Operator Algebras

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Introduction

Operator algebras are algebras of bounded operators on Hilbert spaces. John von Neumann introduced the concept of an abstract Hilbert space and developed the spectral theory of normal operators in [vN29], where he also invented "operator algebras". With F. J. Murray he introduced in the 1930s the concept of a "ring of operators", nowadays called a *von Neumann algebra*, and developed their structure theory [MvN36]. A key motivation for this work was to put the physical theory of Quantum Mechanics, developed in the second half of the 1920s by the physicists E. Schrödinger, W. Heisenberg and P. Dirac, on a solid axiomatic foundation. Before these inventions, Hilbert spaces were mostly treated quite formally as the concrete Hilbert space ℓ^2 of square-summable sequences and operators thereon were considered as infinite matrices, or as an L^2-space on which the operators were given by kernels ([HvN28], [vN27]). Since then, the structure and representation theory of operator algebras has been developed and many unexpected applications and connections to other fields have been unveiled over the years. Some of these connected fields are the theory of unitary group representations, non-commutative geometry, topology (invariants of knots such as the Jones polynomial) and in particular Quantum Field Theory (QFT).

The *state space* of a quantum mechanical system is modeled by the set

$$\mathbb{P}(\mathcal{H}) := \{[v] = \mathbb{C}v \colon 0 \neq v \in \mathcal{H}\}$$

of one-dimensional subspaces of a complex Hilbert space $\mathcal{H}$, its *projective space*. *Observables O_A* are represented by bounded hermitian operators $A = A^*$ on $\mathcal{H}$ and evaluating an observable O_A in a state $[v]$ yields the real number

$$O_A([v]) := \frac{\langle v, Av\rangle}{\langle v, v\rangle} \in [-\|A\|.\|A\|].^1$$

More precisely, the values $O_A([v])$ lie in the closed convex hull of the spectrum $\sigma(A)$, which is a closed subset of the real interval $[-\|A\|, \|A\|]$. Important classes of observables O_P are those corresponding to "yes-no questions", a property that is linked to the spectral condition $\sigma(P) \subseteq \{0, 1\}$, and this means that P is an orthogonal projection onto a closed subspace $P\mathcal{H}$ of $\mathcal{H}$. In particular, the "vector states" $[v]$ correspond to rank-1-projections

$$P_v(w) = \frac{\langle v, w\rangle}{\langle v, v\rangle} v \quad \text{and} \quad O_A([v]) = \operatorname{tr}(AP_v),$$

where tr is the trace of the rank-1-operator AP_v.[2] The expression $\operatorname{tr}(AS)$ makes sense for a large class of operators S and any $A \in B(\mathcal{H})$, namely for all *trace class operators* [3] (because the product AS is trace class). Typical trace class operators are the so-called *"density matrices"*

$$S = \sum_{n=1}^{\infty} \lambda_n P_{v_n} \quad \text{with} \quad \|v_n\| = 1, \lambda_n \geq 0, \sum_n \lambda_n = 1.$$

[1] We refer to [Ha11] for a collection of interesting articles discussing mathematical structures related to Quantum Mechanics and Quantum Field Theory. Other excellent sources are [Em72] and [BR02, BR96].

[2] On any vector space V, the trace of a finite rank operator $A \colon V \to V$ is defined by $\operatorname{tr}(A) = \sum_{j\in J} e_j^*(Ae_j)$, where $(e_j)_{j\in J}$ is a linear basis of V.

[3] These are the compact operators S for which $A \mapsto \operatorname{tr}(AS)$ defines a continuous linear functional on the space $K(\mathcal{H})$ of compact operators. Then, for any ONB $(e_j)_{j\in J}$, the trace is defined by $\operatorname{tr}(S) = \sum_{j\in J}\langle e_j, Se_j\rangle$.

These operators S are hermitian and positive (non-negative spectrum) with $\operatorname{tr} S = 1$. They are called *mixed states* (or density matrices). They form a convex set whose extreme points are precisely the rank-1-projections P_v. In this sense they are "superpositions" of pure (vector) states.

For any such operator S, the linear functional

$$\omega_S \colon B(\mathcal{H}) \to \mathbb{C}, \quad \omega_S(A) := \operatorname{tr}(AS)$$

satisfies

$$\omega_S(\mathbf{1}) = 1 \quad \text{and} \quad \omega_S(A^*A) \geq 0. \tag{1}$$

To a large extent the theory of operator algebras is concerned with unital $*$-invariant subalgebras $\mathcal{A} \subseteq B(\mathcal{H})$ and their *states*, i.e., functionals satisfying (1). [4]

Typical examples that also play a central role in Analysis arise for the algebra $\mathcal{A} = C(X, \mathbb{C})$ of complex-valued continuous functions on the compact space X and a probability measure μ on X. Then

$$\mu(f) := \int_X f(x)\, d\mu(x)$$

is a state of $\mathcal{A}$. From this perspective, states on operator algebras are non-commutative analogs of probability measures. A bridge between the two aspects of positive measures as functionals (integrals) and measures on a σ-algebra is established by the Riesz Representation Theorem. As we shall see, restricting states to abelian subalgebras thus provides a natural link between operator algebras and measure theory. One can actually use these ideas to develop a "functional calculus" which allows us to define $f(A)$ for any bounded measurable function and any normal operator A on $\mathcal{H}$. This is also very natural from the perspective of the interpretation of hermitian operators as observables.

We shall see that norm closed $*$-invariant subalgebras $\mathcal{A} \subseteq B(\mathcal{H})$ can be characterized axiomatically as C^*-algebras, whose norm satisfies the relations

$$\|a^*a\| = \|a\|^2 \quad \text{and} \quad \|a^*\| = \|a\| \quad \text{for every} \quad a \in \mathcal{A}$$

(Gelfand–Naimark Theorem 3.12). As commutative unital C^*-algebras are of the form $C(X, \mathbb{C})$ for a uniquely determined compact space X (Gelfand Representation Theorem 2.2), one may consider the theory of commutative C^*-algebras as another perspective on topology. Accordingly, the theory of operator algebras is sometimes called non-commutative topology, or, if one takes some additional structures, so-called spectral triples, into account, one speaks of non-commutative geometry ([Co94]).

Notation and Conventions

- $\mathbb{N} := \{1, 2, 3, \ldots\}$
- $\mathbb{R}_+ := \{x \in \mathbb{R} \colon x \geq 0\} = [0, \infty)$.
- $\mathbb{R}^\times := \mathbb{R} \setminus \{0\}$, $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$, $\mathbb{T} := \{z \in \mathbb{C} \colon |z| = 1\}$.

[4] The main reason for studying $*$-invariant algebras $\mathcal{A}$ is that they leave a closed subspace $\mathcal{K} \subseteq \mathcal{H}$ invariant if and only if they leave its orthogonal complement $\mathcal{K}^\perp$ invariant, and this in turn means that $\mathcal{A}$ commutes with the corresponding orthogonal projection $P_\mathcal{K}$.

For two sets J and Y we write Y^J for the set of maps $f\colon J \to Y$. If J is a set and S an abelian semigroup with zero element 0, then we also write $S^{(J)} \subseteq S^J$ for the subset of finitely supported functions.

For a metric space (X, d), we write

$$B_r(x) := \{y \in X\colon d(x, y) < r\}$$

for the open ball of radius r around x.

If $\mathcal{H}$ is a complex Hilbert space, then its scalar product is written $\langle \cdot, \cdot \rangle$ or $\langle \cdot | \cdot \rangle$. It is assumed to be antilinear in the first and linear in the second argument

$$\lambda\langle v, w\rangle = \langle \overline{\lambda} v, w\rangle = \langle v, \lambda w\rangle,$$

and $\|v\| := \sqrt{\langle v, v\rangle}$ is the corresponding norm. The linearity in the second argument is customary in physics, where (following Dirac) one writes elements of a Hilbert space $\mathcal{H}$ as $|w\rangle$ (so-called kets) and elements of the dual space (continuous linear functionals) as $\langle v|$ (so-called bras). Then evaluation of the linear functional $\langle v|$ in the vector $|w\rangle$ yields the "bra(c)ket" $\langle v|w\rangle$.

For a subset S of a Banach space X, we write

$$[\![S]\!] := \overline{\operatorname{span} S}$$

for the closed linear subspace generated by S.

For Banach spaces X and Y we write

$$B(X, Y) := \{A\colon X \to Y\colon A \text{ linear}, \ \|A\| < \infty\}$$

for the Banach space of bounded linear operators from X to Y. For $X = Y$ we abbreviate $B(X) := B(X, X)$ and write $\mathrm{GL}(X)$ for the group of invertible elements in $B(X)$. If $\mathcal{H}$ is a complex Hilbert space, then we have an antilinear isometric map $B(\mathcal{H}) \to B(\mathcal{H}), A \mapsto A^*$, determined uniquely by

$$\langle Av, w\rangle = \langle v, A^* w\rangle \quad \text{for } v, w \in \mathcal{H}.$$

1 Banach algebras

In this first section we introduce Banach algebras, Banach-$*$-algebras and in particular C^*-algebras.

1.1 Basics definitions

Definition 1.1. A *Banach algebra* is a triple $(\mathcal{A}, m_{\mathcal{A}}, \|\cdot\|)$ of a Banach space $(\mathcal{A}, \|\cdot\|)$, together with an associative bilinear multiplication

$$m_{\mathcal{A}}\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}, \quad (a, b) \mapsto ab$$

for which the norm $\|\cdot\|$ is *submultiplicative*, i.e.,

$$\|ab\| \leq \|a\| \cdot \|b\| \quad \text{for} \quad a, b \in \mathcal{A}.$$

By abuse of notation, we call $\mathcal{A}$ a Banach algebra, if the norm and the multiplication are clear from the context.

A *unital Banach algebra* is a pair $(\mathcal{A}, \mathbf{1})$ of a Banach algebra $\mathcal{A}$ and an element $\mathbf{1} \in \mathcal{A}$ satisfying $\mathbf{1}a = a\mathbf{1} = a$ for each $a \in \mathcal{A}$ and $\|\mathbf{1}\| = 1$. The subset

$$\mathcal{A}^\times := \{a \in \mathcal{A} \colon (\exists b \in \mathcal{A})\ ab = ba = \mathbf{1}\}$$

is called the *unit group of* $\mathcal{A}$ (cf. Exercise 1.4).

Example 1.2. (a) If $(X, \|\cdot\|)$ is a Banach space, then the space $B(X)$ of continuous linear operators $A \colon X \to X$ is a unital Banach algebra with respect to the *operator norm*

$$\|A\| := \sup\{\|Ax\| \colon x \in X, \|x\| \leq 1\}$$

and composition of maps. Note that the submultiplicativity of the operator norm, i.e.,

$$\|AB\| \leq \|A\| \cdot \|B\|,$$

is an immediate consequence of the estimate

$$\|ABx\| \leq \|A\| \cdot \|Bx\| \leq \|A\| \cdot \|B\| \cdot \|x\| \quad \text{for} \quad x \in X.$$

In this case the unit group is also denoted $\mathrm{GL}(X) := B(X)^\times$.

(b) If X is a compact space and $\mathcal{A}$ a Banach algebra, then the space $C(X, \mathcal{A})$ of $\mathcal{A}$-valued continuous functions on X is a Banach algebra with respect to pointwise multiplication $(fg)(x) := f(x)g(x)$ and the norm

$$\|f\| := \sup_{x \in X} \|f(x)\|$$

(Exercise 1.3).

Example 1.3. For any norm $\|\cdot\|$ on $\mathbb{C}^n$, the canonical basis $e_1, \ldots, e_n$ of $\mathbb{C}^n$ yields an isomorphism of algebras $M_n(\mathbb{C}) \cong B(\mathbb{C}^n)$, so that $\mathrm{GL}_n(\mathbb{C}) \cong \mathrm{GL}(\mathbb{C}^n)$.

Remark 1.4. In a Banach algebra $\mathcal{A}$, the multiplication is continuous because $a_n \to a$ and $b_n \to b$ implies $\|b_n\| \to \|b\|$ and therefore

$$\|a_n b_n - ab\| = \|a_n b_n - ab_n + ab_n - ab\| \leq \|a_n - a\| \cdot \|b_n\| + \|a\| \cdot \|b_n - b\| \to 0.$$

In particular, left and right multiplications

$$\lambda_a \colon \mathcal{A} \to \mathcal{A}, x \mapsto ax, \quad \text{and} \quad \rho_a \colon \mathcal{A} \to \mathcal{A}, x \mapsto xa,$$

are continuous with

$$\|\lambda_a\| \leq \|a\| \quad \text{and} \quad \|\rho_a\| \leq \|a\|. \tag{2}$$

Proposition 1.5. *The unit group $\mathcal{A}^\times$ of a unital Banach algebra is an open subset.*

Proof. The proof is based on the convergence of the Neumann series $\sum_{n=0}^\infty x^n$ for $\|x\| < 1$. For any such x we have

$$(\mathbf{1} - x) \sum_{n=0}^\infty x^n = \Big(\sum_{n=0}^\infty x^n\Big)(\mathbf{1} - x) = \mathbf{1},$$

so that $\mathbf{1} - x \in \mathcal{A}^\times$. We conclude that the open unit ball $B_1(\mathbf{1})$ is contained in $\mathcal{A}^\times$.

Next we note that left multiplications $\lambda_g \colon \mathcal{A} \to \mathcal{A}$ with elements $g \in \mathcal{A}^\times$ are continuous (Remark 1.4), hence homeomorphisms because $\lambda_g^{-1} = \lambda_{g^{-1}}$ is also continuous. Therefore $gB_1(\mathbf{1}) = \lambda_g B_1(\mathbf{1}) \subseteq \mathcal{A}^\times$ is an open subset, showing that g is an interior point of $\mathcal{A}^\times$. Hence $\mathcal{A}^\times$ is open. $\square$

Definition 1.6. (a) An *involutive algebra*, or $*$-*algebra* $\mathcal{A}$ is a pair $(\mathcal{A}, *)$ of a complex algebra $\mathcal{A}$ and a map $\mathcal{A} \to \mathcal{A}, a \mapsto a^*$, satisfying

(I1) $(a^*)^* = a$ (Involutivity)

(I2) $(\lambda a + \mu b)^* = \overline{\lambda} a^* + \overline{\mu} b^*$ (Anti-linearity).

(I3) $(ab)^* = b^* a^*$ ($*$ is an antiautomorphism of $\mathcal{A}$).

Then $*$ is called an *involution* on $\mathcal{A}$.

(b) A *Banach-$*$-algebra* is an involutive algebra $(\mathcal{A}, *)$, where $\mathcal{A}$ is a Banach algebra and $\|a^*\| = \|a\|$ holds for each $a \in \mathcal{A}$. If, in addition,

$$\|a^* a\| = \|a\|^2 \quad \text{for all} \quad a \in \mathcal{A},$$

then $(\mathcal{A}, *)$ is called a *C^*-algebra*.

Example 1.7. (a) The algebra $B(\mathcal{H})$ of bounded operators on a complex Hilbert space $\mathcal{H}$ is a C^*-algebra. Here the main point is that, for each $A \in B(\mathcal{H})$, we have

$$\|A\| = \sup\{|\langle v, Aw\rangle| \colon \|v\|, \|w\| \leq 1\},$$

which immediately implies that $\|A^*\| = \|A\|$. It also implies that

$$\|A^* A\| = \sup\{|\langle Av, Aw\rangle| \colon \|v\|, \|w\| \leq 1\} \geq \sup\{\|Av\|^2 \colon \|v\| \leq 1\} = \|A\|^2,$$

and since $\|A^* A\| \leq \|A^*\| \cdot \|A\| = \|A\|^2$ by Example 1.2, we see that $B(\mathcal{H})$ is a C^*-algebra.

(b) From (a) it immediately follows that every closed $*$-invariant subalgebra of $\mathcal{A} \subseteq B(\mathcal{H})$ also is a C^*-algebra. [5]

(c) If X is a compact space, then the Banach space $C(X) := C(X, \mathbb{C})$, endowed with

$$\|f\| := \sup_{x \in X} |f(x)|$$

is a C^*-algebra with respect to $f^*(x) := \overline{f(x)}$. In this case $\|f^* f\| = \||f|^2\| = \|f\|^2$ is trivial.

For the finite set $X = \mathbf{n} := \{1, \ldots, n\}$, we obtain in particular the C^*-algebra

$$C(\mathbf{n}, \mathbb{C}) = \mathbb{C}^{\mathbf{n}} \cong \mathbb{C}^n$$

with

$$(z_1, \ldots, z_n)(w_1, \ldots, w_n) = (z_1 w_1, \ldots, z_n w_n), \qquad (z_1, \ldots, z_n)^* = (\overline{z_1}, \ldots, \overline{z_n})$$

and

$$\|(z_1, \ldots, z_n)\| = \max\{|z_j| \colon j = 1, \ldots, n\}.$$

(d) If X is a locally compact space, then we say that a continuous function $f \colon X \to \mathbb{C}$ *vanishes at infinity* if, for each $\varepsilon > 0$, there exists a compact subset $K \subseteq X$ with $|f(x)| \leq \varepsilon$ for $x \notin K$. We write $C_0(X) := C_0(X, \mathbb{C})$ for the set of all continuous functions vanishing at infinity and endow it with the norm

$$\|f\| := \sup_{x \in X} |f(x)|.$$

[5] We shall show later that every C^*-algebra can be realized as a closed $*$-subalgebra of some $B(\mathcal{H})$ (Gelfand–Naimark Theorem 3.12).

(cf. Exercise 1.5). Then $C_0(X,\mathbb{C})$ is a C^*-algebra with respect the involution $f^*(x) := \overline{f(x)}$.[6]

(e) Let J be a set and $\ell^\infty(J,\mathbb{C})$ be the Banach space of all bounded complex-valued functions $x\colon J \to \mathbb{C}, j \mapsto x_j$, endowed with the norm

$$\|x\|_\infty := \sup\{|x_j|\colon j \in J\}.$$

Then $\ell^\infty(J,\mathbb{C})$ is a C^*-algebra with respect to pointwise multiplication and the involution defined by $(x^*)_j := \overline{x_j}$.

The following class of examples of Banach-$*$-algebras is rarely C^*, but it provides important constructions of C^*-algebras. In this example we use the Banach space

$$\ell^1(S) := \ell^1(S,\mathbb{C}) := \Big\{a\colon S \to \mathbb{C}\colon \sum_{s\in S} |a(s)| < \infty\Big\}$$

which carries the norm

$$\|a\|_1 := \sum_{s\in S} |a(s)| := \sup\Big\{\sum_{s\in F} |a(s)|\colon F \subseteq S \text{ finite}\Big\}.$$

In measure theoretic terms this is the L^1-space with respect to the counting measure on the measurable space $(S, 2^S)$ defined by

$$\mu(E) = |E| \quad \text{for} \quad E \subseteq S.$$

Note that, for every $a \in \ell^1(S)$, the set

$$\{s \in S\colon a(s) \neq 0\} = \bigcup_{N\in\mathbb{N}} \Big\{s \in S\colon |a(s)| \geq \frac{1}{N}\Big\}$$

is a countable union of finite sets, hence countable or finite. If it is infinite, then any enumeration $(s_n)_{n\in\mathbb{N}}$ of this set leads to the same absolutely convergent series

$$\sum_{s\in S} a(s) := \sum_{n\in\mathbb{N}} a(s_n) \tag{3}$$

(the integral with respect to the counting measure). In this sense we shall always interpret sums over arbitrary index sets.

Example 1.8. (a) Let $(S,*)$ be an *involutive semigroup*, i.e., a semigroup, endowed with an involution $s \mapsto s^*$ satisfying $(st)^* = t^*s^*$ for $s,t \in S$. We consider the complex Banach space $\ell^1(S)$ which carries a natural Banach $*$-algebra structure given by the *convolution product*

$$(a * b)(s) := \sum_{s_1s_2=s} a(s_1)b(s_2) \quad \text{and} \quad a^*(s) := \overline{a(s^*)} \quad \text{for} \quad s \in S, a, b \in \ell^1(S).$$

Here the sum defining $a * b$ has to be understood as a sum over all pairs $(s_1,s_2) \in S \times S$ satisfying $s_1s_2 = s$. It is absolutely convergent in the set of (3) because

$$\sum_{s_1s_2=s} |a(s_1)b(s_2)| \leq \sum_{(s_1,s_2)\in S\times S} |a(s_1)b(s_2)| \leq \sum_{s_1\in S} |a(s_1)| \cdot \sum_{s_2\in S} |b(s_2)| = \|a\|_1\|b\|_1$$

[6]We shall see later that every commutative C^*-algebra is isomorphic to some $C_0(X)$ (Gelfand Representation Theorem 2.2).

holds for every $s \in S$. We even obtain

$$\|a * b\|_1 = \sum_{s \in S} \sum_{s_1 s_2 = s} |a(s_1)b(s_2)| \leq \sum_{(s_1,s_2) \in S \times S} |a(s_1)| \cdot |b(s_2)| = \|a\|_1 \|b\|_1.$$

The nature of these operations is best understood in terms of the "basis elements" $(\delta_s)_{s \in S}$, given by $\delta_s(t) = \delta_{s,t}$ (Kronecker delta). Then

$$\delta_s * \delta_t = \delta_{st} \quad \text{and} \quad \delta_s^* = \delta_{s^*},$$

so that $\delta\colon S \to \ell^1(S), s \mapsto \delta_s$ is a homomorphism of involutive semigroups. Further, each $a \in \ell^1(S)$ can be expressed as the norm convergent sum of the basis elements

$$a = \sum_{s \in S} a(s)\delta_s, \quad a^* = \sum_{s \in S} \overline{a(s)}\delta_{s^*} \quad \text{and} \quad a * b = \sum_{s,t \in S} a(s)b(t)\delta_{st}.$$

(b) An important special case arises if G is a group and $g^* := g^{-1}$. Then $\ell^1(G)$ is called the *ℓ^1-group algebra of the (discrete) group G*. It contains a "copy" of the group through the basis elements $(\delta_g)_{g \in G}$, satisfying

$$\delta_g \delta_h = \delta_{gh} \quad \text{and} \quad \delta_g^{-1} = \delta_g^* = \delta_{g^{-1}} \quad \text{for} \quad g, h \in G.$$

1.2 Some spectral theory

We now take a closer look at C^*-algebras. We shall need some results from spectral theory, in particular the concept of the spectrum of an element of a Banach algebra and some of its properties. We shall continue our investigations in Section 2 with a complete description of commutative C^*-algebras.

We recall some basic facts on spectra of operators. The most natural context for spectral theory is a unital complex Banach algebra $\mathcal{A}$ but the definition of the spectrum and the resolvent make sense in any unital algebra. For $a \in \mathcal{A}$, its *spectrum* is defined as

$$\sigma(a) := \{\lambda \in \mathbb{C}\colon \lambda\mathbf{1} - a \notin \mathcal{A}^\times\}$$

and

$$\rho(a) := \{\lambda \in \mathbb{C}\colon \lambda\mathbf{1} - a \in \mathcal{A}^\times\} = \mathbb{C} \setminus \sigma(a)$$

is called the *resolvent set of* a. Since the unit group $\mathcal{A}^\times$ is open (Proposition 1.5), $\sigma(a)$ is a closed subset of $\mathbb{C}$. Further, $\lambda > \|a\|$ implies that $\lambda\mathbf{1} - a = \lambda(\mathbf{1} - a/\lambda)$ is invertible because $\|\lambda^{-1}a\| < 1$. Therefore $\sigma(a)$ is also bounded, hence compact. The first non-trivial fact on the spectrum is that it is always non-empty.[7]

If the algebra $\mathcal{A}$ is **not** unital, then we enlarge it to the Banach algebra $\mathcal{A}_+ := \mathcal{A} \times \mathbb{C}$ with

$$(a,t)(a',t') := (aa' + ta' + t'a, tt') \quad \text{and} \quad \|(a,t)\| := \|a\| + |t|,$$

(Exercise 1.4) and define, for $a \in \mathcal{A}$ the spectrum as

$$\sigma(a) := \sigma(a, 0) = \{\lambda \in \mathbb{C}\colon \lambda\mathbf{1} - (a,0) = (-a, \lambda) \in \mathcal{A}_+^\times\}.$$

[7]The idea of the proof is quite simple. If $\sigma(a) = \emptyset$, then the resolvent $R(\lambda) := (\lambda\mathbf{1} - a)^{-1}$ defines an entire function $R\colon \mathbb{C} \to \mathcal{A}$. Easy estimates based on the Neumann series show that R is bounded, hence constant by Liouville's Theorem, and this leads to a contradiction.

Then $0 \in \sigma(a)$ because elements of the form $(a, 0)$ are contained in the ideal $\mathcal{A} \times \{0\} \trianglelefteq \mathcal{A}_+$, so that they are never invertible.

We shall also need that the *spectral radius*

$$r(a) := \max\{z \in \mathbb{C}\colon z \in \sigma(a)\}$$

and can be calculated by *Gelfand's Formula:*

$$r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = \inf\{\|a^n\|^{\frac{1}{n}} : n \in \mathbb{N}\}. \tag{4}$$

Example 1.9. (a) Let X be a compact space and $\mathcal{A} = C(X)$ the commutative C^*-algebra of continuous functions $f\colon X \to \mathbb{C}$. Its unit group

$$\mathcal{A}^\times = \{f \in \mathcal{A}\colon f(X) \subseteq \mathbb{C}^\times\}$$

consists of functions with no zeros because, for any such function, the pointwise inverse is continuous (Exercise 1.3). This implies that $\lambda\mathbf{1} - f$ is invertible if and only if $\lambda \notin f(X)$, and thus the spectrum

$$\sigma(f) = f(X)$$

is the set of all values of a function.

(b) For $\mathcal{A} = M_n(\mathbb{C}) \cong B(\mathbb{C}^n)$ and $A \in \mathcal{A}$, we have

$$\sigma(A) = \{\lambda \in \mathbb{C}\colon \det(\lambda\mathbf{1} - A) = 0\},$$

which is the set of eigenvalues of A.

Proposition 1.10. (Gelfand–Mazur)[8] *If $\mathcal{A}$ is a unital complex Banach algebra in which every non-zero element is invertible, then $\mathcal{A} \cong \mathbb{C}$.*

Proof. Let $a \in \mathcal{A}$. As $\sigma(a) \neq \emptyset$, there exist a $\lambda \in \mathbb{C}$ for which $\lambda\mathbf{1} - a$ is not invertible, and this implies $a = \lambda\mathbf{1}$. □

Lemma 1.11. *Let $\varphi\colon \mathcal{A} \to \mathcal{B}$ a homomorphism of unital complex algebras. Then*

$$\sigma\big(\varphi(a)\big) \subseteq \sigma(a) \quad \textit{for every} \quad a \in \mathcal{A}.$$

Proof. We show that $\rho(a) \subseteq \rho\big(\varphi(a)\big)$. Let $\lambda \in \mathbb{C}$ with $\lambda\mathbf{1} - a \in \mathcal{A}^\times$ and $b = (\lambda\mathbf{1} - a)^{-1}$. Then

$$\varphi(\lambda\mathbf{1} - a)\varphi(b) = \varphi((\lambda\mathbf{1} - a)b) = \varphi(\mathbf{1}) = \mathbf{1}$$

and likewise $\varphi(b)\varphi(\lambda\mathbf{1} - a) = \mathbf{1}$. Therefore $\varphi(\lambda\mathbf{1} - a)$ is invertible in $\mathcal{B}$, i.e., $\lambda \in \rho\big(\varphi(a)\big)$. □

Definition 1.12. Let $\mathcal{A}$ be a $*$-algebra and $a \in \mathcal{A}$. We say that a is

- *hermitian* if $a = a^*$. We write $\mathcal{A}_h \subseteq \mathcal{A}$ of the real linear subspace of hermitian elements.
- *skew-hermitian* if $a = -a^*$.
- *normal* if $aa^* = a^*a$.

[8] This result was announced without proof in 1938 by Mazur in a short note published in the C. R. Acad. Sci. Paris. The first proof was given in 1941 in a paper by I. M. Gelfand in which he laid the foundations for the theory of Banach algebras.

- *unitary* if $aa^* = a^*a = \mathbf{1}$ (if $\mathcal{A}$ is unital).

Lemma 1.13. *Let $\mathcal{A}$ be a Banach algebra.*

(i) *Hermitian elements are normal.*

(ii) *For every $a \in \mathcal{A}$, the element aa^* is hermitian.*

(iii) *If a and b are hermitian, then ab is hermitian if and only if $ab = ba$.*

(iv) *$\mathcal{A} = \mathcal{A}_h \oplus i\mathcal{A}_h$, every element $a \in \mathcal{A}$ has a unique decomposition $a = b + ic$ with b, c hermitian.*

(v) *If b, c are hermitian, then $b + ic$ is normal if and only if $bc = cb$.*

(vi) *If $\mathbf{1} \in \mathcal{A}$ is a left identity, i.e., $\mathbf{1}a = a$ for every $a \in \mathcal{A}$, then it is an identity and*

$$(a^{-1})^* = (a^*)^{-1} \quad \text{for} \quad a \in \mathcal{A}^\times.$$

Proof. (i)-(iv) are very elementary.

