because a group homomorphism $\rho : \mathbb{Z} \to \mathrm{Aut}_{\mathbb{F}}(V)$ is determined uniquely by the automorphism $\rho(1) = \phi$, and every automorphism $\phi \in \mathrm{Aut}_{\mathbb{F}}(V)$ determines a representation of $\mathbb{Z}$ given by $\rho(z) = \phi^z$.

5. For any $\mathbb{F}$-vector space V there is a representation of S_n on $V^{\otimes n}$, which is given by $\rho : S_n \to \mathrm{Aut}_{\mathbb{F}}(V^{\otimes n})$ with $\rho(\sigma)(v_1 \otimes ... \otimes v_n) = v_{\sigma(1)} \otimes .... \otimes v_{\sigma(n)}$ for all $v_1, .., v_n \in V$.

6. A representation of the polynomial algebra $\mathbb{F}[x]$ on an $\mathbb{F}$-vector space V amounts to the choice of an endomorphism of V.

   This follows because any algebra homomorphism $\phi : \mathbb{F}[x] \to \mathrm{End}_{\mathbb{F}}(V)$ is determined uniquely by $\rho(x) \in \mathrm{End}_{\mathbb{F}}(V)$ and any element $\phi \in \mathrm{End}_{\mathbb{F}}(V)$ determines an algebra homomorphism $\rho : \mathbb{F}[x] \to \mathrm{End}_{\mathbb{F}}(V)$, $\Sigma_{n\in\mathbb{N}_0} a_n x^n \mapsto \Sigma_{n\in\mathbb{N}_0} a_n \phi^n$.

7. Let V, W be vector spaces over $\mathbb{F}$. The representations of the tensor algebra $T(V)$ on W correspond bijectively to $\mathbb{F}$-linear maps $\phi : V \to \mathrm{End}_{\mathbb{F}}(W)$.

   This follows because the restriction of a representation $\rho : T(V) \to \mathrm{End}_{\mathbb{F}}(W)$ to $\iota_V(V) \subset T(V)$ defines an $\mathbb{F}$-linear map from V to $\mathrm{End}_{\mathbb{F}}(W)$. Conversely, every $\mathbb{F}$-linear map $\phi : V \to \mathrm{End}_{\mathbb{F}}(W)$ induces an algebra homomorphism $\rho : T(V) \to \mathrm{End}_{\mathbb{F}}(W)$ with $\rho \circ \iota = \phi$ by the universal property of the tensor algebra.

8. Let $\mathfrak{g}$ be a Lie algebra and V a vector space over $\mathbb{F}$. Representations of the universal enveloping algebra $U(\mathfrak{g})$ on V correspond bijectively to representations of $\mathfrak{g}$ on V, i. e. Lie algebra homomorphisms $\phi : \mathfrak{g} \to \mathrm{End}_{\mathbb{F}}(V)$.

   This follows because any algebra homomorphism $\rho : U(\mathfrak{g}) \to \mathrm{End}_{\mathbb{F}}(V)$ satisfies

   $$[\rho \circ \iota_{\mathfrak{g}}(x), \rho \circ \iota_{\mathfrak{g}}(y)] = \rho(\iota_{\mathfrak{g}}(x) \cdot \iota_{\mathfrak{g}}(y) - \iota_{\mathfrak{g}}(y) \cdot \iota_{\mathfrak{g}}(x)) = \rho \circ \iota_{\mathfrak{g}}([x, y]) \quad \forall x, y \in \mathfrak{g}.$$

   Hence $\rho \circ \iota_{\mathfrak{g}} : \mathfrak{g} \to \mathrm{End}_{\mathbb{F}}(V)$ is a Lie algebra morphism. Conversely, for any Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathrm{End}_{\mathbb{F}}(V)$ there is a unique algebra homomorphism $\rho : U(\mathfrak{g}) \to \mathrm{End}_{\mathbb{F}}(V)$ with $\rho \circ \iota_{\mathfrak{g}} = \phi$ by the universal property of $U(\mathfrak{g})$.

9. Any algebra is a left module over itself with the module structure given by left multiplication $\rhd : A \otimes A \to A$, $a \otimes b \mapsto a \cdot b$ and a left module over A^{op} with respect to right multiplication $\rhd : A^{op} \otimes A \to A$, $a \otimes b \mapsto b \cdot a$. Combining the two yields an $A \otimes A^{op}$-left module structure on A with $\rhd : (A \otimes A^{op}) \otimes A \to A$, $(a \otimes b) \otimes c \mapsto a \cdot c \cdot b$.

10. If $\phi : A \to B$ is an algebra homomorphism, then every B-module V becomes an A-module with the module structure given by $a \rhd v := \phi(a) \rhd v$ for all $v \in V$. This is called the **pullback** of the B-module structure on V by ϕ.

11. In particular, for any subalgebra $U \subset A$, the inclusion map $\iota : U \to A$, $u \mapsto u$ is an injective algebra homomorphism and induces a U-left module structure on any A-left module V. This is called the **restriction** of the A-module structure to U.

12. If $(V, \rhd_V)$ and $(W, \rhd_W)$ are modules over $\mathbb{F}$-algebras A and B, then

    $$\rhd : (A \otimes B) \otimes (V \otimes W) \to V \otimes W, \qquad (a \otimes b) \rhd (v \otimes w) = (a \rhd_V v) \otimes (b \rhd_W w)$$

    defines an $A \otimes B$ module structure on $V \otimes W$.

Simple, semisimple and indecomposable modules over algebras are defined in the same way as over general rings. However, the fact that they are vector spaces over a field allows one to draw additional conclusions about their structure. We recall the standard definitions and results on semisimple modules over rings for the case where the ring in question is an algebra. We also summarise the main results on modules over rings for the case of algebras. For results that hold for modules over general rings, we omit the proofs, but we prove the additional results that are specific to algebras.

Definition A.16: Let A be an algebra over $\mathbb{F}$.

1. A module M over A is called **simple** if $M \neq \{0\}$ and M has no non-trivial submodules, i. e. the only submodules of M are M and $\{0\}$.
2. A module over A is called **semisimple**, if it is the direct sum $M = \oplus_{i \in I} M_i$ of simple submodules M_i.
3. The algebra A is called **simple** or **semisimple** if it is simple or semisimple as a left module over itself with the left multiplication.

Note that the trivial module $\{0\}$ is not simple by definition, but it is semisimple, since it is given by the direct sum over an empty index set. The following proposition gives an alternative criterion for the semisimplicity of a module that is very useful in practice.

Proposition A.17: Let A be an algebra over $\mathbb{F}$ and M a module over A. Then the following are equivalent:

(i) M is semisimple.

(ii) Every submodule $U \subset M$ is a (not necessarily direct) sum of simple submodules.

(iii) Every submodule $U \subset M$ has a **complement**, i. e. there is a submodule $V \subset M$ with $M = U \oplus V$.

In particular, Proposition A.19 implies that semisimplicity is a property that is inherited by submodules and quotients of modules. This follows by choosing appropriate complements and by considering the images of the simple summands under the canonical surjection for a quotient.

Corollary A.18: Every submodule and every quotient of a semisimple module is semisimple.

As every module over an algebra A can be described as a quotient of a free module over A, that is, as a quotient of a direct sum $\oplus_{i\in I} A$ for some index set I, Corollary A.18 relates the semisimplicity of A to the semisimplicity of A-modules.

Proposition A.19: Let A be an algebra over $\mathbb{F}$. Then the following are equivalent:

(i) A is semisimple,

(ii) every module over A is semisimple.

Schur's lemma for modules over a ring R states that for a simple module M every R-linear map $f : M \to N$ is injective or zero, every R-linear map $f : N \to M$ is surjective or zero and, consequently, every R-linear map $f : M \to M$ is an isomorphism or zero. This states that the ring $\mathrm{End}_R(M)$ is a skew field. If one considers for R an algebra over an *algebraically closed field* $\mathbb{F}$ and for M a *finite-dimensional* simple module, one has a stronger result.

Lemma A.20: (**Schur's lemma**) Let A be an algebra over an algebraically closed field $\mathbb{F}$. Then for every finite-dimensional simple A-module M one has $\mathrm{End}_A(M) \cong \mathbb{F}$.

Proof:
As $\mathbb{F}$ is algebraically closed and M finite-dimensional, every A-linear map $f : M \to M$ has at least one eigenvalue $\lambda \in \mathbb{F}$ with an associated eigenvector. This yields a non-trivial submodule $0 \neq \ker(f - \lambda\mathrm{id}_M) \subset M$. As M is simple, it follows that $f = \lambda\mathrm{id}_M$ and hence $\mathrm{End}_A(M) \cong \mathbb{F}$. □

By definition, non-semisimple modules over an algebra A cannot be expressed as direct sums of simple modules. However, in many cases they can be characterised in terms of submodules such that the quotients with respect to these submodules are simple. This leads to the notion of modules of finite length and Jordan-Hölder series.

Definition A.21: Let$A$ be an algebra over $\mathbb{F}$.

An A-module N is called a module of **finite length** if there is a finite chain of submodules $0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$ such that all quotient modules M_i/M_{i-1} are simple.

Such a chain is called a **Jordan-Hölder series** and the modules M_i/M_{i-1} are called **subquotients**. The **length** $l(M)$ is the minimal length of a Jordan-Hölder series for M.

Example A.22:

1. Every finite direct sum of simple modules is a module of finite length: if $M = \oplus_{i=1}^n U_i$ with simple modules $U_1, ..., U_n$, then $0 \subset M_1 \subset M_2 \subset \ldots \subset M_{n-1} \subset M_n = M$ with $M_k := \oplus_{i=1}^k U_i$ is a Jordan-Hölder series for M with subquotients $M_i/M_{i-1} \cong U_i$.
2. Let V be a finite-dimensional complex vector. Then $\mathbb{C}[x]$-module structures $\rhd$ on V are in bijection with linear endomorphisms $\phi : V \to V$, $v \mapsto x \rhd v$. If $B = (v_1, ..., v_n)$ is a Jordan basis of V, then $0 = M_0 \subset M_1 \subset \ldots \subset M_n = V$ with $M_i = \mathrm{span}_{\mathbb{C}}\{v_1, ..., v_i\}$ is a Jordan-Hölder series for (V, ϕ).

To make sure that the length of a module of finite length and the subquotients in a Jordan-Hölder series are useful quantities that characterise the module, one needs to establish that they are independent of the choice of the Jordan-Hölder series.

Theorem A.23: (Jordan-Hölder theorem)
Let A be an algebra over $\mathbb{F}$ and M an A-module of finite length. Then:

1. Every submodule $U \subset M$ and quotient module M/U is an A-module of finite length with $l(M/U) + l(U) = l(M)$.
2. All Jordan-Hölder series of M have the same length and isomorphic subquotients, up to the ordering.

As modules over an algebra A over $\mathbb{F}$ are vector spaces over $\mathbb{F}$ and simple modules have dimension at least one, the dimension of the modules in a Jordan Hölder series must increase by at least one in each step. it follows that each A-module of finite length is finite-dimensional. Conversely, by iteratively choosing proper submodules of maximal dimension, one can show that each finite-dimensional A-module has finite length.

Lemma A.24: Let A be an algebra over $\mathbb{F}$. Then every finite-dimensional A-module M is a module of finite length $l(M) \leq \dim_{\mathbb{F}} M$.

Proof:
If $M = 0$, it is a module of length 0. If M is a simple A-module, then it has a Jordan-Hölder series of length 1. Otherwise, choose a proper submodule $M_1 \subsetneq M$ of maximal dimension. Then M/M_1 is simple, since proper submodules of M/M_1 correspond bijectively to submodules $M_1 \subsetneq P \subsetneq M$ and consequently with dimension $\dim_{\mathbb{F}} M_1 < \dim_{\mathbb{F}} P < \dim_{\mathbb{F}} M$.

Iterating this procedure yields a sequence of submodules $M = M_0 \supsetneq M_1 \supsetneq M_2 \supsetneq M_3 \supsetneq \ldots$ such that M_i/M_{i+1} is simple for all $i \geq 0$. As $\dim_{\mathbb{F}} M > \dim_{\mathbb{F}} M_1 > \dim_{\mathbb{F}} M_2 > \ldots$, this terminates after at most $\dim_{\mathbb{F}} M$ steps with a Jordan-Hölder series for M. $\square$

By applying the Jordan-Hölder theorem to an algebra A that is a module of finite length as a left module over itself, one can show that every simple module occurs as a subquotient in a composition series of A. This follows, because every element $0 \neq m \in M$ defines an A-linear map $\rhd_m : A \to M$, $a \mapsto a \rhd m$ that is surjective, because M is simple. By extending a Jordan-Hölder series for $\ker(\rhd_m) \subset A$ to a Jordan-Hölder series of A, one obtains a Jordan-Hölder series of A with subquotient $A/\ker(\rhd_m) \cong M$.

Corollary A.25: Let A be an algebra over $\mathbb{F}$ that has finite length as a left module over itself. Then every simple A-module is isomorphic to a quotient of A and is a subquotient of every Jordan-Hölder series of A.

In particular, we can apply Corollary A.25 to a finite-dimensional algebra A over $\mathbb{F}$, which is a module of finite length as a left module over itself by Lemma A.24. As every simple A-module occurs as a subquotient in each Jordan-Hölder series of A and is of dimension at least one, this imposes a restriction on the number of isomorphism classes of simple A-modules.

Corollary A.26: Let A be a finite-dimensional algebra over $\mathbb{F}$. Then every simple A-module is a quotient of A and occurs in each Jordan-Hölder series of A, and there are at most $\dim_{\mathbb{F}}$ isomorphism classes of simple A-modules.

To conclude our discussion of modules over algebras, we recall the Artin-Wedderburn theorem, which characterises them as matrix algebras. While the Wedderburn theorem for rings just states that every semisimple ring is isomorphic to a product of matrix algebras over skew fields, these skew fields inherit an algebra structure and hence become division algebras for a semisimple algebra. If the underlying field is algebraically closed, then Schur's Lemma implies that these division algebras are isomorphic to $\mathbb{F}$.

Theorem A.27: (**Artin-Wedderburn theorem**)
Let A be a semisimple algebra over $\mathbb{F}$. Then there are division algebras $D_1, ..., D_r$ over $\mathbb{F}$ $n_1, ..., n_r \in \mathbb{N}$, unique up to their order, such that

$$A \cong \mathrm{Mat}(n_1, D_1) \times \ldots \times \mathrm{Mat}(n_r, D_r).$$

If $\mathbb{F}$ is algebraically closed, then $D_1 = ... = D_r = \mathbb{F}$.

B Categories

B.1 Categories, functors and natural transformations

In this section we summarise the relevant background on categories, functors and natural transformations. Categories and functors arise whenever one relates different mathematical structures such as sets, topological spaces, modules over a ring, groups or algebras. Whenever one relates different mathematical structures, one needs to take into account not only the structures, but also the structure preserving maps between them. The reason for this is that one usually does not and cannot distinguish mathematical structures that are related by isomorphisms, such as homeomorphic topological spaces, or isomorphic vector spaces, groups or modules. To respect this principle, any relation between different mathematical structures must thus send structure preserving isomorphisms to structure preserving isomorphisms. For this reason, one organises mathematical objects and the structure preserving maps between them into a common framework, namely a *category*.

