Lie Groups–WS 2011/12

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Notation

$\mathbb{N} := \{1, 2, 3, ...\}$ natural numbers
$\mathbb{K}^\times := \{x \in \mathbb{K} \colon x \neq 0\}$, $\mathbb{K}$ a field
$\mathcal{A}^\times := \{x \in \mathcal{A} \colon (\exists y \in R) xy = yx = \mathbf{1}\}$, unit group of a unital algebra

For subsets $A, B \subseteq G$ of a group:
$A^{-1} := \{a^{-1} \colon a \in A\}$
$AB := \{ab \colon a \in A, b \in B\}$
The identity element of a group G is usually denoted $\mathbf{1}$. If G is abelian and the product is written as addition, we write 0 for the identity element.
For $A = (a_{ij})_{i,j=1,\ldots,n} \in M_n(\mathbb{C})$: $A^\top = (a_{ji})$, $\overline{A} = (\overline{a_{ij}})$, $A^* = \overline{A}^\top = (\overline{a_{ji}})$.

Contents

Chapter 1

Concrete Matrix Groups

In this chapter we mainly study the general linear group $\mathrm{GL}_n(\mathbb{K})$ of invertible $n \times n$-matrices with entries in $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and introduce some of its subgroups. In particular, we discuss some of the connections between matrix groups and also introduce certain symmetry groups of geometric structures like bilinear or sesquilinear forms.

1.1 The General Linear Group

We start with some notation. We write $\mathrm{GL}_n(\mathbb{K})$ for the group of invertible matrices in $M_n(\mathbb{K})$ and note that

$$\mathrm{GL}_n(\mathbb{K}) = \{g \in M_n(\mathbb{K})\colon (\exists h \in M_n(\mathbb{K}))\, hg = gh = \mathbf{1}\}.$$

Since the invertibility of a matrix can be tested with its determinant,

$$\mathrm{GL}_n(\mathbb{K}) = \{g \in M_n(\mathbb{K})\colon \det g \neq 0\}.$$

This group is called the *general linear group*.

On the vector space $\mathbb{K}^n$ we consider the *euclidian norm*

$$\|x\| := \sqrt{|x_1|^2 + \ldots + |x_n|^2}, \quad x \in \mathbb{K}^n,$$

and on $M_n(\mathbb{K})$ the corresponding *operator norm*

$$\|A\| := \sup\{\|Ax\|\colon x \in \mathbb{K}^n, \|x\| \leq 1\}$$

which turns $M_n(\mathbb{K})$ into a Banach space. On every subset $S \subseteq M_n(\mathbb{K})$ we shall always consider the subspace topology inherited from $M_n(\mathbb{K})$ (otherwise we shall say so). In this sense $\mathrm{GL}_n(\mathbb{K})$ and all its subgroups carry a natural topology.

Lemma 1.1.1. *The group* $\mathrm{GL}_n(\mathbb{K})$ *has the following properties:*

(i) $\mathrm{GL}_n(\mathbb{K})$ *is open in* $M_n(\mathbb{K})$.

(ii) *The multiplication map* $m\colon \mathrm{GL}_n(\mathbb{K}) \times \mathrm{GL}_n(\mathbb{K}) \to \mathrm{GL}_n(\mathbb{K})$ *and the inversion map* $\eta\colon \mathrm{GL}_n(\mathbb{K}) \to \mathrm{GL}_n(\mathbb{K})$ *are smooth and in particular continuous.*

Proof. (i) Since the determinant function

$$\det\colon M_n(\mathbb{K}) \to \mathbb{K}, \quad \det(a_{ij}) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$$

is continuous and $\mathbb{K}^\times := \mathbb{K} \setminus \{0\}$ is open in $\mathbb{K}$, the set $\mathrm{GL}_n(\mathbb{K}) = \det^{-1}(\mathbb{K}^\times)$ is open in $M_n(\mathbb{K})$.

(ii) For $g \in \mathrm{GL}_n(\mathbb{K})$ we define $b_{ij}(g) := \det(g_{mk})_{m \neq j, k \neq i}$. According to Cramer's Rule, the inverse of g is given by

$$(g^{-1})_{ij} = \frac{(-1)^{i+j}}{\det g} b_{ij}(g).$$

The smoothness of the inversion therefore follows from the smoothness of the determinant (which is a polynomial) and the polynomial functions b_{ij} defined on $M_n(\mathbb{K})$.

For the smoothness of the multiplication map, it suffices to observe that

$$(ab)_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$$

is the (ik)-entry in the product matrix. Since all these entries are quadratic polynomials in the entries of a and b, the product is a smooth map. □

Definition 1.1.2. A *topological group* G is a Hausdorff space G, endowed with a group structure, such that the multiplication map $m_G\colon G \times G \to G$ and the inversion map $\iota_G\colon G \to G$ are continuous, when $G \times G$ is endowed with the product topology.

Lemma 1.1.1(ii) says in particular that $\mathrm{GL}_n(\mathbb{K})$ is a topological group. It is clear that the continuity of group multiplication and inversion is inherited by every subgroup $G \subseteq \mathrm{GL}_n(\mathbb{K})$, so that every subgroup G of $\mathrm{GL}_n(\mathbb{K})$ also is a topological group.

We write a matrix $A = (a_{ij})_{i,j=1,\ldots,n}$ also as (a_{ij}) and define

$$A^\top := (a_{ji}), \quad \overline{A} := (\overline{a_{ij}}), \quad \text{and} \quad A^* := \overline{A}^\top = (\overline{a_{ji}}).$$

Note that $A^* = A^\top$ is equivalent to $\overline{A} = A$, which means that all entries of A are real. Now we can define the most important classes of matrix groups.

Definition 1.1.3. We introduce the following notation for some important subgroups of $\mathrm{GL}_n(\mathbb{K})$:

(1) The *special linear group* : $\mathrm{SL}_n(\mathbb{K}) := \{g \in \mathrm{GL}_n(\mathbb{K})\colon \det g = 1\}$.

(2) The *orthogonal group* : $\mathrm{O}_n(\mathbb{K}) := \{g \in \mathrm{GL}_n(\mathbb{K})\colon g^\top = g^{-1}\}$.

(3) The *special orthogonal group* : $\mathrm{SO}_n(\mathbb{K}) := \mathrm{SL}_n(\mathbb{K}) \cap \mathrm{O}_n(\mathbb{K})$.

(4) The *unitary group* : $\mathrm{U}_n(\mathbb{K}) := \{g \in \mathrm{GL}_n(\mathbb{K})\colon g^* = g^{-1}\}$. Note that $\mathrm{U}_n(\mathbb{R}) = \mathrm{O}_n(\mathbb{R})$, but $\mathrm{O}_n(\mathbb{C}) \neq \mathrm{U}_n(\mathbb{C})$.

(5) The *special unitary group* : $\mathrm{SU}_n(\mathbb{K}) := \mathrm{SL}_n(\mathbb{K}) \cap \mathrm{U}_n(\mathbb{K})$.

One easily verifies that these are indeed subgroups. One simply has to use that $(ab)^\top = b^\top a^\top$, $\overline{ab} = \overline{a}\overline{b}$ and that

$$\det\colon \mathrm{GL}_n(\mathbb{K}) \to (\mathbb{K}^\times, \cdot)$$

is a group homomorphism.

Lemma 1.1.4. *The groups*

$$\mathrm{U}_n(\mathbb{C}), \quad \mathrm{SU}_n(\mathbb{C}), \quad \mathrm{O}_n(\mathbb{R}) \quad \textit{and} \quad \mathrm{SO}_n(\mathbb{R})$$

are compact.

Proof. Since all these groups are subsets of $M_n(\mathbb{C}) \cong \mathbb{C}^{n^2}$, by the Heine–Borel Theorem we only have to show that they are closed and bounded.

Bounded: In view of

$$\mathrm{SO}_n(\mathbb{R}) \subseteq \mathrm{O}_n(\mathbb{R}) \subseteq \mathrm{U}_n(\mathbb{C}) \quad \text{and} \quad \mathrm{SU}_n(\mathbb{C}) \subseteq \mathrm{U}_n(\mathbb{C}),$$

it suffices to see that $\mathrm{U}_n(\mathbb{C})$ is bounded. Let $g_1, \ldots, g_n$ denote the rows of the matrix $g \in M_n(\mathbb{C})$. Then $g^* = g^{-1}$ is equivalent to $gg^* = \mathbf{1}$, which means that $g_1, \ldots, g_n$ form an orthonormal basis for $\mathbb{C}^n$ with respect to the scalar product $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w_j}$ which induces the norm $\|z\| = \sqrt{\langle z, z\rangle}$. Therefore $g \in \mathrm{U}_n(\mathbb{C})$ implies $\|g_j\| = 1$ for each j, so that $\mathrm{U}_n(\mathbb{C})$ is bounded.

Closed: The functions

$$f, h\colon M_n(\mathbb{K}) \to M_n(\mathbb{K}), \quad f(A) := AA^* - \mathbf{1} \quad \text{and} \quad h(A) := AA^\top - \mathbf{1}$$

are continuous. Therefore the groups

$$\mathrm{U}_n(\mathbb{K}) := f^{-1}(\mathbf{0}) \quad \text{and} \quad \mathrm{O}_n(\mathbb{K}) := h^{-1}(\mathbf{0})$$

are closed. Likewise $\mathrm{SL}_n(\mathbb{K}) = \det^{-1}(\mathbf{1})$ is closed, and therefore the groups $\mathrm{SU}_n(\mathbb{C})$ and $\mathrm{SO}_n(\mathbb{R})$ are also closed because they are intersections of closed subsets. ☐

1.1.1 The Polar Decomposition

We write $\mathrm{Herm}_n(\mathbb{K}) := \{A \in M_n(\mathbb{K})\colon A^* = A\}$ for the set of *hermitian matrices*. For $\mathbb{K} = \mathbb{C}$ this is not a vector subspace of $M_n(\mathbb{K})$, but it is always a real subspace. A matrix $A \in \mathrm{Herm}_n(\mathbb{K})$ is called *positive definite* if for each $0 \neq z \in \mathbb{K}^n$ we have $\langle Az, z\rangle > 0$, where

$$\langle z, w\rangle := \sum_{j=1}^n z_j \overline{w_j}$$

is the natural scalar product on $\mathbb{K}^n$. We write $\mathrm{Pd}_n(\mathbb{K}) \subseteq \mathrm{Herm}_n(\mathbb{K})$ for the subset of positive definite matrices.

Proposition 1.1.5. (Polar decomposition) *The multiplication map*

$$m\colon \mathrm{U}_n(\mathbb{K}) \times \mathrm{Pd}_n(\mathbb{K}) \to \mathrm{GL}_n(\mathbb{K}), \quad (u,p) \mapsto up$$

is a homeomorphism. In particular, each invertible matrix g can be written in a unique way as a product $g = up$ of a unitary matrix u and a positive definite matrix p.

Proof. We know from linear algebra that for each hermitian matrix A there exists an orthonormal basis $v_1, \ldots, v_n$ for $\mathbb{K}^n$ consisting of eigenvectors of A, and that all the corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ are real. From that it is obvious that A is positive definite if and only if $\lambda_j > 0$ holds for each j. For a positive definite matrix A, this has two important consequences:

(1) A is invertible, and A^{-1} satisfies $A^{-1}v_j = \lambda_j^{-1}v_j$.

(2) There exists a unique positive definite matrix B with $B^2 = A$ which will be denoted $\sqrt{A}$: We define B with respect to the basis $(v_1, \ldots, v_n)$ by $Bv_j = \sqrt{\lambda_j}v_j$. Then $B^2 = A$ is obvious and since all λ_j are real and the v_j are orthonormal, B is positive definite because

$$\Big\langle B\Big(\sum_i \mu_i v_i\Big), \sum_j \mu_j v_j\Big\rangle = \sum_{i,j} \mu_i \overline{\mu_j} \langle Bv_i, v_j\rangle = \sum_{j=1}^n |\mu_j|^2 \sqrt{\lambda_j} > 0$$

for $\sum_j \mu_j v_j \neq 0$. It remains to verify the uniqueness. So assume that C is positive definite with $C^2 = A$. Pick an orthonormal basis $w_1, \ldots, w_m$ of C-eigenvectors, so that $Cw_j = \mu_j w_j$ with positive numbers $\mu_j > 0$. Then $Aw_j = C^2 w_j = \mu_j^2 w_j$ show that, for $\lambda_j := \mu_j^2$, the matrix C acts on the λ_j-eigenspace of A by multiplication with $\sqrt{\lambda_j} = \mu_j$. This implies $B = C$.

From (1) we derive that the image of the map m is contained in $\mathrm{GL}_n(\mathbb{K})$.
m **is surjective:** Let $g \in \mathrm{GL}_n(\mathbb{K})$. For $0 \neq v \in \mathbb{K}^n$ we then have

$$0 < \langle gv, gv\rangle = \langle g^*gv, v\rangle,$$

showing that g^*g is positive definite. Let $p := \sqrt{g^*g}$ and define $u := gp^{-1}$. Then

$$uu^* = gp^{-1}p^{-1}g^* = gp^{-2}g^* = g(g^*g)^{-1}g^* = gg^{-1}(g^*)^{-1}g^* = \mathbf{1}$$

implies that $u \in \mathrm{U}_n(\mathbb{K})$, and it is clear that $m(u,p) = g$.
m **is injective:** If $m(u,p) = m(w,q) = g$, then $g = up = wq$ implies that

$$p^2 = p^*p = (up)^*up = g^*g = (wq)^*wq = q^2,$$

so that p and q are positive definite square roots of the same positive definite matrix g^*g, hence coincide by (2) above. Now $p = q$, and therefore $u = gp^{-1} = gq^{-1} = w$.

It remains to show that m is a homeomorphism. Its continuity is obvious, so that it remains to prove the continuity of the inverse map m^{-1}. Let $g_j = u_j p_j \to g = up$. We have to show that $u_j \to u$ and $p_j \to p$. Since $\mathrm{U}_n(\mathbb{K})$ is compact, the sequence (u_j) has a subsequence (u_{j_k}) converging to some $w \in \mathrm{U}_n(\mathbb{K})$. Then $p_{j_k} = u_{j_k}^{-1} g_{j_k} \to w^{-1}g =: q \in \mathrm{Herm}_n(\mathbb{K})$ and $g = wq$. For each $v \in \mathbb{K}^n$ we then have

$$0 \leq \langle p_{j_k}v, v\rangle \to \langle qv, v\rangle,$$

showing that all eigenvalues of q are ≥ 0. Moreover, $q = w^{-1}g$ is invertible, and therefore q is positive definite. Now $m(u,p) = m(w,q)$ yields $u = w$ and $p = q$. Since each convergent subsequence of (u_j) converges to u, the sequence itself converges to u (Exercise 1.1.9), and therefore $p_j = u_j^{-1}g_j \to u^{-1}g = p$. □

We shall see later that the set $\mathrm{Pd}_n(\mathbb{K})$ is homeomorphic to a vector space (Proposition 2.3.5), so that, topologically, the group $\mathrm{GL}_n(\mathbb{K})$ is a product of the compact group $\mathrm{U}_n(\mathbb{K})$ and a vector space. Therefore the "interesting" part of the topology of $\mathrm{GL}_n(\mathbb{K})$ is contained in the compact group $\mathrm{U}_n(\mathbb{K})$.

Remark 1.1.6. [Normal forms of unitary and orthogonal matrices] We recall some facts from linear algebra:

(a) For each $u \in \mathrm{U}_n(\mathbb{C})$, there exists an orthonormal basis $v_1, \ldots, v_n$ consisting of eigenvectors of g. This means that the unitary matrix s whose columns are the vectors $v_1, \ldots, v_n$ satisfies

$$s^{-1}us = \mathrm{diag}(\lambda_1, \ldots, \lambda_n),$$

where $uv_j = \lambda_j v$ and $|\lambda_j| = 1$.

The proof of this normal form is based on the existence of an eigenvector v_1 of u which in turn follows from the existence of a zero of the characteristic polynomial. Since u is unitary, it preserves the hyperplane $v_1^\perp$ of dimension $n-1$. Now one uses induction to obtain an orthonormal basis $v_2, \ldots, v_n$ consisting of eigenvectors.

(b) For elements of $\mathrm{O}_n(\mathbb{R})$, the situation is more complicated because real matrices do not always have real eigenvectors.

Let $A \in M_n(\mathbb{R})$ and consider it as an element of $M_n(\mathbb{C})$. We assume that A does not have a real eigenvector. Then there exists an eigenvector $z \in \mathbb{C}^n$ corresponding to some eigenvalue $\lambda \in \mathbb{C}$. We write $z = x + iy$ and $\lambda = a + ib$. Then

$$Az = Ax + iAy = \lambda z = (ax - by) + i(ay + bx).$$

Comparing real and imaginary part yields

$$Ax = ax - by \quad \text{and} \quad Ay = ay + bx.$$

Therefore the two-dimensional subspace generated by x and y in $\mathbb{R}^n$ is invariant under A.

This can be applied to $g \in \mathrm{O}_n(\mathbb{R})$ as follows. The argument above implies that there exists an invariant subspace $W_1 \subseteq \mathbb{R}^n$ with $\dim W_1 \in \{1, 2\}$. Then

$$W_1^\perp := \{v \in \mathbb{R}^n \colon \langle v, W_1 \rangle = \{0\}\}$$

is a subspace of dimension $n - \dim W_1$ which is also invariant (Exercise 1.1.14), and we apply induction to see that $\mathbb{R}^n$ is a direct sum of g-invariant subspaces $W_1, \ldots, W_k$ of dimension ≤ 2. Therefore the matrix g is conjugate by an orthogonal matrix s to a block matrix of the form

$$d = \mathrm{diag}(d_1, \ldots, d_k),$$

where d_j is the matrix of the restriction of the linear map corresponding to g to W_j.

To understand the structure of the d_j, we have to take a closer look at the case $n \leq 2$. For $n = 1$ the group $\mathrm{O}_1(\mathbb{R}) = \{\pm 1\}$ consists of two elements, and for $n = 2$ an element $r \in \mathrm{O}_2(\mathbb{R})$ can be written as

$$r = \begin{pmatrix} a & \mp b \\ b & \pm a \end{pmatrix} \quad \text{with} \quad \det r = \pm(a^2 + b^2) = \pm 1,$$

because the second column contains a unit vector orthogonal to the first one. With $a = \cos\alpha$ and $b = \sin\alpha$ we get

$$r = \begin{pmatrix} \cos\alpha & \mp\sin\alpha \\ \sin\alpha & \pm\cos\alpha \end{pmatrix}.$$

For $\det r = -1$, we obtain

$$r^2 = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \begin{pmatrix} a & b \\ b & -a \end{pmatrix} = \mathbf{1},$$

but this implies that r is an orthogonal reflection with the two eigenvalues ± 1 (Exercise 1.1.13), hence has two orthogonal eigenvectors.

In view of the preceding discussion, we may therefore assume that the first m of the matrices d_j are of the rotation form

$$d_j = \begin{pmatrix} \cos\alpha_j & -\sin\alpha_j \\ \sin\alpha_j & \cos\alpha_j \end{pmatrix},$$

that $d_{m+1}, \ldots, d_\ell$ are -1, and that $d_{\ell+1}, \ldots, d_n$ are 1:

$$\begin{pmatrix}
\cos\alpha_1 & -\sin\alpha_1 & & & & & & & & \\
\sin\alpha_1 & \cos\alpha_1 & & & & & & & & \\
& & \ddots & & & & & & & \\
& & & \cos\alpha_m & -\sin\alpha_m & & & & & \\
& & & \sin\alpha_m & \cos\alpha_m & & & & & \\
& & & & & -1 & & & & \\
& & & & & & \ddots & & & \\
& & & & & & & -1 & & \\
& & & & & & & & 1 & \\
& & & & & & & & & \ddots \\
& & & & & & & & & & 1
\end{pmatrix}.$$

For $n = 3$ we obtain in particular the normal form

$$d = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}.$$

From this normal form we immediately read off that $\det d = 1$ is equivalent to d describing a rotation around an axis consisting of fixed points (the axis is $\mathbb{R}e_3$ for the normal form matrix).

Proposition 1.1.7. (a) *The group* $\mathrm{U}_n(\mathbb{C})$ *is arcwise connected.*

(b) *The group* $\mathrm{O}_n(\mathbb{R})$ *has the two arc components*

$$\mathrm{SO}_n(\mathbb{R}) \quad \text{and} \quad \mathrm{O}_n(\mathbb{R})_- := \{g \in \mathrm{O}_n(\mathbb{R}) \colon \det g = -1\}.$$

Proof. (a) First we consider $\mathrm{U}_n(\mathbb{C})$. To see that this group is arcwise connected, let $u \in \mathrm{U}_n(\mathbb{C})$. Then there exists an orthonormal basis $v_1, \ldots, v_n$ of eigenvectors of u (Remark 1.1.6(a)). Let $\lambda_1, \ldots, \lambda_n$ denote the corresponding eigenvalues. Then the unitarity of u implies that $|\lambda_j| = 1$, and we therefore find $\theta_j \in \mathbb{R}$ with $\lambda_j = e^{\theta_j i}$. Now we define a continuous curve

$$\gamma \colon [0,1] \to \mathrm{U}_n(\mathbb{C}), \quad \gamma(t)v_j := e^{t\theta_j i}v_j, \ j = 1, \ldots, n.$$

We then have $\gamma(0) = \mathbf{1}$ and $\gamma(1) = u$. Moreover, each $\gamma(t)$ is unitary because the basis $(v_1, \ldots, v_n)$ is orthonormal.

(b) For $g \in \mathrm{O}_n(\mathbb{R})$ we have $gg^\top = \mathbf{1}$ and therefore $1 = \det(gg^\top) = (\det g)^2$. This shows that

$$\mathrm{O}_n(\mathbb{R}) = \mathrm{SO}_n(\mathbb{R}) \dot{\cup} \mathrm{O}_n(\mathbb{R})_-$$

and both sets are closed in $\mathrm{O}_n(\mathbb{R})$ because det is continuous. Therefore $\mathrm{O}_n(\mathbb{R})$ is not connected and hence not arcwise connected. Suppose we knew that $\mathrm{SO}_n(\mathbb{R})$ is arcwise connected and $x, y \in \mathrm{O}_n(\mathbb{R})_-$. Then $\mathbf{1}, x^{-1}y \in \mathrm{SO}_n(\mathbb{R})$ can be connected by an arc $\gamma \colon [0,1] \to \mathrm{SO}_n(\mathbb{R})$, and then $t \mapsto x\gamma(t)$ defines an arc $[0,1] \to \mathrm{O}_n(\mathbb{R})_-$ connecting x to y. So it remains to show that $\mathrm{SO}_n(\mathbb{R})$ is arcwise connected.

Let $g \in \mathrm{SO}_n(\mathbb{R})$. In the normal form of g discussed in Remark 1.1.6, the determinant of each two-dimensional block is 1, so that the determinant is the product of all -1-eigenvalues. Hence their number is even, and we can write each consecutive pair as a block

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\pi & -\sin\pi \\ \sin\pi & \cos\pi \end{pmatrix}.$$

This shows that with respect to some orthonormal basis for $\mathbb{R}^n$ the linear map defined by g has a matrix of the form

$$g = \begin{pmatrix} \cos\alpha_1 & -\sin\alpha_1 & & & & & & \\ \sin\alpha_1 & \cos\alpha_1 & & & & & & \\ & & \ddots & & & & & \\ & & & \cos\alpha_m & -\sin\alpha_m & & & \\ & & & \sin\alpha_m & \cos\alpha_m & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix}.$$

Now we obtain an arc $\gamma\colon [0,1] \to \mathrm{SO}_n(\mathbb{R})$ with $\gamma(0) = \mathbf{1}$ and $\gamma(1) = g$ by

$$\gamma(t) := \begin{pmatrix} \cos t\alpha_1 & -\sin t\alpha_1 & & & & & & \\ \sin t\alpha_1 & \cos t\alpha_1 & & & & & & \\ & & \ddots & & & & & \\ & & & \cos t\alpha_m & -\sin t\alpha_m & & & \\ & & & \sin t\alpha_m & \cos t\alpha_m & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix}.$$

□

Corollary 1.1.8. *The group* $\mathrm{GL}_n(\mathbb{C})$ *is arcwise connected and the group* $\mathrm{GL}_n(\mathbb{R})$ *has two arc-components given by*

$$\mathrm{GL}_n(\mathbb{R})_\pm := \{g \in \mathrm{GL}_n(\mathbb{R})\colon \pm \det g > 0\}.$$

Proof. If $X = A \times B$ is a product space, then the arc-components of X are the sets of the form $C \times D$, where $C \subseteq A$ and $D \subseteq B$ are arc-components (easy Exercise!). The polar decomposition of $\mathrm{GL}_n(\mathbb{K})$ yields a homeomorphism

$$\mathrm{GL}_n(\mathbb{K}) \cong \mathrm{U}_n(\mathbb{K}) \times \mathrm{Pd}_n(\mathbb{K}).$$

Since $\mathrm{Pd}_n(\mathbb{K})$ is an open convex set, it is arcwise connected (Exercise 1.1.6). Therefore the arc-components of $\mathrm{GL}_n(\mathbb{K})$ are in one-to-one correspondence with those of $\mathrm{U}_n(\mathbb{K})$ which have been determined in Proposition 1.1.7. □

1.1.2 Normal Subgroups of $\mathrm{GL}_n(\mathbb{K})$

We shall frequently need some basic concepts from group theory which we recall in the following definition.

Definition 1.1.9. Let G be a group with identity element e.

(a) A subgroup $N \subseteq G$ is called *normal* if $gN = Ng$ holds for all $g \in G$. We write this as $N \trianglelefteq G$. The normality implies that the quotient set G/N (the set of all cosets of the subgroup N) inherits a natural group structure by

$$gN \cdot hN := ghN$$

for which eN is the identity element and the quotient map $q\colon G \to G/N$ is a surjective group homomorphism with kernel $N = \ker q = q^{-1}(eN)$.

On the other hand, all kernels of group homomorphisms are normal subgroups, so that the normal subgroups are precisely those which are kernels of group homomorphisms.

It is clear that G and $\{e\}$ are normal subgroups. We call G *simple* if $G \neq \{e\}$ and these are the only normal subgroups.

(b) The subgroup $Z(G) := \{g \in G \colon (\forall x \in G) gx = xg\}$ is called the *center of* G. It obviously is a normal subgroup of G. For $x \in G$ the subgroup

$$Z_G(x) := \{g \in G \colon gx = xg\}$$

is called its *centralizer*. Note that $Z(G) = \bigcap_{x \in G} Z_G(x)$.

(c) If $G_1, \ldots, G_n$ are groups, then the product set $G := G_1 \times \ldots \times G_n$ has a natural group structure given by

$$(g_1, \ldots, g_n)(g_1', \ldots, g_n') := (g_1 g_1', \ldots, g_n g_n').$$

The group G is called the *direct product* of the groups G_j, $j = 1, \ldots, n$. We identify G_j with a subgroup of G. Then all subgroups G_j are normal subgroups and $G = G_1 \cdots G_n$.

In the following we write $\mathbb{R}_+^\times :=]0, \infty[$.

Proposition 1.1.10. (a) $Z(\mathrm{GL}_n(\mathbb{K})) = \mathbb{K}^\times \mathbf{1}$.
(b) *The multiplication map*

$$\varphi \colon (\mathbb{R}_+^\times, \cdot) \times \mathrm{SL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})_+, \quad (\lambda, g) \mapsto \lambda g$$

is a homeomorphism and a group isomorphism, i.e., an isomorphism of topological groups.

Proof. (a) It is clear that $\mathbb{K}^\times \mathbf{1}$ is contained in the center of $\mathrm{GL}_n(\mathbb{K})$. To see that each matrix $g \in Z(\mathrm{GL}_n(\mathbb{K}))$ is a multiple of $\mathbf{1}$, we consider the elementary matrix $E_{ij} := (\delta_{ij})$ with the only nonzero entry 1 in position (i, j). For $i \neq j$ we then have $E_{ij}^2 = 0$, so that $(\mathbf{1} + E_{ij})(\mathbf{1} - E_{ij}) = \mathbf{1}$, which implies that $T_{ij} := \mathbf{1} + E_{ij} \in \mathrm{GL}_n(\mathbb{K})$. From the relation $gT_{ij} = T_{ij} g$ we immediately get $gE_{ij} = E_{ij} g$ for $i \neq j$, so that for $k, \ell \in \{1, \ldots, n\}$ we get

$$g_{ki} \delta_{j\ell} = (gE_{ij})_{k\ell} = (E_{ij} g)_{k\ell} = \delta_{ik} g_{j\ell}.$$

For $k = i$ and $\ell = j$ we obtain $g_{ii} = g_{jj}$ and for $k = j = \ell$, we get $g_{ji} = 0$. Therefore $g = \lambda \mathbf{1}$ for some $\lambda \in \mathbb{K}$.

(b) It is obvious that φ is a group homomorphism and that φ is continuous. Moreover, the map

$$\psi \colon \mathrm{GL}_n(\mathbb{R})_+ \to \mathbb{R}_+^\times \times \mathrm{SL}_n(\mathbb{R}), \quad g \mapsto ((\det g)^{\frac{1}{n}}, (\det g)^{-\frac{1}{n}} g)$$

is continuous and satisfies $\varphi \circ \psi = \mathrm{id}$ and $\psi \circ \varphi = \mathrm{id}$. Hence φ is a homeomorphism. $\square$

Remark 1.1.11. The subgroups

$$Z(\mathrm{GL}_n(\mathbb{K})) \quad \text{and} \quad \mathrm{SL}_n(\mathbb{K})$$

are normal subgroups of $\mathrm{GL}_n(\mathbb{K})$. Moreover, for $\mathrm{GL}_n(\mathbb{R})$ the subgroup $\mathrm{GL}_n(\mathbb{R})_+$ is a proper normal subgroup and the same holds for $\mathbb{R}_+^\times \mathbf{1}$. One can show that these examples exhaust all normal arcwise connected subgroups of $\mathrm{GL}_n(\mathbb{K})$.

Exercises for Section 1.1

Exercise 1.1.1. Let V be a $\mathbb{K}$-vector space and $A \in \mathrm{End}(V)$. We write $V_\lambda(A) := \ker(A - \lambda\mathbf{1})$ for the *eigenspace of* A corresponding to the eigenvalue λ and $V^\lambda(A) := \bigcup_{n\in\mathbb{N}} \ker(A - \lambda\mathbf{1})^n$ for the *generalized eigenspace of* A corresponding to λ.

(a) If $A, B \in \mathrm{End}(V)$ commute, then

$$BV^\lambda(A) \subseteq V^\lambda(A) \quad \text{and} \quad BV_\lambda(A) \subseteq V_\lambda(A)$$

holds for each $\lambda \in \mathbb{K}$.

(b) If $A \in \mathrm{End}(V)$ is diagonalizable and $W \subseteq V$ is an A-invariant subspace, then $A|_W \in \mathrm{End}(W)$ is diagonalizable.

(c) If $A, B \in \mathrm{End}(V)$ commute and both are diagonalizable, then they are simultaneously diagonalizable, i.e., there exists a basis for V which consists of eigenvectors of A and B.

(d) If $\dim V < \infty$ and $\mathcal{A} \subseteq \mathrm{End}(V)$ is a commuting set of diagonalizable endomorphisms, then $\mathcal{A}$ can be simultaneously diagonalized, i.e., V is a direct sum of simultaneous eigenspaces of $\mathcal{A}$.

(e) For any function $\lambda\colon \mathcal{A} \to V$, we write $V_\lambda(\mathcal{A}) = \bigcap_{a\in\mathcal{A}} V_{\lambda(a)}(a)$ for the corresponding simultaneous eigenspace. Show that the sum $\sum_\lambda V_\lambda(\mathcal{A})$ is direct.

(f) If $\mathcal{A} \subseteq \mathrm{End}(V)$ is a finite commuting set of diagonalizable endomorphisms, then $\mathcal{A}$ can be simultaneously diagonalized.

(g) Find a commuting set of diagonalizable endomorphisms of a vector space V which cannot be diagonalized simultaneously.

Exercise 1.1.2. Let G be a topological group. Let G_0 be the *identity component*, i.e., the connected component of the identity in G. Show that G_0 is a closed normal subgroup of G.

Exercise 1.1.3. $\mathrm{SO}_n(\mathbb{K})$ is a closed normal subgroup of $\mathrm{O}_n(\mathbb{K})$ of index 2 and, for every $g \in \mathrm{O}_n(\mathbb{K})$ with $\det(g) = -1$,

$$\mathrm{O}_n(\mathbb{K}) = \mathrm{SO}_n(\mathbb{K}) \cup g\,\mathrm{SO}_n(\mathbb{K})$$

is a disjoint decomposition.

Exercise 1.1.4. For each subset $M \subseteq M_n(\mathbb{K})$ the *centralizer*

$$Z_{\mathrm{GL}_n(\mathbb{K})}(M) := \{g \in \mathrm{GL}_n(\mathbb{K}) : (\forall m \in M) gm = mg\}$$

is a closed subgroup of $\mathrm{GL}_n(\mathbb{K})$.

Exercise 1.1.5. We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by the map $z = x + iy \mapsto (x, y)$ and write $I(x, y) := (-y, x)$ for the real linear endomorphism of $\mathbb{R}^{2n}$ corresponding to multiplication with i. Then

$$\mathrm{GL}_n(\mathbb{C}) \cong Z_{\mathrm{GL}_{2n}(\mathbb{R})}(\{I\})$$

yields a realization of $\mathrm{GL}_n(\mathbb{C})$ as a closed subgroup of $\mathrm{GL}_{2n}(\mathbb{R})$.

Exercise 1.1.6. A subset C of a real vector space V is called a *convex cone* if C is convex and $\lambda C \subseteq C$ for each $\lambda > 0$.

Show that $\mathrm{Pd}_n(\mathbb{K})$ is an open convex cone in $\mathrm{Herm}_n(\mathbb{K})$.

Exercise 1.1.7. Show that

$$\gamma : (\mathbb{R}, +) \to \mathrm{GL}_2(\mathbb{R}), \quad t \mapsto \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

is a continuous group homomorphism with $\gamma(\pi) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\mathrm{im}\,\gamma = \mathrm{SO}_2(\mathbb{R})$.

Exercise 1.1.8. Show that the group $\mathrm{O}_n(\mathbb{C})$ is homeomorphic to the topological product of the subgroup

$$\mathrm{O}_n(\mathbb{R}) \cong \mathrm{U}_n(\mathbb{C}) \cap \mathrm{O}_n(\mathbb{C}) \quad \text{and the set} \quad \mathrm{Pd}_n(\mathbb{C}) \cap \mathrm{O}_n(\mathbb{C}).$$

Exercise 1.1.9. Let (X, d) be a compact metric space and $(x_n)_{n\in\mathbb{N}}$ a sequence in X. Show that $\lim_{n\to\infty} x_n = x$ is equivalent to the condition that each convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$ converges to x.

Exercise 1.1.10. If $A \in \mathrm{Herm}_n(\mathbb{K})$ satisfies $\langle Av, v\rangle = 0$ for each $v \in \mathbb{K}^n$, then $A = 0$.

Exercise 1.1.11. Show that for a complex matrix $A \in M_n(\mathbb{C})$ the following are equivalent:

(1) $A^* = A$.

(2) $\langle Av, v\rangle \in \mathbb{R}$ for each $v \in \mathbb{C}^n$.

Exercise 1.1.12. (a) Show that a matrix $A \in M_n(\mathbb{K})$ is hermitian if and only if there exists an orthonormal basis $v_1, \ldots, v_n$ for $\mathbb{K}^n$ and real numbers $\lambda_1, \ldots, \lambda_n$ with $Av_j = \lambda_j v_j$.

(b) Show that a complex matrix $A \in M_n(\mathbb{C})$ is unitary if and only if there exists an orthonormal basis $v_1, \ldots, v_n$ for $\mathbb{K}^n$ and $\lambda_j \in \mathbb{C}$ with $|\lambda_j| = 1$ and $Av_j = \lambda_j v_j$.

(c) Show that a complex matrix $A \in M_n(\mathbb{C})$ is *normal*, i.e., satisfies $A^*A = AA^*$, if and only if there exists an orthonormal basis $v_1, \ldots, v_n$ for $\mathbb{K}^n$ and $\lambda_j \in \mathbb{C}$ with $Av_j = \lambda_j v_j$.

Exercise 1.1.13. (a) Let V be a vector space and $\mathbf{1} \neq A \in \mathrm{End}(V)$ with $A^2 = \mathbf{1}$ (A is called an *involution*). Show that

$$V = \ker(A - \mathbf{1}) \oplus \ker(A + \mathbf{1}).$$

(b) Let V be a vector space and $A \in \mathrm{End}(V)$ with $A^3 = A$. Show that

$$V = \ker(A - \mathbf{1}) \oplus \ker(A + \mathbf{1}) \oplus \ker A.$$

(c) Let V be a vector space and $A \in \mathrm{End}(V)$ an endomorphism for which there exists a polynomial p of degree n with n different zeros $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$ and $p(A) = 0$. Show that A is diagonalizable with eigenvalues $\lambda_1, \ldots, \lambda_n$.

Exercise 1.1.14. Let $\beta\colon V \times V \to \mathbb{K}$ be a bilinear map and $g : V \to V$ with $\beta(gv, gw) = \beta(v, w)$ be a β-isometry. For a subspace $E \subseteq V$ we write

$$E^\perp := \{v \in V\colon (\forall w \in E)\ \beta(v, w) = 0\}$$

for its *orthogonal space*. Show that $g(E) = E$ implies that $g(E^\top) = E^\top$.

Exercise 1.1.15. [Iwasawa decomposition of $\mathrm{GL}_n(\mathbb{R})$] Let

$$T_n^+(\mathbb{R}) \subseteq \mathrm{GL}_n(\mathbb{R})$$

denote the subgroup of upper triangular matrices with positive diagonal entries. Show that the multiplication map

$$\mu\colon\ \mathrm{O}_n(\mathbb{R}) \times T_n^+(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R}), \quad (a, b) \mapsto ab$$

is a homeomorphism.

Exercise 1.1.16. Let $\mathbb{K}$ be a field and $n \in \mathbb{N}$. Show that

$$Z(M_n(\mathbb{K})) := \{z \in M_n(\mathbb{K})\colon (\forall x \in M_n(\mathbb{K}))\, zx = xz\} = \mathbb{K}\mathbf{1}.$$

1.2 Groups and Geometry

In Definition 1.1.3 we have defined certain matrix groups by concrete conditions on the matrices. Often it is better to think of matrices as linear maps described with respect to a basis. To do that we have to adopt a more abstract point of view. Similarly, one can study symmetry groups of bilinear forms on a vector space V without fixing a certain basis a priori. Actually it is much more convenient to choose a basis for which the structure of the bilinear form is as simple as possible.

1.2.1 Isometry Groups

Definition 1.2.1. [Groups and bilinear forms]

(a) (The abstract general linear group) Let V be a $\mathbb{K}$-vector space. We write $\mathrm{GL}(V)$ for the group of linear automorphisms of V. This is the group of invertible elements in the ring $\mathrm{End}(V)$ of all linear endomorphisms of V.

If V is an n-dimensional $\mathbb{K}$-vector space and $v_1, \ldots, v_n$ is a basis for V, then the map

$$\Phi\colon M_n(\mathbb{K}) \to \mathrm{End}(V), \quad \Phi(A)v_k := \sum_{j=1}^n a_{jk} v_j$$

is a linear isomorphism which describes the passage between linear maps and matrices. In view of $\Phi(\mathbf{1}) = \mathrm{id}_V$ and $\Phi(AB) = \Phi(A)\Phi(B)$, we obtain a group isomorphism

$$\Phi|_{\mathrm{GL}_n(\mathbb{K})}\colon\ \mathrm{GL}_n(\mathbb{K}) \to \mathrm{GL}(V).$$

(b) Let V be an n-dimensional vector space with basis $v_1, \ldots, v_n$ and $\beta\colon V \times V \to \mathbb{K}$ a bilinear map. Then $B = (b_{jk}) := (\beta(v_j, v_k))_{j,k=1,\ldots,n}$ is an $(n \times n)$-matrix, but this matrix should NOT be interpreted as the matrix of a linear map. It is the matrix of a bilinear map to $\mathbb{K}$, which is something different. It describes β in the sense that

$$\beta\Big(\sum_j x_j v_j, \sum_k y_k v_k\Big) = \sum_{j,k=1}^n x_j b_{jk} y_k = x^\top B y,$$

where $x^\top By$ with column vectors $x, y \in \mathbb{K}^n$ is viewed as a matrix product whose result is a (1×1)-matrix, i.e., an element of $\mathbb{K}$.

We write

$$\mathrm{Aut}(V, \beta) := \{g \in \mathrm{GL}(V) \colon (\forall v, w \in V)\, \beta(gv, gw) = \beta(v, w)\}$$

for the *isometry group of the bilinear form* β. Then it is easy to see that

$$\Phi^{-1}(\mathrm{Aut}(V, \beta)) = \{g \in \mathrm{GL}_n(\mathbb{K}) \colon g^\top B g = B\}.$$

If β is symmetric, we also write $\mathrm{O}(V, \beta) := \mathrm{Aut}(V, \beta)$ and if β is skew-symmetric, we write $\mathrm{Sp}(V, \beta) := \mathrm{Aut}(V, \beta)$.

If $v_1, \ldots, v_n$ is an orthonormal basis for β, i.e., $B = \mathbf{1}$, then

$$\Phi^{-1}(\mathrm{Aut}(V, \beta)) = \mathrm{O}_n(\mathbb{K})$$

is the orthogonal group defined in Section 1.1. Note that orthonormal bases can only exist for symmetric bilinear forms (Why?).

For $V = \mathbb{K}^{2n}$ and the block (2×2)-matrix

$$B := \begin{pmatrix} \mathbf{0} & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0} \end{pmatrix}$$

we see that $B^\top = -B$, and the group

$$\mathrm{Sp}_{2n}(\mathbb{K}) := \{g \in \mathrm{GL}_{2n}(\mathbb{K}) \colon g^\top B g = B\}$$

is called the *symplectic group*. The corresponding skew-symmetric bilinear form on $\mathbb{K}^{2n}$ is given by

$$\beta(x, y) = x^\top B y = \sum_{i=1}^n x_i y_{n+i} - x_{n+i} y_i.$$

(c) A symmetric bilinear form β on V is called *nondegenerate* if $\beta(v, V) = \{0\}$ implies $v = 0$. For $\mathbb{K} = \mathbb{C}$ every nondegenerate symmetric bilinear form β possesses an orthonormal basis (this builds on the existence of square roots of nonzero complex numbers; see Exercise 1.2.1), so that for every such form β we get

$$\mathrm{O}(V, \beta) \cong \mathrm{O}_n(\mathbb{C}).$$

For $\mathbb{K} = \mathbb{R}$ the situation is more complicated, since negative real numbers do not have a square root in $\mathbb{R}$. There might not be an orthonormal basis, but if β is nondegenerate, there always exists an orthogonal basis $v_1, \ldots, v_n$ and $p \in \{1, \ldots, n\}$

such that $\beta(v_j, v_j) = 1$ for $j = 1, \ldots, p$ and $\beta(v_j, v_j) = -1$ for $j = p+1, \ldots, n$. Let $q := n - p$ and $I_{p,q}$ denote the corresponding matrix

$$I_{p,q} = \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & -\mathbf{1}_q \end{pmatrix} \in M_{p+q}(\mathbb{R}).$$

Then $\mathrm{O}(V, \beta)$ is isomorphic to the group

$$\mathrm{O}_{p,q}(\mathbb{R}) := \{g \in \mathrm{GL}_n(\mathbb{R}) \colon g^\top I_{p,q} g = I_{p,q}\},$$

where $\mathrm{O}_{n,0}(\mathbb{R}) = \mathrm{O}_n(\mathbb{R})$.

(d) Let V be an n-dimensional complex vector space and $\beta \colon V \times V \to \mathbb{C}$ a sesquilinear form, i.e., β is linear in the first and antilinear in the second argument. Then we also choose a basis $v_1, \ldots, v_n$ in V and define $B = (b_{jk}) := (\beta(v_j, v_k))_{j,k=1,\ldots,n}$, but now we obtain

$$\beta\Big(\sum_j x_j v_j, \sum_k y_k v_k\Big) = \sum_{j,k=1}^n x_j b_{jk} \overline{y_k} = x^\top B \overline{y}.$$

We write

$$\mathrm{U}(V, \beta) := \{g \in \mathrm{GL}(V) \colon (\forall v, w \in V)\, \beta(gv, gw) = \beta(v, w)\}$$

for the corresponding *unitary group* and find

$$\Phi^{-1}(\mathrm{U}(V, \beta)) = \{g \in \mathrm{GL}_n(\mathbb{C}) \colon g^\top B \overline{g} = B\}.$$

If $v_1, \ldots, v_n$ is an orthonormal basis for β, i.e., $B = \mathbf{1}$, then

$$\Phi^{-1}(\mathrm{U}(V, \beta)) = \mathrm{U}_n(\mathbb{C}) = \{g \in \mathrm{GL}_n(\mathbb{C}) \colon g^* = g^{-1}\}$$

is the unitary group over $\mathbb{C}$. We call β *hermitian* if it is sesquilinear and satisfies $\beta(y, x) = \overline{\beta(x, y)}$. In this case one has to face the same problems as for symmetric forms on real vector spaces, but there always exists an orthogonal basis $v_1, \ldots, v_n$ and $p \in \{1, \ldots, n\}$ with $\beta(v_j, v_j) = 1$ for $j = 1, \ldots, p$ and $\beta(v_j, v_j) = -1$ for $j = p+1, \ldots, n$. With $q := n - p$ and

$$I_{p,q} := \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & -\mathbf{1}_q \end{pmatrix} \in M_n(\mathbb{C})$$

we then define the *indefinite unitary groups* by

$$\mathrm{U}_{p,q}(\mathbb{C}) := \{g \in \mathrm{GL}_n(\mathbb{C}) \colon g^\top I_{p,q} \overline{g} = I_{p,q}\}.$$

Since $I_{p,q}$ has real entries,

$$\mathrm{U}_{p,q}(\mathbb{C}) = \{g \in \mathrm{GL}_n(\mathbb{C}) \colon g^* I_{p,q} g = I_{p,q}\},$$

where $\mathrm{U}_{n,0}(\mathbb{C}) = \mathrm{U}_n(\mathbb{C})$.

Exercises for Section 1.2

Exercise 1.2.1. (a) Let β be a symmetric bilinear form on a finite-dimensional complex vector space V. Show that there exists an orthogonal basis $v_1, \ldots, v_n$ with $\beta(v_j, v_j) = 1$ for $j = 1, \ldots, p$ and $\beta(v_j, v_j) = 0$ for $j > p$.

(b) Show that each invertible symmetric matrix $B \in \mathrm{GL}_n(\mathbb{C})$ can be written as $B = AA^\top$ for some $A \in \mathrm{GL}_n(\mathbb{C})$.

Exercise 1.2.2. Let β be a symmetric bilinear form on a finite-dimensional real vector space V. Show that there exists an orthogonal basis $v_1, \ldots, v_{p+q}$ with $\beta(v_j, v_j) = 1$ for $j = 1, \ldots, p$, $\beta(v_j, v_j) = -1$ for $j = p+1, \ldots, p+q$, and $\beta(v_j, v_j) = 0$ for $j > p+q$.

Exercise 1.2.3. Let β be a skew-symmetric bilinear form on a finite-dimensional vector space V which is nondegenerate in the sense that $\beta(v, V) = \{0\}$ implies $v = 0$. Show that there exists a basis $v_1, \ldots, v_n$, $w_1, \ldots, w_n$ of V with

$$\beta(v_i, w_j) = \delta_{ij} \quad \text{and} \quad \beta(v_i, v_j) = \beta(w_i, w_j) = 0.$$

Exercise 1.2.4. [Metric characterization of midpoints] Let $(X, \|\cdot\|)$ be a normed space and $x, y \in X$ distinct points. Let

$$M_0 := \{z \in X : \|z - x\| = \|z - y\| = \tfrac{1}{2}\|x - y\|\} \quad \text{and} \quad m := \frac{x+y}{2}.$$

For a subset $A \subseteq X$ we define its *diameter*

$$\delta(A) := \sup\{\|a - b\| : a, b \in A\}.$$

Show that:

(1) If X is a pre-Hilbert space (i.e., a vector space with a hermitian scalar product), then $M_0 = \{m\}$ is a one-element set.

(2) $\|z - m\| \leq \frac{1}{2}\delta(M_0) \leq \frac{1}{2}\|x - y\|$ for $z \in M_0$.

(3) For $n \in \mathbb{N}$ we define inductively:

$$M_n := \{p \in M_{n-1} : (\forall z \in M_{n-1})\ \|z - p\| \leq \tfrac{1}{2}\delta(M_{n-1})\}.$$

Then, for each $n \in \mathbb{N}$:

(a) M_n is a convex set.

(b) M_n is invariant under the point reflection $s_m(a) := 2m - a$ in m.

(c) $m \in M_n$.

(d) $\delta(M_n) \leq \frac{1}{2}\delta(M_{n-1})$.

(4) $\bigcap_{n \in \mathbb{N}} M_n = \{m\}$.

Exercise 1.2.5. [Isometries of normed spaces are affine maps] Let $(X, \|\cdot\|)$ be a normed space endowed with the metric $d(x, y) := \|x - y\|$. Show that each isometry $\varphi\colon (X, d) \to (X, d)$ is an affine map by using the following steps:

(1) It suffices to assume that $\varphi(0) = 0$ and to show that this implies that φ is a linear map.

(2) $\varphi(\frac{x+y}{2}) = \frac{1}{2}(\varphi(x) + \varphi(y))$ for $x, y \in X$.

(3) φ is continuous.

(4) $\varphi(\lambda x) = \lambda\varphi(x)$ for $\lambda \in 2^{\mathbb{Z}} \subseteq \mathbb{R}$.

(5) $\varphi(x + y) = \varphi(x) + \varphi(y)$ for $x, y \in X$.

(6) $\varphi(\lambda x) = \lambda\varphi(x)$ for $\lambda \in \mathbb{R}$.

Exercise 1.2.6. Let $\beta\colon V \times V \to V$ be a symmetric bilinear form on the vector space V and

$$q\colon V \to V, \quad v \mapsto \beta(v, v)$$

the corresponding quadratic form. Then for $\varphi \in \mathrm{End}(V)$ the following are equivalent:

(1) $(\forall v \in V)\ q(\varphi(v)) = q(v)$.

(2) $(\forall v, w \in V)\ \beta(\varphi(v), \varphi(w)) = \beta(v, w)$.

Exercise 1.2.7. We consider $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$, where the elements of $\mathbb{R}^4$ are considered as space time events (q, t), $q \in \mathbb{R}^3$, $t \in \mathbb{R}$. On $\mathbb{R}^4$ we have the linear (time) functional

$$\Delta\colon \mathbb{R}^4 \to \mathbb{R}, (x, t) \mapsto t$$

and we endow $\ker \Delta \cong \mathbb{R}^3$ with the euclidian scalar product

$$\beta(x, y) := x_1y_1 + x_2y_2 + x_3y_3.$$

Show that

$$H := \{g \in \mathrm{GL}_4(\mathbb{R})\colon g \ker \Delta \subseteq \ker \Delta, g|_{\ker \Delta} \in \mathrm{O}_3(\mathbb{R})\} \cong \mathbb{R}^3 \rtimes (\mathrm{O}_3(\mathbb{R}) \times \mathbb{R}^\times)$$

and

$$G := \{g \in H\colon \Delta \circ g = \Delta\} \cong \mathbb{R}^3 \rtimes \mathrm{O}_3(\mathbb{R}).$$

In this sense the linear part of the Galilei group (extended by the space reflection S) is isomorphic to the symmetry group of the triple $(\mathbb{R}^4, \beta, \Delta)$, where Δ represents a universal time function and β is the scalar product on $\ker \Delta$. In the relativistic picture the time function is combined with the scalar product in the Lorentz form.

Exercise 1.2.8. On the four-dimensional real vector space $V := \mathrm{Herm}_2(\mathbb{C})$ we consider the symmetric bilinear form β given by

$$\beta(A, B) := \mathrm{tr}(AB) - \mathrm{tr}\, A \,\mathrm{tr}\, B.$$

Show that:

(1) The corresponding quadratic form is given by $q(A) := \beta(A, A) = -2 \det A$.

(2) Show that $(V, \beta) \cong \mathbb{R}^{3,1}$ by finding a basis $E_1, \ldots, E_4$ of $\mathrm{Herm}_2(\mathbb{C})$ with

$$q(a_1 E_1 + \ldots + a_4 E_4) = a_1^2 + a_2^2 + a_3^2 - a_4^2.$$

(3) For $g \in \mathrm{GL}_2(\mathbb{C})$ and $A \in \mathrm{Herm}_2(\mathbb{C})$ the matrix gAg^* is hermitian and satisfies

$$q(gAg^*) = |\det(g)|^2 q(A).$$

(4) For $g \in \mathrm{SL}_2(\mathbb{C})$ we define a linear map $\rho(g) \in \mathrm{GL}(\mathrm{Herm}_2(\mathbb{C}))$ by $\rho(g)(A) := gAg^*$. Then we obtain a homomorphism

$$\rho\colon \mathrm{SL}_2(\mathbb{C}) \to \mathrm{O}(V, \beta) \cong \mathrm{O}_{3,1}(\mathbb{R}).$$

(5) Show that $\ker \rho = \{\pm \mathbf{1}\}$.

Exercise 1.2.9. Let $\beta\colon V \times V \to \mathbb{K}$ be a bilinear form.
(1) Show that there exists a unique symmetric bilinear form β_+ and a unique skew-symmetric bilinear form β_- with $\beta = \beta_+ + \beta_-$.
(2) $\mathrm{Aut}(V, \beta) = \mathrm{O}(V, \beta_+) \cap \mathrm{Sp}(V, \beta_-)$.

Exercise 1.2.10. Let G be a group, $N \subseteq G$ a normal subgroup and

$$q\colon G \to G/N, \quad g \mapsto gN$$

be the quotient homomorphism. Show that:

(1) If $G \cong N \rtimes_\delta H$ for a subgroup H, then $H \cong G/N$.

(2) There exists a subgroup $H \subseteq G$ with $G \cong N \rtimes_\delta H$ if and only if there exists a group homomorphism $\sigma\colon G/N \to G$ with $q \circ \sigma = \mathrm{id}_{G/N}$.

Exercise 1.2.11. Show that $\mathrm{O}_{p,q}(\mathbb{C}) \cong \mathrm{O}_{p+q}(\mathbb{C})$ for $p, q \in \mathbb{N}_0$, $p + q > 0$.

Exercise 1.2.12. Let (V, β) be a euclidian vector space, i.e., a real vector space endowed with a positive definite symmetric bilinear form β. An element $\sigma \in \mathrm{O}(V, \beta)$ is called an *orthogonal reflection* if $\sigma^2 = \mathbf{1}$ and $\ker(\sigma - \mathbf{1})$ is a hyperplane. Show that for any finite-dimensional euclidian vector space (V, β), the orthogonal group $\mathrm{O}(V, \beta)$ is generated by reflections.

Exercise 1.2.13. (i) Show that, if n is odd, each $g \in \mathrm{SO}_n(\mathbb{R})$ has the eigenvalue 1.

(ii) Show that each $g \in \mathrm{O}_n(\mathbb{R})_-$ has the eigenvalue -1.

Exercise 1.2.14. Let V be a $\mathbb{K}$-vector space. An element $\varphi \in \mathrm{GL}(V)$ is called a *transvection* if $\dim_{\mathbb{K}} \big(\mathrm{im}(\varphi - \mathrm{id}_V)\big) = 1$ and $\mathrm{im}(\varphi - \mathrm{id}_V) \subseteq \ker(\varphi - \mathrm{id}_V)$. Show that:

(i) For each transvection φ, there exist a $v_\varphi \in V$ and a $\alpha_\varphi \in V^*$ such that $\varphi(v) = v - \alpha_\varphi(v) v_\varphi$ and $\alpha_\varphi(v_\varphi) = 0$.

(ii) For each transvection φ, there exist a $v_\varphi \in V$ and a $\alpha_\varphi \in V^*$ such that $\varphi(v) = v - \alpha_\varphi(v) v_\varphi$ and $\alpha_\varphi(v_\varphi) = 0$.

(ii) If $\dim V < \infty$, then $\det(\varphi) = 1$ for each transvection φ.

(iii) If $\psi \in \mathrm{GL}(V)$ commutes with all transvections, then every element of V is an eigenvector of ψ, so that $\psi \in \mathbb{K}^\times \mathrm{id}_V$.

(iv) $Z(\mathrm{GL}(V)) = \mathbb{K}^\times \mathbf{1}$.

(v) If $\dim V < \infty$, then $Z(\mathrm{SL}(V)) = \Gamma \mathbf{1}$, where $\Gamma := \{z \in \mathbb{K}^\times : z^n = 1\}$.

Exercise 1.2.15. Let V be a finite-dimensional $\mathbb{K}$-vector space for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and β be a skew symmetric bilinear form on V. Show that:

(i) A transvection $\varphi(v) = v - \alpha_\varphi(v) v_\varphi$ preserves β if and only if

$$(\forall v, w \in V): \quad \alpha_\varphi(v)\beta(v_\varphi, w) = \alpha_\varphi(w)\beta(v_\varphi, v).$$

If, in addition, β is nondegenerate, we call φ a *symplectic transvection.*

(ii) If β is nondegenerate and $\psi \in \mathrm{GL}(V)$ commutes with all symplectic transvections, then every vector in v is an eigenvector of ψ.

Exercise 1.2.16. Let V be a finite-dimensional $\mathbb{K}$-vector space for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and and β be a non-degenerate symmetric bilinear form on V. An involution $\varphi \in \mathrm{O}(V, \beta)$ is called an *orthogonal reflection* if $\dim_{\mathbb{K}} \big(\mathrm{im}(\varphi - \mathrm{id}_V)\big) = 1$. Show that:

(i) For each orthogonal reflection φ, there exists a non-isotropic $v_\varphi \in V$ such that $\varphi(v) = v - 2\frac{\beta(v, v_\alpha)}{\beta(v_\alpha, v_\alpha)}$.

(ii) If $\psi \in \mathrm{GL}(V)$ commutes with all orthogonal reflections, then every non-isotropic vector for β is an eigenvector of ψ, and this implies that $\psi \in \mathbb{K}^\times \mathrm{id}_V$.

(iv) $Z\big(\mathrm{O}(V, \beta)\big) = \{\pm \mathbf{1}\}$.

Chapter 2

The Matrix Exponential Function

In this chapter we study one of the central tools in Lie theory: the matrix exponential function. This function has various applications in the structure theory of matrix groups. First of all, it is naturally linked to the one-parameter subgroups, and it turns out that the local group structure of $\mathrm{GL}_n(\mathbb{K})$ in a neighborhood of the identity is determined by its one-parameter subgroups.

In the first section of this chapter we provide some tools to show that matrix valued functions defined by convergent power series are actually smooth. This is applied in the subsequent sections to the exponential and the logarithm functions. Then we discuss restrictions of the exponential function to certain subsets such as small 0-neighborhoods, the set of nilpotent matrices and the set of hermitian matrices. Finally, we derive the Baker–Campbell–Dynkin–Hausdorff formula expressing the product of two exponentials near the identity in terms of the Hausdorff series which involves only commutator brackets.

In the following chapter, we shall use the matrix exponential function to generalize the polar decomposition given in Proposition 1.1.5 to a larger class of groups. This will lead to topological information on various concrete matrix groups.

2.1 Smooth Functions Defined by Power Series

First we put the structure that we have on the space $M_n(\mathbb{K})$ of $(n \times n)$-matrices into a slightly more general context.

Definition 2.1.1. (a) A vector space A together with a bilinear map $A \times A \to A, (x, y) \mapsto x \cdot y$ (called multiplication) is called an (*associative*) *algebra* if the multiplication is associative in the sense that

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) \quad \text{for} \quad x, y, z \in A.$$

We write $xy := x \cdot y$ for the product of x and y in A.

The algebra A is called *unital* if it contains an element $\mathbf{1}$ satisfying $\mathbf{1}a = a\mathbf{1} = a$ for each $a \in A$.

(b) A norm $\|\cdot\|$ on an algebra A is called *submultiplicative* if

$$\|ab\| \leq \|a\| \cdot \|b\| \quad \text{for all} \quad a, b \in A.$$

Then the pair $(A, \|\cdot\|)$ is called a *normed algebra*. If, in addition, A is a complete normed space, then it is said to be a *Banach algebra*.

Remark 2.1.2. Any finite-dimensional normed space is complete, so that each finite-dimensional normed algebra is a Banach algebra.

Example 2.1.3. Endowing $M_n(\mathbb{K})$ with the operator norm with respect to the euclidian norm on $\mathbb{K}^n$ defines on $M_n(\mathbb{K})$ the structure of a unital Banach algebra.

The following proposition shows in particular that inserting elements of a Banach algebra into power series is compatible with composition.

In the following we write $\mathbb{K}[[z]]$ for the space of all formal power series

$$f(z) := \sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathbb{K}$$

in the variable z. For $r \in [0, \infty[$ we define

$$\|f\|_r := \sum_{n=0}^{\infty} |a_n| r^n \in [0, \infty].$$

We write $\mathbb{K}[[z]]_r$ for the subset of all power series with $\|f\|_r < \infty$. Note that this implies that f converges uniformly to a function on the closed disc of radius r in $\mathbb{K}$.

Proposition 2.1.4. *Let $\mathcal{A}$ be a unital Banach algebra.*

(1) *If $x \in \mathcal{A}$ and $f \in \mathbb{K}[[z]]_r$ for some $r \geq \|x\|$, then $f(x) := \sum_{n=0}^{\infty} a_n x^n$ converges absolutely with*

$$\|f(x)\| \leq \|f\|_r.$$

For two power series $f(z) = \sum_n a_n z^n$ and $g(z) = \sum_n b_n z^n$ with $\|f\|_r, \|g\|_r < \infty$, we also have the product formula

$$(f \cdot g)(x) = f(x)g(x), \quad \text{where} \quad (f \cdot g)(z) := \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} a_k b_{n-k}\Big) z^n \tag{2.1}$$

is the power series defined by the Cauchy product of f and g.

(2) *Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{K}[[z]]_r$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n \in z\mathbb{K}[[z]]$ satisfies $\|g\|_s < r$. We define the power series $f \circ g$ by formal composition:*

$$(f \circ g)(z) := \sum_n c_n z^n, \quad c_n = \sum_{k=0}^{n} a_k \sum_{i_1 + \cdots + i_k = n} b_{i_1} \cdots b_{i_k}.$$

Then $\|f \circ g\|_s \leq \|f\|_r$, and for any $x \in \mathcal{A}$ with $\|x\| \leq s$ the element $g(x)$ exists with $\|g(x)\| < r$, and we have the Composition Formula:

$$f(g(x)) = (f \circ g)(x). \tag{2.2}$$

Proof. (1) The convergence of $f(x)$ follows immediately from

$$\sum_n \|a_n x^n\| \leq \sum_n |a_n| \|x\|^n \leq \sum_n |a_n| r^n = \|f\|_r$$

and the Domination Test for absolutely converging series in a Banach space. We also obtain immediately the estimate $\|f(x)\| \leq \|f\|_r$.

If $\|f\|_r, \|g\|_r < \infty$, then (2.1) follows from the Cauchy Product Formula (Exercise 2.1.3) because the series $f(x)$ and $g(x)$ converge absolutely.