(v) follows from

$$(b + ic)(b - ic) = b^2 + c^2 + i(cb - bc) \quad \text{and} \quad (b - ic)(b + ic) = b^2 + c^2 + i(bc - cb).$$

(vi) If $\mathbf{1}$ is a left identity, then $a\mathbf{1}^* = (\mathbf{1}a^*)^* = (a^*)^* = a$, implies that $\mathbf{1}^*$ is a right identity. Hence $\mathbf{1}^* = \mathbf{1}\mathbf{1}^* = \mathbf{1}$ implies that $\mathbf{1}$ is an identity.

For $a \in \mathcal{A}^\times$, the relation $aa^{-1} = a^{-1}a = \mathbf{1}$ implies $(a^{-1})^*a^* = a^*(a^{-1})^* = \mathbf{1}$, and the assertion follows. □

Proposition 1.14. (Unitization of C^*-algebras) *For a C^*-algebra $\mathcal{A}$, the following assertions hold:*

(i) *For the left multiplication $L_a \colon x \mapsto ax$, we have $\|L_a\| = \|a\|$. In particular, $\|\mathbf{1}\| = 1$ whenever $\mathcal{A}$ has an identity.*

(ii) *If $\mathcal{A}$ has no identity, then the Banach algebra $\mathcal{A}_+ = \mathcal{A} \times \mathbb{C}$ is a C^*-algebra with respect to $(a, \lambda)^* = (a^*, \overline{\lambda})$ and the norm $\|(a, \lambda)\| := \|L_{(a,\lambda)}\|$, where $L_{(a,\lambda)}x = ax + \lambda x$ is the corresponding multiplication operator on $\mathcal{A}$.*

Proof. (i) The submultiplicativity of the norm implies $\|L_a\| \leq \|a\|$. Further, the relation $\|aa^*\| = \|a\|^2 = \|a\| \cdot \|a^*\|$ implies that $\|L_a\| \geq \|a\|$. We thus obtain equality.

If $\mathbf{1} \in \mathcal{A}$ is an identity, then $L_{\mathbf{1}} = \mathrm{id}_{\mathcal{A}}$ implies $\|L_{\mathbf{1}}\| = 1$.

(ii) We consider the algebra homomorphism

$$L \colon \mathcal{A}_+ \to B(\mathcal{A}), \quad (a, \lambda) \mapsto L_{(a,\lambda)} = L_a + \lambda\,\mathrm{id}_{\mathcal{A}}.$$

First we show that L is injective. Suppose that $L_{(a,\lambda)} = 0$. If $\lambda = 0$, then $0 = \|L_{(a,0)}\| = \|L_a\| = \|a\|$ implies $a = 0$. If $\lambda \neq 0$, after replacing a by $-\lambda^{-1}a$, we may w.l.o.g. assume that $\lambda = -1$. Then $ax - x = 0$ for every $x \in \mathcal{A}$, i.e., a is a left identity, hence also an identity by Lemma 1.13(vi); this contradicts our assumption. Therefore L is injective and

$$\|(a, \lambda)\| := \|L_{(a,\lambda)}\| = \sup\{\|(ab + \lambda b)\| \colon b \in \mathcal{A}, \|b\| \leq 1\}$$

defines a norm on $\mathcal{A}_+$. By (i), this norm extends the given norm on $\mathcal{A}$. It is also clear that this norm is submultiplicative (it is an operator norm).

$\mathcal{A}_+$ is complete: Since the inclusion $\mathcal{A} \hookrightarrow \mathcal{A} \times \{0\}, a \mapsto (a, 0)$ is isometric, its image is a complete, hence closed, hyperplane in $\mathcal{A}_+$. Hence the linear functional

$$\chi\colon \mathcal{A}_+ \to \mathbb{C}, \quad \chi(a, \lambda) := \lambda$$

has a closed kernel and therefore is continuous (Exercise 1.1). If $(a_n, \lambda_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{A}_+$, then the continuity of χ implies that $(\lambda_n)_{n\in\mathbb{C}}$ is a Cauchy sequence in $\mathbb{C}$, hence convergent, and therefore $(a_n, 0) = (a_n, \lambda_n) - (0, \lambda_n)$ is a Cauchy sequence in $\mathcal{A} \times \{0\} \cong \mathcal{A}$ by (i). Therefore it converges to a limit $a \in \mathcal{A}$ and now $(a_n, \lambda_n) \to (a, \lambda)$. This shows that $\mathcal{A}_+$ is complete, hence a Banach algebra.

Next we show that

$$\|x\|^2 \leq \|x^*x\| \quad \text{for} \quad x \in \mathcal{A}_+. \tag{5}$$

To verify (5), let $x \in \mathcal{A}_+$. For every $r < \|x\| = \|L_x\|$, there exists a $y \in \mathcal{A}$ with $\|y\| \leq 1$ and $\|xy\| \geq r$. As $xy \in \mathcal{A}$, we obtain

$$\|x^*x\| \geq \|y^*(x^*x)y\| = \|(xy)^*xy\| = \|xy\|^2 \geq r^2,$$

and therefore $\|x^*x\| \geq \|x\|^2$. From (5) we derive

$$\|x\|^2 \leq \|x^*x\| \leq \|x^*\| \cdot \|x\|.$$

For $x \neq 0$ this implies $\|x\| \leq \|x^*\|$, and, replacing x by x^*, also $\|x^*\| \leq \|x\|$, so that $\|x\| = \|x^*\|$. Now we also get the C^*-property from $\|x\|^2 \leq \|x^*x\| \leq \|x^*\|\|x\| = \|x\|^2$. This completes the proof that $\mathcal{A}_+$ is a C^*-algebra. □

Example 1.15. (One point compactification) Let X be a non-compact locally compact space and $\mathcal{A} = C_0(X)$ be the non-unital C^*-algebra of continuous functions vanishing at infinity. For this algebra, the unitization $\mathcal{A}_+$ has a natural concrete realization as the algebra $C(X_\omega)$, where $X_\omega = X \dot\cup \{\omega\}$ is the one-point compactification (or Alexandrov compactification) of X (Exercise A.1).

In fact, we have an open embedding $\eta\colon X \to X_\omega$ whose image is the complement of the point ω. By Exercise 2.2, the map

$$\eta^*\colon C_*(X_\omega) := \{f \in C(X_\omega)\colon f(\omega) = 0\} \to C_0(X), \quad f \mapsto f \circ \eta = f|_X$$

is an isomorphism of C^*-algebras, so that we may identify $\mathcal{A}$ with $C_*(X_\omega)$. Accordingly, the natural map

$$\Phi\colon \mathcal{A}_+ \to C(X_\omega), \quad (f, \lambda) \mapsto f + \lambda\mathbf{1}$$

is a bijective $*$-homomorphism. That it is isometric follows from the relation

$$\|f\| = \sup\{\|fh\|\colon h \in C_*(X_\omega)\colon \|h\| \leq 1\},$$

which can be derived from Urysohn's Theorem A.16.

Example 1.16. If $\mathcal{H}$ is an infinite dimensional Hilbert space, then the algebra $\mathcal{A} := K(\mathcal{H})$ of compact operators on $\mathcal{H}$ is a non-unital C^*-algebra.

We claim that, for every $M \in B(\mathcal{H})$, we have

$$\|M\| = \sup\{\|MA\|\colon A \in K(\mathcal{H}), \|A\| \leq 1\}. \tag{6}$$

In fact, for $v \in \mathcal{H}$ with $\|v\| = 1$, the orthogonal projection $P_v(w) := \langle v, w \rangle v$ onto $\mathbb{C}v$ is a rank-one operator (hence compact), and $\|P_v\| = 1$. Further, $AP_v(w) = \langle v, w \rangle Av$ satisfies $AP_v(v) = Av$, so that $\|Av\| = \|AP_v\|$, and thus $\|A\| = \sup_{\|v\|=1} \|AP_v\|$.

Comparing with the definition of the norm on $\mathcal{A}_+$ in Proposition 1.14, it follows that

$$K(\mathcal{H})_+ \cong K(\mathcal{H}) + \mathbb{C}\mathbf{1} \subseteq B(\mathcal{H})$$

is realized as a C^*-subalgebra of $B(\mathcal{H})$.

We now observe some specific spectral properties of C^*-algebras:

Lemma 1.17. *Let $\mathcal{A}$ be a C^*-algebra and $a \in \mathcal{A}$.*

(i) $\sigma(a^*) = \overline{\sigma(a)} = \{\overline{z} \colon z \in \sigma(a)\}$.

(ii) *If a is normal, then $r(a) = \|a\|$.*

(iii) $\|a\| = \sqrt{r(a^*a)}$.

(iv) *If a is hermitian, then $\sigma(a) \subseteq \mathbb{R}$.*

Proof. In view of Proposition 1.14, we may assume that $\mathcal{A}$ is unital.

(i) According to Lemma 1.13(vi), the element $a - \lambda\mathbf{1}$ is invertible if and only if the element $(a - \lambda\mathbf{1})^* = a^* - \overline{\lambda}\mathbf{1}$ is.

(ii) Since a is normal, we have

$$\|a^2\|^2 = \|a^2(a^2)^*\| = \|aaa^*a^*\| = \|aa^*aa^*\| = \|(aa^*)(aa^*)^*\| = \|aa^*\|^2 = \|a\|^4.$$

Therefore $\|a^2\| = \|a\|^2$. Since all powers of a are normal as well, we inductively obtain

$$\|a^{2^n}\| = \|a\|^{2^n} \quad \text{for} \quad n \in \mathbb{N},$$

so that Gelfand's formula (4) yields

$$r(a) = \lim_{n\to\infty} \|a^{2^n}\|^{\frac{1}{2^n}} = \|a\|.$$

(iii) follows from $\|a\| = \sqrt{\|a^*a\|} = \sqrt{r(a^*a)}$, where the last equality holds by (ii).

(iv) Let $\alpha + i\beta \in \sigma(a)$. We claim that $\beta = 0$. For every $\lambda \in \mathbb{R}$, we have

$$\alpha + i(\beta + \lambda) \in \sigma(a) + i\lambda = \sigma(a + i\lambda\mathbf{1})$$

(Exercise 1.11), so that (ii) yields

$$|\alpha + i(\beta + \lambda)| \leq \|a + i\lambda\mathbf{1}\| \quad \text{for all} \quad \lambda \in \mathbb{R}.$$

Now we calculate

$$\begin{aligned} \alpha^2 + (\beta+\lambda)^2 &= |\alpha + i(\beta+\lambda)|^2 \leq \|a + i\lambda\mathbf{1}\|^2 \\ &= \|(a + i\lambda\mathbf{1})^*(a + i\lambda\mathbf{1})\| = \|a^2 + \lambda^2\mathbf{1}\| \leq \|a\|^2 + \lambda^2. \end{aligned}$$

This implies $\alpha^2 + \beta^2 + 2\beta\lambda \leq \|a\|^2$ for every $\lambda \in \mathbb{R}$, so that $\beta = 0$. ∎

Lemma 1.17(iii) is an important observation which shows that the norm of a C^*-algebra is completely determined by the algebra structure. A particularly surprising consequence is the following:

Proposition 1.18. *Let $\mathcal{A}$ be a Banach-$*$-algebra and $\mathcal{B}$ be a C^*-algebra. Then every homomorphism $\varphi\colon \mathcal{A} \to \mathcal{B}$ of $*$-algebras is continuous with $\|\varphi\| \leq 1$.*

Proof. First we observe that $\varphi(a, \lambda) := (\varphi(a), \lambda)$ defines an extension to $\mathcal{B}_+ \to \mathcal{A}_+$. Since $\mathcal{B}_+$ is a Banach-$*$-algebra and $\mathcal{A}_+$ is a C^*-algebra by Proposition 1.14, we may w.l.o.g. assume that $\mathcal{B}$ and $\mathcal{A}$ are unital. From Lemma 1.11 we know that, for $a \in \mathcal{A}$, we have

$$\sigma(\varphi(a)^*\varphi(a))) = \sigma(\varphi(a^*a)) \subseteq \sigma(a^*a).$$

With Lemma 1.17 this leads to

$$\|\varphi(a)\|^2 = r(\varphi(a)^*\varphi(a)) \leq r(a^*a) \leq \|a^*a\| \leq \|a\|^2. \qquad \square$$

The following categorical construction in an interesting application of the preceding proposition.[9]

Definition 1.19. (The enveloping C^*-algebra) [Sketch] Let $\mathcal{A}$ be a Banach-$*$-algebra. In Proposition 1.18 we have seen that, every homomorphism $\alpha\colon \mathcal{A} \to \mathcal{B}$ into a C^*-algebra is a contraction, i.e., $\|\alpha\| \leq 1$, and in particular continuous. Therefore

$$p(a) := \sup\{\|\alpha(a)\|\colon \alpha\colon \mathcal{A} \to \mathcal{B} \ \text{ a linear } *\text{-homo., } \mathcal{B} \ \text{ a } C^*\text{-alg.}\} \leq \|a\|$$

exists. Then p is a submultiplicative seminorm because all functions $a \mapsto \|\alpha(a)\|$ are submultiplicative seminorms on $\mathcal{A}$. In addition,

$$p(a^*a) = \sup_\alpha \|\alpha(a^*a)\| = \sup_\alpha \|\alpha(a)^*\alpha(a)\| = \sup_\alpha \|\alpha(a)\|^2 = p(a)^2.$$

The function p is called the *maximal C^*-seminorm on $\mathcal{A}$*.

It defines a continuous function on $\mathcal{A}$ because

$$|p(a) - p(b)| \leq p(a - b) \leq \|a - b\| \quad \text{ for } \quad a, b \in \mathcal{A}.$$

Therefore the subspace $\mathcal{I} := p^{-1}(0) = \bigcap_\alpha \ker(\alpha)$ is a closed $*$-ideal, so that $\mathcal{A}/\mathcal{I}$ inherits the structure of a Banach-$*$-algebra through which all homomorphisms to C^*-algebras factorize (Exercise 2.3). Further p defines a norm p' on $\mathcal{A}/\mathcal{I}$. We write $C^*(\mathcal{A})$ for the completion of $(\mathcal{A}/\mathcal{I}, p')$. Extending the multiplication, the involution and the norm by continuity to $C^*(\mathcal{A})$, we see that the relation $p'(a^*a) = p'(a)^2$ for $a \in C^*(\mathcal{A})$ implies that $C^*(\mathcal{A})$ actually is a C^*-algebra (Exercise 1.9). Further,

$$\eta\colon \mathcal{A} \to C^*(\mathcal{A}), \quad a \mapsto a + \mathcal{I}$$

is a morphism of Banach-$*$-algebras through which all morphisms $\varphi\colon \mathcal{A} \to \mathcal{B}$ to a C^*-algebra $\mathcal{B}$ factor because any such φ satisfies $\mathcal{I} \subseteq \ker \varphi$.

Example 1.20. The following example shows that there are Banach-$*$-algebras for which the map $\eta\colon \mathcal{A} \to C^*(\mathcal{A})$ into their enveloping C^*-algebra is not injective.

The most extreme cases arise if $\mathcal{A}$ is a Banach-$*$-algebra with the zero multiplication $ab = 0$ for all $a, b \in \mathcal{A}$. Then every $*$-homomorphism $\alpha\colon \mathcal{A} \to \mathcal{B}$, $\mathcal{B}$ a C^*-algebra, vanishes because of

$$\|\alpha(a)\|^2 = \|\alpha(a^*a)\| = 0.$$

Hence $C^*(\mathcal{A}) = \{0\}$.

[9] In the language of categories, it provides an adjoint to the forgetful functor from the category of C^*-algebras to the category of Banach-$*$-algebras.

Exercises for Section 1

Exercise 1.1. Let X be a Banach space over $\mathbb{K}$ and $\alpha\colon X \to \mathbb{K}$ be a linear functional. Show that α is continuous if and only if $\ker \alpha$ is closed.
Hint: If $0 \neq \alpha$ has a closed kernel and $\alpha(x_0) = 1$, then there exists a ball $B_r^X(x_0)$ intersecting $\ker \alpha$ trivially; conclude that $\alpha(B_r^X(0)) \subseteq B_1^{\mathbb{K}}(0)$ and further that $\|\alpha\| \leq r^{-1}$.

Exercise 1.2. Consider the 3-element group $G := \mathbb{Z}/3\mathbb{Z}$. We consider G as an involutive semigroup with $g^* = g^{-1}$. Show that the corresponding Banach-$*$-algebra $\ell^1(G)$ from Example 1.8 is not a C^*-algebra.
Hint: Consider the element $a := \delta_0 - \delta_1 - \delta_2$.

Exercise 1.3. Let X be a compact space and $\mathcal{A}$ be a Banach algebra. Show that:

(a) The space $C(X, \mathcal{A})$ of $\mathcal{A}$-valued continuous functions on X is a complex associative algebra with respect to pointwise multiplication $(fg)(x) := f(x)g(x)$.

(b) $\|f\| := \sup_{x \in X} \|f(x)\|$ is a submultiplicative norm on $C(X, \mathcal{A})$ for which $C(X, \mathcal{A})$ is complete, hence a Banach algebra. Hint: Continuous functions on compact spaces are bounded and uniform limits of sequences of continuous functions are continuous.

(c) $C(X, \mathcal{A})^\times = C(X, \mathcal{A}^\times)$.

(d) If $\mathcal{A}$ is a C^*-algebra, then $C(X, \mathcal{A})$ is also a C^*-algebra with respect to the involution $f^*(x) := f(x)^*$, $x \in X$.

Exercise 1.4. Let $\mathcal{A}$ be a Banach algebra over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. If $\mathcal{A}$ has no unit, we cannot directly associate a "unit group" to $\mathcal{A}$. However, there is a natural way to do that by considering on $\mathcal{A}$ the multiplication

$$x * y := x + y + xy.$$

Show that:

(a) The space $\mathcal{A}_+ := \mathcal{A} \times \mathbb{K}$ is a unital Banach algebra with respect to the multiplication

$$(a, t)(a', t') := (aa' + ta' + t'a, tt') \quad \text{and} \quad \|(a, t)\| = \|a\| + |t|.$$

(b) The map $\eta\colon \mathcal{A} \to \mathcal{A}_+, x \mapsto (x, 1)$ is injective and satisfies $\eta(x * y) = \eta(x)\eta(y)$. Conclude in particular that $(\mathcal{A}, *, 0)$ is a monoid, i.e., a semigroup with neutral element 0.

(c) An element $a \in \mathcal{A}$ is said to be *quasi-invertible* if it is an invertible element in the monoid $(\mathcal{A}, *, 0)$. Show that the set $\mathcal{A}^\times$ of quasi-invertible elements of $\mathcal{A}$ is an open subset and that $(\mathcal{A}^\times, *, 0)$ is a group.

Exercise 1.5. Let X be a locally compact space. We say that a continuous function $f\colon X \to \mathbb{C}$ *vanishes at infinity* if, for each $\varepsilon > 0$, there exists a compact subset $K \subseteq X$ with $|f(x)| \leq \varepsilon$ for $x \notin K$. We write $C_0(X)$ for the set of all continuous $\mathbb{C}$-valued functions vanishing at infinity. Show that:

(a) The space $C_0(X)$ is a commutative complex $*$-algebra with respect to pointwise multiplication $(fg)(x) := f(x)g(x)$ and $f^*(x) := \overline{f(x)}$.

(b) $\|f\| := \sup_{x \in X} \|f(x)\|$ is a submultiplicative norm on $C_0(X)$ for which $C_0(X)$ is a C^*-algebra.

Exercise 1.6. Let $\mathcal{A} \subseteq B(\mathcal{H})$ be a $*$-invariant subspace and $\mathcal{A}_h := \{a \in \mathcal{A}\colon a^* = a\}$ the subspace of hermitian elements in $\mathcal{A}$. Show that

$$\mathcal{A} = \mathcal{A}_h \oplus i\mathcal{A}_h.$$

In particular $\mathcal{A} = \operatorname{span}_{\mathbb{C}} \mathcal{A}_h$.

Exercise 1.7. Let V be a $\mathbb{K}$-vector space and $A \in \mathrm{End}(V)$. We write $V_\lambda(A) := \ker(A - \lambda\mathbf{1})$ for the *eigenspace of* A corresponding to the eigenvalue λ and $V^\lambda(A) := \bigcup_{n\in\mathbb{N}} \ker(A - \lambda\mathbf{1})^n$ for the *generalized eigenspace of* A corresponding to λ. Show that, if $A, B \in \mathrm{End}(V)$ commute, then

$$BV^\lambda(A) \subseteq V^\lambda(A) \quad \text{and} \quad BV_\lambda(A) \subseteq V_\lambda(A)$$

holds for each $\lambda \in \mathbb{K}$.

Exercise 1.8. Let V be a vector space which is the direct sum

$$V = V_1 \oplus \cdots \oplus V_n$$

of the subspaces V_i, $i = 1, \ldots, n$. Accordingly, we write $v \in V$ as a sum $v = v_1 + \cdots + v_n$ with $v_i \in V$. To each $\varphi \in \mathrm{End}(V)$ we associate the map $\varphi_{ij} \in \mathrm{Hom}(V_j, V_i)$, defined by $\varphi_{ij}(v) = \varphi(v)_i$ for $v \in V_j$. Show that

(a) $\varphi(v)_i = \sum_{j=1}^n \varphi_{ij}(v_j)$ for $v = \sum_{j=1}^n v_j \in V$.

(b) The map

$$\Gamma\colon \bigoplus_{i,j=1}^n \mathrm{Hom}(V_j, V_i) \to \mathrm{End}(V), \quad \Gamma((\psi_{ij}))(v) := \sum_{i,j=1}^n \psi_{ij}(v_j)$$

is a linear isomorphism. In this sense we may identify endomorphisms of V with $(n \times n)$-matrices with entries in $\mathrm{Hom}(V_j, V_i)$ in position (i, j).

(c) If V is a Banach space and each V_i is a closed subspace, then the map

$$S\colon V_1 \times \cdots \times V_n \to V, \quad (v_1, \ldots, v_n) \mapsto \sum_{i=1}^n v_i$$

is a homeomorphism. Moreover, a linear endomorphism $\varphi\colon V \to V$ is continuous if and only if each φ_{ij} is continuous. Hint: For the first assertion use the Open Mapping Theorem. Conclude that if $\iota_i\colon V_i \to V$ denotes the inclusion map and $p_j\colon V \to V_j$ the projection map, then both are continuous. Then use that $\varphi_{ij} = p_i \circ \varphi \circ \eta_j$.

Exercise 1.9. Let $\mathcal{A}$ be a Banach-$*$-algebra. A norm q on $\mathcal{A}$ is called a *C^*-norm* if

$$q(ab) \leq q(a)q(b), \qquad q(a^*) = q(a) \quad \text{and} \quad q(a^*a) = q(a)^2 \quad \text{for} \quad a, b \in \mathcal{A}.$$

Let $\mathcal{A}_q$ denote the completion of $\mathcal{A}$ with respect to the norm q and show that:

(i) The multiplication on $\mathcal{A}$ extends continuously to a bilinear associative multiplication on $\mathcal{A}_q$.

(ii) The involution $*$ extends isometrically to an antilinear involution on $\mathcal{A}_q$.

(iii) $\mathcal{A}_q$ is a C^*-algebra.

How do we have to modify the construction if q is only assumed to be a seminorm?

Exercise 1.10. (Consistency of the definition of the spectrum) Let $\mathcal{A}$ be a complex algebra and $\mathcal{A}_+ = \mathcal{A} \times \mathbb{C}$ denote its unitization with the multiplication

$$(a, t)(a', t') = (aa' + ta' + t'a, tt'), \qquad a, a' \in \mathcal{A}, t, t' \in \mathbb{C}.$$

Suppose that $\mathcal{A}$ has a unit $\mathbf{1}$ and show that, in this case, for every element $a \in \mathcal{A}$, we have

$$\sigma_{\mathcal{A}}(a) \cap \mathbb{C}^\times = \sigma_{\mathcal{A}_+}(a, 0) \cap \mathbb{C}^\times.$$

Hint: Show that $\mathcal{A}_+ = \mathcal{A} \oplus \mathbb{C}(-\mathbf{1}, 1) \cong \mathcal{A} \oplus \mathbb{C}$ is a direct sum of unital algebras in which the multiplication of the elements $[a, t] := (a, 0) + t(-\mathbf{1}, 1)$ takes the form $[a, t] * [a', t'] = [aa', tt']$.

Exercise 1.11. Let $\mathcal{A}$ be a unital complex algebra and $a \in \mathcal{A}$. Show that

$$\sigma(a + \lambda\mathbf{1}) = \sigma(a) + \lambda \quad \text{for all} \quad \lambda \in \mathbb{C}.$$

2 C^*-algebras

In this section we take a closer look at C^*-algebras. First we describe the structure of commutative C^*-algebras by showing that every commutative C^*-algebra $\mathcal{A}$ is isomorphic to $C_0(X)$ for some locally compact space X which is compact if $\mathcal{A}$ has a unit (Gelfand Representation Theorem) (Subsection 2.1). Since every normal element a in a non-commutative C^*-algebra $\mathcal{A}$ generates a commutative subalgebra $C^*(a)$, Gelfand's theorem applies to this subalgebra, and we actually show that $C^*(a) \cong C(\sigma(a))$ (Subsection 2.2). From that we further derive that the spectrum of a is the same in all C^*-subalgebras $\mathcal{B} \subseteq \mathcal{A}$ containing a. This permits us in Subsection 2.3 to say that a hermitian element a is positive ($a \geq 0$) if $\sigma(a) \subseteq [0, \infty)$. By Kaplansky's Theorem, the positive elements of a C^*-algebra $\mathcal{A}$. are precisely the products b^*b, $b \in \mathcal{A}$. This characterization further leads to the notion of a positive functional studied in Subsections 2.4 and 2.5.

2.1 Commutative C^*-algebras

Let $\mathcal{A}$ be a commutative Banach algebra. We write $\mathrm{Hom}(\mathcal{A}, \mathbb{C})$ for the set of all algebra homomorphism $\chi \colon \mathcal{A} \to \mathbb{C}$, i.e., χ satisfies the following relations

$$\chi(a+b) = \chi(a) + \chi(b), \quad \chi(\lambda a) = \lambda\chi(a) \quad \text{and} \quad \chi(ab) = \chi(a)\chi(b) \tag{7}$$

for all $a, b \in \mathcal{A}, \lambda \in \mathbb{C}$.

To obtain a suitable topology on $\mathrm{Hom}(\mathcal{A}, \mathbb{C})$, we consider it as a subset of the space $\mathbb{C}^{\mathcal{A}}$ of all functions $\mathcal{A} \to \mathbb{C}$, which we endow with the coarsest topology for which all evaluation maps

$$\widehat{a} \colon \mathrm{Hom}(\mathcal{A}, \mathbb{C}) \to \mathbb{C}, \quad \chi \mapsto \chi(a)$$

for $a \in \mathcal{A}$, are continuous (Section A.1).[10] It coincides with the subspace topology inherited from the product topology of $\mathbb{C}^{\mathcal{A}}$. As the functions $\widehat{a}$ separate the points of $\mathrm{Hom}(\mathcal{A}, \mathbb{C})$, it is a Hausdorff space.

Since the set $\mathrm{Hom}(\mathcal{A}, \mathbb{C})$ is defined by the equations (7), it coincides with

$$\{\chi \in \mathbb{C}^{\mathcal{A}} \colon (\forall a, b \in \mathcal{A}, \lambda \in \mathbb{C})\ \widehat{a}(\chi) + \widehat{b}(\chi) = \widehat{a+b}(\chi), \widehat{\lambda a}(\chi) = \lambda\widehat{a}(\chi), \widehat{a}(\chi)\widehat{b}(\chi) = \widehat{ab}\,(\chi)\},$$

and this is a closed subset of $\mathbb{C}^{\mathcal{A}}$ because all the functions $\widehat{a}$ are continuous on $\mathbb{C}^{\mathcal{A}}$.

Further, $\|\chi\| \leq 1$ for any $\chi \in \mathrm{Hom}(\mathcal{A}, \mathbb{C})$ by Exercise 2.1, so that

$$\mathrm{Hom}(\mathcal{A}, \mathbb{C}) \subseteq \prod_{a \in \mathcal{A}} \{z \in \mathbb{C} \colon |z| \leq \|a\|\} = \{\chi \in \mathbb{C}^{\mathcal{A}} \colon (\forall a \in \mathcal{A})|\chi(a)| \leq \|a\|\}.$$

This subset is compact by Tychonov's Theorem. Therefore the topology of pointwise convergence turns $\mathrm{Hom}(\mathcal{A}, \mathbb{C})$ into a compact space.

Lemma 2.1. *With respect to the topology of pointwise convergence, the set* $\mathrm{Hom}(\mathcal{A}, \mathbb{C})$ *is compact and the subset*

$$\widehat{\mathcal{A}} := \mathrm{Hom}(\mathcal{A}, \mathbb{C}) \setminus \{0\}$$

is locally compact.