Definition B.1: A **category** $\mathcal{C}$ consists of:

- a class $\mathrm{Ob}\,\mathcal{C}$ of **objects**,
- for each pair of objects $X, Y \in \mathrm{Ob}\,\mathcal{C}$ a set[2] $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of **morphisms**,
- for each triple of objects X, Y, Z a **composition map**

$$\circ : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z),$$

such that the following axioms are satisfied:

(C1) The sets $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of morphisms are pairwise disjoint,

(C2) The composition is associative: $f\circ(g\circ h) = (f\circ g)\circ h$ for all morphisms $h \in \mathrm{Hom}_{\mathcal{C}}(W, X)$, $g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, $f \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$,

(C3) For every object X there is a morphism $1_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$, the **identity morphism** on X, with $1_X \circ f = f$ and $g \circ 1_X = g$ for all $f \in \mathrm{Hom}_{\mathcal{C}}(W, X)$, $g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

Instead of $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, we also write $f : X \to Y$. The object X is called the **source** of f, and the object Y the **target** of f. A morphism $f : X \to X$ is called an **endomorphism**.

A morphism $f : X \to Y$ is called an **isomorphism**, if there is a morphism $g : Y \to X$ with $g \circ f = 1_X$ and $f \circ g = 1_Y$. In this case, we call the objects X and Y **isomorphic**.

Example B.2:

1. The category Set: the objects of Set are sets, and the morphisms are maps $f : X \to Y$. The composition is the composition of maps and the identity morphisms are the identity maps. Isomorphisms are bijective maps.

   Note that the definition of a category requires that the *morphisms* between any two objects in a category form a set, but not that the *objects* form a set. Requiring that the

[2] This condition is sometimes relaxed in the literature on category theory. Categories whose morphisms form sets are called *locally small* in these references.

objects of a category form a set would force one to consider sets of sets when defining the category Set, which leads to a contradiction. A category whose objects form a set is called a **small category**.

2. The category Top of topological spaces. Objects are topological spaces, morphisms $f : X \to Y$ are continuous maps, isomorphisms are homeomorphisms.

3. The category Top* of **pointed topological spaces**: Objects are pairs (X, x) of a topological space X and a point $x \in X$, morphisms $f : (X, x) \to (Y, y)$ are continuous maps $f : X \to Y$ with $f(x) = y$.

4. The category Top(2) of **pairs of topological spaces**: Objects are pairs (X, A) of a topological space X and a subspace $A \subset X$, morphisms $f : (X, A) \to (Y, B)$ are continuous maps $f : X \to Y$ with $f(A) \subset B$. Isomorphisms are homeomorphisms $f : X \to Y$ with $f(A) = B$.

5. Many examples of categories we will use in the following are categories of algebraic structures. This includes the following:
   - the category $\mathrm{Vect}_{\mathbb{F}}$ of vector spaces over a field $\mathbb{F}$:
     objects: vector spaces over $\mathbb{F}$, morphisms: $\mathbb{F}$-linear maps,
   - the category $\mathrm{Vect}^{fin}_{\mathbb{F}}$ of finite dimensional vector spaces over a field $\mathbb{F}$:
     objects: vector spaces over $\mathbb{F}$, morphisms: $\mathbb{F}$-linear maps,
   - the category Grp of groups:
     objects: groups, morphisms: group homomorphisms,
   - the category Ab of abelian groups:
     objects: abelian groups, morphisms: group homomorphisms,
   - the category Ring of rings:
     objects: rings, morphisms: ring homomorphisms,
   - the category URing of unital rings:
     objects: unital rings, morphisms: unital ring homomorphisms,
   - the category Field of fields:
     objects: fields, morphisms: field monomorphisms,
   - the category $\mathrm{Alg}_{\mathbb{F}}$ of algebras over a field $\mathbb{F}$:
     objects: algebras over $\mathbb{F}$, morphisms: algebra homomorphisms,
   - the categories R-Mod and Mod-R of left and right modules over a ring R:
     objects: R-left or right modules, morphisms: R-left or right module homomorphisms.
   - the category R-Mod-S of (R, S)-bimodules:
     objects: (R, S)-bimodules, morphisms: (R, S)-bimodule homomorphisms.

In all of the categories in Example B.2 the morphisms are *maps*. A category for which this is the case is called a **concrete category**. A category that is not concrete is the following.

Example B.3: The category Rel has sets as objects. Morphisms from A to B are **relations** from A to B, that is subsets $R \subset A \times B$. The composite of a relation $R \subset A \times B$ and $S \subset B \times C$

is the relation $S \circ R = \{(a,c) \in A \times C \mid \exists b \in B : (a,b) \in R, (b,c) \in S\} \subset A \times C$ and the identity morphism on a set A is the relation $\Delta(A) = \{(a,a) \mid a \in A\}$.

Other important examples and basic constructions for categories are the following.

Example B.4:

1. A small category $\mathcal{C}$ in which all morphisms are isomorphisms is called a **groupoid**.
2. A category with a single object X is a **monoid**, and a groupoid $\mathcal{C}$ with a single object X is a group. Group elements are identified with endomorphisms $f : X \to X$ and the composition of morphisms is the group multiplication. More generally, for any object X in a groupoid $\mathcal{C}$, the set $\mathrm{End}_{\mathcal{C}}(X) = \mathrm{Hom}_{\mathcal{C}}(X,X)$ with the composition $\circ : \mathrm{End}_{\mathcal{C}}(X) \times \mathrm{End}_{\mathcal{C}}(X) \to \mathrm{End}_{\mathcal{C}}(X)$ is a group.
3. For every category $\mathcal{C}$, one has an **opposite category** $\mathcal{C}^{op}$, which has the same objects as $\mathcal{C}$, whose morphisms are given by $\mathrm{Hom}_{\mathcal{C}^{op}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(Y,X)$ and in which the order of the composition is reversed.
4. The **cartesian product** of categories $\mathcal{C}, \mathcal{D}$ is the category $\mathcal{C} \times \mathcal{D}$ with pairs (C,D) of objects in $\mathcal{C}$ and $\mathcal{D}$ as objects, with $\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((C,D),(C',D')) = \mathrm{Hom}_{\mathcal{C}}(C,C') \times \mathrm{Hom}_{\mathcal{D}}(D,D')$ and the composition of morphisms $(h,k) \circ (f,g) = (h \circ f, k \circ g)$.
5. A **subcategory** of a category $\mathcal{C}$ is a category $\mathcal{D}$, such that $\mathrm{Ob}(\mathcal{D}) \subset \mathrm{Ob}(\mathcal{C})$ is a subclass, $\mathrm{Hom}_{\mathcal{D}}(D,D') \subset \mathrm{Hom}_{\mathcal{C}}(D,D')$ for all objects D, D' in $\mathcal{D}$ and the composition of morphisms of $\mathcal{D}$ coincides with their composition in $\mathcal{C}$. A subcategory $\mathcal{D}$ of $\mathcal{C}$ is called a **full subcategory** if $\mathrm{Hom}_{\mathcal{D}}(D,D') = \mathrm{Hom}_{\mathcal{C}}(D,D')$ for all objects D, D' in $\mathcal{D}$.
6. **Quotient categories:** Let $\mathcal{C}$ be a category with an equivalence relation $\sim_{X,Y}$ on each morphism set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ that is compatible with the composition of morphisms: $f \sim_{X,Y} g$ and $h \sim_{Y,Z} k$ implies $h \circ f \sim_{X,Z} k \circ g$.

   Then one obtains a category $\mathcal{C}'$, the **quotient category** of $\mathcal{C}$, with the same objects as $\mathcal{C}$ and equivalence classes of morphisms in $\mathcal{C}$ as morphisms.

   The composition of morphisms in $\mathcal{C}'$ is given by $[h] \circ [f] = [h \circ f]$, and the identity morphisms by $[1_X]$. Isomorphisms in $\mathcal{C}'$ are equivalence classes of morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ for which there exists a morphism $g \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$ with $f \circ g \sim_{Y,Y} 1_Y$ and $g \circ f \sim_{X,X} 1_X$.

The construction in the last example plays an important role in classification problems, in particular in the context of topological spaces. Classifying the objects of a category $\mathcal{C}$ usually means classifying them up to isomorphism, i. e. giving a list of objects in $\mathcal{C}$ such that every object in $\mathcal{C}$ is isomorphic to exactly one object in this list.

While this is possible in some contexts - for the category $\mathrm{Vect}_{\mathbb{F}}^{fin}$ of finite dimensional vector spaces over $\mathbb{F}$, the list contains the vector spaces $\mathbb{F}^n$ with $n \in \mathbb{N}_0$ - it is often too difficult to solve this problem in full generality. In this case, it is sometimes simpler to consider instead a quotient category $\mathcal{C}'$ and to attempt a partial classification. If two objects are isomorphic in $\mathcal{C}$, they are by definition isomorphic in $\mathcal{C}'$ since for any objects X, Y in $\mathcal{C}$ and any isomorphism $f : X \to Y$ with inverse $g : Y \to X$, one has $[g] \circ [f] = [g \circ f] = [1_X]$ and $[f] \circ [g] = [f \circ g] = [1_Y]$. However, the converse does not hold in general - $\mathcal{C}'$ yields a weaker classification result than $\mathcal{C}$.

To relate different categories, one must not only relate their objects but also their morphisms, in a way that is compatible with source and target objects, the composition of morphisms and the identity morphisms. This leads to the concept of a *functor*.

Definition B.5: Let $\mathcal{C}, \mathcal{D}$ be categories. A **functor** $F : \mathcal{C} \to \mathcal{D}$ consists of:

- an assignment of an object $F(C)$ in $\mathcal{D}$ to every object C in $\mathcal{C}$,
- for each pair of objects C, C' in $\mathcal{C}$, a map

$$\mathrm{Hom}_{\mathcal{C}}(C, C') \to \mathrm{Hom}_{\mathcal{D}}(F(C), F(C')), \quad f \mapsto F(f),$$

that is compatible with the composition of morphisms and with the identity morphisms

$$\begin{aligned} &F(g \circ f) = F(g) \circ F(f) && \forall f \in \mathrm{Hom}_{\mathcal{C}}(C, C'), g \in \mathrm{Hom}_{\mathcal{C}}(C', C'') \\ &F(1_C) = 1_{F(C)} && \forall C \in \mathrm{Ob}\,\mathcal{C}. \end{aligned}$$

A functor $F : \mathcal{C} \to \mathcal{C}$ is called an **endofunctor**. A functor $F : \mathcal{C}^{op} \to \mathcal{D}$ is sometimes called a **contravariant functor** from $\mathcal{C}$ to $\mathcal{D}$. The **composite** of two functors $F : \mathcal{B} \to \mathcal{C}$, $G : \mathcal{C} \to \mathcal{D}$ is the functor $GF : \mathcal{B} \to \mathcal{D}$ given by the assignment $B \mapsto GF(B)$ for all objects B in $\mathcal{B}$ and the maps $\mathrm{Hom}_{\mathcal{B}}(B, B') \to \mathrm{Hom}_{\mathcal{D}}(GF(B), GF(B'))$, $f \mapsto G(F(f))$.

Example B.6:

1. For any category $\mathcal{C}$, identity functor $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$, that assigns each object and morphism in $\mathcal{C}$ to itself is an endofunctor of $\mathcal{C}$.

2. The functor $\mathrm{Vect}_{\mathbb{F}} \to \mathrm{Ab}$ that assigns to each vector space the underlying abelian group and to each linear map the associated group homomorphism, and the functors $\mathrm{Vect}_{\mathbb{F}} \to$ Set, Ring $\to$ Set, Grp $\to$ Set, Top$\to$ Set etc that assign to each vector space, ring, group, topological space the underlying set and to each morphism the underlying map are functors. A functor of this type is called **forgetful functor**.

3. The functor $* :\ \mathrm{Vect}_{\mathbb{F}} \to\ \mathrm{Vect}_{\mathbb{F}}^{op}$, which assigns to a vector space V its dual V^* and to a linear map $f : V \to W$ its adjoint $f^* : W^* \to V^*$, $\alpha \mapsto \alpha \circ f$.

4. For a group G, consider the category BG with a single object, with elements of G as morphisms, and with the multiplication of G as the composition. Then functors $F : BG \to \mathrm{Set}$ correspond to G-sets $X = F(\bullet)$ with the group action $\rhd : G \times X \to X$, $g \rhd x = F(g)(x)$. Functors $F : BG \to \mathrm{Vect}_{\mathbb{F}}$ correspond to representations of G over $\mathbb{F}$, with the representation space $V = F(\bullet)$ and $\rho = F(g) : G \to \mathrm{Aut}_{\mathbb{F}}V$.

5. **Restriction functor**: Let $\phi : R \to S$ a ring homomorphism. The restriction functor $\mathrm{Res} : S\text{-Mod} \to R\text{-Mod}$ sends an S-module $(M, \rhd)$ to the R-module $(M, \rhd_\phi)$ with the pullback module structure $r \rhd_\phi m = \phi(r) \rhd m$ and every S-linear map $f : M \to M'$ to itself.

6. **Tensor products**: Let R be a ring, M an R-right module and N an R-left module.

   • The functor $M \otimes_R -\ : R\text{-Mod} \to \mathrm{Ab}$ assigns to an R-left module N the abelian group $M \otimes_R N$ and to an R-linear map $f : N \to N'$ the group homomorphism $\mathrm{id}_M \otimes f : M \otimes_R N \to M \otimes_R N'$.

   • The functor $- \otimes_R N : R^{op}\text{-Mod} \to \mathrm{Ab}$ assigns to an R-right module M the abelian group $M \otimes_R N$ and to an R-linear map $f : M \to M'$ the group homomorphism

$f \otimes \mathrm{id}_N : M \otimes_R N \to M \otimes_R N'$.

• The functor $\otimes_R : R^{op}\text{-Mod} \times R\text{-Mod} \to \mathrm{Ab}$ assigns to a pair (M, N) of an R-right module M and an R-left module N the abelian group $M \otimes_R N$ and to a pair of R-linear maps $f : M \to M'$ and $g : N \to N'$ the group homomorphism $f \otimes g : M \otimes_R N \to M' \otimes_R N'$.

That these are indeed functors follows from Proposition **??**, 2. Note also that for *commutative* rings R, any R-left module is an (R, R)-bimodule and these functors can be defined to take values in R-Mod instead of Ab.

7. The **Hom-functors**: Let $\mathcal{C}$ be a category and C an object in $\mathcal{C}$.

   • The functor $\mathrm{Hom}(C, -) : \mathcal{C} \to \mathrm{Set}$ assigns to an object C' the set $\mathrm{Hom}_{\mathcal{C}}(C, C')$ and to a morphism $f : C' \to C''$ the map $\mathrm{Hom}(C, f) : \mathrm{Hom}_{\mathcal{C}}(C, C') \to \mathrm{Hom}_{\mathcal{C}}(C, C'')$, $g \mapsto f \circ g$.

   • The functor $\mathrm{Hom}(-, C) : \mathcal{C}^{op} \to \mathrm{Set}$ assigns to an object C' the set $\mathrm{Hom}_{\mathcal{C}}(C', C)$ and to a morphism $f : C' \to C''$ the map $\mathrm{Hom}(f, C) : \mathrm{Hom}_{\mathcal{C}}(C'', C) \to \mathrm{Hom}_{\mathcal{C}}(C', C)$, $g \mapsto g \circ f$.