(2) To see that $\|f \circ g\|_s < \infty$, we calculate

$$\begin{aligned}
\sum_n |c_n| s^n &\leq \sum_n \sum_{k=0}^{n} |a_k| \sum_{i_1+\cdots+i_k=n} |b_{i_1}| \cdots |b_{i_k}| s^n \\
&\leq \sum_{k=0}^{\infty} |a_k| \sum_n \sum_{i_1+\cdots+i_k=n} |b_{i_1}| \cdots |b_{i_k}| s^n = \sum_{k=0}^{\infty} |a_k| \|g\|_s^n \\
&\leq \sum_{k=0}^{\infty} |a_k| r^n = \|f\|_r.
\end{aligned}$$

For $\|x\| \leq s$ we obtain from (1) the relation $\|g(x)\| \leq \|g\|_s$, so that

$$f(g(x)) = \sum_{n=0}^{\infty} a_n g(x)^n$$

is defined. Applying the Product Formula to the powers of g, we further obtain $g(x)^n = (g^n)(x)$, so that the polynomials $f_N(z) := \sum_{n=0}^{N} a_n z^n$ satisfy

$$f_N(g(x)) = \sum_{n=0}^{N} a_n g(x)^n = (f_N \circ g)(x).$$

Next we observe that

$$\|f \circ g - f_N \circ g\|_s = \|(f - f_N) \circ g\|_s \leq \|f - f_N\|_r \to 0,$$

so that

$$f_N(g(x)) = (f_N \circ g)(x) \to (f \circ g)(x).$$

Since we also have $f_N(g(x)) \to f(g(x))$ by definition, the Composition Formula is proved. $\square$

Lemma 2.1.5. *If $\mathcal{A}$ is a unital Banach algebra, then we endow the vector space $T\mathcal{A} := \mathcal{A} \oplus \mathcal{A}$ with the norm $\|(a,b)\| := \|a\| + \|b\|$ and the multiplication*

$$(a,b)(a',b') := (aa', ab' + ba').$$

Then $T\mathcal{A}$ is a unital Banach algebra with identity $(\mathbf{1}, 0)$.

We put $\varepsilon := (0,1)$. Then each element of $T\mathcal{A}$ can be written in a unique fashion as $(a,b) = a + b\varepsilon$ and the multiplication satisfies

$$(a + b\varepsilon)(a' + b'\varepsilon) = aa' + (ab' + ba')\varepsilon.$$

In particular, $\varepsilon^2 = 0$.

Proof. That $T\mathcal{A}$ is a unital algebra is a trivial verification. That the norm is submultiplicative follows from

$$\begin{aligned}\|(a,b)(a',b')\| &= \|aa'\| + \|ab' + ba'\| \leq \|a\| \cdot \|a'\| + \|a\| \cdot \|b'\| + \|b\| \cdot \|a'\| \\ &\leq (\|a\| + \|b\|)(\|a'\| + \|b'\|) = \|(a,b)\| \cdot \|(a',b')\|.\end{aligned}$$

This proves that $(T\mathcal{A}, \|\cdot\|)$ is a unital normed algebra, the unit being $\mathbf{1} = (\mathbf{1}, 0)$. The completeness of $T\mathcal{A}$ follows easily from the completeness of $\mathcal{A}$ (Exercise). $\square$

Lemma 2.1.6. *Let $f = \sum_{n=0}^{\infty} c_n z^n \in \mathbb{K}[[z]]_r$ and $\mathcal{A}$ be a finite-dimensional unital Banach algebra. Then*

$$f\colon B_r(0) := \{x \in \mathcal{A}\colon \|x\| < r\} \to \mathcal{A}, \quad x \mapsto \sum_{n=0}^{\infty} c_n x^n$$

defines a smooth function. Its derivative is given by

$$\mathtt{d}f(x) = \sum_{n=0}^{\infty} c_n \mathtt{d}p_n(x),$$

where $p_n(x) = x^n$ is the n^{th} power map whose derivative is given by

$$\mathtt{d}p_n(x)y = x^{n-1}y + x^{n-2}yx + \ldots + xyx^{n-2} + yx^{n-1}.$$

For $\|x\| < r$ and $y \in \mathcal{A}$ with $xy = yx$ we obtain in particular

$$\mathtt{d}p_n(x)y = nx^{n-1}y \quad \text{and} \quad \mathtt{d}f(x)y = \sum_{n=1}^{\infty} c_n n x^{n-1} y.$$

Proof. First we observe that the series defining $f(x)$ converges for $\|x\| < r$ by the Comparison Test (for series in Banach spaces). We shall prove by induction over $k \in \mathbb{N}$ that all such functions f are C^k-functions.

Step 1: First we show that f is a C^1-function. We define $\alpha_n\colon \mathcal{A} \to \mathcal{A}$ by

$$\alpha_n(h) := x^{n-1}h + x^{n-2}hx + \ldots + xhx^{n-2} + hx^{n-1}.$$

Then α_n is a continuous linear map with $\|\alpha_n\| \leq n\|x\|^{n-1}$. Furthermore

$$p_n(x+h) = (x+h)^n = x^n + \alpha_n(h) + r_n(h),$$

where

$$\begin{aligned}\|r_n(h)\| &\leq \binom{n}{2}\|h\|^2\|x\|^{n-2} + \binom{n}{3}\|h\|^3\|x\|^{n-3} + \ldots + \|h\|^n \\ &= \sum_{k\geq 2}\binom{n}{k}\|h\|^k\|x\|^{n-k}.\end{aligned}$$

In particular $\lim_{h\to 0}\frac{\|r_n(h)\|}{\|h\|} = 0$, and therefore p_n is differentiable in x with $\mathtt{d}p_n(x) = \alpha_n$. The series

$$\beta := \sum_{n=0}^{\infty} c_n\alpha_n = \sum_{n=0}^{\infty} c_n \mathtt{d}p_n(x)$$

converges absolutely in $\mathrm{End}(\mathcal{A})$ by the Ratio Test since $\|x\| < r$:

$$\sum_{n=0}^{\infty}|c_n|\|\alpha_n\| \leq \sum_{n=0}^{\infty}|c_n|\cdot n\cdot\|x\|^{n-1} < \infty.$$

We thus obtain a linear map $\beta(x) \in \mathrm{End}(\mathcal{A})$ for each x with $\|x\| < r$.

Now let h satisfy $\|x\| + \|h\| < r$, i.e., $\|h\| < r - \|x\|$. Then

$$f(x+h) = f(x) + \beta(x)(h) + r(h), \quad r(h) := \sum_{n=2}^{\infty} c_n r_n(h),$$

where

$$\begin{aligned}\|r(h)\| &\leq \sum_{n=2}^{\infty}|c_n|\|r_n(h)\| \leq \sum_{n=2}^{\infty}|c_n|\sum_{k=2}^{n}\binom{n}{k}\|h\|^k\|x\|^{n-k} \\ &\leq \sum_{k=2}^{\infty}\Big(\sum_{n=k}^{\infty}|c_n|\binom{n}{k}\|x\|^{n-k}\Big)\|h\|^k < \infty\end{aligned}$$

follows from $\|x\| + \|h\| < r$ because

$$\sum_k\sum_{n\geq k}|c_n|\binom{n}{k}\|x\|^{n-k}\|h\|^k = \sum_n |c_n|(\|x\|+\|h\|)^n \leq \sum_n |c_n| r^n < \infty.$$

Therefore the continuity of real-valued functions represented by a power series yields

$$\lim_{h\to 0}\frac{\|r(h)\|}{\|h\|} = \sum_{k=2}^{\infty}\Big(\sum_{n=k}^{\infty}|c_n|\binom{n}{k}\|x\|^{n-k}\Big)0^{k-1} = 0.$$

This proves that f is a C^1-function with the required derivative.

Step 2: To complete our proof by induction, we now show that if all functions f as above are C^k, then they are also C^{k+1}. In view of Step 1, this implies that they are smooth.

To set up the induction, we consider the Banach algebra $T\mathcal{A}$ from Lemma 2.1.5 and apply Step 1 to this algebra to obtain a smooth function

$$F\colon \{x+\varepsilon h \in T\mathcal{A}\colon \|x\|+\|h\| = \|x+\varepsilon h\| < r\} \to T\mathcal{A}$$
$$F(x+\varepsilon h) = \sum_{n=0}^{\infty} c_n \cdot (x+\varepsilon h)^n,$$

We further note that $(x+\varepsilon h)^n = x^n + \mathtt{d}p_n(x)h \cdot \varepsilon$. This implies the formula

$$F(x+\varepsilon h) = f(x) + \varepsilon \mathtt{d}f(x)h,$$

i.e., that the extension F of f to $T\mathcal{A}$ describes the first order Taylor expansion of f in each point $x \in \mathcal{A}$. Our induction hypothesis implies that F is a C^k-function.

Let $x_0 \in \mathcal{A}$ with $\|x_0\| < r$ and pick a basis $h_1, \ldots, h_d$ for $\mathcal{A}$ with $\|h_i\| < r - \|x_0\|$. Then all functions $x \mapsto \mathtt{d}f(x)h_i$ are defined and C^k on a neighborhood of x_0, and this implies that the function

$$B_r(0) \to \mathrm{Hom}(\mathcal{A},\mathcal{A}), \quad x \mapsto \mathtt{d}f(x)$$

is C^k. This in turn implies that f is C^{k+1}. □

Exercises for Section 2.1

Exercise 2.1.1. Let $X_1, \ldots, X_n$ be finite-dimensional normed spaces and $\beta\colon X_1 \times \ldots \times X_n \to Y$ an n-linear map.

(a) Show that there exists a constant $C \geq 0$ with

$$\|\beta(x_1, \ldots, x_n)\| \leq C\|x_1\| \cdots \|x_n\| \quad \text{for} \quad x_i \in X_i.$$

(b) Show that β is continuous.

(c) Show that β is differentiable with

$$\mathtt{d}\beta(x_1, \ldots, x_n)(h_1, \ldots, h_n) = \sum_{j=1}^{n} \beta(x_1, \ldots, x_{j-1}, h_j, x_{j+1}, \ldots, x_n).$$

Exercise 2.1.2. Let Y be a Banach space and $a_{n,m}$, $n, m \in \mathbb{N}$, elements in Y with

$$\sum_{n,m} \|a_{n,m}\| := \sup_{N \in \mathbb{N}} \sum_{n,m \leq N} \|a_{n,m}\| < \infty.$$

(a) Show that

$$A := \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{n,m} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{n,m}$$

and that both iterated sums exist.

(b) Show that for each sequence $(S_n)_{n\in\mathbb{N}}$ of finite subsets $S_n \subseteq \mathbb{N}\times\mathbb{N}$, $n \in \mathbb{N}$, with $S_n \subseteq S_{n+1}$ and $\bigcup_n S_n = \mathbb{N}\times\mathbb{N}$ we have

$$A = \lim_{n\in\mathbb{N}} \sum_{(j,k)\in S_n} a_{j,k}.$$

Exercise 2.1.3. [Cauchy Product Formula] Let X, Y, Z be Banach spaces and $\beta\colon X\times Y \to Z$ a continuous bilinear map. Suppose that $x := \sum_{n=0}^\infty x_n$ is absolutely convergent in X and that $y := \sum_{n=0}^\infty y_n$ is absolutely convergent in Y. Then

$$\beta(x,y) = \sum_{n=0}^\infty \sum_{k=0}^n \beta(x_k, y_{n-k}).$$

2.2 Elementary Properties of the Exponential Function

After the preparations of the preceding section, it is now easy to see that the matrix exponential function defines a smooth map on $M_n(\mathbb{K})$. In this section we describe some elementary properties of this function. As a group theoretic consequence for $\mathrm{GL}_n(\mathbb{K})$, we show that it has no small subgroups, i.e. has a neighborhood of $\mathbf{1}$ containing only the trivial subgroup. Moreover, we show that all one-parameter groups are smooth and given by the exponential function.

For $x \in M_n(\mathbb{K})$, we define

$$e^x :\ = \sum_{k=0}^\infty \frac{1}{k!}x^k. \tag{2.3}$$

The absolute convergence of the series on the right follows directly from the estimate

$$\sum_{k=0}^\infty \frac{1}{k!}\|x^k\| \leq \sum_{k=0}^\infty \frac{1}{k!}\|x\|^k = e^{\|x\|}$$

and the Comparison Test for absolute convergence of a series in a Banach space. We define the *exponential function of* $M_n(\mathbb{K})$ by

$$\exp\colon M_n(\mathbb{K}) \to M_n(\mathbb{K}), \quad \exp(x) := e^x.$$

Proposition 2.2.1. *The exponential function* $\exp\colon M_n(\mathbb{K}) \to M_n(\mathbb{K})$ *is smooth. For* $xy = yx$ *it satisfies*

$$\mathtt{d}\exp(x)y = \exp(x)y = y\exp(x) \tag{2.4}$$

and in particular

$$\mathtt{d}\exp(0) = \mathrm{id}_{M_n(\mathbb{K})}.$$

Proof. To verify the formula for the differential, we note that for $xy = yx$, Lemma 2.1.6 implies that

$$\mathtt{d}\exp(x)y = \sum_{k=1}^\infty \frac{1}{k!}kx^{k-1}y = \sum_{k=0}^\infty \frac{1}{k!}x^k y = \exp(x)y.$$

For $x = 0$, the relation $\exp(0) = \mathbf{1}$ now implies in particular that $\mathtt{d}\exp(0)y = y$. $\square$

Lemma 2.2.2. *Let $x, y \in M_n(\mathbb{K})$.*

(i) *If $xy = yx$, then $\exp(x+y) = \exp x \exp y$.*

(ii) $\exp(M_n(\mathbb{K})) \subseteq \mathrm{GL}_n(\mathbb{K})$, $\exp(0) = \mathbf{1}$, *and* $(\exp x)^{-1} = \exp(-x)$.

(iii) *For $g \in \mathrm{GL}_n(\mathbb{K})$ the relation $g\exp(x)g^{-1} = \exp(gxg^{-1})$ holds.*

Proof. (i) Using the general form of the Cauchy Product Formula (Exercise 2.1.3), we obtain

$$\begin{aligned}\exp(x+y) &= \sum_{k=0}^{\infty} \frac{(x+y)^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\ell=0}^{k} \binom{k}{\ell} x^\ell y^{k-\ell} \\ &= \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \frac{x^\ell}{\ell!} \frac{y^{k-\ell}}{(k-\ell)!} = \Big(\sum_{p=0}^{\infty} \frac{x^p}{p!}\Big)\Big(\sum_{\ell=0}^{\infty} \frac{y^\ell}{\ell!}\Big).\end{aligned}$$

(ii) From (i) we derive in particular $\exp x \exp(-x) = \exp 0 = \mathbf{1}$, which implies (ii).

(iii) is a consequence of $gx^ng^{-1} = (gxg^{-1})^n$ and the continuity of the conjugation map $c_g(x) := gxg^{-1}$ on $M_n(\mathbb{K})$. $\square$

Remark 2.2.3. (a) For $n = 1$, the exponential function

$$\exp\colon M_1(\mathbb{R}) \cong \mathbb{R} \to \mathbb{R}^\times \cong \mathrm{GL}_1(\mathbb{R}), \quad x \mapsto e^x$$

is injective, but this is not the case for $n > 1$. In fact,

$$\exp\begin{pmatrix} 0 & -2\pi \\ 2\pi & 0 \end{pmatrix} = \mathbf{1}$$

follows from

$$\exp\begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \quad t \in \mathbb{R}.$$

This example is nothing but the real picture of the relation $e^{2\pi i} = 1$.

Proposition 2.2.4. *For each sufficiently small open neighborhood U of 0 in $M_n(\mathbb{K})$, the map*

$$\exp|_U : U \to \mathrm{GL}_n(\mathbb{K})$$

is a diffeomorphism onto an open neighborhood of $\mathbf{1}$ in $\mathrm{GL}_n(\mathbb{K})$.

Proof. We have already seen that exp is a smooth map, and that $\mathtt{d}\exp(\mathbf{0}) = \mathrm{id}_{M_d(\mathbb{K})}$. Therefore the assertion follows from the Inverse Function Theorem. $\square$

If U is as in Proposition 2.2.4 and $V = \exp(U)$, we define

$$\log_V \colon = (\exp|_U)^{-1} \colon V \to U \subseteq M_d(\mathbb{K}).$$

We shall see below why this function deserves to be called a *logarithm function.*

Theorem 2.2.5. (No Small Subgroup Theorem) *There exists an open neighborhood V of $\mathbf{1}$ in $\mathrm{GL}_n(\mathbb{K})$ such that $\{\mathbf{1}\}$ is the only subgroup of $\mathrm{GL}_n(\mathbb{K})$ contained in V.*

Proof. Let U be as in Proposition 2.2.4 and assume further that U is convex and bounded. We set $U_1 := \frac{1}{2}U$. Let $G \subseteq V := \exp U_1$ be a subgroup of $\mathrm{GL}_n(\mathbb{K})$ and $g \in G$. Then we write $g = \exp x$ with $x \in U_1$ and assume that $x \neq 0$. Let $k \in \mathbb{N}$ be maximal with $kx \in U_1$ (the existence of k follows from the boundedness of U). Then

$$g^{k+1} = \exp((k+1)x) \in G \subseteq V$$

implies the existence of $y \in U_1$ with $\exp(k+1)x = \exp y$. Since $(k+1)x \in 2U_1 = U$ follows from $\frac{k+1}{2}x \in [0,k]x \subseteq U_1$, and $\exp|_U$ is injective, we obtain $(k+1)x = y \in U_1$, contradicting the maximality of k. Therefore $g = \mathbf{1}$. $\square$

A *one-parameter (sub)group* of a group G is a group homomorphism $\gamma\colon (\mathbb{R},+) \to G$. The following result describes the differentiable one-parameter subgroups of $\mathrm{GL}_n(\mathbb{K})$.

Theorem 2.2.6. *[One-parameter Group Theorem] For each $x \in M_n(\mathbb{K})$, the map*

$$\gamma\colon (\mathbb{R},+) \to \mathrm{GL}_n(\mathbb{K}), \quad t \mapsto \exp(tx)$$

is a smooth group homomorphism solving the initial value problem

$$\gamma(0) = \mathbf{1} \quad \text{and} \quad \gamma'(t) = \gamma(t)x \quad \text{for } t \in \mathbb{R}.$$

Conversely, every continuous one-parameter group $\gamma\colon \mathbb{R} \to \mathrm{GL}_n(\mathbb{K})$ is of this form.

Proof. In view of Lemma 2.2.2(i) and the differentiability of exp, we have

$$\lim_{h\to 0} \frac{1}{h}\big(\gamma(t+h) - \gamma(t)\big) = \lim_{h\to 0} \frac{1}{h}\big(\gamma(t)\gamma(h) - \gamma(t)\big) = \gamma(t) \lim_{h\to 0} \frac{1}{h}\big(e^{hx} - \mathbf{1}\big) = \gamma(t)x.$$

Hence γ is differentiable with $\gamma'(t) = x\gamma(t) = \gamma(t)x$. From that it immediately follows that γ is smooth with $\gamma^{(n)}(t) = x^n\gamma(t)$ for each $n \in \mathbb{N}$.

Although we will not need it for the completeness of the proof, we first show that each one-parameter group $\gamma\colon \mathbb{R} \to \mathrm{GL}_n(\mathbb{K})$ which is differentiable in 0 has the required form. For $x := \gamma'(0)$, the calculation

$$\gamma'(t) = \lim_{s\to 0} \frac{\gamma(t+s) - \gamma(t)}{s} = \lim_{s\to 0} \gamma(t)\frac{\gamma(s) - \gamma(0)}{s} = \gamma(t)\gamma'(0) = \gamma(t)x$$

implies that γ is differentiable and solves the initial value problem

$$\gamma'(t) = \gamma(t)x, \qquad \gamma(0) = \mathbf{1}.$$

Therefore the Uniqueness Theorem for Linear Differential Equations implies that $\gamma(t) = \exp tx$ for all $t \in \mathbb{R}$.

It remains to show that each continuous one-parameter group γ of $\mathrm{GL}_n(\mathbb{K})$ is differentiable in 0. As in the proof of Theorem 2.2.5, let U be a convex symmetric (i.e., $U = -U$) 0-neighborhood in $M_n(\mathbb{K})$ for which $\exp|_U$ is a diffeomorphism onto an open subset (Proposition 2.2.4) and $U_1 := \frac{1}{2}U$. Since γ is continuous in 0, there exists an $\varepsilon > 0$ such that $\gamma([-\varepsilon, \varepsilon]) \subseteq \exp(U_1)$. Then $\alpha(t) := (\exp|_U)^{-1}(\gamma(t))$ defines a continuous curve $\alpha\colon [-\varepsilon, \varepsilon] \to U_1$ with $\exp(\alpha(t)) = \gamma(t)$ for $|t| \leq \varepsilon$. For any such t we then have

$$\exp\left(2\alpha(\tfrac{t}{2})\right) = \exp(\alpha(\tfrac{t}{2}))^2 = \gamma(\tfrac{t}{2})^2 = \gamma(t) = \exp(\alpha(t)),$$

so that the injectivity of exp on U yields

$$\alpha(\tfrac{t}{2}) = \tfrac{1}{2}\alpha(t) \quad \text{for} \quad |t| \leq \varepsilon.$$

Inductively we thus obtain

$$\alpha(\tfrac{t}{2^k}) = \tfrac{1}{2^k}\alpha(t) \quad \text{for} \quad |t| \leq \varepsilon, k \in \mathbb{N}. \tag{2.5}$$

In particular, we obtain

$$\alpha(t) \in \frac{1}{2^k}U_1 \quad \text{for} \quad |t| \leq \frac{\varepsilon}{2^k}.$$

For $m \in \mathbb{Z}$ with $|m| \leq 2^k$ and $|t| \leq \frac{\varepsilon}{2^k}$ we now have $|mt| \leq \varepsilon$, $m\alpha(t) \in \frac{m}{2^k}U_1 \subseteq U_1$, and

$$\exp(m\alpha(t)) = \gamma(t)^m = \gamma(mt) = \exp(\alpha(mt)).$$

Therefore the injectivity of exp on U_1 yields

$$\alpha(mt) = m\alpha(t) \quad \text{for} \quad m \leq 2^k, |t| \leq \frac{\varepsilon}{2^k}. \tag{2.6}$$

Combining (2.5) and (2.6), leads to

$$\alpha(\tfrac{m}{2^k}t) = \tfrac{m}{2^k}\alpha(t) \quad \text{for} \quad |t| \leq \varepsilon, k \in \mathbb{N}, |m| \leq 2^k.$$

Since the set of all numbers $\frac{mt}{2^k}$, $m \in \mathbb{Z}$, $k \in \mathbb{N}$, $|m| \leq 2^k$, is dense in the interval $[-t, t]$, the continuity of α implies that

$$\alpha(t) = \frac{t}{\varepsilon}\alpha(\varepsilon) \quad \text{for} \quad |t| \leq \varepsilon.$$

In particular, α is smooth and of the form $\alpha(t) = tx$ for some $x \in M_n(\mathbb{K})$. Hence $\gamma(t) = \exp(tx)$ for $|t| \leq \varepsilon$, but then $\gamma(mt) = \exp(mtx)$ for $m \in \mathbb{N}$ leads to $\gamma(t) = \exp(tx)$ for each $t \in \mathbb{R}$. □

Exercises for Section 2.2

Exercise 2.2.1. Let $D \in M_n(\mathbb{K})$ be a diagonal matrix. Calculate its operator norm with respect to the euclidean norm on $\mathbb{K}^n$.

Exercise 2.2.2. If $g \in M_n(\mathbb{K})$ satisfies $\|g - \mathbf{1}\| < 1$, then $g \in \mathrm{GL}_n(\mathbb{K})$. Find a formula for $(\mathbf{1} - g)^{-1}$ using a geometric series.

Exercise 2.2.3. (a) Calculate e^{tN} for $t \in \mathbb{K}$ and the matrix

$$N = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \cdot & 0 & 1 & 0 & \cdot \\ \cdot & & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & 1 \\ 0 & & \dots & & 0 \end{pmatrix} \in M_n(\mathbb{K}).$$

(b) If A is a block diagonal matrix $\operatorname{diag}(A_1, \ldots, A_k)$, then e^A is the block diagonal matrix $\operatorname{diag}(e^{A_1}, \ldots, e^{A_k})$.
(c) Calculate e^{tA} for a matrix $A \in M_n(\mathbb{C})$ given in Jordan Normal Form.

Exercise 2.2.4. Recall that a matrix x is said to be *nilpotent* if $x^d = 0$ for some $d \in \mathbb{N}$ and y is called *unipotent* if $y - \mathbf{1}$ is nilpotent.

Let $a, b \in M_n(\mathbb{K})$ be commuting matrices.

(a) If a and b are nilpotent, then $a + b$ is nilpotent.

(b) If a and b are unipotent, then ab is unipotent.

Exercise 2.2.5. [Jordan decomposition]
(a) (Additive Jordan decomposition) Show that each complex matrix $X \in M_n(\mathbb{C})$ can be written in a unique fashion as

$$X = X_s + X_n \quad \text{with} \quad [X_s, X_n] = 0,$$

where X_n is nilpotent and X_s diagonalizable.
(b) $A \in M_n(\mathbb{C})$ commutes with a diagonalizable matrix D if and only if A preserves all eigenspaces of D.
(c) $A \in M_n(\mathbb{C})$ commutes with X if and only if it commutes with X_s and X_n.

Exercise 2.2.6. [Multiplicative Jordan decomposition] (a) Show that each invertible complex matrix $g \in \mathrm{GL}_n(\mathbb{C})$ can be written in a unique fashion as

$$g = g_s g_u, \quad \text{with} \quad g_s g_u = g_u g_s,$$

where g_u is unipotent and g_s diagonalizable.
(b) If $X = X_s + X_n$ is the additive Jordan decomposition, then $e^X = e^{X_s} e^{X_n}$ is the multiplicative Jordan decomposition of e^X.

Exercise 2.2.7. Let $A \in M_n(\mathbb{C})$. Show that the set $e^{\mathbb{R}A} = \{e^{tA} : t \in \mathbb{R}\}$ is bounded in $M_n(\mathbb{C})$ if and only if A is diagonalizable with purely imaginary eigenvalues.

Exercise 2.2.8. Let $U \in \mathrm{GL}_n(\mathbb{C})$. Then the set $\{U^n : n \in \mathbb{Z}\}$ is bounded if and only if U is diagonalizable and $\operatorname{Spec}(U) \subseteq \{z \in \mathbb{C} : |z| = \mathbf{1}\}$.

Exercise 2.2.9. Show that:
(a) $\exp(M_n(\mathbb{R}))$ is contained in the identity component $\mathrm{GL}_n(\mathbb{R})_+$ of $\mathrm{GL}_n(\mathbb{R})$. In particular the exponential function of $\mathrm{GL}_n(\mathbb{R})$ is not surjective because this group is not connected.
(b) The exponential function $\exp : M_2(\mathbb{R}) \to \mathrm{GL}_2(\mathbb{R})_+$ is not surjective.

Exercise 2.2.10. Let $V \subseteq M_n(\mathbb{C})$ be a commutative subspace, i.e., an abelian Lie subalgebra. Then $A := e^V$ is an abelian subgroup of $\mathrm{GL}_n(\mathbb{C})$ and

$$\exp : (V, +) \to (A, \cdot)$$

is a group homomorphism whose kernel consists of diagonalizable elements whose eigenvalues are contained in $2\pi i\mathbb{Z}$.

Exercise 2.2.11. For $X, Y \in M_n(\mathbb{C})$ the following are equivalent:

(1) $e^X = e^Y$.

(2) $X_n = Y_n$ (the nilpotent Jordan components) and $e^{X_s} = e^{Y_s}$.

Exercise 2.2.12. For $A \in M_n(\mathbb{C})$ the relation $e^A = \mathbf{1}$ holds if and only if A is diagonalizable with all eigenvalues contained in $2\pi i\mathbb{Z}$.

2.3 The Logarithm Function

In this section we apply the tools from Section 2.1 to the logarithm series. Since its radius of convergence is 1, it defines a smooth function $\mathrm{GL}_n(\mathbb{K}) \supseteq B_1(\mathbf{1}) \to M_n(\mathbb{K})$, and we shall see that it thus provides a smooth inverse of the exponential function.

Lemma 2.3.1. *The series* $\log(\mathbf{1} + x) := \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}$ *converges for* $x \in M_n(\mathbb{K})$ *with* $\|x\| < 1$ *and defines a smooth function*

$$\log \colon B_1(\mathbf{1}) \to M_n(\mathbb{K}).$$

For $\|x\| < 1$ *and* $y \in M_n(\mathbb{K})$ *with* $xy = yx$*,*

$$(\mathtt{d}\log)(\mathbf{1} + x)y = (\mathbf{1} + x)^{-1}y.$$

Proof. The convergence follows from

$$\sum_{k=1}^{\infty} \frac{r^k}{k} = -\log(1 - r) < \infty$$

for $0 < r < 1$, so that the smoothness follows from Lemma 2.1.6.

If x and y commute, then the formula for the derivative in Lemma 2.1.6 leads to

$$(\mathtt{d}\log)(\mathbf{1} + x)y = \sum_{k=1}^{\infty} (-1)^{k+1} x^{k-1} y = (\mathbf{1} + x)^{-1} y$$

(here we used the Neumann series; cf. Exercise 2.2.2). □

Proposition 2.3.2. (a) *For* $x \in M_n(\mathbb{K})$ *with* $\|x\| < \log 2$,

$$\log(\exp x) = x.$$

(b) *Every* $a \in \mathrm{GL}_n(\mathbb{K})$ *with* $\|a - \mathbf{1}\| < 1$ *satisfies* $\exp(\log a) = a$.

Proof. (a) We apply Proposition 2.1.4(2) with $g(z) = \exp(z) - 1$, $s = \|x\| < \log 2$, $f(z) = \log(1+z)$ and $r < 1$. We thus obtain $\log(\exp x) = x$.

(b) Next we apply Proposition 2.1.4(2) with $f(z) = \exp(z) - 1$ and $g(z) = \log(1+z)$ to obtain $\exp(\log(a) = a$ for $a = \mathbf{1} + x$, $\|x\| < 1$. □

2.3.1 The Exponential Function on Nilpotent Matrices

Proposition 2.3.3. *Let*

$$U := \{g \in \mathrm{GL}_d(\mathbb{K}) \colon (g - \mathbf{1})^d = 0\}$$

be the set of unipotent matrices and

$$N := \{x \in M_d(\mathbb{K}) \colon x^d = 0\} = U - \mathbf{1}$$

the set of nilpotent matrices. Then $\exp_U := \exp|_N \colon N \to U$ *is a homeomorphism whose inverse is given by*

$$\log_U \colon g \mapsto \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(g-\mathbf{1})^k}{k} = \sum_{k=1}^{d-1} (-1)^{k+1} \frac{(g-\mathbf{1})^k}{k}.$$

Proof. First we observe that for $x \in N$ we have

$$e^x - \mathbf{1} = xa \quad \text{with} \quad a := \sum_{n=1}^{d} \frac{1}{n!} x^{n-1}.$$

In view of $xa = ax$, this leads to $(e^x - \mathbf{1})^d = x^d a^d = 0$. Therefore $\exp_U(N) \subseteq U$. Similarly we obtain for $g \in U$ that

$$\log_U(g) = (g - \mathbf{1}) \sum_{k=1}^{d} (-1)^{k+1} \frac{(g-\mathbf{1})^{k-1}}{k} \in N.$$

For $x \in N$, the curve

$$F \colon \mathbb{R} \to M_d(\mathbb{K}), \quad t \mapsto \log_U \exp_U(tx)$$

is a polynomial function and Proposition 2.3.2 implies that $F(t) = tx$ for $\|tx\| < \log 2$. This implies that $F(t) = tx$ for each $t \in \mathbb{R}$ and hence that $\log_U \exp_U(x) = F(1) = x$.

Likewise we see that for $g = \mathbf{1} + x \in U$ the curve

$$G \colon \mathbb{R} \to M_d(\mathbb{K}), \quad t \mapsto \exp_U \log_U(\mathbf{1} + tx)$$

is polynomial with $G(t) = \mathbf{1} + tx$ for $\|tx\| < 1$. Therefore $\exp_U \log_U(g) = F(1) = \mathbf{1} + x = g$. This proves that the functions $\exp_U$ and $\log_U$ are inverse to each other. □

Corollary 2.3.4. *Let* $X \in \mathrm{End}(V)$ *be a nilpotent endomorphism of the* $\mathbb{K}$*-vector space* V *and* $v \in V$*. Then the following are equivalent:*

(1) $Xv = 0$.

(2) $e^X v = v$.

Proof. Clearly $Xv = 0$ implies $e^X v = \sum_{n=0}^{\infty} \frac{1}{n!} X^n v = v$. If, conversely, $e^X v = v$, then $Xv = \log(e^X) v = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(e^X - \mathbf{1})^k}{k} v = 0$. □

2.3.2 The Exponential Function on Hermitian Matrices

For the following proof, we recall that for a hermitian $d \times d$-matrix A we have

$$\|A\| = \max\{|\lambda| \colon \det(A - \lambda \mathbf{1}) = 0\}$$

(Exercise 2.2.1).

Proposition 2.3.5. *The restriction*

$$\exp_P := \exp|_{\mathrm{Herm}_d(\mathbb{K})} \colon \mathrm{Herm}_d(\mathbb{K}) \to \mathrm{Pd}_d(\mathbb{K})$$

is a diffeomorphism onto the open subset $\mathrm{Pd}_d(\mathbb{K})$ *of* $\mathrm{Herm}_d(\mathbb{K})$.

Proof. We have $(e^x)^* = e^{x^*}$, which implies that $\exp x$ is hermitian if x is hermitian. Moreover, if $\lambda_1, \ldots, \lambda_d$ are the real eigenvalues of x, then $e^{\lambda_1}, \ldots, e^{\lambda_d}$ are the eigenvalues of e^x. Therefore e^x is positive definite for each hermitian matrix x.

If, conversely, $g \in \mathrm{Pd}_d(\mathbb{K})$, then let $v_1, \ldots, v_d$ be an orthonormal basis of eigenvectors for g with $gv_j = \lambda_j v_j$. Then $\lambda_j > 0$ for each j, and we define $\log_H(g) \in \mathrm{Herm}_d(\mathbb{K})$ by $\log_H(g)v_j := (\log \lambda_j)v_j$, $j = 1, \ldots, d$. From this construction of the logarithm function it is clear that

$$\log_H \circ \exp_P = \mathrm{id}_{\mathrm{Herm}_d(\mathbb{K})} \quad \text{and} \quad \exp_P \circ \log_H = \mathrm{id}_{\mathrm{Pd}_d(\mathbb{K})}.$$

For two real numbers $x, y > 0$, we have $\log(xy) = \log x + \log y$. From this we obtain for $\lambda > 0$ the relation

$$\log_H(\lambda g) = (\log \lambda)\, \mathbf{1} + \log_H(g) \tag{2.7}$$

by following what happens on each eigenspace of g.

The relation

$$\log(x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x - \mathbf{1})^k}{k}$$

for $x \in \mathbb{R}$ with $|x - 1| < 1$ implies that for $\|g - \mathbf{1}\| < 1$ we have

$$\log_H(g) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(g - \mathbf{1})^k}{k}.$$

This proves that $\log_H$ is smooth in $B_1(\mathbf{1}) \cap \mathrm{Herm}_d(\mathbb{K})$, hence in a neighborhood of g_0 if $\|g_0 - \mathbf{1}\| < 1$ (Lemma 2.3.1). This condition means that for each eigenvalue μ of g_0 we have $|\mu - 1| < 1$ (Exercise 2.3.1). If it is not satisfied, then we choose $\lambda > 0$ such that $\|\lambda g\| < 2$. Then $\|\lambda g - \mathbf{1}\| < 1$, and we obtain with (2.7) the formula

$$\log_H(g) = -(\log \lambda)\mathbf{1} + \log_H(\lambda g) = -(\log \lambda)\mathbf{1} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(\lambda g - \mathbf{1})^k}{k}.$$

Therefore $\log_H$ is smooth on the entire open cone $\mathrm{Pd}_d(\mathbb{K})$, so that $\log_H = \exp_P^{-1}$ implies that $\exp_P$ is a diffeomorphism. $\square$

With Proposition 1.1.5, we thus obtain:

Corollary 2.3.6. *The group* $\mathrm{GL}_d(\mathbb{K})$ *is homeomorphic to*

$$\mathrm{U}_d(\mathbb{K}) \times \mathbb{R}^m \quad \textit{with} \quad m := \dim_{\mathbb{R}}(\mathrm{Herm}_d(\mathbb{K})) = \begin{cases} \frac{d(d+1)}{2} & \textit{for } \mathbb{K} = \mathbb{R} \\ d^2 & \textit{for } \mathbb{K} = \mathbb{C}. \end{cases}$$

Exercises for Section 2.3.

Exercise 2.3.1. Show that for a hermitian matrix $A \in \mathrm{Herm}_n(\mathbb{K})$ and the euclidian norm $\|\cdot\|$ on $\mathbb{K}^n$ we have

$$\|A\| := \sup\{\|Ax\| \colon \|x\| \leq 1\} = \max\{|\lambda| \colon \ker(A - \lambda \mathbf{1}) = 0\}.$$

Exercise 2.3.2. The exponential function $\exp : M_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ is surjective.

2.4 The Baker–Campbell–Dynkin–Hausdorff Formula

In this section we derive a formula which expresses the product $\exp x \exp y$ of two sufficiently small elements as the exponential image $\exp(x * y)$ of an element $x * y$ which can be described in terms of iterated commutator brackets. This implies in particular that the group multiplication in a small $\mathbf{1}$-neighborhood of $\mathrm{GL}_n(\mathbb{K})$ is completely determined by the commutator bracket. To obtain these results, we express $\log(\exp x \exp y)$ as a power series $x * y$ in two variables. The (local) multiplication $*$ is called the *Baker–Campbell–Dynkin–Hausdorff Multiplication* and the identity

$$\log(\exp x \exp y) = x * y$$

the *Baker–Campbell–Dynkin–Hausdorff Formula* (BCDH). The derivation of this formula requires some preparation. We start with the *adjoint representation of* $\mathrm{GL}_n(\mathbb{K})$. This is the group homomorphism

$$\mathrm{Ad} \colon \mathrm{GL}_n(\mathbb{K}) \to \mathrm{Aut}(M_n(\mathbb{K})), \quad \mathrm{Ad}(g)x = gxg^{-1},$$

where $\mathrm{Aut}(M_n(\mathbb{K}))$ stands for the group of algebra automorphisms of $M_n(\mathbb{K})$. For $x \in M_n(\mathbb{K})$, we further define a linear map

$$\mathrm{ad}(x) \colon M_n(\mathbb{K}) \to M_n(\mathbb{K}), \quad \mathrm{ad}\, x(y) := [x, y] = xy - yx.$$

Lemma 2.4.1. *For each* $x \in M_n(\mathbb{K})$,

$$\mathrm{Ad}(\exp x) = \exp(\mathrm{ad}\, x). \tag{2.8}$$

Proof. We define the linear maps

$$\lambda_x \colon M_n(\mathbb{K}) \to M_n(\mathbb{K}), \quad y \mapsto xy, \quad \rho_x \colon M_n(\mathbb{K}) \to M_n(\mathbb{K}), \quad y \mapsto yx.$$

Then $\lambda_x \rho_x = \rho_x \lambda_x$ and $\mathrm{ad}\, x = \lambda_x - \rho_x$, so that Lemma 2.2.2(ii) leads to

$$\mathrm{Ad}(\exp x)y = e^x y e^{-x} = e^{\lambda_x} e^{-\rho_x} y = e^{\lambda_x - \rho_x} y = e^{\mathrm{ad}\, x} y.$$

This proves (2.8). $\square$

Proposition 2.4.2. *Let $x \in M_n(\mathbb{K})$ and $\lambda_{\exp x}(y) := (\exp x)y$ the left multiplication by* $\exp x$. *Then*

$$\mathtt{d}\exp(x) = \lambda_{\exp x} \circ \frac{\mathbf{1} - e^{-\operatorname{ad} x}}{\operatorname{ad} x} : M_n(\mathbb{K}) \to M_n(\mathbb{K}),$$

where the fraction on the right means $\Phi(\operatorname{ad} x)$ for the entire function

$$\Phi(z) := \frac{1 - e^{-z}}{z} = \sum_{k=1}^{\infty} \frac{(-z)^{k-1}}{k!} = \sum_{k=0}^{\infty} \frac{(-z)^k}{(k+1)!}.$$

The series $\Phi(x)$ converges for each $x \in M_n(\mathbb{K})$.

Note that for $xy = yx$, we have $(\operatorname{ad} x)^k y = 0$ for $k > 0$, and we obtain the known formular $\mathtt{d}\exp(x)y = y$.

Proof. First let $\alpha\colon [0,1] \to M_n(\mathbb{K})$ be a smooth curve. Then

$$\gamma(t,s) := \exp(-s\alpha(t)) \frac{d}{dt} \exp(s\alpha(t))$$

defines a map $[0,1]^2 \to M_n(\mathbb{K})$ which is C^1 in each argument and satisfies $\gamma(t,0) = 0$ for each t. We calculate

$$\begin{aligned}
\frac{\partial \gamma}{\partial s}(t,s) &= \exp(-s\alpha(t)) \cdot (-\alpha(t)) \frac{d}{dt} \exp(s\alpha(t)) \\
&\quad + \exp(-s\alpha(t)) \cdot \frac{d}{dt}\Big(\alpha(t) \exp(s\alpha(t))\Big) \\
&= \exp(-s\alpha(t)) \cdot (-\alpha(t)) \frac{d}{dt} \exp(s\alpha(t)) \\
&\quad + \exp(-s\alpha(t)) \cdot \Big(\alpha'(t) \exp(s\alpha(t)) + \alpha(t) \frac{d}{dt} \exp(s\alpha(t))\Big) \\
&= \operatorname{Ad}(\exp(-s\alpha(t)))\alpha'(t) = e^{-s \operatorname{ad} \alpha(t)} \alpha'(t).
\end{aligned}$$

Integration over $[0,1]$ with respect to s now leads to

$$\gamma(t,1) = \gamma(t,0) + \int_0^1 e^{-s \operatorname{ad} \alpha(t)} \alpha'(t)\, ds = \int_0^1 e^{-s \operatorname{ad} \alpha(t)}\, ds \cdot \alpha'(t).$$

Next we note that, for $x \in M_n(\mathbb{K})$,

$$\begin{aligned}
\int_0^1 e^{-s \operatorname{ad} x}\, ds &= \int_0^1 \sum_{k=0}^{\infty} \frac{(-\operatorname{ad} x)^k}{k!} s^k\, ds = \sum_{k=0}^{\infty} (-\operatorname{ad} x)^k \int_0^1 \frac{s^k}{k!}\, ds \\
&= \sum_{k=0}^{\infty} \frac{(-\operatorname{ad} x)^k}{(k+1)!} = \Phi(\operatorname{ad} x).
\end{aligned}$$

We thus obtain for $\alpha(t) = x + ty$ with $\alpha(0) = x$ and $\alpha'(0) = y$ the relation

$$\exp(-x)\mathtt{d}\exp(x)y = \gamma(0,1) = \int_0^1 e^{-s \operatorname{ad} x} y\, ds = \Phi(\operatorname{ad} x)y. \qquad \square$$

Lemma 2.4.3. *For*

$$\Phi(z) = \frac{1 - e^{-z}}{z} := \sum_{k=0}^{\infty} \frac{(-z)^k}{(k+1)!}, \qquad z \in \mathbb{C}$$

and

$$\Psi(z) = \frac{z \log z}{z - 1} := z \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}(z-1)^k \qquad \text{for } |z-1| < 1$$

we have

$$\Psi(e^z)\Phi(z) = 1 \qquad \text{for} \quad z \in \mathbb{C}, |z| < \log 2.$$

Proof. If $0 < |z| < \log 2$, then $|e^z - 1| < 1$ and we obtain from $\log(e^z) = z$:

$$\Psi(e^z)\Phi(z) = \frac{e^z z}{e^z - 1} \frac{1 - e^{-z}}{z} = 1. \qquad \square$$

In view of the Composition Formula (2.2) (Proposition 2.1.4), the same identity as in Lemma 2.4.3 holds if we insert linear maps $L \in \mathrm{End}(M_n(\mathbb{K}))$ with $\|L\| < \log 2$ into the power series Φ and Ψ:

$$\Psi(\exp L)\Phi(L) = (\Psi \circ \exp)(L)\Phi(L) = ((\Psi \circ \exp) \cdot \Phi)(L) = \mathrm{id}_{M_n(\mathbb{K})}. \tag{2.9}$$

Here we use that $\|L\| < \log 2$ implies that all expressions are defined and in particular that $\|\exp L - \mathbf{1}\| < 1$, as a consequence of the estimate

$$\|\exp L - \mathbf{1}\| \leq e^{\|L\|} - 1 < 1. \tag{2.10}$$

The derivation of the BCDH formula follows a similar scheme as the proof of Proposition 2.4.2. Here we consider $x, y \in V_o := B(0, \log \sqrt{2})$. For $\|x\|, \|y\| < r$ the estimate (2.10) leads to

$$\begin{aligned}
&\|\exp x \exp y - \mathbf{1}\| = \|(\exp x - \mathbf{1})(\exp y - \mathbf{1}) + (\exp y - \mathbf{1}) + (\exp x - \mathbf{1})\| \\
&\quad \leq \|\exp x - \mathbf{1}\| \cdot \|\exp y - \mathbf{1}\| + \|\exp y - \mathbf{1}\| + \|\exp x - \mathbf{1}\| \\
&\quad < (e^r - 1)^2 + 2(e^r - 1) = e^{2r} - 1.
\end{aligned}$$

For $r < \log \sqrt{2} = \frac{1}{2} \log 2$ and $|t| \leq 1$ we obtain in particular

$$\|\exp x \exp ty - \mathbf{1}\| < e^{\log 2} - 1 = 1.$$

Therefore $\exp x \exp ty$ lies for $|t| \leq 1$ in the domain of the logarithm function (Lemma 2.3.1). We therefore define for $t \in [-1, 1]$:

$$F(t) = \log(\exp x \exp ty).$$

To estimate the norm of $F(t)$, we note that for $g := \exp x \exp ty$, $|t| \leq 1$, and $\|x\|, \|y\| < r$ we have

$$\begin{aligned}
\|\log g\| &\leq \sum_{k=1}^{\infty} \frac{\|g - 1\|^k}{k} = -\log(1 - \|g - \mathbf{1}\|) \\
&< -\log(1 - (e^{2r} - 1)) = -\log(2 - e^{2r}).
\end{aligned}$$

For $r := \frac{1}{2}\log(2 - \frac{\sqrt{2}}{2}) < \frac{\log 2}{2} = \log\sqrt{2}$ and $\|x\|, \|y\| < r$ this leads to

$$\|F(t)\| < -\log(2 - e^{2r}) = \log(\tfrac{2}{\sqrt{2}}) = \log(\sqrt{2}) = \frac{1}{2}\log 2. \tag{2.11}$$

Next we calculate $F'(t)$ with the goal to obtain the BCDH formula as $F(1) = F(0) + \int_0^1 F'(t)\,dt$. For the derivative of the curve $t \mapsto \exp F(t)$ we get

$$\begin{aligned}(\mathtt{d}\exp)\big(F(t)\big)F'(t) &= \frac{d}{dt}\exp(F(t)) = \frac{d}{dt}\exp x \exp ty \\ &= (\exp x \exp ty)y = (\exp F(t))y.\end{aligned}$$

Using Proposition 2.4.2, we obtain

$$\begin{aligned} y &= \big(\exp F(t)\big)^{-1}(\mathtt{d}\exp)\big(F(t)\big)F'(t) \\ &= \frac{\mathbf{1} - e^{-\operatorname{ad} F(t)}}{\operatorname{ad} F(t)}F'(t) = \Phi(\operatorname{ad} F(t))F'(t). \end{aligned} \tag{2.12}$$

We claim that $\|\operatorname{ad}(F(t))\| < \log 2$. From $\|ab - ba\| \leq 2\|a\|\,\|b\|$ we derive

$$\|\operatorname{ad} a\| \leq 2\|a\| \quad \text{for} \quad a \in M_n(\mathbb{K}).$$

Therefore, by (2.11),

$$\|\operatorname{ad} F(t)\| \leq 2\|F(t)\| < 2\log(\sqrt{2}) = \log 2,$$

so that (2.12) and (2.9) lead to

$$F'(t) = \Psi\big(\exp(\operatorname{ad} F(t))\big)y. \tag{2.13}$$

Proposition 2.4.4. *For* $\|x\|, \|y\| < \frac{1}{2}\log(2 - \frac{\sqrt{2}}{2})$ *we have*

$$\log(\exp x \exp y) = x + \int_0^1 \Psi\big(\exp(\operatorname{ad} x)\exp(t \operatorname{ad} y)\big)y\,dt,$$

with Ψ *as in* Lemma 2.4.3.

Proof. With (2.13), Lemma 2.4.1 and the preceding remarks we get

$$\begin{aligned} F'(t) &= \Psi\big(\exp(\operatorname{ad} F(t))\big)y \\ &= \Psi\big(\operatorname{Ad}(\exp F(t))\big)y = \Psi\big(\operatorname{Ad}(\exp x \exp ty)\big)y \\ &= \Psi\big(\operatorname{Ad}(\exp x)\operatorname{Ad}(\exp ty)\big)y = \Psi\big(\exp(\operatorname{ad} x)\exp(\operatorname{ad} ty)\big)y. \end{aligned}$$

Moreover, we have $F(0) = \log(\exp x) = x$. By integration we therefore obtain

$$\log(\exp x \exp y) = x + \int_0^1 \Psi\big(\exp(\operatorname{ad} x)\exp(t \operatorname{ad} y)\big)y\,dt. \qquad \square$$

Proposition 2.4.5. *For $x, y \in M_n(\mathbb{K})$ and $\|x\|, \|y\| < \frac{1}{2}\log(2 - \frac{\sqrt{2}}{2})$,*

$$\begin{aligned} x * y &:= \log(\exp x \exp y) \\ &= x + \\ &\sum_{\substack{k,m\geq 0\\ p_i+q_i>0}} \frac{(-1)^k}{(k+1)(q_1+\ldots+q_k+1)} \frac{(\operatorname{ad} x)^{p_1}(\operatorname{ad} y)^{q_1}\ldots(\operatorname{ad} x)^{p_k}(\operatorname{ad} y)^{q_k}(\operatorname{ad} x)^m}{p_1!q_1!\ldots p_k!q_k!m!} y. \end{aligned}$$

Proof. We only have to rewrite the expression in Proposition 2.4.4:

$$\begin{aligned} &\int_0^1 \Psi\big(\exp(\operatorname{ad} x)\exp(\operatorname{ad} ty)\big) y\, dt \\ &= \int_0^1 \sum_{k=0}^{\infty} \frac{(-1)^k\big(\exp(\operatorname{ad} x)\exp(\operatorname{ad} ty) - \operatorname{id}\big)^k}{(k+1)} \big(\exp(\operatorname{ad} x)\exp(\operatorname{ad} ty)\big) y\, dt \\ &= \int_0^1 \sum_{\substack{k\geq 0\\ p_i+q_i>0}} \frac{(-1)^k}{(k+1)} \frac{(\operatorname{ad} x)^{p_1}(\operatorname{ad} ty)^{q_1}\ldots(\operatorname{ad} x)^{p_k}(\operatorname{ad} ty)^{q_k}}{p_1!q_1!\ldots p_k!q_k!} \exp(\operatorname{ad} x) y\, dt \\ &= \sum_{\substack{k,m\geq 0\\ p_i+q_i>0}} \frac{(-1)^k}{(k+1)} \frac{(\operatorname{ad} x)^{p_1}(\operatorname{ad} y)^{q_1}\ldots(\operatorname{ad} x)^{p_k}(\operatorname{ad} y)^{q_k}(\operatorname{ad} x)^m}{p_1!q_1!\ldots p_k!q_k!m!} y \int_0^1 t^{q_1+\ldots+q_k}\, dt \\ &= \sum_{\substack{k,m\geq 0\\ p_i+q_i>0}} \frac{(-1)^k(\operatorname{ad} x)^{p_1}(\operatorname{ad} y)^{q_1}\ldots(\operatorname{ad} x)^{p_k}(\operatorname{ad} y)^{q_k}(\operatorname{ad} x)^m y}{(k+1)(q_1+\ldots+q_k+1)p_1!q_1!\ldots p_k!q_k!m!}. \end{aligned}$$

□

The power series in Proposition 2.4.5 is called the *Hausdorff Series*. We observe that it does not depend on n. For practical purposes it often suffices to know the first terms of the Hausdorff Series:

Corollary 2.4.6. *For $x, y \in M_n(\mathbb{K})$ and $\|x\|, \|y\| < \frac{1}{2}\log(2 - \frac{\sqrt{2}}{2})$,*

$$x * y = x + y + \frac{1}{2}[x, y] + \frac{1}{12}[x, [x, y]] + \frac{1}{12}[y, [y, x]] + \ldots$$

Proof. One has to collect the summands in Proposition 2.4.5 corresponding to $p_1 + q_1 + \ldots + p_k + q_k + m \leq 2$. □

Product and Commutator Formula

We have seen in Proposition 2.2.1 that the exponential image of a sum $x + y$ can be computed easily if x and y commute. In this case we also have for the commutator $[x, y] := xy - yx = 0$ the formula $\exp[x, y] = \mathbf{1}$. The following proposition gives a formula for $\exp(x + y)$ and $\exp[x, y]$ in the general case.

If g, h are elements of a group G, then $(g, h) := ghg^{-1}h^{-1}$ is called their *commutator*. On the other hand, we call for two matrices $A, B \in M_n(\mathbb{K})$ the expression

$$[A, B] := AB - BA$$

their *commutator bracket*.

Proposition 2.4.7. *For $x, y \in M_n(\mathbb{K})$ the following assertions hold:*

(i) (Trotter Product Formula) $\lim_{k\to\infty} \big(e^{\frac{1}{k}x}e^{\frac{1}{k}y}\big)^k = e^{x+y}$.

(ii) (Commutator Formula) $\lim_{k\to\infty} \big(e^{\frac{1}{k}x}e^{\frac{1}{k}y}e^{-\frac{1}{k}x}e^{-\frac{1}{k}y}\big)^{k^2} = e^{xy-yx}$.

Proof. (i) From Corollary 2.4.6 we obtain that $\lim_{k\to\infty} k \cdot \big(\frac{x}{k} * \frac{y}{k}\big) = x + y$. Applying the exponential function, we obtain (i).

(ii) We consider the function

$$\gamma(t) := tx * ty * (-tx) * (-ty),$$

which is defined and smooth on some interval $[-\varepsilon, \varepsilon] \subseteq \mathbb{R}$, $\varepsilon > 0$. In view of

$$\exp(x * y * (-x)) = \exp x \exp y \exp(-x) = \exp\big(\mathrm{Ad}(\exp x)y\big) = \exp(e^{\mathrm{ad}\, x}y)$$

for x, y small enough (Lemma 2.4.1), we have

$$x * y * (-x) = e^{\mathrm{ad}\, x}y, \tag{2.14}$$

and therefore Taylor expansion with respect to t yields

$$\begin{aligned}\gamma(t) &= tx * ty * (-tx) * (-ty) = e^{t\,\mathrm{ad}\, x}ty * (-ty)\\ &= (ty + t^2[x,y] + \frac{t^3}{2}[x,[x,y]] + \ldots) * (-ty)\\ &= ty + t^2[x,y] - ty + [ty, -ty] + t^2r(t) = t^2[x,y] + t^2r(t),\end{aligned}$$

where $\lim_{t\to 0} r(t) = 0$. We now have

$$\gamma(0) = \gamma'(0) = 0 \quad \text{and} \quad \frac{\gamma''(0)}{2} = [x,y].$$

This leads to

$$\lim_{k\to\infty} k^2 \cdot \big(\frac{1}{k}x\big) * \big(\frac{1}{k}y\big) * \big(-\frac{1}{k}x\big) * \big(-\frac{1}{k}y\big) = \frac{\gamma''(0)}{2} = [x,y]. \tag{2.15}$$

Applying exp leads to the Commutator Formula. □

Here is an alternative proof of product and commutator formula, not using the complicated BCDH formula. It is based on the following elementary lemma.

Lemma 2.4.8. *Let $\varepsilon > 0$ and $\gamma\colon [0,\varepsilon] \to \mathrm{GL}_n(\mathbb{K})$ be a continuous curve with $\gamma(0) = \mathbf{1}$. If $\gamma'(0)$ exists, then*

$$\lim_{n\to\infty} \gamma\Big(\frac{1}{n}\Big)^n = e^{\gamma'(0)}.$$

If, in addition, $\gamma'(0) = 0$, γ is C^1, and $\gamma''(0)$ exists, then

$$\lim_{n\to\infty} \gamma\Big(\frac{1}{n}\Big)^{n^2} = e^{\frac{\gamma''(0)}{2}}.$$

Proof. Since exp maps a neighborhood of 0 diffeomorphically onto a neighborhood of $\mathbf{1}$ and $\mathtt{d}\exp(0) = \mathrm{id}$, we can, after possibly shrinking ε, write $\gamma(t) = e^{\beta(t)}$ with $\beta(0) = 0$ and $\beta'(0) = \gamma'(0)$. Then

$$\lim_{n\to\infty} \gamma\Big(\frac{1}{n}\Big)^n = \lim_{n\to\infty} e^{n\beta(\frac{1}{n})} \to e^{\gamma'(0)}$$

follows from $\beta(\frac{1}{n})n \to \beta'(0)$ and the continuity of exp.

If, in addition, $\gamma'(0) = 0$, γ is C^1 and $y := \gamma''(0)$ exists, then we put $\delta(t) := \gamma(\sqrt{t})$. Then the Fundamental Theorem of Calculus implies that

$$\delta(t) = \int_0^{\sqrt{t}} \gamma'(\tau)\, d\tau = \int_0^t \frac{1}{2\sqrt{s}}\gamma'(\sqrt{s})\, ds,$$

and since the continuous integrand converges to $y/2$ for $s \to 0$, we obtain for its mean value $\lim_{t\to 0} \delta(t)/t = y/2$. This shows that $\delta'(0) = y/2$ exists. From above we now obtain

$$\lim_{n\to\infty} \gamma\Big(\frac{1}{n}\Big)^{n^2} = \lim_{n\to\infty} \delta\Big(\frac{1}{n^2}\Big)^{n^2} = e^{y/2}. \qquad \square$$

Example 2.4.9. Applying the preceding lemma to the smooth curve $\gamma(t) := \mathbf{1} + tx$, we obtain the well-known formula

$$\lim_{n\to\infty} \Big(\mathbf{1} + \frac{x}{n}\Big)^n = e^x$$

for the exponential function.

Proof. (of Proposition 2.4.7) To obtain the product formula, we consider the smooth curve $\gamma(t) := e^{tx}e^{ty}$ with $\gamma(0) = \mathbf{1}$ and $\gamma'(0) = x + y$ (Product Rule). The assertion now follows from Lemma 2.4.8.

For the commutator formula, we consider the smooth curve $\gamma(t) := e^{tx}e^{ty}e^{-tx}e^{-ty}$. Then $e^{tx} = \mathbf{1} + tx + \frac{t^2}{2}x^2 + O(t^3)$ leads to

$$\begin{aligned}
\gamma(t) &= \big(\mathbf{1} + tx + \frac{t^2}{2}x^2 + O(t^3)\big)\big(\mathbf{1} + ty + \frac{t^2}{2}y^2 + O(t^3)\big) \\
&\quad \cdot \big(\mathbf{1} - tx + \frac{t^2}{2}x^2 + O(t^3)\big)\big(\mathbf{1} - ty + \frac{t^2}{2}y^2 + O(t^3)\big) \\
&= \mathbf{1} + t(x + y - x - y) + t^2(x^2 + y^2 + xy - x^2 - xy - yx - y^2 + xy) + O(t^3) \\
&= \mathbf{1} + t^2(xy - yx) + O(t^3).
\end{aligned}$$

This implies that $\gamma'(0) = 0$ and $\gamma''(0) = 2(xy - yx)$. Therefore the Commutator Formula follows from the second part of Lemma 2.4.8. $\square$

Chapter 3

Linear Lie Groups

We call a closed subgroup $G \subseteq \mathrm{GL}_n(\mathbb{K})$ a *linear Lie group*. In this section we shall use the exponential function to assign to each linear Lie group G a vector space

$$\mathbf{L}(G) := \{x \in M_n(\mathbb{K})\colon \exp(\mathbb{R}x) \subseteq G\},$$

called the *Lie algebra of* G. This subspace carries an additional algebraic structure because, for $x, y \in \mathbf{L}(G)$, the commutator $[x, y] = xy - yx$ is contained in $\mathbf{L}(G)$, so that $[\cdot,\cdot]$ defines a skew-symmetric bilinear operation on $\mathbf{L}(G)$. As a first step, we shall see how to calculate $\mathbf{L}(G)$ for concrete groups and to use it to generalize the polar decomposition to a large class of linear Lie groups.

3.1 The Lie Algebra of a Linear Lie Group

We start with the introduction of the concept of a Lie algebra.

Definition 3.1.1. (a) Let k be a field and L a k-vector space. A bilinear map $[\cdot,\cdot]\colon L \times L \to L$ is called a *Lie bracket* if

(L1) $[x, x] = 0$ for $x \in L$ and

(L2) $\big[x, [y, z]\big] = \big[[x, y], z\big] + \big[y, [x, z]\big]$ for $x, y, z \in L$ (Jacobi identity).[1]

A *Lie algebra* [2] (over k) is a k-vector space L, endowed with a Lie bracket. A subspace $E \subseteq L$ of a Lie algebra is called a *subalgebra* if $[E, E] \subseteq E$. A *homomorphism* $\varphi\colon L_1 \to L_2$ of Lie algebras is a linear map with $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ for $x, y \in L_1$. A Lie algebra is said to be *abelian* if $[x, y] = 0$ holds for all $x, y \in L$.

The following lemma shows that each associative algebra also carries a natural Lie algebra structure.

[1] Carl Gustav Jacob Jacobi (1804–1851), mathematician in Berlin and Königsberg (Kaliningrad). He found his famous identity about 1830 in the context of Poisson brackets, which are related to Hamiltonian Mechanics and Symplectic Geometry.

[2] The notion of a Lie algebra was coined in the 1920s by Hermann Weyl.

Lemma 3.1.2. *Each associative algebra A is a Lie algebra A_L with respect to the commutator bracket*

$$[a,b] := ab - ba.$$

Proof. (L1) is obvious. For (L2) we calculate

$$[a, bc] = abc - bca = (ab - ba)c + b(ac - ca) = [a,b]c + b[a,c],$$

and this implies

$$[a,[b,c]] = [a,b]c + b[a,c] - [a,c]b - c[a,b] = [[a,b],c] + [b,[a,c]].$$

□

Definition 3.1.3. A closed subgroup $G \subseteq \mathrm{GL}_n(\mathbb{K})$ is called a *linear Lie group*. For each subgroup $G \subseteq \mathrm{GL}_n(\mathbb{K})$ we define the set

$$\mathbf{L}(G) := \{x \in M_n(\mathbb{K})\colon \exp(\mathbb{R}x) \subseteq G\}$$

and observe that $\mathbb{R}\,\mathbf{L}(G) \subseteq \mathbf{L}(G)$ follows immediately from the definition.

We could also define this notion in more abstract terms by considering a finite-dimensional $\mathbb{K}$-vector space V and call a closed subgroup $G \subseteq \mathrm{GL}(V)$ a linear Lie group. Then

$$\mathbf{L}(G) = \{x \in \mathrm{End}(V)\colon \exp(\mathbb{R}x) \subseteq G\}.$$

In the following we shall use both pictures.

From Lemma 3.1.2 we know that the associative algebra $M_n(\mathbb{K})$ is a Lie algebra with respect to the matrix commutator $[x,y] := xy - yx$. We denote this Lie algebra by $\mathfrak{gl}_n(\mathbb{K}) := M_n(\mathbb{K})_L$. We likewise write $\mathfrak{gl}(V) := \mathrm{End}(V)_L$ for a vector space V.

The next proposition assigns a Lie algebra to each linear Lie group.

Proposition 3.1.4. *If $G \subseteq \mathrm{GL}(V)$ is a closed subgroup, then $\mathbf{L}(G)$ is a real Lie subalgebra of $\mathfrak{gl}(V)$ and we obtain a map*

$$\exp_G\colon \mathbf{L}(G) \to G, \quad x \mapsto e^x.$$

We call $\mathbf{L}(G)$ the *Lie algebra of G* and

$$\exp_G\colon \mathbf{L}(G) \to G, \quad x \mapsto e^x$$

the *exponential function of G*.