[10] This topology is also called the topology of pointwise convergence.

Since each function $\widehat{a}\colon \operatorname{Hom}(\mathcal{A}, \mathbb{C}) \to \mathbb{C}$ is continuous and vanishes in the 0-functional,

$$\widehat{a} \in \{f \in C(\operatorname{Hom}(\mathcal{A}, \mathbb{C}))\colon f(0) = 0\} \cong C_0(\widehat{\mathcal{A}})$$

(Exercise 2.2). Further,

$$|\widehat{a}(\chi)| = |\chi(a)| \leq \|\chi\|\|a\| \leq \|a\| \tag{8}$$

(Exercise 2.1). We thus obtain a map

$$\mathcal{G}\colon \mathcal{A} \to C_0(\widehat{\mathcal{A}}), \quad a \mapsto \widehat{a},$$

called the *Gelfand transform.* For $a, b \in \mathcal{A}$ and $\chi \in \widehat{\mathcal{A}}$, we have

$$\mathcal{G}(ab)(\chi) = \chi(ab) = \chi(a)\chi(b) = \mathcal{G}(a)(\chi)\mathcal{G}(b)(\chi),$$

so that $\mathcal{G}$ is a morphism of Banach algebras, i.e., a continuous homomorphism compatible with the involution. By (8), we have $\|\mathcal{G}\| \leq 1$ (cf. also Proposition 1.18 if $\mathcal{A}$ is a Banach-$*$-algebra).

The following theorem provides a concrete description of all commutative C^*-algebras. It is a key tool for all deeper results on operator algebras.

Theorem 2.2. (Gelfand Representation Theorem) *If $\mathcal{A}$ is a commutative C^*-algebra, then the Gelfand transform*

$$\mathcal{G}\colon \mathcal{A} \to C_0(\widehat{\mathcal{A}})$$

is an isometric isomorphism. If $\mathcal{A}$ is unital, then $\widehat{\mathcal{A}}$ is compact and $\mathcal{A} \cong C(\widehat{\mathcal{A}})$.

Proof. First we assume that $\mathcal{A}$ **is unital.**

For any $\chi \in \widehat{\mathcal{A}}$ we then have $\chi(\mathbf{1}) = 1$ (Exercise 2.1), so that

$$\widehat{\mathcal{A}} = \{\chi \in \operatorname{Hom}(\mathcal{A}, \mathbb{C})\colon \chi(\mathbf{1}) = 1\} = \widehat{\mathbf{1}}^{-1}(1)$$

is a closed subset of the compact set $\operatorname{Hom}(\mathcal{A}, \mathbb{C})$ and therefore compact.

If $\chi\colon \mathcal{A} \to \mathbb{C}$ is a non-zero homomorphism and $\chi(a) = \lambda$, then $\lambda \in \sigma(a)$ follows from $\{\lambda\} = \sigma_{\mathbb{C}}(\chi(a)) \subseteq \sigma_{\mathcal{A}}(a)$ (Lemma 1.11). For $a = a^*$, Lemma 1.17(iv) implies $\sigma(a) \subseteq \mathbb{R}$, so that $\chi(a) \in \mathbb{R}$. For two hermitian elements $b, c \in \mathcal{A}$, this leads to

$$\chi((b + ic)^*) = \chi(b - ic) = \chi(b) - i\chi(c) = \overline{\chi(b) + i\chi(c)} = \overline{\chi(b + ic)}.$$

Therefore every homomorphism $\chi\colon \mathcal{A} \to \mathbb{C}$ is automatically compatible with the involution and the Gelfand transform is a homomorphism of C^*-algebras.

Let $a \in \mathcal{A}$. Since every element of $\mathcal{A}$ is normal, Lemma 1.17(ii) implies that $r(a) = \|a\|$. Next we show that

$$\sigma(a) = \{\chi(a)\colon \chi \in \widehat{\mathcal{A}}\}.$$

We have already seen that $\supseteq$ holds, so it remains to verify the converse. If $\lambda \in \sigma(a)$, then $\lambda\mathbf{1}-a$ is not invertible, so that $\mathcal{I} := (\lambda\mathbf{1}-a)\mathcal{A}$ is a proper ideal. Let $\mathcal{J} \supseteq \mathcal{I}$ be a maximal ideal containing $\mathcal{I}$. The existence of such ideals follows from Zorn's Lemma because, for each chain of ideals $(\mathcal{I}_j)_{j \in J}$, the union $\bigcup_j \mathcal{I}_j$ is a an ideal which does not intersect $\mathcal{A}^\times$, so it is proper. As $\mathcal{A}^\times$ is open, the closure of $\mathcal{J}$ also is an ideal intersecting $\mathcal{A}^\times$ trivially; by maximality it follows that $\mathcal{J}$ is closed. We thus obtain a quotient Banach algebra $\mathcal{Q} := \mathcal{A}/\mathcal{J}$ (Exercise 2.3). Let $q\colon \mathcal{A} \to \mathcal{Q}$ denote the quotient homomorphism. Then the inverse image of every ideal in $\mathcal{Q}$ is an ideal in $\mathcal{A}$. As $\mathcal{J}$ is maximal, the ideal $\{0\}$ is maximal in $\mathcal{Q}$, and therefore every non-zero element is invertible. By the Gelfand–Mazur Theorem, there exists an isomorphism $\varphi\colon \mathcal{Q} \to \mathbb{C}$. Now $\chi := \varphi \circ q\colon \mathcal{A} \to \mathbb{C}$ is a homomorphism of unital Banach algebras with $\lambda\mathbf{1} - a \in \mathcal{J} = \ker q \subseteq \ker\chi$, hence $\chi(a) = \lambda$. We conclude that $\sigma(a) \subseteq \{\chi(a)\colon \chi \in \widehat{\mathcal{A}}\}$, hence that equality holds.

This proves that

$$\|a\| = r(a) = \|\widehat{a}\|_\infty,$$

and therefore $\mathcal{G}\colon \mathcal{A} \to C(\widehat{\mathcal{A}})$ is isometric.

It remains to show that $\mathcal{G}$ **is surjective.** The image $\mathcal{G}(\mathcal{A})$ of the Gelfand transform is a unital $*$-subalgebra of $C(\widehat{\mathcal{A}})$ separating the points of $\widehat{\mathcal{A}}$. Therefore the Stone–Weierstraß Theorem (Corollary A.26) implies that $\mathcal{G}(\mathcal{A})$ is dense in $C(\widehat{\mathcal{A}})$. Since $\mathcal{G}$ is isometric, its range is complete, hence closed, and this finally shows that $\mathcal{G}(\mathcal{A}) = C(\widehat{\mathcal{A}})$.

If $\mathcal{A}$ **is not unital**, then we embed $\mathcal{A}$ it into the commutative C^*-algebra $\mathcal{A}_+$ (Proposition 1.14). By $\chi_+(a,\lambda) := \chi(a) + \lambda$, every homomorphism $\mathcal{A} \to \mathbb{C}$ extends uniquely to a unital homomorphism $\mathcal{A}_+ \to \mathbb{C}$. Now

$$\widehat{\mathcal{A}_+} = \mathrm{Hom}(\mathcal{A}_+, \mathbb{C}) \setminus \{0\} = \mathrm{Hom}_1(\mathcal{A}_+, \mathbb{C}) := \{\chi \in \mathrm{Hom}(\mathcal{A}_+, \mathbb{C})\colon \chi(\mathbf{1}) = 1\},$$

and the embedding

$$\mathrm{Hom}(\mathcal{A}, \mathbb{C}) \to \widehat{\mathcal{A}_+}, \quad \chi \mapsto \chi_+$$

is a homeomorphism (Exercise 2.15). From the unital case we know that $\mathcal{G}_{\mathcal{A}_+}\colon \mathcal{A}_+ \to C(\widehat{\mathcal{A}_+})$ is an isomorphism of C^*-algebras. It restricts to an isomorphism from $\mathcal{A} \cong \mathcal{A} \times \{0\} = \ker 0_+$ onto

$$C_*(\widehat{\mathcal{A}_+}) = \{f \in C(\widehat{\mathcal{A}_+})\colon f(0_+) = 0\},$$

which is isomorphic to $C_0(\widehat{\mathcal{A}_+} \setminus \{0_+\}) \cong C_0(\widehat{\mathcal{A}})$ (Exercise 2.2). Finally, the relation

$$\mathcal{G}_{\mathcal{A}_+}(a,0)(\chi_+) = \chi(a) = \mathcal{G}_{\mathcal{A}}(a)(\chi)$$

shows that this isomorphism is the Gelfand transform of $\mathcal{A}$. $\square$

The following corollary will be the key to Schur's Lemma (Theorem 3.13) in the representation theory of $*$-algebras.

Corollary 2.3. *If $\mathcal{A}$ is a unital C^*-algebra and $\dim\mathcal{A} > 1$, then there exist non-zero commuting elements $a, b \in \mathcal{A}$ with $ab = 0$.*

Proof. Since $\mathcal{A} \neq \mathbb{C}\mathbf{1}$, there exists an element $x \in \mathcal{A} \setminus \mathbb{C}\mathbf{1}$. Writing $x = y + iz$ with $y^* = y$ and $z^* = z$, it follows immediately that $\mathcal{A}$ contains a hermitian element $a \in \mathcal{A} \setminus \mathbb{C}\mathbf{1}$. Let $C^*(a) \subseteq \mathcal{A}$ be the closed unital subalgebra of $\mathcal{A}$ generated by a. Then $C^*(a)$ is commutative and larger than $\mathbb{C}\mathbf{1}$, hence isomorphic to $C(X)$ for some compact space X (Theorem 2.2). Then X contains at least two points $x \neq y$, and Urysohn's Theorem A.16 implies the existence of non-zero elements $a, b \in C(X) \cong C^*(a)$ with $ab = 0$. $\square$

By Gelfand's Representation Theorem, commutative unital C^*-algebras $\mathcal{A}$ are of the form $C(X)$ for a compact space X. The following proposition shows that $\widehat{C(X)} \cong X$, so that the compact space in the Gelfand construction is unique.

Proposition 2.4. *If X is a compact space, then the map*

$$\Phi\colon X \to \mathrm{Hom}_1(C(X), \mathbb{C}), \quad \Phi(x) = \delta_x, \quad \delta_x(f) = f(x)$$

is a homeomorphism. In particular, $X \cong \widehat{C(X)}$.

Proof. By definition of the topology on $\mathrm{Hom}(C(X), \mathbb{C})$, the map Φ is continuous because, for every $f \in C(X)$, the map

$$X \to \mathbb{C}, \quad x \mapsto \Phi(x)(f) = \delta_x(f) = f(x)$$

is continuous. Since the continuous functions on X separate the points by Urysohn's Theorem A.16, it is injective. As X is compact and $\mathrm{Hom}(C(X), \mathbb{C})$ is Hausdorff, Φ is a homeomorphism onto its image. It therefore remains to show that Φ is surjective.

So let $\chi\colon C(X) \to \mathbb{C}$ be a unital algebra homomorphism. Then its kernel $\mathcal{N}$ is a hyperplane and an algebra ideal and thus $C(X) = \mathcal{N} \oplus \mathbb{C}\mathbf{1}$. If there exists an $x \in X$ with $\mathcal{N} \subseteq \ker \delta_x$, then the above direct sum decomposition implies that $\delta_x = \chi$ because $\chi(\mathbf{1}) = \delta_x(\mathbf{1}) = 1$ and $\delta_x(\mathcal{N}) = \chi(\mathcal{N}) = \{0\}$.

We now assume that no such x exists and derive a contradiction. For each $x \in X$, there exists a function $f_x \in \mathcal{N}$ with $f_x(x) \neq 0$. Since X is compact, there exist finitely many $x_1, \ldots, x_n \in X$ with

$$X \subseteq \bigcup_{j=1}^{n} \{x \in X\colon f_{x_j}(x) \neq 0\}.$$

Then

$$f := \sum_{j=1}^{n} f_{x_j} \overline{f_{x_j}} = \sum_{j=1}^{n} |f_{x_j}|^2 \in \mathcal{N}$$

satisfies $f(x) > 0$ for every $x \in X$ and therefore f is invertible. This leads to the contradiction that the proper ideal $\mathcal{N} = \ker \chi$ contains an invertible element. $\square$

Corollary 2.5. *For a locally compact space X, the map*

$$\Phi\colon X \to C_0(X)\widehat{\ }, \quad \Phi(x) = \delta_x, \quad \delta_x(f) = f(x)$$

is a homeomorphism.

Proof. Let $\mathcal{A} := C_0(X)$. We have seen in the last part of the proof of the Gelfand Representation Theorem that the map $\chi \mapsto \chi_+$ leads to a homeomorphism $\mathcal{A}_+ \to \mathrm{Hom}_1(\mathcal{A}_+, \mathbb{C}) \setminus \{0_+\}$. For $x \in X$, we have $\Phi(\delta_x)_+ = \Phi_{X_\omega}(\delta_x)$, and by Proposition 2.4, the map Φ_{X_ω} is a homeomorphism. Therefore the restriction to X yields the homeomorphism Φ onto $\widehat{\mathcal{A}} \cong C(X_\omega)\widehat{\ } \setminus \{0_+\}$. $\square$

2.2 Spectral calculus for normal elements

In this section we describe several applications of the Gelfand Representation Theorem to not necessarily commutative unital C^*-algebras $\mathcal{A}$. Here a key observation is that, for any normal element $a \in \mathcal{A}$, the closed subalgebra

$$C^*(a) := \overline{\mathrm{span}\{a^n (a^*)^m \colon n, m \in \mathbb{N}_0\}} \tag{9}$$

is a commutative unital C^*-subalgebra of $\mathcal{A}$. Gelfand's Theorem asserts that it is isomorphic to some $C(X)$ and we now identify the compact space X with the spectrum $\sigma(a)$ and draw several important conclusion. One is that, for any unital C^*-subalgebra $\mathcal{B} \subseteq \mathcal{A}$ containing a, the spectrum $\sigma_{\mathcal{B}}(a)$ of a with respect to $\mathcal{B}$ coincides with $\sigma(a)$. Another application is that, we can define $f(a) \in \mathcal{A}$ for every continuous function $f \in C(\sigma(a))$. This is called the *continuous functional calculus.*

Proposition 2.6. (Continuous spectral calculus for normal elements) *Let $a \in \mathcal{A}$ be a normal element of the unital C^*-algebra $\mathcal{A}$ and let $C^*(a) \subseteq \mathcal{A}$ be the commutative unital C^*-subalgebra generated by a. Then the exists a unique isomorphism of C^*-algebras*

$$\Phi_a \colon C(\sigma(a)) \to C^*(a) \quad \textit{with} \quad \Phi_a(\mathrm{id}) = a.$$

In particular, $C^*(a) \cong C(\sigma(a))$ and $\widehat{C^*(a)} \cong \sigma(a)$.

In the following we write

$$f(a) := \Phi_a(f).$$

Note that this notation is compatible with $f(a) = \sum_{j=0}^n c_j a^j$ if $f(x) = \sum_{j=0}^n c_j x^j$, resp., $f = \sum_{j=0}^n c_j \,\mathrm{id}^j$ is a polynomial.

Proof. From (9) and the continuity of the multiplication it follows that $C^*(a)$ is a commutative C^*-algebra. Therefore Gelfand's Representation Theorem 2.2 implies the existence of a compact space $X = \mathrm{Hom}_1(C^*(a), \mathbb{C})$ for which the Gelfand transform

$$\mathcal{G} \colon C^*(a) \to C(X), \quad b \mapsto \widehat{b}$$

is an isomorphism of C^*-algebras. We claim that the function

$$\widehat{a} \colon X \to \widehat{a}(X) \subseteq \mathbb{C}$$

is a homeomorphism. Since $C(X)$ separates the points of X and is generated by $\widehat{a}$, the function $\widehat{a}$ is injective. As X is compact, $\widehat{a} \colon X \to \mathbb{C}$ is a topological embedding, so that we may, from now on, identify X with a compact subset of $\mathbb{C}$.

It remain to show that $\widehat{a}(X) = \sigma(a)$. The inclusion $\sigma(a) \subseteq \sigma_{C^*(a)}(a) = \widehat{a}(X)$ follows from Lemma 1.11 and Example 1.9. For the converse inclusion, let $\lambda \in X \setminus \sigma(a)$ and put $b := (a - \lambda \mathbf{1})^{-1} \in \mathcal{A}$. Let $m > \|b\|$ and consider the continuous function

$$f \colon X \to [0, m] \subseteq \mathbb{R}, \quad f(z) := m \max(0, 1 - m|\lambda - z|).$$

Then $f(\lambda) = m$ and f vanishes outside the circle of radius m^{-1} around λ, which leads to

$$|f(z)(z - \lambda)| \leq 1 \quad \text{for} \quad z \in X.$$

With the function $g := f \cdot (\lambda - \widehat{a})$ on X, we now obtain

$$m \leq \|f\|_\infty = \|\mathcal{G}^{-1}(f)\| = \|\mathcal{G}^{-1}(f)(\lambda \mathbf{1} - a)b\| = \|\mathcal{G}^{-1}(g)b\| \leq \|g\|_\infty \cdot \|b\| \leq \|b\|.$$

This contradicts the choice of m and shows that $\widehat{a}(X) = \sigma(a)$.

Therefore the Gelfand transform defines an isomorphism $\mathcal{G} \colon C^*(a) \to C(\sigma(a))$ mapping a onto $\mathrm{id}_{\sigma(a)}$ and thus $\Phi_a := \mathcal{G}^{-1}$ is an isomorphism of C^*-algebras mapping $\mathrm{id}_{\sigma(a)}$ onto a. That the latter relation determines this isomorphism uniquely follows from the fact that $\mathrm{id}_{\sigma(a)} = \mathcal{G}(a)$ generated the C^*-algebra $C(\sigma(a))$ because a generates $C^*(a)$. $\square$

Corollary 2.7. *If $\mathcal{A}$ is a unital C^*-algebra, $\mathcal{B}$ is a unital C^*-subalgebra of $\mathcal{A}$ and $a \in \mathcal{B}$ a normal element, then $\sigma_{\mathcal{B}}(a) = \sigma_{\mathcal{A}}(a)$.*

Proof. With Lemma 1.11 and Proposition 2.6, we now obtain

$$\sigma_{\mathcal{A}}(a) \subseteq \sigma_{\mathcal{B}}(a) \subseteq \sigma_{C^*(a)}(a) = \sigma_{\mathcal{A}}(a),$$

which proves equality. $\square$

Remark 2.8. (a) In the special case where $\mathcal{A} = C(X)$ for a compact space X, we have $\sigma(a) = a(X)$ and

$$\Phi_a \colon C(a(X)) \to \mathcal{A}, \quad \Phi_a(f) = f(a) = f \circ a.$$

This shows in particular that

$$\sigma(f(a)) = \sigma(f \circ a) = f(a(X)). \tag{10}$$

(b) More generally, we consider a normal element $a \in \mathcal{A}$ ($\mathcal{A}$ unital) and $f \in C(\sigma(a))$ and the corresponding element $f(a) = \Phi_a(f) \in C^*(a)$. Then

$$\sigma_{\mathcal{A}}(f(a)) = \sigma_{C^*(a)}(f(a)) = \sigma_{C(\sigma(a))}(f) = f(\sigma(a)).$$

This relation is also called the *Spectral Mapping Theorem.*

Theorem 2.9. (The Closed Range Theorem) *Let $\varphi \colon \mathcal{A} \to \mathcal{B}$ be an injective $*$-homomorphism of C^*-algebras. Then φ is isometric. In particular its range is closed and therefore a C^*-subalgebra of $\mathcal{B}$.*

Proof. Extending φ to the $*$-homomorphism

$$\varphi_+ \colon \mathcal{A}_+ \to \mathcal{B}_+, \quad (a, \lambda) \mapsto (\varphi(a), \lambda)$$

of C^*-algebras (Proposition 1.14) which is obviously injective, we may w.l.o.g. assume that $\mathcal{A}$ and $\mathcal{B}$ are unital.

In view of $\|a\|^2 = \|a^*a\|$ for $a \in \mathcal{A}$, it suffices to show that, for $a = a^* \in \mathcal{A}$, we have $\|\varphi(a)\| = \|a\|$. Since φ maps $C^*(a)$ into $C^*(\varphi(a))$, we may w.l.o.g. assume that $\mathcal{A} = C^*(a)$ and $\mathcal{B} = C^*(\varphi(a))$. Using the isomorphisms

$$\Phi_a \colon C(\sigma(a)) \to C^*(a) \quad \text{and} \quad \Phi_{\varphi(a)} \colon C(\sigma(\varphi(a))) \to C^*(\varphi(a))$$

(Proposition 2.6), we may further assume that φ is the unique homomorphism

$$\psi \colon C(\sigma(a)) \to C(\sigma(\varphi(a))) \quad \text{with} \quad \psi(\mathrm{id}_{\sigma(a)}) = \mathrm{id}_{\sigma(\varphi(a))}.$$

As $\sigma(\varphi(a))$ is a subset of $\sigma(a)$ (Lemma 1.11), it follows that

$$\psi(f) = f|_{\sigma(\varphi(a))},$$

i.e., ψ is simply the restriction map (Exercise 2.20). If $\sigma(\varphi(a))$ is a proper subset of $\sigma(a)$, then there exists continuous functions on $\sigma(a)$ vanishing on the closed subset $\sigma(\varphi(a))$. For any $z_0 \in \sigma(a) \setminus \sigma(\varphi(a))$, the distance function $f(z) := \mathrm{dist}(z, \sigma(\varphi(a)))$ is such a function (Exercise!). Therefore the injectivity of φ implies that $\sigma(a) = \varphi(\sigma(a))$ and hence that $\|\varphi(a)\| = r(\varphi(a)) = r(a) = \|a\|$. $\square$

2.3 Positive elements in C^* algebras

Definition 2.10. A hermitian element a of a C^*-algebra $\mathcal{A}$ is called *positive* if $\sigma(a) \subseteq [0, \infty)$. We then write $a \geq 0$. For two hermitian elements $a, b \in \mathcal{A}$, we write

$$a \leq b \quad \text{if} \quad b - a \geq 0.$$

We denote the set of positive elements in $\mathcal{A}$ by $\mathcal{A}^+$ (not to be confused with the unitization $\mathcal{A}_+$).

Example 2.11. (a) If $\mathcal{A} = C(X)$ for a compact space and $f^* = \overline{f}$, then $f \in \mathcal{A}$ is hermitian if and only if $f(X) \subseteq \mathbb{R}$. Since $f(X) = \sigma(f)$ by Example 1.9, the function f is a positive element of $C(X)$ if and only if $f(x) \geq 0$ for every $x \in X$.

(b) If $\mathcal{A} = B(\mathcal{H})$ and $A \in B(\mathcal{H})$ is a hermitian operator, then $A \geq 0$ means that $\mathrm{Spec}(A) \subseteq [0, \infty)$. We claim that that this is equivalent to

$$\langle v, Av \rangle \geq 0 \quad \text{for all} \quad v \in \mathcal{H}. \tag{11}$$

If $A \geq 0$, then Exercise 2.5 implies that

$$\langle e^{-2tA} v, v \rangle = \langle e^{-tA} v, e^{-tA} v \rangle = \|e^{-tA} v\|^2 \leq \|v\|^2 \quad \text{for} \quad t \geq 0.$$

Since the derivative $-2\langle Av, v \rangle$ in $t_0 = 0$ exists, it must be ≤ 0, and therefore $\langle Av, v \rangle \geq 0$. If, conversely, $\langle Av, v \rangle \geq 0$ for every $v \in \mathcal{H}$, then the differentiable function

$$f(t) := \|e^{-tA} v\|^2 = \langle e^{-tA} v, e^{-tA} v \rangle = \langle e^{-2tA} v, v \rangle \quad \text{satisfies} \quad f(0) = \|v\|^2$$

and

$$f'(t) = -2\langle Ae^{-2tA} v, v \rangle = -2\langle Ae^{-tA} v, e^{-tA} v \rangle \leq 0 \quad \text{for} \quad t \geq 0.$$

Therefore $f(t) \leq \|v\|^2$ for $t \geq 0$, and this implies $\|e^{-tA}\| \leq 1$. Using Exercise 2.5 again, we see that $A \geq 0$.

For $\mathcal{H} = \mathbb{C}^n$ and $\mathcal{A} = B(\mathbb{C}^n) \cong M_n(\mathbb{C})$, the equivalence of $A \geq 0$ and (11) follows from the well-known characterization of positive semidefinite matrices A by the property that all its eigenvalues are non-negative.

One may expect that sums of positive elements are positive. This is true, but far from obvious at this point. So we have to take a short detour to prove this fact.

Lemma 2.12. *In every complex unital algebra $\mathcal{A}$, we have*

$$\sigma(xy) \setminus \{0\} = \sigma(yx) \setminus \{0\} \quad \textit{for} \quad x, y \in \mathcal{A}.$$

Proof. We have to show that, for $\lambda \neq 0$, the element $\lambda\mathbf{1} - xy$ is invertible if and only if $\lambda\mathbf{1} - yx$ invertible. To this end, we may w.l.o.g. assume that $\lambda = \mathbf{1}$.

Suppose that $\mathbf{1} - xy$ is invertible. Then

$$\begin{aligned}(\mathbf{1} + y(\mathbf{1} - xy)^{-1}x)(\mathbf{1} - yx) &= \mathbf{1} - yx + y(\mathbf{1} - xy)^{-1}(x - xyx) \\ &= \mathbf{1} - yx + y(\mathbf{1} - xy)^{-1}(\mathbf{1} - xy)x = \mathbf{1} - yx + yx = \mathbf{1}\end{aligned}$$

and likewise

$$(\mathbf{1} - yx)(\mathbf{1} + y(\mathbf{1} - xy)^{-1}x) = \mathbf{1} - yx + y(\mathbf{1} - xy)(\mathbf{1} - xy)^{-1}x = \mathbf{1} - yx + yx = \mathbf{1}.$$

This proves the assertion. $\square$

Lemma 2.13. *In a unital C^*-algebra $\mathcal{A}$, the following assertions hold:*

(i) *For every positive element $a \in \mathcal{A}^+$, there exists a unique $b \in \mathcal{A}^+$ with $b^2 = a$.*

(ii) *For every hermitian element $a \in \mathcal{A}$ there exist uniquely determined positive elements $a_+, a_- \in \mathcal{A}$ with $a = a_+ - a_-$ and $a_+a_- = a_-a_+ = 0$.*

(iii) *If $a \geq 0$ and $0 \leq f \in C(\sigma(a))$, then $\Phi_a(f) = f(a) \geq 0$.*

(iv) *A hermitian element $a \in \mathcal{A}$ is positive if and only if $\big\|\|a\|\mathbf{1} - a\big\| \leq \|a\|$.*

(v) *$a \geq 0$ and $b \geq 0$ implies $a + b \geq 0$.*

(vi) *$\pm a \geq 0$ implies $a = 0$.*

(vii) *$-aa^* \geq 0$ implies $a = 0$.*

Proof. (i) As $\sigma(a) \subseteq [0, \infty)$, the function $S(x) := \sqrt{x}$ is well-defined and non-negative on $\sigma(a)$. Therefore $S(a) := \Phi_a(S) \in C^*(a)$ is a positive square root of a.

For any other positive element b with $b^2 = a$, the element b commutes with a, so that the unital C^*-subalgebra $C^*(a, b)$ generated by a and b is commutative, hence isomorphic to some $C(X)$. But any positive elements $f \in C(X)$ has a unique positive square root given by $x \mapsto \sqrt{f(x)}$.

(ii) For any decomposition $a = a_+ - a_-$ with $a_\pm \geq 0$ and $a_+a_- = 0$, both summands commute with each other and hence also with a, so that a, a_+ and a_- are always contained in a commutative C^*-subalgebra. We may therefore assume w.l.o.g. that $\mathcal{A}$ is commutative, hence $\mathcal{A} \cong C(X)$ by Gelfand's Theorem.

If $a \in C(X)$ is hermitian, i.e., real-valued, then

$$a = a_+ - a_- \quad \text{and} \quad a_+a_- = a_-a_+ = 0 \quad \text{holds for} \quad a_\pm := \max(\pm a, 0).$$

For the uniqueness of such a decomposition, write $a = b_+ - b_-$ with $b_\pm \geq 0$ and $b_+b_- = b_-b_+ = 0$. If $\pm a(p) > 0$, then $b_\pm(p) > 0$ implies $b_\mp(p) = 0$, so that $b_\pm(p) = \mp a(p) = a_\pm(p)$. If $a(p) = 0$, then $b_+(p) = b_-(p) = 0$ follows from $b_\pm \geq 0$. Therefore $b_\pm = a_\pm$.

(iii) Since a is hermitian, it is contained in a commutative subalgebra, so that we may assume that $\mathcal{A} = C(X)$. Then $\sigma(a) = a(X)$ and $0 \leq f \in C(\sigma(a))$ implies that the function $f(a) = f \circ a$ is non-negative (cf. Remark 2.8).