8. The **path component functor** $\pi_0 : \mathrm{Top} \to \mathrm{Set}$ assigns to a topological space X the set $\pi_0(X)$ of its path components $P(x)$ and to a continuous map $f : X \to Y$ the map $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$, $P(x) \mapsto P(f(x))$.

9. The **fundamental group** defines a functor $\pi_1 : \mathrm{Top}^* \to \mathrm{Grp}$ that assigns to a pointed topological space (x, X) its fundamental group $\pi_1(x, X)$ and to a morphism $f : (x, X) \to (y, Y)$ of pointed topological spaces the group homomorphism $\pi_1(f) : \pi_1(x, X) \to \pi_1(y, Y)$, $[\gamma] \mapsto [f \circ \gamma]$.

10. **Abelisation**: The abelisation functor $F : \mathrm{Grp} \to \mathrm{Ab}$ assigns to a group G the abelian group $F(G) = G/[G, G]$, where $[G, G]$ is the normal subgroup generated by the set of all elements $ghg^{-1}h^{-1}$ for $g, h \in G$, and to a group homomorphism $f : G \to H$ the induced group homomorphism $F(f) : G/[G, G] \to H/[H, H]$, $g + [G, G] \mapsto f(g) + [H, H]$.

When dealing with categories, it is not sufficient to consider functors between different categories. There is another structure that relates different functors. As a functor $F : \mathcal{C} \to \mathcal{D}$ involves maps between the sets $\mathrm{Hom}_{\mathcal{C}}(C, C')$ and $\mathrm{Hom}_{\mathcal{D}}(F(C), F(C'))$, a structure that relates two functors $F, G : \mathcal{C} \to \mathcal{D}$ must in particular relate the sets $\mathrm{Hom}_{\mathcal{D}}(F(C), F(C'))$ and $\mathrm{Hom}_{\mathcal{D}}(G(C), G(C'))$. The simplest way to do this is to assign to each object C in $\mathcal{C}$ a morphism $\eta_C : F(C) \to G(C)$ in $\mathcal{D}$. One then requires compatibility with the morphisms $F(f) : F(C) \to G(C')$ and $G(f) : G(C) \to G(C')$ for all morphisms $f : C \to C'$ in $\mathcal{C}$.

Definition B.7: A **natural transformation** $\eta : F \to G$ between functors $F, G : \mathcal{C} \to \mathcal{D}$ is an assignment of a morphism $\eta_C : F(C) \to G(C)$ in $\mathcal{D}$ to every object C in $\mathcal{C}$ such that the following diagram commutes for all morphisms $f : C \to C'$ in $\mathcal{C}$

$$\begin{tikzcd} F(C) \arrow[r, "\eta_C"] \arrow[d, "F(f)"] & G(C) \arrow[d, "G(f)"] \\ F(C') \arrow[r, "\eta_{C'}"] & G(C'). \end{tikzcd}$$

A **natural isomorphism** is a natural transformation $\eta : F \to G$, for which all morphisms $\eta_X : F(X) \to G(X)$ are isomorphisms. Two functors that are related by a natural isomorphism are called **naturally isomorphic**.

Example B.8:

1. For any functor $F : \mathcal{C} \to \mathcal{D}$ the identity natural transformation $\mathrm{id}_F : F \to F$ with component morphisms $(\mathrm{id}_F)_X = 1_{F(X)} : F(X) \to F(X)$ is a natural isomorphism.

2. Consider the functors $\mathrm{id} : \mathrm{Vect}_{\mathbb{F}} \to \mathrm{Vect}_{\mathbb{F}}$ and $** : \mathrm{Vect}_{\mathbb{F}} \to \mathrm{Vect}_{\mathbb{F}}$. Then there is a canonical natural transformation can : id $\to **$, whose component morphisms $\eta_V : V \to V^{**}$ assign to a vector $v \in V$ the unique vector $v^{**} \in V^{**}$ with $v^{**}(\alpha) = \alpha(v)$ for all $\alpha \in V^*$.

3. Consider the category CRing of commutative unital rings and unital ring homomorphisms and the category Grp of groups and group homomorphisms.

   Let F : CRing $\to$ Grp the functor that assigns to a commutative unital ring k the group $\mathrm{GL}_n(k)$ of invertible $n \times n$-matrices with entries in k and to a unital ring homomorphism $f : k \to l$ the group homomorphism

   $$\mathrm{GL}_n(f) : \mathrm{GL}_n(k) \to \mathrm{GL}_n(l), \quad M = (m_{ij})_{i,j=1,\ldots,n} \mapsto f(M) = (f(m_{ij}))_{i,j=1,..,n}.$$

   Let G : CRing $\to$ Grp be the functor that assigns to a commutative unital ring k the group $G(k) = k^\times$ of units in k and to a unital ring homomorphism $f : k \to l$ the induced group homomorphism $G(f) = f|_{k^\times} : k^\times \to l^\times$.

   The determinant defines a natural transformation $\det : F \to G$ with component morphisms $\det_k : \mathrm{GL}_n(k) \to k^\times$, since the following diagram commutes for every unital ring homomorphism $f : k \to l$

   $$\begin{tikzcd} \mathrm{GL}_n(k) \arrow[r, "\det_k"] \arrow[d, "\mathrm{GL}_n(f)"'] & k^\times \arrow[d, "f|_{k^\times}"] \\ \mathrm{GL}_n(l) \arrow[r, "\det_l"] & l^\times. \end{tikzcd}$$

4. For a group G, denote by BG the groupoid with a single object $\bullet$, with group elements $g \in G$ as morphisms and the group multiplication as composition.

   Then by Example B.6, 4. functors $F : BG \to$ Set are G-sets, and natural transformations between them are G-equivariant maps. Every natural transformation $\eta : F \to F'$ is given by a single component morphism $\eta_\bullet : F(\bullet) \to F'(\bullet)$. The naturality condition states that $\eta_\bullet(g \rhd x) = g \rhd' \eta_\bullet(x)$ for all $g \in G$, $x \in X$.

   Similarly, by Example B.6, 4. functors $F : BG \to \mathrm{Vect}_{\mathbb{F}}$ are representations of G over $\mathbb{F}$, and natural transformations between them are homomorphisms of representations.

Remark B.9:

1. For any small category $\mathcal{C}$ and category $\mathcal{D}$, the functors $\mathcal{C} \to \mathcal{D}$ and natural transformations between them form a category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, the **functor category**. The composite of two natural transformations $\eta : F \to G$ and $\kappa : G \to H$ is the natural transformation $\kappa \circ \eta : F \to H$ with component morphisms $(\kappa \circ \eta)_X = \kappa_X \circ \eta_X : F(X) \to H(X)$ and the identity morphisms are the identity natural transformations $1_F = \mathrm{id}_F : F \to F$.

2. Natural transformations can be composed with functors.

   If $F, F' : \mathcal{C} \to \mathcal{D}$ are functors and $\eta : F \to F'$ a natural transformation, then for any functor $G : \mathcal{B} \to \mathcal{C}$ one obtains a natural transformation $\eta G : FG \to F'G$ with component morphisms $(\eta G)_B = \eta_{G(B)} : FG(B) \to F'G(B)$. Similarly, any functor $E : \mathcal{D} \to \mathcal{E}$ defines a natural transformation $E\eta : EF \to EF'$ with $(E\eta)_C = E(\eta_C) : EF(C) \to EF'(C)$.

The notions of natural transformations and natural isomorphisms are particularly important as they allow one to generalise the notion of an inverse map and of a bijection to functors. While it is possible to define an *inverse* of a functor $F : \mathcal{C} \to \mathcal{D}$ as a functor $G : \mathcal{D} \to \mathcal{C}$ with $GF = \mathrm{id}_{\mathcal{C}}$ and $FG = \mathrm{id}_{\mathcal{D}}$, it turns out that this is too strict. There are very few non-trivial examples of functors with an inverse. A more useful generalisation is obtained by weakening this requirement. Instead of requiring $FG = \mathrm{id}_{\mathcal{D}}$ and $GF = \mathrm{id}_{\mathcal{C}}$, one requires only that these functors are *naturally isomorphic* to the identity functors. This leads to the concept of an equivalence of categories.

Definition B.10: A functor $F : \mathcal{C} \to \mathcal{D}$ is called an **equivalence of categories** if there is a functor $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\kappa : GF \to \mathrm{id}_{\mathcal{C}}$ and $\eta : FG \to \mathrm{id}_{D}$. In this case, the categories $\mathcal{C}$ and $\mathcal{D}$ are called **equivalent**.

Sometimes it is easier to use a more direct characterisation of an equivalences of categories in terms of its behaviour on objects and morphisms. The proof of the following lemma makes use of the axiom of choice and an be found for instance in [Ka], Chapter XI, Prop XI.1.5.

Lemma B.11: A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if it is:

1. **essentially surjective**:
   for every object D in $\mathcal{D}$ there is an object C of $\mathcal{C}$ such that D is isomorphic to $F(C)$.
2. **fully faithful**:
   all maps $\mathrm{Hom}_{\mathcal{C}}(C, C') \to \mathrm{Hom}_{\mathcal{D}}(F(C), F(C'))$, $f \mapsto F(f)$ are bijections.

Example B.12:

1. The category $\mathrm{Vect}_{\mathbb{F}}^{fin}$ of finite-dimensional vector spaces over $\mathbb{F}$ is equivalent to the category $\mathcal{C}$, whose objects are non-negative integers $n \in \mathbb{N}_0$, whose morphisms $f : n \to m$ are $m \times n$-matrices with entries in $\mathbb{F}$ and with the matrix multiplication as composition of morphisms.
2. The category Set^{fin} of finite sets is equivalent to the category Ord^{fin}, whose objects are finite **ordinal numbers** $\underline{n} = \{0, 1, ..., n-1\}$ for all $n \in \mathbb{N}_0$ and whose morphisms $f : \underline{m} \to \underline{n}$ are maps $f : \{0, 1, ..., m-1\} \to \{0, 1, ..., n-1\}$ with the composition of maps as the composition of morphisms.

Many concepts and constructions from topological or algebraic settings can be generalised straightforwardly to categories. This is true whenever it is possible to characterise them in terms of *universal properties* involving only the *morphisms* in the category. In particular, there is a concept of categorical product and coproduct that generalise cartesian products and disjoint unions of sets and products and sums of topological spaces.

Definition B.13: Let $\mathcal{C}$ be a category and $(C_i)_{i\in I}$ a family of objects in $\mathcal{C}$.

1. A **product** of the family $(C_i)_{i\in I}$ is an object $\Pi_{i\in I} C_i$ in $\mathcal{C}$ together with a family of morphisms $\pi_i : \Pi_{j\in I} C_j \to C_i$, such that for all families of morphisms $f_i : W \to C_i$ there

is a unique morphism $f : W \to \Pi_{i\in I} C_i$ such that the diagram

$$\begin{tikzcd} W \arrow[r, dashed, "\exists! f"] \arrow[dr, "f_i"'] & \Pi_{j\in I} C_j \arrow[d, "\pi_i"] \\ & C_i \end{tikzcd} \tag{93}$$

commutes for all $i \in I$. This is called the **universal property** of the product.

2. A **coproduct** of the family $(C_i)_{i\in I}$ is an object $\amalg_{i\in I} C_i$ in $\mathcal{C}$ with a family $(\iota_i)_{i\in I}$ of morphisms $\iota_i : C_i \to \amalg_{j\in I} C_j$, such that for every family $(f_i)_{i\in I}$ of morphisms $f_i : C_i \to Y$ there is a unique morphism $f : \amalg_{i\in I} C_i \to Y$ such that the diagram

$$\begin{tikzcd} Y & \amalg_{j\in I} C_j \arrow[l, dashed, "\exists! f"'] \\ & C_i \arrow[ul, "f_i"] \arrow[u, "\iota_i"'] \end{tikzcd} \tag{94}$$

commutes for all $i \in I$. This is called the **universal property** of the coproduct.

Remark B.14: Products or coproducts do not necessarily exist for a given family of objects $(C_i)_{i\in I}$ in a category $\mathcal{C}$, but if they exist, they are **unique up to unique isomorphism**:

If $(\Pi_{i\in I} C_i, (\pi_i)_{i\in I})$ and $(\Pi'_{i\in I} C_i, (\pi'_i)_{i\in I})$ are two products for a family of objects $(C_i)_{i\in I}$ in $\mathcal{C}$, then there is a unique morphism $\pi' : \Pi'_{i\in I} C_i \to \Pi_{i\in I} C_i$ with $\pi_i \circ \pi' = \pi'_i$ for all $i \in I$, and this morphism is an isomorphism.

This follows directly from the universal property of the products: By the universal property of the product $\Pi_{i\in I} C_i$ applied to the family of morphisms $\pi'_i : \Pi'_{i\in I} C_i \to C_i$, there is a unique morphism $\pi' : \Pi'_{i\in I} C_i \to \Pi_{i\in I} C_i$ such that $\pi_i \circ \pi' = \pi'_i$ for all $i \in I$. Similarly, the universal property of $\Pi'_{i\in I} C_i$ implies that for the family of morphisms $\pi_i : \Pi_{i\in I} C_i \to C_i$ there is a unique morphism $\pi : \Pi_{i\in I} C_i \to \Pi'_{i\in I} C_i$ with $\pi'_i \circ \pi = \pi_i$ for all $i \in I$. It follows that $\pi' \circ \pi : \Pi_{i\in I} C_i \to \Pi_{i\in I} C_i$ is a morphism with $\pi_i \circ \pi \circ \pi' = \pi'_i \circ \pi = \pi_i$ for all $i \in I$. Since the identity morphism on $\Pi_{i\in I} C_i$ is another morphism with this property, the uniqueness implies $\pi' \circ \pi = 1_{\Pi_{i\in I} C_i}$. By the same argument one obtains $\pi \circ \pi' = 1_{\Pi'_{i\in I} C_i}$ and hence π' is an isomorphism with inverse π.

$$\begin{tikzcd} \Pi'_{i\in I} C_i \arrow[r, "\pi'"] \arrow[rr, bend left, "1_{\Pi'_{i\in I} C_i}"] \arrow[dr, "\pi'_i"'] & \Pi_{i\in I} C_i \arrow[r, "\pi"] \arrow[d, "\pi_i"] & \Pi'_{i\in I} C_i \arrow[dl, "\pi'_i"] \\ & C_i. & \end{tikzcd} \qquad \begin{tikzcd} \Pi_{i\in I} C_i \arrow[r, "\pi"] \arrow[rr, bend left, "1_{\Pi_{i\in I} C_i}"] \arrow[dr, "\pi_i"'] & \Pi'_{i\in I} C_i \arrow[r, "\pi'"] \arrow[d, "\pi'_i"] & \Pi_{i\in I} C_i \arrow[dl, "\pi_i"] \\ & C_i. & \end{tikzcd}$$

Example B.15:

1. The cartesian product of sets is a product in Set, and the disjoint union of sets is a coproduct in Set. The product of topological spaces is a product in Top and the topological sum is a coproduct in Top. In Set and Top, products and coproducts exist for all families of objects.

2. The direct sum of vector spaces is a coproduct and the direct product of vector spaces a product in $\mathrm{Vect}_{\mathbb{F}}$. More generally, direct sums and products of R-left (right) modules over a unital ring R are coproducts and products in R-Mod (Mod-R). Again, products and coproducts exist for all families of objects in R-Mod (Mod-R).