In particular,

$$\mathbf{L}(\mathrm{GL}(V)) = \mathfrak{gl}(V) \quad \text{and} \quad \mathbf{L}(\mathrm{GL}_n(\mathbb{K})) = \mathfrak{gl}_n(\mathbb{K}).$$

Proof. Let $x, y \in \mathbf{L}(G)$. For $k \in \mathbb{N}$ and $t \in \mathbb{R}$ we then have $\exp \frac{t}{k}x, \exp \frac{t}{k}y \in G$ and with the Trotter Formula (Proposition 2.4.7), we get for all $t \in \mathbb{R}$:

$$\exp(t(x+y)) = \lim_{k\to\infty} \left(\exp \frac{tx}{k} \exp \frac{ty}{k}\right)^k \in G$$

because G is closed. Therefore $x + y \in \mathbf{L}(G)$.

Similarly we use the Commutator Formula to get

$$\exp t[x,y] = \lim_{k\to\infty} \left(\exp\frac{tx}{k}\exp\frac{y}{k}\exp-\frac{tx}{k}\exp-\frac{y}{k}\right)^{k^2} \in G,$$

hence $[x,y] \in \mathbf{L}(G)$. □

Lemma 3.1.5. *Let $G \subseteq \mathrm{GL}_n(\mathbb{K})$ be a subgroup. If $\mathrm{Hom}(\mathbb{R}, G)$ denotes the set of all continuous group homomorphisms $(\mathbb{R}, +) \to G$, then the map*

$$\Gamma\colon\ \mathbf{L}(G) \to \mathrm{Hom}(\mathbb{R}, G), \quad x \mapsto \gamma_x, \quad \gamma_x(t) = \exp(tx)$$

is a bijection.

Proof. For each $x \in \mathbf{L}(G)$, the map γ_x is a continuous group homomorphism (Theorem 2.2.6), and since $x = \gamma_x'(0)$, the map Γ is injective. To see that it is surjective, let $\gamma\colon \mathbb{R} \to G$ be a continuous group homomorphism and $\iota\colon G \to \mathrm{GL}_n(\mathbb{K})$ the natural embedding. Then $\iota \circ \gamma\colon \mathbb{R} \to \mathrm{GL}_n(\mathbb{K})$ is a continuous group homomorphism, so that there exists an $x \in \mathfrak{gl}_n(\mathbb{K})$ with $\gamma(t) = \iota(\gamma(t)) = e^{tx}$ for all $t \in \mathbb{R}$ (Theorem 2.2.6). This implies that $x \in \mathbf{L}(G)$, and therefore that $\gamma_x = \gamma$. □

Remark 3.1.6. The preceding lemma implies in particular that for a linear Lie group the set $\mathbf{L}(G)$ can also be defined in terms of the topological group structure on G as $\mathcal{L}(G) := \mathrm{Hom}(\mathbb{R}, G)$, the set of continuous one-parameter groups. From the Trotter Formula and the Commutator Formula we also know that the Lie algebra structure on $\mathcal{L}(G)$ can be defined intrinsically by

$$(\lambda\gamma)(t) := \gamma(\lambda t),$$

$$(\gamma_1 + \gamma_2)(t) := \lim_{n\to\infty}\left(\gamma_1(\tfrac{t}{n})\gamma_2(\tfrac{t}{n})\right)^{\frac{1}{n}}$$

and

$$[\gamma_1, \gamma_2](t) := \lim_{n\to\infty}\left(\gamma_1(\tfrac{t}{n})\gamma_2(\tfrac{1}{n})\gamma_1(-\tfrac{t}{n})\gamma_2(-\tfrac{1}{n})\right)^{\frac{1}{n^2}}.$$

This shows that the Lie algebra $\mathbf{L}(G)$ does not depend on the special realization of G as a group of matrices.

Example 3.1.7. We consider the homomorphism

$$\Phi\colon \mathbb{K}^n \to \mathrm{GL}_{n+1}(\mathbb{K}), \quad x \mapsto \begin{pmatrix} \mathbf{1} & x \\ 0 & 1 \end{pmatrix}$$

and observe that Φ is an isomorphism of the topological group $(\mathbb{K}^n, +)$ onto a linear Lie group.

The continuous one-parameter groups $\gamma\colon \mathbb{R} \to \mathbb{K}^n$ are easily determined because $\gamma(nt) = n\gamma(t)$ for all $n \in \mathbb{Z}$, $t \in \mathbb{R}$, implies further $\gamma(q) = q\gamma(1)$ for all $q \in \mathbb{Q}$ and hence, by continuity, $\gamma(t) = t\gamma(1)$ for all $t \in \mathbb{R}$. Since $(\mathbb{K}^n, +)$ is abelian, the Lie bracket on the Lie algebra $\mathbf{L}(\mathbb{K}^n, +)$ vanishes, and we obtain

$$\mathbf{L}(\mathbb{K}^n, +) = (\mathbb{K}^n, 0) \cong \mathbf{L}(\Phi(\mathbb{K}^n)) = \left\{ \begin{pmatrix} \mathbf{0} & x \\ 0 & 0 \end{pmatrix} : x \in \mathbb{K}^n \right\}$$

(Exercise).

3.1.1 Functorial Properties of the Lie Algebra

So far we have assigned to each linear Lie group G its Lie algebra $\mathbf{L}(G)$. We shall also see that this assignment can be extended to continuous homomorphisms between linear Lie groups in the sense that we assign to each such homomorphism $\varphi\colon G_1 \to G_2$ a homomorphism $\mathbf{L}(\varphi)\colon \mathbf{L}(G_1) \to \mathbf{L}(G_2)$ of Lie algebras, and this assignment satisfies

$$\mathbf{L}(\mathrm{id}_G) = \mathrm{id}_{\mathbf{L}(G)} \quad \text{and} \quad \mathbf{L}(\varphi_2 \circ \varphi_1) = \mathbf{L}(\varphi_2) \circ \mathbf{L}(\varphi_1)$$

for a composition $\varphi_2 \circ \varphi_1$ of two continuous homomorphisms $\varphi_1\colon G_1 \to G_2$ and $\varphi_2\colon G_2 \to G_3$. In the language of category theory, this means that $\mathbf{L}$ defines a functor from the category of linear Lie groups (where the morphisms are the continuous group homomorphisms) to the category of real Lie algebras.

Proposition 3.1.8. *Let $\varphi\colon G_1 \to G_2$ be a continuous group homomorphism of linear Lie groups. Then the derivative*

$$\mathbf{L}(\varphi)(x) := \frac{d}{dt}\Big|_{t=0} \varphi(\exp_{G_1}(tx))$$

exists for each $x \in \mathbf{L}(G_1)$ and defines a homomorphism of Lie algebras $\mathbf{L}(\varphi)\colon \mathbf{L}(G_1) \to \mathbf{L}(G_2)$ with

$$\exp_{G_2} \circ \mathbf{L}(\varphi) = \varphi \circ \exp_{G_1}, \tag{3.1}$$

i.e., the following diagram commutes

$$\begin{array}{ccc} G_1 & \xrightarrow{\varphi} & G_2 \\ \uparrow{\scriptstyle \exp_{G_1}} & & \uparrow{\scriptstyle \exp_{G_2}} \\ \mathbf{L}(G_1) & \xrightarrow{\mathbf{L}(\varphi)} & \mathbf{L}(G_2). \end{array}$$

Furthermore, $\mathbf{L}(\varphi)$ is the uniquely determined linear map satisfying (3.1).

Proof. For $x \in \mathbf{L}(G_1)$ we consider the homomorphism $\gamma_x \in \mathrm{Hom}(\mathbb{R}, G_1)$ given by $\gamma_x(t) = e^{tx}$. According to Lemma 3.1.5, we have

$$\varphi \circ \gamma_x(t) = \exp_{G_2}(ty)$$

for some $y \in \mathbf{L}(G_2)$, because $\varphi \circ \gamma_x\colon \mathbb{R} \to G_2$ is a continuous group homomorphism. Then clearly $y = (\varphi \circ \gamma_x)'(0) = \mathbf{L}(\varphi)x$. For $t = 1$ we obtain in particular

$$\exp_{G_2}(\mathbf{L}(\varphi)x) = \varphi(\exp_{G_1}(x),$$

which is (3.1).

For every linear map $\psi\colon \mathbf{L}(G_1) \to \mathbf{L}(G_2)$ with

$$\exp_{G_2} \circ\, \psi = \varphi \circ \exp_{G_1}$$

satisfies

$$\varphi \circ \exp_{G_1}(tx) = \exp_{G_2}(\psi(tx)) = \exp_{G_2}(t\psi(x)),$$

and therefore

$$\mathbf{L}(\varphi)x = \frac{d}{dt}\Big|_{t=0} \exp_{G_2}(t\psi(x)) = \psi(x).$$

Next we show that $\mathbf{L}(\varphi)$ is a homomorphism of Lie algebras. From the definition of $\mathbf{L}(\varphi)$ we immediately get for $x \in \mathbf{L}(G_1)$:

$$\exp_{G_2}(s\,\mathbf{L}(\varphi)(tx)) = \varphi(\exp_{G_1}(stx)) = \exp_{G_2}(ts\,\mathbf{L}(\varphi)(x)), \quad s, t \in \mathbb{R},$$

which leads to $\mathbf{L}(\varphi)(tx) = t\,\mathbf{L}(\varphi)(x)$.

Since φ is continuous, the Trotter Formula implies that

$$\begin{aligned}
&\exp_{G_2}(\mathbf{L}(\varphi)(x+y)) = \varphi\big(\exp_{G_1}(x+y)\big)\\
&= \lim_{k\to\infty} \varphi\Big(\exp_{G_1}\frac{1}{k}x \exp_{G_1}\frac{1}{k}y\Big)^k = \lim_{k\to\infty}\Big(\varphi\Big(\exp_{G_1}\frac{1}{k}x\Big)\varphi\Big(\exp_{G_1}\frac{1}{k}y\Big)\Big)^k\\
&= \lim_{k\to\infty}\Big(\exp_{G_2}\frac{1}{k}\mathbf{L}(\varphi)(x)\exp_{G_2}\frac{1}{k}\mathbf{L}(\varphi)(y)\Big)^k\\
&= \exp_{G_2}\big(\mathbf{L}(\varphi)(x) + \mathbf{L}(\varphi)(y)\big)
\end{aligned}$$

for all $x, y \in \mathbf{L}(G_1)$. Therefore $\mathbf{L}(\varphi)(x+y) = \mathbf{L}(\varphi)(x) + \mathbf{L}(\varphi)(y)$ because the same formula holds with tx and ty instead of x and y. Hence $\mathbf{L}(\varphi)$ is additive and therefore linear.

We likewise obtain with the Commutator Formula

$$\varphi(\exp[x,y]) = \exp[\mathbf{L}(\varphi)(x), \mathbf{L}(\varphi)(y)]$$

and thus $\mathbf{L}(\varphi)([x,y]) = [\mathbf{L}(\varphi)(x), \mathbf{L}(\varphi)(y)]$. □

Corollary 3.1.9. *If $\varphi_1 : G_1 \to G_2$ and $\varphi_2\colon G_2 \to G_3$ are continuous homomorphisms of linear Lie groups, then*

$$\mathbf{L}(\varphi_2 \circ \varphi_1) = \mathbf{L}(\varphi_2) \circ \mathbf{L}(\varphi_1).$$

Moreover, $\mathbf{L}(\mathrm{id}_G) = \mathrm{id}_{\mathbf{L}(G)}$.

Proof. We have the relations

$$\varphi_1 \circ \exp_{G_1} = \exp_{G_2} \circ \mathbf{L}(\varphi_1) \quad \text{and} \quad \varphi_2 \circ \exp_{G_2} = \exp_{G_3} \circ \mathbf{L}(\varphi_2),$$

which immediately lead to

$$(\varphi_2 \circ \varphi_1) \circ \exp_{G_1} = \varphi_2 \circ \exp_{G_2} \circ \mathbf{L}(\varphi_1) = \exp_{G_3} \circ (\mathbf{L}(\varphi_2) \circ \mathbf{L}(\varphi_1)),$$

and the uniqueness assertion of Proposition 3.1.8 implies that

$$\mathbf{L}(\varphi_2 \circ \varphi_1) = \mathbf{L}(\varphi_2) \circ \mathbf{L}(\varphi_1).$$

Clearly $\mathrm{id}_{\mathbf{L}(G)}$ is a linear map satisfying $\exp_G \circ \mathrm{id}_{\mathbf{L}(G)} = \mathrm{id}_G \circ \exp_G$, so that the uniqueness assertion of Proposition 3.1.8 implies $\mathbf{L}(\mathrm{id}_G) = \mathrm{id}_{\mathbf{L}(G)}$. □

Corollary 3.1.10. *If $\varphi\colon G_1 \to G_2$ is an isomorphism of linear Lie groups, then $\mathbf{L}(\varphi)$ is an isomorphism of Lie algebras.*

Proof. Since φ is an isomorphism of linear Lie groups, it is bijective and $\psi := \varphi^{-1}$ also is a continuous homomorphism. We then obtain with Corollary 3.1.9 the relations $\mathrm{id}_{\mathbf{L}(G_2)} = \mathbf{L}(\mathrm{id}_{G_2}) = \mathbf{L}(\varphi \circ \psi) = \mathbf{L}(\varphi) \circ \mathbf{L}(\psi)$ and likewise

$$\mathrm{id}_{\mathbf{L}(G_1)} = \mathbf{L}(\psi) \circ \mathbf{L}(\varphi).$$

Hence $\mathbf{L}(\varphi)$ is an isomorphism with $\mathbf{L}(\varphi)^{-1} = \mathbf{L}(\psi)$. □

Definition 3.1.11. If V is a vector space and G a group, then a homomorphism $\varphi\colon G \to \mathrm{GL}(V)$ is called a *representation of G on V*. If $\mathfrak{g}$ is a Lie algebra, then a homomorphism of Lie algebras $\varphi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is called a *representation of $\mathfrak{g}$ on V*.

As a consequence of Proposition 3.1.8, we obtain

Corollary 3.1.12. *If $\varphi\colon G \to \mathrm{GL}(V)$ is a continuous representation of the linear Lie group G, then $\mathbf{L}(\varphi)\colon \mathbf{L}(G) \to \mathfrak{gl}(V)$ is a representation of the Lie algebra $\mathbf{L}(G)$.*

The representation $\mathbf{L}(\varphi)$ obtained in Corollary 3.1.12 from the group representation φ is called the *derived representation*. This is motivated by the fact that for each $x \in \mathbf{L}(G)$ we have

$$\mathbf{L}(\varphi)x = \frac{d}{dt}\Big|_{t=0} e^{t\,\mathbf{L}(\varphi)x} = \frac{d}{dt}\Big|_{t=0} \varphi(\exp tx).$$

3.1.2 The Adjoint Representation

Let $G \subseteq \mathrm{GL}(V)$ be a linear Lie group and $\mathbf{L}(G) \subseteq \mathfrak{gl}(V)$ the corresponding Lie algebra. For $g \in G$ we define the conjugation automorphism $c_g \in \mathrm{Aut}(G)$ by $c_g(x) := gxg^{-1}$. Then

$$\begin{aligned}\mathbf{L}(c_g)(x) &= \frac{d}{dt}\Big|_{t=0} c_g(\exp tx) = \frac{d}{dt}\Big|_{t=0} g(\exp tx)g^{-1} \\ &= \frac{d}{dt}\Big|_{t=0} \exp(tgxg^{-1}) = gxg^{-1}\end{aligned}$$

(Proposition 2.2.1), and therefore $\mathbf{L}(c_g) = c_g|_{\mathbf{L}(G)}$. We define the *adjoint representation of G on $\mathbf{L}(G)$* by

$$\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathbf{L}(G)), \quad \mathrm{Ad}(g)(x) := \mathbf{L}(c_g)x = gxg^{-1}.$$

(That this is a representation follows immediately from the explicit formula).

For each $x \in \mathbf{L}(G)$, the map $G \to \mathbf{L}(G), g \mapsto \mathrm{Ad}(g)(x) = gxg^{-1}$ is continuous and each $\mathrm{Ad}(g)$ is an automorphism of the Lie algebra $\mathbf{L}(G)$. Therefore Ad is a continuous homomorphism from the linear Lie group G to the linear Lie group $\mathrm{Aut}(\mathbf{L}(G)) \subseteq \mathrm{GL}(\mathbf{L}(G))$. The derived representation

$$\mathbf{L}(\mathrm{Ad})\colon\ \mathbf{L}(G) \to \mathfrak{gl}(\mathbf{L}(G))$$

is a representation of $\mathbf{L}(G)$ on $\mathbf{L}(G)$. The following lemma gives a formula for this representation. First we define for $x \in \mathbf{L}(G)$:

$$\mathrm{ad}(x)\colon\ \mathbf{L}(G) \to \mathbf{L}(G), \quad \mathrm{ad}\, x(y) := [x, y] = xy - yx.$$

Lemma 3.1.13. $\mathbf{L}(\mathrm{Ad}) = \mathrm{ad}$.

Proof. In view of Proposition 3.1.8 this is an immediate consequence of the relation $\mathrm{Ad}(\exp x) = e^{\mathrm{ad}\, x}$ (Lemma 2.4.1). □

Exercises for Section 3.1

Exercise 3.1.1. (a) If $(G_j)_{j\in J}$ is a family of linear Lie groups in $\mathrm{GL}_n(\mathbb{K})$, then their intersection $G := \bigcap_{j\in J} G_j$ also is a linear Lie group.

(b) If $(G_j)_{j\in J}$ is a family of subgroups of $\mathrm{GL}_n(\mathbb{K})$, then

$$\mathbf{L}(\bigcap_{j\in J} G_j) = \bigcap_{j\in J} \mathbf{L}(G_j).$$

Exercise 3.1.2. Let $G := \mathrm{GL}_n(\mathbb{K})$ and $V := P_k(\mathbb{K}^n)$ the space of homogeneous polynomials of degree k in $x_1, \ldots, x_n$, considered as functions $\mathbb{K}^n \to \mathbb{K}$. Show that:

(1) $\dim V = \binom{k+n-1}{n-1}$.

(2) We obtain a continuous representation $\rho\colon G \to \mathrm{GL}(V)$ of G on V by $(\rho(g)f)(x) := f(g^{-1}x)$.

(3) The elementary matrix E_{ij} with $E_{ij}e_k = \delta_{jk}e_i$ satisfies

$$\mathbf{L}(\rho)(E_{ij}) = -x_j \frac{\partial}{\partial x_i}.$$

Exercise 3.1.3. If $X \in \mathrm{End}(V)$ is nilpotent, then $\mathrm{ad}\, X \in \mathrm{End}(\mathrm{End}(V))$ is also nilpotent.

Exercise 3.1.4. If $X, Y \in M_n(\mathbb{K})$ are nilpotent, then the following are equivalent:

(1) $\exp X \exp Y = \exp Y \exp X$.

(2) $[X, Y] = 0$.

Exercise 3.1.5. If $(V, \cdot)$ is an associative algebra, then $\mathrm{Aut}(V, \cdot) \subseteq \mathrm{Aut}(V, [\cdot, \cdot])$.

Exercise 3.1.6. Let V be a finite-dimensional vector space, $F \subseteq V$ a subspace and $\gamma : [0,T] \to V$ a continuous curve with $\gamma([0,T]) \subseteq F$. Then for all $t \in [0,T]$:

$$I_t := \int_0^t \gamma(\tau)\, d\tau \in F.$$

Exercise 3.1.7. On each finite-dimensional Lie algebra $\mathfrak{g}$ there exists a norm with

$$\|[x,y]\| \leq \|x\|\|y\| \qquad \forall x,y \in \mathfrak{g},$$

i.e., $\|\operatorname{ad} x\| \leq \|x\|$.

Exercise 3.1.8. Let $\mathfrak{g}$ be a Lie algebra with a norm as in Exercise 3.1.7. Then for $\|x\| + \|y\| < \ln 2$ the Hausdorff series

$$x * y = x + \sum_{\substack{k,m \geq 0 \\ p_i + q_i > 0}} \frac{(-1)^k}{(k+1)(q_1 + \ldots + q_k + 1)} \frac{(\operatorname{ad} x)^{p_1} (\operatorname{ad} y)^{q_1} \ldots (\operatorname{ad} x)^{p_k} (\operatorname{ad} y)^{q_k} (\operatorname{ad} x)^m}{p_1! q_1! \ldots p_k! q_k! m!} y$$

converges absolutely.

Exercise 3.1.9. Let V and W be vector spaces and $q : V \times V \to W$ a skew-symmetric bilinear map. Then

$$[(v,w),(v',w')] := \big(0, q(v,v')\big)$$

is a Lie bracket on $\mathfrak{g} := V \times W$. For $x,y,z \in \mathfrak{g}$ we have $\big[x,[y,z]\big] = 0$.

Exercise 3.1.10. Let $\mathfrak{g}$ be a Lie algebra with $\big[x,[y,z]\big] = 0$ for $x,y,z \in \mathfrak{g}$. Then

$$x * y := x + y + \frac{1}{2}[x,y]$$

defines a group structure on $\mathfrak{g}$. An example for such a Lie algebra is the three-dimensional *Heisenberg algebra*

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} : x,y,z \in \mathbb{K} \right\}.$$

3.2 Calculating Lie Algebras of Linear Lie Groups

In this section we shall see various techniques to determine the Lie algebra of a linear Lie group.

Example 3.2.1. The group $G := \mathrm{SL}_n(\mathbb{K}) = \det^{-1}(\mathbf{1}) = \ker \det|_{\mathrm{GL}_n(\mathbb{K})}$ is a linear Lie group. To determine its Lie algebra, we first claim that

$$\det(e^x) = e^{\operatorname{Tr} x} \tag{3.2}$$

holds for $x \in M_n(\mathbb{K})$. To verify this claim, we consider

$$\det\colon M_n(\mathbb{K}) \cong (\mathbb{K}^n)^n \to \mathbb{K}$$

as a multilinear map, where each matrix x is considered as an n-tuple of its column vectors $x_1, \ldots, x_n$. Then Exercise 2.1.1(c) implies that

$$\begin{aligned}&(\mathtt{d}\det)(\mathbf{1})(x) = (\mathtt{d}\det)(e_1, \ldots, e_n)(x_1, \ldots, x_n)\\ &= \det(x_1, e_2, \ldots, e_n) + \ldots + \det(e_1, \ldots, e_{n-1}, x_n) = x_{11} + \ldots + x_{nn} = \operatorname{Tr} x.\end{aligned}$$

Now we consider the curve $\gamma\colon \mathbb{R} \to \mathbb{K}^\times \cong \mathrm{GL}_1(\mathbb{K}), t \mapsto \det(e^{tx})$. Then γ is a continuous group homomorphism, hence of the form $\gamma(t) = e^{at}$ for $a = \gamma'(0)$ (Theorem 2.2.6). On the other hand the Chain Rule implies

$$a = \gamma'(0) = \mathtt{d}\det(\mathbf{1})\big(\mathtt{d}\exp(\mathbf{0})(x)\big) = \operatorname{Tr}(x),$$

and this implies (3.2). We conclude that

$$\begin{aligned}\mathfrak{sl}_n(\mathbb{K}) &:= \mathbf{L}(\mathrm{SL}_n(\mathbb{K})) = \{x \in M_n(\mathbb{K})\colon (\forall t \in \mathbb{R})\, 1 = \det(e^{tx}) = e^{t \operatorname{Tr} x}\}\\ &= \{x \in M_n(\mathbb{K})\colon \operatorname{Tr} x = 0\}.\end{aligned}$$

Lemma 3.2.2. *Let V and W be finite-dimensional vector spaces and $\beta\colon V \times V \to W$ a bilinear map. For $(x, y) \in \mathrm{End}(V) \times \mathrm{End}(W)$ the following are equivalent:*

(1) *$e^{ty}\beta(v, v') = \beta(e^{tx}v, e^{tx}v')$ for all $t \in \mathbb{R}$ and all $v, v' \in V$.*

(2) *$y\beta(v, v') = \beta(xv, v') + \beta(v, xv')$ for all $v, v' \in V$.*

Proof. (1) $\Rightarrow$ (2): Taking the derivative in $t = 0$, the relation (1) leads to

$$y\beta(v, v') = \beta(xv, v') + \beta(v, xv'),$$

where we use the Product and the Chain Rule (Exercise 2.1.1(c)).

(2) $\Rightarrow$ (1): If (2) holds, then we obtain inductively

$$y^n.\beta(v, v') = \sum_{k=0}^{n} \binom{n}{k} \beta(x^k v, x^{n-k} v').$$

For the exponential series this leads with the general Cauchy Product Formula (Exercise 2.1.3) to

$$\begin{aligned}e^y\beta(v, v') &= \sum_{n=0}^{\infty} \frac{1}{n!} y^n.\beta(v, v') = \sum_{n=0}^{\infty} \frac{1}{n!} \Big(\sum_{k=0}^{n} \binom{n}{k} \beta(x^k v, x^{n-k} v)\Big)\\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \beta\left(\frac{1}{k!} x^k v, \frac{1}{(n-k)!} x^{n-k} v'\right)\\ &= \beta\left(\sum_{k=0}^{\infty} \frac{1}{k!} x^k v, \sum_{m=0}^{\infty} \frac{1}{m!} x^m v'\right) = \beta\left(e^x v, e^x v'\right).\end{aligned}$$

Since (2) also holds for the pair (tx, ty) for all $t \in \mathbb{R}$, this completes the proof. $\square$

Proposition 3.2.3. *Let V and W be finite-dimensional vector spaces and $\beta\colon V\times V\to W$ a bilinear map. For the group*

$$\mathrm{Aut}(V,\beta)=\{g\in \mathrm{GL}(V)\colon (\forall v,v'\in V)\ \beta(gv,gv')=\beta(v,v')\},$$

we then have

$$\mathfrak{aut}(V,\beta):=\mathbf{L}(\mathrm{Aut}(V,\beta))=\{x\in \mathfrak{gl}(V)\colon (\forall v,v'\in V)\ \beta(xv,v')+\beta(v,xv')=0\}.$$

Proof. We only have to observe that $X\in\mathbf{L}(\mathrm{Aut}(V,\beta))$ is equivalent to the pair $(X,0)$ satisfying condition (1) in Lemma 3.2.2. $\square$

Example 3.2.4. (a) Let $B\in M_n(\mathbb{K})$, $\beta(v,w)=v^\top Bw$, and

$$G:=\{g\in \mathrm{GL}_n(\mathbb{K})\colon g^\top Bg=B\}\cong \mathrm{Aut}(\mathbb{K}^n,\beta).$$

Then Proposition 3.2.3 implies that

$$\begin{aligned}\mathbf{L}(G)&=\{x\in\mathfrak{gl}_n(\mathbb{K})\colon (\forall v,v'\in V)\ \beta(xv,v')+\beta(v,xv')=0\}\\ &=\{x\in\mathfrak{gl}_n(\mathbb{K})\colon (\forall v,v'\in V)\ v^\top x^\top Bv'+v^\top Bxv'=0\}\\ &=\{x\in\mathfrak{gl}_n(\mathbb{K})\colon x^\top B+Bx=0\}.\end{aligned}$$

In particular, we obtain

$$\mathfrak{o}_n(\mathbb{K}):=\mathbf{L}(\mathrm{O}_n(\mathbb{K}))=\{x\in\mathfrak{gl}_n(\mathbb{K})\colon x^\top=-x\}=:\mathrm{Skew}_n(\mathbb{K}),$$

$$\mathfrak{o}_{p,q}(\mathbb{K}):=\mathbf{L}(\mathrm{O}_{p,q}(\mathbb{K}))=\{x\in\mathfrak{gl}_{p+q}(\mathbb{K})\colon x^\top I_{p,q}+I_{p,q}x=0\},$$

and

$$\mathfrak{symp}_{2n}(\mathbb{K}):=\mathbf{L}(\mathrm{Sp}_{2n}(\mathbb{K})):=\{x\in\mathfrak{gl}_{2n}(\mathbb{K})\colon x^\top B+Bx=0\},$$

where $B=\begin{pmatrix}0 & -\mathbf{1}_n\\ \mathbf{1}_n & 0\end{pmatrix}$.

(b) Applying Proposition 3.2.3 with $V=\mathbb{C}^n$ and $W=\mathbb{C}$, considered as real vector spaces, we also obtain for a hermitian form $\beta\colon\mathbb{C}^n\times\mathbb{C}^n\to\mathbb{C}$, $(z,w)\mapsto w^*I_{p,q}z$:

$$\begin{aligned}\mathfrak{u}_{p,q}(\mathbb{C})&:=\mathbf{L}(\mathrm{U}_{p,q}(\mathbb{C}))\\ &=\{x\in\mathfrak{gl}_n(\mathbb{C})\colon (\forall z,w\in\mathbb{C}^n)\, w^*I_{p,q}xz+w^*x^*I_{p,q}z=0\}\\ &=\{x\in\mathfrak{gl}_n(\mathbb{C})\colon I_{p,q}x+x^*I_{p,q}=0\}.\end{aligned}$$

In particular, we get

$$\mathfrak{u}_n(\mathbb{C}):=\mathbf{L}(\mathrm{U}_n(\mathbb{C}))=\{x\in\mathfrak{gl}_n(\mathbb{C})\colon x^*=-x\}=:\mathrm{Aherm}_n(\mathbb{C}).$$

Example 3.2.5. Let $\mathfrak{g}$ be a finite-dimensional $\mathbb{K}$-Lie algebra and

$$\mathrm{Aut}(\mathfrak{g}):=\{g\in\mathrm{GL}(\mathfrak{g})\colon (\forall x,y\in\mathfrak{g})\ g[x,y]=[gx,gy]\}.$$

To calculate the Lie algebra of G, we use Lemma 3.2.2 with $V = W = \mathfrak{g}$ and $\beta(x, y) = [x, y]$. Then we see that $D \in \mathfrak{aut}(\mathfrak{g}) := \mathbf{L}(\mathrm{Aut}(\mathfrak{g}))$ is equivalent to (D, D) satisfying the conditions in Lemma 3.2.2, and this leads to

$$\mathfrak{aut}(\mathfrak{g}) = \mathbf{L}(\mathrm{Aut}(\mathfrak{g})) = \{D \in \mathfrak{gl}(\mathfrak{g})\colon (\forall x, y \in \mathfrak{g})\, D([x, y]) = [D(x), y] + [x, D(y)]\}$$

The elements of this Lie algebra are called *derivations of* $\mathfrak{g}$, and $\mathfrak{aut}(\mathfrak{g})$ is also denoted $\mathrm{der}(\mathfrak{g})$. Note that the condition on an endomorphism of $\mathfrak{g}$ to be a derivation resembles the Leibniz Rule (Product Rule).

Remark 3.2.6. We call a linear Lie group $G \subseteq \mathrm{GL}_n(\mathbb{C})$ a *complex linear Lie group* if $\mathbf{L}(G) \subseteq \mathfrak{gl}_n(\mathbb{C})$ is a complex subspace, i.e., $i\,\mathbf{L}(G) \subseteq \mathbf{L}(G)$. Since Proposition 3.1.4 only ensures that $\mathbf{L}(G)$ is a real subspace, this definition makes sense.

For example $\mathrm{U}_n(\mathbb{C})$ is not a complex linear Lie group because

$$i\mathfrak{u}_n(\mathbb{C}) = \mathrm{Herm}_n(\mathbb{C}) \not\subseteq \mathfrak{u}_n(\mathbb{C}).$$

On the other hand $\mathrm{O}_n(\mathbb{C})$ is a complex linear Lie group because

$$\mathfrak{o}_n(\mathbb{C}) = \mathrm{Skew}_n(\mathbb{C})$$

is a complex subspace of $\mathfrak{gl}_n(\mathbb{C})$.

Exercises for Section 3.2

Exercise 3.2.1. (a) $\mathrm{Ad} : \mathrm{GL}_n(\mathbb{K}) \to \mathrm{Aut}\left(\mathfrak{gl}_n(\mathbb{K})\right)$ is a group homomorphism.

(b) For each Lie algebra $\mathfrak{g}$, the operators $\mathrm{ad}\, x(y) := [x, y]$ are derivations and the map $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a homomorphism of Lie algebras.

Exercise 3.2.2. Show that the following groups are linear Lie groups and determine their Lie algebras.

(1) $N := \{g \in \mathrm{GL}_n(\mathbb{R})\colon (\forall i > j)\ g_{ij} = 0, g_{ii} = 1\}$.

(2) $B := \{g \in \mathrm{GL}_n(\mathbb{R})\colon (\forall i > j)\ g_{ij} = 0\}$.

(3) $D := \{g \in \mathrm{GL}_n(\mathbb{R})\colon (\forall i \neq j)\ g_{ij} = 0\}$.
Note that $B \cong N \rtimes D$ is a semidirect product.

(4) A a finite-dimensional associative algebra and

$$G := \mathrm{Aut}(A) := \{g \in \mathrm{GL}(A)\colon (\forall a, b \in A)\ g(ab) = g(a)g(b)\}.$$

Exercise 3.2.3. Realize the two groups $\mathrm{Mot}_n(\mathbb{R})$ and $\mathrm{Aff}_n(\mathbb{R})$ as linear Lie groups in $\mathrm{GL}_{n+1}(\mathbb{R})$.

(1) Determine their Lie algebras $\mathfrak{mot}_n(\mathbb{R})$ and $\mathfrak{aff}_n(\mathbb{R})$.

(2) Calculate the exponential function $\exp\colon \mathfrak{aff}_n(\mathbb{R}) \to \mathrm{Aff}_n(\mathbb{R})$ in terms of the exponential function of $M_n(\mathbb{R})$.

Exercise 3.2.4. Let V be a finite-dimensional $\mathbb{K}$-vector space and $W \subseteq V$ a subspace. Show that
$$\mathrm{GL}(V)_W := \{g \in \mathrm{GL}(V) \colon gW = W\}$$
is a closed subgroup of $\mathrm{GL}(V)$ with
$$\mathbf{L}(\mathrm{GL}(V)_W) = \mathfrak{gl}(V)_W := \{X \in \mathfrak{gl}(V) \colon X.W \subseteq W\}.$$

Exercise 3.2.5. Show that for $n = p + q$ we have
$$\mathrm{O}_{p,q}(\mathbb{K}) \cap \mathrm{O}_n(\mathbb{K}) \cong \mathrm{O}_p(\mathbb{K}) \times \mathrm{O}_q(\mathbb{K}).$$

3.3 Polar Decomposition of Certain Algebraic Lie Groups

In this subsection we show that the polar decomposition of $\mathrm{GL}_n(\mathbb{R})$ can be used to obtain polar decompositions of many subgroups.

Let $G \subseteq \mathrm{GL}_n(\mathbb{K})$ be a linear Lie group. If $g = ue^x \in G$ (u unitary and x hermitian) implies that $u \in G$ and $e^x \in G$, then $g^* = e^x u^{-1} \in G$. Therefore a necessary condition for G to be adapted to the polar decomposition of $\mathrm{GL}_n(\mathbb{K})$ is that G is invariant under the map $g \mapsto g^*$. So we assume that this condition is satisfied. For $x \in \mathbf{L}(G)$, we then obtain from $(e^{tx})^* = e^{tx^*}$ that $x^* \in \mathbf{L}(G)$. Hence each element $x \in \mathbf{L}(G)$ can be written as
$$x = \frac{1}{2}(x - x^*) + \frac{1}{2}(x + x^*),$$
where both summands are in $\mathbf{L}(G)$. This implies that
$$\mathbf{L}(G) = \mathfrak{k} \oplus \mathfrak{p}, \quad \text{where} \quad \mathfrak{k} := \mathbf{L}(G) \cap \mathfrak{u}_n(\mathfrak{k}), \quad \mathfrak{p} := \mathbf{L}(G) \cap \mathrm{Herm}_n(\mathbb{K}).$$

We also need a condition which ensures that $e^x \in G$, $x \in \mathrm{Herm}_n(\mathbb{K})$, implies $x \in \mathbf{L}(G)$.

Definition 3.3.1. We call a subgroup $G \subseteq \mathrm{GL}_n(\mathbb{R})$ *algebraic* if there exists a family $(p_j)_{j \in J}$ of real polynomials
$$p_j(x) = p_j(x_{11}, x_{12}, \ldots, x_{nn}) \in \mathbb{R}[x_{11}, \ldots, x_{nn}]$$
in the entries of the matrix $x \in M_n(\mathbb{R})$ such that
$$G = \{x \in \mathrm{GL}_n(\mathbb{R}) \colon (\forall j \in J)\ p_j(x) = 0\}.$$

Lemma 3.3.2. *Let $G \subseteq \mathrm{GL}_n(\mathbb{R})$ be an algebraic subgroup, $y \in M_n(\mathbb{R})$ diagonalizable and $e^y \in G$. Then $y \in \mathbf{L}(G)$, i.e., $e^{\mathbb{R}y} \subseteq G$.*

Proof. Suppose that $A \in \mathrm{GL}_n(\mathbb{R})$ is such that AyA^{-1} is a diagonal matrix. Then $\widetilde{p}_j(x) = p_j(A^{-1}xA)$, $j \in J$, is also a set of polynomials in the entries of x and $e^y \in G$ is equivalent to
$$e^{AyA^{-1}} = Ae^yA^{-1} \in \widetilde{G} := AGA^{-1} = \{g \in \mathrm{GL}_n(\mathbb{R}) \colon (\forall j)\, \widetilde{p}_j(g) = 0\}.$$

Therefore we may assume that $y = \operatorname{diag}(y_1, \ldots, y_n)$ is a diagonal matrix. Now the polynomial $q_j(t) := p_j(e^{ty})$ has the form

$$\begin{aligned} q_j(t) &= \sum_{(k_1,\ldots,k_n)\in\mathbb{N}_0^n} a_{k_1,\ldots,k_n}(e^{ty_1})^{k_1}\cdots(e^{ty_n})^{k_n} \\ &= \sum_{(k_1,\ldots,k_n)\in\mathbb{N}_0^n} a_{k_1,\ldots,k_n} e^{t(k_1y_1+\ldots+k_ny_n)} \end{aligned}$$

(a finite sum). Therefore it can be written as $q_j(t) = \sum_{k=1}^m \lambda_k e^{tb_k}$, with $b_1 > \ldots > b_m$, where each b_k is a sum of the entries y_l of y. If q_j does not vanish identically on $\mathbb{R}$, then we may assume that $\lambda_1 \neq 0$. This leads to

$$\lim_{t\to\infty} e^{-tb_1} q_j(t) = \lambda_1 \neq 0,$$

which contradicts $q_j(\mathbb{Z}) = \{0\}$, which in turn follows from $e^{\mathbb{Z}y} \subseteq G$. Therefore each polynomial q_j vanishes identically, and hence $e^{\mathbb{R}y} \subseteq G$. □

Proposition 3.3.3. *[Polar decomposition for real algebraic groups] Let $G \subseteq \mathrm{GL}_n(\mathbb{R})$ be an algebraic subgroup with $G = G^\top$. We define $K := G \cap \mathrm{O}_n(\mathbb{R})$ and $\mathfrak{p} := \mathbf{L}(G) \cap \mathrm{Sym}_n(\mathbb{R})$. Then the map*

$$\varphi\colon K \times \mathfrak{p} \to G, \quad (k, x) \mapsto ke^x$$

is a homeomorphism.

Proof. Let $g \in G$ and write it as $g = ue^x$ with $u \in \mathrm{O}_n(\mathbb{R})$ and $x \in \mathrm{Sym}_n(\mathbb{R})$ (Proposition 2.3.5 and the polar decomposition). Then

$$e^{2x} = g^\top g \in G,$$

where $x \in \mathrm{Sym}_n(\mathbb{R})$ is diagonalizable. Therefore Lemma 3.3.2 implies that $e^{\mathbb{R}x} \subseteq G$, so that $x \in \mathfrak{p}$. Hence $u = ge^{-x} \in G \cap \mathrm{O}_n(\mathbb{R}) = K$. We conclude that φ is a surjective map. Furthermore Proposition 1.1.5 on the polar decomposition of $\mathrm{GL}_n(\mathbb{R})$ implies that φ is injective, hence bijective. The continuity of φ^{-1} also follows from the continuity of the inversion in $\mathrm{GL}_n(\mathbb{R})$ (cf. Proposition 1.1.5). □

Example 3.3.4. Proposition 3.3.3 applies to the following groups:

(a) $G = \mathrm{SL}_n(\mathbb{R})$ is $p^{-1}(0)$ for the polynomial $p(x) = \det x - 1$, and we obtain

$$\mathrm{SL}_n(\mathbb{R}) = K \exp \mathfrak{p} \cong K \times \mathfrak{p}$$

with

$$K = \mathrm{SO}_n(\mathbb{R}) \quad \text{and} \quad \mathfrak{p} = \{x \in \mathrm{Sym}_n(\mathbb{R}) \colon \operatorname{Tr} x = 0\}.$$

For $\mathrm{SL}_2(\mathbb{R})$, we obtain in particular a homeomorphism

$$\mathrm{SL}_2(\mathbb{R}) \cong \mathrm{SO}_2(\mathbb{R}) \times \mathbb{R}^2 \cong \mathbb{S}^1 \times \mathbb{R}^2.$$

(b) $G = \mathrm{O}_{p,q} := \mathrm{O}_{p,q}(\mathbb{R})$ is defined by the condition $g^\top I_{p,q} g = I_{p,q}$. These are n^2 polynomial equations, one for each entry of the matrix. Moreover, $g \in \mathrm{O}_{p,q}$ implies

$$I_{p,q} = I_{p,q}^{-1} = (g^\top I_{p,q} g)^{-1} = g^{-1} I_{p,q} (g^\top)^{-1}$$

and hence $g I_{p,q} g^\top = I_{p,q}$, i.e., $g^\top \in \mathrm{O}_{p,q}$. Therefore $\mathrm{O}_{p,q}^\top = \mathrm{O}_{p,q}$, and all the assumptions of Proposition 3.3.3 are satisfied. In this case,

$$K = \mathrm{O}_{p,q} \cap \mathrm{O}_n \cong \mathrm{O}_p \times \mathrm{O}_q,$$

(Exercise 3.2.5) and topologically we obtain

$$\mathrm{O}_{p,q} \cong \mathrm{O}_p \times \mathrm{O}_q \times (\mathfrak{o}_{p,q} \cap \mathrm{Sym}_n(\mathbb{R})).$$

In particular, we see that for $p, q > 0$ the group $\mathrm{O}_{p,q}$ has four arc-components because O_p and O_q have two arc-components (Proposition 1.1.7).

For the subgroup $\mathrm{SO}_{p,q}$ we have one additional polynomial equation, so that it is also algebraic. Here we have

$$\begin{aligned} K_S := K \cap \mathrm{SO}_{p,q} &\cong \{(a,b) \in \mathrm{O}_p \times \mathrm{O}_q \colon \det(a)\det(b) = 1\} \\ &\cong \big(\mathrm{SO}_p \times \mathrm{SO}_q\big) \dot\cup \big(\mathrm{O}_{p,-} \times \mathrm{O}_{q,-}\big), \end{aligned}$$

so that $\mathrm{SO}_{p,q}$ has two arc-components if $p, q > 0$ (cf. the discussion of the Lorentz group in Example 1.2.7).

(c) We can also apply Proposition 3.3.3 to the subgroup $\mathrm{GL}_n(\mathbb{C}) \subseteq \mathrm{GL}_{2n}(\mathbb{R})$ which is defined by the condition $gI = Ig$, where $I \colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ corresponds to the componentwise multiplication with i on $\mathbb{C}^n$. These are $4n^2 = (2n)^2$ polynomial equations defining $\mathrm{GL}_n(\mathbb{C})$. In this case we obtain a new proof of the polar decomposition of $\mathrm{GL}_n(\mathbb{C})$ because

$$K = \mathrm{GL}_n(\mathbb{C}) \cap \mathrm{O}_{2n}(\mathbb{R}) = \mathrm{U}_n(\mathbb{C})$$

and

$$\mathfrak{p} = \mathfrak{gl}_n(\mathbb{C}) \cap \mathrm{Sym}_{2n}(\mathbb{R}) = \mathrm{Herm}_n(\mathbb{C}).$$

Example 3.3.5. Let $X \in \mathrm{Sym}_n(\mathbb{R})$ be a nonzero symmetric matrix and consider the subgroup $G := \exp(\mathbb{Z}X) \subseteq \mathrm{GL}_n(\mathbb{R})$. Since $\exp X$ is symmetric, we then have $G^\top = G$. Moreover, if $\lambda_1 \leq \ldots \leq \lambda_k$ are the eigenvalues of X, then

$$\|\exp(nX) - \mathbf{1}\| = \max(|e^{n\lambda_k} - 1|, |e^{n\lambda_1} - 1|) \geq \max(|e^{\lambda_k} - 1|, |e^{\lambda_1} - 1|)$$

implies that G is a discrete subset of $\mathrm{GL}_n(\mathbb{R})$, hence a closed subgroup, and therefore a linear Lie group. On the other hand, the fact that G is discrete implies that $\mathbf{L}(G) = \{0\}$. This example shows that the assumption that G is algebraic is indispensable for Proposition 3.3.3 because

$$G \cap O_n(\mathbb{R}) = \{\mathbf{1}\} \quad \text{and} \quad \mathbf{L}(G) \cap \mathrm{Sym}_n(\mathbb{R}) = \{0\}.$$

Exercises for Section 3.3

Exercise 3.3.1. Show that the groups $\mathrm{O}_n(\mathbb{C})$, $\mathrm{SO}_n(\mathbb{C})$ and $\mathrm{Sp}_{2n}(\mathbb{R})$ have polar decompositions and describe their intersections with $\mathrm{O}_{2n}(\mathbb{R})$.

Exercise 3.3.2. Let $B \in \mathrm{Herm}_n(\mathbb{K})$ with $B^2 = \mathbf{1}$ and consider the automorphism $\tau(g) = Bg^{-\top}B^{-1}$ of $\mathrm{GL}_n(\mathbb{K})$. Show that:
(1) $\mathrm{Aut}(\mathbb{C}^n, B) = \{g \in \mathrm{GL}_n(\mathbb{K}) \colon \tau(g) = g\}$.
(2) $\mathrm{Aut}(\mathbb{C}^n, B)$ is adapted to the polar decomposition by showing that if $g = ue^x$ is the polar decomposition of g, then $\tau(g) = g$ is equivalent to $\tau(u) = u$ and $\tau(x) = x$.
(3) $\mathrm{Aut}(\mathbb{C}^n, B)$ is adapted to the polar decomposition by using that it is an algebraic group.

3.4 Linear Lie groups as submanifolds

The goal of this section is the following theorem.

The Identity Neighborhood Theorem

Theorem 3.4.1. *Let $G \subseteq \mathrm{GL}_n(\mathbb{K})$ be a closed subgroup. Then there exists an open 0-neighborhood $V \subseteq \mathbf{L}(G)$ such that $\exp|_V \colon V \to W := \exp(V) \subseteq G$ is a homeomorphism onto an open subset of G.*

Proof. First we use Proposition 2.2.4 to find an open 0-neighborhood $V_o \subseteq \mathfrak{gl}_n(\mathbb{K})$ such that

$$\exp_{V_o} \colon = \exp|_{V_o} \colon V_o \to W_o := \exp(V_o)$$

is a diffeomorphism between open sets. In the following we write $\log_{W_o} := (\exp_{V_o})^{-1}$ for the inverse function. Then the following assertions hold:

- $V_o \cap \mathbf{L}(G)$ is a 0-neighborhood in $\mathbf{L}(G)$.
- $W_o \cap G$ is a $\mathbf{1}$-neighborhood in G.
- $\exp(V_o \cap \mathbf{L}(G)) \subseteq W_o \cap G$
- $\exp|_{V_o \cap \mathbf{L}(G)}$ is injective.

If G is not closed, then it need not be true that

$$\exp(V_o \cap \mathbf{L}(G)) = W_o \cap G$$

because it might be the case that $W_o \cap G$ is much larger than $\exp(V_o \cap \mathbf{L}(G))$ (see "the dense wind" discussed below (cf. Lemma 3.4.8). We do not even know whether $\exp(V_o \cap \mathbf{L}(G))$ is open in G. Before we can complete the proof, we need three lemmas.

Lemma 3.4.2. *Let $(g_k)_{k\in\mathbb{N}}$ be a sequence in $G \cap W_o$ with $g_k \neq \mathbf{1}$ for all $k \in \mathbb{N}$ and $g_k \to \mathbf{1}$. We put $y_k := \log_{W_o} g_k$. Then every cluster point of the sequence*

$$\left\{\frac{y_k}{\|y_k\|} \colon k \in \mathbb{N}\right\}$$

is contained in $\mathbf{L}(G)$.

Proof. Let x be such a cluster point. By replacing the original sequence by a subsequence, we may assume that

$$x_k := \frac{y_k}{\|y_k\|} \to x \in \mathfrak{gl}_n(\mathbb{K})$$

and note that this implies $\|x\| = 1$. Let $t \in \mathbb{R}$ and put $p_k := \frac{t}{\|y_k\|}$. Then $tx_k = p_k y_k$, $y_k \to \log_{W_o} \mathbf{1} = 0$,

$$\exp tx = \lim_{k\to\infty} \exp(tx_k) = \lim_{k\to\infty} \exp(p_k y_k),$$

and

$$\exp(p_k y_k) = \exp(y_k)^{[p_k]} \exp\big((p_k - [p_k])y_k\big),$$

where $[p_k] = \max\{l \in \mathbb{Z} \colon l \leq p_k\}$ is the *Gauß function.* We therefore have

$$\|(p_k - [p_k])y_k\| \leq \|y_k\| \to 0$$

and eventually

$$\exp tx = \lim_{k\to\infty} (\exp y_k)^{[p_k]} = \lim_{k\to\infty} g_k^{[p_k]} \in G,$$

because G is closed. This implies $x \in \mathbf{L}(G)$. □

Lemma 3.4.3. *Let $E \subseteq \mathfrak{gl}_n(\mathbb{K})$ be a real vector subspace complementing the real subspace $\mathbf{L}(G)$. Then there exists a 0-neighborhood $U_E \subseteq E$ with*

$$G \cap \exp U_E = \{\mathbf{1}\}.$$

Proof. We argue by contradiction. If a neighborhood U_E with the required properties does not exist, then we find for each $k \in \mathbb{N}$ an element $0 \neq y_k \in E$ with $\|y_k\| \leq \frac{1}{k}$ and $g_k := \exp y_k \in G$. Note that $y_k \to 0$ implies that $g_k \to \mathbf{1}$. Now let $x \in E$ be a cluster point of the sequence $\frac{y_k}{\|y_k\|}$ which lies in the compact set $S_E := \{z \in E \colon \|z\| = 1\}$, so that at least one cluster point exists. According to Lemma 3.4.2, we have $x \in \mathbf{L}(G) \cap E = \{0\}$ because Lemma 3.4.2 applies since $g_k \in G \cap W_o$ holds for k sufficiently large. We arrive at a contradiction to $\|x\| = 1$. This proves the lemma. □

Lemma 3.4.4. *Let $V_1, V_2 \subseteq \mathfrak{gl}_n(\mathbb{K})$ be vector subspaces with $V_1 \oplus V_2 = \mathfrak{gl}_n(\mathbb{K})$. Then the map*

$$\Phi \colon V_1 \times V_2 \to \mathrm{GL}_n(\mathbb{K}), \quad (x_1, x_2) \mapsto (\exp x_1)(\exp x_2)$$

restricts to a diffeomorphism of a neighborhood of $(0,0)$ to an open $\mathbf{1}$-neighborhood in $\mathrm{GL}_n(\mathbb{K})$.

Proof. Let $\mu \colon M_n(\mathbb{K}) \times M_n(\mathbb{K}) \to M_n(\mathbb{K})$ be the multiplication map

$$(x_1, x_2) \mapsto x_1 x_2.$$

This map is bilinear, so that its derivative is given by

$$\mathtt{d}\mu(x_1, x_2)(h_1, h_r) = h_1 x_2 + x_1 h_2.$$

In particular we have

$$\mathtt{d}\mu(\mathbf{1},\mathbf{1})(h_1,h_2) = h_1 + h_2.$$

Now the Chain Rule and $\mathtt{d}\exp(0) = \mathrm{id}$ lead to

$$\begin{aligned}\mathtt{d}\Phi(0,0)(h_1,h_2) &= \mathtt{d}\mu(\mathbf{1},\mathbf{1})(\mathtt{d}\exp(0)h_1, \mathtt{d}\exp(0)h_2)\\ &= d\mu(\mathbf{1},\mathbf{1})(h_1,h_2) = h_1 + h_2.\end{aligned}$$

Since this map is bijective, the Inverse Function Theorem implies that Φ restricts to a diffeomorphism of a 0-neighborhood in $V_1 \times V_2$ onto a $\mathbf{1}$-neighborhood in $\mathrm{GL}_n(\mathbb{K})$. $\square$

Now we are ready to complete the proof of Theorem 3.4.1. We choose E as above, a vector space complement to $\mathbf{L}(G)$, and define

$$\Phi\colon\ \mathbf{L}(G) \times E \to \mathrm{GL}_n(\mathbb{K}), \quad (x,y) \mapsto \exp x \exp y.$$

According to Lemma 3.4.4, there exist open 0-neighborhoods $U_E \subseteq E$ and $U_o \subseteq \mathbf{L}(G)$ such that

$$\Phi|_{U_o \times U_E} : U_o \times U_E \to \exp(U_o)\exp(U_E)$$

is a diffeomorphism onto an open $\mathbf{1}$-neighborhood in $\mathrm{GL}_n(\mathbb{K})$. Moreover, in view of Lemma 3.4.3, we may choose U_E so small that $\exp(U_E) \cap G = \{\mathbf{1}\}$.

Since $\exp(U_o) \subseteq G$, the condition $g = \exp x \exp y \in G \cap (\exp(U_o)\exp(U_E))$ implies $\exp y = (\exp x)^{-1} g \in G \cap \exp U_E = \{\mathbf{1}\}$. Therefore

$$\exp(U_o) = G \cap (\exp(U_o)\exp(U_E))$$

is an open $\mathbf{1}$-neighborhood in G. This completes the proof of Theorem 3.4.1. $\square$

Linear Lie groups as submanifolds

The Identity Neighborhood Theorem has important consequences for the structure of linear Lie groups. One of them is that they are submanifolds of the real vector space $M_n(\mathbb{K}) \cong \mathbb{K}^{(n^2)}$.

Definition 3.4.5. Let V be a finite-dimensional real vector space. A subset $M \subseteq V$ is called a *k-dimensional submanifold* if for each $x \in M$ there exists an open neighborhood U_x of x in V, a k-dimensional subspace $F \subseteq V$ and a diffeomorphism $\varphi\colon U_x \to W$ onto an open neighborhood W of 0 in V such that

$$\varphi(U_x \cap M) = W \cap F.$$

Geometrically this means that a piece of M (such as $U_x \cap M$) looks like a piece of a vector subspace F of V (such as $W \cap F$). In this sense φ “straightens” the curved structure of M.

Proposition 3.4.6. *Every closed subgroup G of $\mathrm{GL}_n(\mathbb{K})$ is a submanifold of $M_n(\mathbb{K})$ of dimension $\dim_{\mathbb{R}} \mathbf{L}(G)$.*

Proof. We recall the diffeomorphism

$$\Phi\colon U_o \times U_E \to \exp(U_o)\exp(U_E)$$

from the proof of Theorem 3.4.1, where $U_o \subseteq \mathbf{L}(G)$ and $U_E \subseteq E$ are open 0-neighborhoods and $M_n(\mathbb{K}) = \mathbf{L}(G) \oplus E$. We also recall that

$$\Phi(U_o \times U_E) \cap G = \exp(U_o) = \Phi(U_o \times \{0\}).$$

For $g \in G$ we write $\lambda_g\colon\ \mathrm{GL}_n(\mathbb{K}) \to \mathrm{GL}_n(\mathbb{K})$ for the left multiplication $\lambda_g(h) = gh$ and observe that λ_g is a linear automorphism of $M_n(\mathbb{K})$. Therefore $U_g := \lambda_g(\mathrm{im}(\Phi)) = g\,\mathrm{im}(\Phi)$ is an open neighborhood of g in $M_n(\mathbb{K})$. Moreover, the map

$$\varphi_g\colon U_g \to \mathbf{L}(G) \oplus E = M_n(\mathbb{K}), \quad x \mapsto \Phi^{-1}(g^{-1}x)$$

is a diffeomorphism onto the open subset $U_o \times U_E$ of $M_n(\mathbb{K})$, and we have

$$\begin{aligned}\varphi_g(U_g \cap G) &= \varphi_g(g\,\mathrm{im}(\Phi) \cap G) = \varphi_g(g(\mathrm{im}(\Phi) \cap G)) \\ &= \varphi_g(g\exp(U_o)) = U_o \times \{0\} = (U_o \times U_E) \cap (\mathbf{L}(G) \times \{0\}).\end{aligned}$$

Therefore the family $(\varphi_g, U_g)_{g\in G}$ satisfies the assumptions of Definition 3.4.5, so that G is a submanifold of $M_n(\mathbb{K})$ of dimension $\dim_{\mathbb{R}} \mathbf{L}(G)$. □

Remark 3.4.7. (a) Every submanifold M of a vector space V is *locally closed* in the sense that for each $x \in M$ there exists a neighborhood U of x in V for which $U \cap M$ is closed in U.

(b) We shall see later that each locally closed subgroup H of a topological group G is closed. Therefore each subgroup $G \subseteq \mathrm{GL}_n(\mathbb{K})$ which is a submanifold of $M_n(\mathbb{K})$ is automatically closed, hence a linear Lie group. This means that the linear Lie groups are precisely those subgroups of $\mathrm{GL}_n(\mathbb{K})$ which are submanifolds of $\mathrm{GL}_n(\mathbb{K})$.

(c) For each submanifold $M \subseteq V$ and each $x \in M$ we define the *geometric tangent space* $T_x(M) \subseteq V$ as the set of all $v \in V$ for which there exists a differentiable curve $\gamma\colon]-\epsilon,\epsilon[\to M \subseteq V$ with $\gamma(0) = x$ and $\gamma'(0) = v$. In terms of Definition 3.4.5, it is not hard to see that $T_x(M) = \mathtt{d}\varphi(x)^{-1}(E)$. In particular T_xM is a k-dimensional vector subspace of V.

(d) If G is a linear Lie group, then

$$T_{\mathbf{1}}(G) = \mathbf{L}(G).$$

In fact, $\gamma(t) := \exp tx \subseteq G$ for $x \in \mathfrak{g}$, $t \in \mathbb{R}$ imply that $x = \gamma'(0) \in T_{\mathbf{1}}(G)$ and hence $\mathbf{L}(G) \subseteq T_{\mathbf{1}}(G)$. Since the spaces $\mathbf{L}(G)$ and $T_{\mathbf{1}}(G)$ have the same dimension (Proposition 3.4.6, both are equal.

The dense wind

In this short subsection we discuss an important example of a subgroup of the 2-torus $\mathbb{T}^2$ which is not closed. It is the simplest example of a non-closed, arcwise connected subgroup.

Let

$$A = \left\{ \begin{pmatrix} e^{it\sqrt{2}} & 0 \\ 0 & e^{it} \end{pmatrix} : t \in \mathbb{R} \right\} \subseteq \mathbb{T}^2 := \left\{ \begin{pmatrix} e^{ir} & 0 \\ 0 & e^{is} \end{pmatrix} : r, s \in \mathbb{R} \right\},$$

where $\mathbb{T}^2$ is the *two-dimensional torus*. We endow $\mathbb{T}^2$ with the subspace topology inherited from $M_2(\mathbb{C})$.

Lemma 3.4.8. *A is a dense subgroup of the 2-torus $\mathbb{T}^2$.*

Proof. We consider the map

$$\Phi \colon \mathbb{R}^2 \to \mathbb{T}^2, \quad (r, s) \mapsto \begin{pmatrix} e^{2\pi ir} & 0 \\ 0 & e^{2\pi is} \end{pmatrix}$$

which is a surjective continuous group homomorphism with kernel $\mathbb{Z}^2$. For $L := \mathbb{R}(\sqrt{2}, 1)$ and $V = \mathbb{R}(1, 0)$ we have $\mathbb{R}^2 \cong V \oplus L$. In view of

$$A = \Phi(L) = \Phi(L + \mathbb{Z}^2),$$

it suffices to show that $L+\mathbb{Z}^2$ is dense in $\mathbb{R}^2$. From the direct decomposition $\mathbb{R}^2 \cong V \oplus L$ and $L \subseteq L + \mathbb{Z}^2$ we derive

$$L + \mathbb{Z}^2 = L + ((L + \mathbb{Z}^2) \cap V),$$

and if $p \colon \mathbb{R}^2 \to V$ denote the projection map with kernel L, then

$$(L + \mathbb{Z}^2) \cap V = p(L + \mathbb{Z}^2) = p(\mathbb{Z}^2).$$

It therefore suffices to show that $p(\mathbb{Z}^2)$ is dense in V. From $p(1, 0) = (1, 0)$ and $p(0, 1) = p((0, 1) - (\sqrt{2}, 1)) = -(\sqrt{2}, 0)$ we obtain $p(\mathbb{Z}^2) = \mathbb{Z} + \sqrt{2}\mathbb{Z}$, so that the density of $p(\mathbb{Z}^2)$ is a consequence of Lemma 3.4.10 below. ☐

Example 3.4.9. We consider the topological group $G = (\mathbb{R}, +)$. Suppose that $\{0\} \neq \Gamma \subseteq \mathbb{R}$ is a subgroup. Then two cases occur:

Case 1: $\inf(\mathbb{R}_+^\times \cap \Gamma) = 0$, i.e., there exists a sequence $0 < x_n \in \Gamma$ with $x_n \to 0$. Then $\mathbb{Z}x_n \subseteq \Gamma$ holds for each n. For each open interval $]a, b[\subseteq \mathbb{R}$ and $x_n < b - a$ we then obtain

$$\emptyset \neq \mathbb{Z}x_n \cap]a, b[\subseteq \Gamma \cap]a, b[,$$

so that Γ is dense, i.e., $\overline{\Gamma} = \mathbb{R}$.

Case 2: $d := \inf(\mathbb{R}_+^\times \cap \Gamma) > 0$. Then $]-d, d[\cap \Gamma = \{0\}$ implies that Γ is discrete and therefore closed. If $d \notin \Gamma$, then there exists a $d' \in]d, 2d[\cap \Gamma$ and likewise a $d'' \in]d, d'[\cap \Gamma$. Then $0 < d' - d'' < d$ contradicts the definition of d. This implies that $d \in \Gamma$, and hence that $\mathbb{Z}d \subseteq \Gamma$. To see that we actually have equality, let $\gamma \in \Gamma$ and $k := \max\{n \in \mathbb{Z} \colon nd \leq \gamma\}$. Then $\gamma - nd \in [0, d[\cap \Gamma = \{0\}$ implies $\gamma = nd$. We conclude that $\Gamma = \mathbb{Z}d$ is a cyclic group.

In particular, we have shown that all non-trivial closed subgroups of $\mathbb{R}$ are cyclic and isomorphic to $\mathbb{Z}$.

Lemma 3.4.10. *Let $\theta \in \mathbb{R}$. Then $\mathbb{Z} + \mathbb{Z}\theta$ is dense in $\mathbb{R}$ if and only if θ is irrational.*

Proof. Suppose first that $\mathbb{Z}+\mathbb{Z}\theta$ is not dense in $\mathbb{R}$. Then it is discrete by Example 3.4.9, hence of the form $\mathbb{Z}x_o$ for some $x_o > 0$. Then there exist $k, m \in \mathbb{Z}$ with

$$1 = kx_o \quad \text{and} \quad \theta = mx_o.$$

We then obtain $\theta = \frac{m}{k} \in \mathbb{Q}$. If, conversely, $\theta = \frac{m}{k} \in \mathbb{Q}$, then $\mathbb{Z} + \mathbb{Z}\theta \subseteq \frac{1}{k}\mathbb{Z}$ is not dense in $\mathbb{R}$. $\square$

Chapter 4

Smooth Manifolds

Contrary to submanifolds of some vector space, a differentiable manifold is described without specifying any surrounding space. In spite of the fact that one can show that each finite dimensional smooth manifold can be realized as a closed submanifold of some $\mathbb{R}^n$ (Whitney's Embedding Theorem), these embeddings are not canonical, and it is therefore much more natural to think of differentiable manifolds as spaces for which no embedding is specified. The concept of a differentiable manifold permits us to define a Lie group as a differentiable manifold for which the group operations are smooth maps. We shall verify below that this approach is compatible with what we have learned previously on linear Lie groups.