(iv) Again, we may assume that $\mathcal{A} = C(X)$. Then $(\|a\|\mathbf{1} - a)(x) = \|a\| - a(x) \geq 0$, and it is $\leq \|a\|$ if and only if $a(x) \geq 0$. This proves (iv).

(v) The element $c := a + b$ is hermitian and thus $\sigma(c) \subseteq \mathbb{R}$ (Lemma 1.17). From (iv) we derive

$$\big\|(\|a\| + \|b\|)\mathbf{1} - c\big\| = \big\|\|a\|\mathbf{1} - a + \|b\|\mathbf{1} - b\big\| \leq \|a\| + \|b\|. \tag{12}$$

Further, $\lambda \in \sigma(c)$ implies

$$\|a\| + \|b\| - \lambda \in \sigma\big((\|a\| + \|b\|)\mathbf{1} - c\big),$$

so that (12) entails

$$\big|\|a\| + \|b\| - \lambda\big| \leq \|a\| + \|b\|$$

This implies $\lambda \geq 0$.

(vi) follows immediately from the corresponding assertion in $\mathcal{A} = C(X)$.

(vii) Lemma 2.12 implies $\sigma(-a^*a) \setminus \{0\} = \sigma(-aa^*) \setminus \{0\} \subseteq \mathbb{R}$, so that also $-a^*a \geq 0$. Writing $a = b + ic$ with hermitian elements $b, c \in \mathcal{A}$, we obtain with (v)

$$aa^* + a^*a = (b + ic)(b - ic) + (b - ic)(b + ic) = 2(b^2 + c^2) \geq 0,$$

because $b^2, c^2 \geq 0$ follows from (iii). This leads with (v) to $aa^* = (aa^* + a^*a) - a^*a \geq 0$, so that $aa^* = 0$ follows from (vi). Now $\|a\|^2 = \|a^*a\| = 0$ yields $a = 0$. □

Theorem 2.14. (Kaplansky, 1953) *An element a of a unital C^*-algebra $\mathcal{A}$ is positive if and only if there exists an element $b \in \mathcal{A}$ with $a = b^*b$.*

Proof. If $a \geq 0$ and b is the positive square root of a, then $a = b^2 = bb^*$ (Lemma 2.12(ii)).

For the converse, let $b \in \mathcal{A}$. We have to show that $a := b^*b \geq 0$. Let $a = a_+ - a_-$ as in Lemma 2.12(iii). Then it remains to verify that $a_- = 0$. From $a_+a_- = 0$ we derive

$$(ba_-)^*(ba_-) = a_-^*b^*ba_- = a_-(a_+ - a_-)a_- = -a_-^3 \leq 0,$$

hence $-(ba_-)^*(ba_-)$ is positive by Lemma 2.13(iii) and thus $ba_- = 0$ by Lemma 2.13(vii). This shows that $a_-^3 = 0$ and hence $a_- = 0$, by $a_- \in C^*(a_-) \cong C(\sigma(a_-))$. □

Kaplansky's Theorem will turn out to be a key between positivity in a C^*-algebra and its representation theory. The next step in the development of this connection is the concept of a positive functional that we introduce in the next subsection.

Proposition 2.15. *The set $\mathcal{A}^+ = \{a \in \mathcal{A} : a \geq 0\}$ of positive elements of a C^*-algebras is a closed convex cone.*

Proof. Lemma 2.13(v) implies that $\mathcal{A}^+$ is closed under addition. For $\lambda \geq 0$ and $a \geq 0$ we further have $\sigma(\lambda a) = \lambda\sigma(a) \subseteq [0, \infty)$, so that $\lambda a \geq 0$. This shows that $\mathcal{A}^+$ is a convex cone.

For the closedness we recall from Exercise 2.5 that

$$\mathcal{A}^+ = \{a \in \mathcal{A} : a^* = a, (\forall t \geq 0)\ \|e^{-ta}\| \leq 1\}.$$

For $a_n \geq 0$ and $a_n \to a$, we thus obtain for any $t \geq 0$:

$$\|e^{-ta}\| = \lim_{n\to\infty} \|e^{-ta_n}\| \leq 1$$

and therefore $a \geq 0$. □

2.4 Positive functionals

Definition 2.16. Let $\mathcal{A}$ be a $*$-algebra. For a linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ we define $\varphi^*(a) := \overline{\varphi(a^*)}$ and note that this is again complex linear. We say that φ is *hermitian* if $\varphi^* = \varphi$.

A linear functional φ is called *positive* if

$$\varphi(aa^*) \geq 0 \quad \text{for every} \quad a \in \mathcal{A}.$$

Remark 2.17. (a) If $\mathcal{A}$ is a C^*-algebra, then Kaplansky's Theorem immediately implies that a linear functional $f : \mathcal{A} \to \mathbb{C}$ is positive if and only if

$$a \geq 0 \quad \Rightarrow \quad f(a) \geq 0. \tag{13}$$

This implies in particular that $f(a) \in \mathbb{R}$ for $a = a^*$ (Lemma 2.13(ii)). For $c = a + ib \in \mathcal{A}$, $a, b \in \mathcal{A}_h$, we then obtain

$$\overline{f(c)} = \overline{f(a) + if(b)} = f(a) - if(b) = f(a - ib) = f(c^*),$$

so that f is hermitian.

(b) The direct sum decomposition $\mathcal{A} = \mathcal{A}_h \oplus i\mathcal{A}_h$ of the real vector space $\mathcal{A}$ shows that a complex linear function $\varphi\colon \mathcal{A} \to \mathbb{C}$ is hermitian if and only if $\varphi(\mathcal{A}_h) \subseteq \mathbb{R}$. Conversely, every real linear functional $\psi\colon \mathcal{A}_h \to \mathbb{R}$ extends by $\varphi(a + ib) := \psi(a) + i\psi(b)$ to a hermitian complex linear functional $\varphi\colon \mathcal{A} \to \mathbb{C}$.

Lemma 2.18. *Let $\mathcal{A}$ be a C^*-algebra and $f\colon \mathcal{A} \to \mathbb{C}$ be a positive functional.*

(i) $|f(ab^*)|^2 \leq f(aa^*)f(bb^*)$.

(ii) *f is continuous with*

$$\|f\| = \sup\{f(a)\colon a \in \mathcal{A}^+, \|a\| \leq 1\}. \tag{14}$$

If $\mathcal{A}$ is unital, then $\|f\| = f(\mathbf{1})$, and if $\mathcal{A}$ is not unital, then f extends to a positive functional f_+ on the unital C^*-algebra $\mathcal{A}_+$ with $f_+(\mathbf{1}) = \|f\|$.

(iii) $f(b^*a^*ab) \leq f(b^*b)\|a\|^2$ *for* $a, b \in \mathcal{A}$.

Proof. (i) Positivity means that the hermitian form $f(ab^*)$ is positive semidefinite, and (ii) simply is the Cauchy–Schwarz inequality for this form.

(ii) We consider the subset

$$M := \{a \in \mathcal{A}^+\colon \|a\| \leq 1\}.$$

We claim that $f(M)$ is bounded. If this is not the case, then there exists a sequence $x_n \in M$ with $f(x_n) \to \infty$. For any non-negative sequence $(\lambda_n)_{n\in\mathbb{N}}$ with $\sum_n \lambda_n < \infty$, we consider the element $x := \sum_{n=1}^\infty \lambda_n x_n$ and observe that the sequence converges absolutely because $\sum_n \|\lambda_n x_n\| \leq \sum_n \lambda_n < \infty$. The positivity of x and all partial sums $\sum_{n>m} \lambda_n x_n$ follows from Proposition 2.15. For every $m \in \mathbb{N}$ we therefore obtain with Lemma 2.13(v)

$$\sum_{j=1}^m \lambda_j x_j \leq x \quad \text{and thus} \quad \sum_{j=1}^m \lambda_j f(x_j) = f\Big(\sum_{j=1}^m \lambda_j x_j\Big) \leq f(x).$$

As $f(x_j) \geq 0$ for each j, the convergence of the series $\sum_{j=1}^\infty \lambda_j f(x_j)$ follows. Since the sequence $\lambda \in \ell^1$ was arbitrary positive, we obtain $\big(f(x_j)\big)_{j\in\mathbb{N}} \in \ell^\infty$ (Exercise), and therefore a contradiction. We conclude that $C := \sup f(M) < \infty$.

If $\mathcal{A}$ is unital and $a \in \mathcal{A}^+$ with $\|a\| \leq 1$, then $0 \leq a \leq \|a\|\mathbf{1}$ implies $f(a) \leq \|a\| f(\mathbf{1}) \leq f(\mathbf{1})$, so that $C = f(\mathbf{1})$ in this case.

If $\mathcal{A}$ is not unital, we extend f to a linear functional

$$f_+\colon \mathcal{A}_+ \to \mathbb{C} \quad \text{by} \quad f_+(a, \lambda) := f(a) + \lambda C.$$

If $(a, \lambda) \geq 0$ with $\lambda \neq 0$, then $\lambda \in \sigma(a, \lambda)$ implies $\lambda > 0$ and $\sigma(a) = \sigma(a, \lambda) - \lambda \subseteq [-\lambda, \infty)$. For $a = a_+ - a_-$ this means that $\|a_-\| \leq \lambda$, so that

$$f_+(a, \lambda) = f(a) + \lambda C = f(a_+) - f(a_-) + \lambda C \geq f(a_+) - \lambda C + \lambda C = f(a_+) \geq 0.$$

Therefore f_+ is a positive functional on $\mathcal{A}_+$. With (i) we now obtain for every $x \in \mathcal{A}_+$:

$$|f_+(x)|^2 \leq f_+(xx^*)f_+(\mathbf{1}) \leq \|xx^*\|C \cdot C = \|x\|^2C^2.$$

Hence f_+ is continuous with $\|f_+\| \leq C \leq \|f\|$, and this proves that $\|f\| = \|f_+\| = C$.

(iii) First we note that the functional $f_b(a) := f(b^*ab)$ on $\mathcal{A}_+$ is positive because $f_b(a^*a) = f(b^*a^*ab) = f((ab)^*ab) \geq 0$. Now (ii) implies that f_b is continuous with $\|f_b\| = f_b(\mathbf{1}) = f(b^*b)$, and now

$$f(b^*a^*ab) = f_b(a^*a) \leq \|f_b\|\|a^*a\| = f(b^*b)\|a\|^2$$

proves (iii). □

Positive functionals on commutative algebras

Since we shall use is later on, we quote the Riesz Representation Theorem describing the positive functionals on commutative C^*-algebras $C_0(X)$ in terms of Radon measures. For the proof we refer to [Ru86, Thm. 2.14]. First we recall the concept of a Radon measure.

Definition 2.19. A Borel measure μ on a locally compact space X is called *a Radon measure* if

(i) $\mu(K) < \infty$ for each compact subset K of X.

(ii) (Outer regularity) For each Borel subset $E \subseteq X$, we have

$$\mu(E) = \inf\{\mu(U) \colon E \subseteq U, U \text{open}\}.$$

(iii) (Inner regularity) If $E \subseteq X$ is open or E is a Borel set with $\mu(E) < \infty$, then

$$\mu(E) = \sup\{\mu(K) \colon K \subseteq E, K \text{ compact}\}.$$

The measure μ is called *regular* if (ii) and (iii) are satisfied. These two properties ensure that μ is uniquely determined by the integral of compactly supported continuous functions.

Theorem 2.20. (Riesz Representation Theorem) *Let X be a locally compact space. Then, for every positive functional $I \colon C_0(X) \to \mathbb{C}$, there exists a uniquely determined finite regular Radon measure μ on X with*

$$I(f) = \int_X f(x)\, d\mu(x) \quad \textit{for} \quad f \in C_0(X).$$

Proposition 2.21. *If μ is a Radon measure on a locally compact space X, then $C_c(X)$ is dense in $L^2(X,\mu)$.*

Proof. Since the characteristic functions χ_E, $\mu(E) < \infty$, span the dense subspace of step functions, i.e., functions with finitely many values, it suffices to show that such function can be approximated by elements of $C_c(X)$ in the L^2-norm. Since every such Borel set is inner regular, we may w.l.o.g. assume that E is compact. Then the outer regularity implies for each $\varepsilon > 0$ the existence of an open subset $U \subseteq X$ with $\mu(U \setminus E) < \varepsilon$. Next we use Urysohn's Theorem A.16 to find a continuous function $f \in C_c(X)$ with $0 \leq f \leq 1$, $f|_E = 1$, and $\mathrm{supp}(f) \subseteq U$. Then

$$\|f - \chi_E\|_2^2 = \int_X |f(x) - \chi_E(x)|^2 \, d\mu(x) = \int_{U \setminus E} |f(x)|^2 \, d\mu(x) \leq \mu(U \setminus E) < \varepsilon,$$

and this completes the proof. □

Remark 2.22. In many cases the regularity of a Borel measure μ on a locally compact space X for which all compact subspaces have finite measure is automatic.

In [Ru86, Thm. 2.18] one finds the convenient criterion that this is the case whenever every open subset $O \subseteq X$ is a countable union of compact subsets.

This is in particular the case for $\mathbb{R}^n$, because we can write

$$O = \bigcup_{n \in \mathbb{N}} O_n \quad \text{with} \quad O_n := \Big\{x \in O \colon \operatorname{dist}(x, O^c) \geq \frac{1}{n}, \|x\| \leq n\Big\}.$$

2.5 States

In this subsection we introduce states of C^*-algebras. This terminology is an abstraction that is due to the interpretation of states as states of quantum mechanical systems discussed in the introduction.

Definition 2.23. A positive functional φ on a C^*-algebra is called a *state* if $\|\omega\| = 1$. We write $\mathfrak{S}(\mathcal{A})$ for the set of all states of $\mathcal{A}$.

Remark 2.24. Since every state $\omega \in \mathfrak{S}(\mathcal{A})$ on a non-unital C^*-algebra $\mathcal{A}$ extends in a unique fashion to a state ω_+ on $\mathcal{A}_+$ by $\omega_+(\mathbf{1}) = 1$, we may identify the states of $\mathcal{A}$ and $\mathcal{A}_+$ (Lemma 2.18):

$$\mathfrak{S}(\mathcal{A}) = \mathfrak{S}(\mathcal{A}_+).$$

Lemma 2.25. *For a unital C^*-algebra $\mathcal{A}$, we have*

$$\mathfrak{S}(\mathcal{A}) = \{\omega \in \mathcal{A}' \colon \omega^* = \omega, \|\omega\| = \omega(\mathbf{1}) = 1\}.$$

This set is convex and compact in the weak-$*$-topology, i.e., the topology of pointwise convergence. [11]

Proof. According to the Alaoglu–Bourbaki Theorem, the dual unit ball

$$B' := \{\alpha \in \mathcal{A}' \colon \|\alpha\| \leq 1\}$$

is convex and compact with respect to the weak-$*$-topology. Since the subspace

$$(\mathcal{A}')_h := \{\omega \in \mathcal{A}' \colon \omega^* = \omega\} = \{\omega \in \mathcal{A}' \colon (\forall a \in \mathcal{A})\ \overline{\omega(a)} = \omega(a^*)\}$$

is specified by pointwise conditions, it is a weak-$*$-closed subset of $\mathcal{A}'$, so that $B' \cap (\mathcal{A}')_h$ is also weak-$*$-compact. As the linear functional

$$\operatorname{ev}_{\mathbf{1}} \colon B' \cap (\mathcal{A}')_h \to [-1, 1] \subseteq \mathbb{R}, \quad \alpha \mapsto \alpha(\mathbf{1})$$

is continuous, the subset

$$S := \{\alpha \in \mathcal{A}' \colon \alpha^* = \alpha, \|\alpha\| \leq 1, \alpha(\mathbf{1}) = 1\} = \{\alpha \in \mathcal{A} \colon \alpha^* = \alpha, \|\alpha\| = \alpha(\mathbf{1}) = 1\}$$

is also weak-$*$-compact and convex. Now $S \supseteq \mathfrak{S}(\mathcal{A})$ by Lemma 2.18, and we next show the converse. So let $\omega \in S$. We have to show that ω is a positive functional. So let $a \in \mathcal{A}^+$. Then $\big\|\|a\|\mathbf{1} - a\big\| \leq \|a\|$ (Lemma 2.13(iv)) implies that

$$\|a\| - \omega(a) = \omega(\|a\|\mathbf{1} - a) \leq \|a\|,$$

and thus $\omega(a) \geq 0$. $\square$

[11] One can show that the condition $\|\omega\| = 1 = \omega(\mathbf{1})$ already implies $\omega = \omega^*$, but this requires more elaborate arguments.

Example 2.26. (a) For the C^*-algebra $\mathcal{A} = C(X)$ of continuous functions on the compact space X, Riesz' Representation Theorem 2.20 identifies the positive functionals with Radon measures μ on X. Among these, the states are the probability measures because

$$\mu(X) = \int_X 1\, d\mu(x).$$

Particular states are the *Dirac measures* corresponding to the evaluation functionals $\delta_x \colon f \mapsto f(x)$. According to Proposition 2.4, these are precisely the characters of $C(X)$. We shall see in Section 3 that these are precisely the pure states (Corollary 3.20).

(b) Consider $\mathcal{A} = B(\mathcal{H})$ for a complex Hilbert space $\mathcal{H}$ and $\Omega \in \mathcal{H}$ with $\|\Omega\| = 1$. Then

$$\omega(A) := \langle \Omega, A\Omega \rangle$$

defines a state of $B(\mathcal{H})$ because ω is a positive functional, $\omega(\mathbf{1}) = 1$ and $\|\omega\| \leq 1$ by the Cauchy–Schwarz inequality. These states are called *vector states.*

Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ satisfying $\sum_{n\in\mathbb{N}} \|\Omega_n\|^2 = 1$. Then

$$\omega(A) := \sum_{n=1}^{\infty} \langle \Omega_n, A\Omega_n \rangle$$

also defines a state of $B(\mathcal{H})$. These states are called *mixed states.* They are not pure, i.e., extreme points in $\mathfrak{S}(B(\mathcal{H}))$ if the Ω_n span a subspace of dimension at least 2 (Exercise 2.21). We shall return to such states below when we discuss trace class operators.

Proposition 2.27. *Let $\mathcal{A}$ be a unital C^*-algebra and $x = x^* \in \mathcal{A}$.*

(i) *For every $\lambda \in \sigma(x)$, there exists a state ω with $\omega(x) = \lambda$.*

(ii) *x is positive if and only if $\omega(x) \geq 0$ for all states $\omega \in \mathfrak{S}(\mathcal{A})$.*

Proof. (i) Let $\lambda \in \sigma(x)$ and consider the unital C^*-subalgebra $C^*(x)$ generated by x. Corollary 2.7 implies that $\sigma_{\mathcal{A}}(x) = \sigma_{C^*(x)}(x)$, and Proposition 2.6 further yields $C^*(x) \cong C\big(\sigma_{\mathcal{A}}(x)\big)$, where $x \in C^*(x)$ corresponds to the identity.

Let $\delta_\lambda \colon C^*(x) \cong C(\sigma(x)) \to \mathbb{C}$ be the evaluation in $\lambda \in \sigma(x)$. Then $\delta_\lambda \in \mathfrak{S}(C^*(x))$. With the Hahn–Banach Extension Theorem, we find an extension $\widetilde{\nu} \in \mathcal{A}_h'$ with $\|\widetilde{\nu}\| = \|\delta_\lambda\| = 1$. We put $\nu := (\widetilde{\nu} + \widetilde{\nu}^*)/2$. Then $\nu^* = \nu$, $\nu(\mathbf{1}) = 1$ and $\|\nu\| \leq 1$ implies that ν is a state (Lemma 2.25). Further, $\nu(x) = \delta_\lambda(x) = \widehat{x}(\lambda) = \lambda$.

(ii) Suppose that x is positive. By definition, all states take non-negative values on x. The converse follows from (i). $\square$

Exercises for Section 2

Exercise 2.1. Let $\mathcal{A}$ be a Banach algebra and $\chi \colon \mathcal{A} \to \mathbb{C}$ be an algebra homomorphism. Show that:

(a) χ extends to the unital Banach algebra $\mathcal{A}_+ := \mathcal{A} \times \mathbb{C}$ with the multiplication

$$(a,t)(a',t') := (aa' + ta' + t'a, tt')$$

(cf. Exercise 1.4).

(b) If $\mathcal{A}$ is unital and $\chi \neq 0$, then $\chi(\mathbf{1}) = 1$ and $\chi(\mathcal{A}^\times) \subseteq \mathbb{C}^\times$. Conclude further that $\chi(B_1(\mathbf{1})) \subseteq \mathbb{C}^\times$ and derive that χ is continuous with $\|\chi\| \leq 1$.

Exercise 2.2. Suppose that Y is a compact space $y_0 \in Y$ and $X := Y \setminus \{y_0\}$. Show that the restriction map yields an isometric isomorphism of C^*-algebras:

$$r \colon C_*(Y, \mathbb{C}) := \{f \in C(Y, \mathbb{C}) \colon f(y_0) = 0\} \to C_0(X, \mathbb{C}).$$

Exercise 2.3. Let $\mathcal{A}$ be a Banach algebra and $\mathcal{I} \subseteq \mathcal{A}$ be a closed two-sided ideal. Show that the quotient space $\mathcal{A}/\mathcal{I}$ carries a natural Banach algebra structure for which the quotient map $q \colon \mathcal{A} \to \mathcal{A}/\mathcal{I}$ is a homomorphism.

If $\mathcal{A}$ is a Banach-$*$-algebra and $\mathcal{I}$ is $*$-invariant, then $\mathcal{A}/\mathcal{I}$ even inherits the structure of a Banach-$*$-algebra.

Exercise 2.4. Let $\mathcal{A}$ be a unital C^*-algebra and $a = a^* \in \mathcal{A}$ with $\|a\| < 1$. Show that

$$u := a + i\sqrt{\mathbf{1} - a^2} \in \mathrm{U}(\mathcal{A})$$

and conclude that $\mathcal{A} = \operatorname{span} \mathrm{U}(\mathcal{A})$.

Exercise 2.5. Let a be a hermitian element of the C^*-algebra $\mathcal{A}$. Show that

$$a \geq 0 \qquad \Longleftrightarrow \qquad (\forall t \geq 0)\ \|e^{-ta}\| \leq 1.$$

Hint: Consider first the case $\mathcal{A} = C(X)$; why does this imply the general case.

Exercise 2.6. (Enveloping C^*-algebra) Let $\mathcal{A}$ be a Banach-$*$-algebra and $\eta_{\mathcal{A}} \colon \mathcal{A} \to C^*(\mathcal{A})$ be the enveloping C^*-algebra. Show that:

(i) (Universal property implies uniqueness up to isomorphism) Suppose that $\zeta \colon \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism to a C^*-algebra which also has the universal property that, for any homomorphism $\alpha \colon \mathcal{A} \to \mathcal{C}$, $\mathcal{C}$ a C^*-algebra, there exists a unique homomorphism $\widetilde{\alpha} \colon \mathcal{B} \to \mathcal{C}$ with $\widetilde{\alpha} \circ \zeta = \alpha$. Then there exists an isomorphism $\Phi \colon C^*(\mathcal{A}) \to \mathcal{B}$ with $\Phi \circ \eta_{\mathcal{A}} = \zeta$.

(ii) For every representations of $\mathcal{A}$, i.e., every $*$-homomorphism $\varphi \colon \mathcal{A} \to B(\mathcal{H})$, $\mathcal{H}$ a complex Hilbert space, there exists a unique representation $\widetilde{\varphi} \colon C^*(\mathcal{A}) \to B(\mathcal{H})$ with $\widetilde{\varphi} \circ \eta_{\mathcal{A}} = \varphi$.

Exercise 2.7. Let G be a group and $\ell^1(G)$ denote the Banach-$*$-algebra from Example 1.8(b), so that we have a homomorphism $\eta \colon G \to \ell^1(G), \eta(g) = \delta_g$ whose range generates a dense $*$-subalgebra of $\ell^1(G)$. Show that:

(i) For every homomorphism $U \colon G \to \mathrm{U}(\mathcal{A})$, where $\mathcal{A}$ is a unital C^*-algebra and $\mathrm{U}(\mathcal{A}) = \{u \in \mathcal{A} \colon uu^* = u^*u = \mathbf{1}\}$ denotes its unital group, there exists a unique $*$-homomorphism $\widetilde{U} \colon \ell^1(G) \to \mathcal{A}$ with $\widetilde{U} \circ \eta = U$.

(ii) Consider the complex Hilbert space $\mathcal{H} := \ell^2(G)$ with the ONB $(e_g)_{g \in G}$. Show that the homomorphism $U \colon G \to \mathrm{U}(\ell^2(G))$ defined by $U(g)e_h = e_{gh}$, $h \in G$, leads to an injective homomorphism $\widetilde{U} \colon \ell^1(G) \to B(\ell^2(G))$.

Exercise 2.8. Let G be a group. Show that there exists a C^*-algebra $C^*(G)$ and a group homomorphism

$$\eta_G \colon G \to \mathrm{U}(C^*(G)) := \{u \in C^*(G) \colon uu^* = u^*u = \mathbf{1}\}$$

with the following universal property. For every unitary representation $U \colon G \to \mathrm{U}(\mathcal{H})$, there exists a unique $*$-homomorphism $\widetilde{U} \colon C^*(G) \to B(\mathcal{H})$ of C^*-algebras with $\widetilde{U} \circ \eta_G = U$.
Hint: Define $C^*(G)$ as $C^*(\ell^1(G))$, where $\ell^1(G)$ is the Banach-$*$-algebra from Example 1.8(b), and put $\eta_G(g) = \delta_g$.

Exercise 2.9. Show that the homomorphism $\eta_{\ell^1(G)} \colon \ell^1(G) \to C^*(\ell^1(G)) = C^*(G)$ is injective.
Hint: Exercise 2.7.

Exercise 2.10. (Initial topology) Let X be a set and $(f_i)_{i\in I}$ be a family of maps $f_i\colon X \to Y_i$, where the Y_i are topological spaces. Let τ be the topology on X generated by the subsets $f_i^{-1}(O)$, $O \subseteq Y_i$ open. Show that:

(i) τ is the coarsest (minimal) topology on X for which all maps f_i are continuous.

(ii) A map $g\colon Z \to X$, Z a topological space, is continuous if and only if all compositions $f_i \circ g$ are continuous.

Exercise 2.11. (Topology of pointwise convergence) Let A and B be topological spaces and B^A be the set of all maps $f\colon A \to B$. On B^A we consider the coarsest topology for which all maps

$$\mathrm{ev}_a\colon B^A \to B, \quad f \mapsto f(a)$$

are continuous. Show that a map $F\colon X \to B^A$, X a topological space, is continuous if and only if all maps

$$F_a\colon X \to B, \quad F_a(x) := F(x)(a)$$

are continuous.

Exercise 2.12. (Gelfand isomorphism for group algebras) Let G be a commutative group and $\eta_G\colon G \to C^*(G) = C^*(\ell^1(G))$ be the corresponding C^*-algebra from Exercise 3.5. It has the universal property that, for every C^*-algebra $\mathcal{A}$, the map

$$\eta_G^*\colon \mathrm{Hom}_{\mathbf{1}}(C^*(G),\mathcal{A}) \to \mathrm{Hom}_{\mathrm{grp}}(G,\mathrm{U}(\mathcal{A})), \quad \varphi \mapsto \varphi \circ \eta_G$$

is a bijection, where $\mathrm{Hom}_{\mathbf{1}}$ denotes the set of C^*-homomorphisms preserving $\mathbf{1}$. Show that:

(a) $C^*(G)$ is a commutative C^*-algebra. Hint: $\eta(G)$ spans a dense subspace of $C^*(G)$.

(b) $\eta_G^*\colon \widehat{C^*(G)} \to \widehat{G} := \mathrm{Hom}_{\mathrm{grp}}(G,\mathbb{T})$ is a bijection.

(c) $\widehat{G}$ is a group with respect to the pointwise multiplication of characters

$$(\chi_1\chi_2)(g) := \chi_1(g)\chi_2(g).$$

It is called the *character group of G*.

(d) Determine the character groups $\widehat{G}$ for the abelian groups $G = \mathbb{Z}, \mathbb{Z}^n, \mathbb{Z}/n\mathbb{Z}$.

(e)* $\widehat{G}$ is a compact topological group with respect to the topology of pointwise convergence on elements of G. Recall that a topological group is a group H with a topology for which the map $H \times H \to H, (g,h) \mapsto gh^{-1}$ is continuous.

Exercise 2.13. (Another exotic Banach-$*$-algebra) Let $\mathcal{A}$ be a unital C^*-algebra and $b \in \mathcal{A}$ be a unitary involution, i.e., $b^2 = \mathbf{1}$ and $b^* = b$. Show that:

(a) $\mathcal{A}$ is a Banach-$*$-algebra with respect to the involution $a^\sharp = ba^*b$.

(b) $b = b_+ - b_-$ with hermitian elements $b_\pm$ satisfying $b_+b_- = b_-b_+ = 0$, $b_\pm^2 = b_\pm$ and $b_+ + b_- = \mathbf{1}$. Hint: $(b_+ + b_-)^2 = \mathbf{1}$.