3. The wedge sum is a coproduct in the category Top^* of pointed topological spaces. It exists for all families of pointed topological spaces.

4. The direct product of groups is a product in Grp and the free product of groups is a coproduct in Grp. They exist for all families of groups.

In particular, we can consider categorical products and coproducts over empty index sets I. By definition, a categorical product for an empty family of objects is an object $T = \Pi_\emptyset$ such that for every object C in $\mathcal{C}$ there is a unique morphism $t_C : C \to T$. (This is the morphism associated to the empty family of morphisms from C to the objects in the empty family by the universal property of the product). Similarly, a coproduct over an empty index set I is an object $I := \amalg_\emptyset$ in $\mathcal{C}$ such that for every object C in $\mathcal{C}$, there is a unique morphism $i_C : I \to C$. Such objects are called, respectively, *terminal* and *initial* objects in $\mathcal{C}$.

Initial and terminal objects do not exist in every category $\mathcal{C}$, but if they exist they are unique up to unique isomorphism by the universal property of the products and coproducts.

An object that is both, terminal and initial, is called a *zero object*. If it exists, it is unique up to unique isomorphism, and it gives rise to a distinguished morphism, the *zero morphism* $0 = i_{C'} \circ t_C : C \to C'$ between objects C, C' in $\mathcal{C}$.

Definition B.16: Let $\mathcal{C}$ be a category. An object X in a category $\mathcal{C}$ is called:

1. A **final** or **terminal object** in $\mathcal{C}$ is an object T in $\mathcal{C}$ such that for every object C in $\mathcal{C}$ there is a unique morphism $t_C : C \to T$.
2. A **cofinal** or **initial object** in $\mathcal{C}$ is an object I in $\mathcal{C}$ such that for every object C in $\mathcal{C}$ there is a unique morphism $i_C : I \to C$,
3. A **null object** or **zero object** in $\mathcal{C}$ is an object 0 in $\mathcal{C}$ that is both final and initial: for every object C in $\mathcal{C}$ there are a unique morphisms $t_C : C \to 0$ and $i_C : 0 \to C$.
4. If $\mathcal{C}$ has a zero object, then the morphism $0 = i_{C'} \circ t_C : C \to 0 \to C'$ is called the **trivial morphism** or **zero morphism** from C to C'.

Example B.17:

1. The empty set is an initial object in Set and the empty topological space an initial object in Top. Any set with one element is a final object in Set and any one point space an initial object in Top. The categories Set and Top do not have null objects.

2. The null vector space $\{0\}$ is a null object in the category $\mathrm{Vect}_{\mathbb{F}}$. More generally, for any ring R, the trivial R-module $\{0\}$ is a null object in R-Mod (Mod-R).

3. The trivial group $G = \{e\}$ is a null object in Grp and in Ab.

4. The ring $\mathbb{Z}$ is an initial object in the category URing, since for every unital ring R, there is exactly one ring homomorphism $f : \mathbb{Z} \to R$, namely the one determined by $f(0) = 0_R$

and $f(1) = 1_R$. The zero ring $R = \{0\}$ with $0 = 1$ is a final object in URing, but not an initial one. The category URing has no zero object.

5. The category Field does not have initial or final objects. As any ring homomorphism $f : \mathbb{F} \to \mathbb{K}$ between fields is injective, an initial object $\mathbb{F}$ in Field would be a subfield of all other fields, and every field would be a subfield of a final object $\mathbb{F}$. This would imply $\mathrm{char}(\mathbb{F}) = \mathrm{char}(\mathbb{K})$ for all other fields $\mathbb{K}$, a contradiction.

Besides forming an equivalence of categories, there is another important way in which two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ can be related, namely being *adjoints* of each other. Adjoint functors encode universal properties of algebraic constructions such as products and coproducts, freely generated modules or abelisation of groups. The constructions are encoded in the functors and their universal properties in bijections between certain Hom-sets in the categories $\mathcal{C}$ and $\mathcal{D}$.

Definition B.18: A functor $F : \mathcal{C} \to \mathcal{D}$ is called **left adjoint** to a functor $G : \mathcal{D} \to \mathcal{C}$ and G **right adjoint** to F, in formulas $F \dashv G$, if the functors $\mathrm{Hom}(F(-), -) : \mathcal{C}^{op} \times \mathcal{D} \to \mathrm{Set}$ and $\mathrm{Hom}(-, G(-)) : \mathcal{C}^{op} \times \mathcal{D} \to \mathrm{Set}$ are naturally isomorphic.

In other words, there is a family of bijections $\phi_{C,D} : \mathrm{Hom}_{\mathcal{C}}(C, G(D)) \to \mathrm{Hom}_{\mathcal{D}}(F(C), D)$, indexed by objects C in $\mathcal{C}$ and D in $\mathcal{D}$, such that the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(C, G(D)) \arrow[r, "\mathrm{Hom}(f,G(g))", "h\mapsto G(g)\circ h\circ f"'] \arrow[d, "\phi_{C,D}"] & \mathrm{Hom}_{\mathcal{C}}(C', G(C')) \arrow[d, "\phi_{C',D'}"] \\
\mathrm{Hom}_{\mathcal{D}}(F(C), D) \arrow[r, "\mathrm{Hom}(F(f),g)", "h\mapsto g\circ h\circ F(f)"'] & \mathrm{Hom}_{\mathcal{D}}(F(C'), D').
\end{tikzcd} \tag{95}$$

commutes for all morphisms $f : C' \to C$ in $\mathcal{C}$ and $g : D \to D'$ in $\mathcal{D}$.

Example B.19:

1. **Forgetful functors and freely generated modules:**
For a ring R, the forgetful functor $G : R\text{-Mod} \to \mathrm{Set}$ is right adjoint to the functor $F : \mathrm{Set} \to R\text{-Mod}$ that assigns to a set A the free R-module $F(A) = \langle A \rangle_R$ generated by A and to a map $f : A \to B$ the R-linear map $F(f) : \langle A \rangle_R \to \langle B \rangle_R$ with $F(f) \circ \iota_A = \iota_B \circ f$.
For every map $f : A \to M$ into an R-module M, there is a unique R-linear map $\langle f \rangle_R : \langle A \rangle_R \to M$ with $\langle f \rangle_R \circ \iota_A = f$ for the inclusion $\iota_A : A \to \langle A \rangle_R$. This defines bijections
$$\phi_{A,M} : \mathrm{Hom}_{\mathrm{Set}}(A, G(M)) \to \mathrm{Hom}_{R-\mathrm{Mod}}(F(A), M), \quad f \mapsto \langle f \rangle_R.$$
For all maps $f : A' \to A$, $h : A \to M$ and R-linear maps $g : M \to M'$ we have
$$g \circ \langle h \rangle_R \circ F(f) \circ \iota_{A'} = g \circ \langle h \rangle_R \circ \iota_A \circ f = g \circ h \circ f = \langle g \circ h \circ f \rangle_R \circ \iota_{A'}.$$
This implies $\langle g \circ h \circ f \rangle_R = g \circ \langle h \rangle_R \circ F(f)$.

2. **Discrete and indiscrete topology**: The forgetful functor $F : \mathrm{Top} \to \mathrm{Set}$ is left adjoint to the indiscrete topology functor $I : \mathrm{Set} \to \mathrm{Top}$ that assigns to a set X the topological space $(X, \mathcal{O}_{ind})$ with the indiscrete topology and to a map $f : X \to Y$ the continuous map $f : (X, \mathcal{O}_{ind}) \to (Y, \mathcal{O}_{ind})$.

It is right adjoint to the discrete topology functor $D : \mathrm{Set} \to \mathrm{Top}$ that assigns to a set X the topological space $(X, \mathcal{O}_{disc})$ with the discrete topology and to a map $f : X \to Y$ the continuous map $f : (X, \mathcal{O}_{disc}) \to (Y, \mathcal{O}_{disc})$. The bijections between the Hom-Sets are

$$\begin{aligned}
\Phi_{(W,\mathcal{O}),X} &: \mathrm{Hom}_{\mathrm{Top}}((W,\mathcal{O}),(X,\mathcal{O}_{ind})) \to \mathrm{Hom}_{\mathrm{Set}}(W,X), \quad f \mapsto f\\
\Phi_{X,(W,\mathcal{O})} &: \mathrm{Hom}_{\mathrm{Set}}(X,W) \to \mathrm{Hom}_{\mathrm{Top}}((X,\mathcal{O}_{disc}),(W,\mathcal{O})), \quad f \mapsto f.
\end{aligned}$$

The statement that these are bijections expresses the fact that any map $f : W \to X$ from a topological space $(W, \mathcal{O})$ into a set X becomes continuous when X is equipped with the indiscrete topology and any map $f : X \to W$ becomes continuous when X is equipped with the discrete topology. The naturality condition in (95) follows directly.

3. **Forgetful functors without left or right adjoints**:
The forgetful functor $V : \mathrm{Field} \to \mathrm{Set}$ has no right or left adjoint. If it had a left adjoint $F : \mathrm{Set} \to \mathrm{Field}$ or a right adjoint $G : \mathrm{Set} \to \mathrm{Field}$ there would be bijections

$$\Phi_{\emptyset,\mathbb{K}} : \mathrm{Hom}_{\mathrm{Set}}(\emptyset,\mathbb{K}) \to \mathrm{Hom}_{\mathrm{Field}}(F(\emptyset),\mathbb{F}). \quad \Phi_{\mathbb{F},\{x\}} : \mathrm{Hom}_{\mathrm{Field}}(\mathbb{F},G(\{x\})) \to \mathrm{Hom}_{\mathrm{Set}}(\mathbb{F},\{x\})$$

for any field $\mathbb{F}$. This would imply that $F(\emptyset)$ is an initial object in Field and hence a subfield of any other field $\mathbb{F}$ and that $G(\{p\})$ is a terminal object in Field and hence contains any field $\mathbb{F}$ as a subfield. It follows that $\mathrm{char}\,\mathbb{F} = \mathrm{char}\,F(\emptyset) = \mathrm{char}\,G(\{x\})$ for all fields $\mathbb{F}$, a contradiction.

4. **Inclusion functor and abelisation**: The inclusion functor $G : \mathrm{Ab} \to \mathrm{Grp}$ is right adjoint to the abelisation functor $F : \mathrm{Grp} \to \mathrm{Ab}$ from Example B.6, 10. (Exercise **??**).

5. **Products, coproducts and diagonal functors**:
• Let $\mathcal{C}$ be a category and I a set such that products and coproducts in $\mathcal{C}$ exist for all families of objects indexed by I.

• Let $\mathcal{C}_I$ be the category with families $(C_i)_{i\in I}$ and $(f_i)_{i\in I} : (C_i)_{i\in i} \to (C_i')_{i\in I}$ of objects and morphisms in $\mathcal{C}$ as objects and morphisms, with componentwise composition.

• Let $\Delta : \mathcal{C} \to \mathcal{C}_I$ be the diagonal functor that assigns to an object C and a morphism $f : C \to C'$ in $\mathcal{C}$ the constant families $(C)_{i\in I}$ and $(f)_{i\in I}$.

• Let $\Pi_I : \mathcal{C}_I \to \mathcal{C}$ be the product functor that assigns to a family $(C_i)_{i\in I}$ the product $\Pi_{i\in I}C_i$ and to a family $(f_i)_{i\in I} : (C_i)_{i\in I} \to (C_i')_{i\in I}$ the morphism $\Pi_{i\in I}f_i : \Pi_{i\in I}C_i \to \Pi_{i\in I}C_i'$ with $\pi_i' \circ (\Pi_{i\in I}f_i) = f_i \circ \pi_i$ induced by the universal property of the product.

• Let $\amalg_I : \mathcal{C}_I \to \mathcal{C}$ be the coproduct functor that assigns to a family $(C_i)_{i\in I}$ the coproduct $\amalg_{i\in I}C_i$ and to a family $(f_i)_{i\in I} : (C_i)_{i\in I} \to (C_i')_{i\in I}$ the morphism $\amalg_{i\in I}f_i : \Pi_{i\in I}C_i \to \Pi_{i\in I}C_i'$ with $(\amalg_{i\in I}f_i) \circ \iota_i = \iota_i' \circ f_i$ induced by the universal property of the coproduct.

Then $\Pi_I : \mathcal{C}_I \to \mathcal{C}$ is right adjoint to Δ and $\amalg_I : \mathcal{C}_I \to \mathcal{C}$ is left adjoint to Δ. The bijections between the Hom-sets are given by

$$\begin{aligned}
\Phi_{C,(C_i)_{i\in I}} &: \mathrm{Hom}_{\mathcal{C}}(C, \Pi_{i\in I}C_i) \to \mathrm{Hom}_{\mathcal{C}_I}((C)_{i\in I}, \Pi_{i\in I}C_i), \quad f \mapsto (\pi_i \circ f)_{i\in I}\\
\Phi^{-1}_{(C_i)_{i\in I},C} &: \mathrm{Hom}_{\mathcal{C}}(\amalg_{i\in I}C_i, C) \to \mathrm{Hom}_{\mathcal{C}_I}((C_i)_{i\in I}, (C)_{i\in I}), \quad f \mapsto (f \circ \iota_i)_{i\in I}.
\end{aligned}$$

The universal property of the (co)product implies that they are bijections, and a short computation shows that they satisfy the naturality condition in (95).

6. **Tensor products and Hom-functors**:
• For any R-right module M, the functor $M\otimes_R - : R\text{-Mod} \to \mathrm{Ab}$ is left adjoint to the functor $\mathrm{Hom}(M,-) : \mathrm{Ab} \to R\text{-Mod}$.

• For any R-left module N the functor $-\otimes_R N : R^{op}\text{-Mod} \to \text{Ab}$ is left adjoint to the functor $\text{Hom}(N,-) : \text{Ab} \to R^{op}\text{-Mod}$.

We prove the claim for R-right modules M. For an abelian group A and R-left module L we equip $\text{Hom}_{\text{Ab}}(M,A)$ with the R-module structure $(r \rhd \phi)(m) = \phi(m \lhd r)$ and define

$$\begin{aligned} \phi_{L,A} :\ & \text{Hom}_{R\text{-Mod}}(L, \text{Hom}_{\text{Ab}}(M,A)) && \to && \text{Hom}_{\text{Ab}}(M \otimes_R L, A) \\ & \psi : L \to \text{Hom}_{\text{Ab}}(M,A),\ l \mapsto \psi_l && \mapsto && \chi : M\otimes_R L \to A,\ m\otimes l \mapsto \psi_l(m). \end{aligned}$$

The map $\chi : M\otimes_R L \to A$, $m\otimes l \mapsto \psi_l(m)$ is well defined, since the R-linearity of the map $\psi : L \to \text{Hom}_{\text{Ab}}(M,A)$ implies that $\chi' : M \times L \to A$, $(m,l) \to \psi_l(m)$ is R-bilinear: $\chi'(m, r \rhd l) = \psi_{r\rhd l}(m) = (r \rhd \psi_l)(m) = \psi_l(m \lhd r) = \chi'(m \lhd r, l)$ for all $r \in R$, $l \in L$ and $m \in M$. By the universal property of the tensor product, it induces a unique group homomorphism $\chi : M \otimes_R L \to A$ with $\chi(m\otimes l) = \chi'(m,l)$. The inverse of $\phi_{L,A}$ is given by

$$\begin{aligned} \phi^{-1}_{L,A} :\ & \text{Hom}_{\text{Ab}}(M \otimes_R L, A) && \to && \text{Hom}_{R\text{-Mod}}(L, \text{Hom}_{\text{Ab}}(M,A)) \\ & \chi : M\otimes_R L \to A, && \mapsto && \psi : L \to \text{Hom}_{\text{Ab}}(M,A),\ l \mapsto \psi_l \text{ with } \psi_l(m) = \chi(m \otimes l). \end{aligned}$$

As we have $\psi_{r\rhd l}(m) = \chi(m \otimes (r \rhd l)) = \chi((m \lhd r) \otimes l) = \psi_l(m \lhd r)$, the map ψ_l is indeed R-linear, and a short computation shows that the diagram (95) commutes for all R-linear maps $f : L' \to L$ and all group homomorphisms $g : A \to A'$.