4.1 Manifolds and Smooth Maps

Before we turn to the concept of a smooth manifold, we recall the concept of a Hausdorff space. We assume, however, some familiarity with basic topological constructions and concepts, such as the quotient topology. A topological space (X, τ) is called a *Hausdorff space* if for two different points $x, y \in X$ there exist disjoint open subsets O_x, O_y with $x \in O_x$ and $y \in O_y$. Recall that each metric space (X, d) is Hausdorff.

Definition 4.1.1. Let M be a topological space.

(a) A pair (φ, U), consisting of an open subset $U \subseteq M$ and a homeomorphism $\varphi\colon U \to \varphi(U) \subseteq \mathbb{R}^n$ of U onto an open subset of $\mathbb{R}^n$ is called an *n-dimensional chart of M*.

(b) Two n-dimensional charts (φ, U) and (ψ, V) of M are said to be *C^k-compatible* $(k \in \mathbb{N} \cup \{\infty\})$ if $U \cap V = \emptyset$ or the map

$$\psi \circ \varphi^{-1}|_{\varphi(U\cap V)}\colon \varphi(U \cap V) \to \psi(U \cap V)$$

is a C^k-diffeomorphism. Since $\varphi\colon U \to \varphi(U)$ is a homeomorphism onto an open subset of $\mathbb{R}^n$, $\varphi(U \cap V)$ is an open subset of $\varphi(U)$ and hence of $\mathbb{R}^n$.

(c) An *n-dimensional C^k-atlas of M* is a family $\mathcal{A} := (\varphi_i, U_i)_{i\in I}$ of n-dimensional charts of M with the following properties:

(A1) $\bigcup_{i\in I} U_i = M$, i.e., $(U_i)_{i\in I}$ is an open covering of M.

(A2) All charts (φ_i, U_i), $i \in I$, are pairwise C^k-compatible. For $U_{ij} := U_i \cap U_j$, this means that all maps

$$\varphi_{ji} := \varphi_j \circ \varphi_i^{-1}|_{\varphi_i(U_{ij})} \colon \varphi_i(U_{ij}) \to \varphi_j(U_{ij})$$

are C^k-maps because $\varphi_{ji}^{-1} = \varphi_{ij}$.

(d) A chart (φ, U) is called *compatible* with a C^k-atlas $(\varphi_i, U_i)_{i\in I}$ if it is C^k-compatible with all charts of the atlas $\mathcal{A}$. A C^k-atlas $\mathcal{A}$ is called *maximal* if it contains all charts compatible with it. A maximal C^k-atlas is also called a *C^k-differentiable structure on M*. For $k = \infty$ we also call it a *smooth structure*.

Remark 4.1.2. (a) In Definition 4.1.1(b) we required that the map

$$\psi \circ \varphi^{-1}|_{\varphi(U\cap V)} \colon \varphi(U \cap V) \to \psi(U \cap V)$$

is a C^k-diffeomorphism. Since φ and ψ are homeomorphisms, this map always is a homeomorphism between open subsets of $\mathbb{R}^n$. The differentiability is an additional requirement.

(b) For $M = \mathbb{R}$ the maps (M, φ) and (M, ψ) with $\varphi(x) = x$ and $\psi(x) = x^3$ are 1-dimensional charts. These charts are not C^1-compatible: the map

$$\psi \circ \varphi^{-1} \colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto x^3$$

is smooth, but not a diffeomorphism, since its inverse $\varphi \circ \psi^{-1}$ is not differentiable.

(c) Every atlas $\mathcal{A}$ is contained in a unique maximal atlas: We simply add all charts compatible with $\mathcal{A}$, and thus obtain a maximal atlas. This atlas is unique (Exercise 4.1.2).

(d) A given topological space M may carry different differentiable structures. Examples are the exotic differentiable structures on $\mathbb{R}^4$ (the only $\mathbb{R}^n$ carrying exotic differentiable structures) and the 7-sphere $\mathbb{S}^7$.

Definition 4.1.3. An *n-dimensional C^k-manifold* is a pair $(M, \mathcal{A})$ of a Hausdorff space M and a maximal n-dimensional C^k-atlas $\mathcal{A}$ for M. For $k = \infty$ we call it a *smooth manifold*.

To specify a manifold structure, it suffices to specify a C^k-atlas $\mathcal{A}$ because this atlas is contained in a unique maximal one (Exercise 4.1.2). In the following we shall never describe a maximal atlas. We shall always try to keep the number of charts as small as possible. For simplicity, we always assume in the following that $k = \infty$.

Example 4.1.4. [Open subsets of $\mathbb{R}^n$] Let $U \subseteq \mathbb{R}^n$ be an open subset. Then U is a Hausdorff space with respect to the induced topology. The inclusion map $\varphi \colon U \to \mathbb{R}^n$ defines a chart (φ, U) which already defines a smooth atlas of U, turning U into an n-dimensional smooth manifold.

Example 4.1.5. [The n-dimensional sphere] We consider the unit sphere

$$\mathbb{S}^n := \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \colon x_0^2 + x_1^2 + \ldots + x_n^2 = 1\}$$

in $\mathbb{R}^n$, endowed with the subspace topology, turning it into a compact space.

(a) To specify a smooth manifold structure on $\mathbb{S}^n$, we consider the open subsets

$$U_i^\varepsilon := \{x \in \mathbb{S}^n \colon \varepsilon x_i > 0\}, \quad i = 0, \ldots, n, \quad \varepsilon \in \{\pm 1\}.$$

These $2(n+1)$ subsets form a covering of $\mathbb{S}^n$. We have homeomorphisms

$$\varphi_i^\varepsilon \colon U_i^\varepsilon \to B := \{x \in \mathbb{R}^n \colon \|x\|_2 < 1\}$$

onto the open unit ball in $\mathbb{R}^n$, given by

$$\varphi_i^\varepsilon(x) = (x_0, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

and with continuous inverse map

$$(y_1, \ldots, y_n) \mapsto \Big(y_1, \ldots, y_i, \varepsilon\sqrt{1 - \|y\|_2^2}, y_{i+1}, \ldots, y_n\Big).$$

This leads to charts $(\varphi_i^\varepsilon, U_i^\varepsilon)$ of $\mathbb{S}^n$.

It is easy to see that these charts are pairwise compatible. We have $\varphi_i^\varepsilon \circ (\varphi_i^{\varepsilon'})^{-1} = \mathrm{id}_B$, and for $i < j$, we have

$$\varphi_i^\varepsilon \circ (\varphi_j^{\varepsilon'})^{-1}(y) = \Big(y_1, \ldots, y_i, y_{i+2}, \ldots, y_j, \varepsilon'\sqrt{1 - \|y\|_2^2}, y_{j+1}, \ldots, y_n\Big),$$

which is a smooth map

$$\varphi_j^{\varepsilon'}(U_i^\varepsilon \cap U_j^{\varepsilon'}) \to \varphi_i^\varepsilon(U_i^\varepsilon \cap U_j^{\varepsilon'}).$$

(b) There is another atlas of $\mathbb{S}^n$ consisting only of two charts, where the maps are slightly more complicated.

We call the unit vector $e_0 := (1, 0, \ldots, 0)$ the *north pole* of the sphere and $-e_0$ the *south pole*. We then have the corresponding *stereographic projection maps*

$$\varphi_+ \colon U_+ := \mathbb{S}^n \setminus \{e_0\} \to \mathbb{R}^n, \quad (y_0, y) \mapsto \frac{1}{1 - y_0} y$$

and

$$\varphi_- \colon U_- := \mathbb{S}^n \setminus \{-e_0\} \to \mathbb{R}^n, \quad (y_0, y) \mapsto \frac{1}{1 + y_0} y.$$

Both maps are bijective with inverse maps

$$\varphi_\pm^{-1}(x) = \Big(\pm \frac{\|x\|_2^2 - 1}{\|x\|_2^2 + 1}, \frac{2x}{1 + \|x\|_2^2}\Big)$$

(Exercise 4.1.4). This implies that (φ_+, U_+) and (φ_-, U_-) are charts of $\mathbb{S}^n$. That both are smoothly compatible, hence a smooth atlas, follows from

$$(\varphi_+ \circ \varphi_-^{-1})(x) = (\varphi_- \circ \varphi_+^{-1})(x) = \frac{x}{\|x\|^2}, \quad x \in \mathbb{R}^n \setminus \{0\},$$

which is the inversion at the unit sphere.

Example 4.1.6. [Submanifolds of $\mathbb{R}^n$] We claim that a d-dimensional submanifold $M \subseteq \mathbb{R}^n$ carries a natural d-dimensional manifold structure when endowed with the topology inherited from $\mathbb{R}^n$, which obviously turns it into a Hausdorff space.

In fact, for each submanifold chart (φ, U) (cf. Definition 3.4.5) with $\varphi(U \cap M) = (\mathbb{R}^d \times \{0\}) \cap \varphi(U)$, we obtain a d-dimensional chart

$$(\varphi|_{U\cap M}, U \cap M),$$

where we have identified $\mathbb{R}^d$ with $\mathbb{R}^d \times \{0\}$. For two such charts coming from (φ, U) and (ψ, V), we have

$$\psi \circ \varphi^{-1}|_{\varphi(U\cap V\cap M)} = (\psi|_{V\cap M}) \circ (\varphi|_{U\cap M})^{-1}|_{\varphi(U\cap V\cap M)},$$

which is a smooth map onto an open subset of $\mathbb{R}^d$. We thus obtain a smooth atlas of M.

Example 4.1.7. Let E be an n-dimensional real vector space. We know from Linear Algebra that E is isomorphic to $\mathbb{R}^n$, and that for each ordered basis $B := (b_1, \ldots, b_n)$ for E, the linear map

$$\varphi_B \colon \mathbb{R}^n \to E, \quad x = (x_1, \ldots, x_n) \mapsto \sum_{j=1}^{n} x_j b_j$$

is a linear isomorphism. Using such a linear isomorphism φ_B, we define a topology on E in such a way that φ_B is a homeomorphism, i.e., $O \subseteq E$ is open if and only if $\varphi_B^{-1}(O)$ is open in $\mathbb{R}^n$.

For any other choice of a basis $C = (c_1, \ldots, c_m)$ in E we recall from linear algebra that $m = n$ and that the map

$$\varphi_C^{-1} \circ \varphi_B \colon \mathbb{R}^n \to \mathbb{R}^n$$

is a linear isomorphism, hence a homeomorphism. This implies that for a subset $O \subseteq E$ the condition that $\varphi_B^{-1}(O)$ is open is equivalent to $\varphi_C^{-1}(O) = \varphi_C^{-1} \circ \varphi_B \circ \varphi_B^{-1}(O)$ being open. We conclude that the topology introduced on E by φ_B does not depend on the choice of a basis.

We thus obtain on E a natural topology for which it is homeomorphic to $\mathbb{R}^n$, hence in particular a Hausdorff space. From each coordinate map $\kappa_B := \varphi_B^{-1}$ we obtain a chart (κ_B, E) which already defines an atlas of E. We thus obtain on E the structure of an n-dimensional smooth manifold. That all these charts are compatible follows from the smoothness of the linear maps $\kappa_C \circ \kappa_B^{-1} = \varphi_C^{-1} \circ \varphi_B \colon \mathbb{R}^n \to \mathbb{R}^n$.

New Manifolds from Old Ones

Definition 4.1.8. [Open subsets are manifolds] Let M be a smooth manifold and $N \subseteq M$ an open subset. Then N carries a natural smooth manifold structure.

Let $\mathcal{A} = (\varphi_i, U_i)_{i\in I}$ be an atlas of M. Then $V_i := U_i \cap N$ and $\psi_i := \varphi_i|_{V_i}$ define a smooth atlas $\mathcal{B} := (\psi_i, V_i)_{i\in I}$ of N (Exercise).

Definition 4.1.9. [Products of manifolds] Let M and N be smooth manifolds of dimensions d, resp., k and

$$M \times N = \{(m,n) \colon m \in M, n \in N\}$$

the product set, which we endow with the product topology.

We show that $M \times N$ carries a natural structure of a smooth $(d+k)$-dimensional manifold. Let $\mathcal{A} = (\varphi_i, U_i)_{i \in I}$ be an atlas of M and $\mathcal{B} = (\psi_j, V_j)_{j \in J}$ an atlas of N. Then the product sets $W_{ij} := U_i \times V_j$ are open in $M \times N$ and the maps

$$\gamma_{ij} := \varphi_i \times \psi_j \colon U_i \times V_j \to \mathbb{R}^d \times \mathbb{R}^k \cong \mathbb{R}^{d+k}, \quad (x,y) \mapsto (\varphi_i(x), \psi_j(y))$$

are homeomorphisms onto open subsets of $\mathbb{R}^{d+k}$. On $\gamma_{i'j'}(W_{ij} \cap W_{i'j'})$ we have

$$\gamma_{ij} \circ \gamma_{i'j'}^{-1} = (\varphi_i \circ \varphi_{i'}^{-1}) \times (\psi_j \circ \psi_{j'}^{-1}),$$

which is a smooth map. Therefore $(\varphi_{ij}, W_{ij})_{(i,j) \in I \times J}$ is a smooth atlas on $M \times N$.

Smooth maps

Definition 4.1.10. (a) Let M and N be differentiable manifolds. We call a continuous map $f \colon M \to N$ *smooth in* $p \in M$ if, for some chart (φ, U) of M with $p \in U$ and some chart (ψ, V) of N with $f(p) \in V$, the map

$$\psi \circ f \circ \varphi^{-1} \colon \varphi(f^{-1}(V)) \to \psi(V), \quad \varphi(x) \mapsto \psi(f(x)) \tag{4.1}$$

between open subsets of a vector space is smooth in a neighborhood of $\varphi(p)$. Note that the assumption that f is continuous implies that $f^{-1}(U)$ is open in M, so that the set $\psi(f^{-1}(U))$ is open. We call a continuous map $f \colon M \to N$ *smooth* if it is smooth in each point of M. We write $C^\infty(M,N)$ for the set of smooth maps $f \colon M \to N$.

(b) A smooth map $f \colon M \to N$ is called a *smooth isomorphism* or a *diffeomorphism* if there exists a smooth map $g \colon N \to M$ with $g \circ f = \mathrm{id}_M$ and $f \circ g = \mathrm{id}_N$. We write $\mathrm{Diff}(M,N)$ for the set of diffeomorphisms of M to N and $\mathrm{Diff}(M) := \mathrm{Diff}(M,M)$.

Remark 4.1.11. (a) If $f \colon M \to N$ and $g \colon N \to Q$ are continuous maps and $p \in M$ is such that f is smooth in p and g is smooth in $f(p)$, then the composition $g \circ f$ is smooth in p. In fact, for charts (φ, U), (ψ, V), resp., (η, W) of M, N, resp., Q, we have

$$\eta \circ (g \circ f) \circ \varphi^{-1} = (\eta \circ g \circ \psi^{-1}) \circ (\psi \circ f \circ \varphi^{-1}),$$

on its natural domain, which contains a neighborhood of $\varphi(p)$.

(b) From (a) it follows in particular that, if $f \colon M \to N$ is smooth in p and $(\widetilde{\varphi}, \widetilde{U})$ is any chart of M with $p \in \widetilde{U}$, then, for any chart $(\widetilde{\psi}, \widetilde{V})$ of N with $f(p) \in \widetilde{V}$, the map

$$\widetilde{\psi} \circ f \circ \widetilde{\varphi}^{-1} \colon \widetilde{\varphi}(f^{-1}(\widetilde{V})) \to \widetilde{\psi}(\widetilde{V})$$

is smooth.

(c) The map $f \colon \mathbb{R} \to \mathbb{R}, x \mapsto x^3$ is smooth and invertible, but it is not a smooth isomorphism because $f^{-1}(x) = x^{1/3}$ is not differentiable in 0.

Remark 4.1.12. (a) If $I \subseteq \mathbb{R}$ is an open interval, then a smooth map $\gamma\colon I \to M$ is called a *smooth curve*.

For a not necessarily open interval $I \subseteq \mathbb{R}$, a map $\gamma\colon I \to \mathbb{R}^n$ is called smooth if all derivatives $\gamma^{(k)}$ exist in all points of I and define continuous functions $I \to \mathbb{R}^n$. Based on this generalization of smoothness for curves, a curve $\gamma\colon I \to M$ is said to be smooth, if for each chart (φ, U) of M the curves

$$\varphi \circ \gamma\colon \gamma^{-1}(U) \to \mathbb{R}^n$$

are smooth.

A curve $\gamma\colon [a,b] \to M$ is called *piecewise smooth* if γ is continuous and there exists a subdivision $x_0 = a < x_1 < \ldots, < x_N = b$ such that $\gamma|_{[x_i,x_{i+1}]}$ is smooth for $i = 0, \ldots N-1$.

(b) Smoothness of maps $f\colon M \to \mathbb{R}^n$ can be checked more easily. Since the identity is a chart of $\mathbb{R}^n$, the smoothness condition simply means that for each chart (φ, U) of M the map

$$f \circ \varphi^{-1}\colon \varphi(f^{-1}(V) \cap U) \to \mathbb{R}^n$$

is smooth.

(c) If U is an open subset of $\mathbb{R}^n$, then a map $f\colon U \to M$ to a smooth m-dimensional manifold M is smooth if and only if for each chart (φ, V) of M the map

$$\varphi \circ f\colon f^{-1}(V) \to \mathbb{R}^n$$

is smooth.

(d) Any chart (φ, U) of a smooth n-dimensional manifold M defines a diffeomorphism $U \to \varphi(U) \subseteq \mathbb{R}^n$, when U is endowed with the canonical manifold structure as an open subset of M.

In fact, by definition, we may use (φ, U) as an atlas of U. Then the smoothness of φ is equivalent to the smoothness of the map $\varphi \circ \varphi^{-1} = \mathrm{id}_{\varphi(U)}$, which is trivial. Likewise, the smoothness of $\varphi^{-1}\colon \varphi(U) \to U$ is equivalent to the smoothness of $\varphi \circ \varphi^{-1} = \mathrm{id}_{\varphi(U)}$.

Remark 4.1.13. If M and N are differentiable manifolds, then the product manifold $M \times N$ has the following properties:

(a) The projection maps $p_M\colon M \times N \to M$ and $p_N\colon M \times N \to N$ are smooth.

(b) For $x \in M$, the embedding

$$i_x\colon N \to M \times N, \quad y \mapsto (x, y)$$

is smooth and, for $y \in N$, the embedding

$$i^y\colon M \to M \times N, \quad x \mapsto (x, y)$$

is smooth.

(c) The diagonal embedding

$$\Delta_M\colon M \to M \times M, \quad x \mapsto (x, x)$$

is smooth.

Exercises for Section 4.1

Exercise 4.1.1. Let $M := \mathbb{R}$, endowed with its standard topology. Show that C^k-compatibility of 1-dimensional charts is not an equivalence relation.

Exercise 4.1.2. Show that each n-dimensional C^k-atlas is contained in a unique maximal one.

Exercise 4.1.3. Let If M_i, $i = 1, \ldots, n$, be smooth manifolds of dimension d_i. Show that the product space $M := M_1 \times \ldots \times M_n$ carries the structure of a $(d_1 + \ldots + d_n)$-dimensional manifold.

Exercise 4.1.4. (a) Verify the details in Example 4.1.5, where we describe an atlas of $\mathbb{S}^n$ by stereographic projections.

(b) Show that the two atlasses of $\mathbb{S}^n$ constructed in Example 4.1.5 and the atlas obtained from the realization of $\mathbb{S}^n$ as a quadric in $\mathbb{R}^{n+1}$ define the same differentiable structure.

Exercise 4.1.5. Show that the set $A := C^\infty(M, \mathbb{R})$ of smooth real-valued functions on M is a real algebra. If $g \in A$ is nonzero and $U := g^{-1}(\mathbb{R}^\times)$, then $\frac{1}{g} \in C^\infty(U, \mathbb{R})$.

Exercise 4.1.6. Let $f_1 \colon M_1 \to N_1$ and $f_2 \colon M_2 \to N_2$ be smooth maps. Show that the map

$$f_1 \times f_2 \colon M_1 \times M_2 \to N_1 \times N_2, \quad (x, y) \mapsto (f_1(x), f_2(y))$$

is smooth.

Exercise 4.1.7. Let $f_1 \colon M \to N_1$ and $f_2 \colon M \to N_2$ be smooth maps. Show that the map

$$(f_1, f_2) \colon M \to N_1 \times N_2, \quad x \mapsto (f_1(x), f_2(x))$$

is smooth.

Exercise 4.1.8. Let N be an open subset of the smooth manifold M. Show that if $\mathcal{A} = (\varphi_i, U_i)_{i \in I}$ is a smooth atlas of M, $V_i := U_i \cap N$ and $\psi_i := \varphi_i|_{V_i}$, then $\mathcal{B} := (\psi_i, V_i)_{i \in I}$ is a smooth atlas of N.

Exercise 4.1.9. Let $V_1, \ldots, V_k$ and V be finite-dimensional real vector space and

$$\beta \colon V_1 \times \ldots \times V_k \to V$$

be a k-linear map. Show that β is smooth with

$$\mathtt{d}\beta(x_1, \ldots, x_k)(h_1, \ldots, h_k) = \sum_{j=1}^{k} \beta(x_1, \ldots, x_{j-1}, h_j, x_{j+1}, \ldots, x_k).$$

Exercise 4.1.10. Let M be a compact smooth manifold containing at least two points. Then each atlas of M contains at least two charts. In particular the atlas of $\mathbb{S}^n$ obtained from stereographic projections is minimal.

Exercise 4.1.11. Let X and Y be topological spaces and $q\colon X \to Y$ a quotient map, i.e., q is surjective and $O \subseteq Y$ is open if and only if $q^{-1}(O)$ is open in X. Show that a map $f\colon Y \to Z$ (Z a topological space) is continuous if and only if the map $f \circ q\colon X \to Z$ is continuous.

Exercise 4.1.12. Show that a function $f\colon \mathbb{R} \to \mathbb{R}$ is a diffeomorphism if and only if either

(1) $f' > 0$ and $\lim_{x\to\pm\infty} f(x) = \pm\infty$.

(2) $f' < 0$ and $\lim_{x\to\pm\infty} f(x) = \mp\infty$.

4.2 The Tangent Bundle

The real strength of the theory of smooth manifolds is due to the fact that it permits to analyze differentiable structures in terms of their derivatives. To model these derivatives appropriately, we introduce the tangent bundle TM of a smooth manifold, tangent maps of smooth maps and smooth vector fields.

We start with the definition of a tangent vector of a smooth manifold. The subtle point of this definition is that tangent vectors and the vector space structure can only be defined rather indirectly. The most straight forward way is to construct tangent vectors as "tangents" to smooth curves.

4.2.1 Tangent Vectors and Tangent Maps

Definition 4.2.1. Let M be a smooth manifold, $p \in M$ and (φ, U) a chart of M with $p \in U$. Let $\gamma\colon I \to M$ be a smooth curve, where $I \subseteq \mathbb{R}$ is an interval containing 0 and $\gamma(0) = p$. We call two such curves $\gamma_i\colon I_i \to M$, $i = 1, 2$, *equivalent*, denoted $\gamma_1 \sim \gamma_2$, if

$$(\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0).$$

Clearly, this defines an equivalence relation. The equivalence classes are called *tangent vectors in* p. We write $T_p(M)$ for the set of all tangent vectors in p and $[\gamma] \in T_p(M)$ for the equivalence class of the curve γ. The disjoint union

$$T(M) := \coprod_{p\in M} T_p(M)$$

is called the *tangent bundle of* M and we write $\pi_{TM}\colon TM \to M$ for the projection, mapping $T_p(M)$ to $\{p\}$.

Remark 4.2.2. (a) The equivalence relation defining tangent vectors does not depend on the chart (φ, U). If (ψ, V) is a second chart with $p \in V$ and $\gamma\colon I \to M$ a smooth curve with $\gamma(0) = p$, then

$$(\psi \circ \gamma)'(0) = \mathtt{d}(\psi \circ \varphi^{-1})(\varphi(p))(\varphi \circ \gamma)'(0),$$

so that we obtain the same equivalence relation on curves through p.

(b) If $U \subseteq \mathbb{R}^n$ is an open subset and $p \in U$, then each smooth curve $\gamma\colon I \to U$ with $\gamma(0) = p$ is equivalent to the curve $\eta_v(t) := p + tv$ for $v = \gamma'(0)$. Hence each equivalence class contains exactly one curve η_v. We may therefore think of a tangent vector in $p \in U$ as a vector $v \in \mathbb{R}^n$ attached to the point p, and the map

$$\mathbb{R}^n \to T_p(U), \quad v \mapsto [\eta_v]$$

is a bijection. In this sense, we identify all tangent spaces $T_p(U)$ with $\mathbb{R}^n$, so that we obtain a bijection

$$T(U) \cong U \times \mathbb{R}^n.$$

As an open subset of the product space $T(\mathbb{R}^n) \cong \mathbb{R}^{2n}$, the tangent bundle $T(U)$ inherits a natural manifold structure.

(c) For each $p \in M$ and any chart (φ, U) with $p \in U$, the map

$$T_p(\varphi)\colon T_p(M) \to \mathbb{R}^n, \quad [\gamma] \mapsto (\varphi \circ \gamma)'(0)$$

is well-defined and injective by the definition of the equivalence relation. Moreover, the curve

$$\gamma(t) := \varphi^{-1}(\varphi(p) + tv),$$

which is smooth and defined on some neighborhood of 0, satisfies $(\varphi \circ \gamma)'(0) = v$. Hence $T_p(\varphi)$ is a bijection.

Definition 4.2.3. Let M be an n-dimensional smooth manifold.

(a) Each tangent space $T_p(M)$ carries the unique structure of an n-dimensional vector space with the property that for each chart (φ, U) of M with $p \in U$, the map

$$T_p(\varphi)\colon T_p(M) \to \mathbb{R}^n, \quad [\gamma] \mapsto (\varphi \circ \gamma)'(0)$$

is a linear isomorphism.

In fact, since $T_p(\varphi)$ is a bijection, we may define a vector space structure on $T_p(M)$ by

$$v + w := T_p(\varphi)^{-1}(T_p(\varphi)v + T_p(\varphi)w) \quad \text{and} \quad \lambda v := T_p(\varphi)^{-1}(\lambda T_p(\varphi)v)$$

for $\lambda \in \mathbb{R}$, $v, w \in T_p(M)$. For any other chart (ψ, V) with $p \in V$ we then have

$$T_p(\psi) = \mathtt{d}(\psi \circ \varphi^{-1})(\varphi(p)) \circ T_p(\varphi),$$

and since $\mathtt{d}(\psi \circ \varphi^{-1})(\varphi(p))$ is a linear automorphism of $\mathbb{R}^n$, the vector space structure on $T_p(M)$ does not depend on the chart we use for its definition.

(b) If $f\colon M \to N$ is a smooth map and $p \in M$, then we obtain a linear map

$$T_p(f)\colon T_p(M) \to T_{f(p)}(N), \quad [\gamma] \mapsto [f \circ \gamma].$$

In fact, we only have to observe that for any chart (φ, U) of N with $f(p) \in U$ and any chart (ψ, V) of M with $p \in V$, we have

$$\begin{aligned} T_{f(p)}(\varphi)[f \circ \gamma] &= (\varphi \circ f \circ \gamma)'(0) = \mathtt{d}(\varphi \circ f \circ \psi^{-1})(\psi(p))(\psi \circ \gamma)'(0) \\ &= \mathtt{d}(\varphi \circ f \circ \psi^{-1})(\psi(p)) T_p(\psi)[\gamma]. \end{aligned}$$

This relation shows that $T_p(f)$ is well-defined, and a linear map.

The collection of all these maps defines a map

$$T(f)\colon T(M) \to T(N) \quad \text{with} \quad T_p(f) = T(f)|_{T_p(M)}, p \in M.$$

It is called the *tangent map of* f.

(c) If $M \subseteq \mathbb{R}$ is an open subset, then $f\colon M \to N$ is a smooth curve in N, and its tangent vector is $f'(t) := T_t(f)(1)$, where $1 \in T_t(\mathbb{R}) \cong \mathbb{R}$ is considered as a tangent vector.

(d) If V is a vector space, then we identify $T(V)$ in a natural way with $V \times V$. Accordingly we have

$$T_p(f)(v) = (f(p), \mathtt{d}f(p)v),$$

for a map $\mathtt{d}f\colon T(M) \to V$ with $\mathtt{d}f(p) := \mathtt{d}f|_{T_p(M)}$.

Remark 4.2.4. (a) For an open subset $U \subseteq \mathbb{R}^n$ and $p \in U$, the vector space structure on $T_p(U) = \{p\} \times \mathbb{R}^n$ is simply given by

$$(p, v) + (p, w) := (p, v + w) \quad \text{and} \quad \lambda(p, v) := (p, \lambda v)$$

for $v, w \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$.

(b) If $f\colon U \to V$ is a smooth map between open subsets $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$, $p \in U$, and $\eta_v(t) = p + tv$, then the tangent map satisfies

$$T(f)(p, v) = [f \circ \eta_v] = (f \circ \eta_v)'(0) = (f(p), \mathtt{d}f(p)\eta_v'(0)) = (f(p), \mathtt{d}f(p)v).$$

The main difference to the map $\mathtt{d}f$ is the book keeping; here we keep track of what happens to the point p and the tangent vector v. We may also write

$$T(f) = (f \circ \pi_{TU}, \mathtt{d}f)\colon TU \cong U \times \mathbb{R}^n \to TV \cong V \times \mathbb{R}^n,$$

where $\pi_{TU}\colon TU \to U, (p, v) \mapsto p$, is the projection map.

(c) If (φ, U) is a chart of M and $p \in U$, then we identify $T(\varphi(U))$ with $\varphi(U) \times \mathbb{R}^n$ and obtain for $[\gamma] \in T_p(M)$:

$$T(\varphi)([\gamma]) = (\varphi(p), [\varphi \circ \gamma]) = (\varphi(p), (\varphi \circ \gamma)'(0)),$$

which is consistent with our previously introduced notation $T_p(\varphi)$ (Remark 4.2.2).

Lemma 4.2.5. *[Chain Rule for Tangent Maps] For smooth maps $f\colon M \to N$ and $g\colon N \to L$, the tangent maps satisfy*

$$T(g \circ f) = T(g) \circ T(f).$$

Proof. We recall from Remark 4.1.11 that $g \circ f\colon M \to L$ is a smooth map, so that $T(g \circ f)$ is defined. For $p \in M$ and $[\gamma] \in T_p(M)$, we further have

$$T_p(g \circ f)[\gamma] = [g \circ f \circ \gamma] = T_{f(p)}(g)[f \circ \gamma] = T_{f(p)}(g)T_p(f)[\gamma].$$

Since p was arbitrary, this implies the lemma. $\square$

So far we only considered the tangent bundle $T(M)$ of a smooth manifold M as a set, but this set also carries a natural topology and a smooth manifold structure.

Definition 4.2.6. [Manifold structure on $T(M)$] Let M be a smooth manifold. First we introduce a topology on $T(M)$.

For each chart (φ, U) of M, we have a tangent map

$$T(\varphi)\colon T(U) \to T(\varphi(U)) \cong \varphi(U) \times \mathbb{R}^n,$$

where we consider $T(U) = \bigcup_{p\in U} T_p(M)$ as a subset of $T(M)$. We define a topology on $T(M)$ by declaring a subset $O \subseteq T(M)$ to be open if for each chart (φ, U) of M, the set $T(\varphi)(O \cap T(U))$ is an open subset of $T(\varphi(U))$. It is easy to see that this defines indeed a Hausdorff topology on $T(M)$ for which all the subsets $T(U)$ are open and the maps $T(\varphi)$ are homeomorphisms onto open subsets of $\mathbb{R}^{2n}$ (Exercise 4.2.1).

Since for two charts (φ, U), (ψ, V) of M, the map

$$T(\varphi \circ \psi^{-1}) = T(\varphi) \circ T(\psi)^{-1}\colon T(\psi(V)) \to T(\varphi(U))$$

is smooth, for each atlas $\mathcal{A}$ of M, the collection $(T(\varphi), T(U))_{(\varphi,U)\in\mathcal{A}}$ is a smooth atlas of $T(M)$. We thus obtain on $T(M)$ the structure of a smooth manifold.

Lemma 4.2.7. *If $f\colon M \to N$ is a smooth map, then its tangent map $T(f)$ is smooth.*

Proof. Let $p \in M$ and choose charts (φ, U) and (ψ, V) of M, resp., N with $p \in U$ and $f(p) \in V$. Then the map

$$T(\psi) \circ T(f) \circ T(\varphi)^{-1} = T(\psi \circ f \circ \varphi^{-1})\colon T(\varphi(f^{-1}(V) \cap U)) \to T(V)$$

is smooth, and this implies that $T(f)$ is a smooth map. □

Remark 4.2.8. For smooth manifolds $M_1, \ldots, M_n$, the projection maps

$$\pi_i\colon M_1 \times \cdots \times M_n \to M_i, \quad (p_1, \ldots, p_n) \mapsto p_i$$

induce a diffeomorphism

$$(T(\pi_1), \ldots, T(\pi_n))\colon T(M_1 \times \cdots \times M_n) \to TM_1 \times \cdots \times TM_n$$

(Exercise 4.2.2).

Exercises for Section 4.2

Exercise 4.2.1. Let M be a smooth manifold. We call a subset $O \subseteq T(M)$ open if for each chart (φ, U) of M, the set $T(\varphi)(O \cap T(U))$ is an open subset of $T(\varphi(U))$. Show that:

(1) This defines a topology on $T(M)$.

(2) All subsets $T(U)$ are open (Remark 4.2.4(b)).

(3) The maps $T(\varphi)\colon TU \to T(\varphi(U)) \cong \varphi(U) \times \mathbb{R}^n$ are homeomorphisms onto open subsets of $\mathbb{R}^{2n} \cong T(\mathbb{R}^n)$.

(4) The projection $\pi_{TM}\colon T(M) \to M$ is continuous.

(5) $T(M)$ is Hausdorff.

Exercise 4.2.2. For smooth manifolds $M_1, \ldots, M_n$, the projection maps

$$\pi_i\colon M_1 \times \cdots \times M_n \to M_i, \quad (p_1, \ldots, p_n) \mapsto p_i$$

induce a diffeomorphism

$$(T(\pi_1), \ldots, T(\pi_n))\colon T(M_1 \times \cdots \times M_n) \to TM_1 \times \cdots \times TM_n.$$

Exercise 4.2.3. Let N and $M_1, \ldots, M_n$ be a smooth manifolds. Show that a map

$$f\colon N \to M_1 \times \cdots \times M_n$$

is smooth if and only if all its component functions $f_i\colon N \to M_i$ are smooth.

Exercise 4.2.4. Let $f\colon M \to N$ be a smooth map between manifolds, $\pi_{TM}\colon TM \to M$ the tangent bundle projection and $\sigma_M\colon M \to TM$ the zero section. Show that for each smooth map $f\colon M \to N$ we have

$$\pi_{TN} \circ Tf = f \circ \pi_{TM} \quad \text{and} \quad \sigma_N \circ f = Tf \circ \sigma_M.$$

Exercise 4.2.5. [Inverse Function Theorem for manifolds] Let $f\colon M \to N$ be a smooth map and $p \in M$ such that $T_p(f)\colon T_p(M) \to T_{f(p)}(N)$ is a linear isomorphism. Show that there exists an open neighborhood U of p in M such that the restriction $f|_U\colon U \to f(U)$ is a diffeomorphism onto an open subset of N.

Exercise 4.2.6. Let $\mu\colon E \times F \to W$ be a bilinear map and M a smooth manifold. For $f \in C^\infty(M, E)$, $g \in C^\infty(M, F)$ and $p \in M$ set $h(p) := \mu\big(f(p), g(p)\big)$. Show that h is smooth with

$$T(h)v = \mu\big(T(f)v, g(p)\big) + \mu\big(f(p), T(g)v\big) \quad \text{for } v \in T_p(M).$$

4.3 Vector Fields

Vector fields are maps which associate with each point in a manifold a tangent vector at this point. They can be interpreted as a geometric way to formulate first order differential equations on a manifold, a point of view we will elaborate below. First we introduce the Lie algebra structure on the space of smooth vector fields.

4.3.1 The Lie Algebra of Vector Fields

Definition 4.3.1. (a) Let $\pi_{TM} \colon TM \to M$ denote the canonical projection mapping $T_p(M)$ to p. A *(smooth) vector field* X on M is a smooth section of the tangent bundle TM, i.e., a smooth map $X \colon M \to TM$ with $\pi_{TM} \circ X = \mathrm{id}_M$. We write $\mathcal{V}(M)$ for the space of all vector fields on M.

(b) If $f \in C^\infty(M, V)$ is a smooth function on M with values in some finite-dimensional vector space V and $X \in \mathcal{V}(M)$, then we obtain a smooth function on M via

$$\mathcal{L}_X f := \mathtt{d}f \circ X \colon M \to TM \to V.$$

We thus obtain for each $X \in \mathcal{V}(M)$ a linear operator $\mathcal{L}_X$ on $C^\infty(M, V)$. The function $\mathcal{L}_X f$ is also called the *Lie derivative* of f with respect to X.

Remark 4.3.2. (a) If U is an open subset of $\mathbb{R}^n$, then $TU = U \times \mathbb{R}^n$ with the bundle projection

$$\pi_{TU} \colon U \times \mathbb{R}^n \to U, \quad (x, v) \mapsto x.$$

Therefore each smooth vector field is of the form $X(x) = (x, \widetilde{X}(x))$ for some smooth function $\widetilde{X} \colon U \to \mathbb{R}^n$, and we may thus identify $\mathcal{V}(U)$ with the space $C^\infty(U, \mathbb{R}^n)$ of smooth $\mathbb{R}^n$-valued functions on U.

(b) The space $\mathcal{V}(M)$ carries a natural vector space structure given by

$$(X + Y)(p) := X(p) + Y(p), \quad (\lambda X)(p) := \lambda X(p).$$

More generally, we can multiply vector fields with smooth functions

$$(fX)(p) := f(p)X(p), \quad f \in C^\infty(M, \mathbb{R}), X \in \mathcal{V}(M).$$

Before we turn to the Lie bracket on the space $\mathcal{V}(M)$ of smooth vector fields on a manifold M, we take a closer look at the local level.

Lemma 4.3.3. *Let $U \subseteq \mathbb{R}^n$ be an open subset. Then we obtain a Lie bracket on the space $C^\infty(U, \mathbb{R}^n)$ by*

$$[X, Y](p) := \mathtt{d}Y(p)X(p) - \mathtt{d}X(p)Y(p) \quad \textit{for} \quad p \in U.$$

With respect to this Lie bracket, the map

$$\mathcal{L} \colon C^\infty(U, \mathbb{R}^n) \to \mathrm{End}(C^\infty(U, \mathbb{R})), \quad X \mapsto \mathcal{L}_X, \quad \mathcal{L}_X(f)(p) := \mathtt{d}f(p)X(p)$$

is an injective homomorphism of Lie algebras, i.e., $\mathcal{L}_{[X,Y]} = [\mathcal{L}_X, \mathcal{L}_Y]$.

Proof. (L1) is obvious from the definition. To verify the Jacobi identity, we first observe that the map $X \mapsto \mathcal{L}_X$ is injective. In fact, if $\mathcal{L}_X = 0$, then we have for each linear function $f \colon \mathbb{R}^n \to \mathbb{R}$ the relation $0 = (\mathcal{L}_X f)(p) = \mathtt{d}f(p)X(p) = f(X(p))$, and therefore $X(p) = 0$.

Next we observe that

$$\begin{aligned}\mathcal{L}_X\mathcal{L}_Y(f)(p) &= \mathtt{d}(\mathcal{L}_Y f)(p)X(p) = \mathtt{d}(\mathtt{d}f \circ Y)(p)X(p) \\ &= (\mathtt{d}^2 f)(p)(X(p), Y(p)) + \mathtt{d}f(p)\mathtt{d}Y(p)X(p),\end{aligned}$$

so that the Schwarz Lemma implies $\mathcal{L}_X\mathcal{L}_Y f - \mathcal{L}_Y\mathcal{L}_X f = \mathcal{L}_{[X,Y]}f$. Clearly, the bracket on $\mathcal{V}(U)$ is skew-symmetric. That it also satisfies the Jacobi identity follows from the injectivity of $\mathcal{L}$, the Jacobi identity in $\mathrm{End}(C^\infty(U,\mathbb{R}))$ (Lemma 3.1.2) and $\mathcal{L}_{[X,Y]} = [\mathcal{L}_X, \mathcal{L}_Y]$:

$$\mathcal{L}_{[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]} = [\mathcal{L}_X,[\mathcal{L}_Y,\mathcal{L}_Z]] + [\mathcal{L}_Y,[\mathcal{L}_Z,\mathcal{L}_X]] + [\mathcal{L}_Z,[\mathcal{L}_X,\mathcal{L}_Y]]. \quad \square$$

Remark 4.3.4. For any open subset $U \subseteq \mathbb{R}^n$, the map

$$\mathcal{V}(U) \to C^\infty(U,\mathbb{R}^n), \quad X \to \widetilde{X}$$

with $X(p) = (p, \widetilde{X}(p))$ is a linear isomorphism. We use this map to transfer the Lie bracket on $C^\infty(U,\mathbb{R}^n)$, defined in Lemma 4.3.3, to a Lie bracket on $\mathcal{V}(U)$, determined by

$$[X,Y]\widetilde{\ }(p) := [\widetilde{X},\widetilde{Y}](p) = \mathtt{d}\widetilde{Y}(p)\widetilde{X}(p) - \mathtt{d}\widetilde{X}(p)\widetilde{Y}(p).$$

Our goal is to use the Lie brackets on the space $\mathcal{V}(U)$ and local charts to define a Lie bracket on $\mathcal{V}(M)$. The following lemma will be needed to ensure consistency in this process.

First, we introduce the concept or related vector fields. If $\varphi\colon M \to N$ is a smooth map, then we call two vector fields $X' \in \mathcal{V}(M)$ and $X \in \mathcal{V}(N)$ *φ-related* if

$$X \circ \varphi = T\varphi \circ X' \colon M \to TN. \tag{4.2}$$

With respect to the pullback map

$$\varphi^*\colon C^\infty(N,\mathbb{R}) \to C^\infty(M,\mathbb{R}), \quad f \mapsto f\circ\varphi,$$

the φ-relatedness of X and X' implies that

$$\mathcal{L}_{X'}(\varphi^* f) = \mathcal{L}_{X'}(f\circ\varphi) = \mathtt{d}(f\circ\varphi)\circ X' = \mathtt{d}f\circ T\varphi\circ X' = \mathtt{d}f\circ X\circ\varphi = \varphi^*(\mathcal{L}_X f),$$

i.e.,

$$\mathcal{L}_{X'}\circ\varphi^* = \varphi^*\circ\mathcal{L}_X. \tag{4.3}$$

Lemma 4.3.5. *Let $M \subseteq \mathbb{R}^n$ and $N \subseteq \mathbb{R}^m$ be open subsets. Suppose that X', resp., $Y' \in \mathcal{V}(M)$ is φ-related to X, resp., $Y \in \mathcal{V}(N)$. Then $[X',Y']$ is φ-related to $[X,Y]$.*

Proof. In view of (4.3), we have

$$\mathcal{L}_{X'}\circ\varphi^* = \varphi^*\circ\mathcal{L}_X \quad \text{and} \quad \mathcal{L}_{Y'}\circ\varphi^* = \varphi^*\circ\mathcal{L}_Y$$

as linear maps $C^\infty(N,\mathbb{R}) \to C^\infty(M,\mathbb{R})$. Therefore

$$\begin{aligned}[\mathcal{L}_{X'},\mathcal{L}_{Y'}]\circ\varphi^* &= \mathcal{L}_{X'}\circ\mathcal{L}_{Y'}\circ\varphi^* - \mathcal{L}_{Y'}\circ\mathcal{L}_{X'}\circ\varphi^* \\ &= \varphi^*\circ\mathcal{L}_X\circ\mathcal{L}_Y - \varphi^*\circ\mathcal{L}_Y\circ\mathcal{L}_X = \varphi^*\circ[\mathcal{L}_X,\mathcal{L}_Y].\end{aligned}$$

For any $f \in C^\infty(N,\mathbb{R})$, we thus obtain

$$\mathtt{d}f\circ T\varphi\circ[X',Y'] = \mathcal{L}_{[X',Y']}(f\circ\varphi) = (\mathcal{L}_{[X,Y]}f)\circ\varphi = \mathtt{d}f\circ[X,Y]\circ\varphi.$$

Since each linear functional on the space $T_x(N) \cong \mathbb{R}^m$ is of the form $\mathtt{d}f(x)$ for some linear map $f\colon \mathbb{R}^m \to \mathbb{R}$, the assertion follows. $\square$

Proposition 4.3.6. *For a vector field $X \in \mathcal{V}(M)$ and a chart (φ, U) of M, we write $X_\varphi := T\varphi \circ X \circ \varphi^{-1}$ for the corresponding vector field on the open subset $\varphi(U) \subseteq \mathbb{R}^n$.*

For $X, Y \in \mathcal{V}(M)$, there exists a vector field $[X, Y] \in \mathcal{V}(M)$ which is uniquely determined by the property that for each chart (φ, U) of M, the following equation holds

$$[X, Y]_\varphi = [X_\varphi, Y_\varphi]. \tag{4.4}$$

Proof. If (φ, U) and (ψ, V) are charts of M, the vector fields X_φ on $\varphi(U)$ and X_ψ on $\psi(V)$ are $(\psi \circ \varphi^{-1})$-related. Therefore Lemma 4.3.5 implies that $[X_\varphi, Y_\varphi]$ is $(\psi \circ \varphi^{-1})$-related to $[X_\psi, Y_\psi]$. This in turn is equivalent to

$$T(\varphi)^{-1} \circ [X_\varphi, Y_\varphi] \circ \varphi = T(\psi)^{-1} \circ [X_\psi, Y_\psi] \circ \psi,$$

which is an identity of vector fields on the open subset $U \cap V$.

Hence there exists a unique vector field $[X, Y] \in \mathcal{V}(M)$, satisfying

$$[X, Y]|_U = T(\varphi)^{-1} \circ [X_\varphi, Y_\varphi] \circ \varphi$$

for each chart (φ, U), i.e., $[X, Y]_\varphi = [X_\varphi, Y_\varphi]$ on $\varphi(U)$. ☐

Lemma 4.3.7. *For $f \in C^\infty(M, \mathbb{R})$ and $X, Y \in \mathcal{V}(M)$, the following equation holds*

$$\mathcal{L}_{[X,Y]} f = \mathcal{L}_X(\mathcal{L}_Y f) - \mathcal{L}_Y(\mathcal{L}_X f).$$

Proof. It suffices to show that this relation holds on U for any chart (φ, U) of M. For $f_\varphi := f \circ \varphi^{-1}$, we then obtain with (4.3)

$$\begin{aligned}
\mathcal{L}_{[X,Y]} f &= \mathtt{d}f \circ [X, Y] = \mathtt{d}f \circ T(\varphi^{-1}) \circ [X, Y]_\varphi \circ \varphi \\
&= \mathtt{d}f_\varphi \circ [X_\varphi, Y_\varphi] \circ \varphi = \varphi^*\big(\mathcal{L}_{[X_\varphi, Y_\varphi]} f_\varphi\big) \\
&= \varphi^*\big(\mathcal{L}_{(X_\varphi)} \mathcal{L}_{(Y_\varphi)} f_\varphi - \mathcal{L}_{(Y_\varphi)} \mathcal{L}_{(X_\varphi)} f_\varphi\big) \\
&= \mathcal{L}_X(\mathcal{L}_Y f) - \mathcal{L}_Y(\mathcal{L}_X f),
\end{aligned}$$

because $\varphi^* f_\varphi = f$. ☐

Proposition 4.3.8. $(\mathcal{V}(M), [\cdot,\cdot])$ *is a Lie algebra.*

Proof. Clearly (L1) is satisfied. To verify the Jacobi identity, let $X, Y, Z \in \mathcal{V}(M)$ and (φ, U) be a chart of M. For the vector field $J(X, Y, Z) := \sum_{\text{cyc.}} [X, [Y, Z]] \in \mathcal{V}(M)$ we then obtain from the definition of the bracket, Remark 4.3.4 and Proposition 4.3.6:

$$J(X, Y, Z)_\varphi = J(X_\varphi, Y_\varphi, Z_\varphi) = 0$$

because $[\cdot,\cdot]$ is a Lie bracket on $\mathcal{V}(\varphi(U))$. This means that $J(X, Y, Z)$ vanishes on U, but since the chart (φ, U) was arbitrary, $J(X, Y, Z) = 0$. ☐

We shall see later that the following lemma is an extremely important tool.

Lemma 4.3.9. *[Related Vector Field Lemma] Let M and N be smooth manifolds, $\varphi\colon M \to N$ a smooth map, $X, Y \in \mathcal{V}(N)$ and $X', Y' \in \mathcal{V}(M)$. If X' is φ-related to X and Y' is φ-related to Y, then the Lie bracket $[X', Y']$ is φ-related to $[X, Y]$.*

Proof. We have to show that for each $p \in M$ we have

$$[X,Y](\varphi(p)) = T_p(\varphi)[X',Y'](p).$$

Let (ρ, U) be a chart of M with $p \in U$ and (ψ, V) a chart of N with $\varphi(p) \in V$. Then the vector fields X'_ρ and X_ψ are $\psi \circ \varphi \circ \rho^{-1}$-related on the domain $\rho(\varphi^{-1}(V) \cap U)$:

$$\begin{aligned}
&T(\psi \circ \varphi \circ \rho^{-1})X'_\rho = T(\psi \circ \varphi \circ \rho^{-1})(T(\rho) \circ X' \circ \rho^{-1}) \\
&= T(\psi) \circ T(\varphi) \circ X' \circ \rho^{-1} = T(\psi) \circ X \circ \varphi \circ \rho^{-1} = X_\psi \circ (\psi \circ \varphi \circ \rho^{-1}),
\end{aligned}$$

and the same holds for the vector fields Y'_ρ and Y_ψ, hence for their Lie brackets (Lemma 4.3.5).

Now the definition of the Lie bracket on $\mathcal{V}(N)$ and $\mathcal{V}(M)$ implies that

$$\begin{aligned}
&T(\psi) \circ T(\varphi) \circ [X',Y'] = T(\psi \circ \varphi \circ \rho^{-1}) \circ [X',Y']_\rho \circ \rho \\
&= T(\psi \circ \varphi \circ \rho^{-1}) \circ [X'_\rho, Y'_\rho] \circ \rho = [X_\psi, Y_\psi] \circ \psi \circ \varphi \circ \rho^{-1} \circ \rho \\
&= [X_\psi, Y_\psi] \circ \psi \circ \varphi = [X,Y]_\psi \circ \psi \circ \varphi = T(\psi) \circ [X,Y] \circ \varphi,
\end{aligned}$$

and since $T(\psi)$ is injective, the assertion follows. □

Example 4.3.10. Let (φ, U) be a chart of M and $x_1, \ldots, x_n \colon U \to \mathbb{R}$ the corresponding coordinate functions. Then we obtain on U the vector fields X_j, $j = 1, \ldots, n$, defined by

$$X_j(p) := T_p(\varphi)^{-1} e_j := \frac{\partial}{\partial x_j}(p) := \frac{\partial}{\partial x_j}\Big|_p,$$

where $e_1, \ldots, e_n$ is the standard basis for $\mathbb{R}^n$. We call these vector fields the φ-*basic vector fields* on U. The expression basic vector field is doubly justified. On the one hand, $\big(X_1(p), \ldots, X_n(p)\big)$ is a basis for $T_p(M)$ for every $p \in U$. On the other hand, the definition shows that every $X \in \mathcal{V}(U)$ can be written as

$$X = \sum_{j=1}^n a_j \cdot X_j \quad \text{for} \quad a_j \in C^\infty(U).$$

Since basic vector fields are φ-related with the constant vector fields on $\mathbb{R}^n$, they commute (Related Vector Field Lemma 4.3.9), i.e., $[X_j, X_k] = 0$.

Exercises for Section 4.3

Exercise 4.3.1. Let M be a smooth manifold, $X, Y \in \mathcal{V}(M)$ and $f, g \in C^\infty(M, \mathbb{R})$. Show that

(1) $\mathcal{L}_X(f \cdot g) = \mathcal{L}_X(f) \cdot g + f \cdot \mathcal{L}_X(g)$, i.e., the map $f \mapsto \mathcal{L}_X(f)$ is a derivation.

(2) $\mathcal{L}_{fX}(g) = f \cdot \mathcal{L}_X(g)$.

Exercise 4.3.2. Let $\mathcal{A}$ be a $\mathbb{K}$-algebra (not necessarily associative). Show that

(i) $\mathrm{der}(\mathcal{A}) := \{D \in \mathrm{End}(\mathcal{A})\colon (\forall a, b \in \mathcal{A})\, D(ab) = Da \cdot b + a \cdot Db\}$ is a Lie subalgebra of $\mathfrak{gl}(\mathcal{A}) = \mathrm{End}(\mathcal{A})_L$.

(ii) If, in addition, $\mathcal{A}$ is commutative, then for $D \in \mathrm{der}(\mathcal{A})$ and $a \in \mathcal{A}$, the map $aD\colon \mathcal{A} \to \mathcal{A}, x \mapsto aDx$ also is a derivation.

Exercise 4.3.3. Let U be an open subset of $\mathbb{R}^{2n}$ and $\mathcal{P} = C^\infty(U, \mathbb{R})$ be the set of smooth functions on U and write $q_1, \ldots, q_m, p_1, \ldots, p_m$ for the coordinates with respect to a basis. Then $\mathfrak{g}$ is a Lie algebra with respect to the *Poisson bracket*

$$\{f, g\} := \sum_{i=1}^{n} \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i}.$$

Exercise 4.3.4. To each $A \in \mathfrak{gl}_n(\mathbb{R})$, we associate the linear vector field $X_A(x) := Ax$ on $\mathbb{R}^n$ Show that, for $A, B \in M_n(\mathbb{R})$, we have $X_{[A,B]} = -[X_A, X_B]$.

4.4 Integral Curves and Local Flows

In this section we turn to the geometric nature of vector fields as infinitesimal generators of local flows on manifolds. This provides a natural perspective on (autonomous) ordinary differential equations.

4.4.1 Integral Curves

Throughout this subsection M denotes an n-dimensional smooth manifold.

Definition 4.4.1. Let $X \in \mathcal{V}(M)$ and $I \subseteq \mathbb{R}$ an open interval containing 0. A differentiable map $\gamma\colon I \to M$ is called an *integral curve* of X if

$$\gamma'(t) = X(\gamma(t)) \qquad \text{for each} \quad t \in I.$$

Note that the preceding equation implies that γ' is continuous and further that if γ is C^k, then γ' is also C^k. Therefore integral curves are automatically smooth.

If $J \supseteq I$ is an interval containing I, then an integral curve $\eta\colon J \to M$ is called an *extension* of γ if $\eta|_I = \gamma$. An integral curve γ is said to be *maximal* if it has no proper extension.

Remark 4.4.2. (a) If $U \subseteq \mathbb{R}^n$ is an open subset of $\mathbb{R}^n$, then we write a vector field $X \in \mathcal{V}(U)$ as $X(x) = (x, F(x))$, where $F\colon U \to \mathbb{R}^n$ is a smooth function. A curve $\gamma\colon I \to U$ is an integral curve of X if and only if it satisfies the ordinary differential equation

$$\gamma'(t) = F(\gamma(t)) \quad \text{for all} \quad t \in I.$$

(b) If (φ, U) is a chart of the manifold M and $X \in \mathcal{V}(M)$, then a curve $\gamma\colon I \to M$ is an integral curve of X if and only if the curve $\eta := \varphi \circ \gamma$ is an integral curve of the vector field $X_\varphi := T(\varphi) \circ X \circ \varphi^{-1} \in \mathcal{V}(\varphi(U))$ because

$$X_\varphi(\eta(t)) = T_{\gamma(t)}(\varphi) X(\gamma(t)) \quad \text{and} \quad \eta'(t) = T_{\gamma(t)}(\varphi)\gamma'(t).$$

Remark 4.4.3. A curve $\gamma\colon I \to M$ is an integral curve of X if and only if $\widetilde{\gamma}(t) := \gamma(-t)$ is an integral curve of the vector field $-X$.

More generally, for $a, b \in \mathbb{R}$, the curve $\eta(t) := \gamma(at + b)$ is an integral curve of the vector field aX.

Definition 4.4.4. Let $a < b \in [-\infty, \infty]$. For a continuous curve $\gamma\colon]a, b[\to M$ we say that

$$\lim_{t\to b} \gamma(t) = \infty$$

if for each compact subset $K \subseteq M$ there exists a $c < b$ with $\gamma(t) \notin K$ for $t > c$. Similarly, we define

$$\lim_{t\to a} \gamma(t) = \infty.$$

Theorem 4.4.5. *[Existence and Uniqueness of Integral Curves] Let $X \in \mathcal{V}(M)$ and $p \in M$. Then there exists a unique maximal integral curve $\gamma_p\colon I_p \to M$ with $\gamma_p(0) = p$. If $a := \inf I_p > -\infty$, then $\lim_{t\to a} \gamma_p(t) = \infty$ and if $b := \sup I_p < \infty$, then $\lim_{t\to b} \gamma_p(t) = \infty$.*

Proof. We have seen in Remark 4.4.2 that in local charts, integral curves are solutions of an ordinary differential equation with a smooth right hand side. We now reduce the proof to the Local Existence- and Uniqueness Theorem for ODE's.

Uniqueness: Let $\gamma, \eta\colon I \to M$ be two integral curves of X with $\gamma(0) = \eta(0) = p$. The continuity of the curves implies that

$$0 \in J := \{t \in I\colon \gamma(t) = \eta(t)\}$$

is a closed subset of I. In view of the Local Uniqueness Theorem for ODE's, for each $t_0 \in J$ there exists an $\varepsilon > 0$ with $[t_0, t_0 + \varepsilon] \subseteq J$, and likewise $[t_0 - \varepsilon, t_0] \subseteq J$ (Remark 4.4.3). Therefore J is also open. Now the connectedness of I implies $I = J$, so that $\gamma = \eta$.

Existence: The Local Existence Theorem implies the existence of some integral curve $\gamma\colon I \to M$ on some open interval containing 0. For any other integral curve $\eta\colon J \to M$, the intersection $I \cap J$ is an interval containing 0, so that the uniqueness assertion implies that $\eta = \gamma$ on $I \cap J$.

Let $I_p \subseteq \mathbb{R}$ be the union of all open intervals I_j containing 0 on which there exists an integral curve $\gamma_j\colon I_j \to M$ of X with $\gamma_j(0) = p$. Then the preceding argument shows that

$$\gamma(t) := \gamma_j(t) \quad \text{for} \quad t \in I_j$$

defines an integral curve of X on I_p, which is maximal by definition. The uniqueness of the maximal integral curve also follows from its definition.

Limit condition: Suppose that $b := \sup I_p < \infty$. If $\lim_{t\to b} \gamma(t) = \infty$ does not hold, then there exists a compact subset $K \subseteq M$ and a sequence $t_m \in I_p$ with $t_m \to b$ and $\gamma(t_m) \in K$. As K can be covered with finitely many closed subsets homeomorphic to a closed subset of a ball in $\mathbb{R}^n$, after passing to a suitable subsequence, we may w.l.o.g. assume that K itself is homeomorphic to a compact subset of $\mathbb{R}^n$. Then a subsequence of $(\gamma(t_m))_{m\in\mathbb{N}}$ converges, and we may replace the original sequence by this subsequence, hence assume that $q := \lim_{m\to\infty} \gamma(t_m)$ exists.

The Local Existence Theorem for ODE's implies the existence of a compact neighborhood $V \subseteq M$ of q and $\varepsilon > 0$ such that the initial value problem

$$\eta(0) = x, \quad \eta' = X \circ \eta$$

has a solution on $[-\varepsilon, \varepsilon]$ for each $x \in V$. Pick $m \in \mathbb{N}$ with $t_m > b - \varepsilon$ and $\gamma(t_m) \in V$. Further let $\eta \colon [-\varepsilon, \varepsilon] \to M$ be an integral curve with $\eta(0) = \gamma(t_m)$. Then

$$\gamma(t) := \eta(t - t_m) \quad \text{for} \quad t \in [t_m - \varepsilon, t_m + \varepsilon],$$

defines an extension of γ to the interval $I_p \cup]t_m, t_m + \varepsilon[$ strictly containing $]a, b[$, hence contradicting the maximality of I_p. This proves that $\lim_{t \to b} \gamma(t) = \infty$. Replacing X by $-X$, we also obtain $\lim_{t \to a} \gamma(t) = \infty$. $\square$

If $q = \gamma_p(t)$ is a point on the unique maximal integral curve of X through $p \in M$, then $I_q = I_p - t$ and

$$\gamma_q(s) := \gamma_p(t + s)$$

is the unique maximal integral curve through q. Here I_p is the domain of definition of the maximal integral curve through p and I_q is the domain of definition of the maximal integral curve through q.

Example 4.4.6. (a) On $M = \mathbb{R}$ we consider the vector field X given by the function $F(s) = 1 + s^2$, i.e., $X(s) = (s, 1 + s^2)$. The corresponding ODE is

$$\gamma'(s) = X(\gamma(s)) = 1 + \gamma(s)^2.$$

For
$\gamma(0) = 0$ the function $\gamma(s) := \tan(s)$ on $I :=] - \frac{\pi}{2}, \frac{\pi}{2}[$ is the unique maximal solution because

$$\lim_{t \to \frac{\pi}{2}} \tan(t) = \infty \quad \text{and} \quad \lim_{t \to -\frac{\pi}{2}} \tan(t) = -\infty.$$

(b) Let $M :=]-1, 1[$ and $X(s) = (s, 1)$, so that the corresponding ODE is $\gamma'(s) = 1$. Then the unique maximal solution is

$$\gamma(s) = s, \quad I =] - 1, 1[.$$

Note that we also have in this case

$$\lim_{s \to \pm 1} \gamma(s) = \infty$$

if we consider γ as a curve in the noncompact manifold M.

For $M = \mathbb{R}$ the same vector field has the maximal integral curve

$$\gamma(s) = s, \quad I = \mathbb{R}.$$

(c) For $M = \mathbb{R}$ and $X(s) = (s, -s)$, the differential equation is $\gamma'(t) = -\gamma(t)$, so that we obtain the maximal integral curves $\gamma(t) = \gamma_0 e^{-t}$. For $\gamma_0 = 0$ this curve is constant, and for $\gamma_0 \neq 0$ we have $\lim_{t \to \infty} \gamma(t) = 0$, hence $\lim_{t \to \infty} \gamma(t) \neq \infty$. This shows that maximal integral curves do not always leave every compact subset of M if they are defined on an interval unbounded from above.

The preceding example shows in particular that the global existence of integral curves can also be destroyed by deleting parts of the manifold M, i.e., by considering $M' := M \setminus K$ for some closed subset $K \subseteq M$.

Definition 4.4.7. A vector field $X \in \mathcal{V}(M)$ is said to be *complete* if all its maximal integral curves are defined on all of $\mathbb{R}$.