From now on we assume that $\mathcal{A} = M_2(\mathbb{C})$ and $b := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(c) The unitary group $\mathrm{U}(\mathcal{A},\sharp)$ is unbounded. Hint: Consider the elements $u_t = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$.

(d) For the matrix $Y := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ we have $Y^\sharp = Y$ and $\sigma(Y) = \{\pm i\}$.

(e) Every $*$-homomorphism $\varphi\colon (\mathcal{A},\sharp) \to \mathcal{C}$ into a C^*-algebra $\mathcal{C}$ vanishes. In particular, we have $C^*(\mathcal{A},\sharp) = \{0\}$.
Hint: $\varphi(\mathbf{1}) = -\varphi(Y)^2 \leq 0$.

Exercise 2.14. Let $\chi\colon \mathcal{A} \to \mathbb{C}$ be a homomorphism of a unital algebra to $\mathbb{C}$ and $\mathbf{1} \in \mathcal{A}$ be a unit element. Show that either $\chi = 0$ or $\chi(\mathbf{1}) = 1$. This means that, for every unital algebra $\mathcal{A}$,

$$\mathrm{Hom}_1(\mathcal{A}, \mathbb{C}) = \{\chi \in \mathrm{Hom}(\mathcal{A}, \mathbb{C}) \colon \chi(\mathbf{1}) = 1\} = \mathrm{Hom}(\mathcal{A}, \mathbb{C}) \setminus \{0\} =: \widehat{\mathcal{A}}.$$

Exercise 2.15. Let $\mathcal{A}$ be a Banach algebra. For an algebra homomorphism $\chi\colon \mathcal{A} \to \mathbb{C}$, let $\chi_+\colon \mathcal{A}_+ \to \mathbb{C}$ be the canonical unital extension defined by

$$\chi_+(a, t) = \chi(a) + t\mathbf{1}.$$

Show that the map $\iota\colon \mathrm{Hom}(\mathcal{A}, \mathbb{C}) \hookrightarrow \mathrm{Hom}_1(\mathcal{A}_+, \mathbb{C}), \chi \mapsto \chi_+$ is a homeomorphism with respect to the topology of pointwise convergence on both spaces. Conclude that $\widehat{\mathcal{A}}$ is homeomorphic to $\widehat{\mathcal{A}_+} \setminus \{\varepsilon\}$, where $\varepsilon(a, t) = t$ for $a \in \mathcal{A}, t \in \mathbb{C}$.

Exercise 2.16. (Strange Banach-$*$-algebras) We consider the two-dimensional commutative complex Banach algebra $\mathcal{A} := \mathbb{C}^2$ with the multiplication

$$(z_1, z_2)(w_1, w_2) := (z_1 w_1, z_2 w_2) \quad \text{and} \quad \|(z_1, z_2)\| = \max\{|z_1|, |z_2|\}.$$

Show that:

(a) $\mathcal{A}$ is a C^*-algebra with respect to $(z_1, z_2)^* := (\overline{z_1}, \overline{z_2})$. Determine $\widehat{\mathcal{A}}$.

(b) $\mathcal{A}$ is a Banach-$*$-algebra with respect to the involution $(z_1, z_2)^\sharp := (\overline{z_2}, \overline{z_1})$.

(c) The Banach-$*$-algebra $(\mathcal{A}, \sharp)$ has the following properties:

   (i) No character $\chi \in \mathrm{Hom}(\mathcal{A}, \mathbb{C})$ is a $*$-homomorphism.

   (ii) The Gelfand transform $\mathcal{G}\colon (\mathcal{A}, \sharp) \to C(\widehat{\mathcal{A}})$ is an isomorphism of Banach algebras but not a $*$-homomorphism.

   (iii) Every $*$-homomorphism $\varphi\colon \mathcal{A} \to \mathcal{C}$ ($\mathcal{C}$ a C^*-algebra) is zero. In particular $C^*(\mathcal{A}, \sharp) = \{0\}$. Hint: $(1, 0)^\sharp(1, 0) = 0$.

   (iv) The unitary group $\mathrm{U}(\mathcal{A}, \sharp) = \{a \in \mathcal{A}^\times \colon a^\sharp = a^{-1}\}$ is not a bounded subset.

Exercise 2.17. Let X be a set and $\ell^\infty(X)$ be the vector space of all bounded functions $f\colon X \to \mathbb{C}$. Show that $\ell^\infty(X)$ is a commutative C^*-algebra with respect to

$$(fg)(x) := f(x)g(x), \qquad f^*(x) := \overline{f(x)}, \qquad \|f\|_\infty = \sup\{|f(x)| \colon x \in X\}.$$

If $(X, \mathfrak{S})$ is a measurable space, resp., (X, τ) is a topological space, then the bounded measurable, resp., bounded continuous functions are closed $*$-subalgebras, hence also C^*-algebras.

Exercise 2.18. (Stone–Čech compactification) Let X be a topological space. We consider the commutative unital C^*-algebra $C^b(X)$ of bounded continuous functions (Exercise 2.17). Its spectrum is denoted $\beta(X) := \widehat{C^b(X)} = \mathrm{Hom}_1(C^b(X), \mathbb{C})$. This is a compact space with $C(\beta(X)) \cong C^b(X)$. Show that:

(a) The map $\eta_X\colon X \to \beta(X), \eta_X(x) = \delta_x$, $\delta_x(f) := f(x)$, is continuous and its range is dense. The pair $(\beta(X), \eta_X)$ is called the *Stone–Čech compactification of* X.

(b) η_X is a homeomorphism if and only if X is compact.

(c) For a sequence $(x_n)_{n\in\mathbb{N}}$ in X, the limit of the sequence $(\delta_{x_n})_{n\in\mathbb{N}}$ exists in $\beta(X)$ if and only if, for every bounded continuous function $f \in C^b(X)$, the limit $\lim_{n\to\infty} f(x_n)$ exists.

(d) If X is discrete, then $C^b(X) = \ell^\infty(X)$. Here the case $X = \mathbb{N}$ is particularly interesting:

   (i) If the sequence δ_{x_n} converges in $\beta(\mathbb{N})$, then it is eventually constant, i.e., there exists an $n_0 \in \mathbb{N}$ with $x_n = x_{n_0}$ for $n \geq n_0$. So $\eta_{\mathbb{N}}(\mathbb{N})$ is sequentially closed in $\beta(\mathbb{N})$.

(ii) $\eta_{\mathbb{N}}$ is not surjective (so that $\beta(\mathbb{N})$ contains elements that cannot be reached by sequences in $\mathbb{N}$). Hint: The set $Y := \bigcap_{k\in\mathbb{N}} \overline{\eta_{\mathbb{N}}(\{k, k+1, \ldots\})} \subseteq \beta(X)$ is non-empty and vanishes on the subalgebra $c_0(\mathbb{N}) \subseteq \ell^\infty(\mathbb{N})$.

Exercise 2.19. (Spectral calculus for normal matrices) We consider the concrete unital C^*-algebra $\mathcal{A} = M_n(\mathbb{C}) \cong B(\mathbb{C}^n)$. Let $a \in \mathcal{A}$ be a normal element, so that there exists an ONB $(v_1, \ldots, v_n)$ of eigenvectors for a: $av_j = \lambda_j v_j$. Show that:

(i) $\sigma(a) = \{\lambda_1, \ldots, \lambda_n\}$.

(ii) For every function $f \in C(\sigma(a)) = \mathbb{C}^{\{\lambda_1,\ldots,\lambda_n\}}$, the element $f(a) \in \mathcal{A}$ is diagonalizable with $f(a)v_j = f(\lambda_j)v_j$, $j = 1, \ldots, n$.

Exercise 2.20. (Uniqueness of Functional Calculus) Let $X \subseteq \mathbb{C}$ be a compact space and $Y \subseteq X$ be a closed subset. Show that the restriction map $R\colon C(X) \to C(Y), R(f) = f|_Y$ is the only homomorphism of C^*-algebras mapping id_X to id_Y.

Exercise 2.21. Let $\mathbf{v} := (v_n)_{n\in\mathbb{N}}$ be a sequence in the Hilbert space $\mathcal{H}$ satisfying $\sum_n \|v_n\|^2 = 1$ and consider the corresponding state

$$\omega_{\mathbf{v}}(A) := \sum_n \langle v_n, Av_n\rangle$$

of $B(\mathcal{H})$. Show that, if $\operatorname{span}\{v_n \colon n \in \mathbb{N}\}$, is not one-dimensional, then $\omega_{\mathbf{v}}$ is not a pure state, i.e., it can be write as $\omega_{\mathbf{v}} = \lambda\omega_1 + (1-\lambda)\omega_2$ with states $\omega_j \neq \omega_{\mathbf{v}}$ and $0 < \lambda < 1$.
Hint: Consider the state $\omega_1(A) = \frac{\langle v_1, Av_1\rangle}{\|v_1\|^2}$.

Exercise 2.22. Let $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers. Show that $\mathbf{x}$ is bounded if and only if, for all non-negative sequences $\lambda = (\lambda_n)_{n\in\mathbb{N}}$ with $\sum_n \lambda_n = 1$, the series $\sum_n x_n\lambda_n$ converges absolutely.

Exercise 2.23. Consider the C^*-algebra $\mathcal{A} = C_0(X)$, where X is a locally compact space. Let $\mathfrak{S}$ denote the σ-algebra of Borel subsets of X, i.e., the smallest σ-algebra containing the open subsets. We write $L^\infty(X)$ for the C^*-algebra of all bounded measurable functions $f\colon X \to \mathbb{C}$ with the norm

$$\|f\|_\infty := \sup\{|f(x)| \colon x \in X\}.$$

Show that every non-degenerate representation $(\pi, \mathcal{H})$ of $C_0(X)$ has an extension to a representation of the larger C^*-algebra $L^\infty(X)$.
Hint: Decompose first into cyclic representations and recall that these are equivalent to multiplication representations on spaces $L^2(X, \mu)$, μ a finite positive Radon measure on X. [12]

Exercise 2.24. (States of the algebra of compact operators) We consider the C^*-algebra $\mathcal{A} := K(\mathcal{H})$ of compact operators on the Hilbert space $\mathcal{H}$. Show that:

(i) The sesquilinear map

$$\Gamma\colon \mathcal{H} \times \mathcal{H} \to K(\mathcal{H}), \quad \Gamma(x, y) := |y\rangle\langle x|, \qquad \Gamma(x, y)z := \langle x, z\rangle y$$

is continuous; more precisely $\|\Gamma(x, y)\| \leq \|x\|\|y\|$ for $x, y \in \mathcal{H}$.

(ii) For every continuous linear functional $\omega \in \mathcal{A}'$, there exists a uniquely determined bounded operator $S \in B(\mathcal{H})$ with $\omega(\Gamma(x, y)) = \langle x, Sy\rangle$ for $x, y \in \mathcal{H}$. We then write $\omega = \omega_S$.

(c) $\omega_S^* = \omega_{S^*}$.

[12] We shall see later in the context of commutative von Neumann algebras how to specify this extension uniquely in terms of continuity properties.

(d) $\omega_S \geq 0$ if and only if $S \geq 0$.
Hint: $A \geq 0$ if and only if $A = \sum_n \lambda_n \Gamma(x_n, x_n)$ for an orthonormal sequence x_n in $\mathcal{H}$ (Spectral Theorem for compact hermitian operators).

(e) S is a compact operator.
Hint: Use Exercise 2.27(i)(b) to show that S is Hilbert–Schmidt, specifically, $\sum_j \|Se_j\|^2 \leq \|\omega_S\|^2$ for any orthonormal basis $(e_j)_{j\in J}$ of $\mathcal{H}$.

(f) If $S \geq 0$, then there exists an orthogonal sequence $(v_n)_{n\in\mathbb{N}}$ in $\mathcal{H}$ and $\lambda_n \geq 0$ with $\sum_n \lambda_n < \infty$ such that $S = \sum_n \lambda_n \Gamma(v_n, v_n) = \sum_n \lambda_n |v_n\rangle\langle v_n|$ (S is a "trace class operator").
Hint: Use the compactness and the Spectral Theorem for compact operators to diagonalize S, so that $S = \sum_n \lambda_n \Gamma(v_n, v_n)$. Then show that $\|\omega_S\| = \sum_n |\lambda_n|$ (Exercise 2.26).

Exercise 2.25. We consider the Hilbert space $\mathcal{H} := M_n(\mathbb{C})$ with the Hilbert–Schmidt scalar product

$$\langle A, B\rangle := \operatorname{tr}(A^*B) = \sum_{j=1}^n \langle Ae_j Be_j\rangle.$$

Show that $\pi(A)B := AB$ defines a $*$-representation of the C^*-algebra $\mathcal{A} := M_n(\mathbb{C})$ on $\mathcal{H}$ for which $\Omega := \frac{1}{\sqrt{n}}\mathbf{1}$ is a cyclic unit vector with

$$\langle \Omega, \pi(A)\Omega\rangle = \frac{1}{n}\operatorname{tr}(A) \quad \text{and} \quad \operatorname{End}(\pi, \mathcal{H}) \cong M_n(\mathbb{C}).$$

Can you characterize the cyclic elements in $\mathcal{H}$? How do the corresponding states look like?

Exercise 2.26. (Linear functionals on finite dimensional commutative algebras) We consider the commutative C^*-algebra $\mathcal{A} := \mathbb{C}^n \cong C(\{1, 2, \ldots, n\})$, endowed with pointwise multiplication and conjugation. We consider the linear isomorphism

$$\Phi\colon \mathbb{C}^n \to \mathcal{A}', \quad \Phi(s)(a) := \sum_{j=1}^n s_j a_j$$

with $\|\Phi(s)\| = \sum_{j=1}^n |s_j|$. Show that the linear functional $\Phi(s)$ is

(a) hermitian, i.e., $\Phi(s)^* = \Phi(s)$, if and only if $s \in \mathbb{R}^n$.

(b) positive if and only if $s \geq 0$ componentwise.

(c) a state if and only if $s \geq 0$ and $\sum_j s_j = 1$.

(d) a extreme point in the state space $\mathfrak{S}(\mathcal{A})$ if and only if $s_j = \delta_{jk}$ for some $k \in \{1, \ldots, n\}$.

Exercise 2.27. (Linear functionals on matrix algebras) We consider the C^*-algebra $\mathcal{A} := M_n(\mathbb{C})$ of complex $n \times n$-matrices. Show that:

(i) The map $\Phi\colon M_n(\mathbb{C}) \to \mathcal{A}', \Phi(S)(A) := \operatorname{tr}(SA)$ is a linear isomorphism satisfying

  (a) $\Phi(S)^* = \Phi(S^*)$.

  (b) $\|\Phi(S)\| \geq \|S\|_2 := \sqrt{\operatorname{tr}(S^*S)} = \sqrt{\sum_{j,k} |s_{jk}|^2}$ (Hilbert–Schmidt norm).
  Hint: $\|A\|_2 := \sqrt{\operatorname{tr}(A^*A)} \geq \|A\|$ and $\|S\|_2 = \sup\{|\operatorname{tr}(S^*A)|\colon \|A\|_2 \leq 1\}$.

(ii) The linear functional $\Phi(S)$ is

  (a) hermitian, i.e., $\Phi(S)^* = \Phi(S)$, if and only if $S^* = S$.

  (b) positive if and only if $S \geq 0$.

  (c) a state if and only if $S \geq 0$ and $\operatorname{tr}(S) = 1$.

(d) an extreme point of the state space $\mathfrak{S}(\mathcal{A})$ if and only if S is an orthogonal projection onto a one-dimensional subspace.
Hint: The set $\mathfrak{S}(\mathcal{A})$ is invariant under conjugation with unitary operators; Exercise 2.26.

Exercise 2.28. Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{B} \subseteq \mathcal{A}$ be a unital C^*-subalgebra. Show that, for every state $\omega \in \mathfrak{S}(\mathcal{B})$ there exists a state $\widetilde{\omega} \in \mathfrak{S}(\mathcal{A})$ with $\widetilde{\omega}|_{\mathcal{B}} = \omega$.

Exercise 2.29. Let $\mathcal{A}$ be a unital C^*-algebra and $x \in \mathcal{A}$ be a normal element. Show that

$$\mathfrak{S}(\mathcal{A})(x) = \operatorname{conv}(\sigma(x)).$$

Hint: Use Exercise 2.28 to reduce to the case where $\mathcal{A} = C^*(x) \cong C(\sigma(x))$. Then note that $\operatorname{conv}(\sigma(x))$ is compact because $\sigma(x)$ is compact and that $\int_{\sigma(x)} x\, d\mu(x) \in \operatorname{conv}(\sigma(x))$ for every probability measure μ on $\sigma(x)$.

Exercise 2.30.) Let $\mathcal{A}$ be a unital C^*-algebra and $A \in \mathcal{A}$. Show that $\|A\|^2 \in \sigma(A^*A)$.
Hint: For $0 \leq f \in C(X)$, we have $\|f\| = \max\{f(x) \colon x \in X\}$.

3 Representations of C^*-algebras

In this section we study representations of C^*-algebras. After introducing the relevant terminology, we explain the Gelfand–Naimark–Segal (GNS) construction that relates states and cyclic representations. As an application of the GNS construction, we obtain the Gelfand–Naimark Theorem asserting that every C^*-algebra is isomorphic to a closed subalgebra of some $B(\mathcal{H})$. We conclude this section with some facts on irreducible representations, such as their characterization in terms of pure states and their characterization in terms of their endomorphism algebra (Schur's Lemma, Theorem 3.13).

3.1 Basic terminology

Definition 3.1. Let $\mathcal{A}$ be a $*$-algebra. A *representation* $(\pi, \mathcal{H})$ *of* $\mathcal{A}$ is a homomorphism $\pi \colon \mathcal{A} \to B(\mathcal{H})$ of $*$-algebras, i.e., π is linear, multiplicative and satisfies $\pi(a^*) = \pi(a)^*$ for each $a \in \mathcal{A}$.

A representation $(\pi, \mathcal{H})$ is called

- *irreducible* if $\mathcal{H} \neq \{0\}$ and $\{0\}$ and $\mathcal{H}$ are the only closed $\pi(\mathcal{A})$-invariant subspaces of $\mathcal{H}$.
- *non-degenerate*, if $[\![\pi(\mathcal{A})\mathcal{H}]\!] = \mathcal{H}$, i.e., if $\pi(\mathcal{A})\mathcal{H}$ spans a dense subspace. Note that this condition is trivially satisfied if $\mathbf{1} \in \mathcal{A}$ and $\pi(\mathbf{1}) = \mathbf{1}$.
- *cyclic* if there exists a unit vector $\Omega \in \mathcal{H}$ for which $\pi(\mathcal{A})\Omega$ is dense in $\mathcal{H}$. Then Ω is called a *cyclic vector*. To specify the cyclic vector in the notation, we often write cyclic representations as triples $(\pi, \mathcal{H}, \Omega)$.

Definition 3.2. (a) If $(\pi, \mathcal{H})$ is a representation of $\mathcal{A}$ and $\mathcal{K} \subseteq \mathcal{H}$ a closed $\pi(\mathcal{A})$-invariant subspace, then $\rho(a) := \pi(a)|_{\mathcal{K}}^{\mathcal{K}}$ defines a representation $(\rho, \mathcal{K})$ which is called a *subrepresentation of* $(\pi, \mathcal{H})$.

(b) If $(\pi, \mathcal{H})$ and $(\rho, \mathcal{K})$ are representations of $\mathcal{A}$, then a bounded operator $T \colon \mathcal{K} \to \mathcal{H}$ satisfying

$$T \circ \rho(a) = \pi(a) \circ T \quad \text{for all} \quad a \in \mathcal{A}$$

is called an *intertwining operator*. We write

$$\mathrm{Hom}((\rho, \mathcal{K}), (\pi, \mathcal{H}))$$

for the set of all intertwining operators. It is a closed subspace of the Banach space $B(\mathcal{K}, \mathcal{H})$ (Exercise 3.7). We also put

$$\mathrm{End}(\pi, \mathcal{H}) := \mathrm{Hom}((\pi, \mathcal{H}), (\pi, \mathcal{H})).$$

(c) Two representations $(\pi, \mathcal{H})$ and $(\rho, \mathcal{K})$ of $\mathcal{A}$ are said to be *equivalent*, written $\pi \simeq \rho$ if there exists a *unitary* intertwining operator $T: \mathcal{K} \to \mathcal{H}$. It is easy to see that this defines indeed an equivalence relation on the class of all representations. We write $[\pi]$ for the equivalence class of the representation $(\pi, \mathcal{H})$.

Two cyclic representation $(\pi, \mathcal{H}, \Omega)$ and $(\pi', \mathcal{H}', \Omega')$ are said to be *equivalent* if there exists a unitary intertwining operator $U: \mathcal{H} \to \mathcal{H}'$ with $U\Omega = \Omega'$.

(d) We write

- $\rho \leq \pi$ if ρ is equivalent to a subrepresentation of π.
- $\pi \perp \rho$ if no subrepresentation of $(\pi, \mathcal{H})$ is equivalent to a subrepresentation of $(\rho, \mathcal{K})$. Then π and ρ are then called *disjoint*.
- $\rho \prec \pi$ if no subrepresentation of ρ is disjoint from π.
- $\pi \sim \rho$ if $\pi \prec \rho$ and $\rho \prec \pi$ (π and ρ are *quasi-equivalent*).

3.2 Non-degenerate and cyclic representations

We start with an easy observation on invariant subspaces:

Lemma 3.3. *Let $\mathcal{K} \subseteq \mathcal{H}$ be a closed subspace, $P \in B(\mathcal{H})$ be the orthogonal projection on $\mathcal{K}$ and $\mathcal{S} \subseteq B(\mathcal{H})$ be a $*$-invariant subset. Then the following are equivalent*

(i) *$\mathcal{K}$ is $\mathcal{S}$-invariant.*

(ii) *$\mathcal{K}^\perp$ is $\mathcal{S}$-invariant.*

(iii) *P commutes with $\mathcal{S}$.*

Proof. (i) $\Rightarrow$ (ii): If $w \in \mathcal{K}^\perp$ and $v \in \mathcal{K}$, we have for any $S \in \mathcal{S}$ the relation $\langle Sw, v\rangle = \langle w, S^*v\rangle = 0$ because $S^*v \in S\mathcal{K} \subseteq \mathcal{K}$.

(ii) $\Rightarrow$ (iii): The same argument as above implies that the invariance of $\mathcal{K}^\perp$ entails the invariance of $\mathcal{K} = (\mathcal{K}^\perp)^\perp$.

We write $v = v_0 + v_1$, according to the decomposition $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}^\perp$. Then we have for any $S \in \mathcal{S}$:

$$SPv = Sv_0 = PSv_0 = P(Sv_0 + Sv_1) = PSv,$$

so that P commutes with $\mathcal{S}$.

(iii) $\Rightarrow$ (i) follows from the fact that $\mathcal{K} = \ker(P - \mathbf{1})$ is an eigenspace of P, hence invariant under every operator commuting with P. $\square$

Lemma 3.4. (Characterization of non-degenerate representations) *For a representation $(\pi, \mathcal{H})$ of a $*$-subalgebra $\mathcal{A}$, the following are equivalent:*

(i) $(\pi, \mathcal{H})$ *is non-degenerate.*

(ii) $\pi(\mathcal{A})v = \{0\}$ *implies* $v = 0$.

(iii) *For every* $v \in \mathcal{H}$*, we have* $v \in \overline{\pi(\mathcal{A})v}$.

Proof. (i) $\Rightarrow$ (ii): Let $v \in \mathcal{H}$ with $\pi(\mathcal{A})v = \{0\}$. As $\pi(\mathcal{A})$ is $*$-invariant, it follows that $\langle v, \pi(\mathcal{A})\mathcal{H}\rangle = \langle \pi(\mathcal{A})v, \mathcal{H}\rangle = \{0\}$, so that $v \in [\![\pi(\mathcal{A})\mathcal{H}]\!]^\perp = \{0\}$.

(ii) $\Rightarrow$ (iii): We write $v = v_0 + v_1$ with $v_0 \in \big(\pi(\mathcal{A})v\big)^\perp$ and $v_1 \in [\![\pi(\mathcal{A})v]\!]$, according to the orthogonal decomposition $\mathcal{H} = [\![\pi(\mathcal{A})v]\!] \oplus (\pi(\mathcal{A})v)^\perp$. Then the invariance of $[\![\pi(\mathcal{A})v]\!]$ under $\pi(\mathcal{A})$ implies that $\pi(\mathcal{A})v_1 \subseteq [\![\pi(\mathcal{A})v]\!]$ and therefore

$$\pi(\mathcal{A})v_0 = \pi(\mathcal{A})(v - v_1) \subseteq \pi(\mathcal{A})v + \pi(\mathcal{A})v_1 \subseteq [\![\pi(\mathcal{A})v]\!].$$

By Lemma 3.3, the subspace $(\pi(\mathcal{A})v)^\perp$ is also $\pi(\mathcal{A})$-invariant, hence $\pi(\mathcal{A})v_0 \subseteq (\pi(\mathcal{A})v)^\perp$. This shows that

$$\pi(\mathcal{A})v_0 \subseteq (\pi(\mathcal{A})v)^\perp \cap [\![\pi(\mathcal{A})v]\!] = \{0\}.$$

Hence (ii) implies $v_0 = 0$ and thus $v = v_1 \in [\![\pi(\mathcal{A})v]\!]$.

(iii) $\Rightarrow$ (i) is trivial. $\square$

To understand the decomposition of representations into smaller pieces, we also need infinite "direct sums" of representations, hence the concept of a direct sum of Hilbert spaces which in turn requires the somewhat subtle concept of summability in Banach spaces (Subsection A.5).

Definition 3.5. For a family of $(\mathcal{H}_j)_{j\in J}$ of Hilbert spaces, we define

$$\widehat{\bigoplus_{j\in J}}\mathcal{H}_j := \Big\{(x_j)_{j\in J} \in \prod_{j\in J}\mathcal{H}_j \colon \sum_{j\in J}\|x_j\|^2 < \infty\Big\}$$

with the scalar product

$$\langle x, y\rangle = \sum_{j\in J}\langle x_j, y_j\rangle \quad \text{and} \quad \|x\|^2 = \sum_{j\in J}\|x_j\|^2$$

(Lemma A.29). We call this space the *Hilbert space direct sum* of the spaces $(\mathcal{H}_j)_{j\in J}$. This space is larger than the direct vector space sum of the $\mathcal{H}_j$, which is a dense subspace of $\widehat{\bigoplus}_{j\in J}\mathcal{H}_j$ (Exercise 3.5). In the following we always identify $\mathcal{H}_i$ with the subspace

$$\mathcal{H}_i \cong \{(x_j)_{j\in J} \colon (\forall j \neq i)\ x_j = 0\}.$$

Note that the requirement that $(\|x_j\|^2)_{j\in J}$ is summable implies in particular that, for each $x \in \mathcal{H}$, only countably many x_j are non-zero, even if J is uncountable (Example 1.8).

Example 3.6. (a) If $\mathcal{H}_j = \mathbb{C}$ for each $j \in J$, we also write

$$\ell^2(J, \mathbb{C}) := \widehat{\bigoplus}_{j\in J}\mathbb{C} = \Big\{(x_j)_{j\in J} \in \mathbb{C}^J \colon \sum_{j\in J}|x_j|^2 < \infty\Big\}.$$

On this space we have

$$\langle x, y\rangle = \sum_{j\in J}\overline{x_j}y_j \quad \text{and} \quad \|x\|^2 = \sum_{j\in J}|x_j|^2.$$

For $\mathbf{n} = \{1, \ldots, n\}$, we obtain in particular the Hilbert space $\mathbb{C}^n \cong \ell^2(\mathbf{n}, \mathbb{C})$.

(b) If all Hilbert spaces $\mathcal{H}_j = \mathcal{K}$ are equal, we put

$$\ell^2(J, \mathcal{K}) := \widehat{\bigoplus}_{j\in J} \mathcal{K} = \Big\{(x_j)_{j\in J} \in \mathcal{K}^J \colon \sum_{j\in J} \|x_j\|^2 < \infty\Big\}.$$

Proposition 3.7. (Existence of direct sums of representations) (a) *Let $(\pi_j, \mathcal{H}_j)_{j\in J}$ be a family of representation of the $*$-algebra $\mathcal{A}$, for which*

$$\sup_{j\in J} \|\pi_j(a)\| < \infty \quad \textit{for every} \quad a \in \mathcal{A}.$$

Then

$$\pi(a)(v_j)_{j\in J} := \big(\pi_j(a)v_j\big)_{j\in J}$$

defines on $\mathcal{H} := \widehat{\bigoplus}_{j\in J} \mathcal{H}_j$ a representation of $\mathcal{A}$. The representation $(\pi, \mathcal{H})$ is called the direct sum of the representations $(\pi_j)_{j\in J}$. It is also denoted $\pi = \sum_{j\in J} \pi_j$.