7. **Restriction, induction and coinduction**:
Let $\phi : R \to S$ be a ring homomorphism and $\text{Res} : S\text{-Mod} \to R\text{-Mod}$ the **restriction functor** from Example B.6, 5. that sends an S-module $(M, \rhd_S)$ to the R-module $(M, \rhd_R)$ with $r \rhd_R m = \phi(r) \rhd_S m$ and every S-linear map $f : M \to M'$ to itself. Then:

• The **induction functor** $\text{Ind} = S\otimes_R - : R\text{-Mod} \to S\text{-Mod}$ is left adjoint to Res. It sends
- an R-module M to the S-module $\text{Ind}(M) = S \otimes_R M$ with $s \rhd (s'\otimes m) = (ss')\otimes m$,
- an R-linear map $f : M \to M'$ to the S-linear map $\text{Ind}(f) = \text{id}_S \otimes f$.

• The **coinduction functor** $\text{Coind} = \text{Hom}_R(S,-) : R\text{-Mod} \to S\text{-Mod}$ is right adjoint to Res. It sends
- an R-module M to the S-module $\text{Hom}_R(S,M)$ with $(s \rhd f)(s') = f(s' \cdot s)$,
- an R-linear map $f : M \to M'$ to $\text{Hom}_R(S,f) : g \mapsto f \circ g$.

To see that Ind is left adjoint to Res, note that by Lemma **??** the (S,R)-bimodule structure on S given by $s \rhd s' = s \cdot s'$ and $s \lhd r = s \cdot \phi(r)$ defines an S-left-module structure on the abelian group $S\otimes_R M$ given by $s \rhd (s'\otimes m) = (s \cdot s')\otimes m$. For all R-modules M and S-modules N the group homomorphisms

$$\begin{aligned} &\phi_{M,N} : \text{Hom}_R(M, \text{Res}(N)) \to \text{Hom}_S(\text{Ind}(M), N), && \phi_{M,N}(f)(s\otimes m) = s \rhd f(m) \\ &\psi_{M,N} : \text{Hom}_S(\text{Ind}(M), N) \to \text{Hom}_R(M, \text{Res}(N)), && \psi_{M,N}(g)(m) = g(1\otimes m). \end{aligned}$$

are mutually inverse and hence bijections. To prove that the diagram (95) commutes, we compute for all R-linear maps $f : M' \to M$, $h : M \to N$ and S-linear maps $g : N \to N'$

$$\begin{aligned} g \circ \phi_{M,N}(h) \circ (\text{id}_S\otimes f)(s \otimes m') &= g \circ \phi_{M,N}(h)(s\otimes f(m')) = g(s \rhd h \circ f(m')) \\ &= s \rhd (g \circ h \circ f(m')) = \phi_{M',N'}(g \circ h \circ f)(s \otimes m'). \end{aligned}$$

To show that Coind is right adjoint to Res we consider the ring S with the R-left module structure $r \rhd s := \phi(r) \cdot s$ and the abelian group $\text{Hom}_R(S,M)$ with the S-left module structure $(s \triangleright f)(s') = f(s' \cdot s)$ and note that the maps

$$\begin{aligned}
\phi_{M,N} &: \mathrm{Hom}_R(\mathrm{Res}(N), M) \to \mathrm{Hom}_S(N, \mathrm{Hom}_R(S, M)), &\qquad \phi_{M,N}(f)(s) &= f(s \triangleright n)\\
\psi_{M,N} &: \mathrm{Hom}_S(N, \mathrm{Hom}_R(S, M)) \to \mathrm{Hom}_R(\mathrm{Res}(N), M), &\qquad \psi_{M,N}(g)(n) &= g(n)(1).
\end{aligned}$$

are mutually inverse and hence bijections. A short computation shows that $\phi_{M,N}$ makes the diagram (95) commute.

8. **Induction, coinduction and forgetful functor**:
   For every ring S, the induction functor $\mathrm{Ind} = S\otimes_{\mathbb{Z}} - : \mathrm{Ab} \to S\text{-Mod}$ is left adjoint and the coinduction functor $\mathrm{Coind} = \mathrm{Hom}_{\mathbb{Z}}(S, -) : \mathrm{Ab} \to S\text{-Mod}$ is right adjoint to the forgetful functor $\mathrm{Res} : S\text{-Mod} \to \mathrm{Ab}$.

   This is Example B.19, 7. for $R = \mathbb{Z}$, where $\mathrm{Res} : S\text{-Mod} \to \mathrm{Ab}$ is the forgetful functor.

These examples show that adjoint functors arise in many contexts in algebra and topology and are often related to certain canonical constructions such as forgetful functors, freely generated modules or tensoring over a ring. Example B.19, 3. shows that a functor need not have left and right adjoints. However, it seems plausible that if they exist, left or right adjoint functors should be unique, at least up to natural isomorphisms. To address this question, it is advantageous to work with an alternative characterisation of left and right adjoints in terms of natural transformations.

Proposition B.20: A functor $F : \mathcal{C} \to \mathcal{D}$ is left adjoint to $G : \mathcal{D} \to \mathcal{C}$ if and only if there are natural transformations $\epsilon : FG \to \mathrm{id}_{\mathcal{D}}$ and $\eta : \mathrm{id}_{\mathcal{C}} \to GF$ such that

$$(G\epsilon) \circ (\eta G) = \mathrm{id}_G, \qquad (\epsilon F) \circ (F\eta) = \mathrm{id}_F. \tag{96}$$

Proof:
1. Let $F : \mathcal{C} \to \mathcal{D}$ be left adjoint to $G : \mathcal{D} \to \mathcal{C}$. Then there are bijections

$$\begin{aligned}
\phi_{G(D),D} &: \mathrm{Hom}_{\mathcal{C}}(G(D), G(D)) \to \mathrm{Hom}_{\mathcal{D}}(FG(D), D)\\
\phi^{-1}_{C,F(C)} &: \mathrm{Hom}_{\mathcal{D}}(F(C), F(C)) \to \mathrm{Hom}_{\mathcal{C}}(C, GF(C)).
\end{aligned}$$

We define the natural transformations $\epsilon : FG \to \mathrm{id}_{\mathcal{D}}$ and $\eta : \mathrm{id}_{\mathcal{C}} \to GF$ by specifying their component morphisms:

$$\epsilon_D := \phi_{G(D),D}(1_{G(D)}) : FG(D) \to D \qquad \eta_C := \phi^{-1}_{C,F(C)}(1_{F(C)}) : C \to GF(C).$$

The commuting diagram (95) in Definition B.18 implies for all morphisms $f : D \to D'$ in $\mathcal{D}$:

$$\begin{aligned}
\epsilon_{D'} \circ FG(f) &= \phi_{G(D'),D'}(1_{G(D')}) \circ FG(f) \overset{(95)}{=} \phi_{G(D),D}(1_{G(D')} \circ G(f)) = \phi_{G(D),D}(G(f))\\
&= \phi_{G(D),D}(G(f) \circ 1_{G(D)}) \overset{(95)}{=} f \circ \phi_{G(D),D}(1_{G(D)}) = f \circ \epsilon_D.
\end{aligned}$$

This shows that the morphisms $\epsilon_D : FG(D) \to D$ define a natural transformation $\epsilon : FG \to \mathrm{id}_{\mathcal{D}}$. Diagram (95) then implies for all objects C in $\mathcal{C}$

$$\begin{aligned}
\epsilon_{F(C)} \circ F(\eta_C) &= \phi_{GF(C),F(C)}(1_{GF(C)}) \circ F(\phi^{-1}_{C,F(C)}(1_{F(C)}))\\
&\overset{(95)}{=} \phi_{C,F(C)}(1_{GF(C)} \circ \phi^{-1}_{C,F(C)}(1_{F(C)})) = \phi_{C,F(C)} \circ \phi^{-1}_{C,F(C)}(1_{F(C)}) = 1_{F(C)}.
\end{aligned}$$

The proofs for $\eta : \mathrm{id}_{\mathcal{C}} \to GF$ and of the identity $G(\epsilon_D) \circ \eta_{G(D)} = 1_{G(D)}$ are analogous.

2. Let $\epsilon : FG \to \mathrm{id}_{\mathcal{D}}$ and $\eta : \mathrm{id}_{\mathcal{C}} \to GF$ be natural transformations that satisfy (96). Consider for all objects C in $\mathcal{C}$ und D in $\mathcal{D}$ the maps

$$\begin{aligned}
\phi_{C,D} &= \mathrm{Hom}(1_{F(C)}, \epsilon_D) \circ F : \mathrm{Hom}_{\mathcal{C}}(C, G(D)) \to \mathrm{Hom}_{\mathcal{D}}(F(C), D), \quad f \mapsto \epsilon_D \circ F(f)\\
\psi_{C,D} &= \mathrm{Hom}(\eta_C, 1_{G(D))}) \circ G : \mathrm{Hom}_{\mathcal{D}}(F(C), D) \to \mathrm{Hom}_{\mathcal{C}}(C, G(D)), \quad g \mapsto G(g) \circ \eta_C.
\end{aligned}$$

Then we have for all morphisms $f : C \to G(D)$ in $\mathcal{C}$ and $g : F(C) \to D$ in $\mathcal{D}$

$$\begin{aligned}
\psi_{C,D} \circ \phi_{C,D}(f) &= G(\epsilon_D) \circ GF(f) \circ \eta_C \overset{\text{nat}}{=} G(\epsilon_D) \circ \eta_{G(D)} \circ f \overset{(96)}{=} f\\
\phi_{C,D} \circ \psi_{C,D}(g) &= \epsilon_D \circ FG(g) \circ F(\eta_C) \overset{\text{nat}}{=} g \circ \epsilon_{F(C)} \circ F(\eta_C) \overset{(96)}{=} g.
\end{aligned}$$

This shows that $\psi_{C,D} = \phi_{C,D}^{-1}$ and $\phi_{C,D} : \mathrm{Hom}_{\mathcal{C}}(C, G(D)) \to \mathrm{Hom}_{\mathcal{D}}(F(C), D)$ is a bijection. To verify that the diagram (95) in Definition B.18 commutes, consider morphisms $f : C' \to C$, $h : C \to G(D)$ in $\mathcal{C}$ and $g : D \to D'$ in $\mathcal{D}$ and compute

$$\phi_{C',D'}(G(g) \circ h \circ f) = \epsilon_{D'} \circ FG(g) \circ F(h) \circ F(f) \overset{\text{nat}}{=} g \circ \epsilon_D \circ F(h) \circ F(f) = g \circ \phi_{C,D}(h) \circ F(f).$$

□

Theorem B.21: Left and right adjoint functors are unique up to natural isomorphisms.

Proof:
Let $F, F' : \mathcal{C} \to \mathcal{D}$ be left adjoint to $G : \mathcal{D} \to \mathcal{C}$. Then by Proposition B.20 there are natural transformations $\epsilon : FG \to \mathrm{id}_{\mathcal{D}}$, $\eta : \mathrm{id}_{\mathcal{C}} \to GF$ and $\epsilon' : F'G \to \mathrm{id}_{\mathcal{D}}$, $\eta' : \mathrm{id}_{\mathcal{C}} \to GF'$ satisfying (96). Consider the natural transformations $\kappa = (\epsilon F') \circ (F\eta') : F \to F'$, $\kappa' = (\epsilon' F) \circ (F'\eta) : F' \to F$ with component morphisms $\kappa_C = \epsilon_{F'(C)} \circ F(\eta'_C)$ and $\kappa'_C = \epsilon'_{F(C)} \circ F'(\eta_C)$. Then κ_C und κ'_C are inverse to each other since

$$\begin{aligned}
\kappa_C \circ \kappa'_C &\overset{\text{def}\,\kappa}{=} \epsilon_{F'(C)} \circ F(\eta'_C) \circ \kappa'_C \overset{\text{nat}\,\kappa'}{=} \epsilon_{F'(C)} \circ \kappa'_{GF'(C)} \circ F'(\eta'_C)\\
&\overset{\text{def}\,\kappa'}{=} \epsilon_{F'(C)} \circ \epsilon'_{FGF'(C)} \circ F'(\eta_{GF'(C)}) \circ F'(\eta'_C) \overset{\text{nat}\,\epsilon'}{=} \epsilon'_{F'(C)} \circ F'G(\epsilon_{F'(C)}) \circ F'(\eta_{GF'(C)}) \circ F'(\eta'_C)\\
&= \epsilon'_{F'(C)} \circ F'(G(\epsilon_{F'(C)}) \circ \eta_{GF'(C)}) \circ F'(\eta'_C) \overset{(96)}{=} \epsilon'_{F'(C)} \circ F'(\eta'_C) \overset{(96)}{=} 1_{F'(C)},
\end{aligned}$$

and an analogous computation yields $\kappa'_C \circ \kappa_C = 1_{F(C)}$. This shows that κ and κ' are natural isomorphisms and that F is naturally isomorphic to F'. The proof for right adjoints is analogous. □

B.2 Abelian categories

In this section, we summarise the required background on abelian categories and exact functors for Section 8. Roughly speaking, an abelian category is a category $\mathcal{A}$ in which

- products and coproducts exist for all finite families of objects,
- the morphism sets $\mathrm{Hom}_{\mathcal{A}}(A, B)$ have the structure of abelian groups, and the composition of morphisms is biadditive,
- there is a notion of a kernel and the dual notion of a cokernel for morphisms in $\mathcal{A}$ that behave like kernels of module homomorphisms and quotients by their images.

and an exact functor is a functor that respects the abelian group structures on the Hom-sets and preserves kernels and cokernels.

The first two requirements on an abelian category lead to the concept of an *additive category*. Functors between *additive categories* that respect these conditions are called *additive functors*.

Definition B.22: A category $\mathcal{C}$ is called **additive** if

(Add1) For all objects C, C' of $\mathcal{C}$ the set of morphisms $\mathrm{Hom}_{\mathcal{C}}(C, C')$ has the structure of an abelian group, and the composition of morphisms is $\mathbb{Z}$-bilinear: $g\circ(f+f') = g\circ f+g\circ f'$ and $(g+g')\circ f = g\circ f + g'\circ f$ for all morphisms $f, f' : X \to Y$ and $g, g' : Y \to Z$.

(Add2) Products and coproducts exist for all finite families of objects in $\mathcal{C}$.

A functor $F : \mathcal{C} \to \mathcal{D}$ between additive categories $\mathcal{C}$, $\mathcal{D}$ is called **additive** if for all objects C, C' in $\mathcal{C}$ the map $F : \mathrm{Hom}_{\mathcal{C}}(C, C') \to \mathrm{Hom}_{\mathcal{D}}(F(C), F(C'))$ is a group homomorphism.