Corollary 4.4.8. *All vector fields on a compact manifold M are complete.*

4.4.2 Local Flows

Definition 4.4.9. Let M be a smooth manifold. A *local flow on M* is a smooth map

$$\Phi\colon U \to M,$$

where $U \subseteq \mathbb{R} \times M$ is an open subset containing $\{0\} \times M$, such that for each $x \in M$ the intersection $I_x := U \cap (\mathbb{R} \times \{x\})$ is an interval containing 0 and

$$\Phi(0,x) = x \quad \text{and} \quad \Phi\big(t, \Phi(s,x)\big) = \Phi(t+s,x)$$

hold for all t, s, x for which both sides are defined. The maps

$$\alpha_x\colon I_x \to M, \qquad t \mapsto \Phi(t,x)$$

are called the *flow lines*. The flow Φ is said to be *global* if $U = \mathbb{R} \times M$.

Lemma 4.4.10. *If $\Phi\colon U \to M$ is a local flow, then*

$$X^\Phi(x) := \frac{d}{dt}\Big|_{t=0} \Phi(t,x) = \alpha_x'(0)$$

defines a smooth vector field.

It is called the *velocity field* or the *infinitesimal generator* of the local flow Φ.

Lemma 4.4.11. *If $\Phi\colon U \to M$ is a local flow on M, then the flow lines are integral curves of the vector field X^Φ. In particular, the local flow Φ is uniquely determined by the vector field X^Φ.*

Proof. Let $\alpha_x\colon I_x \to M$ be a flow line and $s \in I_x$. For sufficiently small $t \in \mathbb{R}$ we then have

$$\alpha_x(s+t) = \Phi(s+t,x) = \Phi\big(t, \Phi(s,x)\big) = \Phi\big(t, \alpha_x(s)\big),$$

so that taking derivatives in $t = 0$ leads to $\alpha_x'(s) = X^\Phi(\alpha_x(s))$.

That Φ is uniquely determined by the vector field X^Φ follows from the uniqueness of integral curves (Theorem 4.4.5). □

Theorem 4.4.12. *Each smooth vector field X is the velocity field of a unique local flow defined by*

$$\mathcal{D}_X := \bigcup_{x \in M} I_x \times \{x\} \quad \text{and} \quad \Phi(t,x) := \gamma_x(t) \quad \text{for} \quad (t,x) \in \mathcal{D}_X,$$

where $\gamma_x\colon I_x \to M$ is the unique maximal integral curve through $x \in M$.

Proof. If $(s,x), \big(t,\Phi(s,x)\big)$ and $(s+t,x) \in \mathcal{D}_X$, the relation

$$\Phi(s+t,x) = \Phi\big(t,\Phi(s,x)\big) \quad \text{and} \quad I_{\Phi(s,x)} = I_{\gamma_x(s)} = I_x - s$$

follow from the fact that both curves

$$t \mapsto \Phi(t+s,x) = \gamma_x(t+s) \quad \text{and} \quad t \mapsto \Phi\big(t,\Phi(s,x)\big) = \gamma_{\Phi(s,x)}(t)$$

are integral curves of X with the initial value $\Phi(s,x)$, hence coincide.

We claim that all maps

$$\Phi_t \colon M_t := \{x \in M \colon (t,x) \in \mathcal{D}_X\} \to M, \quad x \mapsto \Phi(t,x)$$

are injective. In fact, if $p := \Phi_t(x) = \Phi_t(y)$, then $\gamma_x(t) = \gamma_y(t)$, and on $[0,t]$ the curves $s \mapsto \gamma_x(t-s), \gamma_y(t-s)$ are integral curves of $-X$, starting in p. Hence the Uniqueness Theorem 4.4.5 implies that they coincide in $s = t$, which mans that $x = \gamma_x(0) = \gamma_y(0) = y$. From this argument it further follows that $\Phi_t(M_t) = M_{-t}$ and $\Phi_t^{-1} = \Phi_{-t}$.

It remains to show that $\mathcal{D}_X$ is open and Φ smooth. The local Existence Theorem provides for each $x \in M$ an open neighborhood U_x diffeomorphic to a cube and some $\varepsilon_x > 0$, as well as a smooth map

$$\varphi_x \colon]-\varepsilon_x, \varepsilon_x[\times U_x \to M, \quad \varphi_x(t,y) = \gamma_y(t) = \Phi(t,y).$$

Hence $]-\varepsilon_x,\varepsilon_x[\times U_x \subseteq \mathcal{D}_X$, and the restriction of Φ to this set is smooth. Therefore Φ is smooth on a neighborhood of $\{0\} \times M$ in $\mathcal{D}_X$.

Now let J_x be the set of all $t \in [0,\infty[$, for which $\mathcal{D}_X$ contains a neighborhood of $[0,t] \times \{x\}$ on which Φ is smooth. The interval J_x is open in $\mathbb{R}^+ := [0,\infty[$ by definition. We claim that $J_x = I_x \cap \mathbb{R}^+$. This entails that $\mathcal{D}_X$ is open because the same argument applies to $I_x \cap]-\infty,0]$.

We assume the contrary and find a minimal $\tau \in I_x \cap \mathbb{R}^+ \setminus J_x$, because this interval is closed. Put $p := \Phi(\tau,x)$ and pick a product set $I \times W \subseteq \mathcal{D}_X$, where W is an open neighborhood of p and $I =]-2\varepsilon, 2\varepsilon[$ a 0-neighborhood, such that $2\varepsilon < \tau$ and $\Phi : I \times W \to M$ is smooth. By assumption, there exists an open neighborhood V of x such that Φ is smooth on $[0,\tau-\varepsilon] \times V \subseteq \mathcal{D}_X$. Then $\Phi_{\tau-\varepsilon}$ is smooth on V and

$$V' := \Phi_{\tau-\varepsilon}^{-1}\big(\Phi_\varepsilon^{-1}(W)\big) \cap V$$

is a neighborhood of x. Further,

$$V' = \Phi_{\tau-\varepsilon}^{-1}\big(\Phi_\varepsilon^{-1}(W)\big) \cap V = \Phi_\tau^{-1}(W) \cap V,$$

and Φ is smooth on $]\tau-2\varepsilon, \tau+2\varepsilon[\times V'$, because it is a composition of smooth maps:

$$]\tau-2\varepsilon,\tau+2\varepsilon[\times V' \to M, \quad (t,y) \mapsto \Phi\big(t-\tau, \Phi(\varepsilon, \Phi(\tau-\varepsilon,y))\big).$$

We thus arrive at the contradiction $\tau \in J_x$.

This completes the proof of the openness of $\mathcal{D}_X$ and the smoothness of Φ. The uniqueness of the flow follows from the uniqueness of the integral curves. $\square$

Remark 4.4.13. Let $X \in \mathcal{V}(M)$ be a complete vector field. If

$$\Phi^X : \mathbb{R} \times M \to M$$

is the corresponding global flow, then the maps $\Phi_t^X : x \mapsto \Phi^X(t,x)$ satisfy

(A1) $\Phi_0^X = \mathrm{id}_M$.

(A2) $\Phi_{t+s}^X = \Phi_t^X \circ \Phi_s^X$ for $t, s \in \mathbb{R}$.

It follows in particular that $\Phi_t^X \in \mathrm{Diff}(M)$ with $(\Phi_t^X)^{-1} = \Phi_{-t}^X$, so that we obtain a group homomorphism

$$\gamma_X : \mathbb{R} \to \mathrm{Diff}(M), \quad t \mapsto \Phi_t^X.$$

With respect to the terminology introduced below, (A1) and (A2) mean that Φ^X defines a *smooth action of* $\mathbb{R}$ *on* M. As Φ^X is determined by the vector field X, we call X the infinitesimal generator of this action. In this sense the smooth $\mathbb{R}$-actions on a manifold M are in one-to-one correspondence with the complete vector fields on M.

Remark 4.4.14. Let $\Phi^X : \mathcal{D}_X \to M$ be the maximal local flow of a vector field X on M. Let $M_t = \{x \in M : (t,x) \in \mathcal{D}_X\}$, and observe that this is an open subset of M. We have already seen in the proof of Theorem 4.4.12 above, that all the smooth maps $\Phi_t^X : M_t \to M$ are injective with $\Phi_t^X(M_t) = M_{-t}$ and $(\Phi_t^X)^{-1} = \Phi_{-t}^X$ on the image. It follows in particular, that $\Phi_t^X(M_t) = M_{-t}$ is open, and that

$$\Phi_t^X : M_t \to M_{-t}$$

is a diffeomorphism whose inverse is Φ_{-t}^X.

Proposition 4.4.15. (Smooth Dependence Theorem) *Let M and Λ be smooth manifolds and $\Psi : \Lambda \to \mathcal{V}(M)$ be a map for which the map*

$$\Lambda \times M \to T(M), \quad (\lambda, p) \mapsto \Psi_\lambda(p)$$

is smooth (the vector field Ψ_λ depends smoothly on the parameter λ). Then the subset

$$\mathcal{D} := \{(t, \lambda, p) \in \mathbb{R} \times \Lambda \times M : (t,p) \in \mathcal{D}_{\Phi_\lambda}\}$$

of $\mathbb{R} \times \Lambda \times M$ is open and the map $\mathcal{D} \to M, (t,\lambda,p) \mapsto \Phi^{\Psi_\lambda}(t,p)$ is smooth.

Proof. The parameters do not cause any additional problems, as can be seen by the following trick: On the product manifold $\Lambda \times M$ we consider the smooth vector field Y, given by

$$Y(\lambda, p) := (0_\lambda, \Psi_\lambda(p)) \in T_\lambda(\Lambda) \times T_p(M) \cong T_{(\lambda,p)}(\Lambda \times M).$$

Then the integral curves of Y are of the form $\gamma(t) = (\lambda, \gamma_p(t))$, where γ_p is an integral curve of the smooth vector field Ψ_λ on M. Therefore the assertion is an immediate consequence on the smoothness of the flow of Y on $\Lambda \times M$ (Theorem 4.4.12). □

4.4.3 Lie Derivatives

We take a closer look at the interaction of local flows and vector fields. It will turn out that this leads to a new concept of a directional derivative which works for general tensor fields.

Let $X \in \mathcal{V}(M)$ and $\Phi^X : \mathcal{D}_X \to M$ its maximal local flow. For $f \in C^\infty(M)$ and $t \in \mathbb{R}$ we set

$$(\Phi_t^X)^* f := f \circ \Phi_t^X \in C^\infty(M_t).$$

Then we find

$$\lim_{t \to 0} \frac{1}{t}((\Phi_t^X)^* f - f) = \mathtt{d}f(X) = \mathcal{L}_X f \in C^\infty(M).$$

For a second vector field $Y \in \mathcal{V}(M)$, we define a smooth vector field on the open subset $M_{-t} \subseteq M$ by

$$(\Phi_t^X)_* Y := T(\Phi_t^X) \circ Y \circ \Phi_{-t}^X = T(\Phi_t^X) \circ Y \circ (\Phi_t^X)^{-1}$$

(cf. Remark 4.4.14) and define the *Lie derivative* by

$$\mathcal{L}_X Y := \lim_{t \to 0} \frac{1}{t}((\Phi_{-t}^X)_* Y - Y) = \frac{d}{dt}\Big|_{t=0} (\Phi_{-t}^X)_* Y,$$

which is defined on all of M since for each $p \in M$ the vector $((\Phi_t^X)_* Y)(p)$ is defined for sufficiently small t and depends smoothly on t.

Theorem 4.4.16. $\mathcal{L}_X Y = [X, Y]$ *for* $X, Y \in \mathcal{V}(M)$.

Proof. Fix $p \in M$. It suffices to show that $\mathcal{L}_X Y$ and $[X, Y]$ coincide in p. We may therefore work in a local chart, hence assume that $M = U$ is an open subset of $\mathbb{R}^n$.

Identifying vector fields with smooth $\mathbb{R}^n$-valued functions, we then have

$$[X, Y](x) = \mathtt{d}Y(x)X(x) - \mathtt{d}X(x)Y(x), \quad x \in U.$$

On the other hand,

$$\begin{aligned}((\Phi_{-t}^X)_* Y)(x) &= T(\Phi_{-t}^X) \circ Y \circ \Phi_t^X(x) \\ &= \mathtt{d}(\Phi_{-t}^X)(\Phi_t^X(x))Y(\Phi_t^X(x)) = \big(\mathtt{d}(\Phi_t^X)(x)\big)^{-1} Y(\Phi_t^X(x)).\end{aligned}$$

To calculate the derivative of this expression with respect to t, we first observe that it does not matter if we first take derivatives with respect to t and then with respect to x or vice versa. This leads to

$$\frac{d}{dt}\Big|_{t=0} \mathtt{d}(\Phi_t^X)(x) = \mathtt{d}\Big(\frac{d}{dt}\Big|_{t=0} \Phi_t^X\Big)(x) = \mathtt{d}X(x).$$

Next we note that for any smooth curve $\alpha : [-\varepsilon, \varepsilon] \to \mathrm{GL}_n(\mathbb{R})$ with $\alpha(0) = \mathbf{1}$ we have

$$(\alpha^{-1})'(t) = -\alpha(t)^{-1}\alpha'(t)\alpha(t)^{-1},$$

and in particular $(\alpha^{-1})'(0) = -\alpha'(0)$. Combining all this, we obtain with the Product Rule

$$\mathcal{L}_X(Y)(x) = -\mathtt{d}X(x)Y(x) + \mathtt{d}Y(x)X(x) = [X, Y](x). \qquad \square$$

Corollary 4.4.17. *If $X, Y \in \mathcal{V}(M)$ are complete vector fields, then their global flows $\Phi^X, \Phi^Y : \mathbb{R} \to \mathrm{Diff}(M)$ commute if and only if X and Y commute, i.e., $[X,Y] = 0$.*

Proof. (1) Suppose first that Φ^X and Φ^Y commute, i.e.,

$$\Phi^X(t) \circ \Phi^Y(s) = \Phi^Y(s) \circ \Phi^X(t) \quad \text{ for } t, s \in \mathbb{R}.$$

Let $p \in M$ and $\gamma_p(s) := \Phi^Y_s(p)$ the Y-integral curve through p. We then have

$$\gamma_p(s) = \Phi^Y_s(p) = \Phi^X_t \circ \Phi^Y_s \circ \Phi^X_{-t}(p),$$

and passing to the derivative in $s = 0$ yields

$$Y(p) = \gamma_p'(0) = T(\Phi^X_t)Y(\Phi^X_{-t}(p)) = \big((\Phi^X_t)_* Y\big)(p).$$

Passing now to the derivative in $t = 0$, we arrive at $[X, Y] = \mathcal{L}_X(Y) = 0$.

(2) Now we assume $[X, Y] = 0$. First we show that $(\Phi^X_t)_* Y = Y$ holds for all $t \in \mathbb{R}$. For $t, s \in \mathbb{R}$ we have

$$(\Phi^X_{t+s})_* Y = (\Phi^X_t)_* (\Phi^X_s)_* Y,$$

so that

$$\frac{d}{dt}(\Phi^X_t)_* Y = -(\Phi^X_t)_* \mathcal{L}_X(Y) = 0$$

for each $t \in \mathbb{R}$. Since for each $p \in M$ the curve

$$\mathbb{R} \to T_p(M), \quad t \mapsto \big((\Phi^X_t)_* Y\big)(p)$$

is smooth, and its derivative vanishes, it is constant $Y(p)$. This shows that $(\Phi^X_t)_* Y = Y$ for each $t \in \mathbb{R}$.

For $\gamma(s) := \Phi^X_t \Phi^Y_s(p)$ we now have $\gamma(0) = \Phi^X_t(p)$ and

$$\gamma'(s) = T(\Phi^X_t) \circ Y(\Phi^Y_s(p)) = Y(\Phi^X_t \Phi^Y_s(p)) = Y(\gamma(s)),$$

so that γ is an integral curve of Y. We conclude that $\gamma(s) = \Phi^Y_s(\Phi^X_t(p))$, and this means that the flows of X and Y commute. $\square$

Remark 4.4.18. Let $X, Y \in \mathcal{V}(M)$ be two complete vector fields and Φ^X, resp., Φ^Y their global flows. We then consider the commutator map

$$F : \mathbb{R}^2 \to \mathrm{Diff}(M), \quad (t, s) \mapsto \Phi^X_t \circ \Phi^Y_s \circ \Phi^X_{-t} \circ \Phi^Y_{-s}.$$

We know from Corollary 4.4.17 that it vanishes if and only if $[X, Y] = 0$, but there is also a more direct way from F to the Lie bracket. In fact, we first observe that

$$\frac{\partial F}{\partial s}(t, 0) = (\Phi^X_t)_* Y - Y,$$

and hence that

$$\frac{\partial^2 F}{\partial t \partial s}(0, 0) = [Y, X].$$

Here we use that for a smooth function of the form $G(t, s) = H(t, s, -t, -s)$ we have

$$\frac{\partial}{\partial s} G(t, 0) = \frac{\partial}{\partial x_2} H(t, 0, -t, 0) - \frac{\partial}{\partial x_4} H(t, 0, -t, 0)$$

by the Chain Rule.

Exercises for Section 4.4

Exercise 4.4.1. Let $M := \mathbb{R}^n$. For a matrix $A \in M_n(\mathbb{R})$, we consider the linear vector field $X_A(x) := Ax$. Determine the maximal flow Φ^X of this vector field.

Exercise 4.4.2. Let M be a smooth manifold and $Y \in \mathcal{V}(M)$ a smooth vector field on M. Suppose that Y generates a local flow $\Phi^Y : \mathcal{D}_Y \to M$ which is defined on an entire box of the form $[-\varepsilon, \varepsilon] \times M \subseteq \mathcal{D}_Y$. Show that this implies the completeness of Y.

Exercise 4.4.3. Let $\varphi \colon M \to N$ be a smooth map and $X \in \mathcal{V}(M)$, $Y \in \mathcal{V}(N)$ be φ-related vector fields. Show that for any integral curve $\gamma \colon I \to M$ of X, the curve $\varphi \circ \gamma \colon I \to N$ is an integral curve of Y.

Exercise 4.4.4. Let $X \in \mathcal{V}(M)$ be a vector field and write $X^{\mathbb{R}} \in \mathcal{V}(\mathbb{R})$ for the vector field on $\mathbb{R}$, given by $X^{\mathbb{R}}(t) = (t, 1)$. Show that, for an open interval $I \subseteq \mathbb{R}$, a smooth curve $\gamma \colon I \to M$ is an integral curve of X if and only if $X^{\mathbb{R}}$ and X are γ-related.

Exercise 4.4.5. Let $X \in \mathcal{V}(M)_c$ be a complete vector field and $\varphi \in \mathrm{Diff}(M)$. Then $\varphi_* X$ is also complete and

$$\Phi_t^{\varphi_* X} = \varphi \circ \Phi_t^X \circ \varphi^{-1} \quad \text{for} \quad t \in \mathbb{R}.$$

Exercise 4.4.6. Let M be a smooth manifold, $\varphi \in \mathrm{Diff}(M)$ and $X \in \mathcal{V}(M)_c$ be a complete vector field. Show that the following are equivalent:

(1) φ commutes with the flow maps Φ_t^X.

(2) For each integral curve $\gamma \colon I \to M$ of X, the curve $\varphi \circ \gamma$ also is an integral curve of X.

(3) $X = \varphi_* X = T(\varphi) \circ X \circ \varphi^{-1}$, i.e., X is φ-invariant.

Exercise 4.4.7. Let $X, Y \in \mathcal{V}(M)$ be two commuting complete vector fields, i.e., $[X, Y] = 0$. Show that the vector field $X + Y$ is complete and that its flow is given by

$$\Phi_t^{X+Y} = \Phi_t^X \circ \Phi_t^Y \quad \text{for all} \quad t \in \mathbb{R}.$$

Exercise 4.4.8. Let V be a finite-dimensional vector space and $\mu_t(v) := tv$ for $t \in \mathbb{R}^\times$. Show that:

(1) A vector field $X \in \mathcal{V}(V)$ is linear if and only if $(\mu_t)_* X = X$ holds for all $t \in \mathbb{R}^\times$.

(2) A diffeomorphism $\varphi \in \mathrm{Diff}(V)$ is linear if and only if it commutes with all the maps μ_t, $t \in \mathbb{R}^\times$.

Chapter 5

General Lie Groups

In the context of smooth manifolds, the natural class of groups are those endowed with a manifold structure compatible with the group structure. Such groups will be called Lie groups.

5.1 First Examples and the Tangent Group

Definition 5.1.1. A *Lie group* is a group G, endowed with the structure of a smooth manifold, such that the group operations

$$m_G \colon G \times G \to G, \quad (x, y) \mapsto xy \quad \text{and} \quad \iota_G \colon G \to G, \quad x \mapsto x^{-1}$$

are smooth.

Throughout this chapter, G denotes a Lie group with multiplication map $m_G \colon G \times G \to G, (x, y) \mapsto xy$, inversion map $\iota_G \colon G \to G, x \mapsto x^{-1}$, and neutral element $\mathbf{1}$. For $g \in G$ we write $\lambda_g \colon G \to G, x \mapsto gx$ for the left multiplication map, $\rho_g \colon G \to G, x \mapsto xg$ for the right multiplication map, and $c_g \colon G \to G, x \mapsto gxg^{-1}$ for the conjugation with g. A *morphism of Lie groups* is a smooth homomorphism of Lie groups $\varphi \colon G_1 \to G_2$.

Remark 5.1.2. All maps λ_g, ρ_g and c_g are smooth. Moreover, they are bijective with $\lambda_{g^{-1}} = \lambda_g^{-1}$, $\rho_{g^{-1}} = \rho_g^{-1}$ and $c_{g^{-1}} = c_g^{-1}$, so that they are diffeomorphisms of G onto itself.

In addition, the maps c_g are automorphisms of G, so that we obtain a group homomorphism

$$C \colon G \to \operatorname{Aut}(G), \quad g \mapsto c_g,$$

where $\operatorname{Aut}(G)$ stands for the *group of automorphisms of the Lie group* G, i.e., the group automorphisms which are diffeomorphisms. The automorphisms of the form c_g are called *inner automorphisms* of G. The group of inner automorphisms of G is denoted by $\operatorname{Inn}(G)$.

One can show that the requirement of ι_G being smooth is redundant (Exercise 5.4.4).

Example 5.1.3. We consider the additive group $G := (\mathbb{R}^n, +)$, endowed with the natural n-dimensional manifold structure. A corresponding chart is given by $(\mathrm{id}_{\mathbb{R}^n}, \mathbb{R}^n)$, which shows that the corresponding product manifold structure on $\mathbb{R}^n \times \mathbb{R}^n$ is given by the chart $(\mathrm{id}_{\mathbb{R}^n} \times \mathrm{id}_{\mathbb{R}^n}, \mathbb{R}^n \times \mathbb{R}^n) = (\mathrm{id}_{\mathbb{R}^{2n}}, \mathbb{R}^{2n})$, hence coincides with the natural manifold structure on $\mathbb{R}^{2n}$. Therefore the smoothness of addition and inversion follows from the smoothness of the maps

$$\mathbb{R}^{2n} \to \mathbb{R}^n, \quad (x, y) \mapsto x + y \quad \text{and} \quad \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto -x.$$

Example 5.1.4. Next we consider the group $G := \mathrm{GL}_n(\mathbb{R})$ of invertible $(n \times n)$-matrices. As an open subset of $M_n(\mathbb{R})$ it carries a natural manifold structure, and since its multiplication and inversion are smooth by Lemma 1.1.1, it is a Lie group.

Example 5.1.5. (a) (The circle group) We have already seen how to endow the circle

$$\mathbb{S}^1 := \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 = 1\}$$

with a manifold structure (Example 4.1.5). Identifying it with the unit circle

$$\mathbb{T} := \{z \in \mathbb{C} \colon |z| = 1\}$$

in $\mathbb{C}$, it also inherits a group structure, given by

$$(x, y) \cdot (x', y') := (xx' - yy', xy' + x'y) \quad \text{and} \quad (x, y)^{-1} = (x, -y).$$

With these explicit formulas, it is easy to verify that $\mathbb{T}$ is a Lie group (Exercise 5.4.1).

(b) (The n-dimensional torus) In view of (a), we have a natural manifold structure on the *n-dimensional torus* $\mathbb{T}^n := (\mathbb{S}^1)^n$. The corresponding direct product group structure

$$(t_1, \ldots, t_n)(s_1, \ldots, s_n) := (t_1 s_1, \ldots, t_n s_n)$$

turns $\mathbb{T}^n$ into a Lie group (Exercise 5.4.2).

Lemma 5.1.6. (a) *As usual, we identify $T(G \times G)$ with $T(G) \times T(G)$. Then the tangent map*

$$T(m_G) \colon T(G \times G) \cong T(G) \times T(G) \to T(G), \quad (v, w) \mapsto v \cdot w := Tm_G(v, w)$$

defines a Lie group structure on $T(G)$ with identity element $0_{\mathbf{1}} \in T_{\mathbf{1}}(G)$ and inversion $T(\iota_G)$. For $v \in T_g(G)$ and $w \in T_h(G)$, we have

$$v \cdot w = T_g(\rho_h)v + T_h(\lambda_g)w = v \cdot 0_h + 0_g \cdot w. \tag{5.1}$$

(b) *The canonical projection $\pi_{T(G)} \colon T(G) \to G$ is a morphism of Lie groups with kernel $(T_{\mathbf{1}}(G), +)$ and the zero section $\sigma \colon G \to T(G), g \mapsto 0_g \in T_g(G)$ is a homomorphism of Lie groups with $\pi_{T(G)} \circ \sigma = \mathrm{id}_G$.*

(c) *The map*

$$\Phi \colon G \times T_{\mathbf{1}}(G) \to T(G), \quad (g, x) \mapsto g.x := 0_g \cdot x = T(\lambda_g)x$$

is a diffeomorphism.

Proof. (a) Since the multiplication map $m_G \colon G \times G \to G$ is smooth, the same holds for its tangent map

$$Tm_G \colon T(G \times G) \cong T(G) \times T(G) \to T(G).$$

Let $\varepsilon_G \colon G \to G, g \mapsto \mathbf{1}$ be the constant homomorphism. Then the group axioms for G are encoded in the relations

(1) $m_G \circ (m_G \times \mathrm{id}_G) = m_G \circ (\mathrm{id}_G \times m_G)$ (associativity),

(2) $m_G \circ (\iota_G, \mathrm{id}_G) = m_G \circ (\mathrm{id}_G, \iota_G) = \varepsilon_G$ (inversion), and

(3) $m_G \circ (\varepsilon_G, \mathrm{id}_G) = m_G \circ (\mathrm{id}_G, \varepsilon_G) = \mathrm{id}_G$ (unit element).

Using the functoriality (cf. Lemma 4.2.5) of T and its compatibility with products, we see that these properties carry over to the corresponding maps on $T(G)$:

(1) $T(m_G) \circ T(m_G \times \mathrm{id}_G) = T(m_G) \circ (T(m_G) \times \mathrm{id}_{T(G)})$
$= T(m_G) \circ (\mathrm{id}_{T(G)} \times T(m_G))$ (associativity),

(2) $T(m_G) \circ (T(\iota_G), \mathrm{id}_{T(G)}) = T(m_G) \circ (\mathrm{id}_{T(G)}, T(\iota_G)) = T(\varepsilon_G)$ (inversion), and

(3) $T(m_G) \circ (T(\varepsilon_G), \mathrm{id}_{T(G)}) = T(m_G) \circ (\mathrm{id}_{T(G)}, T(\varepsilon_G)) = \mathrm{id}_{T(G)}$ (unit element).

Here we only have to observe that the tangent map $T(\varepsilon_G)$ maps each $v \in T(G)$ to $0_{\mathbf{1}} \in T_{\mathbf{1}}(G)$, which is the neutral element of $T(G)$. We conclude that $T(G)$ is a Lie group with multiplication $T(m_G)$, inversion $T(\iota_G)$, and unit element $0_{\mathbf{1}} \in T_{\mathbf{1}}(G)$.

For $v \in T_g(G)$ and $w \in T_h(G)$, the linearity of $T_{(g,h)}(m_G)$ implies that

$$\begin{aligned} Tm_G(v,w) &= T_{(g,h)}(m_G)(v,w) = T_{(g,h)}(m_G)(v,0) + T_{(g,h)}(m_G)(0,w) \\ &= T_g(\rho_h)v + T_h(\lambda_g)w, \end{aligned}$$

(b) The definition of the tangent map implies that the zero section $\sigma \colon G \to T(G)$ satisfies

$$Tm_G \circ (\sigma \times \sigma) = \sigma \circ m_G \quad \text{and} \quad Tm_G(0_g, 0_h) = 0_{m_G(g,h)} = 0_{gh},$$

which means that it is a morphism of Lie groups. That $\pi_{T(G)}$ also is a morphism of Lie groups follows likewise from the relation

$$\pi_{T(G)} \circ Tm_G = m_G \circ (\pi_{T(G)} \times \pi_{T(G)}),$$

which also is an immediate consequence of the definition of the tangent map T_{m_G}: it maps $T_g(G) \times T_h(G)$ into $T_{gh}(G)$. From (5.1), we obtain in particular that the multiplication on the normal subgroup $\ker \pi_{T(G)} = T_{\mathbf{1}}(G)$ is simply given by addition.

(c) The smoothness of Φ follows from the smoothness of the multiplication of $T(G)$ and the smoothness of the zero section $\sigma \colon G \to T(G), g \mapsto 0_g$. That Φ is a diffeomorphism follows from the following explicit formula for its inverse: $\Phi^{-1}(v) = (\pi_{T(G)}(v), \pi_{T(G)}(v)^{-1}v)$, so that its smoothness follows from the smoothness of $\pi_{T(G)}$ (its first component), and the smoothness of the multiplication on $T(G)$. ☐

Definition 5.1.7. In the following we shall mostly use the simplified notation

$$g.v := 0_g \cdot v \quad \text{for} \quad g \in G, v \in TG.$$

We likewise write

$$v.g := v \cdot 0_g \quad \text{for} \quad g \in G, v \in TG.$$

5.2 The Lie Functor

The Lie functor assigns a Lie algebra to each Lie group and a Lie algebra homomorphism to each morphism of Lie groups. It is the key tool to translate Lie group problems into problems in linear algebra.

Definition 5.2.1. [The Lie algebra of G] A vector field $X \in \mathcal{V}(G)$ is called *left invariant* if

$$X = (\lambda_g)_* X := T(\lambda_g) \circ X \circ \lambda_g^{-1}$$

holds for each $g \in G$, i.e., $(\lambda_g)_* X = X$. We write $\mathcal{V}(G)^l$ for the set of left invariant vector fields in $\mathcal{V}(G)$. Clearly $\mathcal{V}(G)^l$ is a linear subspace of $\mathcal{V}(G)$.

Writing the left invariance as $X = T(\lambda_g) \circ X \circ \lambda_g^{-1}$, we see that it means that X is λ_g-related to itself. Therefore the Related Vector Field Lemma 4.3.9 implies that if X and Y are left-invariant, their Lie bracket $[X, Y]$ is also λ_g-related to itself for each $g \in G$, hence left invariant. We conclude that the vector space $\mathcal{V}(G)^l$ is a Lie subalgebra of $(\mathcal{V}(G), [\cdot, \cdot])$.

Next we observe that the left invariance of a vector field X implies that for each $g \in G$ we have $X(g) = g.X(\mathbf{1})$ (Lemma 5.1.6(b)), so that X is completely determined by its value $X(\mathbf{1}) \in T_{\mathbf{1}}(G)$. Conversely, for each $x \in T_{\mathbf{1}}(G)$, we obtain a left invariant vector field $x_l \in \mathcal{V}(G)^l$ with $x_l(\mathbf{1}) = x$ by $x_l(g) := g.x$. That this vector field is indeed left invariant follows from

$$x_l \circ \lambda_h(g) = x_l(hg) = (hg).x = h.(g.x) = T(\lambda_h) x_l(g)$$

for all $h, g \in G$. Hence

$$T_{\mathbf{1}}(G) \to \mathcal{V}(G)^l, \quad x \mapsto x_l$$

is a linear bijection. We thus obtain a Lie bracket $[\cdot, \cdot]$ on $T_{\mathbf{1}}(G)$ satisfying

$$[x, y]_l = [x_l, y_l] \quad \text{for all} \quad x, y \in T_{\mathbf{1}}(G). \tag{5.2}$$

The Lie algebra

$$\mathbf{L}(G) := (T_{\mathbf{1}}(G), [\cdot, \cdot]) \cong \mathcal{V}(G)^l$$

is called *the Lie algebra of G*.

Proposition 5.2.2. (Functoriality of the Lie algebra) *If $\varphi\colon G \to H$ is a morphism of Lie groups, then the tangent map*

$$\mathbf{L}(\varphi) := T_{\mathbf{1}}(\varphi)\colon \mathbf{L}(G) \to \mathbf{L}(H)$$

is a homomorphism of Lie algebras.

Proof. Let $x, y \in \mathbf{L}(G)$ and x_l, y_l be the corresponding left invariant vector fields. Then $\varphi \circ \lambda_g = \lambda_{\varphi(g)} \circ \varphi$ for each $g \in G$ implies that

$$T(\varphi) \circ T(\lambda_g) = T(\lambda_{\varphi(g)}) \circ T(\varphi),$$

and applying this relation to $x, y \in T_{\mathbf{1}}(G)$, we get

$$T\varphi \circ x_l = \big(\mathbf{L}(\varphi)x\big)_l \circ \varphi \quad \text{and} \quad T\varphi \circ y_l = \big(\mathbf{L}(\varphi)y\big)_l \circ \varphi, \tag{5.3}$$

i.e., x_l is φ-related to $\big(\mathbf{L}(\varphi)x\big)_l$ and y_l is φ-related to $\big(\mathbf{L}(\varphi)y\big)_l$. Therefore the Related Vector Field Lemma implies that

$$T\varphi \circ [x_l, y_l] = [\big(\mathbf{L}(\varphi)x\big)_l, \big(\mathbf{L}(\varphi)y\big)_l] \circ \varphi.$$

Evaluating at $\mathbf{1}$, we obtain $\mathbf{L}(\varphi)[x, y] = [\mathbf{L}(\varphi)(x), \mathbf{L}(\varphi)(y)]$, showing that $\mathbf{L}(\varphi)$ is a homomorphism of Lie algebras. □

Remark 5.2.3. We obviously have $\mathbf{L}(\mathrm{id}_G) = \mathrm{id}_{\mathbf{L}(G)}$, and for two morphisms $\varphi_1 : G_1 \to G_2$ and $\varphi_2 : G_2 \to G_3$ of Lie groups, we obtain

$$\mathbf{L}(\varphi_2 \circ \varphi_1) = \mathbf{L}(\varphi_2) \circ \mathbf{L}(\varphi_1),$$

from the Chain Rule:

$$T_{\mathbf{1}}(\varphi_2 \circ \varphi_1) = T_{\varphi_{\mathbf{1}}(\mathbf{1})}(\varphi_2) \circ T_{\mathbf{1}}(\varphi_1) = T_{\mathbf{1}}(\varphi_2) \circ T_{\mathbf{1}}(\varphi_1).$$

The preceding lemma implies that the assignments $G \mapsto \mathbf{L}(G)$ and $\varphi \mapsto \mathbf{L}(\varphi)$ define a functor, called the *Lie functor*,

$$\mathbf{L} : \underline{LieGrp} \to \underline{LieAlg}$$

from the category $\underline{LieGrp}$ of Lie groups to the category $\underline{LieAlg}$ of (finite-dimensional) Lie algebras.

Corollary 5.2.4. *For each isomorphism of Lie groups $\varphi : G \to H$, the map $\mathbf{L}(\varphi)$ is an isomorphism of Lie algebras, and for each $x \in \mathbf{L}(G)$, the following equation holds*

$$\varphi_* x_l := T(\varphi) \circ x_l \circ \varphi^{-1} = \big(\mathbf{L}(\varphi)x\big)_l. \tag{5.4}$$

Proof. Let $\psi : H \to G$ be the inverse of φ. Then $\varphi \circ \psi = \mathrm{id}_H$ and $\psi \circ \varphi = \mathrm{id}_G$ leads to $\mathbf{L}(\varphi) \circ \mathbf{L}(\psi) = \mathrm{id}_{\mathbf{L}(H)}$ and $\mathbf{L}(\psi) \circ \mathbf{L}(\varphi) = \mathrm{id}_{\mathbf{L}(G)}$ (Remark 5.2.3). Further (5.4) follows from (5.3) in the proof of Proposition 5.2.2. □

5.3 Smooth Actions of Lie Groups

We already encountered smooth flows on manifolds in Chapter 4. These can be viewed as actions of the one-dimensional Lie group $(\mathbb{R}, +)$. In particular, we have seen that these actions are in one-to-one correspondence with complete vector fields, which is the corresponding Lie algebra picture. Now we describe the corresponding concept for general Lie groups.

Definition 5.3.1. Let M be a smooth manifold and G a Lie group. A *(smooth) action of G on M* is a smooth map

$$\sigma\colon G \times M \to M, \quad (g, m) \mapsto g.m = \sigma_g(m)$$

with the following properties:

(A1) $\sigma(\mathbf{1}, m) = m$ for all $m \in M$.

(A2) $\sigma(g_1, \sigma(g_2, m)) = \sigma(g_1 g_2, m)$ for $g_1, g_2 \in G$ and $m \in M$.

We also write

$$g.m := \sigma(g, m), \quad \sigma_g(m) := \sigma(g, m), \quad \sigma^m(g) := \sigma(g, m) = g.m.$$

The map σ^m is called the *orbit map*.

For each smooth action σ, the map

$$\widehat{\sigma}\colon G \to \mathrm{Diff}(M), \quad g \mapsto \sigma_g$$

is a group homomorphism and any homomorphism $\gamma\colon G \to \mathrm{Diff}(M)$ for which the map

$$\sigma_\gamma\colon G \times M \to M, \quad (g, m) \mapsto \gamma(g)(m)$$

is smooth defines a smooth action of G on M.

Remark 5.3.2. What we call an action is sometimes called a *left action*. Likewise one defines a *right action* as a smooth map $\sigma_R\colon M \times G \to M$ with

$$\sigma_R(m, \mathbf{1}) = m, \quad \sigma_R(\sigma_R(m, g_1), g_2) = \sigma_R(m, g_1 g_2).$$

For $m.g := \sigma_R(m, g)$, this takes the form

$$m.(g_1 g_2) = (m.g_1).g_2$$

of an associativity condition.

If σ_R is a smooth right action of G on M, then

$$\sigma_L(g, m) := \sigma_R(m, g^{-1})$$

defines a smooth left action of G on M. Conversely, if σ_L is a smooth left action, then

$$\sigma_R(m, g) := \sigma_L(g^{-1}, m)$$

defines a smooth left action. This translation is one-to-one, so that we may freely pass from one type of action to the other.

Examples 5.3.3. (a) If $X \in \mathcal{V}(M)$ is a complete vector field (cf. Definition 4.4.7) and $\Phi\colon \mathbb{R} \times M \to M$ its global flow, then Φ defines a smooth action of $G = (\mathbb{R}, +)$ on M.
(b) If G is a Lie group, then the multiplication map $\sigma := m_G\colon G \times G \to G$ defines a smooth left action of G on itself. In this case the $(m_G)_g = \lambda_g$ are the left multiplications.

The multiplication map also defines a smooth right action of G on itself. The corresponding left action is

$$\sigma\colon G \times G \to G, \quad (g,h) \mapsto hg^{-1} \quad \text{with} \quad \sigma_g = \rho_g^{-1}.$$

There is a third action of G on itself, the *conjugation action*:

$$\sigma\colon G \times G \to G, \quad (g,h) \mapsto ghg^{-1} \quad \text{with} \quad \sigma_g = c_g.$$

(c) We have a natural smooth action of the Lie group $\mathrm{GL}_n(\mathbb{R})$ on $\mathbb{R}^n$:

$$\sigma\colon \mathrm{GL}_n(\mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}^n, \quad \sigma(g,x) := gx.$$

We further have an action of $\mathrm{GL}_n(\mathbb{R})$ on $M_n(\mathbb{R})$:

$$\sigma\colon \mathrm{GL}_n(\mathbb{R}) \times M_n(\mathbb{R}) \to M_n(\mathbb{R}), \quad \sigma(g,A) = gAg^{-1}.$$

(d) On the set $M_{p,q}(\mathbb{R})$ of $(p \times q)$-matrices we have an action of the direct product Lie group $G := \mathrm{GL}_p(\mathbb{R}) \times \mathrm{GL}_q(\mathbb{R})$ by $\sigma((g,h),A) := gAh^{-1}$.

The following proposition generalizes the passage from flows of vector fields to actions of general Lie groups.

Proposition 5.3.4. (Derived action) *Let G be a Lie group and $\sigma\colon G \times M \to M$ a smooth action of G on M. Then the assignment*

$$\dot\sigma\colon \mathbf{L}(G) \to \mathcal{V}(M), \qquad \dot\sigma(x)_m := -T_{\mathbf{1}}(\sigma^m)(x)$$

is a homomorphism of Lie algebras.

Proof. First we observe that for each $x \in \mathbf{L}(G)$ the map $\dot\sigma(x)$ defines a smooth map $M \to T(M)$, and since $\dot\sigma(x)_m \in T_{\sigma(\mathbf{1},m)}(M) = T_m(M)$, it is a smooth vector field on M.

To see that $\dot\sigma$ is a homomorphism of Lie algebras, we pick $m \in M$ and write

$$\varphi^m := \sigma^m \circ \iota_G\colon G \to M, \quad g \mapsto g^{-1}.m$$

for the reversed orbit map. Then

$$\varphi^m(gh) = (gh)^{-1}.m = h^{-1}.(g^{-1}.m) = \varphi^{g^{-1}.m}(h),$$

which can be written as

$$\varphi^m \circ \lambda_g = \varphi^{g^{-1}.m}.$$

Taking the differential in $\mathbf{1} \in G$, we obtain for each $x \in \mathbf{L}(G) = T_{\mathbf{1}}(G)$:

$$\begin{aligned} T_g(\varphi^m)x_l(g) &= T_g(\varphi^m)T_{\mathbf{1}}(\lambda_g)x = T_{\mathbf{1}}(\varphi^m \circ \lambda_g)x = T_{\mathbf{1}}(\varphi^{g^{-1}.m})x \\ &= T_{\mathbf{1}}(\sigma^{g^{-1}.m})T_{\mathbf{1}}(\iota_G)x = -T_{\mathbf{1}}(\sigma^{\varphi^m(g)})x = \dot{\sigma}(x)_{\varphi^m(g)}. \end{aligned}$$

This means that the left invariant vector field x_l on G is φ^m-related to the vector field $\dot{\sigma}(x)$ on M. Therefore the Related Vector Field Lemma 4.3.9 implies that for $x, y \in \mathbf{L}(G)$ the vector field $[x_l, y_l]$ is φ^m-related to $[\dot{\sigma}(x), \dot{\sigma}(y)]$, which leads for each $m \in M$ to

$$\begin{aligned} \dot{\sigma}([x,y])_m &= T_{\mathbf{1}}(\varphi^m)[x,y]_l(\mathbf{1}) = T_{\mathbf{1}}(\varphi^m)[x_l, y_l](\mathbf{1}) \\ &= [\dot{\sigma}(x), \dot{\sigma}(y)]_{\varphi^m(\mathbf{1})} = [\dot{\sigma}(x), \dot{\sigma}(y)]_m. \end{aligned}$$ □

5.4 Basic Topology of Lie Groups

In this section we collect some basic topological properties of Lie groups.

Proposition 5.4.1. *The topology of a Lie group G has the following properties:*

(i) *G is a locally compact space, i.e., each neighborhood of an element of g contains a compact one.*

(ii) *The identity component G_0 of G is an open normal subgroup which coincides with the arc-component of $\mathbf{1}$.*

(iii) *For a subgroup H of G the following are equivalent:*

  (a) *H is a neighborhood of $\mathbf{1}$.*

  (b) *H is open.*

  (c) *H is open and closed.*

  (d) *H contains G_0.*

(iv) *If the set $\pi_0(G) := G/G_0$ of connected components of G is countable, then, in addition, the following statements hold:*

  (a) *G is countable at infinity, i.e., a countable union of compact subsets.*

  (b) *For each $\mathbf{1}$-neighborhood U in G there exists a sequence $(g_n)_{n\in\mathbb{N}}$ in G with $G = \bigcup_{n\in\mathbb{N}} g_n U$.*

  (c) *G is second countable, i.e., the topology of G has a countable basis.*

  (d) *If $(U_i)_{i\in I}$ is a pairwise disjoint collection of open subsets of G, then I is countable.*

Proof. (i) This is true for any smooth n-dimensional manifold M. If $m \in M$, V is a neighborhood of m and (φ, U) is a chart with $m \in M$, then $\varphi(U \cap V)$ is a neighborhood of $\varphi(m)$ in $\mathbb{R}^n$. If $B \subseteq \varphi(U \cap V)$ is a closed ball around $\varphi(m)$, which is compact due to the Heine–Borel Theorem, its inverse image $\varphi^{-1}(B)$ is a compact neighborhood of m, contained in V. Here we use that M is Hausdorff to see that $\varphi^{-1}(B)$ is compact.

(ii) Since G is a smooth manifold, each point has an open neighborhood U homeomorphic to an open ball in some $\mathbb{R}^n$. Then U is in particular arcwise connected. This implies that the arc-components of G are open, hence that they coincide with the connected components.

To see that the identity component G_0 of G is a subgroup, we first note that G_0G_0 is the image of the connected set $G_0 \times G_0$ under the multiplication map, hence connected. Since it contains $\mathbf{1}$, we find $G_0G_0 \subseteq G_0$. Similarly, we see that the inversion preserves G_0, i.e., $G_0^{-1} \subseteq G_0$, showing that G_0 is a subgroup of G. Each conjugation $c_g(x) := gxg^{-1}$ fixes the identity element $\mathbf{1}$, hence maps the identity component G_0 into itself. Thus G_0 is normal.

(iii) (a) $\Rightarrow$ (b): If H is a neighborhood of $\mathbf{1}$, then each coset gH is a neighborhood of g because the left multiplication maps $\lambda_g \colon G \to G$ are homeomorphism. Hence all left cosets of H are open. In particular, H is open.

(b) $\Rightarrow$ (c): If H is an open subgroup, then its complement is the union of all cosets gH, $g \notin H$, hence also open. Therefore H is also closed.

(c) $\Rightarrow$ (d): If H is open and closed, then the connectedness of G_0 implies $G_0 \subseteq H$.

(d) $\Rightarrow$ (a) is trivial.

(iv) (a) In view of (i), there exists a compact identity neighborhood U in G. Replacing U by $U \cap U^{-1}$, we may w.l.o.g. assume that $U = U^{-1}$. Then each set

$$U^n := \{u_1 \cdots u_n \colon u_i \in U\}$$

is also compact, because it is the image of the compact topological product space $U^{\times n}$ under the n-fold multiplication map which is continuous.

Now $H := \bigcup_{n\in\mathbb{N}} U^n$ is a subgroup of G which is a $\mathbf{1}$-neighborhood, and (iii) implies $G_0 \subseteq H$. Hence the set of H-cosets is countable, and since each coset gH is a union of the countably many compact subsets gU^n, we see that G also is a countable union of compact subsets.

(b) In view of (a), we have $G = \bigcup_{n\in\mathbb{N}} K_n$, where each K_n is a compact subset. For each n, the open sets kU°, $k \in K_n$, cover the compact set K_n, so that there exist finitely many $k_{n,1}, \ldots, k_{n,m_n}$ with $K_n \subseteq \bigcup_{j=1}^{m_n} k_{n,j}U$. Then $G \subseteq \bigcup_{n\in\mathbb{N}} \bigcup_{j=1}^{m_n} k_{n,j}U$.

(c) Let $(U_n)_{n\in\mathbb{N}}$ be a countable basis of open $\mathbf{1}$-neighborhoods, we may take $U_n = \varphi(\frac{1}{n}B)$, where $B \subseteq \mathbf{L}(G)$ is an open ball with respect to some norm and $\varphi \colon B \to G$ is a diffeomorphism onto an open subset of G with $\varphi(0) = \mathbf{1}$. In view of (b), there exists for each $n \in \mathbb{N}$ a sequence $(g_{n,k})_{k\in\mathbb{N}}$ in G with $G = \bigcup_{k\in\mathbb{N}} g_{n,k}U_n$. We claim that $\{g_{n,k}U_n \colon n, k \in \mathbb{N}\}$ is a basis for the topology of G. In fact, if $O \subseteq G$ is an open subset and $g \in O$, then there exists some n with $gU_n \subseteq O$. Next we pick m such that $U_m^{-1}U_m \subseteq U_n$ and some $k \in \mathbb{N}$ with $g \in g_{m,k}U_m$. Then $g_{m,k}U_m \subseteq gU_m^{-1}U_m \subseteq gU_n \subseteq O$, and this proves our claim.

(d) follows immediately from (c). $\square$

Exercises for Chapter 5

Exercise 5.4.1. Show that the natural group structure on $\mathbb{T} \cong \mathbb{S}^1 \subseteq \mathbb{C}^\times$ turns it into a Lie group.

Exercise 5.4.2. Let $G_1, \ldots, G_n$ be Lie groups and $G := G_1 \times \ldots \times G_n$, endowed with the direct product group structure

$$(g_1, \ldots, g_n)(g_1', \ldots, g_n') := (g_1 g_1', \ldots, g_n g_n')$$

and the product manifold structure. Show that G is a Lie group with

$$\mathbf{L}(G) \cong \mathbf{L}(G_1) \times \ldots \times \mathbf{L}(G_n).$$

Exercise 5.4.3. Let V and W be finite-dimensional real vector spaces and $\beta \colon V \times V \to W$ a bilinear map. Show that $G := W \times V$ is a Lie group with respect to

$$(w, v)(w', v') := (w + w' + \beta(v, v'), v + v').$$

For $(w, v) \in \mathbf{L}(G) \cong T_{(0,0)}(G)$, find a formula for the corresponding left invariant vector field $(w, v)_l$, considered as a smooth function $G \to W \times V$.

Exercise 5.4.4. [Automatic smoothness of the inversion] Let G be an n-dimensional smooth manifold, endowed with a group structure for which the multiplication map m_G is smooth. Show that the inversion is also smooth, so that G is a Lie group. Proceed along the following steps:

(1) $T_{(g,h)}(m_G)(v, w) = T_g(\rho_h)v + T_h(\lambda_g)w$ for $\lambda_g(x) = gx$ and $\rho_h(x) = xh$.

(2) $T_{(\mathbf{1},\mathbf{1})}(m_G)(v, w) = v + w$.

(3) The inverse map $\iota_G \colon G \to G, g \mapsto g^{-1}$ is smooth if it is smooth in a neighborhood of $\mathbf{1}$. Hint: Left and right translations are smooth.

(4) The inverse map ι_G is smooth. Hint: Apply the Inverse Function Theorem to the map $F \colon G \times G \to G \times G, F(g, h) = (g, gh)$.

Exercise 5.4.5. Let $\mathcal{A}$ be a finite-dimensional unital real algebra and $\mathcal{A}^\times$ its group of units. We write $\lambda_a(b) := ab$ for the left multiplication with $a \in \mathcal{A}$. Show that:

(1) $\mathcal{A}^\times = \{a \in \mathcal{A} \colon \det(\lambda_a) \neq 0\}$.

(2) $\mathcal{A}^\times$ is an open subset of $\mathcal{A}$ and with respect to the corresponding manifold structure it is a Lie group.

(3) Identifying vector fields on the open subset $\mathcal{A}^\times$ with smooth $\mathcal{A}$-valued functions, a vector field $X \in \mathcal{V}(\mathcal{A}^\times) \cong C^\infty(\mathcal{A}^\times, \mathcal{A})$ is left invariant if and only if there exists an element $x \in \mathcal{A}$ with $X(a) = ax$ for $a \in \mathcal{A}^\times$.

Exercise 5.4.6. Consider the three-dimensional Heisenberg group

$$G = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

Determine the space of (left) invariant vector fields in the coordinates (x, y, z).

Chapter 6

The Exponential Function of a Lie Group

In the preceding chapter we have introduced the Lie functor which assigns to a Lie group G its Lie algebra $\mathbf{L}(G)$ and to a morphism φ of Lie groups its tangent morphism $\mathbf{L}(\varphi)$ of Lie algebras. In this section, we introduce a key tool of Lie theory which will allow us to also go in the opposite direction: the exponential function $\exp_G\colon \mathbf{L}(G) \to G$. It is a natural generalization of the matrix exponential map, which is obtained for $G = \mathrm{GL}_n(\mathbb{R})$ and its Lie algebra $\mathbf{L}(G) = \mathfrak{gl}_n(\mathbb{R})$.

6.1 Basic Properties of the Exponential Function

Proposition 6.1.1. *Each left invariant vector field X on G is complete.*

Proof. Let $g \in G$ and $\gamma\colon I \to G$ be the unique maximal integral curve (cf. Theorem 4.4.5) of $X \in \mathcal{V}(G)^l$ with $\gamma(0) = g$.

For each $h \in G$ we have $(\lambda_h)_* X = X$, which implies that $\eta := \lambda_h \circ \gamma$ also is an integral curve of X (cf. Exercise 4.4.3). Put $h = \gamma(s)g^{-1}$ for some $s > 0$. Then

$$\eta(0) = (\lambda_h \circ \gamma)(0) = h\gamma(0) = hg = \gamma(s),$$

and the uniqueness of integral curves implies that $\gamma(t) = \eta(t-s)$ for all t in the interval $I \cap (I + s)$ which is nonempty because it contains s. We thus obtain an extension of γ to the interval $I \cup (I + s)$, and the maximality of I thus leads to $I + s \subseteq I$, and hence to $I + ns \subseteq I$ for each $n \in \mathbb{N}$. Therefore the interval I is unbounded from below. Applying the same argument to some $s < 0$, we see that I is also unbounded from above. Hence $I = \mathbb{R}$, which means that X is complete. $\square$

Definition 6.1.2. We now define the *exponential function*

$$\exp_G\colon \mathbf{L}(G) \to G, \quad \exp_G(x) := \gamma_x(1),$$

where $\gamma_x\colon \mathbb{R} \to G$ is the unique maximal integral curve of the left invariant vector field x_l, satisfying $\gamma_x(0) = \mathbf{1}$. This means that γ_x is the unique solution of the initial value problem

$$\gamma(0) = \mathbf{1}, \quad \gamma'(t) = x_l(\gamma(t)) = \gamma(t).x \quad \text{for all} \quad t \in \mathbb{R}.$$

Example 6.1.3. (a) Let $G := (V, +)$ be the additive group of a finite-dimensional vector space. The left invariant vector fields on V are given by

$$x_l(w) := \frac{d}{dt}\Big|_{t=0} w + tx = x,$$

so that they are simply the constant vector fields. Hence (cf. Lemma 4.3.3)

$$[x_l, y_l](0) = \mathtt{d}y_l(x_l(0)) - \mathtt{d}x_l(y_l(0)) = \mathtt{d}y_l(x) - \mathtt{d}x_l(y) = 0.$$

Therefore $\mathbf{L}(V)$ is an abelian Lie algebra.

For each $x \in V$, the flow of x_l is given by $\Phi^{x_l}(t, v) = v + tx$, so that

$$\exp_V(x) = \Phi^{x_l}(1, 0) = x, \quad \text{i.e.,} \quad \exp_V = \mathrm{id}_V .$$

(b) Now let $G := \mathrm{GL}_n(\mathbb{R})$ be the Lie group of invertible $(n \times n)$-matrices, which inherits its manifold structure from the embedding as an open subset of the vector space $M_n(\mathbb{R})$.

The left invariant vector field A_l corresponding to a matrix A is given by

$$A_l(g) = T_{\mathbf{1}}(\lambda_g)A = gA$$

because $\lambda_g(h) = gh$ extends to a linear endomorphism of $M_n(\mathbb{R})$. The unique solution $\gamma_A\colon \mathbb{R} \to \mathrm{GL}_n(\mathbb{R})$ of the initial value problem

$$\gamma(0) = \mathbf{1}, \quad \gamma'(t) = A_l(\gamma(t)) = \gamma(t)A$$

is the curve describing the fundamental system of the linear differential equation defined by the matrix A:

$$\gamma_A(t) = e^{tA} = \sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k.$$

It follows that $\exp_G(A) = e^A$ is the matrix exponential function.

The Lie algebra $\mathbf{L}(G)$ of G is determined from

$$\begin{aligned}[A, B] = [A_l, B_l](\mathbf{1}) &= \mathtt{d}B_l(\mathbf{1})A_l(\mathbf{1}) - \mathtt{d}A_l(\mathbf{1})B_l(\mathbf{1}) \\ &= \mathtt{d}B_l(\mathbf{1})A - \mathtt{d}A_l(\mathbf{1})B = AB - BA.\end{aligned}$$

Therefore the Lie bracket on $\mathbf{L}(G) = T_{\mathbf{1}}(G) \cong M_n(\mathbb{R})$ is given by the commutator bracket. This Lie algebra is denoted $\mathfrak{gl}_n(\mathbb{R})$, to express that it is the Lie algebra of $\mathrm{GL}_n(\mathbb{R})$.

(c) If V is a finite-dimensional real vector space, then $V \cong \mathbb{R}^n$, so that we can immediately use (b) to see that $\mathrm{GL}(V)$ is a Lie group with Lie algebra $\mathfrak{gl}(V) := (\mathrm{End}(V), [\cdot, \cdot])$ and exponential function

$$\exp_{\mathrm{GL}(V)}(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

Lemma 6.1.4. (a) *For each $x \in \mathbf{L}(G)$, the curve $\gamma_x : \mathbb{R} \to G$ is a smooth homomorphism of Lie groups with $\gamma_x'(0) = x$.*

(b) *The global flow of the left invariant vector field x_l is given by*

$$\Phi_t^{x_l}(g) = \Phi(t, g) = g\gamma_x(t) = g \exp_G(tx).$$

(c) *If $\gamma : \mathbb{R} \to G$ is a smooth homomorphism of Lie groups and $x := \gamma'(0)$, then $\gamma = \gamma_x$. In particular, the map*

$$\mathrm{Hom}(\mathbb{R}, G) \to \mathbf{L}(G), \quad \gamma \mapsto \gamma'(0)$$

is a bijection, where $\mathrm{Hom}(\mathbb{R}, G)$ stands for the set of morphisms, i.e., smooth homomorphisms, of Lie groups $\mathbb{R} \to G$.

Proof. (a), (b) Since γ_x is an integral curve of the smooth vector field x_l, it is a smooth curve. Hence the smoothness of the multiplication in G implies that $\Phi(t, g) := g\gamma_x(t)$ defines a smooth map $\mathbb{R} \times G \to G$. In view of the left invariance of x_l, we have for each $g \in G$ and $\Phi^g(t) := \Phi(t, g)$ the relation

$$(\Phi^g)'(t) = T(\lambda_g)\gamma_x'(t) = T(\lambda_g)x_l(\gamma_x(t)) = x_l(g\gamma_x(t)) = x_l(\Phi^g(t)).$$

Therefore Φ^g is an integral curve of x_l with $\Phi^g(0) = g$, and this proves that Φ is the unique maximal flow of the complete vector field x_l.

In particular, we obtain for $t, s \in \mathbb{R}$:

$$\gamma_x(t + s) = \Phi(t + s, \mathbf{1}) = \Phi(t, \Phi(s, \mathbf{1})) = \Phi(s, \mathbf{1})\gamma_x(t) = \gamma_x(s)\gamma_x(t). \tag{6.1}$$

Hence γ_x is a group homomorphism $(\mathbb{R}, +) \to G$.

(c) If $\gamma : (\mathbb{R}, +) \to G$ is a smooth group homomorphism, then

$$\Phi(t, g) := g\gamma(t)$$

defines a global flow on G whose infinitesimal generator is the vector field given by

$$X(g) = \frac{d}{dt}\Big|_{t=0} \Phi(t, g) = T(\lambda_g)\gamma'(0).$$

We conclude that $X = x_l$ for $x = \gamma'(0)$, so that X is a left invariant vector field. Since γ is its unique integral curve through 0, it follows that $\gamma = \gamma_x$. In view of (a), this proves (c). $\square$

Proposition 6.1.5. *For a Lie group G, the exponential function*

$$\exp_G : \mathbf{L}(G) \to G$$

is smooth and satisfies $T_0(\exp_G) = \mathrm{id}_{\mathbf{L}(G)}$. In particular, $\exp_G$ is a local diffeomorphism in 0 in the sense that it maps some 0-neighborhood in $\mathbf{L}(G)$ diffeomorphically onto some $\mathbf{1}$-neighborhood in G.

Proof. The map $\Psi\colon \mathbf{L}(G) \to \mathcal{V}(G), x \mapsto x_l$ satisfies the assumptions of Proposition 4.4.15 because the map

$$\mathbf{L}(G) \times G \to T(G), \quad (x, g) \mapsto x_l(g) = g.x$$

is smooth (Lemma 5.1.6). In the terminology of Proposition 4.4.15, it now follows that the map

$$\Phi\colon \mathbb{R} \times \mathbf{L}(G) \times G \to G, \quad (t, x, g) \mapsto g\gamma_x(t) = g \exp_G(tx)$$

is smooth, and this implies the smoothness of $\exp_G$. Finally, we observe that

$$T_0(\exp_G)(x) = \frac{d}{dt}\Big|_{t=0} \exp_G(tx) = \gamma_x'(0) = x,$$

so that $T_0(\exp_G) = \mathrm{id}_{\mathbf{L}(G)}$. □

Lemma 6.1.6. (Canonical Coordinates) *Let G be a Lie group and $b_1, \ldots, b_n$ be a basis for its Lie algebra $\mathbf{L}(G)$. Then the following maps restrict to diffeomorphisms of some 0-neighborhood in $\mathbb{R}^n$ to some open $\mathbf{1}$-neighborhood in G:*

(i) $x \mapsto \exp_G(x_1 b_1 + \ldots + x_n b_n)$ (*Canonical coordinates of the first kind*).

(ii) $x \mapsto \exp_G(x_1 b_1) \cdot \ldots \cdot \exp_G(x_n b_n)$ (*Canonical coordinates of the second kind*).

Proof. (i) This is immediate from Proposition 6.1.5.

(ii) In view of Proposition 6.1.5, $T_0(\exp_G) = \mathrm{id}_{\mathbf{L}(G)}$, and further $T_{\mathbf{1}}(m_G)(x, y) = x + y$ by Lemma 5.1.6. Therefore

$$\Phi\colon \mathbb{R}^n \to G, \quad x \mapsto \exp_G(x_1 b_1) \cdot \ldots \cdot \exp_G(x_n b_n)$$

satisfies $T_0(\Phi)(x) = \sum_{i=1}^n x_i b_i$. Hence the claim follows from the Inverse Function Theorem. □

Lemma 6.1.7. *If $\sigma\colon G \times M \to M$ is a smooth action and $x \in \mathbf{L}(G)$, then the global flow of the vector field $\dot\sigma(x)$ is given by $\Phi^x(t, m) = \exp_G(-tx).m$. In particular,*

$$\dot\sigma(x)_m = \frac{d}{dt}\Big|_{t=0} \exp_G(-tx).m = -T_{(\mathbf{1},m)}(\sigma)(x, 0).$$

Proof. Clearly Φ^x defines a smooth global flow on M, and its infinitesimal generator is given by

$$\frac{d}{dt}\Big|_{t=0} \exp_G(-tx).m = -T_{(\mathbf{1},m)}(\sigma)(x, 0) = \dot\sigma(x)_m.$$

This proves the lemma. □

Lemma 6.1.8. *If $x, y \in \mathbf{L}(G)$ commute, i.e., $[x, y] = 0$, then*

$$\exp_G(x + y) = \exp_G(x)\exp_G(y).$$

Proof. If x and y commute, then the corresponding left invariant vector fields commute, and Corollary 4.4.17 implies that their flows commute. We conclude that for all $t, s \in \mathbb{R}$ we have

$$\exp_G(tx)\exp_G(sy) = \exp_G(sy)\exp_G(tx). \tag{6.2}$$

Therefore

$$\gamma(t) := \exp_G(tx)\exp_G(ty)$$

is a smooth group homomorphism. In view of

$$\gamma'(0) = T_{(\mathbf{1},\mathbf{1})}(m_G)(x,y) = x + y$$

(Lemma 5.1.6), Lemma 6.1.4(c) leads to $\gamma(t) = \exp_G(t(x+y))$, and for $t = 1$ we obtain the lemma. □

Lemma 6.1.9. *The subgroup $\langle \exp_G(\mathbf{L}(G))\rangle$ of G generated by $\exp_G(\mathbf{L}(G))$ coincides with the identity component G_0 of G, i.e., the connected component containing $\mathbf{1}$.*

Proof. Since $\exp_G$ is a local diffeomorphism in 0 (Proposition 6.1.5), the Inverse Function Theorem (see Exercise 4.2.5) implies that $\exp_G(\mathbf{L}(G))$ is a neighborhood of $\mathbf{1}$. We conclude that the subgroup $H := \langle \exp_G(\mathbf{L}(G))\rangle$ generated by the exponential image is a $\mathbf{1}$-neighborhood, hence contains G_0 (Proposition 5.4.1(iii)(d)). On the other hand, $\exp_G$ is continuous, so that it maps the connected space $\mathbf{L}(G)$ into the identity component G_0 of G, which leads to $H \subseteq G_0$, and hence to equality. □

6.2 Naturality of the Exponential Function

In this subsection we study how the exponential function is related to the Lie functor.