(b) *Let $(\pi, \mathcal{H})$ be a representation of the $*$-algebra $\mathcal{A}$ and $(\mathcal{H}_j)_{j\in J}$ be a mutually orthogonal family of $\pi(\mathcal{A})$-invariant closed subspaces of $\mathcal{H}$. Then $(\pi, \mathcal{H})$ is equivalent to the direct sum of the representations $(\pi_j, \mathcal{H}_j)_{j\in J}$.*

Proof. (a) For any $v = (v_j)_{j\in J} \in \mathcal{H}$, we have

$$\sum_{j\in J} \|\pi_j(a)v_j\|^2 \leq \sum_{j\in J} \|\pi_j(a)\|^2 \|v_j\|^2 \leq \Big(\sup_{j\in J} \|\pi_j(a)\|^2\Big) \sum_{j\in J} \|v_j\|^2 < \infty.$$

Therefore each $\pi(a)$ defines a bounded operator on $\mathcal{H}$ and we thus obtain a representation $(\pi, \mathcal{H})$ of $\mathcal{A}$.

(b) follows directly from Exercise 3.5 which ensures that the summation map

$$\bigoplus_{j\in J} \mathcal{H}_j \to \mathcal{H}, \quad (x_j)_{j\in J} \mapsto \sum_j x_j$$

extends to an isometry

$$\Phi \colon \widehat{\bigoplus}_{j\in J} \mathcal{H}_j \to \mathcal{H}, \quad \Phi(x) := \sum_{j\in J} x_j.$$

Since its range is dense and complete, it is also surjective. Moreover, $\pi(a)\Phi((x_j)) = \Phi((\pi_j(a)x_j))$ implies that Φ is an equivalence of representations. $\square$

Proposition 3.8. (Non-degenerate and cyclic representations) *A representation $(\pi, \mathcal{H})$ of an involutive algebra $\mathcal{A}$ is non-degenerate if and only if it is a direct sum of cyclic subrepresentations $(\pi_j, \mathcal{H}_j)_{j\in J}$.*

Proof. First we observe that every direct sum $\oplus_{j\in J}(\pi_j, \mathcal{H}_j)$ of cyclic representations $(\pi_j, \mathcal{H}_j, v_j)$ is non-degenerate because the subspace $\sum_{j\in J} \pi_j(\mathcal{A})v_j$ is dense in the Hilbert space direct sum $\widehat{\oplus}_{j\in J}\mathcal{H}_j$.

We now show the converse. So let $(\pi, \mathcal{H})$ be a non-degenerate representation. The proof is a typical application of Zorn's Lemma. We order the set $\mathcal{M}$ of all sets $\{\mathcal{H}_j \colon j \in J\}$ of mutually orthogonal closed $\pi(\mathcal{A})$-invariant subspaces on which the representation is cyclic by set inclusion. Each chain $\mathcal{K}$ in this ordered space has an upper bound given by the union $\bigcup \mathcal{K} \in \mathcal{M}$. Now Zorn's Lemma yields a maximal element $Z := \{\mathcal{H}_j : j \in J\}$ in $\mathcal{M}$.

Let $\mathcal{K} := \overline{\sum_{j\in J} \mathcal{H}_j}$. Since each $\mathcal{H}_j$ is $\mathcal{A}$-invariant and each $\pi(a)$ is continuous, $\mathcal{K}$ is also $\mathcal{A}$-invariant. In view of Lemma 3.3, the orthogonal complement $\mathcal{K}^\perp$ is also $\mathcal{A}$-invariant and $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}^\perp$. If $\mathcal{K}^\perp$ is non-zero, we pick $0 \neq v \in \mathcal{K}^\perp$. Then $\mathcal{C} := [\![\pi(A)v]\!]$ is a closed $\mathcal{A}$-invariant subspace containing v (Lemma 3.4). Hence the representation on $\mathcal{C}$ is cyclic. Therefore $\{\mathcal{C}\} \cup Z \in \mathcal{M}$. This contradicts the maximality of Z. We thus obtain $\mathcal{K}^\perp = \{0\}$, which proves that $\mathcal{K} = \mathcal{H}$. Now the assertion follows from Proposition 3.7(b). $\square$

3.3 The Gelfand–Naimark–Segal construction

Let $\mathcal{A}$ be a unital C^*-algebra and $(\pi, \mathcal{H})$ be a representation of $\mathcal{A}$ on $\mathcal{H}$. For every unit vector $\Omega \in \mathcal{H}$ we thus obtain a state by

$$\omega_\Omega(a) := \langle \Omega, \pi(a)\Omega\rangle$$

because $\omega_\Omega(\mathbf{1}) = \|\Omega\|^2 = 1$ and $\omega_\Omega(a^*a) = \|\pi(a)\Omega\|^2 \geq 0$ (cf. Lemma 2.25 and Example 2.26).

In this section we introduce the so-called Gelfand–Naimark–Segal (GNS) construction, which provides, conversely, for every state ω of a C^*-algebra $\mathcal{A}$, a cyclic representation $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$, which is unique up to unitary equivalence. This provides a representation without relying on a concrete model of the Hilbert space or the scalar product. The correspondence between cyclic representations and states is one of the cornerstones of the representation theory of C^*-algebras and has numerous applications in other fields of mathematics, such as harmonic analysis. Its power stems from the fact that it connects two fields of mathematics: the representation theory of $\mathcal{A}$ and the convex geometry of the compact convex state space $\mathfrak{S}(\mathcal{A})$. This permits us to translate problems in representation theory into convex geometry.

We now turn to the GNS construction.

Theorem 3.9. (Gelfand–Naimark–Segal (GNS) Theorem)*For every state $\omega \in \mathfrak{S}(\mathcal{A})$, there exists a cyclic representation $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ of $\mathcal{A}$ with*

$$\omega(a) = \langle \Omega_\omega, \pi_\omega(a)\Omega_\omega\rangle \quad \text{for} \quad a \in \mathcal{A}.$$

If, conversely, $(\pi, \mathcal{H}, \Omega)$ is a cyclic representation with

$$\omega(a) = \langle \Omega, \pi(a)\Omega\rangle \quad \text{for every} \quad a \in \mathcal{A}, \tag{15}$$

then there exists a unique unitary intertwining operator $U : \mathcal{H} \to \mathcal{H}_\omega$ with $U\Omega = \Omega_\omega$, i.e., an equivalence of cyclic representations $(\pi, \mathcal{H}, \Omega) \to (\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$.

Proof. **Existence:** For $\varphi \in \mathcal{A}'$ and $a \in \mathcal{A}$, we define the right translates $R_a\varphi \in \mathcal{A}'$ by $(R_a\varphi)(b) := \varphi(ba)$. We consider the subspace

$$\mathcal{D} := R_{\mathcal{A}}\omega = \{R_a\omega : a \in \mathcal{A}\} \subseteq \mathcal{A}'.$$

On $\mathcal{A}$ we consider the positive semidefinite hermitian form

$$\beta(a, b) := \omega(a^*b)$$

and observe that $\beta(a,a) = 0$ if and only if $\omega(a^*a) = 0$, which, by the Cauchy–Schwarz inequality (cf. Lemma 2.18(i)), is equivalent to $\omega(b^*a) = 0$ for all $b \in \mathcal{A}$, i.e., to $R_a\omega = 0$. Therefore we obtain on $\mathcal{D}$ a well-defined positive definite hermitian inner product

$$\langle R_a\omega, R_b\omega\rangle := \omega(a^*b) = (R_b\omega)(a^*). \tag{16}$$

Therefore $(\mathcal{D}, \langle\cdot,\cdot\rangle)$ is a pre-Hilbert space.

Since $R_aR_b = R_{ab}$ on $\mathcal{A}'$, we obtain by restriction linear maps $R_a\colon \mathcal{D} \to \mathcal{D}$. Then

$$\|R_aR_b\omega\|^2 = \|R_{ab}\omega\|^2 = \omega(b^*a^*ab) \le \omega(b^*b)\|a\|^2 = \|R_b\omega\|^2\|a\|^2$$

by Lemma 2.18(iii). Therefore the linear operator $R_a\colon \mathcal{D} \to \mathcal{D}$ extends to a bounded linear operator $\pi_\omega(a)$ on $\mathcal{H}_\omega$ with $\|\pi_\omega(a)\| \le \|a\|$. For $a, b, c \in \mathcal{A}$, we then have

$$\langle R_b\omega, \pi_\omega(a)R_c\omega\rangle = \langle R_b\omega, R_{ac}\omega\rangle = \omega(b^*ac) = \langle R_{a^*b}\omega, R_c\omega\rangle = \langle \pi_\omega(a^*)R_b\omega, R_c\omega\rangle.$$

We conclude that $\pi_\omega(a^*) = \pi_\omega(a)^*$. Further,

$$\pi_\omega(a)\pi_\omega(b)R_c\omega = R_{abc}\omega = \pi_\omega(ab)R_c\omega$$

yields $\pi_\omega(a)\pi_\omega(b) = \pi_\omega(ab)$. We thus obtain a cyclic representation $(\pi_\omega, \mathcal{H}_\omega, \omega)$ of $\mathcal{A}$ because the element ω is cyclic by construction. Finally, we note that

$$\langle \omega, \pi_\omega(a)\omega\rangle = \langle \omega, R_a\omega\rangle = \omega(a) \quad \text{for} \quad a \in \mathcal{A}.$$

Uniqueness: Now let $(\pi, \mathcal{H}, \Omega)$ be any cyclic representation satisfying (15). Then

$$\langle \pi(b)\Omega, \pi(a)\Omega\rangle = \langle \Omega, \pi(b^*a)\Omega\rangle = \omega(b^*a) = \langle \pi_\omega(b)\Omega_\omega, \pi_\omega(a)\Omega_\omega\rangle$$

implies the existence of complex linear isometry

$$U_0\colon \pi(\mathcal{A})\Omega \to \pi_\omega(\mathcal{A})\Omega_\omega \quad \text{with} \quad U_0(\pi(a)\Omega) = \pi_\omega(a)\Omega_\omega.$$

As $\pi(\mathcal{A})\Omega$ is dense in $\mathcal{H}$ and $\pi_\omega(\mathcal{A})\Omega_\omega$ is dense in $\mathcal{H}_\omega$, we obtain by continuous extension of U_0 a unitary operator $U\colon \mathcal{H} \to \mathcal{H}_\omega$. By construction, we then have $U\Omega = \Omega_\omega$ and $U \circ \pi(a) = \pi_\omega(a) \circ U$ for every $a \in \mathcal{A}$. These two conditions determine U uniquely. □

Remark 3.10. Sometimes it is convenient to have a more concrete picture of the space $\mathcal{H}_\omega$. As $\mathcal{D}$ is dense in $\mathcal{H}_\omega$, we have an injective linear map

$$\Phi\colon \mathcal{H}_\omega \to \mathcal{A}', \quad \Phi(\xi)(b) := \langle \omega, \pi(b)\xi\rangle = \langle R_{b^*}\omega, \xi\rangle$$

(cf. Exercise 3.13). This map satisfies

$$\Phi(R_a\omega)(b) = \langle \omega, R_bR_a\omega\rangle = \langle \omega, R_{ba}\omega\rangle = \omega(ba) = (R_a\omega)(b),$$

so that

$$\Phi(R_a\omega) = R_a\omega.$$

We may therefore identify $\mathcal{H}_\omega$ with a linear subspace of the dual space $\mathcal{A}'$.

Example 3.11. It is instructive to consider some examples.

(a) We first consider the commutative case $\mathcal{A} = C(X)$, where X is a compact space. For $\omega \in \mathfrak{S}(C(X))$, let μ_ω denote the corresponding Radon probability measure on X (Theorem 2.20, Example 2.26). On $L^2(X, \mu_\omega)$ we then have a representation π of $C(X)$, defined by the multiplication operators.

$$\pi(f)h := M_f h := fh$$

(Exercise 3.8). As $C(X)$ is dense in $L^2(X, \mu_\omega)$ (Proposition 2.21), the constant function $\Omega = 1$ is a cyclic vector and

$$\langle \Omega, M_f \Omega \rangle = \int_X f\, d\mu_\omega = \omega(f).$$

We further note that

$$\mathrm{End}(\pi, L^2(X, \mu_\omega)) = \{M_f \colon f \in L^\infty(X, \mu_\omega)\} \cong L^\infty(X, \mu_\omega)$$

(b) Let $\mathcal{H}$ be a Hilbert space and $\Omega \in \mathcal{H}$ be a unit vector. Then the representation of $\mathcal{A} = B(\mathcal{H})$ on $\mathcal{H}$ is cyclic and $\omega(A) := \langle \Omega, A\Omega \rangle$ is a corresponding state.

(c) Let $\mathcal{A} = B(\mathbb{C}^n)$ and $\omega(A) = \frac{1}{n} \operatorname{tr} A$. Then ω is a state of $\mathcal{A}$. The corresponding Hilbert space is $\mathcal{H} := B(\mathbb{C}^n)$ with the Hilbert–Schmidt scalar product

$$\langle A, B \rangle = \operatorname{tr}(A^*B) \quad \text{and} \quad \pi(A)B = AB.$$

The cyclic unit vector is $\Omega = \frac{1}{\sqrt{n}}\mathbf{1}$ because

$$\langle \Omega, \pi(A)\Omega \rangle = \frac{1}{n}\langle \mathbf{1}, A \rangle = \frac{1}{n} \operatorname{tr} A = \omega(A).$$

(Exercise 2.25).

Combing the preceding theorem with Example 1.7(b), it follows that C^*-algebras are precisely the closed $*$-subalgebras of $B(\mathcal{H})$ (with respect to the norm topology).

Theorem 3.12. (Gelfand–Naimark Theorem) *Every C^*-algebra $\mathcal{A}$ is isomorphic to a closed subalgebra of some $B(\mathcal{H})$.*

Proof. In view of Proposition 1.14, we may w.l.o.g. assume that $\mathcal{A}$ is unital. For every state $\omega \in \mathfrak{S}(\mathcal{A})$, we have the cyclic GNS representation $(\pi_\omega, \mathcal{H}_\omega)$ from Theorem 3.9. We now consider the direct sum representation

$$\pi := \widehat{\bigoplus}_{\omega \in \mathfrak{S}(\mathcal{A})} \pi_\omega \quad \text{on} \quad \widehat{\bigoplus}_{\omega \in \mathfrak{S}(\mathcal{A})} \mathcal{H}_\omega$$

(Proposition 3.7). This representation exists because

$$\sup\{\|\pi_\omega(A)\| \colon \omega \in \mathfrak{S}(\mathcal{A})\} \leq \|A\| \quad \text{for} \quad A \in \mathcal{A}$$

(Proposition 1.18). Pick $A \in \mathcal{A}$. It remains to show that $\|\pi(A)\| = \|A\|$ for every $A \in \mathcal{A}$. Since $\|A\|^2 = \|A^*A\| \in \sigma(A^*A)$ (Exercise 2.30), Proposition 2.27 implies the existence of a state $\omega \in \mathfrak{S}(\mathcal{A})$ with $\omega(A^*A) = \|A^*A\| = \|A\|^2$. Then

$$\|\pi_\omega(A)\|^2 \geq \|\pi_\omega(A)\Omega_\omega\|^2 = \omega(A^*A) = \|A\|^2$$

implies that $\|\pi(A)\| \geq \|\pi_\omega(A)\| \geq \|A\| \geq \|\pi(A)\|$. This shows that π is isometric. Hence $\mathrm{im}(\pi)$ is complete and therefore closed. ◻

3.4 Irreducible representations and Schur's Lemma

A key result in this context is Schur's Lemma, asserting that $\mathrm{End}(\pi, \mathcal{H}) = \mathbb{C}\mathbf{1}$ if and only if $(\pi, \mathcal{H})$ is irreducible.

Theorem 3.13. (Schur's Lemma) *A representation $(\pi, \mathcal{H})$ of an involutive algebra $\mathcal{A}$ is irreducible if and only if* $\mathrm{End}(\pi, \mathcal{H}) = \mathbb{C}\mathbf{1}$.

Proof. If $(\pi, \mathcal{H})$ is not irreducible and $\mathcal{K} \subseteq \mathcal{H}$ is a proper closed invariant subspace, then the orthogonal projection P onto $\mathcal{K}$ commutes with $\pi(\mathcal{A})$ (Lemma 3.3) and $P \notin \mathbb{C}\mathbf{1}$. Therefore $\mathrm{End}(\pi, \mathcal{H}) \neq \mathbb{C}\mathbf{1}$ if π is not irreducible.

Suppose, conversely, that $\mathrm{End}(\pi, \mathcal{H}) \neq \mathbb{C}\mathbf{1}$. Then Corollary 2.3 applies to the C^*-algebra $\mathrm{End}(\pi, \mathcal{H})$ (Exercise 3.7), so that there exist non-zero commuting $A, B \in \mathrm{End}(\pi, \mathcal{H})$ with $AB = 0$. Then $\mathcal{K} := \overline{A(\mathcal{H})}$ is a non-zero closed subspace invariant under $\pi(\mathcal{A})$ and satisfying $B\mathcal{K} = \{0\}$. Therefore $(\pi, \mathcal{H})$ is not irreducible. $\square$

Corollary 3.14. *Every irreducible representation $(\pi, \mathcal{H})$ of a commutative involutive algebra $\mathcal{A}$ is one-dimensional.*

Proof. If $\mathcal{A}$ is commutative, then $\pi(\mathcal{A}) \subseteq \mathrm{End}(\pi, \mathcal{H})$. If $(\pi, \mathcal{H})$ is irreducible, then

$$\mathrm{End}(\pi, \mathcal{H}) = \mathbb{C}\mathbf{1}$$

by Schur's Lemma, and therefore $\pi(\mathcal{A}) \subseteq \mathbb{C}\mathbf{1}$, so that the irreducibility further implies $\dim \mathcal{H} = 1$. $\square$

Corollary 3.15. *Suppose that $(\pi, \mathcal{H})$ is an irreducible representation of the involutive algebra $\mathcal{A}$ and $(\rho, \mathcal{K})$ any representation of $\mathcal{A}$.*

(a) *If* $\mathrm{Hom}((\pi, \mathcal{H}), (\rho, \mathcal{K})) \neq \{0\}$*, then $(\pi, \mathcal{H})$ is equivalent to a subrepresentation of $(\rho, \mathcal{K})$.*

(b) $\mathrm{Hom}((\pi, \mathcal{H}), (\rho, \mathcal{K})) = \{0\}$ *if $(\rho, \mathcal{K})$ is also irreducible and not equivalent to $(\pi, \mathcal{H})$.*

Proof. (a) Let $A \in \mathrm{Hom}((\pi, \mathcal{H}), (\rho, \mathcal{K}))$ be a non-zero intertwining operator. Then $A^*A \in \mathrm{End}(\pi, \mathcal{H}) = \mathbb{C}\mathbf{1}$ by Schur's Lemma. Further

$$\langle v, A^*Av \rangle = \|Av\|^2 \geq 0 \quad \text{for} \quad v \in \mathcal{H}$$

implies that $A^*A = \lambda\mathbf{1}$ for some $\lambda > 0$ (because $A \neq 0$). Then $B := \lambda^{-1/2}A$ is another intertwining operator with $B^*B = \mathbf{1}$. Hence $B \colon \mathcal{H} \to \mathcal{K}$ is an isometric embedding. In particular, its image $\mathcal{K}_0$ is a closed non-zero invariant subspace on which the representation induced by ρ is equivalent to $(\pi, \mathcal{H})$.

(b) If $(\rho, \mathcal{K})$ is also irreducible and $A \neq 0$, the preceding argument shows that $\rho \cong \pi$. $\square$

Corollary 3.16. *If $(\pi, \mathcal{H})$ is a representation of an involutive algebra $\mathcal{A}$ and $(\pi_j, \mathcal{H}_j)_{j=1,2}$ are non-equivalent irreducible subrepresentations, then $\mathcal{H}_1 \perp \mathcal{H}_2$.*

Proof. Let $P \colon \mathcal{H} \to \mathcal{H}_1$ denote the orthogonal projection onto $\mathcal{H}_1$. Since $\mathcal{H}_1$ is invariant under $\pi(\mathcal{A})$, Lemma 3.3 implies that $P \in \mathrm{Hom}((\pi, \mathcal{H}), (\pi_1, \mathcal{H}_1))$. Hence

$$P|_{\mathcal{H}_2} \in \mathrm{Hom}((\pi_2, \mathcal{H}_2), (\pi_1, \mathcal{H}_1)) = \{0\}$$

by Corollary 3.15. This means that $\mathcal{H}_1 \perp \mathcal{H}_2$. $\square$

3.5 Pure states

We have already seen that the GNS construction provides a natural correspondence between states and cyclic representations. As the state space $\mathfrak{S}(\mathcal{A})$ is a convex set, it is natural to ask which representation theoretic property corresponds to a state being an extreme point, i.e., a so-called pure state. In this subsection we show that the pure states correspond to irreducible representations.

Definition 3.17. Let C be a convex subset of the real vector space V. We call $x \in C$ an *extreme point* if

$$x = \lambda y + (1-\lambda)z, \qquad x, z \in C, 0 < \lambda < 1 \quad \text{implies} \quad y = z = x.$$

This means that x does not lie in the interior of a proper line segment generated by two different points of C. It is also equivalent to the convexity of the complement $C \setminus \{x\}$. We write $\mathrm{Ext}(C)$ for the set of extreme points of C.

Definition 3.18. A state $\omega \in \mathfrak{S}(\mathcal{A})$ of a C^*-algebra $\mathcal{A}$ is called *pure* if ω is an extreme point of the convex set $\mathfrak{S}(\mathcal{A})$ of all states. We write

$$\mathfrak{S}_p(\mathcal{A}) = \mathrm{Ext}(\mathfrak{S}(\mathcal{A}))$$

for the set of pure states of $\mathcal{A}$.

Theorem 3.19. (Pure State Theorem) *Let $\mathcal{A}$ be a unital C^*-algebra. A cyclic representation $(\pi, \mathcal{H}, \Omega)$ of $\mathcal{A}$ is irreducible if and only if the state defined by $\omega(a) := \langle \Omega, \pi(a)\Omega\rangle$ is pure.*

Proof. Suppose first that $(\pi, \mathcal{H}, \Omega)$ is not irreducible. Then there exists a non-trivial orthogonal $\pi(\mathcal{A})$-invariant decomposition

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$$

(Lemma 3.3). Since Ω is cyclic in $\mathcal{H}$, it is contained neither in $\mathcal{H}_1$ nor in $\mathcal{H}_2$. Accordingly, $\Omega = \Omega_1 + \Omega_2$ with non-zero vectors $\Omega_j \in \mathcal{H}_j$, and we write

$$\omega_j(a) := \frac{1}{\|\Omega_j\|^2}\langle \Omega_j, \pi(a)\Omega_j\rangle = \langle \widetilde{\Omega}_j, \pi(a)\widetilde{\Omega}_j\rangle \quad \text{for} \quad \widetilde{\Omega}_j := \frac{1}{\|\Omega_j\|}\Omega_j$$

for the corresponding states. Then

$$\omega(a) = \langle \Omega, \pi(a)\Omega\rangle = \langle \Omega_1, \pi(a)\Omega_1\rangle + \langle \Omega_2, \pi(a)\Omega_2\rangle = \|\Omega_1\|^2\omega_1(a) + \|\Omega_2\|^2\omega_2(a)$$

shows that

$$\omega = \|\Omega_1\|^2\omega_1 + \|\Omega_2\|^2\omega_2 \quad \text{with} \quad 1 = \|\Omega\|^2 = \|\Omega_1\|^2 + \|\Omega_2\|^2.$$

To verify that ω is not extreme, it remains to show that we do not have $\omega_1 = \omega_2 = \omega$. So let us assume that this is the case. Then the GNS Theorem implies that $(\pi_j, \mathcal{H}_j, \widetilde{\Omega}_j) \cong (\pi, \mathcal{H}, \Omega)$ for $j = 1, 2$ and it follows that in the direct sum representation

$$(\pi^{\oplus 2}, \mathcal{H}^{\oplus 2}) \cong (\pi_1, \mathcal{H}_1) \oplus (\pi_2, \mathcal{H}_2) \cong (\pi, \mathcal{H}),$$

the vector $(\|\Omega_1\|^2\Omega, \|\Omega_2\|^2\Omega)$ is cyclic. This contradicts the fact that it is contained in the closed invariant diagonal subspace $\mathcal{K} := \{(\|\Omega_1\|^2 v, \|\Omega_2\|^2 v) \colon v \in \mathcal{H}\}$.

Now we assume that $(\pi, \mathcal{H}, \Omega)$ is irreducible. Let $0 < \lambda_j < 1$ with $\lambda_1 + \lambda_2 = 1$ and $\omega_1, \omega_2 \in \mathfrak{S}(\mathcal{A})$ with

$$\omega = \lambda_1\omega_1 + \lambda_2\omega_2.$$

We consider the cyclic representations $(\pi_j, \mathcal{H}_j, \Omega_j)$ corresponding to the states ω_j, $j = 1, 2$. In the direct sum of these two representations, we consider the unit vector

$$\Omega_0 := (\sqrt{\lambda_1}\Omega_1, \sqrt{\lambda_2}\Omega_2) \in \mathcal{H}_1 \oplus \mathcal{H}_2$$

and observe that, for $a \in \mathcal{A}$, we have

$$\begin{aligned}\langle \Omega_0, (\pi_1 \oplus \pi_2)(a)\Omega_0\rangle &= \langle \sqrt{\lambda_1}\Omega_1, \sqrt{\lambda_1}\pi_1(a)\Omega_1\rangle + \langle \sqrt{\lambda_2}\Omega_2, \sqrt{\lambda_2}\pi_2(a)\Omega_2 1\rangle \\ &= \lambda_1\omega_1(a) + \lambda_2\omega_2(a) = \omega(a).\end{aligned}$$

Therefore the GNS Theorem implies that the cyclic subrepresentation $(\pi_0, \mathcal{H}_0, \Omega_0)$ of $\mathcal{H}_1 \oplus \mathcal{H}_2$ generated by Ω_0 is equivalent to $(\pi, \mathcal{H}, \Omega)$, hence irreducible. By construction, the projection

$$\Psi \colon \mathcal{H}_0 \to \mathcal{H}_1, \quad (\xi_1, \xi_2) \mapsto \frac{1}{\sqrt{\lambda_1}}\xi_1$$

is a bounded $\mathcal{A}$-intertwining operator with $\Psi(\Omega_0) = \Omega_1$. Schur's Lemma (Theorem 3.13) implies that the intertwining operator $\Psi^*\Psi \in \mathrm{End}(\pi_0, \mathcal{H}_0) = \mathbb{C}\mathbf{1}$ is a multiple of the identity. Therefore

$$\langle \Omega_0, \Psi^*\Psi\Omega_0\rangle = \|\Psi(\Omega_0)\|^2 = \|\Omega_1\|^2 = 1$$

yields $\Psi^*\Psi = \mathbf{1}$. Hence Ψ is isometric. This implies that, for $a \in \mathcal{A}$, we have

$$\omega_1(a) = \langle \Omega_1, \pi_1(a)\Omega_1\rangle = \langle \Psi(\Omega_0), \Psi\big(\pi_0(a)\Omega_0\big)\rangle = \langle \Omega_0, \pi_0(a)\Omega_0\rangle = \omega(a).$$

We conclude that $\omega_1 = \omega$, and hence also that $\omega_2 = \omega$. This shows that the state ω is pure if the corresponding cyclic representation is irreducible. $\square$

Corollary 3.20. *For a commutative unital C^*-algebra $\mathcal{A}$ the set of pure states coincides with the set $\widehat{\mathcal{A}}$ of unital characters:*

$$\mathrm{Ext}(\mathfrak{S}(\mathcal{A})) = \mathrm{Hom}_1(\mathcal{A}, \mathbb{C}) = \widehat{\mathcal{A}}.$$

Proof. According to Theorem 3.19, a state ω is pure if and only if the corresponding cyclic representation $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ is irreducible. Since $\mathcal{A}$ is commutative, this is equivalent to $\dim \mathcal{H}_\omega = 1$ (Corollary 3.14), and then $\pi(a) = \omega(a)\mathbf{1}$ shows that $\omega \in \widehat{\mathcal{A}}$. Conversely, every character defines a one-dimensional representation which is obviously irreducible. $\square$

Remark 3.21. For a compact space X and $\mathcal{A} = C(X)$, we know that the states of $\mathcal{A}$ correspond to regular probability measures μ on X (Example 2.26) and the elements of $\widehat{\mathcal{A}}$ correspond to the Dirac measures. Therefore the preceding theorem means that the extreme points in the convex set of regular probability measures on X are the Dirac measures.

Theorem 3.22. (Gelfand–Raïkov Theorem) *For every C^*-algebra $\mathcal{A}$, the irreducible representations separate the points, i.e., for every $0 \neq a \in \mathcal{A}$ there exists an irreducible representation $(\pi, \mathcal{H})$ with $\pi(a) \neq 0$.*

Proof. We may w.l.o.g. assume that $\mathcal{A}$ is unital (Proposition 1.14). Fix $0 \neq a \in \mathcal{A}$ and note that this implies that $a^*a \neq 0$. With Proposition 2.27 we now find a state $\omega \in \mathfrak{S}(\mathcal{A})$ with $\omega(a^*a) > 0$.