Remark B.23:

1. In particular, Definition B.22 requires the existence of an *empty* product and an *empty coproduct*, a terminal object $T = \Pi_\emptyset$ and an initial object $I = \amalg_\emptyset$ (see Definition B.16). In an additive category $\mathcal{C}$, these objects are isomorphic and hence zero objects: $I \cong T \cong 0$.

   This follows because one has $\mathrm{Hom}_{\mathcal{C}}(I, I) = \{1_I\} = \{0\}$ by definition of an initial object, where 0 denotes the neutral element of the abelian group $\mathrm{Hom}_{\mathcal{C}}(I, I)$. If $\mathcal{C}$ is additive, this implies $f = 1_I \circ f = 0 \circ f = 0 : C \to I$ for any morphism $f : C \to I$, since the composition of morphisms is $\mathbb{Z}$-bilinear. It follows that $\mathrm{Hom}_{\mathcal{C}}(C, I) = \{0\}$ and hence I is terminal.

2. It follows that for any two objects C, C' in an additive category $\mathcal{C}$, the neutral element of the abelian group $\mathrm{Hom}_{\mathcal{C}}(C, C')$ is given by $0 = i_{C'} \circ t_C : C \to 0 \to C'$.

3. Finite products and coproducts in additive categories are canonically isomorphic: $\Pi_{i\in I} C_i \cong \amalg_{i\in I} C_i$ for all finite index sets $i \in I$ and objects C_i in $\mathcal{C}$.

   The isomorphism is induced by the family $(f_{ij})_{i,j\in I}$ of morphisms $f_{ij} = \delta_{ij} 1_{C_i} : C_i \to C_j$ with $f_{ij} = 0$ for $i \neq j$ and $f_{ii} = 1_{C_i}$. By the universal property of the product and the coproduct, there is a unique morphism $f : \amalg_{k\in I} C_k \to \Pi_{k\in I} C_k$ with $\pi_j \circ f \circ \iota_i = \delta_{ij} 1_{C_i}$. The inverse of this morphism is given by $f^{-1} = \Sigma_{i\in I}\, \iota_i \circ \pi_i : \Pi_{k\in I} C_k \to \amalg_{k\in I} C_k$, since

   $$\begin{aligned} &\pi_k \circ f \circ f^{-1} = \Sigma_{i\in I}\, \pi_k \circ f \circ \iota_i \circ \pi_i = \Sigma_{i\in I}\, \delta_{ik} 1_{C_i} \circ \pi_i = \pi_k \\ &f^{-1} \circ f \circ \iota_k = \Sigma_{i\in I}\, \iota_i \circ \pi_i \circ f \circ \iota_k = \Sigma_{i\in I}\, \iota_i \circ \delta_{ik} 1_{C_i} = \iota_k \qquad \forall k \in I, \end{aligned}$$

   and the universal property of the (co)product implies $f \circ f^{-1} = 1_{\Pi_{i\in I} C_i}$, $f^{-1} \circ f = 1_{\amalg_{i\in I} C_i}$.

4. The abelian group structure on the morphism sets $\mathrm{Hom}_{\mathcal{C}}(C, C')$ in an additive category $\mathcal{C}$ is determined uniquely by its products and coproducts.

   For a finite index set I and an object C in $\mathcal{C}$ we denote by $\phi_C : \amalg_{i\in I} C \to \Pi_{i\in I} C$ the unique morphism with $\pi_i \circ \phi_C \circ \iota_j = \delta_{ij}\, 1_C$ from 3. with inverse $\phi_C^{-1} = \Sigma_{i\in I}\, \iota_i \circ \pi_i : \Pi_{i\in I} C \to \amalg_{i\in I} C$. We also consider the unique morphism $\Delta_C : C \to \Pi_{i\in I} C$ with $\pi_i \circ \Delta_C = 1_C$ for all $i \in I$ and the unique morphism $\nabla_C : \amalg_{i\in I} C \to C$ with $\nabla_C \circ \iota_i = 1_C$ for all $i \in I$. For a finite family $(f_i)_{i\in I}$ of morphisms $f_i : C \to D$, we consider the unique morphism $f : \amalg_{i\in I} C \to \Pi_{i\in I} D$

with $\pi_j \circ f \circ \iota_i = \delta_{ij} f_i$ from 3. Then we have

$$\begin{aligned}\nabla_D \circ \phi_D^{-1} \circ f \circ \phi_C^{-1} \circ \Delta_C &= \Sigma_{i,j\in I} \nabla_D \circ \iota_i \circ \pi_i \circ f \circ \iota_j \circ \pi_j \circ \Delta_C \\ &= \Sigma_{i,j\in I} 1_D \circ (\pi_i \circ f \circ \iota_j) \circ 1_C = \Sigma_{i,j\in I} \delta_{ij}\, 1_D \circ f_i \circ 1_C = \Sigma_{i\in I} f_i.\end{aligned}$$

Hence, we expressed the sum of the morphisms f_i in terms of quantities that are defined in terms of the product and coproduct in an additive category. (This includes the morphism f, since the zero object that enters its definition is the empty coproduct). As products and coproducts are unique up to unique isomorphisms, a given category $\mathcal{C}$ has at most one additive structure. Additivity is a *property*, not a choice of *structure*.

5. An object X in an additive category $\mathcal{C}$ is a product or coproduct of a finite family of objects $(C_i)_{i\in I}$ if and only if there are families $(i_j)_{j\in I}$ and $(p_j)_{j\in I}$ of morphisms $i_j : C_j \to X$ and $p_j : X \to C_j$ with $p_j \circ i_k = \delta_{jk} 1_{C_j}$ and $1_X = \Sigma_{i\in I} \iota_j \circ p_j$ (Exercise **??**).

6. A functor $F : \mathcal{C} \to \mathcal{D}$ between additive categories $\mathcal{C}, \mathcal{D}$ is additive if and only if it preserves finite products or finite coproducts (Exercise **??**):

$F(\Pi_{i\in I} C_i) \cong \Pi_{i\in I} F(C_i)$, $F(\amalg_{i\in I} C_i) \cong \amalg_{i\in I} F(C_i)$ for all finite families of objects $(C_i)_{i\in I}$.

Example B.24:

1. For any ring R the category R-Mod of R-modules und R-linear maps is additive.

   Products and coproducts are products and direct sums of modules and exist for all families of modules. The set $\mathrm{Hom}_R(M, N)$ of R-linear maps $f : M \to N$ is an abelian group with the pointwise addition, and this is compatible with their composition.

2. For any ring homomorphism $\phi : R \to S$, the functor $F_\phi : S\text{-Mod} \to R\text{-Mod}$ that sends an S-module $(M, \rhd)$ to the R-module $(M, \rhd_R)$ with $r \rhd_R m = \phi(r) \rhd m$ and an S-linear map $f : (M, \rhd) \to (M', \rhd')$ to the associated R-linear map $f : (M, \rhd_R) \to (M', \rhd'_R)$ is additive.

3. Every full subcategory of an additive category $\mathcal{C}$ in which finite products and coproducts exist, is an additive category as well.

4. For every small category $\mathcal{C}$ and additive category $\mathcal{A}$, the category $\mathrm{Fun}(\mathcal{C}, \mathcal{A})$ of functors $F : \mathcal{C} \to \mathcal{A}$ and natural transformations between them is an additive category.

   • The product of a family of functors $(F_i)_{i\in I}$ is the functor $\Pi_{i\in I} F_i : \mathcal{C} \to \mathcal{A}$ that assigns to an object C the product $\Pi_{i\in I} F_i(C)$ and to a morphism $\alpha : C \to C'$ the unique morphism $\Pi_{i\in I} F_i(\alpha) : \Pi_{i\in I} F_i(C) \to \Pi_{i\in I} F_i(C')$ with $\pi_{iC'} \circ \Pi_{i\in I} F_i(\alpha) = F_i(\alpha) \circ \pi_{iC}$, where $\pi_{iC} : \Pi_{i\in I} F_i(C) \to F_i(C)$ are the projection morphisms for the product in $\mathcal{A}$.

   • The projection morphisms for $\Pi_{i\in I} F_i$ are the natural transformations $\pi_i : \Pi_{i\in I} F_i \to F_i$ with component morphisms $\pi_{iC} : \Pi_{i\in I} F_i(C) \to F_j(C)$.

   • Coproducts of functors are defined analogously, and the sum of two natural transformations $\eta, \kappa : F \to G$ is the natural transformation $\eta + \kappa : F \to G$ with component morphisms $(\eta + \kappa)_C = \eta_C + \kappa_C : F(C) \to G(C)$.

We now develop a notion of kernels and images for morphisms in a category $\mathcal{C}$. In contrast to the standard definition of a kernel and image of an R-module morphism $f : M \to N$, as *subsets* of the modules M and N, a sensible categorical notion of a kernel and image must be formulated purely in terms of morphisms and universal properties. These morphisms and their universal properties must thus generalise the inclusion maps $\iota : \ker(f) \to M$ and $\iota' : \mathrm{im}(f) \to N$. There are also two canonical surjection associated with each R-linear map $f : M \to N$, the canonical surjection $\pi : N \to N/\mathrm{im}(f)$ and the canonical surjection $\pi' : M \to M/\ker(f)$. The former is generalised by the dual notion of a cokernel in a category $\mathcal{C}$ and the latter by the notion of a coimage. These categorical concepts due not require additivity, but they require a zero object 0 in $\mathcal{C}$ and the associated zero morphisms $0 = i_{C'} \circ t_C : C \to 0 \to C'$.

Definition B.25: Let $\mathcal{C}$ be a category with a zero object and $f : X \to Y$ a morphism in $\mathcal{C}$.

1. A **kernel** of f is a morphism $\iota : \ker(f) \to X$ with the following **universal property**: $f \circ \iota = 0 : \ker(f) \to Y$, and for every morphism $g : W \to X$ with $f \circ g = 0 : W \to Y$ there is a unique morphism $g' : W \to \ker(f)$ with $\iota \circ g' = g$.

$$\begin{tikzcd}
\ker(f) \arrow[r, "\iota"] \arrow[rr, bend left, "0"] & X \arrow[r, "f"] & Y \\
 & W \arrow[ul, dashed, "\exists! g'"] \arrow[u, "g"'] \arrow[ur, "0"'] &
\end{tikzcd}$$

2. A **cokernel** of f is a morphism $\pi : Y \to \mathrm{coker}(f)$ with the following **universal property**: $\pi \circ f = 0 : X \to \mathrm{coker}(f)$, and for every morphism $g : Y \to W$ with $g \circ f = 0 : X \to W$ there is a unique morphism $g' : \mathrm{coker}(f) \to W$ with $g' \circ \pi = g$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend left, "0"] \arrow[dr, "0"'] & Y \arrow[r, "\pi"] \arrow[d, "g"] & \mathrm{coker}(f) \arrow[dl, dashed, "\exists! g'"] \\ & W & \end{tikzcd}$$

3. A kernel of a cokernel of f is called an **image** of f and denoted $\iota' : \mathrm{im}(f) \to Y$. A cokernel of a kernel of f is called a **coimage** of f and denoted $\pi' : X \to \mathrm{coim}(f)$.

Remark B.26: As (co)kernels and (co)images are defined by a universal property, they are unique up to unique isomorphism: If $\iota : \ker(f) \to X$, $\eta : \ker(f)' \to X$ are two kernels for $f : X \to Y$, then there is a unique morphism $\phi : \ker(f) \to \ker(f)'$ with $\eta \circ \phi = \iota$, and this morphism is an isomorphism. Analogous statements hold for cokernels, images and coimages.

Example B.27: Let R be a ring and $f : M \to N$ an R-linear map.

- The inclusion map $\iota : \ker(f) \to M$ is a kernel of f in R-Mod.
- The canonical surjection $\pi : N \to N/\mathrm{im}(f)$ is a cokernel of f in R-Mod.
- The canonical inclusion $\iota' : \mathrm{im}(f) \to N$ is an image of f in R-Mod.
- The canonical surjection $\pi' : M \to M/\ker(f)$ is a coimage of f in R-Mod.

That $\iota : \ker(f) \to M$ is a kernel of f follows, because $f \circ \iota = 0$ and for any R-linear map $\phi : L \to M$ with $f \circ \phi = 0$, one has $\mathrm{im}(\phi) \subset \ker(f)$. The corestriction $\phi' : L \to \ker(f)$, $l \mapsto \phi(l)$ is an R-linear map with $\iota \circ \phi' = \phi$. As ι is injective, it is the only one.

That $\pi : N \to N/\mathrm{im}(f)$ is a cokernel of f follows, because $\pi \circ f = 0$ and for any R-linear map $\psi : N \to P$ with $\psi \circ f = 0$ one has $\mathrm{im}(f) \subset \ker(\psi)$. By the characteristic property of the quotient, there is a unique R-linear map $\psi' : N/\mathrm{im}(f) \to P$, $[n] \mapsto \psi(n)$ with $\psi' \circ \pi = \psi$.

That the inclusion map $\iota' : \mathrm{im}(f) \to N$ is a kernel of $\pi : N \to N/\mathrm{im}(f)$ follows, because $\pi \circ \iota' = 0$, and for any R-linear map $\chi : L \to N$ with $\pi \circ \chi = 0$, one has $\mathrm{im}(\chi) \subset \ker(\pi) = \mathrm{im}(f)$. The corestriction $\chi' : L \to \mathrm{im}(f)$, $l \mapsto \chi(l)$ satisfies $\chi' \circ \iota' = \chi$ and is the only R-linear map with this property, since ι' is injective.

That the canonical surjection $\pi' : M \to M/\ker(f)$ is a cokernel of $\iota : \ker(f) \to M$ follows because $\pi' \circ \iota = 0$ and because any R-linear map $\xi : M \to P$ with $\xi \circ \iota = 0$ satisfies $\mathrm{im}(\iota) = \ker(f) \subset \ker(\xi)$. By the characteristic property of the quotient, there is a unique R-linear map $\xi' : M/\ker(f) \to P$ with $\xi' \circ \pi' = \xi$.

In addition to kernels and cokernels, we also require an appropriate concept of injectivity and surjectivity and must relate it to kernels and cokernels. Just as for kernels and cokernels, the appropriate notion of injectivity and surjectivity in a category needs to be formulated purely in terms of morphisms and universal properties. It is obtained from the observation that a map $\iota : X \to Y$ is injective (a map $\pi : X \to Y$ is surjective) if and only if $\iota \circ f = \iota \circ g$ ($f \circ \pi = g \circ \pi$) implies $f = g$ for all maps $f, g : W \to X$ (for all maps $f, g : Y \to Z$). This notion of injectivity and surjectivity in Set generalises to any category.

Definition B.28: Let $\mathcal{C}$ be a category.

1. A morphism $\iota : X \to Y$ in $\mathcal{C}$ is called a **monomorphism**, if $\iota \circ f = \iota \circ g$ for morphisms $f, g : W \to X$ implies $f = g$.
2. A morphism $\pi : X \to Y$ in $\mathcal{C}$ is called an **epimorphism**, if $f \circ \pi = g \circ \pi$ for morphisms $f, g : Y \to Z$ implies $f = g$.

In diagrams, monomorphisms $\iota : X \to Y$ are denoted $X \xhookrightarrow{\iota} Y$ and epimorphisms $\pi : X \to Y$ are denoted $X \xrightarrow{\pi} \!\!\!\!\!\rightarrow Y$.