Proposition 6.2.1. *Let $\varphi\colon G_1 \to G_2$ be a morphism of Lie groups and $\mathbf{L}(\varphi)\colon \mathbf{L}(G_1) \to \mathbf{L}(G_2)$ its differential in $\mathbf{1}$. Then*

$$\exp_{G_2} \circ \mathbf{L}(\varphi) = \varphi \circ \exp_{G_1}, \tag{6.3}$$

i.e., the following diagram commutes

$$\begin{tikzcd} G_1 \arrow[r, "\varphi"] & G_2 \\ \mathbf{L}(G_1) \arrow[u, "\exp_{G_1}"'] \arrow[r, "\mathbf{L}(\varphi)"] & \mathbf{L}(G_2). \arrow[u, "\exp_{G_2}"'] \end{tikzcd}$$

Proof. For $x \in \mathbf{L}(G_1)$ we consider the smooth homomorphism

$$\gamma_x \in \mathrm{Hom}(\mathbb{R}, G_1), \quad \gamma_x(t) = \exp_{G_1}(tx).$$

According to Lemma 6.1.4, we have

$$\varphi \circ \gamma_x(t) = \exp_{G_2}(ty)$$

for $y = (\varphi\circ\gamma_x)'(0) = \mathbf{L}(\varphi)x$, because $\varphi\circ\gamma_x : \mathbb{R} \to G_2$ is a smooth group homomorphism. For $t = 1$ we obtain in particular

$$\exp_{G_2}(\mathbf{L}(\varphi)x) = \varphi(\exp_{G_1}(x)),$$

which we had to show. □

Corollary 6.2.2. *Let G_1 and G_2 be Lie groups and $\varphi\colon G_1 \to G_2$ be a group homomorphism. Then the following are equivalent:*

(a) *φ is smooth in an identity neighborhood of G_1.*

(b) *φ is smooth.*

(c) *There exists a linear map $\psi\colon \mathbf{L}(G_1) \to \mathbf{L}(G_2)$ satisfying*

$$\exp_{G_2} \circ \psi = \varphi \circ \exp_{G_1} . \tag{6.4}$$

Proof. (a) $\Rightarrow$ (b): Let U be an open $\mathbf{1}$-neighborhood of G_1 such that $\varphi|_U$ is smooth. Since each left translation λ_g is a diffeomorphism, $\lambda_g(U) = gU$ is an open neighborhood of g, and we have

$$\varphi(gx) = \varphi(g)\varphi(x), \quad \text{i.e.,} \quad \varphi \circ \lambda_g = \lambda_{\varphi(g)} \circ \varphi.$$

Hence the smoothness of φ on U implies the smoothness of φ on gU, and therefore that φ is smooth.

(b) $\Rightarrow$ (c): If φ is smooth, then $\psi := \mathbf{L}(\varphi)$ satisfies (6.4).

(c) $\Rightarrow$ (a): If ψ is a linear map satisfying (6.4), then the fact that the exponential functions $\exp_{G_1}$ and $\exp_{G_2}$ are local diffeomorphisms and the smoothness of the linear map ψ implies (a). $\square$

Corollary 6.2.3. *If $\varphi_1, \varphi_2\colon G_1 \to G_2$ are morphisms of Lie groups with $\mathbf{L}(\varphi_1) = \mathbf{L}(\varphi_2)$, then φ_1 and φ_2 coincide on the identity component of G_1.*

Proof. In view of Proposition 6.2.1, we have for $x \in \mathbf{L}(G_1)$:

$$\varphi_1(\exp_{G_1}(x)) = \exp_{G_2}(\mathbf{L}(\varphi_1)x) = \exp_{G_2}(\mathbf{L}(\varphi_2)x) = \varphi_2(\exp_{G_1}(x)),$$

so that φ_1 and φ_2 coincide on the image of $\exp_{G_1}$, hence on the subgroup generated by this set. Now the assertion follows from Lemma 6.1.9. $\square$

Proposition 6.2.4. *For a morphism $\varphi\colon G_1 \to G_2$ of Lie groups, the following assertions hold:*

(1) $\ker \mathbf{L}(\varphi) = \{x \in \mathbf{L}(G_1)\colon \exp_{G_1}(\mathbb{R}x) \subseteq \ker \varphi\}$.

(2) *φ is an open map if and only if $\mathbf{L}(\varphi)$ is surjective.*

(3) *If $\mathbf{L}(\varphi)$ is a linear isomorphism and φ is bijective, then φ is an isomorphism of Lie groups.*

Proof. (1) The condition $x \in \ker \mathbf{L}(\varphi)$ is equivalent to

$$\{\mathbf{1}\} = \exp_{G_2}(\mathbb{R}\,\mathbf{L}(\varphi)x) = \varphi(\exp_{G_1}(\mathbb{R}x)).$$

(2) Suppose first that φ is an open map. Since $\exp_{G_i}$, $i = 1, 2$, are local diffeomorphisms,

$$\exp_{G_2} \circ \mathbf{L}(\varphi) = \varphi \circ \exp_{G_1} \tag{6.5}$$

implies that there exists some 0-neighborhood in $\mathbf{L}(G_1)$ on which $\mathbf{L}(\varphi)$ is an open map, hence that $\mathbf{L}(\varphi)$ is surjective.

If, conversely, $\mathbf{L}(\varphi)$ is surjective, then $\mathbf{L}(\varphi)$ is an open map, so that the relation (6.5) implies that there exists an open $\mathbf{1}$-neighborhood U_1 in G_1 such that $\varphi|_{U_1}$ is an open map. We claim that this implies that φ is an open map. In fact, suppose that $O \subseteq G_1$ is open and $g \in O$. Then there exists an open $\mathbf{1}$-neighborhood U_2 of G_1 with $gU_2 \subseteq O$ and $U_2 \subseteq U_1$. Then

$$\varphi(O) \supseteq \varphi(gU_2) = \varphi(g)\varphi(U_2),$$

and since $\varphi(U_2)$ is open in G_2, we see that $\varphi(O)$ is a neighborhood of $\varphi(g)$, hence that $\varphi(O)$ is open because $g \in O$ was arbitrary.

(3) From the relation $\exp_{G_2} \circ \mathbf{L}(\varphi) = \varphi \circ \exp_{G_1}$ and the bijectivity of φ we derive that the group homomorphism φ^{-1} satisfies

$$\varphi^{-1} \circ \exp_{G_2} = \exp_{G_1} \circ \mathbf{L}(\varphi)^{-1},$$

so that Corollary 6.2.2 implies that φ^{-1} is also smooth. $\square$

Theorem 6.2.5. (One-parameter Group Theorem) *Let G be a Lie group. For each $x \in \mathfrak{g} := \mathbf{L}(G)$, the map $\gamma_x \colon (\mathbb{R}, +) \to G, t \mapsto \exp_G(tx)$ is a smooth group homomorphism. Conversely, every continuous one-parameter group $\gamma \colon \mathbb{R} \to G$ is of this form.*

Proof. The first assertion is an immediate consequence of Lemma 6.1.4(c). It therefore remains to show that each continuous one-parameter group γ of G is a γ_x for some $x \in \mathfrak{g}$. Let $U = -U$ be a convex 0-neighborhood in $\mathfrak{g}$ for which $\exp_G|_U$ is a diffeomorphism onto an open subset of G and put $U_1 := \frac{1}{2}U$. Since γ is continuous in 0, there exists an $\varepsilon > 0$ such that $\gamma([-\varepsilon, \varepsilon]) \subseteq \exp_G(U_1)$. Then $\alpha(t) := (\exp_G|_U)^{-1}(\gamma(t))$ defines a continuous curve $\alpha \colon [-\varepsilon, \varepsilon] \to U_1$ with $\exp(\alpha(t)) = \gamma(t)$ for $|t| \leq \varepsilon$. With the same arguments as in the proof of Theorem 2.2.6, we see that $\alpha(t) = tx$ for some $x \in \mathfrak{g}$. Hence $\gamma(t) = \exp_G(tx)$ for $|t| \leq \varepsilon$, but then $\gamma(nt) = \exp_G(ntx)$ for $n \in \mathbb{N}$ leads to $\gamma(t) = \exp_G(tx)$ for each $t \in \mathbb{R}$. $\square$

Proposition 6.2.6. *Let G be a Lie group with Lie algebra $\mathbf{L}(G)$. For $x, y \in \mathbf{L}(G)$ we have the Product Formula*

$$\exp_G(x + y) = \lim_{k\to\infty} \left(\exp_G(\tfrac{1}{k}x)\exp_G(\tfrac{1}{k}y)\right)^k.$$

Proof. To obtain the product formula, we consider the smooth curve

$$\gamma \colon \mathbb{R} \to G, \quad \gamma(t) := \exp(tx)\exp(ty) \quad \text{with} \quad \gamma(0) = \mathbf{1}, \gamma'(0) = x + y$$

(cf. Lemma 5.1.6). The assertion now follows from the relation

$$\exp(\gamma'(0)) = \lim_{n\to\infty} \gamma(1/n)^n$$

which can be proved with the same argument as in the proof of Lemma 2.4.8. $\square$

Theorem 6.2.7. (Automatic Smoothness Theorem) *Each continuous homomorphism $\varphi\colon G \to H$ of Lie groups is smooth.*

Proof. From Theorem 6.2.5 we know that the map

$$\mathbf{L}(G) \to \operatorname{Hom}_c(\mathbb{R}, G), \quad x \mapsto \gamma_x, \quad \gamma_x(t) := \exp_G(tx)$$

is a bijection, where $\operatorname{Hom}_c(\mathbb{R}, G)$ denotes the set of all continuous one-parameter groups of G. For $x \in \mathbf{L}(G_1)$ we consider the continuous homomorphism $\varphi \circ \gamma_x \in \operatorname{Hom}_c(\mathbb{R}, G_2)$. Since this one-parameter group is smooth (Theorem 6.2.5), it is of the form

$$\varphi \circ \gamma_x(t) = \exp_{G_2}(ty)$$

for $y = (\varphi \circ \gamma_x)'(0) \in \mathbf{L}(G_2)$. We define a map $\mathbf{L}(\varphi)\colon \mathbf{L}(G_1) \to \mathbf{L}(G_2)$ by $\mathbf{L}(\varphi)x := (\varphi \circ \gamma_x)'(0)$. For $t = 1$ we then obtain

$$\exp_{G_2} \circ \mathbf{L}(\varphi) = \varphi \circ \exp_{G_1} \colon \mathbf{L}(G_1) \to G_2. \tag{6.6}$$

Next we show that $\mathbf{L}(\varphi)$ is a linear map. Our definition immediately shows that $\mathbf{L}(\varphi)\lambda x = \lambda\, \mathbf{L}(\varphi)x$ for each $x \in \mathbf{L}(G_1)$. Further, the Product Formula (Proposition 6.2.6) yields

$$\begin{aligned}
&\exp_{G_2}(\mathbf{L}(\varphi)(x+y)) = \varphi\big(\exp_{G_1}(x+y)\big)\\
&= \lim_{k\to\infty} \varphi\Big(\exp_{G_1}\big(\tfrac{1}{k}x\big)\exp_{G_1}\big(\tfrac{1}{k}y\big)\Big)^k\\
&= \lim_{k\to\infty} \Big(\exp_{G_2}\big(\tfrac{1}{k}\,\mathbf{L}(\varphi)x\big)\exp_{G_2}\big(\tfrac{1}{k}\,\mathbf{L}(\varphi)y\big)\Big)^k\\
&= \exp_{G_2}\big(\mathbf{L}(\varphi)x + \mathbf{L}(\varphi)y\big).
\end{aligned}$$

This proves that $\mathbf{L}(\varphi)(x+y) = \mathbf{L}(\varphi)x + \mathbf{L}(\varphi)y$, so that $\mathbf{L}(\varphi)$ is indeed a linear map. Now the smoothness of φ follows from (6.6) and Corollary 6.2.2. $\square$

Corollary 6.2.8. *A topological group G carries at most one Lie group structure.*

Proof. If G_1 and G_2 are two Lie groups which are isomorphic as topological groups, then the Automatic Smoothness Theorem applies to each topological isomorphism $\varphi\colon G_1 \to G_2$ and shows that φ is smooth. It likewise applies to φ^{-1}, so that φ is an isomorphism of Lie groups. $\square$

6.3 The Adjoint Representation

The Lie functor associates linear automorphisms of the Lie algebra with conjugations on the Lie group. The resulting representation of the Lie group is called the adjoint representation. Its interplay with the exponential function will be important in the entire theory.

Definition 6.3.1. We know that for each finite-dimensional vector space V, the group $\mathrm{GL}(V)$ carries a natural Lie group structure. For a Lie group G, a smooth homomorphism $\pi\colon G \to \mathrm{GL}(V)$ is a called a *representation of G on V* (cf. Exercise 6.5.3).

Any representation defines a smooth action of G on V via

$$\sigma(g, v) := \pi(g)(v).$$

In this sense, representations are the same as *linear actions*, i.e., actions on vector spaces for which the σ_g are linear.

As a consequence of Proposition 5.2.2, we obtain

Proposition 6.3.2. *If $\varphi\colon G \to \mathrm{GL}(V)$ is a representation of G, then $\mathbf{L}(\varphi)\colon \mathbf{L}(G) \to \mathfrak{gl}(V)$ is a representation of its Lie algebra $\mathbf{L}(G)$.*

The representation $\mathbf{L}(\varphi)$ obtained in Proposition 6.3.2 from the group representation φ is called the *derived representation.* This is motivated by the fact that for each $x \in \mathbf{L}(G)$ we have

$$\mathbf{L}(\varphi)(x) = \frac{d}{dt}\Big|_{t=0} e^{t\,\mathbf{L}(\varphi)x} = \frac{d}{dt}\Big|_{t=0} \varphi(\exp_G tx).$$

Let G be a Lie group and $\mathbf{L}(G)$ its Lie algebra. For $g \in G$ we recall the conjugation automorphism $c_g \in \mathrm{Aut}(G), c_g(x) = gxg^{-1}$, and define

$$\mathrm{Ad}(g) := \mathbf{L}(c_g) \in \mathrm{Aut}(\mathbf{L}(G)).$$

Then

$$\mathrm{Ad}(g_1 g_2) = \mathbf{L}(c_{g_1 g_2}) = \mathbf{L}(c_{g_1}) \circ \mathbf{L}(c_{g_2}) = \mathrm{Ad}(g_1)\,\mathrm{Ad}(g_2)$$

shows that $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathbf{L}(G))$ is a group homomorphism. It is called the *adjoint representation.* To see that it is smooth, we observe that for each $x \in \mathbf{L}(G)$ we have

$$\mathrm{Ad}(g)x = T_{\mathbf{1}}(c_g)x = T_{\mathbf{1}}(\lambda_g \circ \rho_{g^{-1}})x = T_{g^{-1}}(\lambda_g)T_{\mathbf{1}}(\rho_{g^{-1}})x = 0_g \cdot x \cdot 0_{g^{-1}}$$

in the Lie group $T(G)$ (Lemma 5.1.6). Since the multiplication in $T(G)$ is smooth, the representation Ad of G on $\mathbf{L}(G)$ is smooth (cf. Exercise 6.5.3), and

$$\mathbf{L}(\mathrm{Ad})\colon \mathbf{L}(G) \to \mathfrak{gl}(\mathbf{L}(G))$$

is a representation of $\mathbf{L}(G)$ on $\mathbf{L}(G)$. The following lemma gives a formula for this representation.

Lemma 6.3.3. $\mathbf{L}(\mathrm{Ad}) = \mathrm{ad}$*, i.e.,* $\mathbf{L}(\mathrm{Ad})(x)(y) = [x, y]$.

Proof. Let $x, y \in \mathbf{L}(G)$ and x_l, y_l be the corresponding left invariant vector fields. Corollary 5.2.4 implies for $g \in G$ the relation

$$(c_g)_* y_l = \big(\mathbf{L}(c_g)y\big)_l = \big(\mathrm{Ad}(g)y\big)_l.$$

On the other hand, the left invariance of y_l leads to

$$(c_g)_* y_l = (\rho_g^{-1} \circ \lambda_g)_* y_l = (\rho_g^{-1})_* (\lambda_g)_* y_l = (\rho_g^{-1})_* y_l.$$

Next we observe that $\Phi_t^{x_l} = \rho_{\exp_G(tx)}$ is the flow of the vector field x_l (Lemma 6.1.4), so that Theorem 4.4.16 implies that

$$\begin{aligned}[x_l, y_l] = \mathcal{L}_{x_l} y_l &= \frac{d}{dt}\Big|_{t=0} (\Phi_{-t}^{x_l})_* y_l = \frac{d}{dt}\Big|_{t=0} (c_{\exp_G(tx)})_* y_l \\ &= \frac{d}{dt}\Big|_{t=0} \big(\operatorname{Ad}(\exp_G(tx))y\big)_l.\end{aligned}$$

Evaluating in $\mathbf{1}$, we get

$$[x, y] = [x_l, y_l](\mathbf{1}) = \frac{d}{dt}\Big|_{t=0} \operatorname{Ad}(\exp_G(tx))y = \mathbf{L}(\operatorname{Ad})(x)(y). \qquad \square$$

Combining Proposition 6.2.1 with Lemma 6.3.3, we obtain the important formula

$$\operatorname{Ad} \circ \exp_G = \exp_{\operatorname{Aut}(\mathbf{L}(G))} \circ \operatorname{ad},$$

i.e.,

$$\operatorname{Ad}(\exp_G(x)) = e^{\operatorname{ad} x} \quad \text{for} \quad x \in \mathbf{L}(G). \tag{6.7}$$

Lemma 6.3.4. *For a Lie group G, the kernel of the adjoint representation* $\operatorname{Ad}\colon G \to \operatorname{Aut}(\mathbf{L}(G))$, *is given by*

$$Z_G(G_0) := \{g \in G \colon (\forall x \in G_0)\, gx = xg\},$$

where G_0 is the connected component of the identity in G. If, in addition, G is connected, then

$$\ker \operatorname{Ad} = Z(G).$$

Proof. Since G_0 is connected, the automorphism $c_g|_{G_0}$ of G_0 is trivial if and only if $\mathbf{L}(c_g) = \operatorname{Ad}(g)$ is trivial. This implies the lemma. $\square$

As we shall see later, in many situations it is important to have some information on the center of (simply) connected Lie groups. Below we shall use Lemma 6.3.4 to determine the kernel of the adjoint representation for various Lie groups. For that we have to know their center.

Example 6.3.5. (a) Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. First we recall from Proposition 1.1.10 that $Z(\operatorname{GL}_n(\mathbb{K})) = \mathbb{K}^\times \mathbf{1}$ and from Exercise 1.2.14(v) that

$$Z(\operatorname{SL}_n(\mathbb{K})) = \{z\mathbf{1} \colon z \in \mathbb{K}^\times, z^n = 1\}.$$

In particular,

$$Z(\operatorname{SL}_n(\mathbb{C})) = \{z\mathbf{1} \colon z^n = 1\} \cong C_n$$

and

$$Z(\operatorname{SL}_n(\mathbb{R})) = \begin{Bmatrix} \mathbf{1} & \text{for } n \in 2\mathbb{N}_0 + 1 \\ \{\pm\mathbf{1}\} & \text{for } n \in 2\mathbb{N}. \end{Bmatrix}$$

(b) For $g \in Z(\mathrm{SU}_n(\mathbb{C})) = \ker \mathrm{Ad}$ we likewise have $gx = xg$ for all $x \in \mathfrak{su}_n(\mathbb{C})$. From

$$\mathfrak{gl}_n(\mathbb{C}) = \mathfrak{u}_n(\mathbb{C}) + i\mathfrak{u}_n(\mathbb{C}) = \mathfrak{su}_n(\mathbb{C}) + i\,\mathfrak{su}_n(\mathbb{C}) + \mathbb{C}\mathbf{1},$$

we derive that $g \in Z(\mathrm{GL}_n(\mathbb{C})) = \mathbb{C}^\times \mathbf{1}$. From that we immediately get

$$Z(\mathrm{SU}_n(\mathbb{C})) = \{z\mathbf{1} \colon z^n = 1\} \cong C_n$$

and similarly we obtain

$$Z(\mathrm{U}_n(\mathbb{C})) = \{z\mathbf{1} \colon |z| = 1\} \cong \mathbb{T}.$$

(c) (cf. also Exercise 1.2.16) Next we show that

$$Z(\mathrm{O}_n(\mathbb{R})) = \{\pm\mathbf{1}\} \quad \text{and} \quad Z(\mathrm{SO}_n(\mathbb{R})) = \begin{cases} \mathrm{SO}_2(\mathbb{R}) & \text{for } n = 2 \\ \mathbf{1} & \text{for } n \in 2\mathbb{N}+1 \\ \{\pm\mathbf{1}\} & \text{for } n \in 2\mathbb{N}+2. \end{cases}$$

If $g \in Z\big(\mathrm{O}_n(\mathbb{R})\big)$, then g commutes with each orthogonal reflection

$$\sigma_v \colon \mathbb{R}^n \to \mathbb{R}^n, \quad w \mapsto w - 2\langle v, w\rangle v$$

in the hyperplane $v^\perp$, where v is a unit vector. Since $\mathbb{R}v$ is the -1-eigenspace of σ_v, this space is invariant under g (Exercise 1.1.1). This implies that for each $v \in \mathbb{R}^n$ we have $gv \in \mathbb{R}v$ which by an elementary argument leads to $g \in \mathbb{R}^\times\mathbf{1}$. We conclude that

$$Z(\mathrm{O}_n(\mathbb{R})) = \mathrm{O}_n(\mathbb{R}) \cap \mathbb{R}^\times\mathbf{1} = \{\pm\mathbf{1}\}.$$

To determine the center of $\mathrm{SO}_n(\mathbb{R})$, we consider for orthogonal unit vectors v_1, v_2 the map $\sigma_{v_1,v_2} := \sigma_{v_1}\sigma_{v_2} \in \mathrm{SO}_n(\mathbb{R})$ (a reflection in the subspace $v_1^\top \cap v_2^\top$). Since an element $g \in Z(\mathrm{SO}_n(\mathbb{R}))$ commutes with σ_{v_1,v_2}, it leaves the plane $\mathbb{R}v_1 + \mathbb{R}v_2 = \ker(\sigma_{v_1,v_2} + \mathbf{1})$ invariant. If a linear map preserves all two-dimensional planes and $n \geq 3$, then it preserves all one-dimensional subspaces. As above, we get $g \in \mathbb{R}^\times\mathbf{1}$, which in turn leads to

$$Z(\mathrm{SO}_n(\mathbb{R})) = \mathrm{SO}_n(\mathbb{R}) \cap \mathbb{R}^\times\mathbf{1},$$

and the assertion follows.

6.4 Semidirect Products

The easiest way to construct a new Lie group from two given Lie groups G and H, is to endow the product manifold $G \times H$ with the multiplication

$$(g_1, h_1)(g_2, h_2) := (g_1g_2, h_1h_2).$$

The resulting group is called the *direct product* of the Lie groups G and H. Here G and H can be identified with normal subgroups of $G \times H$ for which the multiplication map

$$(G \times \{\mathbf{1}\}) \times (\{\mathbf{1}\} \times H) \to G \times H, \quad ((g,\mathbf{1}),(\mathbf{1},h)) \mapsto (g,\mathbf{1})(\mathbf{1},h) = (g,h)$$

is a diffeomorphism. Relaxing this condition in the sense that only one factor is assumed to be normal, leads to the concept of a semidirect product of Lie groups, introduced below.

Definition 6.4.1. Let N and G be Lie groups and $\alpha\colon G \to \mathrm{Aut}(N)$ be a group homomorphism defining a smooth action $(g,n) \mapsto \alpha_g(n)$ of G on N.

(a) Then the product manifold $N \times G$ is a group with respect to the product

$$(n,g)(n',g') := (n\alpha_g(n'), gg')$$

and the inversion

$$(n,g)^{-1} = (\alpha_{g^{-1}}(n^{-1}), g^{-1}).$$

Since multiplication and inversion are smooth, this group is a Lie group, called the *semidirect product of N and G with respect to α.* It is denoted by $N \rtimes_\alpha G$.

(b) On the manifold $G \times N$ we also obtain a Lie group structure by

$$(g,n)(g',n') := (gg', \alpha_{g'}^{-1}(n)n'),$$

and this Lie group is denoted $G \ltimes_\alpha N$. It is easy to verify that the map

$$\Phi\colon N \rtimes_\alpha G \to G \ltimes_\alpha N, \quad (n,g) \mapsto (g, \alpha_g^{-1}(n))$$

is an isomorphism of Lie groups.

Example 6.4.2. A typical example of a semidirect product is the group $\mathrm{Aff}(V)$ of affine automorphisms of the vector space V. It consists of maps of the form $\psi(v) = \varphi(v) + b$, where $\varphi \in \mathrm{GL}(V)$ is an invertible linear map. Writing the elements of $\mathrm{Aff}(V)$ accordingly as pairs $\psi = (b,\varphi)$, we see that composition of affine maps corresponds to the composition formula

$$(b,\varphi)(b',\varphi') = (b + \varphi(b'), \varphi\varphi')$$

for pairs. With $\alpha\colon \mathrm{GL}(V) \to \mathrm{Aut}(V), \alpha_\varphi(v) = \varphi(v)$ we see that

$$\mathrm{Aff}(V) \cong V \rtimes_\alpha \mathrm{GL}(V)$$

is a semidirect product of the Lie groups $\mathrm{GL}(V)$ and $(V,+)$.

Remark 6.4.3. If $\widehat{G} := N \rtimes_\alpha G$ is a semidirect product, then

$$\pi\colon \widehat{G} \to G, \quad (n,g) \mapsto g, \qquad \sigma\colon G \to \widehat{G}, \quad g \mapsto (\mathbf{1}, g)$$

and $\iota\colon N \to \widehat{G}, n \mapsto (n,\mathbf{1})$ are morphisms of Lie groups with $\pi \circ \sigma = \mathrm{id}_G$ and ι is an isomorphism of N onto the closed subgroup $\ker \pi$ of $\widehat{G}$.

Example 6.4.4. Let G be a Lie group and $T(G)$ its tangent Lie group (Lemma 5.1.6). We have already seen that the map $G \times \mathbf{L}(G) \to TG$, $(g,x) \mapsto g.x = 0_g \cdot x$ is a diffeomorphism, and for similar reasons, the map $\mathbf{L}(G) \times G \to TG, (x,g) \mapsto x.g := x \cdot 0_g$ is a diffeomorphism. In these coordinates, the multiplication is given by

$$(x.g) \cdot (x'.g') = x \cdot 0_g \cdot x' \cdot 0_{g'} = x \cdot \mathrm{Ad}(g)x' \cdot 0_g \cdot 0_{g'} = (x + \mathrm{Ad}(g)x').gg'.$$

This shows that the tangent bundle is a semidirect product

$$TG \cong \mathbf{L}(G) \rtimes_{\mathrm{Ad}} G.$$

Similarly, the calculation

$$(g.x) \cdot (g'.x') = 0_{gg'} \cdot \mathrm{Ad}(g')^{-1}x \cdot x' = (gg', \mathrm{Ad}(g')^{-1}x + x')$$

shows that also

$$TG \cong G \ltimes_{\mathrm{Ad}} \mathbf{L}(G).$$

Proposition 6.4.5. *The Lie algebra of the semidirect product group $N \rtimes_\alpha G$ is given by*

$$\mathbf{L}(N \rtimes_\alpha G) \cong \mathbf{L}(N) \rtimes_\beta \mathbf{L}(G),$$

where $\beta \colon \mathbf{L}(G) \to \mathrm{der}(\mathbf{L}(N))$ is the derived representation of $\mathbf{L}(G)$ on $\mathbf{L}(N)$ corresponding to the representation of G on $\mathbf{L}(N)$ given by $g.x := \mathbf{L}(\alpha_g)x$.

Proof. We identify $\mathbf{L}(N)$, resp., $\mathbf{L}(G)$, with a subspace of

$$T_{\mathbf{1}}(N) \oplus T_{\mathbf{1}}(G) = T_{(\mathbf{1},\mathbf{1})}(N \times G) \cong \mathbf{L}(N \rtimes_\alpha G).$$

Since N and G are subgroups, the functoriality of $\mathbf{L}$ implies that $\mathbf{L}(G)$ and $\mathbf{L}(N)$ are Lie subalgebras of $\mathbf{L}(N \rtimes_\alpha G)$. The normal subgroup N is the kernel of the projection $\pi \colon N \rtimes_\alpha G \to G$, so that our identification shows that $\mathbf{L}(N) = \ker \mathbf{L}(\pi)$ is an ideal of $\mathbf{L}(N \rtimes_\alpha G)$. This already implies

$$\mathbf{L}(N \rtimes_\alpha G) \cong \mathbf{L}(N) \rtimes_\beta \mathbf{L}(G)$$

for the homomorphism $\beta \colon \mathbf{L}(G) \to \mathrm{der}(\mathbf{L}(N))$, given by

$$(\beta(x)(y), 0) = [(0, x), (y, 0)].$$

To determine β in terms of α, we note that the smooth action of G on N by automorphisms induces a smooth action of G on the tangent bundle $T(N)$, hence in particular on $T_{\mathbf{1}}(N) \cong \mathbf{L}(N)$. We thus obtain a representation $\pi \colon G \to \mathrm{Aut}(\mathbf{L}(N))$. In $N \rtimes_\alpha G$ we have $(\mathbf{1}, g)(n, \mathbf{1})(\mathbf{1}, g)^{-1} = (\alpha_g(n), \mathbf{1})$, so that

$$\pi(g)y = \mathbf{L}(\alpha_g)y = \mathrm{Ad}(\mathbf{1}, g)(y, 0).$$

Now Lemma 6.3.3 immediately shows that $\mathbf{L}(\pi)x = \mathrm{ad}(0, x) = \beta(x)$. □

6.5 The Baker–Campbell–Dynkin–Hausdorff Formula

In this section we show that the formula

$$\exp_G(x * y) = \exp_G x \exp_G y,$$

where $x * y$, for sufficiently small elements $x, y \in \mathfrak{g} = \mathbf{L}(G)$, is given by the Hausdorff series (cf. Proposition 2.4.5), also holds for the exponential function of a general Lie group G with Lie algebra $\mathfrak{g}$.

Definition 6.5.1. For a smooth function $f\colon M \to G$ of a smooth manifold M with values in the Lie group G, we define its *(left) logarithmic derivative* as the function

$$\delta(f)\colon TM \to \mathfrak{g}, \quad \delta(f)(v) := f(m)^{-1} \cdot T_m(f)v \quad \text{for} \quad v \in T_m(M).$$

This map is a convenient way to describe the derivative of f in terms of a less complex structure than the tangent map $Tf\colon TM \to TG$.

Lemma 6.5.2. *For two smooth maps $f, h\colon M \to G$, the logarithmic derivative of the pointwise products fh and fh^{-1} is given by the*

(1) *Product Rule:* $\delta(fh) = \delta(h) + \mathrm{Ad}(h^{-1})\delta(f)$, *and the*

(2) *Quotient Rule:* $\delta(fh^{-1}) = \mathrm{Ad}(h)(\delta(f) - \delta(h))$.

Proof. Writing $fg = m_G \circ (f, g)$, we obtain from

$$T_{(a,b)}(m_G)(v, w) = v \cdot b + a \cdot w$$

for $a, b \in G$ and $v, w \in \mathbf{L}(G) \subseteq TG$ (Lemma 5.1.6), the relation

$$T(fh) = T(m_G) \circ (T(f), T(h)) = T(f) \cdot h + f \cdot T(h)\colon T(M) \to T(G),$$

where $f \cdot T(h)$, resp., $T(f) \cdot h$ refers to the pointwise product in the group $T(G)$, containing G as the zero section (Lemma 5.1.6). This immediately leads to the Product Rule

$$\delta(fh) = (fh)^{-1} \cdot (T(f) \cdot h + f \cdot T(h)) = h^{-1} \cdot (\delta(f) \cdot h) + \delta(h) = \mathrm{Ad}(h)^{-1}\delta(f) + \delta(h).$$

For $h = f^{-1}$, we then obtain

$$0 = \delta(ff^{-1}) = \mathrm{Ad}(f)\delta(f) + \delta(f^{-1}),$$

hence $\delta(f^{-1}) = -\mathrm{Ad}(f)\delta(f)$. This in turn leads to

$$\delta(fh^{-1}) = \mathrm{Ad}(h)\delta(f) + \delta(h^{-1}) = \mathrm{Ad}(h)\delta(f) - \mathrm{Ad}(h)\delta(h),$$

which is the Quotient Rule. □

Remark 6.5.3. For any $g \in G$ and a smooth function $f\colon M \to G$, the function $g \cdot f = \lambda_g \circ f$ has the same logarithmic derivative as f because

$$\delta(g \cdot f) = \delta(f) + \mathrm{Ad}(f)^{-1}\delta(g) = \delta(f)$$

is a consequence of the Product Rule and the fact that $\delta(g) = 0$ for the constant map with value g.

Proposition 6.5.4. *The logarithmic derivative of* $\exp_G$ *is given by*

$$\delta(\exp_G)(x) = \Phi(\operatorname{ad} x) \in \mathcal{L}(\mathfrak{g}), \quad \textit{where} \quad \Phi(z) := \frac{1 - e^{-z}}{z} = \sum_{k=1}^{\infty} \frac{(-z)^{k-1}}{k!}.$$

With respect to the group structure on TG, this can also be written as

$$T_x(\exp_G)y = \exp_G(x) \cdot \Phi(\operatorname{ad} x)y \quad \textit{for} \quad x, y \in \mathbf{L}(G).$$

Proof. Fix $t, s \in \mathbb{R}$. Then the smooth functions $f, f_t, f_s \colon \mathbf{L}(G) \to G$, given by

$$f(x) := \exp_G((t+s)x), \quad f_t(x) := \exp_G(tx) \quad \text{and} \quad f_s(x) := \exp_G(sx),$$

satisfy $f = f_t f_s$ pointwise on $\mathbf{L}(G)$. The Product Rule (Lemma 6.5.2) therefore implies that

$$\delta(f) = \delta(f_s) + \mathrm{Ad}(f_s)^{-1}\delta(f_t).$$

For the smooth curve $\psi \colon \mathbb{R} \to \mathbf{L}(G), \psi(t) := \delta(\exp_G)_{tx}(ty)$, we now obtain

$$\begin{aligned} \psi(t+s) &= \delta(f)_x(y) = \delta(f_s)_x(y) + \mathrm{Ad}(f_s)^{-1}\delta(f_t)_x(y) \\ &= \psi(s) + \mathrm{Ad}(\exp_G(-sx))\psi(t). \end{aligned}$$

We have $\psi(0) = 0$ and

$$\psi'(0) = \lim_{t\to 0} \delta(\exp_G)_{tx}(y) = \delta(\exp_G)_0(y) = y,$$

so that taking derivatives with respect to t in 0, leads with (6.7) to

$$\psi'(s) = \mathrm{Ad}(\exp_G(-sx))y = e^{-\operatorname{ad}(sx)}y.$$

Now the assertion follows by integration from

$$\delta(\exp_G)_x(y) = \psi(1) = \int_0^1 \psi'(s)\, ds$$

and $\int_0^1 e^{-s \operatorname{ad} x}\, ds = \sum_{k=0}^\infty \frac{(-\operatorname{ad} x)^k}{(k+1)!} = \Phi(\operatorname{ad} x)$, which we saw already in the proof of Proposition 2.4.2. □

Let $U \subseteq \mathfrak{g}$ be a convex 0-neighborhood for which $\exp_G|_U$ is a diffeomorphism onto an open subset of G and $V \subseteq U$ a smaller convex open 0-neighborhood with $\exp_G V \exp_G V \subseteq \exp_G U$. Put $\log_U := (\exp_G|_U)^{-1}$ and define

$$x * y := \log_U(\exp_G x \exp_G y) \quad \text{for} \quad x, y \in V.$$

This defines a smooth map $V \times V \to U$. Fix $x, y \in V$. Then the smooth curve $F(t) := x * ty \in U$ satisfies $\exp_G F(t) = \exp_G(x)\exp_G(ty)$, so that the logarithmic derivative of this curve is

$$y = \delta(\exp_G)_{F(t)}F'(t) = \Phi(\operatorname{ad} F(t))F'(t).$$

We now choose U so small that the power series $\Psi(z) = \frac{z \log z}{z-1}$ from Lemma 2.4.3 satisfies

$$\Psi(e^{\operatorname{ad} z})\Phi(\operatorname{ad} z) = \mathrm{id}_{\mathfrak{g}} \quad \text{for} \quad z \in U$$

(Lemma 2.4.3). For $z = F(t)$, we then arrive with Proposition 6.5.4 at

$$F'(t) = \Psi(e^{\operatorname{ad} F(t)})y.$$

Now the same arguments as in Propositions 2.4.4 and 2.4.5 imply that

$$x * y = F(1) = x + y + \frac{1}{2}[x,y] + \cdots$$

is given by the convergent *Hausdorff series*:

Proposition 6.5.5. *If G is a Lie group, then there exists a convex 0-neighborhood $V \subseteq \mathfrak{g}$ such that for $x, y \in V$ the Hausdorff series*

$$x * y := x + \sum_{\substack{k,m \geq 0 \\ p_i + q_i > 0}} \frac{(-1)^k}{(k+1)(q_1 + \ldots + q_k + 1)} \frac{(\operatorname{ad} x)^{p_1} (\operatorname{ad} y)^{q_1} \ldots (\operatorname{ad} x)^{p_k} (\operatorname{ad} y)^{q_k} (\operatorname{ad} x)^m}{p_1! q_1! \ldots p_k! q_k! m!} y.$$

converges and satisfies

$$\exp_G(x * y) = \exp_G(x) \exp_G(y).$$

With the same proof as for Proposition 2.4.7, we obtain:

Proposition 6.5.6. *Let G be a Lie group. For $x, y \in \mathbf{L}(G)$ we have the following commutator formula holds:*

$$\lim_{k \to \infty} \Big(\exp\Big(\frac{1}{k}x\Big) \exp\Big(\frac{1}{k}y\Big) \exp\Big(-\frac{1}{k}x\Big) \exp\Big(-\frac{1}{k}y\Big)\Big)^{k^2} = \exp([x, y]).$$

Exercises for Chapter 6

Exercise 6.5.1. Let G be a connected Lie group and $x \in \mathfrak{g} = \mathbf{L}(G)$. Show that the corresponding left invariant vector field $x_l \in \mathcal{V}(G)$ is biinvariant, i.e., also invariant under all right multiplications, if and only if $x \in \mathfrak{z}(\mathfrak{g})$.

Exercise 6.5.2. Let $f_1, f_2 \colon G \to H$ be two group homomorphisms. Show that the pointwise product

$$f_1 f_2 \colon G \to H, \quad g \mapsto f_1(g) f_2(g)$$

is a homomorphism if and only if $f_1(G)$ commutes with $f_2(G)$.

Exercise 6.5.3. Let M be a manifold and V a finite-dimensional vector space with a basis $(b_1, \ldots, b_n)$. Let $f \colon M \to \mathrm{GL}(V)$ be a map. Show that the following are equivalent:

(1) f is smooth.

(2) For each $v \in V$ the map $f_v \colon M \to V, m \mapsto f(m)v$ is smooth.

(3) For each i, the map $f \colon M \to V, m \mapsto f(m)b_i$ is smooth.

Exercise 6.5.4. A vector field X on a Lie group G is called *right invariant* if for each $g \in G$ the vector field $(\rho_g)_* X = T(\rho_g) \circ X \circ \rho_g^{-1}$ coincides with X. We write $\mathcal{V}(G)^r$ for the set of right invariant vector fields on G. Show that:

(1) The evaluation map $\mathrm{ev}_{\mathbf{1}} \colon \mathcal{V}(G)^r \to T_{\mathbf{1}}(G)$ is a linear isomorphism.

(2) If X is right invariant, then there exists a unique $x \in T_{\mathbf{1}}(G)$ such that $X(g) = x_r(g) := T_{\mathbf{1}}(\rho_g)x = x \cdot 0_g$ (w.r.t. multiplication in $T(G)$).

(3) If X is right invariant, then $\widetilde{X} := (\iota_G)_*X := T(\iota_G) \circ X \circ \iota_G^{-1}$ is left invariant and vice versa.

(4) Show that $(\iota_G)_*x_r = -x_l$ and $[x_r, y_r] = -[x, y]_r$ for $x, y \in T_{\mathbf{1}}(G)$.

(5) Show that each right invariant vector field is complete and determine its flow.

Exercise 6.5.5. Let G be a Lie group. Show that any map $\varphi\colon G \to G$ commuting with all left multiplications λ_g, $g \in G$, is a right multiplication.

Exercise 6.5.6. (The exponential function of $\mathrm{SU}_2(\mathbb{C})$) Show that:

(a) $\mathrm{U}_2(\mathbb{C}) = \mathbb{T}\,\mathrm{SU}_2(\mathbb{C}) = Z(\mathrm{U}_2(\mathbb{C}))\,\mathrm{SU}_2(\mathbb{C})$.

(b) If $x \in \mathfrak{su}_2(\mathbb{C})$ with eigenvalues $\pm i\lambda$, $\lambda \geq 0$, we have $\|x\| = \lambda$.

(c) For $x, y \in \mathfrak{su}_2(\mathbb{C})$, there exists an element $g \in \mathrm{SU}_2(\mathbb{C})$ with $y = \mathrm{Ad}(g)x$ if and only if $\|x\| = \|y\|$.

(d) No one-parameter group $\gamma : \mathbb{R} \to \mathrm{SU}_2(\mathbb{C})$ is injective, in particular, the image of $\gamma(\mathbb{R})$ is always circle group.

Exercise 6.5.7. Verify the following semidirect decompositions:

(a) $\mathrm{GL}_n(\mathbb{K}) \cong \mathrm{SL}_n(\mathbb{K}) \rtimes_\delta \mathbb{K}^\times$ for a suitable homomorphism $\delta\colon \mathbb{K}^\times \to \mathrm{Aut}(\mathrm{SL}_n(\mathbb{K}))$, where $\mathbb{K}$ is any field.

(b) $\mathrm{U}_n(\mathbb{C}) \cong \mathrm{SU}_n(\mathbb{C}) \rtimes_\delta \mathbb{T}$ for a suitable homomorphism $\delta\colon \mathbb{T} \to \mathrm{Aut}(\mathrm{SU}_n(\mathbb{C}))$.

(c) $B \cong N \rtimes D$ for

$$\begin{aligned} N &:= \{g \in \mathrm{GL}_n(\mathbb{R})\colon (\forall i > j)\ g_{ij} = 0, g_{ii} = 1\} \\ B &:= \{g \in \mathrm{GL}_n(\mathbb{R})\colon (\forall i > j)\ g_{ij} = 0\} \\ D &:= \{g \in \mathrm{GL}_n(\mathbb{R})\colon (\forall i \neq j)\ g_{ij} = 0\}. \end{aligned}$$

Chapter 7

From Local Data to Lie Groups

In this chapter we first introduce a method to obtain the structure of a Lie group on an abstract group G from local data (Section 7.1). We continue in Section 7.2 with a study of closed subgroups of Lie groups. Here the main result is the Closed Subgroup Theorem asserting that any closed subgroup of a Lie group is a submanifold and that it carries a natural Lie group structure. Applying this to subgroups of $\mathrm{GL}_n(\mathbb{R})$, it follows in particular that all linear Lie groups are Lie groups. We conclude this chapter with Section 7.3, where we show that every Lie subalgebra $\mathfrak{h}$ of the Lie algebra $\mathbf{L}(G)$ of a Lie group G "generates" a so-called integral subgroup, but this subgroup need not be closed. The dense wind in the 2-torus is a typical example (cf. Lemma 3.4.8).

7.1 Constructing Lie Group Structures on Groups

In this section we describe some methods to construct Lie group structures on groups, starting from a manifold structure on some "identity neighborhood" for which the group operations are smooth close to $\mathbf{1}$.

7.1.1 Group Topologies from Local Data

We call a Hausdorff topology on a group G a *group topology* if it turns G into a topological group, i.e., the group operations are continuous maps. The following lemma tells us how to construct a group topology on a group G from a filter basis of subsets which then becomes a filter basis of identity neighborhoods for the group topology.

Definition 7.1.1. Let X be a set. A set $\mathcal{F} \subseteq \mathbb{P}(X)$ of subsets of X is called a *filter basis* if the following conditions are satisfied:

(F1) $\mathcal{F} \neq \emptyset$.

(F2) Each set $F \in \mathcal{F}$ is nonempty.

(F3) $A, B \in \mathcal{F} \Rightarrow (\exists C \in \mathcal{F})\ C \subseteq A \cap B$.

Lemma 7.1.2. *Let G be a group and $\mathcal{F} \subseteq \mathbb{P}(G)$ a filter basis satisfying $\bigcap \mathcal{F} = \{\mathbf{1}\}$ and*

(U1) $(\forall U \in \mathcal{F})(\exists V \in \mathcal{F})\ VV \subseteq U$.

(U2) $(\forall U \in \mathcal{F})(\exists V \in \mathcal{F})\ V^{-1} \subseteq U$.

(U3) $(\forall U \in \mathcal{F})(\forall g \in G)(\exists V \in \mathcal{F})\ gVg^{-1} \subseteq U$.

Then there exists a unique group topology on G such that $\mathcal{F}$ is a basis of $\mathbf{1}$-neighborhoods in G. This topology is given by

$$\{U \subseteq G \colon (\forall g \in U)(\exists V \in \mathcal{F})\ gV \subseteq U\}.$$

Proof. Let

$$\tau := \{U \subseteq G \colon (\forall g \in U)(\exists V \in \mathcal{F})\ gV \subseteq U\}.$$

First we show that τ is a topology. Clearly $\emptyset, G \in \tau$. Let $(U_j)_{j\in J}$ be a family of elements of τ and $U := \bigcup_{j\in J} U_j$. For each $g \in U$, there exists a $j_0 \in J$ with $g \in U_{j_0}$ and a $V \in \mathcal{F}$ with $gV \subseteq U_{j_0} \subseteq U$. Thus $U \in \tau$ and we see that τ is stable under arbitrary unions.

If $U_1, U_2 \in \tau$ and $g \in U_1 \cap U_2$, there exist $V_1, V_2 \in \mathcal{F}$ with $gV_i \subseteq U_i$. Since $\mathcal{F}$ is a filter basis, there exists $V_3 \in \mathcal{F}$ with $V_3 \subseteq V_1 \cap V_2$, and then $gV_3 \subseteq U_1 \cap U_2$. We conclude that $U_1 \cap U_2 \in \tau$, and hence that τ is a topology on G.

We claim that the interior U° of a subset $U \subseteq G$ is given by

$$U_1 := \{u \in U \colon (\exists V \in \mathcal{F})\ uV \subseteq U\}.$$

In fact, if there exists a $V \in \mathcal{F}$ with $uV \subseteq U$, then we pick a $W \in \mathcal{F}$ with $WW \subseteq V$ and obtain $uWW \subseteq U$, so that $uW \subseteq U_1$. Hence $U_1 \in \tau$, i.e., U_1 is open, and it clearly is the largest open subset contained in U, i.e., $U_1 = U^\circ$. It follows in particular that U is a neighborhood of g if and only if $g \in U^\circ$, and we see in particular that $\mathcal{F}$ is a neighborhood basis at $\mathbf{1}$. The property $\bigcap \mathcal{F} = \{\mathbf{1}\}$ implies that for $x \neq y$ there exists $U \in \mathcal{F}$ with $y^{-1}x \notin U$. For $V \in \mathcal{F}$ with $VV \subseteq U$ and $W \in \mathcal{F}$ with $W^{-1} \subseteq V$ we then obtain $y^{-1}x \notin VW^{-1}$, i.e., $xW \cap yV = \emptyset$. Thus (G, τ) is a Hausdorff space.

To see that G is a topological group, we have to verify that the map

$$f \colon G \times G \to G, \quad (x, y) \mapsto xy^{-1}$$

is continuous. So let $x, y \in G$, $U \in \mathcal{F}$ and pick $V \in \mathcal{F}$ with $yVy^{-1} \subseteq U$ and $W \in \mathcal{F}$ with $WW^{-1} \subseteq V$. Then

$$f(xW, yW) = xWW^{-1}y^{-1} = xy^{-1}y(WW^{-1})y^{-1} \subseteq xy^{-1}yVy^{-1} \subseteq xy^{-1}U$$

implies that f is continuous in (x, y). $\square$

Before we turn to Lie group structures, it is illuminating to first consider the topological variant.

Lemma 7.1.3. *Let G be a group and $U = U^{-1}$ a symmetric subset containing $\mathbf{1}$. We further assume that U carries a Hausdorff topology for which*

(T1) *$D := \{(x,y) \in U \times U \colon xy \in U\}$ is an open subset of $U \times U$ and the group multiplication $m_U \colon D \to U, (x,y) \mapsto xy$ is continuous,*

(T2) *the inversion map $\iota_U \colon U \to U, u \mapsto u^{-1}$ is continuous, and*

(T3) *for each $g \in G$, there exists an open $\mathbf{1}$-neighborhood U_g in U with $c_g(U_g) \subseteq U$, such that the conjugation map $c_g \colon U_g \to U, x \mapsto gxg^{-1}$ is continuous.*

Then there exists a unique group topology on G for which the inclusion map $U \hookrightarrow G$ is a homeomorphism onto an open subset of G.

If, in addition, U generates G, then (T1/2) *imply* (T3)*.*

Proof. First we consider the filter basis $\mathcal{F}$ of $\mathbf{1}$-neighborhoods in U. Then (T1) implies (U1), (T2) implies (U2), and (T3) implies (U3). Moreover, the assumption that U is Hausdorff implies that $\bigcap \mathcal{F} = \{\mathbf{1}\}$. Therefore Lemma 7.1.2 implies that G carries a unique structure of a (Hausdorff) topological group for which $\mathcal{F}$ is a basis of $\mathbf{1}$-neighborhoods.

We claim that the inclusion map $U \to G$ is an open embedding. So let $x \in U$. Then

$$U_x := U \cap x^{-1}U = \{y \in U \colon (x,y) \in D\} \tag{7.1}$$

is open in U and λ_x restricts to a continuous map $\lambda_x^U \colon U_x \to U$ with image $U_{x^{-1}}$. Its inverse $\lambda_{x^{-1}}^U$ is also continuous. Hence $\lambda_x^U \colon U_x \to U_{x^{-1}}$ is a homeomorphism. We conclude that the sets of the form xV, where V a neighborhood of $\mathbf{1}$, form a basis of neighborhoods of $x \in U$. Hence the inclusion map $U \hookrightarrow G$ is an open embedding.

Suppose, in addition, that G is generated by U. For each $g \in U$, there exists an open $\mathbf{1}$-neighborhood U_g with $gU_g \times \{g^{-1}\} \subseteq D$. Then $c_g(U_g) \subseteq U$, and the continuity of m_U implies that $c_g|_{U_g} \colon U_g \to U$ is continuous.

Hence, for each $g \in U$, the conjugation c_g is continuous in a neighborhood of $\mathbf{1}$. Since the set of all these g is a submonoid of G containing U, it contains U^n for each $n \in \mathbb{N}$, hence all of G because G is generated by $U = U^{-1}$. Therefore (T3) follows from (T1) and (T2). $\square$

7.1.2 Lie Group Structures from Local Data

The following theorem, the smooth version of the preceding lemma, is an important tool to construct Lie group structures on groups.

Theorem 7.1.4. *Let G be a group and $U = U^{-1}$ be a symmetric subset containing $\mathbf{1}$. We further assume that U is a smooth manifold and that*

(L1) *$D := \{(x,y) \in U \times U \colon xy \in U\}$ is an open subset and the multiplication $m_U \colon D \to U, (x,y) \mapsto xy$ is smooth,*

(L2) *the inversion map $\iota_U \colon U \to U, u \mapsto u^{-1}$ is smooth, and*

(L3) *for each $g \in G$ there exists an open $\mathbf{1}$-neighborhood $U_g \subseteq U$ with $c_g(U_g) \subseteq U$ and such that the conjugation map $c_g \colon U_g \to U, x \mapsto gxg^{-1}$ is smooth.*

Then there exists a unique structure of a Lie group on G such that the inclusion map $U \hookrightarrow G$ is a diffeomorphism onto an open subset of G.

If, in addition, U generates G, then (L1/2) *imply* (L3).

Proof. From the preceding Lemma 7.1.3, we immediately obtain a unique group topology on G for which the inclusion map $U \hookrightarrow G$ is an open embedding.

Now we turn to the manifold structure. Let $V = V^{-1} \subseteq U$ be an open $\mathbf{1}$-neighborhood with $VV \times VV \subseteq D$, for which there exists a chart (φ, V) of U. For $g \in G$ we consider the maps

$$\varphi_g \colon gV \to E, \quad \varphi_g(x) = \varphi(g^{-1}x)$$

which are homeomorphisms of gV onto $\varphi(V) \subseteq \mathbb{R}^n$. We claim that $(\varphi_g, gV)_{g \in G}$ is a smooth atlas of G.

Let $g_1, g_2 \in G$ and put $W := g_1V \cap g_2V$. If $W \neq \emptyset$, then $g_2^{-1}g_1 \in VV^{-1} = VV$. The smoothness of the map

$$\psi := \varphi_{g_2} \circ \varphi_{g_1}^{-1}|_{\varphi_{g_1}(W)} \colon \varphi_{g_1}(W) \to \varphi_{g_2}(W)$$

given by

$$\psi(x) = \varphi_{g_2}(\varphi_{g_1}^{-1}(x)) = \varphi_{g_2}(g_1\varphi^{-1}(x)) = \varphi(g_2^{-1}g_1\varphi^{-1}(x))$$

follows from the smoothness of the multiplication $VV \times VV \to U$. This proves that the charts $(\varphi_g, gU)_{g \in G}$ form a smooth atlas of G. Moreover, the construction implies that all left translations of G are smooth maps.

The construction also shows that, for each $g \in G$, the conjugation map $c_g \colon G \to G$ is smooth in a neighborhood of $\mathbf{1}$. Since all left translations are smooth, and

$$c_g \circ \lambda_x = \lambda_{c_g(x)} \circ c_g,$$

the smoothness of c_g in a neighborhood of $x \in G$ follows. Therefore all conjugations and hence also all right multiplications are smooth. The smoothness of the inversion follows from its smoothness on V and the fact that left and right multiplications are smooth. Finally, the smoothness of the multiplication follows from the smoothness in $\mathbf{1} \times \mathbf{1}$ because

$$g_1xg_2y = g_1g_2c_{g_2^{-1}}(x)y.$$

Next we show that the inclusion $U \hookrightarrow G$ of U is a diffeomorphism. So let $x \in U$ and recall the open set $U_x = U \cap x^{-1}U$ from (7.1). Then λ_x restricts to a smooth map $U_x \to U$ with image $U_{x^{-1}}$. Its inverse is also smooth. Hence $\lambda_x^U \colon U_x \to U_{x^{-1}}$ is a diffeomorphism. Since $\lambda_x \colon G \to G$ also is a diffeomorphism, it follows that the inclusion $\lambda_x \circ \lambda_{x^{-1}}^U \colon U_{x^{-1}} \to G$ is a diffeomorphism. As x was arbitrary, the inclusion of U in G is a diffeomorphic embedding.

The uniqueness of the Lie group structure is clear because each locally diffeomorphic bijective homomorphism between Lie groups is a diffeomorphism (Proposition 6.2.4(3)).

Finally, we assume that G is generated by U. We show that, in this case, (L3) is a consequence of (L1) and (L2); the argument is similar to the topological case. Indeed, for each $g \in U$, there exists an open **1**-neighborhood U_g with $gU_g \times \{g^{-1}\} \subseteq D$. Then $c_g(U_g) \subseteq U$, and the smoothness of m_U implies that $c_g|_{U_g} : U_g \to U$ is smooth. Hence, for each $g \in U$, the conjugation c_g is smooth in a neighborhood of **1**. Since the set of all these g is a submonoid of G containing U, it contains U^n for each $n \in \mathbb{N}$, hence all of G because G is generated by $U = U^{-1}$. Therefore (L3) is satisfied. □

Corollary 7.1.5. *Let G be a group and $N \trianglelefteq G$ a normal subgroup of G that carries a Lie group structure. Then there exists a unique Lie group structure on G for which N is an open subgroup if and only if for each $g \in G$ the restriction $c_g|_N$ is a smooth automorphism of N.*

Proof. If N is an open normal subgroup of the Lie group G, then clearly all inner automorphisms of G restrict to smooth automorphisms of N.

Suppose, conversely, that N is a normal subgroup of the group G which is a Lie group and that all inner automorphisms of G restrict to smooth automorphisms of N. Then we can apply Theorem 7.1.4 with $U = N$ and obtain a Lie group structure on G for which the inclusion $N \to G$ is a diffeomorphism onto an open subgroup of G. □

7.2 Closed Subgroups of Lie Groups and their Lie Algebras

In this section, we show that closed subgroups of Lie groups are always submanifolds, which in turn implies that they are Lie groups. For a closed subgroup H of G, its Lie algebra can be computed by

$$\mathbf{L}(H) \cong \{x \in \mathbf{L}(G) \colon \exp_G(\mathbb{R}x) \subseteq H\}.$$

This makes it particularly easy to verify that certain groups are Lie groups and to determine their Lie algebras.

7.2.1 Submanifolds

Definition 7.2.1. [Submanifolds] Let M be a smooth n-dimensional manifold. A subset $S \subseteq M$ is called a *d-dimensional submanifold* if for each $p \in S$ there exists a chart (φ, U) of M with $p \in U$ asuch that

$$\varphi(U \cap S) = \varphi(U) \cap (\mathbb{R}^d \times \{0\}). \tag{7.2}$$

A submanifold of codimension 1, i.e., $\dim S = n - 1$, is called a *smooth hypersurface.*

Remark 7.2.2. (a) Any discrete subset S of M is a 0-dimensional submanifold.

(b) If $n = \dim M$, any open subset $S \subseteq M$ is an n-dimensional submanifold. If, conversely, $S \subseteq M$ is an n-dimensional submanifold, then the definition immediately shows that S is open.

Lemma 7.2.3. *Any submanifold S of a manifold M has a natural manifold structure. If $i_S: S \to M$ denotes the inclusion map, then a map $\varphi: N \to S$ from a smooth manifold N to S is smooth if and only if $i_S \circ \varphi: N \to M$ is smooth.*

Proof. (a) We endow S with the subspace topology inherited from M, which turns it into a Hausdorff space. For each chart (φ, U) satisfying (7.2), we obtain a d-dimensional chart

$$(\varphi|_{U\cap S}, U \cap S)$$

of S. For two such charts coming from (φ, U) and (ψ, V), we have

$$\psi \circ \varphi^{-1}|_{\varphi(U\cap V\cap S)} = (\psi|_{V\cap S}) \circ (\varphi|_{U\cap S})^{-1}|_{\varphi(U\cap V\cap S)},$$

which is a smooth map onto an open subset of $\mathbb{R}^d$. We thus obtain a smooth d-dimensional atlas on S.

(b) To see that i_S is smooth, let $p \in S$ and (φ, U) be a chart satisfying (7.2). Then

$$\varphi \circ i_S \circ (\varphi|_{S\cap U})^{-1}: \varphi(U) \cap (\mathbb{R}^d \times \{0\}) \to \varphi(U) \subseteq \mathbb{R}^n$$

is the inclusion map, hence smooth. This implies that i_S is smooth.

(c) If $f: N \to S$ is smooth, then the composition $i_S \circ f$ is smooth. Suppose, conversely, that $i_S \circ f: N \to M$ is smooth. Let $p \in N$ and choose a chart (φ, U) of M satisfying (7.2) with $f(p) \in U$. Then the map

$$\varphi \circ i_S \circ f|_{f^{-1}(U)}: f^{-1}(U) \to \varphi(U) \subseteq \mathbb{R}^n$$

is smooth, but its values lie in

$$\varphi(U \cap S) = \varphi(U) \cap (\mathbb{R}^d \times \{0\}).$$

Therefore $\varphi \circ i_S \circ f|_{f^{-1}(U)}$ is also smooth as a map into $\mathbb{R}^d$, which means that

$$\varphi|_{U\cap S} \circ f|_{f^{-1}(U)}: f^{-1}(U) \to \varphi(U \cap S) \subseteq \mathbb{R}^d$$

is smooth, and hence that f is smooth as a map $N \to S$. □

7.2.2 The Lie Algebra of a Closed Subgroup

Definition 7.2.4. Let G be a Lie group and $H \leq G$ a closed subgroup. We set

$$\mathbf{L}^e(H) := \{x \in \mathbf{L}(G) : \exp_G(\mathbb{R}x) \subseteq H\}$$

and observe that $\mathbb{R}\,\mathbf{L}^e(H) \subseteq \mathbf{L}^e(H)$ follows immediately from the definition.

Note that, for each $x \in \mathbf{L}(G)$, the set

$$\{t \in \mathbb{R} : \gamma_x(t) = \exp_G(tx) \in H\} = \gamma_x^{-1}(H)$$

is a closed subgroup of $\mathbb{R}$, hence either discrete cyclic or equal to $\mathbb{R}$ (cf. Example 3.4.9).

Example 7.2.5. We consider the Lie group $G := \mathbb{R} \times \mathbb{T}$ (the cylinder) with Lie algebra $\mathbf{L}(G) \cong \mathbb{R}^2$ (Exercise 5.4.2) and the exponential function

$$\exp_G(x, y) = (x, e^{2\pi i y}).$$

For the closed subgroup $H := \mathbb{R} \times \{\mathbf{1}\}$, we then see that $(x, y) \in \mathbf{L}^e(H)$ is equivalent to $y = 0$, but $\exp_G^{-1}(H) = \mathbb{R} \times \mathbb{Z}$.

Proposition 7.2.6. *If $H \leq G$ is a closed subgroup of the Lie group G, then $\mathbf{L}^e(H)$ is a real Lie subalgebra of $\mathbf{L}(G)$.*

Proof. Let $x, y \in \mathbf{L}^e(H)$. For $k \in \mathbb{N}$ we then have $\exp_G(x/k)\exp_G(y/k) \in H$, and with the Product Formula (Proposition 6.2.6), we get

$$\exp_G(x + y) = \lim_{k\to\infty} \Big(\exp_G \frac{x}{k} \exp_G \frac{y}{k}\Big)^k \in H$$

because H is closed. Therefore $\exp_G(x + y) \in H$, and $\mathbb{R}\,\mathbf{L}^e(H) = \mathbf{L}^e(H)$ now implies $\exp_G(\mathbb{R}(x + y)) \subseteq H$, hence $x + y \in \mathbf{L}^e(H)$.

Similarly, we use the Commutator Formula (Proposition 6.5.6) to get

$$\exp_G[x, y] = \lim_{k\to\infty} \Big(\exp_G \frac{x}{k} \exp_G \frac{y}{k} \exp_G -\frac{x}{k} \exp_G -\frac{y}{k}\Big)^{k^2} \in H,$$

hence $\exp_G([x, y]) \in H$, and $\mathbb{R}\,\mathbf{L}^e(H) = \mathbf{L}^e(H)$ yields $[x, y] \in \mathbf{L}^e(H)$. $\square$

7.2.3 The Closed Subgroup Theorem and its Consequences

We now address more detailed information on closed subgroups of Lie groups. We start with three key lemmas providing the main information for the proof of the Closed Subgroup Theorem.