Now we recall the Krein–Milman Theorem which implies that the weak-$*$-compact convex set $\mathfrak{S}(\mathcal{A})$ is the closed convex hull of its extreme points, i.e., the pure states. Hence there even exists a pure state ω with $\omega(a^*a) > 0$. Then the corresponding GNS representation π_ω is irreducible with $\pi_\omega(a) \neq 0$ because $\omega(a^*a) = \|\pi_\omega(a)\Omega\|^2 > 0$. □

The Gelfand–Raïkov Theorem is a remarkably strong result on representations of C^*-algebras. It is based on the Krein–Milman Theorem asserting that a compact convex set is always generated by its extreme points, which implies in particular that extreme points exist. One cannot expect the existence of a faithful irreducible representation because, if $\mathcal{A}$ is commutative, then all irreducible representations are one-dimensional. However, since we can form direct sums of representations, the Gelfand–Raïkov Theorem implies the Gelfand–Naimark Theorem.

Exercises for Section 3

Exercise 3.1. Let $b\colon V \times V \to \mathbb{C}$ be a sesquilinear form on the complex vector space V, i.e., b is linear in the second argument and antilinear in the first.

(i) Show that b satisfies the *polarization identity* which permits to recover all values of b from those on the diagonal:

$$b(x,y) = \frac{1}{4}\sum_{k=0}^{3} i^{-k} b(x + i^k y, x + i^k y).$$

As a consequence, every operator $A \in B(\mathcal{H})$, $\mathcal{H}$ a complex Hilbert space, is uniquely determined by the numbers $\langle v, Av\rangle$, $v \in \mathcal{H}$.

(ii) Show also that, if b is positive semidefinite, then it satisfies the *Cauchy–Schwarz inequality*:

$$|b(x,y)|^2 \leq b(x,x)b(y,y) \quad \text{for} \quad v, w \in V.$$

Exercise 3.2. (Direct sums of Hilbert spaces) For a family of $(\mathcal{H}_j)_{j\in J}$ of Hilbert spaces, we consider the set

$$\mathcal{H} := \widehat{\bigoplus_{j\in J}} \mathcal{H}_j := \Big\{(x_j)_{j\in J} \in \prod_{j\in J} \mathcal{H}_j \colon \sum_{j\in J} \|x_j\|^2 < \infty\Big\}.$$

Show that:

(i) $\mathcal{H}$ is a linear subspace of the linear space $\prod_{j\in J}\mathcal{H}_j$.
Hint: $\|x+y\|^2 \leq 2(\|x\|^2 + \|y\|^2)$ holds in every Hilbert space.

(ii) If $x = (x_j)_{j\in J} \in \mathcal{H}$, then the set of all j with $x_j \neq 0$ is countable.

(iii) For $x, y \in \mathcal{H}$, the absolutely convergent series $\langle x, y\rangle = \sum_{j\in J}\langle x_j, y_j\rangle$ defines a positive definite scalar product on $\mathcal{H}$ for which $\mathcal{H}$ becomes a Hilbert space.
Hint: $|\langle x, y\rangle| \leq \|x\|\|y\| \leq \frac{1}{2}(\|x\|^2 + \|y\|^2)$ holds in every Hilbert space.

(iv) For a family of bounded operators $A_j \in B(\mathcal{H}_j)$, there exist a bounded operator A on $\mathcal{H}$ with $(Ax)_j = A_j x_j$ for $x \in \mathcal{H}, j \in J$, if and only if $\sup_{j\in J}\|A_j\| < \infty$.

Exercise 3.3. Show that, for an orthogonal family $(x_j)_{j\in J}$ in the Hilbert space $\mathcal{H}$, the following are equivalent:

(i) $(x_j)_{j\in J}$ is summable in the following sense: There exists an $x \in \mathcal{H}$ such that, for every $\varepsilon > 0$, there exists a finite subset $J_\varepsilon \subseteq J$ with the property that, for every finite subset $F \supseteq J_\varepsilon$, we have $\big\|\sum_{j\in F} x_j - x\big\| < \varepsilon$.

(ii) $(\|x_j\|^2)_{j\in J}$ is summable in $\mathbb{R}$.

Hint: A series $\sum_{n=1}^\infty x_{j_n}$ converges in $\mathcal{H}$ if and only if the partial sums $s_N := \sum_{n=1}^N x_{j_n}$ form a Cauchy sequence.
Show further that, if this is the case, then $\big\|\sum_{j\in J} x_j\big\|^2 = \sum_{j\in J}\|x_j\|^2$ and the set $\{j \in J : x_j \neq 0\}$ is countable.

Exercise 3.4. Show that, for an orthonormal family $(x_j)_{j\in J}$ in the Hilbert space $\mathcal{H}$, the following assertions hold:

(a) $(\forall x \in \mathcal{H}) \sum_{j\in J} |\langle x_j, x\rangle|^2 \leq \|x\|^2$ (Bessel inequality) and only countably many x_j are non-zero.

(b) $x = \sum_{j\in J}\langle x, x_j\rangle x_j$ holds if and only if $\sum_{j\in J}|\langle x, x_j\rangle|^2 = \|x\|^2$ (Parseval equality).

(c) If $(x_n)_{n\in\mathbb{N}}$ is an orthonormal sequence in $\mathcal{H}$, then the series $\sum_{n=1}^\infty \frac{1}{n}x_n$ converges, but not absolutely.

Exercise 3.5. Let $(\mathcal{H}_j)_{j\in J}$ be an orthogonal family of closed subspaces of the Hilbert space $\mathcal{H}$. Show that, for each $x = (x_j)_{j\in J} \in \widehat{\bigoplus}\mathcal{H}_j$, the sum $\Phi(x) := \sum_{j\in J} x_j$ converges in $\mathcal{H}$ and that $\Phi \colon \widehat{\bigoplus}_{j\in J}\mathcal{H}_j \to \mathcal{H}, (x_j)_{j\in J} \mapsto \sum_{j\in J} x_j$ defines an isometric embedding (cf. Exercise 3.3).

Exercise 3.6. Let $A \colon \mathcal{H}_1 \to \mathcal{H}_2$ be an isometric linear map between two Hilbert spaces. Show that A is unitary if $A(\mathcal{H}_1)$ is dense in $\mathcal{H}_2$. Hint: Subsets of complete metric spaces are complete if and only if they are closed.

Exercise 3.7. (Intertwining operators) Let $(\pi_j, \mathcal{H}_j)$, $j = 1, 2, 3$ be representations of the $*$-algebra $\mathcal{A}$. Show that:

(i) Composition defines a map

$$\mathrm{Hom}((\pi_2, \mathcal{H}_2), (\pi_3, \mathcal{H}_3)) \times \mathrm{Hom}((\pi_1, \mathcal{H}_1), (\pi_2, \mathcal{H}_2)) \to \mathrm{Hom}((\pi_1, \mathcal{H}_1), (\pi_3, \mathcal{H}_3)).$$

(ii) $\mathrm{Hom}((\pi_1, \mathcal{H}_1), (\pi_2, \mathcal{H}_2))$ is a closed subspace of the Banach space $B(\mathcal{H}_1, \mathcal{H}_2)$.

(iii) $A \in \mathrm{Hom}((\pi_1, \mathcal{H}_1), (\pi_2, \mathcal{H}_2))$ implies $A^* \in \mathrm{Hom}((\pi_2, \mathcal{H}_2), (\pi_1, \mathcal{H}_1))$.

(iv) For every representation $(\pi, \mathcal{H})$ of $\mathcal{A}$, the space $\mathrm{End}(\pi, \mathcal{H})$ is a C^*-subalgebra of $B(\mathcal{H})$.

Exercise 3.8. Let $(X, \mathfrak{S}, \mu)$ be a finite measure space and $\mathcal{A} := L^\infty(X, \mathfrak{S}, \mu)$ the C^*-algebra of (equivalence classes of) essentially bounded measurable functions. Show that

$$\pi(f)h := fh$$

defines a $*$-representation of $\mathcal{A}$ on $L^2(X, \mu)$ for which the constant function $\Omega = 1$ is a cyclic vector and

$$\mathrm{End}(\pi, L^2(X, \mu)) = \pi(L^\infty(X, \mu)).$$

Exercise 3.9. (Block matrix picture of subrepresentations) Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ be an orthogonal direct sum of Hilbert spaces. We identify operators $A \in B(\mathcal{H})$ with (2×2)-matrices $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}$ with $a_{ij} \in B(\mathcal{H}_j, \mathcal{H}_i)$. Show that: For a representation $(\pi, \mathcal{H})$ of a $*$-algebra $\mathcal{A}$, the subspace $\mathcal{H}_1$ is invariant if and only if there exist representations $\pi_j \colon \mathcal{A} \to B(\mathcal{H}_j)$, $j = 1, 2$, such that $\pi(a) = \begin{pmatrix} \pi_1(a) & 0 \\ 0 & \pi_2(a)\end{pmatrix}$ for $a \in \mathcal{A}$.

Exercise 3.10. (Direct sums of cyclic representations) Let $(\omega_n)_{n\in\mathbb{N}}$ be a sequence of states of the unital C^*-algebra $\mathcal{A}$ and $\lambda_n \geq 0$ with $\sum_n \lambda_n = 1$. Show that the series $\omega := \sum_n \lambda_n\omega_n$ converges in the Banach space $\mathcal{A}'$, that $\omega \in \mathfrak{S}(\mathcal{A})$, and that the cyclic representation $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ is equivalent to a subrepresentation of the direct sum representation $\oplus_{n\in\mathbb{N}}(\pi_{\omega_n}, \mathcal{H}_{\omega_n})$.

Exercise 3.11. Show that every finite dimensional representation $(\pi, \mathcal{H})$ of an involutive algebra $\mathcal{A}$ is a direct sum of irreducible representations.

Exercise 3.12. Show that every representation $(\pi, \mathcal{H})$ of a finite dimensional $*$-algebra $\mathcal{A}$ is a direct sum of irreducible representations $(\pi_j, \mathcal{H}_j)_{j\in J}$ and that all irreducible representations of $\mathcal{A}$ are finite dimensional.

Exercise 3.13. Let $(\pi, \mathcal{H}, \Omega)$ be a cyclic representation of the $*$-algebra $\mathcal{A}$ and $\mathcal{A}^*$ the algebraic dual space of $\mathcal{A}$. Show that

$$\Phi\colon \mathcal{H} \to \mathcal{A}^*, \quad \Phi(\xi)(a) := \langle \Omega, \pi(a)\xi \rangle$$

defines an injective linear map satisfying

$$\Phi \circ \pi(a) = R_a \circ \Phi \quad \text{for} \quad a \in \mathcal{A} \quad \text{and} \quad (R_a\omega)(b) = \omega(ba).$$

Compare this realization of $(\pi, \mathcal{H})$ as a subspace of $\mathcal{A}^*$ with the GNS construction.

Exercise 3.14. Let $\mathcal{A}$ be a non-unital C^*-algebra and $\mathcal{A}_+$ be its unitization. We consider a state $\nu \in \mathfrak{S}(\mathcal{A}_+)$ and a corresponding cyclic GNS representation $(\pi, \mathcal{H}, \Omega)$ with $\nu(a) = \langle \Omega, \pi(a)\Omega \rangle$ for $a \in \mathcal{A}$. We further put $\omega := \nu|_{\mathcal{A}}$ and write ω_+ for the positive functional on $\mathcal{A}_+$ obtained by $\omega_+(\mathbf{1}) := \|\omega\|$. Show that:

(i) $\nu = \omega_+ + \nu'$, where ν' vanishes on $\mathcal{A}$ and satisfies $\nu'(\mathbf{1}) = 1 - \|\omega\| \geq 0$.

(ii) The following are equivalent:

 (a) The restriction $\pi|_{\mathcal{A}}$ is non-degenerate.

 (b) $\Omega \in \overline{\pi(\mathcal{A})\Omega}$.

 (c) Ω is cyclic for $\pi(\mathcal{A})$.

 (d) $\|\omega\| = 1$, i.e., $\omega_+ = \nu$.
 Hint: Show first the equivalence of (i)-(iii). Write $\Omega = \Omega_0 \oplus \Omega_1$ with $\Omega_0 \in \big(\pi(\mathcal{A})\Omega\big)^\perp$ and $\Omega_1 \in \overline{\pi(\mathcal{A})\Omega}$, so that $\pi(\mathcal{A})\Omega_0 = \{0\}$ (why?). Then $\|\omega\| \leq \|\Omega_1\|^2$ and $1 = \|\Omega_0\|^2 + \|\Omega_1\|^2$. Therefore $\|\omega\| = 1$ implies $\Omega_0 = 0$, hence (b), and, conversely, (a) implies $\Omega_0 = 0$ by Lemma 3.4.

Exercise 3.15. (Reduction to non-degenerate representations) Let $(\pi, \mathcal{H})$ be a representation of the $*$-algebra $\mathcal{A}$. Show that $(\pi, \mathcal{H}) \cong (\pi_0, \mathcal{H}_0) \oplus (\pi_1, \mathcal{H}_1)$, where $\pi_0(\mathcal{A}) = \{0\}$ (this representation is "totally degenerate") and π_1 is non-degenerate.

Exercise 3.16. Let $(\mathcal{H}_j)_{j\in J}$ be a family of Hilbert spaces and $A_j \in B(\mathcal{H}_j)$. Suppose that $\sup_{j\in J} \|A_j\| < \infty$. Then $A(x_j) := (A_j x_j)$ defines a bounded linear operator on $\widehat{\bigoplus}_{j\in J}\mathcal{H}_j$ with

$$\|A\| = \sup_{j\in J} \|A_j\|.$$

If, conversely, $\mathcal{H} = \widehat{\bigoplus}_{j\in J}\mathcal{H}_j$ is a Hilbert space direct sum and $A \in B(\mathcal{H})$ preserves each subspace $\mathcal{H}_j$, then the restrictions $A_j := A|_{\mathcal{H}_j}$ are bounded operators in $B(\mathcal{H}_j)$ satisfying $\|A\| = \sup_{j\in J} \|A_j\|$.

4 Von Neumann Algebras

A fundamental observation about sets of operators on Hilbert spaces is that, for a $*$-invariant subset $S \subseteq B(\mathcal{H})$, a closed subspace $\mathcal{K}$ of $\mathcal{H}$ is S-invariant if and only if the corresponding orthogonal projection $P_{\mathcal{K}}$ onto $\mathcal{K}$ belongs to the commutant

$$S' := \{A \in B(\mathcal{H})\colon (\forall s \in S)\, As = sA\}$$

(Lemma 3.3). For this reason, $*$-invariant subalgebras which are commutants are of particular importance; they are called von Neumann algebras. This is the reason for considering primarily $*$-invariant algebras of operators.

In this section we start with the description of some topologies on the algebra $B(\mathcal{H})$ of bounded operators on a complex Hilbert space which are weaker than the norm topology. In Subsection 4.2 we define von Neumann algebras as $*$-subalgebras $\mathcal{A} \subseteq B(\mathcal{H})$ which coincide with their bicommutant, i.e., $\mathcal{A} = \mathcal{A}''$.

An important difference between C^*- and von Neumann algebras is that every von Neumann algebra $\mathcal{M}$ is generated (as a von Neumann algebra) by the subset $P(\mathcal{M})$ of its projections (=hermitian idempotents). This is shown in Subsection 4.3, where we also explain the natural equivalence relation on $P(\mathcal{M})$ and how it can be used to distinguish three types I, II and III of simple von Neumann algebras (=factors). The structure of type I factors is completely determined in terms of tensor products in Subsection 4.4. We conclude this section with a construction of a factor of type II (Subsection 4.6).

4.1 Topologies on $B(\mathcal{H})$

In this subsection we define some topologies on the space $B(\mathcal{H})$ of all continuous operators which are weaker (coarser) than the norm topology. Accordingly, closedness in these weaker topologies is a stronger condition, so that one may expect it to have stronger impact on the structure of these algebras (cf. Theorem 4.13).

Definition 4.1. Let $\mathcal{H}$ be a Hilbert space. On $B(\mathcal{H})$ we define the *weak operator topology* τ_w as the coarsest topology for which all functions

$$f_{v,w} \colon B(\mathcal{H}) \to \mathbb{C}, \quad A \mapsto \langle v, Aw \rangle, \quad v, w \in \mathcal{H},$$

are continuous (cf. Appendix A.1 and Exercise 4.17). The closure of a subset $\mathcal{S} \subseteq B(\mathcal{H})$ in this topology is denoted $\overline{\mathcal{S}}^w$.

We define the *strong operator topology* τ_s on $B(\mathcal{H})$ as the coarsest topology for which all maps

$$B(\mathcal{H}) \to \mathcal{H}, \quad A \mapsto Av, \quad v \in \mathcal{H},$$

are continuous. This topology is also called the *topology of pointwise convergence*. The closure of a subset $\mathcal{S} \subseteq B(\mathcal{H})$ in this topology is denoted $\overline{\mathcal{S}}^s$.

For the closure in the norm topology, we write $\overline{\mathcal{S}}^n$.

Remark 4.2. (a) Since

$$|f_{v,w}(A) - f_{v,w}(B)| = |\langle v, (A-B)w \rangle| \leq \|(A-B)w\| \cdot \|v\| \leq \|A-B\|\|w\|\|v\|$$

by the Cauchy–Schwarz Inequality, the functions $f_{v,w}$ are continuous on $B(\mathcal{H})$ with respect to the strong operator topology. Therefore the weak operator topology is weaker (=coarser) than the strong one, which in turn in weaker than the norm topology. For a subset $\mathcal{S} \subseteq B(\mathcal{H})$, we therefore have the inclusions

$$\mathcal{S} \subseteq \overline{\mathcal{S}}^n \subseteq \overline{\mathcal{S}}^s \subseteq \overline{\mathcal{S}}^w. \tag{17}$$

(b) If $\dim \mathcal{H} < \infty$, then the norm topology, the strong and the weak operator topology coincide on $B(\mathcal{H})$. In fact, choosing an orthonormal basis $(e_1, \ldots, e_n)$ in $\mathcal{H}$, we represent $A \in$

$B(\mathcal{H})$ by the matrix $A = (a_{ij}) \in M_n(\mathbb{C})$, where $a_{ij} = \langle e_i, Ae_j\rangle = f_{e_i,e_j}(A)$. If $E_{ij} \in M_n(\mathbb{C})$ denote the matrix units, we have $A = \sum_{i,j=1}^n a_{ij}E_{ij}$, so that

$$\|A\| \leq \sum_{i,j=1}^n |a_{ij}|\|E_{ij}\| = \sum_{i,j=1}^n |f_{e_i,e_j}(A)|\|E_{ij}\|,$$

which shows that convergence in the weak topology implies convergence in the norm topology.

(c) If $\dim \mathcal{H} = \infty$, then the Gram–Schmidt process ensures the existence of an orthonormal sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{H}$. We consider the operators

$$P_n(v) = \langle f_n, v\rangle f_n \quad \text{and} \quad Q_n(v) = \langle f_1, v\rangle f_n \quad \text{for} \quad n \in \mathbb{N}.$$

Then P_n is the orthogonal projection onto the one-dimensional subspace $\mathbb{C}f_n$. For any $v \in \mathcal{H}$, we have $\sum_n |\langle v, f_n\rangle|^2 \leq \|v\|^2$ (Bessel inequality), so that $\lim_{n\to\infty}\langle v, f_n\rangle = 0$. This shows that

$$\lim_{n\to\infty} \|P_n(v)\|^2 = \lim_{n\to\infty} |\langle f_n, v\rangle|^2 = 0,$$

so that $P_n \to 0$ in the strong operator topology. As $P_n(f_n) = f_n$, we have $\|P_n\| = 1$ and P_n does not converge in the norm topology.

The sequence Q_n does not converge in the strong operator topology because $Q_n(f_1) = f_n$ does not converge in $\mathcal{H}$. However, $Q_n \to 0$ holds in the weak operator topology because, for every $w \in \mathcal{H}$, we have

$$\langle w, Q_n v\rangle = \langle f_1, v\rangle\langle w, f_n\rangle \to 0.$$

The proof of the following lemma is an easy exercise (cf. Exercise 4.7):

Lemma 4.3. (A neighborhood basis for weak and strong topology) *Let $A \in B(\mathcal{H})$.*

(a) *A basis of neighborhoods of A in the strong operator topology consists of the sets*

$$U = \{S \in B(\mathcal{H}) \colon \|Sv_j - Av_j\| < \varepsilon \ \text{ for } \ j = 1, \ldots, n\},$$

where $\varepsilon > 0$, $n \in \mathbb{N}$ and $v_1, \ldots, v_n \in \mathcal{H}$.

(b) *A basis of neighborhoods of A in the weak operator topology consists of the sets*

$$U = \{S \in B(\mathcal{H}) \colon |\langle w_j, (S - A)v_j\rangle| < \varepsilon \ \text{ for } \ j = 1, \ldots, n\},$$

where $\varepsilon > 0$, $n \in \mathbb{N}$ and $v_1, \ldots, v_n, w_1, \ldots, w_n \in \mathcal{H}$.

Lemma 4.4. *Let $(A_n)_{n\in\mathbb{N}}$ be a bounded sequence of hermitian operators which is increasing ($A_n \leq A_{n+1}$) or decreasing ($A_n \geq A_{n+1}$). Then $A := \lim_{n\to\infty} A_n$ exists in $B(\mathcal{H})$ with respect to the weak operator topology and $A^* = A$.*

Proof. Assume that $\|A_n\| \leq C$ holds for every $n \in \mathbb{N}$. Then, for every $v \in \mathcal{H}$, we have

$$-C\|v\|^2 \leq \langle v, A_n v\rangle \leq C\|v\|^2, \tag{18}$$

and since the sequence $(\langle v, A_n v\rangle)_{n\in\mathbb{N}}$ is monotone and bounded, it converges. The polarization identity

$$\langle v, A_n w\rangle = \frac{1}{4}\sum_{k=0}^{3} i^{-k}\langle v + i^k w, A_n(v + i^k w)\rangle$$

now implies that

$$\beta(v, w) := \lim_{n\to\infty} \langle v, A_n w\rangle$$

exists for any $v, w \in \mathcal{H}$. As β is a pointwise limit of hermitian forms on $\mathcal{H}$, it also is hermitian. Passing to the limit in the estimate $|\langle v, A_n w\rangle| \leq C\|v\|\|w\|$, we obtain

$$|\beta(v, w)| \leq C\|v\|\|w\| \quad \text{for} \quad v, w \in \mathcal{H},$$

so that there exists a bounded hermitian operator $A \in B(\mathcal{H})$ with

$$\beta(v, w) = \langle v, Aw\rangle \quad \text{for} \quad v, w \in \mathcal{H}.$$

(Exercise 4.16; the main point is that $w \mapsto \beta(v, w)$ is a continuous linear map, hence represented by a scalar product by the Riesz–Fischer Theorem). Then $A_n \to A$ holds in the weak operator topology. □

4.2 von Neumann algebras

We now introduce von Neumann algebras as $*$-subalgebra $\mathcal{A} \subseteq B(\mathcal{H})$ which coincide with their bicommutant $\mathcal{A} = \mathcal{A}''$. , which is equivalent to $\mathcal{A}$ being the commutant of some $*$-invariant subset. We also prove von Neumann's Bicommutant Theorem assertion that, for a $*$-subalgebra $\mathcal{A} \subseteq B(\mathcal{H})$ whose representation on $\mathcal{H}$ is non-degenerate, the bicommutant $\mathcal{A}''$ coincides with the closure in the strong and the weak operator topology.

Definition 4.5. For a subset $\mathcal{S} \subseteq B(\mathcal{H})$, we define the *commutant* by

$$\mathcal{S}' := \{A \in B(\mathcal{H}) \colon (\forall S \in \mathcal{S})\, SA = AS\}.$$

Lemma 4.6. *For subsets $E, F \subseteq B(\mathcal{H})$, we have:*

(i) $E \subseteq F' \Leftrightarrow F \subseteq E'$.

(ii) $E \subseteq E''$.

(iii) $E \subseteq F \Rightarrow F' \subseteq E'$.

(iv) $E' = E'''$.

(v) $E = E''$ *if and only if* $E = F'$ *holds for some subset* $F \subseteq B(\mathcal{H})$.

Proof. (i)-(iii) are trivial.

(iv) From (ii) we get $E' \subseteq (E')'' = E'''$. Moreover, (ii) and (iii) imply $E''' \subseteq E'$.

(v) If $E = F'$, then $E'' = F''' = F' = E$ is a consequence of (iv). The converse is trivial, □

Lemma 4.7. *The commutant E' of a subset $E \subseteq B(\mathcal{H})$ has the following properties:*

(i) *If E is commutative, then so is E''.*

(ii) *E' is a unital subalgebra of $B(\mathcal{H})$ which is closed in the weak operator topology, hence in particular closed in the strong and the norm topology.*

(iii) *If $E^* = E$, then E' is also $*$-invariant, hence in particular a C^*-subalgebra of $B(\mathcal{H})$.*

Proof. (i) That E is commutative is equivalent to $E \subseteq E'$, but this implies $E'' \subseteq E' = E'''$ (Lemma 4.6(iv)), which means that E'' is commutative.

(ii) Clearly E' is a linear subspace closed under products, hence a subalgebra of $B(\mathcal{H})$. To see that E' is closed in the weak operator topology, let $v, w \in \mathcal{H}$ and $B \in E$. For $A \in B(\mathcal{H})$ we then have

$$f_{v,w}(AB - BA) = \langle v, ABw\rangle - \langle v, BAw\rangle = (f_{v,Bw} - f_{B^*v,w})(A),$$

which leads to

$$E' = \bigcap_{v,w\in\mathcal{H}, B\in E} \ker(f_{v,Bw} - f_{B^*v,w}),$$

which is subspace of $B(\mathcal{H})$ that is closed in the weak operator topology.

(iii) If $A \in E'$ and $B \in E$, then

$$A^*B - BA^* = (B^*A - AB^*)^* = 0$$

follows from $B^* \in E$. Therefore E' is $*$-invariant. Since it is in particular norm closed by (ii), E' is a C^*-subalgebra of $B(\mathcal{H})$. □

We now come to one of the most fundamental concepts of this course.

Definition 4.8. A unital $*$-subalgebra $\mathcal{M} \subseteq B(\mathcal{H})$ is called a *von Neumann algebra* if $\mathcal{M} = \mathcal{M}''$.

A von Neumann algebra $\mathcal{M}$ is called a *factor* if its center

$$Z(\mathcal{M}) := \{z \in \mathcal{M} \colon (\forall a \in \mathcal{M})\, az = za\} = \mathcal{M} \cap \mathcal{M}'$$

is trivial, i.e., $Z(\mathcal{M}) = \mathbb{C}\mathbf{1}$.

We shall see later that factors are precisely the simple von Neumann algebras in the sense that they have no non-trivial ideal which also is a von Neumann algebra (Exercise 4.11). Factors are the building blocks of general von Neumann algebras, and there is a well-developed decomposition theory according to which any von Neumann algebra is a so-called "direct integral" of factors (cf. [Dix69]). The classification theory of factors is an important branch of noncommutative geometry (cf. [Co94]).

Remark 4.9. (a) In view of Lemma 4.7, any von Neumann algebra $\mathcal{M}$ is closed in the weak operator topology. As the norm topology is finer than the weak operator topology, it follows in particular that $\mathcal{M}$ is norm closed, hence a C^*-algebra. Von Neumann's Bicommutant Theorem 4.13 below shows the converse for unital $*$-invariant algebras: they are von Neumann if and only if they are closed in the weak or strong operator topology.

(b) For every $*$-invariant subset $E \subseteq B(\mathcal{H})$, the commutant E' is a von Neumann algebra because it is also $*$-invariant and $E''' = E'$ (Lemma 4.6).

Lemma 4.10. *For a von Neumann algebra $\mathcal{M} \subseteq B(\mathcal{H})$, the following assertions hold:*

(i) *The commutant $\mathcal{M}'$ also is a von Neumann algebra.*

(ii) *The center $Z(\mathcal{M})$ is a commutative von Neumann algebra*

(iii) *The following are equivalent:*

(a) $\mathcal{M}$ *is a factor.*

(b) $\mathcal{M}'$ *is a factor.*

(c) *The representation of* $\mathrm{span}(\mathcal{M} \cdot \mathcal{M}')$ *on* $\mathcal{H}$ *is irreducible.*

Proof. (i) Remark 4.9(b).

(ii) follows from $Z(\mathcal{M}) = (\mathcal{M} \cup \mathcal{M}')'$ and Remark 4.9(b).