Remark B.29: Clearly, every isomorphism is a monomorphism and an epimorphism. However, a morphism that is a monomorphism and an epimorphism need not be an isomorphism. A counterexample is the inclusion morphism $\iota : \mathbb{Z} \to \mathbb{Q}$ in the category of unital rings.

We now relate epimorphisms and monomorphisms to (co)kernels and (co)images. Example B.27 shows that in the category R-Mod the kernel $\iota : \ker(f) \to M$ of an R-linear map $f : M \to N$ is injective and its cokernel $\pi : N \to N/\mathrm{im}(f)$ is surjective. Moreover, the module morphism $0 \to M$ is a kernel of f if and only if f is injective and the module morphism $N \to 0$ is a cokernel of f if and only if f is surjective. Analogues of this hold in all *additive* categories.

Lemma B.30: Let $\mathcal{C}$ be an additive category.

1. All kernels of morphisms in $\mathcal{C}$ are monomorphisms. A morphism $f : X \to Y$ is a monomorphism if and only if the morphism $i_X : 0 \to X$ is a kernel of f.
2. All cokernels of morphisms in $\mathcal{C}$ are epimorphisms. A morphism $f : X \to Y$ is an epimorphism if and only if the morphism $t_Y : Y \to 0$ is a cokernel of f.

Proof:
We prove the first statement. The proof of the second one is analogous. Let $\iota : \ker(f) \to X$ be a kernel of $f : X \to Y$ and $g_1, g_2 : W \to \ker(f)$ morphisms with $\iota \circ g_1 = \iota \circ g_2$. Then we have $f \circ (\iota \circ g_i) = (f \circ \iota) \circ g_i = 0 \circ g_i = 0 : W \to Y$, and by the universal property of the kernel, there is a *unique* morphism $g' : W \to \ker(f)$ with $\iota \circ g' = \iota \circ g_1 = \iota \circ g_2$. The uniqueness implies $g' = g_1 = g_2$, and hence $\iota : W \to \ker(f)$ is a monomorphism.

Let now $f : X \to Y$ be a monomorphism. We have $f \circ i_X = i_Y = 0 : 0 \to Y$. If $g : W \to X$ is a morphism with $f \circ g = 0 : W \to X$ then $f \circ i_X \circ t_W = 0 : W \to X$ as well, and because f is a monomorphism, it follows that $g = i_X \circ t_W$. Hence, $i_X : 0 \to X$ is a kernel of f.

Conversely, if $i_X : 0 \to X$ is a kernel of f and $g_1, g_2 : W \to X$ are morphisms with $f \circ g_1 = f \circ g_2$, then $f \circ (g_1 - g_2) = 0$ and by the universal property of the kernel, there is a unique morphism $g' : W \to 0$ with $i_X \circ g' = g_1 - g_2 = 0$. Since $g' = t_W : W \to 0$ is the only morphism from W to 0, we have $g_1 - g_2 = i_X \circ t_W = 0 : W \to X$ and $g_1 = g_2$. This shows that f is a monomorphism $\square$

This lemma shows that in any additive category, kernels are monomorphisms and cokernels epimorphisms, as expected from the corresponding statement for R-Mod. However, in R-Mod, the converse also holds. Every injective R-linear map $f : M \to N$ is a kernel, namely of its cokernel $\pi : N \to N/\mathrm{im}(f)$. This follows because $\pi \circ f = 0$, and for every R-linear map $g : L \to N$ with $\pi \circ g = 0$ one has $\mathrm{im}(g) \subset \ker(\pi) = \mathrm{im}(f)$. Hence, by injectivity of f there is a unique R-linear map $g' : L \to M$ with $f \circ g' = g$.

Similarly, every surjective R-linear map $f : M \to N$ is a cokernel of its kernel $\iota : \ker(f) \to M$. One has $f \circ \iota = 0$ and $\ker(f) = \mathrm{im}(\iota) \subset \ker(g)$ for every R-linear map $g : M \to L$ with $g \circ \iota = 0$. As f is surjective, there is a unique R-linear map $g' : N \to L$, $f(m) \mapsto g(m)$ with $g' \circ f = g$.

In contrast to the claims in Lemma B.30, these statements do not hold automatically in an additive category. They require in particular that every monomorphism has a cokernel and every epimorphism has a kernel, which is not guaranteed in an additive category. If we impose these conditions and the existence of kernels and cokernels for all morphisms, we obtain the notion of an *abelian category*, which we will use later as the framework for (co)homology.

We also consider functors between abelian categories that are compatible with the required structures. Clearly, such functors need to be additive and map kernels to kernels and cokernels to cokernels. We will see in the following that there are many additive functors that satisfy only one of the last two conditions and that these functors play an important role in (co)homology. For this reason, they also receive a name.

Definition B.31:

1. An additive category is called **abelian** if it satisfies the following additional conditions:
   (Ab1) Every morphism has a kernel and a cokernel.
   (Ab2) Every monomorphism is a kernel of its cokernel or, equivalently, an image of itself.
   (Ab3) Every epimorphism is a cokernel of its kernel or, equivalently, a coimage of itself.
2. A functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories $\mathcal{A}$, $\mathcal{B}$ is called
   - **left exact** if it is additive and preserves kernels:
     if $\iota : \ker(f) \to X$ is a kernel of $f : X \to Y$, then $F(\iota) : F(\ker(f)) \to F(X)$ is a kernel of $F(f) : F(X) \to F(Y)$.
   - **right exact** if it is additive and preserves cokernels:
     if $\pi : Y \to \mathrm{coker}(f)$ is a cokernel of $f : X \to Y$, then $F(\pi) : F(Y) \to F(\mathrm{coker}(f))$ is a cokernel of $F(f) : F(X) \to F(Y)$.
   - **exact** if it is left exact and right exact.

Example B.32:

1. For any ring R, the category R-Mod is abelian.

   By Example B.24, 1. it is additive, and by Example B.27, every R-linear map $f : M \to N$ has a kernel $\iota : \ker(f) \to M$ and a cokernel $\pi : N \to N/\mathrm{im}(f)$. As shown above, every monomorphism in R-Mod is a kernel of its cokernel and every epimorphism a cokernel of its kernel.

2. The category of finitely generated free abelian groups and group homomorphisms between them is additive, but not abelian.

3. The full subcategory of $\mathrm{Vect}_{\mathbb{F}}$ with even-dimensional $\mathbb{F}$-vector spaces of objects is additive, but not abelian.

4. For any abelian category $\mathcal{A}$, the category $\mathcal{A}^{op}$ is abelian. Kernels and cokernels in $\mathcal{A}^{op}$ correspond to cokernels and kernels in $\mathcal{A}$, respectively.

5. For any small category $\mathcal{C}$ and any abelian category $\mathcal{A}$ the category $\mathrm{Fun}(\mathcal{C}, \mathcal{A})$ of functors $F : \mathcal{C} \to \mathcal{A}$ and natural transformations between them is abelian.

Remark B.33:

1. One can show that in an abelian category $\mathcal{A}$ a morphism that is both a monomorphism and an epimorphism is an isomorphism.
2. Like additivity, being abelian is a *property* of a category and *not a choice of structure*. If all objects in an additive category have kernels and cokernels that satisfy the conditions in Definition B.31, they are unique up to unique isomorphism and determined by the additive structure.
3. **Mitchell's embedding theorem** states that any *small* abelian category $\mathcal{A}$ is equivalent to a full subcategory of the abelian category R-Mod for some ring R, with an exact equivalence of categories. For a proof, see [Mi, p 151].

Although Mitchell's embedding theorem allows one to interpret any *small* abelian category as a subcategory of the abelian category R-Mod for a suitable ring R, it is still advantageous to work with general abelian categories. Firstly, there are also non-small abelian categories. Secondly, the construction of the associated ring R and the subcategory of R-Mod for an abelian category $\mathcal{A}$ in Mitchell's embedding theorem is implicit. It is often simpler to use the general formalism for abelian categories.

In an abelian category kernels and cokernels exist for all morphisms and generalise the inclusion maps $\iota : \ker(f) \to X$ and the canonical surjections $\pi : Y \to Y/\mathrm{im}(f)$ for R-linear maps $f : X \to Y$. Similarly, images and coimages exist for all morphisms and generalise the inclusion $\iota' : \mathrm{im}(f) \to Y$ and the canonical surjection $\pi' : X \to X/\ker(f)$.

A well-known result in R-Mod states that for any R-linear map $f : X \to Y$ one has $\mathrm{im}(f) \cong X/\ker(f)$ and that f factorises as $f = \iota' \circ f'$, where $f' : X \to \mathrm{im}(f)$ is the corestriction of f and $\iota' : \mathrm{im}(f) \to Y$ the inclusion. These statements have counterparts in abelian categories. The first statement translates into the statement that $\mathrm{im}(f) \cong \mathrm{coim}(f)$, the second into the canonical factorisation in an abelian category.

Lemma B.34: Let $\mathcal{A}$ be an abelian category. Every morphism $f : X \to Y$ in $\mathcal{A}$ factorises as $f = \iota'_f \circ \pi'_f$ where $\iota'_f : \mathrm{im}(f) \to Y$ is an image of f and $\pi'_f : X \to \mathrm{im}(f)$ a coimage of f. This is called the **canonical factorisation** of f.

Proof:
1. For any morphism $f : X \to Y$ in $\mathcal{A}$, we have $\pi_f \circ f = 0$ for the cokernel $\pi_f : Y \to \mathrm{coker}(f)$. By the universal property of the image $\iota'_f : \mathrm{im}(f) \to Y$, there is a unique morphism $\pi'_f : X \to \mathrm{im}(f)$ with $\iota'_f \circ \pi'_f = f$. We show that $\pi'_f : X \to \mathrm{im}(f)$ is an epimorphism. The first claim then follows because every epimorphism is its own coimage, or, equivalently, a cokernel of its kernel. As the morphisms π'_f and $f = \iota'_f \circ \pi'_f$ have the same kernel and hence the same coimage $\pi'_f : X \to \mathrm{im}(f)$.

To show that $\pi'_f : X \to \mathrm{im}(f)$ is an epimorphism, let $\phi : \mathrm{im}(f) \to U$ be a morphism with $\phi \circ \pi'_f = 0$. By the universal property of the kernel $\iota_\phi : \ker(\phi) \to \mathrm{im}(f)$, there is a unique morphism f' with $\iota_\phi \circ f' = \pi'_f$:

$$\begin{tikzcd}
\ker(\phi) \arrow[r, hook, "\iota_\phi"] & \mathrm{im}(f) \arrow[r, "\phi"] \arrow[d, hook, "\iota'_f"] & U \\
X \arrow[u, dashed, "\exists! f'"] \arrow[ur, "\pi'_f"] \arrow[r, "f"'] & Y. &
\end{tikzcd}$$

The morphism $\iota'_f \circ \iota_\phi : \ker(\phi) \to Y$ is a monomorphism as a composite of two monomorphisms. Hence, it is a kernel of its cokernel $\pi' : Y \to \mathrm{coker}(\iota'_f \circ \iota_\phi)$. This implies

$$\pi' \circ f = \pi' \circ (\iota'_f \circ \iota_\phi \circ f') = (\pi' \circ \iota'_f \circ \iota_\phi) \circ f' = 0 \circ f' = 0,$$

and by the universal property of the cokernel $\pi_f : Y \to \mathrm{coker}(f)$ there is a unique morphism $\pi'' : \mathrm{coker}(f) \to \mathrm{coker}(\iota'_f \circ \iota_\phi)$ with $\pi'' \circ \pi_f = \pi'$

$$\begin{tikzcd}
\ker(\phi) \arrow[r, hook, "\iota_\phi"] & \mathrm{im}(f) \arrow[rr, "\phi"] \arrow[d, hook, "\iota'_f"] & & U \\
X \arrow[u, "f'"] \arrow[ur, "\pi'_f"] \arrow[r, "f"'] & Y \arrow[rr, two heads, "\pi'"] \arrow[d, "\pi_f"] & & \mathrm{coker}(\iota'_f \circ \iota_\phi) \\
& \mathrm{coker}(f). \arrow[urr, dashed, "\exists! \pi''"'] & &
\end{tikzcd}$$

This implies $\pi' \circ \iota'_f = (\pi'' \circ \pi_f) \circ \iota'_f = \pi'' \circ (\pi_f \circ \iota'_f) = \pi'' \circ 0 = 0$, since $\iota'_f : \mathrm{im}(f) \to Y$ is a kernel of $\pi_f : Y \to \mathrm{coker}(f)$. As $\iota'_f \circ \iota_\phi$ is a kernel of π' and $\pi' \circ \iota'_f = 0$, the universal property of the kernel $\iota'_f \circ \iota_\phi$ implies that there is a unique morphism $\iota'' : \mathrm{im}(f) \to \ker(\phi)$ with $\iota'_f \circ \iota_\phi \circ \iota'' = \iota'_f$. Because ι'_f is a monomorphism, it follows that $\iota_\phi \circ \iota'' = 1_{\mathrm{im}(f)}$. As ι_ϕ is a kernel of ϕ, this implies $\phi = \phi \circ 1_{\mathrm{im}_f} = \phi \circ (\iota_\phi \circ \iota'') = (\phi \circ \iota_\phi) \circ \iota'' = 0 \circ \iota'' = 0$, and $\pi'_f : X \to \mathrm{im}(f)$ is an epimorphism. $\square$

After clarifying the properties of abelian categories, we now focus on functors that are compatible with abelian categories - *exact functors* - and on functors that are partially compatible with it - *left or right exact functors*. It turns out that there are few exact functors, and most of them arise from certain standard constructions. Important examples are the following.

Example B.35:

1. For any abelian category $\mathcal{A}$, the cartesian product $\mathcal{A} \times \mathcal{A}$ is abelian and the functors $\Pi : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ und $\amalg : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ are exact.

2. For any abelian category $\mathcal{A}$, small category $\mathcal{C}$ and object C in $\mathcal{C}$, the evaluation functor $\mathrm{ev}_C : \mathrm{Fun}(\mathcal{C}, \mathcal{A}) \to \mathcal{A}$ that sends a functor $F : \mathcal{C} \to \mathcal{A}$ to the object $F(C)$ and a natural transformation $\eta : F \to G$ to the component morphism $\eta_C : F(C) \to G(C)$ is exact.

One reason why so many functors of interest are left or right exact is that one typically considers functors related to certain constructions, such as tensoring, abelisation or Hom-functors, and such functors tend to have adjoints. The existence of a left (right) adjoint to a given functor is sufficient to ensure its left (right) exactness.

Lemma B.36: Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ additive functors. If F is left adjoint to G, then F is right exact and G left exact.

Proof:
If F is left adjoint to G, by Proposition B.20 there are natural transformations $\epsilon : FG \to \mathrm{id}_{\mathcal{B}}$ and $\eta : \mathrm{id}_{\mathcal{A}} \to GF$ with $(\epsilon F) \circ (F\eta) = \mathrm{id}_F$ and $(G\epsilon) \circ (\eta G) = \mathrm{id}_G$. We show that F is right exact by proving that it sends cokernels to cokernels. The proof that G is left exact is analogous (Exercise).