The following lemma follows from the same arguments as Lemma 3.4.2.

Lemma 7.2.7. *Let $W \subseteq \mathbf{L}(G)$ be an open 0-neighborhood for which $\exp_G|_W$ is a diffeomorphism and $\log_W \colon \exp_G(W) \to W$ its inverse function. Further, let $H \subseteq G$ be a closed subgroup and $(g_k)_{k\in\mathbb{N}}$ be a sequence in $H \cap \exp_G(W)$ with $g_k \neq \mathbf{1}$ for all $k \in \mathbb{N}$ and $g_k \to \mathbf{1}$. We put $y_k := \log_W g_k$ and fix a norm $\|\cdot\|$ on $\mathbf{L}(G)$. Then every cluster point of the sequence $\big\{\frac{y_k}{\|y_k\|} : k \in \mathbb{N}\big\}$ is contained in $\mathbf{L}^e(H)$.*

Proof. Let x be such a cluster point. Replacing the original sequence by a subsequence, we may assume that

$$x_k := \frac{y_k}{\|y_k\|} \to x \in \mathbf{L}(G).$$

Note that this implies $\|x\| = 1$. Let $t \in \mathbb{R}$ and put $p_k := \frac{t}{\|y_k\|}$. Then $tx_k = p_k y_k$ and $y_k \to \log_W \mathbf{1} = 0$, so that

$$\exp_G(tx) = \lim_{k\to\infty} \exp_G(tx_k) = \lim_{k\to\infty} \exp_G(p_k y_k)$$

and

$$\exp_G(p_k y_k) = \exp_G(y_k)^{[p_k]} \exp_G\big((p_k - [p_k])y_k\big),$$

where $[p_k] = \max\{l \in \mathbb{Z} \colon l \leq p_k\}$ is the *Gauß function*. We then have

$$\|(p_k - [p_k])y_k\| \leq \|y_k\| \to 0$$

and

$$\exp_G(tx) = \lim_{k\to\infty} (\exp_G y_k)^{[p_k]} = \lim_{k\to\infty} g_k^{[p_k]} \in H,$$

because H is closed. This implies $x \in \mathbf{L}^e(H)$. $\square$

Lemma 7.2.8. *Let $H \subseteq G$ be a closed subgroup and $E \subseteq \mathbf{L}(G)$ be a vector subspace complementing $\mathbf{L}^e(H)$. Then there exists a 0-neighborhood $U_E \subseteq E$ with*

$$H \cap \exp_G(U_E) = \{\mathbf{1}\}.$$

Proof. Let $\|\cdot\|$ be a norm on $\mathbf{L}(G)$. We argue by contradiction. If a neighborhood U_E with the required properties does not exist, then we find for each $k \in \mathbb{N}$ an element $0 \neq y_k \in E$ with $\|y_k\| \leq \frac{1}{k}$ and $g_k := \exp y_k \in H$. Note that $y_k \to 0$ implies that $g_k \to \mathbf{1}$. Now let $x \in E$ be a cluster point of the sequence $\frac{y_k}{\|y_k\|}$ which lies in the compact set $S_E := \{z \in E \colon \|z\| = 1\}$, so that at least one cluster point exists. According to Lemma 7.2.7, we have $x \in \mathbf{L}(H) \cap E = \{0\}$ because Lemma 7.2.7 applies since $g_k \in H \cap \exp_G(W)$ holds for k sufficiently large. We arrive at a contradiction to $\|x\| = 1$. This proves the lemma. $\square$

Lemma 7.2.9. *Let $E, F \subseteq \mathbf{L}(G)$ be vector subspaces with $E \oplus F = \mathbf{L}(G)$. Then the map*

$$\Phi\colon E \times F \to G, \quad (x,y) \mapsto \exp_G(x)\exp_G(y),$$

restricts to a diffeomorphism of a neighborhood of $(0,0)$ to an open $\mathbf{1}$-neighborhood in G.

Proof. The Chain Rule implies that

$$\begin{aligned} T_{(0,0)}(\Phi)(x,y) &= T_{(\mathbf{1},\mathbf{1})}(m_G) \circ (T_0(\exp_G)|_E, T_0(\exp_G)|_F)(x,y) \\ &= T_{(\mathbf{1},\mathbf{1})}(m_G)(x,y) = x + y, \end{aligned}$$

Since the addition map $E \times F \to \mathbf{L}(G) \cong T_{\mathbf{1}}(G)$ is bijective, the Inverse Function Theorem implies that Φ restricts to a diffeomorphism of an open neighborhood of $(0,0)$ in $E \times F$ onto an open neighborhood of $\mathbf{1}$ in G. $\square$

Theorem 7.2.10. (von Neumann's Closed Subgroup Theorem) *Let H be a closed subgroup of the Lie group G. Then the following assertions hold:*

(i) *Each 0-neighborhood in $\mathbf{L}^e(H)$ contains an open 0-neighborhood V such that $\exp_G|_V \colon V \to \exp_G(V)$ is a homeomorphism onto an open subset of H.*

(ii) *H is a submanifold of G and $m_H := m_G|_{H\times H}$ induces a Lie group structure on H such that the inclusion map $j_H \colon H \to G$ is a morphism of Lie groups for which $\mathbf{L}(j_H) \colon \mathbf{L}(H) \to \mathbf{L}(G)$ is an isomorphism of $\mathbf{L}(H)$ onto $\mathbf{L}^e(H)$.*

(iii) *Let $E \subseteq \mathbf{L}(G)$ be a vector space complement of $\mathbf{L}^e(H)$. Then there exists an open 0-neighborhood $V_E \subseteq E$ such that*

$$\varphi\colon V_E \times H \to \exp_G(V_E)H, \quad (x,h) \mapsto \exp_G(x)h$$

is a diffeomorphism onto an open subset of G.

In view of (ii) above, from now on, we identify $\mathbf{L}(H)$ with the subalgebra $\mathbf{L}^e(H)$ if H is a closed subgroup of G.

Proof. (i) Let $E \subseteq \mathbf{L}(G)$ be a vector space complement of the subspace $\mathbf{L}^e(H)$ of $\mathbf{L}(G)$ and define

$$\Phi\colon E \times \mathbf{L}^e(H) \to G, \quad (x,y) \mapsto \exp_G x \exp_G y.$$

According to Lemma 7.2.9, there exist open 0-neighborhoods $U_E \subseteq E$ and $U_H \subseteq \mathbf{L}^e(H)$ such that

$$\Phi_1 := \Phi|_{U_E \times U_H} \colon U_E \times U_H \to \exp_G(U_E)\exp_G(U_H)$$

is a diffeomorphism onto an open $\mathbf{1}$-neighborhood in G. In view of Lemma 7.2.8, we may even choose U_E so small that $\exp_G(U_E) \cap H = \{\mathbf{1}\}$.

Since $\exp_G(U_H) \subseteq H$, the condition

$$g = \exp_G x \exp_G y \in H \cap (\exp_G(U_E)\exp_G(U_H))$$

implies $\exp_G x = g(\exp_G y)^{-1} \in H \cap \exp_G U_E = \{\mathbf{1}\}$. Therefore

$$\exp_G(U_H) = H \cap (\exp_G(U_E)\exp_G(U_H))$$

is an open $\mathbf{1}$-neighborhood in H. This proves (i).

(ii) Let Φ_1, U_E and U_H be as in (i). For $h \in H$, the set $U_h := \lambda_h(\mathrm{im}(\Phi_1)) = h\,\mathrm{im}(\Phi_1)$ is an open neighborhood of h in G. Moreover, the map

$$\varphi_h\colon U_h \to E \oplus \mathbf{L}^e(H) = \mathbf{L}(G), \quad x \mapsto \Phi_1^{-1}(h^{-1}x)$$

is a diffeomorphism onto the open subset $U_E \times U_H$ of $\mathbf{L}(G)$, and we have

$$\begin{aligned}\varphi_h(U_h \cap H) &= \varphi_h(h\,\mathrm{im}(\Phi_1) \cap H) = \varphi_h(h(\mathrm{im}(\Phi_1) \cap H)) \\ &= \varphi_h(h\exp_G(U_H)) = \{0\} \times U_H = (U_E \times U_H) \cap (\{0\} \times \mathbf{L}^e(H)).\end{aligned}$$

Therefore the family $(\varphi_h, U_h)_{h \in H}$ provides a submanifold atlas for H in G. This defines a manifold structure on H for which $\exp_G|_{U_H}$ is a local chart (cf. Lemma 7.2.3).

The map $m_H\colon H \times H \to H$ is a restriction of the multiplication map m_G of G, hence smooth as a map $H \times H \to G$, and Lemma 7.2.3 implies that m_H is smooth. With a similar argument we see that the inversion ι_H of H is smooth. Therefore H is a Lie group and the inclusion map $j_H\colon H \to G$ a smooth homomorphism. The corresponding morphism of Lie algebras $\mathbf{L}(j_H)\colon \mathbf{L}(H) \to \mathbf{L}(G)$ is injective, and from $\exp_G \circ \mathbf{L}(j_H) = j_H \circ \exp_H$ it follows that its image consists of the set $\mathbf{L}^e(H)$ of all elements $x \in \mathbf{L}(G)$ with $\exp_G(\mathbb{R}x) \subseteq H$ because each element of $\mathbf{L}^e(H)$ defines a smooth one-parameter group of H (cf. Lemma 7.2.3).

(iii) Let E be as in the proof of (i) and consider the smooth map

$$\Psi\colon E \times H \to G, \quad (x,h) \mapsto \exp_G(x)h,$$

where H carries the submanifold structure from (ii). Since $\exp_H\colon \mathbf{L}^e(H) \to H$ is a local diffeomorphism in 0, the proof of (i) implies the existence of a 0-neighborhood $U_E \subseteq E$ and a $\mathbf{1}$-neighborhood $V_H \subseteq H$ such that

$$\Psi_1 := \Psi|_{U_E \times V_H} \colon U_E \times V_H \to \exp_G(U_E)V_H$$

is a diffeomorphism onto an open subset of G. We further recall from Lemma 7.2.8, that we may assume, in addition, that

$$\exp_G(U_E) \cap H = \{\mathbf{1}\}. \tag{7.3}$$

We now pick a small symmetric 0-neighborhood $V_E = -V_E \subseteq U_E$ such that $\exp_G(V_E)\exp_G(V_E) \subseteq \exp_G(U_E)V_H$. Its existence follows from the continuity of the multiplication in G. We claim that the map

$$\varphi := \Psi|_{V_E \times H} \colon V_E \times H \to \exp_G(V_E)H$$

is a diffeomorphism onto an open subset of G. To this end, we first observe that

$$\varphi \circ (\mathrm{id}_{V_E} \times \rho_h) = \rho_h \circ \varphi \quad \text{for each} \quad h \in H,$$

i.e., $\varphi(x, h'h) = \varphi(x, h')h$, so that

$$T_{(x,h)}(\varphi) \circ (\mathrm{id}_E \times T_{\mathbf{1}}(\rho_h)) = T_{\varphi(x,\mathbf{1})}(\rho_h) \circ T_{(x,\mathbf{1})}(\varphi).$$

Since $T_{(x,\mathbf{1})}(\varphi) = T_{(x,\mathbf{1})}(\Psi)$ is invertible for each $x \in V_E$, $T_{(x,h)}(\varphi)$ is invertible for each $(x,h) \in V_E \times H$. This implies that φ is a local diffeomorphism in each point (x,h). To see that φ is injective, we observe that

$$\exp_G(x)h = \varphi(x,h) = \varphi(x',h') = \exp_G(x')h'$$

implies that

$$\exp_G(x)^{-1}\exp_G(x') = h(h')^{-1} \in \exp_G(V_E)^2 \cap H \subseteq (\exp_G(U_E)V_H) \cap H = V_H,$$

where we have used (7.3). We thus obtain $\exp_G(x') \in \exp_G(x)V_H$, so that the injectivity of Ψ_1 yields $x = x'$, which in turn leads to $h = h'$. This proves that φ is injective and a local diffeomorphism, hence a diffeomorphism. $\square$

Example 7.2.11. Since linear Lie groups are closed subgroups $G \subseteq \mathrm{GL}_n(\mathbb{R})$, the Closed Subgroup Theorem implies that a linear Lie group carries a natural Lie group structure with $\mathbf{L}(G) \cong \mathbf{L}^e(G)$.

Example 7.2.12. We take a closer look at closed subgroups of the Lie group $(V,+)$, where V is a finite-dimensional vector space. From Example 6.1.3 we know that $\exp_V = \mathrm{id}_V$. Let $H \subseteq V$ be a closed subgroup. Then

$$\mathbf{L}(H) = \{x \in V : \mathbb{R}x \subseteq H\} \subseteq H$$

is the largest vector subspace contained in H. Let $E \subseteq V$ be a vector space complement for $\mathbf{L}(H)$. Then $V \cong \mathbf{L}(H) \times E$, and we derive from $\mathbf{L}(H) \subseteq H$ that

$$H \cong \mathbf{L}(H) \times (E \cap H).$$

Lemma 7.2.8 implies the existence of some 0-neighborhood $U_E \subseteq E$ with $U_E \cap H = \{0\}$, hence that $H \cap E$ is discrete because 0 is an isolated point of $H \cap E$. Now Exercise 7.3.4 implies the existence of linearly independent elements $f_1, \ldots, f_k \in E$ with

$$E \cap H = \mathbb{Z}f_1 + \ldots + \mathbb{Z}f_k.$$

We conclude that

$$H \cong \mathbf{L}(H) \times \mathbb{Z}^k \cong \mathbb{R}^d \times \mathbb{Z}^k \quad \text{for} \quad d = \dim \mathbf{L}(H).$$

Note that $\mathbf{L}(H)$ coincides with the connected component H_0 of 0 in H.

In view of Corollary 6.2.8, we may think of Lie groups as a special class of topological groups. We may therefore ask, which subgroups of a Lie group G are Lie groups with respect to the subspace topology:

Proposition 7.2.13. *A subgroup of a Lie group is a Lie group with respect to the induced topology if and only if it is closed.*

Proof. If H is closed, then the Closed Subgroup Theorem 7.2.10 implies that H is a submanifold of G which is a Lie group.

Suppose, conversely, that H is a Lie group. Then H possesses a compact identity neighborhood $K \subseteq H$. As K is also compact as a subset of G, it is in particular closed. Therefore H is locally closed, hence closed by Exercise 7.3.3. □

Definition 7.2.14. Let G be a Lie group. A *Lie subgroup* of G is a closed subgroup H together with its Lie group structure provided by Proposition 7.2.13.

7.2.4 Examples

Example 7.2.15. [Closed Subgroups of $\mathbb{T}$] Let $H \subseteq \mathbb{T} \subseteq (\mathbb{C}^\times, \cdot)$ be a closed proper (=different from $\mathbb{T}$) subgroup. Since $\dim \mathbb{T} = 1$, it follows that $\mathbf{L}(H) = \{0\}$, so that the Identity Neighborhood Theorem implies that H is discrete, hence finite because $\mathbb{T}$ is compact.

If $q \colon \mathbb{R} \to \mathbb{T}$ is the covering projection, $q^{-1}(H)$ is a closed proper subgroup of $\mathbb{R}$, hence cyclic (this is a very simple case of Exercise 7.3.4), which implies that $H = q(q^{-1}(H))$ is also cyclic. Therefore H is one of the groups

$$C_n = \{z \in \mathbb{T} \colon z^n = 1\}$$

of n-th roots of unity.

Example 7.2.16. [Subgroups of $\mathbb{T}^2$] (a) Let $H \subseteq \mathbb{T}^2$ be a closed proper subgroup. Then $\mathbf{L}(H) \neq \mathbf{L}(\mathbb{T}^2)$ implies $\dim H < \dim \mathbb{T}^2 = 2$. Further, H is compact, so that the group $\pi_0(H)$ of connected components of H is finite.

If $\dim H = 0$, then H is finite, and for $n := |H|$ it is contained in a subgroup of the form $C_n \times C_n$, where $C_n \subseteq \mathbb{T}$ is the subgroup of n-th roots of unity (cf. Example 7.2.15).

If $\dim H = 1$, then H_0 is a compact connected 1-dimensional Lie group, hence isomorphic to $\mathbb{T}$ (Exercise 7.3.5). Therefore $H_0 = \exp_{\mathbb{T}^2}(\mathbb{R}x)$ for some $x \in \mathbf{L}(H)$ with $\exp_{\mathbb{T}^2}(x) = (e^{2\pi i x_1}, e^{2\pi i x_2}) = (1,1)$, which is equivalent to $x \in \mathbb{Z}^2$. We conclude that the Lie algebras of the closed subgroups are of the form $\mathbf{L}(H) = \mathbb{R}x$ for some $x \in \mathbb{Z}^2$.

(b) For each $\theta \in \mathbb{R} \setminus \mathbb{Q}$ the image of the 1-parameter group

$$\gamma \colon \mathbb{R} \to \mathbb{T}^2, \quad t \mapsto (e^{i\theta t}, e^{it})$$

is not closed because γ is injective. Hence the closure of $\gamma(\mathbb{R})$ is a closed subgroup of dimension at least 2, which shows that $\gamma(\mathbb{R})$ is dense in $\mathbb{T}^2$ (cf. Lemma 3.4.8).

7.3 Existence of a Lie Group for a given Lie Algebra

We have seen in the preceding section that a closed subgroup H of a Lie group G is a Lie group and that its Lie algebra can be identified with a subalgebra of $\mathbf{L}(G)$. As the dense wind in $G = \mathbb{T}^2$ shows, in general, not all Lie subalgebras $\mathfrak{h} \subseteq \mathbf{L}(G)$ correspond to closed subgroups of G. The following theorem shows that we may nonetheless find a Lie group structure on an arcwise connected subgroup H of G for which $\mathbf{L}(H) \cong \mathfrak{h}$.

Theorem 7.3.1. (Integral Subgroup Theorem) *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{h} \subseteq \mathfrak{g}$ a Lie subalgebra. Then the subgroup $H := \langle \exp \mathfrak{h} \rangle$ of G generated by $\exp_G \mathfrak{h}$ carries a Lie group structure with the following properties:*

(a) *The inclusion $j_H \colon H \to G$ is a smooth morphism of Lie groups and $\mathbf{L}(j_H) \colon \mathbf{L}(H) \to \mathfrak{h}$ an isomorphism of Lie algebras. These two properties determine the Lie group structure on H uniquely.*

(b) *H is connected.*

(c) *H is closed in G if and only if j_H is a topological embedding.*

Proof. (a) Let $V \subseteq \mathfrak{g}$ be an open convex symmetric 0-neighborhood for which the Hausdorff series for $x * y$ converges for $x, y \in V$ and satisfies

$$\exp_G(x * y) = \exp_G(x) \exp_G(y)$$

(Proposition 6.5.5). We further assume that $\exp_G|_V$ is a diffeomorphism onto an open subset of G.

Put $W := V \cap \mathfrak{h}$. Then $x * y \in \mathfrak{h}$ for $x, y \in W$ because each summand in the Hausdorff series is an iterated Lie bracket. Further, $x * y$ defines a smooth function $W \times W \to \mathfrak{h}$ because it is the restriction of a smooth function $V \times V \to \mathfrak{g}$. We consider the subset $U := \exp_G(W) \subseteq H$. From $W = -W$ we derive $U = U^{-1}$ and we note that

$\varphi := \exp_G|_W$ is injective. We may thus endow $U \subseteq H$ with the manifold structure turning φ into a diffeomorphism.

Then

$$\widetilde{D} = \{(x,y) \in W \times W : x * y \in W\}$$

is an open subset of $W \times W$ on which the $*$-multiplication is smooth, so that the multiplication $D \to U$ is also smooth. We further observe that

$$\exp_G(-x) = \exp_G(x)^{-1},$$

from which it follows that the inversion on U is smooth. Since U generates H, (L3) follows from (L1) and (L2). Therefore U satisfies all assumptions of Theorem 7.1.4, so that we obtain a Lie group structure on H for which φ, resp., $\exp_{\mathfrak{h}}$ induces a local diffeomorphism in 0.

Since the map $j_H \circ \exp_{\mathfrak{h}} : \mathfrak{h} \to G$ is smooth and $\exp_{\mathfrak{h}}$ is a local diffeomorphism in 0, the inclusion $j_H : H \to G$ is smooth. Now

$$\mathbf{L}(j_H) : \mathbf{L}(H) \to \mathbf{L}(G)$$

is injective, and by construction, its image contains $\mathfrak{h}$ because each element $x \in \mathfrak{h}$ generates a one-parameter group of H. As $\dim \mathbf{L}(H) = \dim H = \dim \mathfrak{h}$, we have $\mathbf{L}(j_H)\,\mathbf{L}(H) = \mathfrak{h}$.

If $\widehat{H}$ denotes another Lie group structure on the subgroup H for which $j_{\widehat{H}} : \widehat{H} \to G$ is smooth and $\mathbf{L}(j_{\widehat{H}}) : \mathbf{L}(\widehat{H}) \to \mathfrak{h} \subseteq \mathbf{L}(G)$ is an isomorphism of Lie algebras, then the relation

$$j_{\widehat{H}} \circ \exp_{\widehat{H}} = \exp_G \circ \mathbf{L}(j_{\widehat{H}}) = \exp_{\mathfrak{h}} \circ \mathbf{L}(j_{\widehat{H}})$$

and (a) imply that the identical map

$$j := j_H^{-1} \circ j_{\widehat{H}} : \widehat{H} \to H$$

is a bijective morphism of connected Lie groups for which $\mathbf{L}(j)$ is an isomorphism, hence an isomorphism (Proposition 6.2.4(3)).

(b) Since H is generated by $\exp_H(\mathbf{L}(H))$, this follows from Lemma 6.1.9.

(c) If H is closed, then the Closed Subgroup Theorem 7.2.10 shows that H is a Lie group with respect to the subspace topology, so that the uniqueness part of (a) implies that j_H is a topological embedding.

If, conversely, j_H is a topological embedding, then H is a Lie group with respect to the subspace topology inherited from G. Therefore Proposition 7.2.13 implies that H is closed. $\square$

Remark 7.3.2. Example 7.2.16, the dense wind in the 2-torus, shows that we cannot expect that the group $H = \langle \exp_G \mathfrak{h} \rangle$ is closed in G or that the inclusion map $H \to G$ (which is a smooth homomorphism) is a topological embedding.

Definition 7.3.3. Let G be a Lie group. An *integral subgroup* H of G is a subgroup that is generated by $\exp \mathfrak{h}$ for a subalgebra $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ of G.

The Integral Subgroup Theorem implies in particular that each Lie subalgebra $\mathfrak{h}$ of the Lie algebra $\mathbf{L}(G)$ of a Lie group G is *integrable* in the sense that it is the Lie algebra of some Lie group H.

Combining this with Ado's Theorem which asserts the existence of an injective homomorphism $\mathfrak{g} \hookrightarrow \mathfrak{gl}_n(\mathbb{R})$ for any finite dimensional Lie algebra $\mathfrak{g}$, we obtain one of the cornerstones of the theory of Lie groups:

Theorem 7.3.4. (Lie's Third Theorem) *Each finite-dimensional Lie algebra $\mathfrak{g}$ is the Lie algebra of a connected Lie group G.*

Proof. Ado's Theorem implies that $\mathfrak{g}$ is isomorphic to a subalgebra of some $\mathfrak{gl}_n(\mathbb{R})$, so that the assertion follows directly from the Integral Subgroup Theorem. □

Exercises for Chapter 7

Exercise 7.3.1. If $(H_j)_{j\in J}$ is a family of subgroups of the Lie group G, then

$$\mathbf{L}\Big(\bigcap_{j\in J} H_j\Big) = \bigcap_{j\in J} \mathbf{L}(H_j).$$

Exercise 7.3.2. Let $\varphi\colon G \to H$ be a morphism of Lie groups. Show that

$$\mathbf{L}(\ker\varphi) = \ker \mathbf{L}(\varphi).$$

Exercise 7.3.3. Show that any locally closed subgroup H of a topological group G is closed. Hint: Proceed as follows. Suppose that U is a $\mathbf{1}$-neighborhood of G for which $U \cap H$ is closed in U and let $g \in \overline{H}$. For any symmetric $\mathbf{1}$-neighborhood V with $VV \subseteq U$ there exists an $h \in gV \cap H$ and then $h^{-1}g \in U$ has the property that $h^{-1}gV \cap H \neq \emptyset$. Conclude that $h^{-1}g \in H$ and hence that $g \in H$.

Exercise 7.3.4. (Structure of discrete subgroups of $\mathbb{R}^n$) Let $D \subseteq \mathbb{R}^n$ be a discrete subgroup. Then there exist linearly independent elements $v_1, \ldots, v_k \in \mathbb{R}^n$ with $D = \sum_{i=1}^k \mathbb{Z}v_i$. Hint: Use induction on $\dim \operatorname{span} D$. If $n > 1$, and D spans $\mathbb{R}^n$, then pick a linear basis $f_1, \ldots, f_n \in D$. Then apply induction on $F \cap D$ for the hyperplane $F := \operatorname{span}\{f_1, \ldots, f_{n-1}\}$. Now show that there exists a $d \in \mathbb{R}f_n$ with $D = \mathbb{Z}d + F \cap D$. This can be done by considering the image of D under the linear projection $p\colon \mathbb{R}^n \to \mathbb{R}f_n$ with kernel F and using the discreteness of $p(D)$ to derive that it is cyclic.

Exercise 7.3.5. [Connected abelian Lie groups] Let A be a connected abelian Lie group. Show that

(1) $\exp_A\colon (\mathbf{L}(A), +) \to A$ is an open morphism of Lie groups.

(2) $\exp_A$ is surjective and open.

(3) $\Gamma_A := \ker \exp_A$ is a discrete subgroup of $(\mathbf{L}(A), +)$.

(4) $\mathbf{L}(A)/\Gamma_A \cong \mathbb{R}^k \times \mathbb{T}^m$ for some $k, m \geq 0$. In particular, it is a Lie group and the quotient map $q_A\colon \mathbf{L}(A) \to \mathbf{L}(A)/\Gamma_A$ is a smooth map and a local diffeomorphism.

(5) $\exp_A$ factors through a diffeomorphism $\varphi\colon \mathbf{L}(A)/\Gamma_A \to A$.

(6) $A \cong \mathbb{R}^k \times \mathbb{T}^m$ as Lie groups.

Exercise 7.3.6. [Divisible groups] An abelian group D is called *divisible* if for each $d \in D$ and $n \in \mathbb{N}$ there exists an $a \in D$ with $a^n = d$. Show that:

(1)* If G is an abelian group, H a subgroup and $f\colon H \to D$ a homomorphism into an abelian divisible group D, then there exists an extension of f to a homomorphism $\widetilde{f}\colon G \to D$.

(2) If G is an abelian group and D a divisible subgroup, then $G \cong D \times H$ for some subgroup H of G.

Exercise 7.3.7. [Nonconnected abelian Lie groups] Let A be an abelian Lie group. Show that:

(1) The identity component of A_0 is isomorphic to $\mathbb{R}^k \times \mathbb{T}^m$ for some $k, m \in \mathbb{N}_0$.

(2) A_0 is divisible.

(3) $A \cong A_0 \times \pi_0(A)$, where $\pi_0(A) := A/A_0$.

(4) There exists a discrete abelian group D with $A \cong \mathbb{R}^k \times \mathbb{T}^m \times D$.

Exercise 7.3.8. If $q\colon G \to H$ is a surjective open morphism of topological groups, then the induced map $G/\ker q \to H$ is an isomorphism of topological groups, where $G/\ker q$ is endowed with the quotient topology, i.e., a subset $O \subseteq G/\ker q$ is open if and only if $q^{-1}(O)$ is open in G.

Exercise 7.3.9. If G is a topological group and $\mathbf{1} \in U \subseteq G$ a connected subset. Then all sets $U^n := U \cdots U$ are connected and so is their union $\bigcup_n U^n$.

Exercise 7.3.10. Let G be a topological group. Then for each open subset $O \subseteq G$ and for each subset $S \subseteq G$ the product sets

$$OS = \{gh\colon g \in O, h \in S\} \quad \text{and} \quad SO = \{hg\colon g \in O, h \in S\}$$

are open.

Exercise 7.3.11. (Refining Lemma 7.1.3) Show that the conclusion of Lemma 7.1.3 is still valid if the assumption (T1) is weakened as follows: There exists an open subset $D \subseteq U \times U$ with $xy \in U$ for all $(x, y) \in D$, containing all pairs (x, x^{-1}), $(x, \mathbf{1})$, $(\mathbf{1}, x)$ for $x \in U$, such that the group multiplication $m\colon D \to U$ is continuous.

Exercise 7.3.12. Let G be an abelian group and $N \leq G$ a subgroup carrying a Lie group structure. Then there exists a unique Lie group structure on G for which N is an open subgroup.

Exercise 7.3.13. Let G be a connected topological group and $\Gamma \trianglelefteq G$ a discrete normal subgroup. Show that Γ is central.

Exercise 7.3.14. Let X be a topological space and $(X_i)_{i\in I}$ connected subspaces of X with $X = \bigcup_{i\in I} X_i$. If $\bigcap_{i\in I} X_i \neq \emptyset$, then X is connected.

Exercise 7.3.15. Let G be a group, endowed with a manifold structure. Show that G is a Lie group if the following conditions are satisfied:

(i) The left multiplication maps $\lambda_g \colon G \to G, x \mapsto gx$ are smooth.

(ii) The right multiplication maps $\rho_g \colon G \to G, x \mapsto xg$ are smooth.

(iii) The inversion map $\eta_G \colon G \to G$ is smooth in $\mathbf{1}$.

(iv) The multiplication $m_G \colon G \times G \to G$ is smooth in a neighborhood of $(\mathbf{1}, \mathbf{1})$.

Chapter 8

Covering Theory for Lie Groups

In this chapter we turn to applications of covering theory to Lie groups. Our goal is to see to which extent the Lie algebra $\mathbf{L}(G)$ and the fundamental group $\pi_1(G)$ determine a connected Lie group G. In the first section we show that each connected Lie group G has a simply connected covering group $\widetilde{G}$ which also carries a Lie group structure. The kernel of the covering morphism $q_G \colon \widetilde{G} \to G$ can be identified with the fundamental group $\pi_1(G)$. Since $\mathbf{L}(q_G)$ is an isomorphism of Lie algebras, we have $\mathbf{L}(G) \cong \mathbf{L}(\widetilde{G})$. We further prove the Monodromy Principle which implies that any Lie algebra morphism $\mathbf{L}(G) \to \mathbf{L}(H)$ can be integrated to a group homomorphism, provided G is 1-*connected*, i.e., connected and simply connected. From that we shall derive in particular that the Lie algebra $\mathbf{L}(G)$ determines the corresponding simply connected group up to isomorphy.

8.1 Simply Connected Coverings of Lie Groups

In the preceding chapter we have seen how to construct Lie group structures on groups from local data. This construction applies in particular to those quotient morphisms $q \colon G \to G/N$, where G is a Lie group and q is a local homeomorphism, i.e., maps some open identity neighborhood homeomorphically to an open identity neighborhood in N. This means that N is a discrete subgroup of G, such as $\mathbb{Z}$ in $\mathbb{R}$. To deal properly with such maps, we recall the concept of a covering map from Definition A.2.1. This concept is particularly important in the theory of Lie groups because it can be used to understand how different connected Lie groups with the same Lie algebra can be.

We start with another corollary of Theorem 7.1.4.

Corollary 8.1.1. *Let $\varphi \colon G \to H$ be a covering of topological groups. If G or H is a Lie group, then the other group has a unique Lie group structure for which φ is a morphism of Lie groups which is a local diffeomorphism.*

Proof. Let $U_G \subseteq G$ be an open symmetric $\mathbf{1}$-neighborhood for which $\varphi|_{U_G}$ is a homeomorphism onto an open subset U_H of H. Since $\ker\varphi = \varphi^{-1}(\mathbf{1})$ is discrete, we may choose U_G so small that $U_G^3 \cap \ker\varphi = \{\mathbf{1}\}$.

Suppose first that G is a Lie group. Then we apply Theorem 7.1.4 to U_H, endowed with the manifold structure for which $\varphi|_{U_G}$ is a diffeomorphism. Then (L2) follows from $\varphi(x)^{-1} = \varphi(x^{-1})$. To verify the smoothness of the multiplication map

$$m_{U_H} \colon D_H := \{(a,b) \in U_H \times U_H \colon ab \in U_H\} \to U_H,$$

we first observe that, if $x, y \in U_G$ satisfy $(\varphi(x), \varphi(y)) \in D_H$, i.e., $\varphi(xy) \in U_H$, then there exists a $z \in U_G$ with $\varphi(xy) = \varphi(z)$, and $xyz^{-1} \in {U_G}^3 \cap \ker(\varphi) = \{\mathbf{1}\}$ yields $xy = z \in U_G$. We thus have $D_H = (\varphi \times \varphi)(D_G)$ for

$$D_G := \{(x,y) \in U_G \times U_G \colon xy \in U_G\}$$

and the smoothness of m_H follows from the smoothness of the multiplication $m_{U_G} \colon D_G \to U_G$ and

$$m_{U_H} \circ (\varphi \times \varphi) = \varphi \circ m_{U_G}.$$

To verify (L3), we note that the surjectivity of φ implies that for each $h \in H$ there is an element $g \in G$ with $\varphi(g) = h$. Now we choose an open $\mathbf{1}$-neighborhood $U_g \subseteq U_G$ with $c_g(U_g) \subseteq U_G$ and put $U_h := \varphi(U_g)$.

If, conversely, H is a Lie group, then we apply Theorem 7.1.4 to U_G, endowed with the manifold structure for which $\varphi|_{U_G}$ is a diffeomorphism onto U_H. Again, (L2) follows right away, and (L1) follows from $(\varphi \times \varphi)(D_G) \subseteq D_H$ and the smoothness of

$$m_{U_H} \circ (\varphi \times \varphi) = \varphi \circ m_{U_G}.$$

For (L3), we choose U_g as any open $\mathbf{1}$-neighborhood in U_G with $c_g(U_g) \subseteq U$. Then the smoothness of $c_g|_{U_g}$ follows from the smoothness the maps of $\varphi \circ c_g = c_{\varphi(g)} \circ \varphi$. □

Proposition 8.1.2. *If G is a connected Lie group and $q_G \colon \widetilde{G} \to G$ its universal covering space, then $\widetilde{G}$ carries a unique Lie group structure for which q_G is a smooth covering map.*

We call this Lie group the simply connected covering group of G.

Proof. We first have to construct a (topological) group structure on the universal covering space $\widetilde{G}$ (cf. Theorem A.2.12). Its existence follows from the fact that G is a manifold, hence in particular locally simply connected. Pick an element $\widetilde{\mathbf{1}} \in q_G^{-1}(\mathbf{1})$. Then the multiplication map $m_G \colon G \times G \to G$ lifts uniquely to a continuous map $\widetilde{m}_G \colon \widetilde{G} \times \widetilde{G} \to \widetilde{G}$ with $\widetilde{m}_G(\widetilde{\mathbf{1}}, \widetilde{\mathbf{1}}) = \widetilde{\mathbf{1}}$. To see that the multiplication map $\widetilde{m}_G$ is associative, we observe that

$$\begin{aligned}
& q_G \circ \widetilde{m}_G \circ (\mathrm{id}_{\widetilde{G}} \times \widetilde{m}_G) = m_G \circ (q_G \times q_G) \circ (\mathrm{id}_{\widetilde{G}} \times \widetilde{m}_G) \\
= \quad & m_G \circ (\mathrm{id}_G \times m_G) \circ (q_G \times q_G \times q_G) = m_G \circ (m_G \times \mathrm{id}_G) \circ (q_G \times q_G \times q_G) \\
= \quad & q_G \circ \widetilde{m}_G \circ (\widetilde{m}_G \times \mathrm{id}_{\widetilde{G}}),
\end{aligned}$$

so that the two continuous maps

$$\widetilde{m}_G \circ (\mathrm{id}_{\widetilde{G}} \times \widetilde{m}_G), \quad \widetilde{m}_G \circ (\widetilde{m}_G \times \mathrm{id}_{\widetilde{G}}) \colon \widetilde{G}^3 \to \widetilde{G},$$

are lifts of the same map $\widetilde{G}^3 \to G$ and both map $(\widetilde{\mathbf{1}}, \widetilde{\mathbf{1}}, \widetilde{\mathbf{1}})$ to $\widetilde{\mathbf{1}}$. Hence the uniqueness of lifts (Theorem A.2.9) implies that $\widetilde{m}_G$ is associative. We likewise obtain that the unique lift $\widetilde{\iota}_G \colon \widetilde{G} \to \widetilde{G}$ of the inversion map $\iota_G \colon G \to G$ with $\widetilde{\iota}_G(\widetilde{\mathbf{1}}) = \widetilde{\mathbf{1}}$ satisfies

$$\widetilde{m}_G \circ (\widetilde{\iota}_G, \mathrm{id}_{\widetilde{G}}) = \widetilde{\mathbf{1}} = \widetilde{m}_G \circ (\mathrm{id}_{\widetilde{G}}, \widetilde{\iota}_G).$$

Finally $\lambda_{\widetilde{\mathbf{1}}}$ lifts $\lambda_{\mathbf{1}} = \mathrm{id}_G$, so that $\lambda_{\widetilde{\mathbf{1}}}(\widetilde{\mathbf{1}}) = \widetilde{\mathbf{1}}$ leads to $\lambda_{\widetilde{\mathbf{1}}} = \mathrm{id}_{\widetilde{G}}$, and likewise one shows that $\rho_{\widetilde{\mathbf{1}}} = \mathrm{id}_{\widetilde{G}}$, so that $\widetilde{\mathbf{1}}$ is a neutral element for the multiplication on $\widetilde{G}$. Therefore $\widetilde{m}_G$ defines on $\widetilde{G}$ a topological group structure such that $q_G \colon \widetilde{G} \to G$ is a covering morphism of topological groups. Now Corollary 8.1.1 applies. □

Proposition 8.1.3. *A surjective morphism $\varphi \colon G \to H$ of Lie groups is a covering if and only if $\mathbf{L}(\varphi) \colon \mathbf{L}(G) \to \mathbf{L}(H)$ is a linear isomorphism.*

If H is connected, then φ is a covering if and only if $\mathbf{L}(\varphi) \colon \mathbf{L}(G) \to \mathbf{L}(H)$ is a linear isomorphism, i.e., the surjectivity of φ is not required.

Proof. If φ is a covering, then it is an open homomorphism with discrete kernel (Exercise 8.5.5), so that $\mathbf{L}(\ker \varphi) = \{0\}$, and Proposition 6.2.4 implies that $\mathbf{L}(\varphi)$ is bijective, hence an isomorphism of Lie algebras.

If, conversely, $\mathbf{L}(\varphi)$ is bijective, then Proposition 6.2.4 implies that

$$\mathbf{L}(\ker \varphi) = \ker \mathbf{L}(\varphi) = \{0\}.$$

Now the Closed Subgroup Theorem 7.2.10 shows that $\ker \varphi$ is discrete. Since $\mathbf{L}(\varphi)$ is surjective, Proposition 6.2.4 implies that φ is an open map. Finally Exercise 8.5.5 shows that φ is a covering.

If, in addition, H is connected, then $H = \langle \exp_H(\mathbf{L}(H)) \rangle$ by Lemma 6.1.9. If $\mathbf{L}(\varphi)$ is surjective, we thus obtain

$$H = \langle \exp_H(\mathbf{L}(H)) \rangle = \langle \exp_H(\mathbf{L}(\varphi)\,\mathbf{L}(G)) \rangle = \langle \varphi(\exp_G(\mathbf{L}(G))) \rangle \subseteq \varphi(G). \qquad \square$$

Proposition 8.1.4. *For a covering $q \colon G_1 \to G_2$ of connected Lie groups, the following equalities hold*

$$q\big(Z(G_1)\big) = Z(G_2) \quad \textit{and} \quad Z(G_1) = q^{-1}(Z(G_2)).$$

Proof. Since q is a covering, $\mathbf{L}(q) \colon \mathbf{L}(G_1) \to \mathbf{L}(G_2)$ is an isomorphism of Lie algebras (Proposition 8.1.3), and the adjoint representations satisfy

$$c_{q(g)} \circ q = q \circ c_g \quad \Rightarrow \quad \mathrm{Ad}_{G_2}(q(g)) \circ \mathbf{L}(q) = \mathbf{L}(q) \circ \mathrm{Ad}_{G_1}(g) \quad \text{for} \quad g \in G_1.$$

Hence

$$Z(G_1) = \ker \mathrm{Ad}_{G_1} = q^{-1} \ker \mathrm{Ad}_{G_2} = q^{-1}(Z(G_2)).$$

Now the claim follows from the surjectivity of q. □

Theorem 8.1.5. (Lifting Theorem for Groups) *Let $q\colon G \to H$ be a covering morphism of Lie groups. If $f\colon L \to H$ is a morphism of Lie groups, where L is 1-connected, then there exists a unique lift $\widetilde{f}\colon L \to G$ which is a morphism of Lie groups.*

Proof. Since Lie groups are locally arcwise connected, the Lifting Theorem A.2.9 implies the existence of a unique lift $\widetilde{f}$ with $\widetilde{f}(\mathbf{1}_L) = \mathbf{1}_G$. Then

$$m_G \circ (\widetilde{f} \times \widetilde{f})\colon L \times L \to G$$

is the unique lift of $m_H \circ (f \times f)\colon L \times L \to H$ mapping $(\mathbf{1}_L, \mathbf{1}_L)$ to $\mathbf{1}_G$. We also have

$$q \circ \widetilde{f} \circ m_L = f \circ m_L = m_H \circ (f \times f),$$

so that $\widetilde{f} \circ m_L$ is another lift of $m_H \circ (f \times f)$ mapping $(\mathbf{1}_L, \mathbf{1}_L)$ to $\mathbf{1}_G$. Therefore

$$\widetilde{f} \circ m_L = m_G \circ (\widetilde{f} \times \widetilde{f}),$$

which means that $\widetilde{f}$ is a group homomorphism.

Since q is a local diffeomorphism and $\widetilde{f}$ is a continuous lift of f, it is also smooth in an identity neighborhood of L, hence smooth by Corollary 6.2.2. □

Theorem 8.1.6. *Let G be a connected Lie group and $q_G\colon \widetilde{G} \to G$ a universal covering homomorphism. Then*

$$\ker q_G \cong \pi_1(G)$$

is a discrete central subgroup and

$$G \cong \widetilde{G}/\ker q_G.$$

Moreover, for any discrete central subgroup $\Gamma \subseteq \widetilde{G}$, the group $\widetilde{G}/\Gamma$ is a connected Lie group with the same universal covering group as G. We thus obtain a bijection from the set of all $\mathrm{Aut}(\widetilde{G})$-orbits in the set of discrete central subgroups of $\widetilde{G}$ onto the set of isomorphy classes of connected Lie groups whose universal covering group is isomorphic to $\widetilde{G}$.

Proof. First we note that $\ker q_G$ is a discrete normal subgroup of the connected Lie group $\widetilde{G}$, hence central by Exercise 7.3.13. Left multiplications by elements of $\ker q_G$ lead to deck transformations of the covering $\widetilde{G} \to G$, and this group of deck transformations acts transitively on the fiber $\ker q_G$ of $\mathbf{1}$. Proposition A.2.15 now shows that

$$\pi_1(G) \cong \ker q_G \tag{8.1}$$

as groups. Since $q_G\colon \widetilde{G} \to G$ is open and surjective, we have $G \cong \widetilde{G}/\ker q_G$ as topological groups (Exercise 7.3.8), hence as Lie groups (Theorem 6.2.7).

If, conversely, $\Gamma \subseteq \widetilde{G}$ is a discrete central subgroup, then the topological quotient group $\widetilde{G}/\Gamma$ is a Lie group (Corollary 8.1.1) whose universal covering group is $\widetilde{G}$. Two such groups $\widetilde{G}/\Gamma_1$ and $\widetilde{G}/\Gamma_2$ are isomorphic if and only if there exists a Lie group automorphism $\varphi \in \mathrm{Aut}(\widetilde{G})$ with $\varphi(\Gamma_1) = \Gamma_2$ (Theorem 8.1.5). Therefore the isomorphism classes of Lie groups with the same universal covering group as G are parameterized by the orbits of the group $\mathrm{Aut}(\widetilde{G})$ in the set of discrete central subgroups of $\widetilde{G}$. □

Remark 8.1.7. (a) Since the normal subgroup $\mathrm{Inn}(\widetilde{G}) := \{c_g : g \in \widetilde{G}\}$ of inner automorphisms acts trivially on the center of $\widetilde{G}$, the action of $\mathrm{Aut}(\widetilde{G})$ on the set of all discrete normal subgroups factors through an action of the group $\mathrm{Out}(\widetilde{G}) := \mathrm{Aut}(\widetilde{G})/\mathrm{Inn}(\widetilde{G})$.

(b) Since each automorphism $\varphi \in \mathrm{Aut}(G)$ lifts to a unique automorphism $\widetilde{\varphi} \in \mathrm{Aut}(\widetilde{G})$ (Theorem 8.1.5), we have a natural embedding $\mathrm{Aut}(G) \hookrightarrow \mathrm{Aut}(\widetilde{G})$, and the image of this homomorphism consists of the stabilizer of the subgroup $\ker q_G \subseteq Z(\widetilde{G})$ in $\mathrm{Aut}(\widetilde{G})$.

Example 8.1.8. [Connected abelian Lie groups] Let A be a connected abelian Lie group and $\exp_A : \mathbf{L}(A) \to A$ its exponential function. Then $\exp_A$ is a morphism of Lie groups with $\mathbf{L}(\exp_A) = \mathrm{id}_{\mathbf{L}(A)}$, hence a covering morphism. Since $\mathbf{L}(A)$ is simply connected, we have $(\mathbf{L}(A), +) \cong \widetilde{A}$ and $\ker \exp_A \cong \pi_1(A)$ is the fundamental group of A (cf. Exercise 7.3.5).

As special cases we obtain in particular the finite-dimensional tori

$$\mathbb{T}^d \cong \mathbb{R}^d/\mathbb{Z}^d \quad \text{with} \quad \pi_1(\mathbb{T}^n) \cong \mathbb{Z}^n.$$

If we want to classify all connected abelian Lie groups A of dimension n, we can now proceed as follows. First we note that $\widetilde{A} \cong \mathbf{L}(A) \cong (\mathbb{R}^n, +)$ as abelian Lie groups. Then $\mathrm{Aut}(\widetilde{A}) \cong \mathrm{GL}_n(\mathbb{R})$ follows from the Automatic Smoothness Theorem 6.2.7. Further, Exercise 7.3.4 implies that the discrete subgroup $\pi_1(A)$ of $\widetilde{A} \cong \mathbb{R}^n$ can be mapped by some $\varphi \in \mathrm{GL}_n(\mathbb{R})$ onto

$$\mathbb{Z}^k \cong \mathbb{Z}^k \times \{0\} \subseteq \mathbb{R}^k \times \mathbb{R}^{n-k} \cong \mathbb{R}^n.$$

Therefore

$$A \cong \mathbb{R}^n/\mathbb{Z}^k \cong \mathbb{T}^k \times \mathbb{R}^{n-k},$$

and it is clear that the number k is an isomorphy invariant of the Lie group A, namely, the rank of its fundamental subgroup. Therefore connected abelian Lie groups A are determined up to isomorphism by the pair (n, k), where $n = \dim A$ and $k = \operatorname{rank} \pi_1(A)$. The case where $n = k$ gives the compact connected abelian Lie groups. The above argument shows that such groups are always of the form $A \cong \mathbf{L}(A)/\Gamma$, where Γ is a discrete subgroup of $\mathbf{L}(T)$ generated by a basis for $\mathbf{L}(T)$. Such discrete subgroups are called *lattices*.

Examples 8.1.9. (a) The group $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$ is homeomorphic to the one-dimensional sphere $\mathbb{S}^1$, which is not simply connected.

The group

$$\mathrm{SU}_2(\mathbb{C}) \cong \left\{ \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}) : |a|^2 + |b|^2 = 1 \right\}$$

is homeomorphic to the 3-sphere

$$\{(a, b) \in \mathbb{C}^2 : \|(a, b)\| = 1\} \cong \mathbb{S}^3$$

which is simply connected (Exercise A.1.3). One can show that the sphere $\mathbb{S}^n$ carries a Lie group structure if and only if $n = 0, 1, 3$.

(b) With some more advanced tools from homotopy theory, one can show that the groups $\mathrm{SU}_n(\mathbb{C})$ are always simply connected. However, this is never the case for the groups $\mathrm{U}_n(\mathbb{C})$.

To see this, consider the group homomorphism

$$\gamma\colon \mathbb{T} \to \mathrm{U}_n(\mathbb{C}), \quad z \mapsto \mathrm{diag}(z, 1, \ldots, 1)$$

and note that $\det \circ \gamma = \mathrm{id}_{\mathbb{T}}$. From that one easily derives that the multiplication map

$$\mu\colon\ \mathrm{SU}_n(\mathbb{C}) \times \mathbb{T} \to \mathrm{U}_n(\mathbb{C}), \quad (g, z) \mapsto g\gamma(z)$$

is a homeomorphism, so that

$$\pi_1(\mathrm{U}_n(\mathbb{C})) \cong \pi(\mathrm{SU}_n(\mathbb{C})) \times \pi_1(\mathbb{T}) \cong \pi_1(\mathbb{T}) \cong \mathbb{Z}.$$

We further derive that the universal covering group is given by

$$\widetilde{\mathrm{U}}_n(\mathbb{C}) \cong \mathrm{SU}_n(\mathbb{C}) \rtimes_\beta \mathbb{R} \quad \text{where} \quad \beta(t)g := \gamma(e^{it})g\gamma(e^{-it}).$$

Example 8.1.10. We show that

$$\pi_1(\mathrm{SO}_3(\mathbb{R})) \cong C_2 = \{\pm 1\}$$

by constructing a surjective homomorphism

$$\varphi\colon\ \mathrm{SU}_2(\mathbb{C}) \to \mathrm{SO}_3(\mathbb{R})$$

with $\ker \varphi = \{\pm \mathbf{1}\}$, so that

$$\mathrm{SO}_3(\mathbb{R}) \cong \mathrm{SU}_2(\mathbb{C})/\{\pm \mathbf{1}\}.$$

Since $\mathrm{SU}_2(\mathbb{C})$ is homeomorphic to $\mathbb{S}^3$, it is simply connected (Exercise A.1.3), so that we obtain $\pi_1(\mathrm{SO}_3(\mathbb{R})) \cong C_2$ (Theorem 8.1.6).

We consider

$$\mathfrak{su}_2(\mathbb{C}) = \{x \in \mathfrak{gl}_2(\mathbb{C})\colon x^* = -x, \mathrm{tr}\, x = 0\} = \left\{ \begin{pmatrix} ai & b \\ -\overline{b} & -ai \end{pmatrix} : b \in \mathbb{C}, a \in \mathbb{R} \right\}$$

and observe that this is a three-dimensional real subspace of $\mathfrak{gl}_2(\mathbb{C})$. We obtain on $E := \mathfrak{su}_2(\mathbb{C})$ the structure of a euclidean vector space by the scalar product

$$\beta(x, y) := -\,\mathrm{tr}(xy) = \mathrm{tr}(xy^*) = \sum_{j,k=1}^{2} x_{ij}\overline{y_{ij}}.$$

Now we consider the adjoint representation

$$\mathrm{Ad}\colon\ \mathrm{SU}_2(\mathbb{C}) \to \mathrm{GL}(E), \quad \mathrm{Ad}(g)(x) = gxg^{-1}.$$

Then we have for $x, y \in E$ and $g \in \mathrm{SU}_2(\mathbb{C})$ the relation

$$\begin{aligned}\beta\big(\mathrm{Ad}(g)x, \mathrm{Ad}(g)y\big) &= \mathrm{tr}(gxg^{-1}(gyg^{-1})^*) = \mathrm{tr}(gxg^{-1}(g^{-1})^*y^*g^*) \\ &= \mathrm{tr}(gxg^{-1}gy^*g^{-1}) = \mathrm{tr}(xy^*) = \beta(x,y).\end{aligned}$$

This means that

$$\mathrm{Ad}(\mathrm{SU}_2(\mathbb{C})) \subseteq \mathrm{O}(E,\beta) \cong \mathrm{O}_3(\mathbb{R}).$$

Since $\mathrm{SU}_2(\mathbb{C})$ is connected, we further obtain $\mathrm{Ad}(\mathrm{SU}_2(\mathbb{C})) \subseteq \mathrm{SO}(E,\beta) \cong \mathrm{SO}_3(\mathbb{R})$, the identity component of $\mathrm{O}(E,\beta)$.

The derived representation is given by

$$\mathbf{L}(\mathrm{Ad}) = \mathrm{ad}\colon\ \mathfrak{su}_2(\mathbb{C}) \to \mathfrak{so}(E,\beta) \cong \mathfrak{so}_3(\mathbb{R}), \quad \mathrm{ad}(x)(y) = [x,y].$$

If $\mathrm{ad}\, x = 0$, then $\mathrm{ad}\, x(i\mathbf{1}) = 0$ implies that $\mathrm{ad}\, x(\mathfrak{u}_2(\mathbb{C})) = \{0\}$, so that $\mathrm{ad}\, x(\mathfrak{gl}_2(\mathbb{C})) = \{0\}$ follows from $\mathfrak{gl}_2(\mathbb{C}) = \mathfrak{u}_2(\mathbb{C}) + i\mathfrak{u}_2(\mathbb{C})$. This implies that $x \in \mathbb{C}\mathbf{1}$ (Exercise 8.5.2), so that $\mathrm{tr}\, x = 0$ leads to $x = 0$. Hence ad is injective, and we conclude with $\dim \mathfrak{so}(E,\beta) = \dim \mathfrak{so}_3(\mathbb{R}) = 3$ that

$$\mathrm{ad}(\mathfrak{su}_2(\mathbb{C})) = \mathfrak{so}(E,\beta)$$

(cf. Exercise 8.5.1). Since $\mathrm{SO}_3(\mathbb{R})$ is connected, Proposition 8.1.3 now implies that

$$\varphi\colon\ \mathrm{SU}_2(\mathbb{C}) \to \mathrm{SO}_3(\mathbb{R})$$

is a covering. We further have

$$\ker\varphi = Z(\mathrm{SU}_2(\mathbb{C})) = \mathrm{SU}_2(\mathbb{C}) \cap \mathbb{C}^\times\mathbf{1} = \{\pm\mathbf{1}\}$$

(Exercise 7.3.8), so that

$$\widetilde{\mathrm{SO}}_3(\mathbb{R}) \cong \mathrm{SU}_2(\mathbb{C}) \quad \text{and} \quad \pi_1(\mathrm{SO}_3(\mathbb{R})) \cong C_2.$$

8.2 The Monodromy Principle and its Applications

To round off the picture, we still have to provide the link between Lie algebras and covering groups. The main point is that, in general, one cannot integrate morphisms of Lie algebras $\mathbf{L}(G) \to \mathbf{L}(H)$ to morphisms of the corresponding groups $G \to H$ if G is not simply connected.

Proposition 8.2.1. (Monodromy Principle) *Let G be a connected simply connected Lie group and H a group. Let V be an open symmetric connected identity neighborhood in G and $f\colon V \to H$ a function with*

$$f(xy) = f(x)f(y) \quad \textit{for} \quad x, y, xy \in V.$$

Then there exists a unique group homomorphism extending f. If, in addition, H is a Lie group and f is smooth, then its extension is also smooth.

Proof. We consider the group $G \times H$ and the subgroup $S \subseteq G \times H$ generated by the subset $U := \{(x, f(x)) : x \in V\}$. We endow U with the topology for which $x \mapsto (x, f(x)), V \to U$ is a homeomorphism. Note that $f(\mathbf{1})^2 = f(\mathbf{1}^2) = f(\mathbf{1})$ implies $f(\mathbf{1}) = \mathbf{1}$, which further leads to $\mathbf{1} = f(xx^{-1}) = f(x)f(x^{-1})$, so that $f(x^{-1}) = f(x)^{-1}$. Hence $U = U^{-1}$.

We now apply Lemma 7.1.3 because S is generated by U, and (T1/2) directly follow from the corresponding properties of V and $(x, f(x))(y, f(y)) = (xy, f(xy))$ for $x, y, xy \in V$. This leads to a group topology on S, for which S is a connected topological group. Indeed, its connectedness follows from $S = \bigcup_{n \in \mathbb{N}} U^n$ and the connectedness of all sets U^n (Exercise 7.3.14). The projection $p_G : G \times H \to G$ induces a covering homomorphism $q : S \to G$ because its restriction to the open $\mathbf{1}$-neighborhood U is a homeomorphism (Exercise A.2.2(c)), and the connectedness of S and the simple connectedness of G imply that q is a homeomorphism (Corollary A.2.8). Now $F := p_H \circ q^{-1} : G \to H$ provides the required extension of f. In fact, for $x \in U$ we have $q^{-1}(x) = (x, f(x))$, and therefore $F(x) = f(x)$.

If, in addition, H is Lie and f is smooth, then the smoothness of the extension follows directly from Corollary 6.2.2. □

Theorem 8.2.2. (Integrability Theorem for Lie Algebra Homomorphisms) *Let G be a connected simply connected Lie group, H a Lie group and $\psi : \mathbf{L}(G) \to \mathbf{L}(H)$ a Lie algebra morphism. Then there exists a unique morphism $\varphi : G \to H$ with $\mathbf{L}(\varphi) = \psi$.*

Proof. Let $U \subseteq \mathbf{L}(G)$ be an open connected symmetric 0-neighborhood such that

- $\exp_G |_U$ is a homeomorphism onto an open subset of G (cf. Proposition 6.1.5).
- the BCH-product is defined by the Hausdorff series on $U \times U$ and $\psi(U) \times \psi(U)$
- $\exp_G(x*y) = \exp_G(x)\exp_G(y)$ and $\exp_H(\psi(x)*\psi(y)) = \exp_H(\psi(x))\exp_H(\psi(y))$ for $x, y \in U$.

The continuity of ψ and the fact that ψ is a Lie algebra homomorphism imply that for $x, y \in U$ the element $\psi(x * y)$ coincides with the convergent Hausdorff series $\psi(x) * \psi(y)$. We define

$$f : \exp_G(U) \to H, \quad f(\exp_G(x)) := \exp_H(\psi(x)).$$

For $x, y, x * y \in U$, we then obtain

$$\begin{aligned} & f(\exp_G(x)\exp_G(y)) = f(\exp_G(x * y)) = \exp_H(\psi(x * y)) \\ = \quad & \exp_H(\psi(x) * \psi(y)) = \exp_H(\psi(x))\exp_H(\psi(y)) = f(\exp_G(x))f(\exp_G(y)). \end{aligned}$$

Then $f : \exp(U) \to H$ satisfies the assumptions of Proposition 8.2.1, and we see that f extends uniquely to a group homomorphism $\varphi : G \to H$. Since $\exp_G$ is a local diffeomorphism, f is smooth in a $\mathbf{1}$-neighborhood, and therefore φ is smooth. We finally observe that φ is uniquely determined by $\mathbf{L}(\varphi) = \psi$ because G is connected (Corollary 6.2.3). □

The following corollary can be viewed as an integrability condition for ψ.

Corollary 8.2.3. *If G is a connected Lie group and H is a Lie group, then for a Lie algebra morphism $\psi\colon \mathbf{L}(G) \to \mathbf{L}(H)$, there exists a morphism $\varphi\colon G \to H$ with $\mathbf{L}(\varphi) = \psi$ if and only if $\pi_1(G) \subseteq \ker \widetilde{\varphi}$, where $\pi_1(G)$ is identified with the kernel of the universal covering map $q_G\colon \widetilde{G} \to G$ and $\widetilde{\varphi}\colon \widetilde{G} \to H$ is the unique morphism with $\mathbf{L}(\widetilde{\varphi}) = \psi \circ \mathbf{L}(q_G)$.*

Proof. If φ exists, then

$$(\varphi \circ q_G) \circ \exp_{\widetilde{G}} = \varphi \circ \exp_G \circ \mathbf{L}(q_G) = \exp_H \circ \psi \circ \mathbf{L}(q_G)$$

and the uniqueness of $\widetilde{\varphi}$ imply that $\widetilde{\varphi} = \varphi \circ q_G$ and hence that $\pi_1(G) = \ker q_G \subseteq \ker \widetilde{\varphi}$.

If, conversely, $\ker q_G \subseteq \ker \widetilde{\varphi}$, then $\varphi(q_G(g)) := \widetilde{\varphi}(g)$ defines a continuous morphism $G \cong \widetilde{G}/\ker q_G \to H$ with $\varphi \circ q_G = \widetilde{\varphi}$ (Exercise 7.3.8) and

$$\varphi \circ \exp_G \circ \mathbf{L}(q_G) = \varphi \circ q_G \circ \exp_{\widetilde{G}} = \widetilde{\varphi} \circ \exp_{\widetilde{G}} = \exp_H \circ \psi \circ \mathbf{L}(q_G). \qquad \square$$

We recall that a Lie group G is called 1-connected if it is connected and simply connected.

Corollary 8.2.4. *If G is a 1-connected Lie group with Lie algebra $\mathfrak{g}$, then the map*

$$\mathbf{L}\colon \operatorname{Aut}(G) \to \operatorname{Aut}(\mathfrak{g})$$

is an isomorphism of groups.

Proof. First, we recall from Corollary 5.2.4 that for each automorphism $\varphi \in \operatorname{Aut}(G)$ the endomorphism $\mathbf{L}(\varphi)$ of $\mathfrak{g}$ also is an automorphism. That $\mathbf{L}$ is injective follows from the connectedness of G (Corollary 6.2.3) and that $\mathbf{L}$ is surjective from the Integrability Theorem 8.2.2. $\square$

8.3 Classification of Lie Groups with given Lie Algebra

Let G and H be linear Lie groups. If $\varphi\colon G \to H$ is an isomorphism, then the functoriality of $\mathbf{L}$ directly implies that $\mathbf{L}(\varphi)\colon \mathbf{L}(G) \to \mathbf{L}(H)$ is an isomorphism. In fact, if $\psi\colon H \to G$ is a morphism with $\varphi \circ \psi = \mathrm{id}_H$ and $\psi \circ \varphi = \mathrm{id}_G$, then

$$\mathrm{id}_{\mathbf{L}(H)} = \mathbf{L}(\mathrm{id}_H) = \mathbf{L}(\varphi \circ \psi) = \mathbf{L}(\varphi) \circ \mathbf{L}(\psi)$$

and likewise $\mathbf{L}(\psi) \circ \mathbf{L}(\varphi) = \mathrm{id}_{\mathbf{L}(G)}$.

In this subsection we ask to which extent a Lie group G is determined by its Lie algebra $\mathbf{L}(G)$.

Theorem 8.3.1. *Two connected Lie groups G and H have isomorphic Lie algebras if and only if their universal covering groups $\widetilde{G}$ and $\widetilde{H}$ are isomorphic.*

Proof. If $\widetilde{G}$ and $\widetilde{H}$ are isomorphic, then we clearly have

$$\mathbf{L}(G) \cong \mathbf{L}(\widetilde{G}) \cong \mathbf{L}(\widetilde{H}) \cong \mathbf{L}(H)$$

(cf. Proposition 8.1.3).

Conversely, let $\psi\colon \mathbf{L}(G) \cong \mathbf{L}(\widetilde{G}) \to \mathbf{L}(H) \cong \mathbf{L}(\widetilde{H})$ be an isomorphism. Using Theorem 8.2.2, we obtain a unique morphism $\varphi\colon \widetilde{G} \to \widetilde{H}$ with $\mathbf{L}(\varphi) = \psi$ and also a unique morphism $\widehat{\varphi}\colon \widetilde{H} \to \widetilde{G}$ with $\mathbf{L}(\widehat{\varphi}) = \psi^{-1}$. Then $\mathbf{L}(\varphi \circ \widehat{\varphi}) = \mathrm{id}_{\mathbf{L}(\widetilde{G})}$ implies $\varphi \circ \widehat{\varphi} = \mathrm{id}_{\widetilde{G}}$, and likewise $\widehat{\varphi} \circ \varphi = \mathrm{id}_{\widetilde{H}}$. Therefore $\widetilde{G}$ and $\widetilde{H}$ are isomorphic Lie groups. □

Combining the preceding theorem with Theorem 8.1.6, we obtain:

Corollary 8.3.2. *Let G be a connected Lie group and $q_G\colon \widetilde{G} \to G$ the universal covering morphism of connected Lie groups. Then for each discrete central subgroup $\Gamma \subseteq \widetilde{G}$, the group $\widetilde{G}/\Gamma$ is a connected Lie group with $\mathbf{L}(\widetilde{G}/\Gamma) \cong \mathbf{L}(G)$ and, conversely, each Lie group with the same Lie algebra as G is isomorphic to some quotient $\widetilde{G}/\Gamma$.*

Example 8.3.3. We now describe a pair of nonisomorphic Lie groups with isomorphism Lie algebras and isomorphic fundamental groups.

Let

$$\widetilde{G} := \mathrm{SU}_2(\mathbb{C}) \times \mathrm{SU}_2(\mathbb{C})$$

whose center is $C_2 \times C_2 = \{\pm\mathbf{1}\} \times \{\pm\mathbf{1}\}$,

$$G := \widetilde{G}/(C_2 \times \{\mathbf{1}\}) \cong \mathrm{SO}_3(\mathbb{R}) \times \mathrm{SU}_2(\mathbb{C})$$

and

$$H := \widetilde{G}/\{(\mathbf{1},\mathbf{1}), (-\mathbf{1},-\mathbf{1})\} \cong \mathrm{SO}_4(\mathbb{R}),$$

where the latter isomorphy follows from Proposition 8.5.1 below. Then $\pi_1(G) \cong \pi_1(H) \cong C_2$, but there is no automorphism of $\widetilde{G}$ mapping $\pi_1(G)$ to $\pi_1(H)$.

Indeed, one can show that the two direct factors are the only nontrivial connected normal subgroups of $\widetilde{G}$, so that each automorphism of $\widetilde{G}$ either preserves both or exchanges them. Since $\pi_1(H)$ is not contained in any of them, it cannot be mapped to $\pi_1(G)$ by an automorphism of $\widetilde{G}$.

Examples 8.3.4. Here are some examples of pairs of linear Lie groups with isomorphic Lie algebras:

(1) $G = \mathrm{SO}_3(\mathbb{R})$ and $\widetilde{G} \cong \mathrm{SU}_2(\mathbb{C})$ (Example 8.1.10).

(2) $G = \mathrm{SO}_{2,1}(\mathbb{R})_0$ and $H = \mathrm{SL}_2(\mathbb{R})$: In this case we actually have a covering morphism $\varphi\colon H \to G$ coming from the adjoint representation

$$\mathrm{Ad}\colon \mathrm{SL}_2(\mathbb{R}) \to \mathrm{GL}(\mathbf{L}(H)) \cong \mathrm{GL}_3(\mathbb{R}).$$

On $\mathbf{L}(H) = \mathfrak{sl}_2(\mathbb{R})$ we consider the symmetric bilinear form given by $\beta(x,y) := \frac{1}{2}\operatorname{tr}(xy)$ and the basis

$$e_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_3 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then the matrix B of β with respect to this basis is

$$B := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

One easily verifies that

$$\operatorname{Im} \operatorname{Ad} \subseteq \mathrm{O}(\mathbf{L}(H), \beta) \cong \mathrm{O}_{2,1}(\mathbb{R}),$$

and since $\operatorname{ad}\colon \mathbf{L}(H) \to \mathfrak{o}_{2,1}(\mathbb{R})$ is injective between spaces of the same dimension 3 (Exercise), it is bijective. Therefore Proposition 8.1.3 implies that

$$\operatorname{Ad}\colon \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SO}_{2,1}(\mathbb{R})_0$$

is a covering morphism. Its kernel is given by $Z(\mathrm{SL}_2(\mathbb{R})) = \{\pm \mathbf{1}\}$.

From the polar decomposition one derives that both groups are homeomorphic to $\mathbb{T} \times \mathbb{R}^2$, and topologically the map Ad is like $(z, x, y) \mapsto (z^2, x, y)$, a two-fold covering.

(3) $G = \mathrm{SL}_2(\mathbb{C})$ and $H = \mathrm{SO}_{3,1}(\mathbb{R})_0$:

Here we show that the universal covering group of the identity component H of the Lorentz group $\mathrm{SO}_{3,1}(\mathbb{R})$ is isomorphic to G. The construction follows a similar scheme as the argument in (2) above.

On the real 4-dimensional vector space $V := \mathrm{Herm}_2(\mathbb{C})$ we consider the representation

$$\sigma\colon G = \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}(V), \quad \sigma(g)(x) := gxg^*.$$

We want to find a symmetric bilinear form β on V invariant under the action of G and with $\mathrm{O}(V, \beta) \cong \mathrm{O}_{3,1}(\mathbb{R})$. We consider the symmetric bilinear form

$$\beta\colon V \times V \to \mathbb{R}, \quad \beta(x, y) := \operatorname{tr}(xy) - \operatorname{tr} x \operatorname{tr} y.$$

It is obvious that this form is symmetric. An orthogonal basis with respect to β is given by

$$e_1 := \mathbf{1}, \quad e_2 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_3 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_4 := \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},$$

and we have

$$\beta(e_1, e_1) = -2, \quad \beta(e_2, e_2) = \beta(e_3, e_3) = \beta(e_4, e_4) = 2.$$

Therefore $\mathrm{O}(V, \beta) \cong \mathrm{O}_{3,1}(\mathbb{R})$.

To see that $\operatorname{im}(\sigma) \subseteq \mathrm{O}(V, \beta)$, we observe that the quadratic form corresponding to β is

$$\beta(x, x) = \operatorname{tr} x^2 - (\operatorname{tr} x)^2 = -2 \det x.$$

Now the invariance of β under G follows from the Polarization Identity and

$$\det(gxg^*) = \det g \det x \det g^* = \det x, \quad g \in \mathrm{SL}_2(\mathbb{C}), x \in \mathrm{Herm}_2(\mathbb{C}).$$

We conclude that $\sigma(G) \subseteq \mathrm{O}(V,\beta)$, and since G is connected, we further obtain $\sigma(G) \subseteq \mathrm{O}(V,\beta)_0 \cong \mathrm{SO}_{3,1}(\mathbb{R})_0$ (see also Exercise 1.2.8). We also write σ for the corresponding homomorphism $G \to H = \mathrm{SO}_{3,1}(\mathbb{R})_0$.

The derived representation is given by

$$\mathbf{L}(\sigma)\colon\ \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{so}_{3,1}(\mathbb{R}), \quad \mathbf{L}(\sigma)(x)(y) = xy + yx^*.$$

If $\sigma(g) = \mathbf{1}$, then $gxg^* = x$ for all $x \in \mathrm{Herm}_2(\mathbb{C})$, and this implies that $g(ix)g^* = ix$, which leads with $M_2(\mathbb{C}) = \mathrm{Herm}_2(\mathbb{C}) + i\,\mathrm{Herm}_2(\mathbb{C})$ to $gxg^* = x$ for all $x \in M_2(\mathbb{C})$. For $x = g^*$ we obtain in particular $g^* = g^{-1}$. This in turn yields $gxg^{-1} = x$ for all $x \in M_2(\mathbb{C})$, so that $g \in \mathbb{C}^\times\mathbf{1}$, and thus $g \in \{\pm\mathbf{1}\}$. We conclude that $\ker\sigma = \{\pm\mathbf{1}\}$ is discrete and therefore $\ker\mathbf{L}(\sigma) \subseteq \mathbf{L}(\ker\sigma) = \{0\}$. Hence $\mathbf{L}(\sigma)$ is injective. Next $\dim\mathfrak{sl}_2(\mathbb{C}) = \dim\mathfrak{so}_{3,1}(\mathbb{R}) = 6$ shows that $\mathbf{L}(\sigma)$ is bijective. Therefore $\sigma\colon G \to H$ is a covering morphism by Proposition 8.1.3. In view of $\pi_1(\mathrm{SL}_2(\mathbb{C})) \cong \pi_1(\mathrm{SU}_2(\mathbb{C})) \cong \pi_1(\mathbb{S}^3) = \{\mathbf{1}\}$, it follows that

$$\mathrm{SL}_2(\mathbb{C}) \cong \widetilde{\mathrm{SO}}_{3,1}(\mathbb{R})_0.$$

Example 8.3.5. Let $G = \mathrm{SL}_2(\mathbb{R})$ and $H = \mathrm{SO}_{2,1}(\mathbb{R})_0$ and recall that $\widetilde{G} \cong \widetilde{H}$ follows from $\mathfrak{sl}_2(\mathbb{R}) \cong \mathfrak{so}_{2,1}(\mathbb{R})$ (cf. Example 8.3.4(2)).

We further have $q_G(Z(\widetilde{G})) \subseteq Z(G) = \{\pm\mathbf{1}\}$ and $\pi_1(G) = \ker q_G \subseteq Z(\widetilde{G})$ (cf. Proposition 8.1.4). Likewise $q_H(Z(\widetilde{G})) \subseteq Z(H) = \{\mathbf{1}\}$ implies

$$Z(\widetilde{G}) \cong \pi_1(H) \cong \pi_1(\mathrm{O}_2(\mathbb{R}) \times \mathrm{O}_1(\mathbb{R})) \cong \mathbb{Z},$$

where the latter is a consequence of the polar decomposition. This implies that $Z(\widetilde{G}) \cong \mathbb{Z}$, where

$$\pi_1(G) \cong 2\mathbb{Z} \quad \text{and} \quad \pi_1(H) \cong \mathbb{Z} = Z(\widetilde{G}).$$

Therefore G and H are not isomorphic, but they have isomorphic Lie algebras and isomorphic fundamental groups.

8.4 Nonlinear Lie Groups

We have already seen how to describe all connected Lie groups with a given Lie algebra. To determine all such groups which are, in addition, linear turns out to be a much more subtle enterprise. If $\widetilde{G}$ is a simply connected group with a given Lie algebra, it means to determine which of the groups $\widetilde{G}/D$ are linear. As the following examples show, the answer to this problem is not easy. In fact, a complete answer requires detailed knowledge of the structure of finite-dimensional Lie algebras.

Example 8.4.1. We show that the universal covering group $G := \widetilde{\mathrm{SL}}_2(\mathbb{R})$ of $\mathrm{SL}_2(\mathbb{R})$ is not a linear Lie group. Moreover, we show that every continuous homomorphism $\varphi\colon G \to \mathrm{GL}_n(\mathbb{R})$ satisfies $D := \pi_1(\mathrm{SL}_2(\mathbb{R})) \subseteq \ker\varphi$, hence factors through $G/D \cong \mathrm{SL}_2(\mathbb{R})$.

We consider the Lie algebra homomorphism $\mathbf{L}(\varphi)\colon\ \mathfrak{sl}_2(\mathbb{R}) \to \mathfrak{gl}_n(\mathbb{R})$. Then it is easy to see that

$$\mathbf{L}(\varphi)_{\mathbb{C}}(x + iy) := \mathbf{L}(\varphi)x + i\,\mathbf{L}(\varphi)y$$

defines an extension of $\mathbf{L}(\varphi)$ to a complex linear Lie algebra homomorphism

$$\mathbf{L}(\varphi)_{\mathbb{C}}\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}_n(\mathbb{C}).$$

Since the group $\mathrm{SL}_2(\mathbb{C})$ is simply connected, there exists a unique group homomorphism $\psi\colon \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ with $\mathbf{L}(\psi) = \mathbf{L}(\varphi)_{\mathbb{C}}$.

Let $\alpha\colon G \to G/D \cong \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{C})$ be the canonical morphism. Then

$$\mathbf{L}(\varphi) = \mathbf{L}(\varphi)_{\mathbb{C}} \circ \mathbf{L}(\alpha) = \mathbf{L}(\psi) \circ \mathbf{L}(\alpha) = \mathbf{L}(\psi \circ \alpha)$$

implies $\varphi = \psi \circ \alpha$. We conclude that $\ker\varphi \supseteq \ker\alpha = D$. Therefore G has no faithful linear representation.

Lemma 8.4.2. *If $\mathcal{A}$ is a Banach algebra with unit $\mathbf{1}$ and $p, q \in \mathcal{A}$ with $[p,q] = \lambda\mathbf{1}$, then $\lambda = 0$.*

Proof. By induction we obtain

$$[p, q^n] = \lambda n q^{n-1} \quad \text{for} \quad n \in \mathbb{N}. \tag{8.2}$$

In fact,

$$[p, q^{n+1}] = [p,q]q^n + q[p,q^n] = \lambda q^n + \lambda n q^n = \lambda(n+1)q^n.$$

Therefore

$$|\lambda| n \|q^{n-1}\| \leq 2\|p\|\|q^n\| \leq 2\|p\|\|q\|\|q^{n-1}\|$$

for each $n \in \mathbb{N}$, which leads to

$$(|\lambda| n - 2\|p\|\|q\|)\|q^{n-1}\| \leq 0.$$

If $\lambda \neq 0$, then we obtain for sufficiently large n that $q^{n-1} = 0$. For $n > 1$ we derive from (8.2) that $q^{n-2} = 0$. Inductively we arrive at the contradiction $q = 0$. ☐

If $\mathcal{A}$ is a finite-dimensional algebra, we may w.l.o.g. assume that it is a subalgebra of some matrix algebra $M_n(\mathbb{K})$, and then $[p,q] = \lambda\mathbf{1}$ implies

$$n\lambda = \mathrm{tr}(\lambda\mathbf{1}) = \mathrm{tr}([p,q]) = 0$$

so that $\lambda = 0$.

Example 8.4.3. We consider the three-dimensional *Heisenberg group*

$$G = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\} \quad \text{with} \quad \mathbf{L}(G) = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

Note that $\exp_G\colon \mathbf{L}(G) \to G$ is a diffeomorphism whose inverse is given by

$$\log(g) = (g - \mathbf{1}) - \frac{1}{2}(g - \mathbf{1})^2$$

(Proposition 2.3.3). Let

$$z := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad p := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad q := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $[p, q] = z$, $[p, z] = [q, z] = 0$, $\exp \mathbb{R}z = \mathbf{1} + \mathbb{R}z \subseteq Z(G)$ and $D := \exp(\mathbb{Z}z)$ is a discrete central subgroup of G. We claim that the group G/D is not a linear Lie group. This will be verified by showing that each homomorphism $\alpha : G \to \mathrm{GL}_n(\mathbb{C})$ with $D \subseteq \ker \alpha$ satisfies $\exp(\mathbb{R}z) \subseteq \ker \alpha$.

The map $\mathbf{L}(\alpha)\colon \mathbf{L}(G) \to \mathfrak{gl}_n(\mathbb{C})$ is a Lie algebra homomorphism and we obtain linear maps

$$P := \mathbf{L}(\alpha)(p), \quad Q := \mathbf{L}(\alpha)(q) \quad \text{and} \quad Z := \mathbf{L}(\alpha)(z)$$

with $[P, Q] = Z$. Now $\exp_G z \in D = \ker \alpha$ implies that $e^Z = \alpha(\exp z) = \mathbf{1}$ and hence that Z is diagonalizable with all eigenvalues contained in $2\pi i\mathbb{Z}$ (Exercise 2.2.12). Let $V_\lambda := \ker(Z - \lambda\mathbf{1})$. Since z is central in $\mathbf{L}(G)$, the space V_λ is invariant under G (Exercise 1.1.1), hence also under $\mathbf{L}(G)$ (Exercise 3.2.4). Therefore the restrictions $P_\lambda := P|_{V_\lambda}$ and $Q_\lambda := Q|_{V_\lambda}$ satisfy $[P_\lambda, Q_\lambda] = \lambda \,\mathrm{id}$ in the Banach algebra $\mathrm{End}(V_\lambda)$. In view of the preceding lemma, we have $\lambda = 0$. Therefore the diagonalizability of Z entails that $Z = 0$ and hence that $\mathbb{R}z \subseteq \ker \mathbf{L}(\alpha)$. It follows in particular that the group G/D has no faithful linear representation.

8.5 The Quaternions, $\mathrm{SU}_2(\mathbb{C})$ and $\mathrm{SO}_4(\mathbb{R})$

In this subsection we shall use the quaternion algebra $\mathbb{H}$ to get some more information on the structure of the group $\mathrm{SO}_4(\mathbb{R})$. Here the idea is to identify $\mathbb{R}^4$ with $\mathbb{H}$.

Proposition 8.5.1. *There exists a covering homomorphism*

$$\varphi\colon \mathrm{SU}_2(\mathbb{C}) \times \mathrm{SU}_2(\mathbb{C}) \to \mathrm{SO}_4(\mathbb{R}) \subseteq \mathrm{GL}(\mathbb{H}), \quad \varphi(a, b)x = axb^{-1}.$$

This homomorphism is a universal covering with $\ker \varphi = \{\pm(\mathbf{1}, \mathbf{1})\}$.

Proof. Since $|a| = |b| = 1$, all the maps $\varphi(a, b)\colon \mathbb{H} \to \mathbb{H}$ are orthogonal, so that φ is a homomorphism

$$\mathrm{SU}_2(\mathbb{C}) \times \mathrm{SU}_2(\mathbb{C}) \to \mathrm{O}_4(\mathbb{R}).$$

Since $\mathrm{SU}_2(\mathbb{C}) \times \mathrm{SU}_2(\mathbb{C})$ is connected, it further follows that $\mathrm{im}(\varphi) \subseteq \mathrm{SO}_4(\mathbb{R})$.

To determine the kernel of φ, suppose that $\varphi(a, b) = \mathrm{id}_{\mathbb{H}}$. Then $axb^{-1} = x$ for all $x \in \mathbb{H}$. For $x = b$ we obtain in particular $a = b$. Hence $ax = xa$ for all $x \in \mathbb{H}$. With $x = I$ and $x = J$ this leads to $a \in \mathbb{R}\mathbf{1}$, and hence to $(a, b) \in \{\pm(\mathbf{1}, \mathbf{1})\}$. This proves the assertion on $\ker \varphi$.

The derived representation is given by

$$\mathbf{L}(\varphi)\colon \mathfrak{su}_2(\mathbb{C}) \times \mathfrak{su}_2(\mathbb{C}) \to \mathfrak{so}_4(\mathbb{R}), \quad \mathbf{L}(\varphi)(x, y)(z) = xz - zy.$$

Since $\ker \varphi$ is discrete, it follows that $\ker \mathbf{L}(\varphi) \subseteq \mathbf{L}(\ker \varphi) = \{0\}$. Hence $\mathbf{L}(\varphi)$ is injective. Next $\dim \mathfrak{so}_4(\mathbb{R}) = 6 = 2 \dim \mathfrak{su}_2(\mathbb{C})$ shows that $\mathbf{L}(\varphi)$ is surjective, and we conclude that

$$\mathrm{im}(\varphi) = \langle \exp \mathrm{im}\, \mathbf{L}(\varphi) \rangle = \mathrm{SO}_4(\mathbb{R}).$$

Therefore φ is a covering morphism (Proposition 8.1.3). Since $\mathrm{SU}_2(\mathbb{C})$ is simply connected, $\widetilde{\mathrm{SO}}_4(\mathbb{R}) \cong \mathrm{SU}_2(\mathbb{C})^2$. □

Let $G := \mathrm{SU}_2(\mathbb{C})^2$. We have just seen that this is the universal covering group of $\mathrm{SO}_4(\mathbb{R})$. On the other hand $\mathrm{SU}_2(\mathbb{C}) \cong \widetilde{\mathrm{SO}}_3(\mathbb{R})$. From $Z(\mathrm{SU}_2(\mathbb{C})) = \{\pm \mathbf{1}\}$ we derive that

$$Z(G) = \{(\mathbf{1}, \mathbf{1}), (\mathbf{1}, -\mathbf{1}), (-\mathbf{1}, \mathbf{1}), (-\mathbf{1}, -\mathbf{1})\} \cong C_2^2.$$

We have

$$G/Z(G) \cong \mathrm{SO}_3(\mathbb{R}) \times \mathrm{SO}_3(\mathbb{R}),$$

and therefore

$$\mathrm{SO}_4(\mathbb{R})/\{\pm \mathbf{1}\} \cong G/Z(G) \cong \mathrm{SO}_3(\mathbb{R}) \times \mathrm{SO}_3(\mathbb{R}).$$

The group $\mathrm{SO}_4(\mathbb{R})$ is a twofold covering group of $\mathrm{SO}_3(\mathbb{R})^2$.

Exercises for Chapter 8

Exercise 8.5.1. Let (E, β) be an n-dimensional euclidean space, i.e., β is a positive definite symmetric bilinear form on E. Show that there exists an isometric isomorphism $\Phi\colon \mathbb{R}^n \to E$, and that

$$\Psi\colon \mathrm{O}_n(\mathbb{R}) \to \mathrm{O}(E, \beta), \quad g \mapsto \Phi \circ g \circ \Phi^{-1}$$

is an isomorphism of Lie groups.

Exercise 8.5.2. Show that the center of the Lie algebra $\mathfrak{gl}_n(\mathbb{K})$ is

$$\mathfrak{z}(\mathfrak{gl}_n(\mathbb{K})) := \{x \in \mathfrak{gl}_n(\mathbb{K})\colon (\forall x \in \mathfrak{gl}_n(\mathbb{K}))\, [x, y] = 0\} = \mathbb{K}\mathbf{1}.$$

Hint: Consider the elementary matrices $E_{ij} := (\delta_{ik}\delta_{jl})_{k,l}$ and note that $T_{ij} := \mathbf{1} + E_{ij} \in \mathrm{GL}_n(\mathbb{K})$.

Exercise 8.5.3. We consider the simply connected covering group $G := \widetilde{\mathrm{SL}}_2(\mathbb{R})$ with $\mathbf{L}(G) = \mathfrak{sl}_2(\mathbb{R})$ and we write $q : G \to \mathrm{SL}_2(\mathbb{R})$ for the covering homomorphism. The map

$$\alpha : \mathbb{R} \to G, \quad t \mapsto \exp_G(t 2\pi u), \quad u =: \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is injective.

Exercise 8.5.4. Show that a subgroup Γ of the topological group G is discrete (with respect to the subspace topology) if and only if there exists a $\mathbf{1}$-neighborhood $U \subseteq G$ with $U \cap \Gamma = \{\mathbf{1}\}$.

Exercise 8.5.5. Let $\varphi\colon G \to H$ be a surjective morphism of topological groups. Show that the following conditions are equivalent:

(1) φ is open with discrete kernel.

(2) φ is a covering, i.e., each $h \in H$ has an open neighborhood U such that $\varphi^{-1}(U) = \bigcup_{i\in I} U_i$ is a disjoint union of open subsets U_i for which all restrictions $\varphi|_{U_i}\colon U_i \to U$ are homeomorphisms.

Hint: If $\Gamma \subseteq G$ is a discrete subgroup, then there exists an open symmetric $\mathbf{1}$-neighborhood U with $UU \cap \Gamma = \{1\}$ and then the sets γU, $\gamma \in \Gamma$, are pairwise disjoint.

Exercise 8.5.6. Let G be a connected Lie group. Show that

$$\mathbf{L}(Z(G)) = \mathfrak{z}(\mathbf{L}(G)) := \{x \in \mathbf{L}(G)\colon (\forall y \in \mathbf{L}(G))\ [x, y] = 0\}.$$

Exercise 8.5.7. Let $q_G\colon \widetilde{G} \to G$ be a simply connected covering of the connected Lie group G.

(1) Show that each automorphism $\varphi \in \mathrm{Aut}(G)$ has a unique lift $\widetilde{\varphi} \in \mathrm{Aut}(\widetilde{G})$.

(2) Derive from (1) that $\mathrm{Aut}(G) \cong \{\widetilde{\varphi} \in \mathrm{Aut}(\widetilde{G})\colon \widetilde{\varphi}(\pi_1(G)) = \pi_1(G)\}$.

(3) Show that, in general, $\{\widetilde{\varphi} \in \mathrm{Aut}(\widetilde{G})\colon \widetilde{\varphi}(\pi_1(G)) \subseteq \pi_1(G)\}$ is not a subgroup of $\mathrm{Aut}(\widetilde{G})$.

Appendix A

Basic Covering Theory

In this appendix we provide the main results on coverings of topological spaces needed to develop coverings of Lie groups and manifolds. In particular, this material is needed to show that, for each finite-dimensional Lie algebra $\mathfrak{g}$, there exists a 1-connected Lie group G with Lie algebra $\mathbf{L}(G) = \mathfrak{g}$ which is unique up to isomorphism.

A.1 The Fundamental Group

To define the notion of a simply connected space, we first have to define its fundamental group. The elements of this group are homotopy classes of loops. The present section develops this concept and provides some of its basic properties.

Definition A.1.1. Let X be a topological space, $I := [0,1]$, and $x_0, x_1 \in X$. We write

$$P(X, x_0) := \{\gamma \in C(I, X) \colon \gamma(0) = x_0\}$$

and

$$P(X, x_0, x_1) := \{\gamma \in P(X, x_0) \colon \gamma(1) = x_1\}.$$

We call two paths $\alpha_0, \alpha_1 \in P(X, x_0, x_1)$ *homotopic*, written $\alpha_0 \sim \alpha_1$, if there exists a continuous map

$$H \colon I \times I \to X \quad \text{with} \quad H_0 = \alpha_0, \quad H_1 = \alpha_1$$

(for $H_t(s) := H(t,s)$) and

$$(\forall t \in I) \quad H(t,0) = x_0, \ H(t,1) = x_1.$$

It is easy to show that $\sim$ is an equivalence relation (Exercise A.1.2), called *homotopy*. The homotopy class of α is denoted by $[\alpha]$.

We write $\Omega(X, x_0) := P(X, x_0, x_0)$ for the set of loops based at x_0. For $\alpha \in P(X, x_0, x_1)$ and $\beta \in P(X, x_1, x_2)$ we define a product $\alpha * \beta$ in $P(X, x_0, x_2)$ as the concatenation

$$(\alpha * \beta)(t) := \begin{cases} \alpha(2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Lemma A.1.2. *If $\varphi\colon [0,1] \to [0,1]$ is a continuous map with $\varphi(0) = 0$ and $\varphi(1) = 1$, then for each $\alpha \in P(X, x_0, x_1)$ we have $\alpha \sim \alpha \circ \varphi$.*

Proof. Use $H(t,s) := \alpha(ts + (1-t)\varphi(s))$. □

Proposition A.1.3. *The following assertions hold:*

(1) *$\alpha_1 \sim \alpha_2$ and $\beta_1 \sim \beta_2$ implies $\alpha_1 * \beta_1 \sim \alpha_2 * \beta_2$, so that we obtain a well-defined product*

$$[\alpha] * [\beta] := [\alpha * \beta]$$

of homotopy classes.

(2) *If x also denotes the constant map $I \to \{x\} \subseteq X$, then*

$$[x_0] * [\alpha] = [\alpha] = [\alpha] * [x_1] \quad \text{for} \quad \alpha \in P(X, x_0, x_1).$$

(3) (Associativity) *$[\alpha * \beta] * [\gamma] = [\alpha] * [\beta * \gamma]$ for $\alpha \in P(X, x_0, x_1)$, $\beta \in P(X, x_1, x_2)$ and $\gamma \in P(X, x_2, x_3)$.*

(4) (Inverse) *For $\alpha \in P(X, x_0, x_1)$ and $\overline{\alpha}(t) := \alpha(1-t)$ we have*

$$[\alpha] * [\overline{\alpha}] = [x_0].$$

(5) (Functoriality) *For any continuous map $\varphi\colon X \to Y$ with $\varphi(x_0) = y_0$ we have*

$$(\varphi \circ \alpha) * (\varphi \circ \beta) = \varphi \circ (\alpha * \beta)$$

and $\alpha \sim \beta$ implies $\varphi \circ \alpha \sim \varphi \circ \beta$.

Proof. (1) If H^α is a homotopy from α_1 to α_2 and H^β a homotopy from β_1 to β_2, then we put

$$H(t,s) := \begin{cases} H^\alpha(t, 2s) & \text{for } 0 \leq s \leq \frac{1}{2} \\ H^\beta(t, 2s-1) & \text{for } \frac{1}{2} \leq s \leq 1 \end{cases}$$

(cf. Exercise A.1.1).

(2) For the first assertion we use Lemma A.1.2 and

$$x_0 * \alpha = \alpha \circ \varphi \quad \text{for} \quad \varphi(t) := \begin{cases} 0 & \text{for } 0 \leq t \leq \frac{1}{2} \\ 2t-1 & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

For the second, we have

$$\alpha * x_1 = \alpha \circ \varphi \quad \text{for} \quad \varphi(t) := \begin{cases} 2t & \text{for } 0 \leq t \leq \frac{1}{2} \\ 1 & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

(3) We have $(\alpha * \beta) * \gamma = (\alpha * (\beta * \gamma)) \circ \varphi$ for

$$\varphi(t) := \begin{cases} 2t & \text{for } 0 \leq t \leq \frac{1}{4} \\ \frac{1}{4} + t & \text{for } \frac{1}{4} \leq t \leq \frac{1}{2} \\ \frac{t+1}{2} & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

(4)
$$H(t,s) := \begin{cases} \alpha(2s) & \text{for } s \leq \frac{1-t}{2} \\ \alpha(1-t) & \text{for } \frac{1-t}{2} \leq s \leq \frac{1+t}{2} \\ \overline{\alpha}(2s-1) & \text{for } s \geq \frac{1+t}{2}. \end{cases}$$

(5) is trivial. □

Definition A.1.4. From the preceding definition, we derive in particular that the set

$$\pi_1(X, x_0) := \Omega(X, x_0)/\sim$$

of homotopy classes of loops in x_0 carries a natural group structure. This group is called the *fundamental group of X with respect to x_0*.

A space X is called *simply connected* if $\pi_1(X, x_0)$ vanishes for all $x_0 \in X$. If X is pathwise connected it suffices to check this for a single $x_0 \in X$ (Exercise A.1.4).

Lemma A.1.5. (Functoriality of the Fundamental Group) *If $f\colon X \to Y$ is a continuous map with $f(x_0) = y_0$, then*

$$\pi_1(f)\colon \pi_1(X, x_0) \to \pi_1(Y, y_0), \quad [\gamma] \mapsto [f \circ \gamma]$$

is a group homomorphism. Moreover, we have

$$\pi_1(\mathrm{id}_X) = \mathrm{id}_{\pi_1(X,x_0)} \qquad \textit{and} \qquad \pi_1(f \circ g) = \pi_1(f) \circ \pi_1(g).$$

Proof. This follows directly from Proposition A.1.3(5). □

Remark A.1.6. The map

$$\sigma\colon \pi_1(X, x_0) \times \big(P(X, x_0)/\sim\big) \to P(X, x_0)/\sim, \quad ([\alpha], [\beta]) \mapsto [\alpha * \beta] = [\alpha] * [\beta]$$

defines an action of the group $\pi_1(X, x_0)$ on the set $P(X, x_0)/\sim$ of homotopy classes of paths starting in x_0 (Proposition A.1.3).

Remark A.1.7. (a) Suppose that the topological space X is contractible, i.e., there exists a continuous map $H\colon I \times X \to X$ and $x_0 \in X$ with $H(0, x) = x$ and $H(1, x) = x_0$ for $x \in X$. Then $\pi_1(X, x_0) = \{[x_0]\}$ is trivial (Exercise).

(b) $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

(c) $\pi_1(\mathbb{R}^n, 0) = \{0\}$ because $\mathbb{R}^n$ is contractible.

More generally, if the open subset $\Omega \subseteq \mathbb{R}^n$ is starlike with respect to x_0, then $H(t, x) := x + t(x - x_0)$ yields a contraction to x_0, and we conclude that $\pi_1(\Omega, x_0) = \{[x_0]\}$.

(d) If $G \subseteq \mathrm{GL}_n(\mathbb{R})$ is a linear Lie group with a polar decomposition, i.e., for $K := G \cap \mathrm{O}_n(\mathbb{R})$ and $\mathfrak{p} := \mathbf{L}(G) \cap \mathrm{Sym}_n(\mathbb{R})$, the polar map

$$p\colon K \times \mathfrak{p} \to G, \quad (k, x) \mapsto ke^x$$

is a homeomorphism, then the inclusion $K \to G$ induces an isomorphism

$$\pi_1(K, \mathbf{1}) \to \pi_1(G, \mathbf{1})$$

because the vector space $\mathfrak{p}$ is contractible.

The following lemma implies in particular, that fundamental groups of topological groups are always abelian.

Lemma A.1.8. *Let G be a topological group and consider the identity element $\mathbf{1}$ as a base point. Then the path space $P(G,\mathbf{1})$ also carries a natural group structure given by the pointwise product $(\alpha \cdot \beta)(t) := \alpha(t)\beta(t)$ and we have*

(1) *$\alpha \sim \alpha'$, $\beta \sim \beta'$ implies $\alpha \cdot \beta \sim \alpha' \cdot \beta'$, so that we obtain a well-defined product*

$$[\alpha][\beta] := [\alpha] \cdot [\beta] := [\alpha \cdot \beta]$$

of homotopy classes, defining a group structure on $P(G,\mathbf{1})/\sim$.

(2) *$\alpha \sim \beta \Longleftrightarrow \alpha \cdot \beta^{-1} \sim \mathbf{1}$, the constant map.*

(3) (Commutativity) *$[\alpha] \cdot [\beta] = [\beta] \cdot [\alpha]$ for $\alpha, \beta \in \Omega(G,\mathbf{1})$.*

(4) (Consistency) *$[\alpha] \cdot [\beta] = [\alpha] * [\beta]$ for $\alpha \in \Omega(G,\mathbf{1})$, $\beta \in P(G,\mathbf{1})$.*

Proof. (1) follows by composing homotopies with the multiplication map m_G.

(2) follows from (1).

(3)

$$[\alpha][\beta] = [\alpha * \mathbf{1}][\mathbf{1} * \beta] = [(\alpha * \mathbf{1})(\mathbf{1} * \beta)] = [(\mathbf{1} * \beta)(\alpha * \mathbf{1})] = [\mathbf{1} * \beta][\alpha * \mathbf{1}] = [\beta][\alpha].$$

(4) $[\alpha][\beta] = [(\alpha * \mathbf{1})(\mathbf{1} * \beta)] = [\alpha * \beta] = [\alpha] * [\beta]$. □

As a consequence of (4), we can calculate the product of homotopy classes as a pointwise product of representatives and obtain:

Proposition A.1.9. (Hilton's Lemma) *For each topological group G, the fundamental group $\pi_1(G) := \pi_1(G,\mathbf{1})$ is abelian.*

Proof. We only have to combine (3) and (4) in Lemma A.1.8 for loops $\alpha, \beta \in \Omega(G,\mathbf{1})$. □

Exercises for Section A.1

Exercise A.1.1. If $f\colon X \to Y$ is a map between topological spaces and

$$X = X_1 \cup \ldots \cup X_n$$

holds with closed subsets $X_1, \ldots, X_n$, then f is continuous if and only if all restrictions $f|_{X_i}$ are continuous.

Exercise A.1.2. Show that the homotopy relation on $P(X, x_0, x_1)$ is an equivalence relation.

Exercise A.1.3. Show that for $n > 1$ the sphere $\mathbb{S}^n$ is simply connected. For the proof, proceed along the following steps:
(a) Let $\gamma : [0,1] \to \mathbb{S}^n$ be continuous. Then there exists an $m \in \mathbb{N}$ such that $\|\gamma(t) - \gamma(t')\| < \frac{1}{2}$ for $|t - t'| < \frac{1}{m}$.
(b) Define $\widetilde{\alpha} : [0,1] \to \mathbb{R}^{n+1}$ as the piecewise affine curve with $\widetilde{\alpha}(\frac{k}{m}) = \gamma(\frac{k}{m})$ for $k = 0, \ldots, m$. Then $\alpha(t) := \frac{1}{\|\widetilde{\alpha}(t)\|}\widetilde{\alpha}(t)$ defines a continuous curve $\alpha : [0,1] \to \mathbb{S}^n$.
(c) $\alpha \sim \gamma$.
(d) α is not surjective. The image of α is the central projection of a polygonal arc on the sphere.
(e) If $\beta \in \Omega(\mathbb{S}^n, y_0)$ is not surjective, then $\beta \sim y_0$ (it is homotopic to a constant map).
(f) $\pi_1(\mathbb{S}^n, y_0) = \{[y_0]\}$ for $n \geq 2$ and $y_0 \in \mathbb{S}^n$.

Exercise A.1.4. Let X be a topological space, $x_0, x_1 \in X$ and $\alpha \in P(X, x_0, x_1)$ a path from x_0 to x_1. Show that the map

$$C\colon \pi_1(X, x_1) \to \pi_1(X, x_0), \quad [\gamma] \mapsto [\alpha * \gamma * \overline{\alpha}]$$

is an isomorphism of groups. In this sense the fundamental group does not depend on the base point if X is arcwise connected.

Exercise A.1.5. Let $\sigma\colon G \times X \to X$ be a continuous action of the topological group G on the topological space X and $x_0 \in X$. Then the orbit map $\sigma^{x_0}\colon G \to X, g \mapsto \sigma(g, x_0)$ defines a group homomorphism

$$\pi_1(\sigma^{x_0})\colon \pi_1(G) \to \pi_1(X, x_0).$$

Show that the image of this homomorphism is central, i.e., lies in the center of $\pi_1(X, x_0)$.

A.2 Coverings

In this section we discuss the concept of a covering map. Its main application in Lie theory is that it provides, for each connected Lie group G, a simply connected covering group $q_G\colon \widetilde{G} \to G$ and hence also a tool to calculate its fundamental group $\pi_1(G) \cong \ker q_G$. In the following chapter we shall investigate to which extent a Lie group is determined by its Lie algebra and its fundamental group.

Definition A.2.1. Let X and Y be topological spaces. A continuous map $q\colon X \to Y$ is called a *covering* if each $y \in Y$ has an open neighborhood U such that $q^{-1}(U)$ is a nonempty disjoint union of open subsets $(V_i)_{i \in I}$, such that for each $i \in I$ the restriction $q|_{V_i}\colon V_i \to U$ is a homeomorphism. We call any such U an *elementary* open subset of X.

Note that this condition implies in particular that q is surjective and that the fibers of q are discrete subsets of X.

Examples A.2.2. (a) The exponential function $\exp\colon \mathbb{C} \to \mathbb{C}^\times, z \mapsto e^z$ is a covering map.

(b) The map $q\colon \mathbb{R} \to \mathbb{T}, x \mapsto e^{ix}$ is a covering.

(c) The power maps $p_k\colon \mathbb{C}^\times \to \mathbb{C}^\times, z \mapsto z^k$ are coverings.

(d) If $q\colon G \to H$ is a surjective open morphism of topological groups with discrete kernel, then q is a covering (Exercise A.2.2). All the examples (a)-(c) are of this type.

Lemma A.2.3. (Lebesgue Number) *Let (X, d) be a compact metric space and $(U_i)_{i\in I}$ an open cover. Then there exists a positive number $\lambda > 0$, called a Lebesgue number of the covering, such that any subset $S \subseteq X$ with diameter $\leq \lambda$ is contained in some U_i.*

Proof. Let us assume that such a number λ does not exist. Then for each $n \in \mathbb{N}$ there exists a subset S_n of diameter $\leq \frac{1}{n}$ which is not contained in some U_i. Pick a point $s_n \in S_n$. The sequence (s_n) has a subsequence converging to some $s \in X$ and s is contained in some U_i. Since U_i is open, there exists an $\varepsilon > 0$ with $U_\varepsilon(s) \subseteq U_i$. If $n \in \mathbb{N}$ is such that $\frac{1}{n} < \frac{\varepsilon}{2}$ and $d(s_n, s) < \frac{\varepsilon}{2}$, we arrive at the contradiction $S_n \subseteq U_{\varepsilon/2}(s_n) \subseteq U_\varepsilon(s) \subseteq U_i$. □

Remark A.2.4. (1) If $(U_i)_{i\in I}$ is an open cover of the unit interval $[0, 1]$, then there exists an $n > 0$ such that all subsets of the form $\left[\frac{k}{n}, \frac{k+1}{n}\right]$, $k = 0, \ldots, n-1$, are contained in some U_i.

(2) If $(U_i)_{i\in I}$ is an open cover of the unit square $[0, 1]^2$, then there exists an $n > 0$ such that all subsets of the form

$$\left[\frac{k}{n}, \frac{k+1}{n}\right] \times \left[\frac{j}{n}, \frac{j+1}{n}\right], \quad k, j = 0, \ldots, n-1,$$

are contained in some U_i.

Theorem A.2.5. (Path Lifting Theorem) *Let $q\colon X \to Y$ be a covering map and $\gamma\colon [0, 1] \to Y$ a path. Let $x_0 \in X$ be such that $q(x_0) = \gamma(0)$. Then there exists a unique path $\widetilde{\gamma}\colon [0, 1] \to X$ such that*

$$q \circ \widetilde{\gamma} = \gamma \quad \textit{and} \quad \widetilde{\gamma}(0) = x_0.$$

Proof. Cover Y by elementary open sets $U_i, i \in I$. By Lemma A.2.3, there exists an $n \in \mathbb{N}$ such that all sets $\gamma(\left[\frac{k}{n}, \frac{k+1}{n}\right])$, $k = 0, \ldots, n-1$, are contained in some U_i. We now use induction to construct $\widetilde{\gamma}$. Let $V_0 \subseteq q^{-1}(U_0)$ be an open subset containing x_0 for which $q|_{V_0}$ is a homeomorphism onto U_0 and define $\widetilde{\gamma}$ on $\left[0, \frac{1}{n}\right]$ by

$$\widetilde{\gamma}(t) := (q|_{V_0})^{-1} \circ \gamma(t).$$

Assume that we have already constructed a continuous lift $\widetilde{\gamma}$ of γ on the interval $\left[0, \frac{k}{n}\right]$ and that $k < n$. Then we pick an open subset $V_k \subseteq X$ containing $\widetilde{\gamma}(\frac{k}{n})$ for which $q|_{V_k}$ is a homeomorphism onto some U_i and define $\widetilde{\gamma}$ for $t \in \left[\frac{k}{n}, \frac{k+1}{n}\right]$ by

$$\widetilde{\gamma}(t) := (q|_{V_k})^{-1} \circ \gamma(t).$$

We thus obtain the required lift $\widetilde{\gamma}$ of γ.

If $\widehat{\gamma}\colon [0, 1] \to X$ is any continuous lift of γ with $\widehat{\gamma}(0) = x_0$, then $\widehat{\gamma}(\left[0, \frac{1}{n}\right])$ is a connected subset of $q^{-1}(U_0)$ containing x_0, hence contained in V_0. This shows that $\widetilde{\gamma}$ coincides with $\widehat{\gamma}$ on $\left[0, \frac{1}{n}\right]$. Applying the same argument at each step of the induction, we obtain $\widehat{\gamma} = \widetilde{\gamma}$, so that the lift $\widetilde{\gamma}$ is unique. □

Theorem A.2.6. (Covering Homotopy Theorem) *Let $I := [0,1]$ and $q\colon X \to Y$ be a covering map and $H\colon I^2 \to Y$ be a homotopy with fixed endpoints of the paths $\gamma := H_0$ and $\eta := H_1$. For any lift $\widetilde{\gamma}$ of γ there exists a unique lift $G\colon I^2 \to X$ of H with $G_0 = \widetilde{\gamma}$. Then $\widetilde{\eta} := G_1$ is the unique lift of η starting in the same point as $\widetilde{\gamma}$ and G is a homotopy from $\widetilde{\gamma}$ to $\widetilde{\eta}$. In particular, lifts of homotopic curves in Y starting in the same point are homotopic in X.*

Proof. Using the Path Lifting Property (Theorem A.2.5), for each $t \in I$ we find a unique continuous lift $I \to X, s \mapsto G(s,t)$, starting in $\widetilde{\gamma}(t)$ with $q(G(s,t)) = H(s,t)$. It remains to show that the map $G\colon I^2 \to X$ obtained in this way is continuous.

So let $s \in I$. Using Lemma A.2.3, we find a natural number n such that for each connected neighborhood W_s of s of diameter $\leq \frac{1}{n}$ and each $i = 0, \ldots, n$, the set $H\big(W_s \times \big[\frac{k}{n}, \frac{k+1}{n}\big]\big)$ is contained in some elementary subset U_k of Y. Assuming that G is continuous in $W_s \times \{\frac{k}{n}\}$, G maps this set into a connected subset of $q^{-1}(U_k)$, hence into some open subset V_k for which $q|_{V_k}$ is a homeomorphism onto U_k. But then the lift G on $W_s \times \big[\frac{k}{n}, \frac{k+1}{n}\big]$ must be contained in V_k, so that it is of the form $(q|_{V_k})^{-1} \circ H$, hence continuous. This means that G is continuous on $U_s \times \big[\frac{k}{n}, \frac{k+1}{n}\big]$. Now an inductive argument shows that G is continuous on $U_s \times I$ and hence on the whole square I^2.

Since the fibers of q are discrete and the curves $s \mapsto H(s,0)$ and $s \mapsto H(s,1)$ are constant, the curves $G(s,0)$ and $G(s,1)$ are also constant. Therefore $\widetilde{\eta}$ is the unique lift of η starting in $\widetilde{\gamma}(0) = G(0,0) = G(1,0)$ and G is a homotopy with fixed endpoints from $\widetilde{\gamma}$ to $\widetilde{\eta}$. $\square$

Corollary A.2.7. *If $q\colon X \to Y$ is a covering with $q(x_0) = y_0$, then the corresponding homomorphism*

$$\pi_1(q)\colon \pi_1(X, x_0) \to \pi_1(Y, y_0), \quad [\gamma] \mapsto [q \circ \gamma]$$

is injective.

Proof. If γ, η are loops in x_0 with $[q \circ \gamma] = [q \circ \eta]$, then the Covering Homotopy Theorem A.2.6 implies that γ and η are homotopic. Therefore $[\gamma] = [\eta]$ shows that $\pi_1(q)$ is injective. $\square$

Corollary A.2.8. *If Y is simply connected and X is arcwise connected, then each covering map $q\colon X \to Y$ is a homeomorphism.*

Proof. Since q is an open continuous map, it remains to show that q is injective. So pick $x_0 \in X$ and $y_0 \in Y$ with $q(x_0) = y_0$. If $x \in X$ also satisfies $q(x) = y_0$, then there exists a path $\alpha \in P(X, x_0, x)$ from x_0 to x. Now $q \circ \alpha$ is a loop in Y, hence contractible because Y is simply connected. Now the Covering Homotopy Theorem implies that the unique lift α of $q \circ \alpha$ starting in x_0 is a loop, and therefore that $x_0 = x$. This proves that q is injective. $\square$

The following theorem provides a powerful tool, from which the preceding corollary easily follows. We recall that a topological space X is called *locally arcwise connected* if each neighborhood U of a point $x \in X$ contains an arcwise connected neighborhood.

Theorem A.2.9. (Lifting Theorem) *Assume that $q\colon X \to Y$ is a covering map with $q(x_0) = y_0$, that W is arcwise connected and locally arcwise connected, and that $f\colon W \to Y$ is a given map with $f(w_0) = y_0$. Then a continuous map $g\colon W \to X$ with*

$$g(w_0) = x_0 \quad \text{and} \quad q \circ g = f \tag{A.1}$$

exists if and only if

$$\pi_1(f)(\pi_1(W, w_0)) \subseteq \pi_1(q)(\pi_1(X, x_0)), \ \textit{i.e.,}\ \operatorname{im}(\pi_1(f)) \subseteq \operatorname{im}(\pi_1(q)). \tag{A.2}$$

If g exists, then it is uniquely determined by (A.1). *Condition* (A.2) *is in particular satisfied if W is simply connected.*

Proof. If g exists, then $f = q \circ g$ implies that the image of the homomorphism $\pi_1(f) = \pi_1(q) \circ \pi_1(g)$ is contained in the image of $\pi_1(q)$.

Let us, conversely, assume that this condition is satisfied. To define g, let $w \in W$ and $\alpha_w\colon I \to W$ be a path from w_0 to w. Then $f \circ \alpha_w\colon I \to Y$ is a path which has a continuous lift $\beta_w\colon I \to X$ starting in x_0. We claim that $\beta_w(1)$ does not depend on the choice of the path α_w. Indeed, if α'_w is another path from w_0 to w, then $\alpha_w * \overline{\alpha'_w}$ is a loop in w_0, so that $(f \circ \alpha_w) * (f \circ \overline{\alpha'_w})$ is a loop in y_0. In view of (A.2), the homotopy class of this loop is contained in the image of $\pi_1(q)$, so that it has a lift $\eta\colon I \to X$ which is a loop in x_0. Since the reverse of the second half $\eta|_{[\frac{1}{2},1]}$ of η is a lift of $f \circ \alpha'_w$, starting in x_0, it is β'_w, and we obtain

$$\beta'_w(1) = \eta\Big(\frac{1}{2}\Big) = \beta_w(1).$$

We now put $g(w) := \beta_w(1)$, and it remains to see that g is continuous. This is where we shall use the assumption that W is locally arcwise connected. Let $w \in W$ and put $y := f(w)$. Further, let $U \subseteq Y$ be an elementary neighborhood of y and V be an arcwise connected neighborhood of w in W such that $f(V) \subseteq U$. Fix a path α_w from w_0 to w as before. For any point $w' \in V$ we choose a path $\gamma_{w'}$ from w to w' in V, so that $\alpha_w * \gamma_{w'}$ is a path from w_0 to w'. Let $\widetilde{U} \subseteq X$ be an open subset of X for which $q|_{\widetilde{U}}$ is a homeomorphism onto U and $g(w) \in \widetilde{U}$. Then the uniqueness of lifts implies that

$$\beta_{w'} = \beta_w * \big((q|_{\widetilde{U}})^{-1} \circ (f \circ \gamma_{w'})\big).$$

We conclude that

$$g(w') = (q|_{\widetilde{U}})^{-1}(f(w')) \in \widetilde{U},$$

hence that $g|_V$ is continuous.

Finally, we show that g is unique. In fact, if $h\colon W \to X$ is another lift of f satisfying $h(w_0) = x_0$, then the set $S := \{w \in W\colon g(w) = h(w)\}$ is nonempty and closed. We claim that it is also open. In fact, let $w_1 \in S$ and U be a connected open elementary neighborhood of $f(w_1)$ and V an arcwise connected neighborhood of w_1 with $f(V) \subseteq U$. If $\widetilde{U} \subseteq q^{-1}(U)$ is the open subset on which q is a homeomorphism containing $g(w_1) = h(w_1)$, then, since V is arcwise connected, we have that $g(V), h(V) \subseteq \widetilde{U}$, whence $V \subseteq S$. Therefore S is open, closed and nonempty. Since W is connected this implies that $S = W$, i.e., $g = h$. □

Corollary A.2.10. *[Uniqueness of Simply Connected Coverings] Suppose that Y is locally arcwise connected. If $q_1 \colon X_1 \to Y$ and $q_2 \colon X_2 \to Y$ are two simply connected arcwise connected coverings, then there exists a homeomorphism $\varphi \colon X_1 \to X_2$ with $q_2 \circ \varphi = q_1$.*

Proof. Since Y is locally arcwise connected, both covering spaces X_1 and X_2 also have this property. Pick points $x_1 \in X_1$, $x_2 \in X_2$ with $y := q_1(x_1) = q_2(x_2)$. According to the Lifting Theorem A.2.9, there exists a unique lift $\varphi \colon X_1 \to X_2$ of q_1 with $\varphi(x_1) = x_2$. We likewise obtain a unique lift $\psi \colon X_2 \to X_1$ of q_2 with $\psi(x_2) = x_1$. Then $\varphi \circ \psi \colon X_1 \to X_1$ is a lift of id_Y fixing x_1, so that the uniqueness of lifts implies that $\varphi \circ \psi = \mathrm{id}_{X_1}$. The same argument yields $\psi \circ \varphi = \mathrm{id}_{X_2}$, so that φ is a homeomorphism with the required properties. $\square$

Definition A.2.11. A topological space X is called *semilocally simply connected* if each point $x_0 \in X$ has a neighborhood U such that each loop $\alpha \in \Omega(U, x_0)$ is homotopic to $[x_0]$ in X, i.e., the natural homomorphism

$$\pi(i_U) \colon \pi_1(U, x_0) \to \pi_1(X, x_0), \quad [\gamma] \mapsto [i_U \circ \gamma]$$

induced by the inclusion map $i_U \colon U \to X$ is trivial.

Theorem A.2.12. (Existence of simply connected coverings) *Let Y be arcwise connected and locally arcwise connected. Then Y has a simply connected covering space if and only if Y is semilocally simply connected.*

Proof. If $q \colon X \to Y$ is a simply connected covering space and $U \subseteq Y$ is a pathwise connected elementary open subset. Then each loop γ in U lifts to a loop $\widetilde{\gamma}$ in X, and since $\widetilde{\gamma}$ is homotopic to a constant map in X, the same holds for the loop $\gamma = q \circ \widetilde{\gamma}$ in Y.

Conversely, let us assume that Y is semilocally simply connected. We choose a base point $y_0 \in Y$ and let

$$\widetilde{Y} := P(Y, y_0)/\sim\, := \bigcup_{y_1 \in Y} P(Y, y_0, y_1)/\sim$$

be the set of homotopy classes of paths starting in y_0. We shall provide $\widetilde{Y}$ with a topology such that the map

$$q \colon \widetilde{Y} \to Y, \quad [\gamma] \mapsto \gamma(1)$$

defines a simply connected covering of Y.

Let $\mathcal{B}$ denote the set of all arcwise connected open subsets $U \subseteq Y$ for which each loop in U is contractible in Y and note that our assumptions on Y imply that $\mathcal{B}$ is a basis of the topology of Y, i.e., each open subset is a union of elements of $\mathcal{B}$. If $\gamma \in P(Y, y_0)$ satisfies $\gamma(1) \in U \in \mathcal{B}$, let

$$U_{[\gamma]} := \{[\eta] \in q^{-1}(U) \colon (\exists \beta \in C(I, U))\ \eta \sim \gamma * \beta\}.$$

We shall now verify several properties of these definitions, culminating in the proof of the theorem.

(1) $[\eta] \in U_{[\gamma]} \Rightarrow U_{[\eta]} = U_{[\gamma]}$.

To prove this, let $[\zeta] \in U_{[\eta]}$. Then $\zeta \sim \eta * \beta$ for some path β in U. Further $\eta \sim \gamma * \beta'$ for some path β' in U. Now $\zeta \sim \gamma * \beta' * \beta$, and $\beta' * \beta$ is a path in U, so that $[\zeta] \in U_{[\gamma]}$. This proves $U_{[\eta]} \subseteq U_{[\gamma]}$. We also have $\gamma \sim \eta * \overline{\beta'}$, so that $[\gamma] \in U_{[\eta]}$, and the first part implies that $U_{[\gamma]} \subseteq U_{[\eta]}$.

(2) q maps $U_{[\gamma]}$ injectively onto U.

That $q(U_{[\gamma]}) = U$ is clear since U and Y are arcwise connected. To show that it is one-to-one, let $[\eta], [\eta'] \in U_{[\gamma]}$, which we know from (1) is the same as $U_{[\eta]}$. Suppose $\eta(1) = \eta'(1)$. Since $[\eta'] \in U_{[\eta]}$, we have $\eta' \sim \eta * \alpha$ for some loop α in U. But then α is contractible in Y, so that $\eta' \sim \eta$, i.e., $[\eta'] = [\eta]$.

(3) $U, V \in \mathcal{B}$, $\gamma(1) \in U \subseteq V$, implies $U_{[\gamma]} \subseteq V_{[\gamma]}$.

This is trivial.

(4) The sets $U_{[\gamma]}$ for $U \in \mathcal{B}$ and $[\gamma] \in \widetilde{Y}$ form a basis of a topology on $\widetilde{Y}$.

Suppose $[\gamma] \in U_{[\eta]} \cap V_{[\eta']}$. Let $W \subseteq U \cap V$ be in $\mathcal{B}$ with $\gamma(1) \in W$. Then $[\gamma] \in W_{[\gamma]} \subseteq U_{[\gamma]} \cap V_{[\gamma]} = U_{[\eta]} \cap V_{[\eta']}$.

(5) q is open and continuous.

We have already seen in (2) that $q(U_{[\gamma]}) = U$, and these sets form a basis of the topology on $\widetilde{Y}$, resp., Y. Therefore q is an open map. We also have for $U \in \mathcal{B}$ the relation

$$q^{-1}(U) = \bigcup_{\gamma(1) \in U} U_{[\gamma]},$$

which is open. Hence q is continuous.

(6) $q|_{U_{[\gamma]}}$ is a homeomorphism.

This is because it is bijective, continuous and open.

At this point we have shown that $q \colon \widetilde{Y} \to Y$ is a covering map. It remains to see that $\widetilde{Y}$ is arcwise connected and simply connected.

(7) Let $H \colon I \times I \to Y$ be a continuous map with $H(t, 0) = y_0$. Then $h_t(s) := H(t, s)$ defines a path in Y starting in y_0. Let $\widetilde{h}(t) := [h_t] \in \widetilde{Y}$. Then $\widetilde{h}$ is a path in $\widetilde{Y}$ covering the path $t \mapsto h_t(1) = H(t, 1)$ in Y. We claim that $\widetilde{h}$ is continuous. Let $t_0 \in I$. We shall prove continuity at t_0. Let $U \in \mathcal{B}$ be a neighborhood of $h_{t_0}(1)$. Then there exists an interval $I_0 \subseteq I$ which is a neighborhood of t_0 with $h_t(1) \in U$ for $t \in I_0$. Then $\alpha(s) := H(t_0 + s(t - t_0), 1)$ is a continuous curve in U with $\alpha(0) = h_{t_0}(1)$ and $\alpha(1) = h_t(1)$, so that $h_{t_0} * \alpha$ is curve with the same endpoint as h_t. Applying Exercise A.2.1 to the restriction of H to the interval between t_0 and t, we see that $h_t \sim h_{t_0} * \alpha$, so that $\widetilde{h}(t) = [h_t] \in U_{[h_{t_0}]}$ for $t \in I_0$. Since $q|_{U_{[h_{t_0}]}}$ is a homeomorphism, $\widetilde{h}$ is continuous in t_0.

(8) $\widetilde{Y}$ is arcwise connected.

For $[\gamma] \in \widetilde{Y}$ put $h_t(s) := \gamma(st)$. By (7), this yields a path $\widetilde{\gamma}(t) = [h_t]$ in $\widetilde{Y}$ from $\widetilde{y}_0 := [y_0]$ (the class of the constant path) to the point $[\gamma]$.

(9) $\widetilde{Y}$ is simply connected.

Let $\widetilde{\alpha} \in \Omega(\widetilde{Y}, \widetilde{y}_0)$ be a loop in $\widetilde{Y}$ and $\alpha := q \circ \widetilde{\alpha}$ its image in Y. Let $h_t(s) := \alpha(st)$. Then we have the path $\widetilde{h}(t) = [h_t]$ in $\widetilde{Y}$ from (7). This path covers α since $h_t(1) = \alpha(t)$.

Further, $\widetilde{h}(0) = \widetilde{y}_0$ is the constant path. Also, by definition, $\widetilde{h}(1) = [\alpha]$. From the uniqueness of lifts we derive that $\widetilde{h} = \widetilde{\alpha}$ is closed, so that $[\alpha] = [y_0]$. Therefore the homomorphism

$$\pi_1(q)\colon \pi_1(\widetilde{Y}, \widetilde{y}_0) \to \pi_1(Y, y_0)$$

vanishes. Since it is also injective (Corollary A.2.7), $\pi_1(\widetilde{Y}, \widetilde{y}_0)$ is trivial, i.e., $\widetilde{Y}$ is simply connected. $\square$

Definition A.2.13. Let $q\colon X \to Y$ be a covering. A homeomorphism $\varphi\colon X \to X$ is called a *deck transformation* of the covering if $q \circ \varphi = q$. This means that φ permutes the elements in the fibers of q. We write $\mathrm{Deck}(X, q)$ for the group of deck transformations.

Example A.2.14. For the covering map $\exp\colon \mathbb{C} \to \mathbb{C}^\times$, the deck transformations have the form

$$\varphi(z) = z + 2\pi i n, \quad n \in \mathbb{Z}.$$

Proposition A.2.15. *Suppose that Y is arcwise connected and locally arcwise connected, that $q_Y : \widetilde{Y} \to Y$ is a simply connected covering of Y, and that $\widetilde{y}_0 \in \widetilde{Y}$ satisfies $q_Y(\widetilde{y}_0) = y_0$. For each $[\gamma] \in \pi_1(Y, y_0)$ we write $\varphi_{[\gamma]} \in \mathrm{Deck}(\widetilde{Y}, q_Y)$ for the unique lift of id_Y mapping $\widetilde{y}_0$ to the endpoint $\widetilde{\gamma}(1)$ of the unique lift $\widetilde{\gamma}$ of γ starting in $\widetilde{y}_0$. Then the map*

$$\Phi\colon \pi_1(Y, y_0) \to \mathrm{Deck}(\widetilde{Y}, q), \quad \Phi([\gamma]) = \varphi_{[\gamma]}$$

is an isomorphism of groups.

Proof. For $\gamma, \eta \in \Omega(Y, y_0)$, the composition $\varphi_{[\gamma]} \circ \varphi_{[\eta]}$ is a deck transformation mapping $\widetilde{y}_0$ to the endpoint of $\varphi_{[\gamma]} \circ \widetilde{\eta}$ which coincides with the endpoint of the lift of η starting in $\widetilde{\gamma}(1)$. Hence it also is the endpoint of the lift of the loop $\gamma * \eta$. Therefore Φ is a group homomorphism.

To see that Φ is injective, we note that $\varphi_{[\gamma]} = \mathrm{id}_{\widetilde{Y}}$ implies that $\widetilde{\gamma}(1) = \widetilde{y}_0$, so that $\widetilde{\gamma}$ is a loop, hence contractible, and therefore $[\gamma] = [q_Y \circ \widetilde{\gamma}] = [y_0]$.

For the surjectivity, let φ be a deck transformation and $y := \varphi(\widetilde{y}_0)$. If α is a path from $\widetilde{y}_0$ to y, then $\gamma := q_Y \circ \alpha$ is a loop in y_0 with $\alpha = \widetilde{\gamma}$, so that $\varphi_{[\gamma]}(\widetilde{y}_0) = y$, and the uniqueness of lifts implies that $\varphi = \varphi_{[\gamma]} = \Phi([\gamma])$. $\square$

Exercises for Section A.2

Exercise A.2.1. Let $F\colon I^2 \to X$ be a continuous map with $F(0, s) = x_0$ for $s \in I$ and define

$$\gamma(t) := F(t, 0), \quad \eta(t) := F(t, 1), \quad \alpha(t) := F(1, t), \quad t \in I.$$

Show that $\gamma * \alpha \sim \eta$.

Exercise A.2.2. Let $q\colon G \to H$ be an morphism of topological groups with discrete kernel Γ. Show that:

(1) If $V \subseteq G$ is an open $\mathbf{1}$-neighborhood with $(V^{-1}V) \cap \Gamma = \{\mathbf{1}\}$ and q is open, then $q|_V : V \to q(V)$ is a homeomorphism.

(2) If q is open and surjective, then q is a covering.

(3) If q is open and H is connected, then q is surjective, hence a covering.

Exercise A.2.3. A map $f\colon X \to Y$ between topological spaces is called a *local homeomorphism* if each point $x \in X$ has an open neighborhood U such that $f|_U : U \to f(U)$ is a homeomorphism onto an open subset of Y.

(1) Show that each covering map is a local homeomorphism.

(2) Find a surjective local homeomorphism which is not a covering. Can you also find an example where X is connected?