Let $\pi : A' \to \mathrm{coker}(f)$ be a cokernel of $f : A \to A'$. Then $\pi \circ f = 0$, and for every morphism $g : A' \to A''$ with $g \circ f = 0$ there is a unique morphism $g' : \mathrm{coker}(f) \to A''$ with $g' \circ \pi = g$.

To show that $F(\pi) : F(A') \to F(\mathrm{coker}(f))$ is a cokernel of $F(f) : F(A) \to F(A')$, note first that the additivity of F implies $F(\pi) \circ F(f) = F(\pi \circ f) = F(0) = 0$.

If $h : F(A') \to B$ is a morphism with $h \circ F(f) = 0$, then $G(h) \circ \eta_{A'} : A' \to G(B)$ can be pre-composed with f, and by additivity of G and naturality of η we have

$$G(h) \circ \eta_{A'} \circ f = G(h) \circ GF(f) \circ \eta_A = G(h \circ F(f)) \circ \eta_A = G(0) \circ \eta_A = 0.$$

By the universal property of the cokernel π there is a unique morphism $k : \mathrm{coker}(f) \to G(B)$ with $G(h) \circ \eta_{A'} = k \circ \pi$. The morphism $h' = \epsilon_B \circ F(k) : F(\mathrm{coker}(f)) \to B$ satisfies

$$h' \circ F(\pi) = \epsilon_B \circ F(k) \circ F(\pi) = \epsilon_B \circ F(k \circ \pi) = \epsilon_B \circ FG(h) \circ F(\eta_{A'}) = h \circ \epsilon_{F(A')} \circ F(\eta_{A'}) = h.$$

If $h'' : F(\mathrm{coker}(f)) \to B$ is another morphism with $h'' \circ F(\pi) = h = h' \circ F(\pi)$, then we have $(h'' - h') \circ F(\pi) = 0$. This implies $0 = G(h'' - h') \circ GF(\pi) \circ \eta_{A'} = G(h'' - h') \circ \eta_{\mathrm{coker}(f)} \circ \pi$ and hence $G(h'' - h') \circ \eta_{\mathrm{coker}(f)} = 0$, because π is an epimorphism. Using the naturality of ϵ and the condition $(\epsilon F) \circ (F\eta) = \mathrm{id}_F$, we obtain

$$h'' - h' = (h'' - h) \circ \epsilon_{F(\mathrm{coker}(f))} \circ F(\eta_{\mathrm{coker}(f)}) = \epsilon_B \circ FG(h'' - h') \circ F(\eta_{\mathrm{coker}(f)}) = \epsilon_B \circ F(0) = 0.$$

This shows that $h'' = h'$ and that h' is the *unique* morphism with $h' \circ F(\pi) = h$. Hence, $F(\pi) : F(A') \to F(\mathrm{coker}(f))$ has the universal property of the cokernel and F is right exact. □

Two of the most important right exact functors are the functors $M \otimes_R - : R\text{-Mod} \to \mathrm{Ab}$ and $- \otimes_R N : R^{op}\text{-Mod} \to \mathrm{Ab}$ for an R-right module M and an R-left module N. They are left adjoint to $\mathrm{Hom}(M, -) : \mathrm{Ab} \to R\text{-Mod}$ and $\mathrm{Hom}(N, -) : \mathrm{Ab} \to R^{op}\text{-Mod}$ by Example B.19, 6.

Corollary B.37: Let R be a ring, M an R-right module and N an R-left module. Then

1. The functors $M \otimes_R - : R\text{-Mod} \to \mathrm{Ab}$ and $- \otimes_R N : R^{op}\text{-Mod} \to \mathrm{Ab}$ are right exact.
2. The functors $\mathrm{Hom}(M, -) : \mathrm{Ab} \to R\text{-Mod}$ and $\mathrm{Hom}(N, -) : \mathrm{Ab} \to R^{op}\text{-Mod}$ are left exact.

While there is no direct analogue of the tensor product over a ring for general abelian categories, the functors $\mathrm{Hom}(A, -) : \mathcal{A} \to \mathrm{Ab}$ and $\mathrm{Hom}(-, A) : \mathcal{A}^{op} \to \mathrm{Ab}$ are defined for any abelian category $\mathcal{A}$.

- The functor $\mathrm{Hom}(A, -)$ sends an object A' to the abelian group $\mathrm{Hom}_{\mathcal{A}}(A, A')$ and a morphism $f : A' \to A''$ to the group homomorphism

$$\mathrm{Hom}(A, f) : \mathrm{Hom}_{\mathcal{A}}(A, A') \to \mathrm{Hom}_{\mathcal{A}}(A, A''), \quad g \mapsto f \circ g.$$

- The functor $\mathrm{Hom}(-, A)$ sends an object A' in $\mathcal{A}$ to the abelian group $\mathrm{Hom}_{\mathcal{A}}(A', A)$ and a morphism $f : A' \to A''$ to the group homomorphism

$$\mathrm{Hom}(f, A) : \mathrm{Hom}_{\mathcal{A}}(A, A'') \to \mathrm{Hom}_{\mathcal{A}}(A, A'), \quad g \mapsto g \circ f.$$

This raises the question if left exactness holds for these generalisations as well. Indeed, it is possible to prove this without Lemma B.36.

Lemma B.38: Let $\mathcal{A}$ be an abelian category and $f : X \to Y$ a morphism in $\mathcal{A}$.

1. For any object A in $\mathcal{A}$ the functor $\mathrm{Hom}(A, -) : \mathcal{A} \to \mathrm{Ab}$ is left exact:

   A morphism $\iota : W \to X$ is a kernel of $f : X \to Y$ in $\mathcal{A}$ if and only if for all objects A in $\mathcal{A}$ the morphism $\iota_* = \mathrm{Hom}(A, \iota)$ in Ab is a kernel of $f_* = \mathrm{Hom}(A, f)$.

2. For any object A in $\mathcal{A}$ the functor $\mathrm{Hom}(-, A) : \mathcal{A}^{op} \to \mathrm{Ab}$ is left exact:

   A morphism $\pi : Y \to Z$ is a cokernel of $f : X \to Y$ in $\mathcal{A}$ if and only if for all objects A in $\mathcal{A}$ the morphism $\pi^* = \mathrm{Hom}(\pi, A)$ in Ab is a kernel of $f^* = \mathrm{Hom}(f, A)$.

Proof:
We prove the first claim. The proof of the second claim is analogous if one takes into account that kernels and cokernels in $\mathcal{A}$ are cokernels and kernels in $\mathcal{A}^{op}$, respectively.

As we work in $\mathrm{Ab} = \mathbb{Z}$-Mod, Example B.27 implies that the group homomorphism ι_* is a kernel of the group homomorphism f_* if and only if (i) ι_* is injective and (ii) $\mathrm{im}(\iota_*) = \ker(f_*)$.

Condition (i) is satisfied if and only if $\iota \circ g = \iota \circ g'$ implies $g = g'$ for all $g, g' \in \mathrm{Hom}_{\mathcal{A}}(A, W)$, and this is equivalent to the statement that ι is a monomorphism. Condition (ii) is satisfied if and only if (iia) $f \circ \iota \circ h = 0$ for all morphisms $h : A \to W$ and (iib) for every morphism $g : A \to X$ with $f \circ g = 0$ there is a morphism $g' : A \to X$ with $\iota \circ g' = g$. Condition (iia) is satisfied if $f \circ \iota = 0$, and by setting $h = 1_A$ we see that (iia) implies $f \circ \iota = 0$. As ι is a monomorphism, condition (iib) then states that ι is a kernel of f. □

Remark B.39:

1. In general the functor $A \otimes_R - : R\text{-Mod} \to \mathrm{Ab}$ is not left exact.

   A counterexample is $R = \mathbb{Z}$, $A = \mathbb{Z}/n\mathbb{Z}$. Then $\iota : \mathbb{Z} \to \mathbb{Z}$, $z \mapsto nz$ is injective with $\mathrm{im}(\iota) = \ker(\pi)$ for $\pi : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, $z \mapsto \bar{z}$ and hence a kernel of π. But $\mathrm{id} \otimes \iota : \mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}$ is given by $(\mathrm{id} \otimes \iota)(\bar{k} \otimes z) = \bar{k} \otimes (nz) = \overline{nk} \otimes z = \bar{0} \otimes z = 0$ for all $\bar{k} \in \mathbb{Z}/n\mathbb{Z}$ and $z \in \mathbb{Z}$. Hence, $\mathrm{id} \otimes \iota = 0$ is not injective and not a kernel of $\mathrm{id} \otimes \pi$.

2. The functors $\mathrm{Hom}(A, -) : \mathcal{A} \to \mathrm{Ab}$, $\mathrm{Hom}(-, A) : \mathcal{A}^{op} \to \mathrm{Ab}$ are in general not right exact.

   A counterexample is $\mathcal{A} = \mathrm{Ab}$ and $A = \mathbb{Z}/n\mathbb{Z}$. Then by 1. the group homomorphism $\iota : \mathbb{Z} \to \mathbb{Z}$, $z \mapsto nz$ is a kernel of $\pi : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, $z \mapsto \bar{z}$ and π a cokernel of ι.

   However, $\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \pi) : \mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \to \mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$, $g \mapsto \pi \circ g$ is not surjective and hence not a cokernel of $\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \iota)$.

   Similarly, $\mathrm{Hom}(\iota, \mathbb{Z}/n\mathbb{Z}) : \mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$, $g \mapsto g \circ \iota$ is trivial since $g \circ \iota(z) = g(nz) = ng(z) = 0$ for all $z \in \mathbb{Z}$ and group homomorphisms $g : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. Hence, $\mathrm{Hom}(\iota, \mathbb{Z}/n\mathbb{Z}) = 0$ is not surjective and not a cokernel of $\mathrm{Hom}(\pi, \mathbb{Z}/n\mathbb{Z})$.

R-right modules M for which the associated functor $A\otimes_R- : R\text{-Mod} \to \text{Ab}$ is not only right exact but exact and objects in an abelian category $\mathcal{A}$ for which the functor $\text{Hom}(A,-)$ or $\text{Hom}(-,A)$ are exact play a special role in representation theory and homology theories. For this reason, they receive a name.

Definition B.40:
A right module A over a ring R is called **flat** if the functor $A\otimes_R- : R\text{-Mod} \to \text{Ab}$ is exact.

Definition B.41: An object A in an abelian category $\mathcal{A}$ is called

- **projective** if the functor $\text{Hom}(A,-) : \mathcal{A} \to \text{Ab}$ is exact,
- **injective** if the functor $\text{Hom}(-,A) : \mathcal{A}^{op} \to \text{Ab}$ is exact.

One can show that for $\mathcal{A} = R^{op}$-Mod any projective R^{op}-module is flat. Hence, projectivity and injectivity are not only more general concepts, but also stronger conditions when both are defined. There is an alternative characterisations of projectivity and injectivity that is easier to handle and generalises to non-abelian categories.

Lemma B.42: Let $\mathcal{A}$ be an abelian category.

1. An object A in $\mathcal{A}$ is projective if and only if for every epimorphism $\pi : X \to Y$ and every morphism $f : A \to Y$ there is a morphism $f' : A \to X$ with $\pi \circ f' = f$

$$\begin{tikzcd} & A \arrow[dl, dashed, "\exists f'"'] \arrow[d, "f"] & \\ X \arrow[r, two heads, "\pi"] & Y \arrow[r] & 0 \end{tikzcd}$$

2. An object A in $\mathcal{A}$ is injective if and only if for every monomorphism $\iota : Y \to X$ and every morphism $f : Y \to A$ there is a morphism $f' : X \to A$ with $f' \circ \iota = f$

$$\begin{tikzcd} & A & \\ X \arrow[ur, dashed, "\exists f'"] & Y \arrow[l, hook', "\iota"'] \arrow[u, "f"'] & 0 \arrow[l] \end{tikzcd}$$

Proof:
We prove the first statement. The proof of the second one is analogous.

$\Rightarrow$ Let A be projective. Then $\text{Hom}(A,-)$ is exact and maps kernels to kernels and cokernels to cokernels. As every epimorphism $\pi : X \to Y$ in $\mathcal{A}$ is a cokernel of its kernel, the morphism $\text{Hom}(A,\pi) : \text{Hom}_{\mathcal{A}}(A,X) \to \text{Hom}_{\mathcal{A}}(A,Y)$ is a cokernel as well and hence an epimorphism in Ab by Lemma B.30. This means that for every morphism $f : A \to Y$, there is a morphism $f' : A \to X$ with $\text{Hom}(A,\pi)(f') = \pi \circ f' = f$.

$\Leftarrow$ Suppose that for every morphism $f : A \to Y$ and epimorphism $\pi : X \to Y$ there is a morphism $f' : A \to X$ with $\pi \circ f' = f$. Then $\text{Hom}(A,\pi) : \text{Hom}_{\mathcal{A}}(A,X) \to \text{Hom}_{\mathcal{A}}(A,Y)$ is an epimorphism for every epimorphism $\pi : X \to Y$.

Let $f : A \to X$ be a morphism with cokernel $\pi : X \to Y$. By Lemma B.34 f has a canonical factorisation $f = \iota' \circ \pi'$ with a monomorphism ι' and an epimorphism π'. By Exercise **??** the morphism $\pi : X \to Y$ is also a cokernel of ι'. As ι' is a monomorphism, it is a kernel of its cokernel $\pi : X \to Y$. By left-exactness of $\text{Hom}(A,-)$, it follows that $\text{Hom}(A,\iota')$ is a kernel of $\text{Hom}(A,\pi)$.

As every epimorphism is a cokernel of its kernel, it follows that $\mathrm{Hom}(A, \pi)$ is a cokernel of $\mathrm{Hom}(A, \iota')$. As $\mathrm{Hom}(A, f) = \mathrm{Hom}(A, \iota' \circ \pi') = \mathrm{Hom}(A, \iota') \circ \mathrm{Hom}(A, \pi')$ and $\mathrm{Hom}(A, \pi')$ is an epimorphism, $\mathrm{Hom}(A, \pi)$ is also a cokernel of $\mathrm{Hom}(A, f)$ and hence $\mathrm{Hom}(A, -)$ is right exact. □

Example B.43:

1. By Remark B.39 the objects $\mathbb{Z}/n\mathbb{Z}$ in Ab are neither projective nor injective.
2. For every ring R, any free R-module is projective.

   If A is a free R-module with basis B, $\pi : X \to Y$ R-linear and surjective and $f : A \to Y$ R-linear, then we can choose for every element $b \in B$ an element $f'(b) \in \pi^{-1}(f(b))$ and obtain an R-linear map $f' : A \to X$, $b \mapsto f'(b)$ with $\pi \circ f' = f$.
3. The object $\mathbb{Z}$ in Ab is projective, but not injective.

   The projectivity of $\mathbb{Z}$ follows from 2. However, $\mathbb{Z}$ is not injective, because for the monomorphism $\iota : \mathbb{Z} \to \mathbb{Z}$, $z \mapsto nz$ with $n > 1$ and the group homomorphism $f = \mathrm{id}_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}$ there is no morphism $f' : \mathbb{Z} \to \mathbb{Z}$ with $f' \circ \iota = f = \mathrm{id}_{\mathbb{Z}}$.

Lemma B.42 not only gives simple criteria for projectivity and injectivity but also allows one to extend the notions of projectivity and injectivity from abelian categories to general categories. As the concepts in Lemma B.42 are defined in any category, we can take the conditions in Lemma B.42 as the definition of projectivity and injectivity there.

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