(iii) The equivalence of (a) and (b) follows from $Z(\mathcal{M}) = Z(\mathcal{M}')$. The equivalence with (c) follows from $(\mathcal{M} \cup \mathcal{M}')' = \mathcal{M}' \cap \mathcal{M} = Z(\mathcal{M})$ and Schur's Lemma (Theorem 3.13). $\square$

Example 4.11. (a) The full algebra $\mathcal{M} = B(\mathcal{H})$ is a von Neumann algebra. In this case $\mathcal{M}' = \mathbb{C}\mathbf{1}$ (Exercise 4.15), which implies that $B(\mathcal{H})$ is a factor. If $n = \dim \mathcal{H} \in \mathbb{N}_0 \cup \{\infty\}$, then $B(\mathcal{H})$ is called a *factor of type* I_n.

(b) For $\mathcal{H} = \mathbb{C}^n$, it follows in particular that $M_n(\mathbb{C}) \cong B(\mathbb{C}^n)$ is a von Neumann algebra with commutant $M_n(\mathbb{C})' = \mathbb{C}\mathbf{1}$.

(c) We consider the subalgebra $\mathcal{A} := M_n(\mathbb{C})$ of the matrix algebra

$$\mathcal{B} := M_n(M_m(\mathbb{C})) \cong M_{nm}(\mathbb{C}) \cong B(\mathbb{C}^{nm})$$

of $(n \times n)$-matrices whose entries are $(m \times m)$-matrices. Then $\mathcal{C} := M_m(\mathbb{C})\mathbf{1} \cong M_m(\mathbb{C})$ (the multiples of the identity) form a subalgebra of $\mathcal{B}$. Using (b), it is easy to see that

$$\mathcal{C}' = \mathcal{A} \quad \text{and} \quad \mathcal{A}' = \mathcal{C}$$

(Exercise 4.13). Therefore $\mathcal{A}$ is a von Neumann algebra whose commutant is $\mathcal{C}$.

Identifying $\mathbb{C}^{nm}$ with $\mathbb{C}^n \otimes \mathbb{C}^m$, we have $\mathcal{A} \cong B(\mathbb{C}^n) \otimes \mathbf{1}$ and $\mathcal{C} \cong \mathbf{1} \otimes B(\mathbb{C}^m)$. We shall develop this point of view in more detail in Subsection 4.4 below.

The following theorem implies in particular that von Neumann algebras are precisely the weakly (strongly) closed $*$-subalgebras of $B(\mathcal{H})$ which are non-degenerate in the following sense.

Definition 4.12. A $*$-algebra $\mathcal{A} \subseteq B(\mathcal{H})$ is said to be *non-degenerate* if $\mathcal{A}\mathcal{H}$ spans a dense subspace of $\mathcal{H}$, i.e., if the identical representation of $\mathcal{A}$ on $\mathcal{H}$ is non-degenerate. Note that this condition is trivially satisfied if $\mathbf{1} \in \mathcal{A}$.

Theorem 4.13. (von Neumann's Bicommutant Theorem) *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{A} \subseteq B(\mathcal{H})$ be a non-degenerate $*$-subalgebra, then the closure of $\mathcal{A}$ in the weak and strong operator topology coincides with $\mathcal{A}''$.*

Proof. As $\mathcal{A}''$ is closed in the weak-operator topology by Lemma 4.7, $\mathcal{A} \subseteq \mathcal{A}''$ implies

$$\mathcal{A} \subseteq \overline{\mathcal{A}}^s \subseteq \overline{\mathcal{A}}^w \subseteq \mathcal{A}''$$

(see (17) in Remark 4.2). It therefore remains to show that $\mathcal{A}'' \subseteq \overline{\mathcal{A}}^s$.

So let $T \in \mathcal{A}''$. A basis of neighborhoods of $\mathcal{A}$ in the strong operator topology is specified by a finite set $v_1, \ldots, v_n \in \mathcal{H}$ and $\varepsilon > 0$ via

$$U = \{S \in B(\mathcal{H}) \colon \|Sv_j - Tv_j\| < \varepsilon \ \text{ for } \ j = 1, \ldots, n\}$$

(Lemma 4.3). Fixing the v_j and $\varepsilon > 0$, we therefore have to find an $S \in \mathcal{A} \cap U$.

We consider the $*$-homomorphism

$$\rho\colon B(\mathcal{H}) \to B(\mathcal{H}^n), \quad A \mapsto \mathrm{diag}(A, \ldots, A).$$

Then a quick calculation shows that, an element $X = (X_{ij}) \in B(\mathcal{H}^n) \cong M_n(B(\mathcal{H}))$ (Exercise 1.8) commutes with $\rho(\mathcal{A})$ if and only of all entries X_{ij} commute with $\mathcal{A}$, i.e.,

$$\rho(\mathcal{A})' \cong M_n(\mathcal{A}'). \tag{19}$$

This in turn implies that

$$\rho(\mathcal{A})'' \cong M_n(\mathcal{A}')' = \mathcal{A}''\mathbf{1}_n \tag{20}$$

(Exercise 4.13).

Next we observe that $\rho(\mathcal{A}) \subseteq B(\mathcal{H}^n)$ is non-degenerate. In fact, if $\mathbf{w} = (w_1, \ldots, w_n) \in \mathcal{H}^n$ satisfies

$$\rho(A)\mathbf{w} = (Aw_1, \ldots, Aw_n) = 0$$

for every $A \in \mathcal{A}$, then, for each j, we have $\mathcal{A}w_j = \{0\}$, so that $w_j = 0$ by Lemma 3.4 because $\mathcal{A}$ is non-degenerate. Applying Lemma 3.4 again, we see that $\rho(\mathcal{A})$ is non-degenerate.

Consider the vector $\mathbf{v} := (v_1, \ldots, v_n) \in \mathcal{H}^n$. Then $E := \overline{\rho(\mathcal{A})\mathbf{v}}$ contains $\mathbf{v}$ by Lemma 3.4 because $\rho(\mathcal{A})$ is non-degenerate. As $\rho(\mathcal{A})$ is $*$-invariant, the orthogonal projection P_E onto E commutes with $\rho(\mathcal{A})$ (Lemma 3.3), hence with $\rho(T)$ by (20). We conclude that $\rho(T)E \subseteq E$ (Lemma 3.3). In particular $\rho(T)\mathbf{v} \in E = \overline{\rho(\mathcal{A})\mathbf{v}}$, and this means that there exists an $A \in \mathcal{A}$ with

$$\sum_{j=1}^n \|Tv_j - Av_j\|^2 = \|\rho(T)\mathbf{v} - \rho(A)\mathbf{v}\|^2 < \varepsilon^2.$$

We thus obtain $\|Tv_j - Av_j\| < \varepsilon$ for $j = 1, \ldots, n$. $\square$

At this point the next step is to determine the structure of a von Neumann algebra more closely. We shall do this shortly by using representation theoretic tools.

4.3 Projections in von Neumann algebras

One of the main differences between C^*-algebras and von Neumann algebras is that von Neumann algebras $\mathcal{M}$ are generated by their projections (Proposition 4.18). In a commutative C^*-algebra $C(X)$, X compact, a projection is a characteristic function χ_E of an open compact subset $E \subseteq X$. If X is connected, then $\chi_\emptyset = 0$ and $\chi_X = \mathbf{1}$ are the only projections in $C(X)$. However, for a commutative von Neumann algebra such as $L^\infty(X, \mu)$, the density of the subspace of step functions shows that the projections, resp., the characteristic functions span a norm-dense subspace.

Definition 4.14. A positive measure μ on $(X, \mathfrak{S})$ is said to be *σ-finite* if $X = \bigcup_{n\in\mathbb{N}} E_n$ with $E_n \in \mathfrak{S}$ and $\mu(E_n) < \infty$. This is an important assumption for many results in measure theory, such as Fubini's Theorem and the Radon–Nikodym Theorem.

A positive measure μ on $(X, \mathfrak{S})$ is called *semi-finite* if, for each $E \in \mathfrak{S}$ with $\mu(E) = \infty$, there exists a measurable subset $F \subseteq E$ satisfying $0 < \mu(F) < \infty$.

The following lemma is a central basic tool to deal with Hilbert spaces with continuous decompositions.

Lemma 4.15. (Multiplication operators) *Let $(X, \mathfrak{S}, \mu)$ be a measure space and $L^\infty(X, \mu)$ be the corresponding $*$-algebra of essentially bounded measurable functions. Then the following assertions hold:*

(i) *For each $f \in L^\infty(X, \mu)$, we obtain a bounded operator $M_f \in B(L^2(X, \mu))$ by $M_f(g) := fg$ satisfying $\|M_f\| \leq \|f\|_\infty$.*[13]

(ii) *The map $M\colon L^\infty(X, \mu) \to B(L^2(X, \mu)), f \mapsto M_f$ is a representation of the $*$-algebra $L^\infty(X, \mu)$, where $f^*(x) := \overline{f(x)}$.*

(iii) *If μ is semi-finite , then M is isometric, i.e., $\|M_f\| = \|f\|_\infty$ for each f, so that we may identify $L^\infty(X, \mu)$ with a subalgebra of $B(L^2(X, \mu))$.*

(iv) *If $(f_n)_{n \in \mathbb{N}}$ is a bounded sequence in $L^\infty(X, \mu)$ converging pointwise μ-almost everywhere to f, then $M_{f_n} \to M_f$ in the weak operator topology.*

(v) *If μ is finite, then $1 \in L^2(X, \mu)$ is a cyclic vector for M and $M_{L^\infty(X,\mu)} = M'_{L^\infty(X,\mu)}$ is its own commutant, hence in particular a von Neumann algebra.*

Proof. (i) Since $|f(x)g(x)| \leq \|f\|_\infty |g(x)|$ holds μ-almost everywhere, M_f defines a bounded operator on $L^2(X, \mu)$ with $\|M_f\| \leq \|f\|_\infty$.

(ii) We clearly have $M_{f+g} = M_f + M_g$, $M_{fg} = M_f M_g$ and $M_f^* = M_{f^*}$, so that M defines a homomorphism of C^*-algebras.

(iii) Now assume that $\|f\|_\infty > c \geq 0$. Then $F := \{|f| \geq c\}$ has positive measure, and since μ is semi-finite, it contains a subset E of positive and finite measure. Then $\chi_E \in L^2(X, \mu)$ and

$$c\|\chi_E\|_2 \leq \|f\chi_E\|_2 \leq \|M_f\|\|\chi_E\|_2$$

lead to $\|M_f\| \geq c$. Since c was arbitrary, we obtain $\|f\|_\infty \leq \|M_f\|$.

(iv) For $g, h \in L^2(X, \mu)$, the function $g\overline{h}$ is integrable and $|f_n g\overline{h}| \leq \|f_n\|_\infty |g\overline{h}|$, so that the Dominated Convergence Theorem implies that

$$\langle M_{f_n} g, h\rangle = \int_X f_n g\overline{h}\, d\mu \to \int_X fg\overline{h}\, d\mu = \langle M_f g, h\rangle. \qquad \square$$

(v) The subspace $\pi(L^\infty(X, \mu))1 = L^\infty(X, \mu)$ is dense in $L^2(X, \mu)$, because, for each $f \in L^2(X, \mu)$, the sequence f_n, defined by

$$f_n(x) := \begin{cases} f(x) & \text{for } |f(x)| \leq n \\ 0 & \text{for } |f(x)| > n \end{cases}$$

converges to f because

$$\|f - f_n\|_2^2 = \int_{\{|f| > n\}} |f(x)|^2 \, d\mu(x) \to 0$$

[13] Although elements of $L^\infty(X, \mu)$ can be represented by bounded functions, they are equivalence classes of functions modulo functions h for which $h^{-1}(\mathbb{C}^\times)$ is a set of measure zero. Accordingly,

$$\|f\|_\infty = \inf\{c \in [0, \infty) \colon \mu(\{|f| > c\}) = 0\}$$

denotes the essential supremum of the function (cf. [Ru86]).

follows from the Monotone Convergence Theorem. Here we use that

$$\{|f| = \infty\} = \bigcap_{n\in\mathbb{N}} \{|f(x)| > n\} = \bigcap_{n\in\mathbb{N}} \{x \in X : |f(x)| > n\}$$

is a set of measure zero.

Since $\pi(L^\infty(X,\mu))$ is commutative, it is contained in its own commutant. Suppose, conversely, that $B \in \pi(L^\infty(X,\mu))'$. Then $h := B(1) \in L^2(X,\mu)$, and, for $f \in L^\infty(X,\mu) \subseteq L^2(X,\mu)$, we have

$$B(f) = B(f \cdot 1) = B(\pi(f)1) = \pi(f)B(1) = fh.$$

If $f = \chi_{E_n}$ is the characteristic function of the set

$$E_n := \{x \in X : n \leq |h(x)| \leq n+1\},$$

then $\|B\|\|f\|_2 \geq \|B(f)\|_2 = \|hf\|_2 \geq n\|f\|_2$, and since B is bounded, it follows that $\|f\|_2^2 = \mu(E_n) = 0$ if n is sufficiently large. This means that $h \in L^\infty(X,\mu)$. Now $\pi(h)$ and B coincide on the dense subspace $L^\infty(X,\mu)$, hence on all of $L^2(X,\mu)$. This proves that $B = \pi(h) \in \pi(L^\infty(X,\mu))$.

Remark 4.16. If $(X,\mathfrak{S},\mu)$ is a finite measure space, then Lemma 4.15 implies in particular that the C^*-algebra $L^\infty(X,\mathfrak{S},\mu)$ is realized as a von Neumann algebra on $L^2(X,\mathfrak{S},\mu)$ by multiplication operators.

Definition 4.17. For a $*$-algebra $\mathcal{M}$, we write

$$P(\mathcal{M}) := \{p \in \mathcal{M} : p = p^2 = p^*\}$$

for the set of all *projections* in $\mathcal{M}$, i.e., the set of hermitian idempotents.

Proposition 4.18. *Every von Neumann algebra $\mathcal{M} \subseteq B(\mathcal{H})$ is generated, as a von Neumann algebra, by $P(\mathcal{M})$, i.e., $\mathcal{M} = P(\mathcal{M})''$.*

Proof. Since $P(\mathcal{M})$ consists of hermitian elements, the von Neumann algebra it generates is $\mathcal{B} := P(\mathcal{M})'' \subseteq \mathcal{M}$. It suffices to show that $\mathcal{B}$ contains every hermitian element $x = x^* \in \mathcal{M}$. Let $C^*(x) \cong C(\sigma(x)) \subseteq \mathcal{M}$ be the unital C^*-algebra generated by x. For $a \in \mathbb{R}$, the sequence $f_n^a(t) := e^{-n(\max(t-a,0))} \in [0,1]$ is bounded and converges in a monotone fashion pointwise to the characteristic function $\chi_{(-\infty,a]}$. Lemma 4.4 now shows that $f_n^a(x) \in C^*(x) \subseteq \mathcal{M}$ converges in the weak operator topology to a hermitian operator $P_a \in C^*(x)'' \subseteq \mathcal{M}$.

To see that P_a is a projection, it suffices to consider its action on the cyclic subspace $\mathcal{H}_v := [\![C^*(x)v]\!]$ generated by an element $v \in \mathcal{H}$. Note that any such subspace is invariant under P_a. In view of Example 3.11, this representation is equivalent to the multiplication representation of $C^*(x) \cong C(\sigma(x))$ on a space $L^2(\sigma(x),\mu)$ for some probability measure μ on the spectrum $\sigma(x) \subseteq \mathbb{C}$. Since $P_a = M_{\chi_{(-\infty,a]}}$ follows from Lemma 4.15(iv) and multiplications with characteristic functions are projections, P_a is a projection.

As the bounded continuous function $\mathrm{id}_{\sigma(x)}$ is a pointwise monotone limit of step function s_n, i.e., of linear combinations of characteristic functions of the form $\chi_{(-\infty,a]}$, we find a sequence $b_n \in \mathcal{B}$ converging to x in the weak operator topology because it holds on every subspace $\mathcal{H}_v$. This shows that $x \in \mathcal{B}$. $\square$

Corollary 4.19. *Let $\mathcal{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra and $A = A^* \in B(\mathcal{H})$. Then $A \in \mathcal{M}$ if and only if A preserves all $\mathcal{M}$-invariant closed subspaces of $\mathcal{H}$.*

Proof. By Lemma 3.3 the orthogonal projections onto the closed $\mathcal{M}$-invariant subspaces $\mathcal{K} \subseteq \mathcal{H}$ are precisely the projections in the commutant $\mathcal{M}'$. If A preserves all $\mathcal{M}$-invariant subspaces, then it commutes with all projections in $\mathcal{M}'$ (Lemma 3.3) and now Proposition 4.18 shows that $A \in P(\mathcal{M}')' = \mathcal{M}'' = \mathcal{M}$. □

The fact that the projections $p \in P(\mathcal{M})$ correspond to the $\mathcal{M}'$-invariant subspaces permits us to relate projections in $\mathcal{M}$ to $\mathcal{M}'$-subrepresentations. This suggests a natural equivalence relation:

Lemma 4.20. *For $p, q \in P(\mathcal{M})$, the following are equivalent:*

(i) *There exists $u \in \mathcal{M}$ with $p = u^*u$ and $q = uu^*$.*

(ii) *The representations of $\mathcal{M}'$ on $p\mathcal{H}$ and $q\mathcal{H}$ are equivalent.*

Proof. (i) $\Rightarrow$ (ii): The relation $p = u^*u$ implies that $u|_{p\mathcal{H}}$ is an isometry:

$$\|upv\|^2 = \langle pv, u^*upv\rangle = \langle pv, p^2v\rangle = \langle pv, pv\rangle = \|pv\|^2$$

and that $\ker u = \ker p = (p\mathcal{H})^\perp$. This means that u is a *partial isometry* (cf. Exercise 4.26).

We claim that $q = uu^*$ is the orthogonal projection onto $up(\mathcal{H})$. In fact, for $v \in \mathcal{H}$, we have

$$qupv = uu^*upv = up^2v = upv \quad \text{and} \quad \ker q = \ker u^* = (u\mathcal{H})^\perp = (up\mathcal{H})^\perp.$$

This implies that u restricts to a unitary operator $p\mathcal{H} \to q\mathcal{H}$ and, as $u \in \mathcal{M}$, it is an intertwining operator for $\mathcal{M}'$.

(ii) $\Rightarrow$ (i): Let $U \colon p\mathcal{H} \to q\mathcal{H}$ be an $\mathcal{M}'$ intertwining operator and extend U to a partial isometry $u \in B(\mathcal{H})$ with kernel $(p\mathcal{H})^\perp$. Then $\ker(u^*) = (u\mathcal{H})^\perp = (q\mathcal{H})^\perp$ and $u^*|_{q\mathcal{H}} \colon q\mathcal{H} \to p\mathcal{H}$ is unitary. In particular, u^* is a partial isometry from $q\mathcal{H}$ to $p\mathcal{H}$. This implies that $u^*u = p$ and $uu^* = q$ (cf. Exercise 4.26). □

Definition 4.21. (a) For two projections p, q in a von Neumann algebra $\mathcal{M}$, we write $p \sim q$ if there exists a $u \in \mathcal{M}$ with $p = u^*u$ and $q = uu^*$. Then p and q are called *equivalent* (in the sense of Murray–von Neumann). We write $[p]$ for the equivalence class of p and $[P(\mathcal{M})]$ for the set of equivalence classes of projections.

(b) There is a natural order on $P(\mathcal{M})$ defined by $p \leq q$ if $p\mathcal{H} \subseteq q\mathcal{H}$. We write $[p] \leq [q]$ if p is equivalent to a projection $\widetilde{p} \leq q$, i.e., if the representation of $\mathcal{M}'$ on $p\mathcal{H}$ is equivalent to a subrepresentation of the representation on $q\mathcal{H}$, a property which depends only on the equivalence classes of p, resp., q (Lemma 4.20). The interpretation of the order relation in terms of representations of $\mathcal{M}'$ shows immediately that $\leq$ is a quasi-order, i.e., a symmetric transitive relation.

Example 4.22. (a) (Equivalence classes of projections in $B(\mathcal{H})$) If $\mathcal{M} = B(\mathcal{H})$, then $\mathcal{M}' = \mathbb{C}\mathbf{1}$, so that two projections $p, q \in \mathcal{M}$ are equivalent if and only if $p\mathcal{H}$ and $q\mathcal{H}$ are isomorphic Hilbert spaces, i.e., if $\dim p\mathcal{H} = \dim q\mathcal{H}$.[14] Therefore the set $[P(B(\mathcal{H}))]$ of equivalence classes is parametrized by the set of all cardinal numbers $\leq \dim \mathcal{H}$.

[14] Here we use the non-trivial result that two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are isomorphic, i.e., there exists a unitary operator $\mathcal{H}_1 \to \mathcal{H}_2$, if and only if the cardinalities of orthonormal bases in $\mathcal{H}_1$ and $\mathcal{H}_2$ (their *Hilbert dimension*) coincide. The necessity of this condition is clear, but the sufficiency requires that, for infinite cardinal numbers $\aleph$, we have $\aleph = \aleph \cdot \aleph_0$ and the Schröder–Bernstein Theorem is required to define the Hilbert dimension.

(b) (Equivalence classes of projections in abelian algebras) If $\mathcal{M}$ is abelian, then two projections $p, q \in P(\mathcal{M})$ are equivalent if and only if $p = q$. For the von Neumann algebra $\mathcal{M} = L^\infty(X, \mathfrak{S}, \mu)$, μ a finite measure, the set $P(\mathcal{M})$ is the set of characteristic functions χ_E, considered as elements of $L^\infty(X, \mathfrak{S}, \mu)$. Therefore $P(\mathcal{M})$ can be identified with $\mathfrak{S}/\sim$, where $E \sim F$ if $\mu(E \triangle F) = 0$.

Definition 4.23. If $\mathcal{M} \subseteq B(\mathcal{H})$ is a factor, we use the representation of $\mathcal{M}'$ on $\mathcal{H}$ to distinguish three types I, II, III:

(I) There exists an irreducible subrepresentation for $\mathcal{M}'$.

(II) There exists no irreducible subrepresentation for $\mathcal{M}'$ but a subrepresentation $(\rho, \mathcal{K})$ for $\mathcal{M}'$ which is not equivalent to a proper subrepresentation of itself.

(III) All subrepresentations $(\rho, \mathcal{K})$ for $\mathcal{M}'$ are equivalent to proper subrepresentations of themselves.

According to Definition 4.23, factors come in three types. We now address the problem to get a better picture of factors of type I and to see how other factors can be constructed. For type I, we need tensor products of Hilbert spaces to understand their structure. We saw already some indication for that in Example 4.11(c).

The immediate examples of factors are of type I:

Example 4.24. (a) If $\dim \mathcal{H} < \infty$, then all factors $\mathcal{M} \subseteq B(\mathcal{H})$ are of type I (Exercise).

(b) The factor $\mathcal{M} = B(\mathcal{H})$ is of type I because $\mathcal{M}' = \mathbb{C}\mathbf{1}$ and every one-dimensional subspace $\mathbb{C}v \subseteq \mathcal{H}$ is an irreducible subrepresentation of $\mathcal{M}'$.

At this point we quote the following theorem without proof (cf. [Bl06, Thm. III.1.7.9]). It describes the ordered sets $([P(\mathcal{M})], \leq)$ for factors on separable Hilbert spaces:

Theorem 4.25. *If $\mathcal{M} \subseteq B(\mathcal{H})$ is a factor and $\mathcal{H}$ is separable, then the order on $[P(\mathcal{M})]$ is linear and the ordered set $([P(\mathcal{M})], \leq)$ is isomorphic to one of the following:*

- *The set $\{0, 1, \ldots, n\}$ (type I_n); a typical example is $\mathcal{M} = M_n(\mathbb{C})$.*
- $\mathbb{N}_0 \cup \{\infty\}$ *(type I_∞); a typical example is $\mathcal{M} = B(\mathcal{H})$, $\dim \mathcal{H} = \infty$.*
- $[0, 1]$ *(type II_1).*
- $[0, \infty]$ *(type II_∞).*
- $\{0, \infty\}$ *(type III).*

Definition 4.26. A positive functional ω on a $*$-algebra is called a *trace* if $\omega(ab) = \omega(ba)$ for $a, b \in \mathcal{A}$. If $\mathcal{A}$ is a C^*-algebra, a *tracial state* is a state which is a trace.

Immediately from the definitions, we obtain:

Lemma 4.27. *For a trace $\tau \colon \mathcal{M} \to \mathbb{C}$ on the von Neumann algebra $\mathcal{M}$, we have*

$$p, q \in P(\mathcal{M}), p \sim q \qquad \Rightarrow \qquad \tau(p) = \tau(q).$$

Example 4.28. (a) On $M_n(\mathbb{C}) \cong B(\mathbb{C}^n)$, $\tau(A) := \frac{1}{n}\operatorname{tr}(A)$ is a tracial state.

(b) If $\mathcal{M}$ is a factor of type II_1, then one can show that there exists a unique tracial state τ ([Tak02, Cor. V.2.32]), and that τ induces an order isomorphism $[P(\mathcal{M})] \to [0, 1], [p] \mapsto \tau(p)$.

(c) If $\dim \mathcal{H} = \infty$, then $B(\mathcal{H})$ has no trace (Exercise 4.22).

4.4 Tensor products of Hilbert spaces

In this subsection we introduce tensor products of Hilbert space. Here our motivation is to understand type I factors. More precise, for any type I factor $\mathcal{M} \subseteq B(\mathcal{H})$, there exists a tensor product factorization $\mathcal{H} \cong \mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2$ with $\mathcal{M} = B(\mathcal{H}_1) \otimes \mathbf{1}$. In this sense type I factors leads to the most obvious pairs of mutually commuting pairs of subalgebras $(\mathcal{M}, \mathcal{M}')$ of $B(\mathcal{H})$ whose union acts irreducibly.

Definition 4.29. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. On the vector space tensor product $\mathcal{H} \otimes \mathcal{K}$, we obtain by

$$\langle x \otimes y, x' \otimes y' \rangle := \langle x, x' \rangle \langle y, y' \rangle$$

a well-defined hermitian form. [15] To see that it is positive definite, we consider an arbitrary element $z = \sum_{j=1}^n x_j \otimes y_j \in \mathcal{H} \otimes \mathcal{K}$. Choosing an orthonormal basis in $\operatorname{span}\{y_1, \ldots, y_n\}$ and expanding accordingly, we may w.l.o.g. assume that $y_1, \ldots, y_n$ are orthonormal. Then

$$\langle z, z \rangle = \sum_{j,k=1}^n \langle x_j, x_k \rangle \langle y_j, y_k \rangle = \sum_{j=1}^n \langle x_j, x_j \rangle = \sum_j \|x_j\|^2 \geq 0,$$

and if $\langle z, z \rangle = 0$, then $x_j = 0$ for each j implies $z = 0$. Therefore the hermitian form on $\mathcal{H} \otimes \mathcal{K}$ is positive definite. We write

$$\mathcal{H} \widehat{\otimes} \mathcal{K}$$

for the completion of $\mathcal{H} \otimes \mathcal{K}$ with respect to $\langle \cdot, \cdot \rangle$ and call it the *tensor product Hilbert space.*

Remark 4.30. (a) One can obtain a more concrete picture of the tensor product by choosing orthonormal bases $(e_j)_{j \in J}$ in $\mathcal{H}$ and $(f_k)_{k \in K}$ in $\mathcal{K}$. Then the family $(e_j \otimes f_k)_{(j,k) \in J \times K}$ is orthonormal in the tensor product and spans a dense subspace, so that it is an orthonormal basis. That it spans a dense subspace follows directly from the continuity of the bilinear map

$$\gamma \colon \mathcal{H} \times \mathcal{K} \to \mathcal{H} \otimes \mathcal{K}, \quad \gamma(x, y) := x \otimes y$$

which follows from $\|\gamma(x, y)\| = \|x\| \|y\|$.

(b) Similarly, we find that the subspaces $e_j \otimes \mathcal{K}$ of $\mathcal{H} \widehat{\otimes} \mathcal{K}$ are pairwise orthogonal and span a dense subspace, so that

$$\mathcal{H} \widehat{\otimes} \mathcal{K} \cong \widehat{\bigoplus}_{j \in J} (e_j \otimes \mathcal{K})$$

(cf. Exercise 3.5). In addition, we have

$$\langle e_j \otimes v, e_j \otimes w \rangle = \langle v, w \rangle,$$

[15] To derive the existence of this hermitian form from the universal property of the tensor product, one first observes that, for every fixed pair (x, y), the map

$$\alpha_{x,y} \colon \mathcal{H} \times \mathcal{K} \to \mathbb{C}, \quad (x', y') \mapsto \langle x, x' \rangle \langle y, y' \rangle$$

is bilinear, hence factors through a linear map

$$\overline{\alpha}_{x,y} \colon \mathcal{H} \otimes \mathcal{K} \to \mathbb{C}, \quad x' \otimes y' \mapsto \langle x, x' \rangle \langle y, y' \rangle.$$

Since, for each $z \in \mathcal{H} \otimes \mathcal{K}$, the map $(x, y) \mapsto \overline{\alpha_{x,y}(z)}$ is complex bilinear, there exists a well-defined sesquilinear map

$$\langle \cdot, \cdot \rangle \colon (\mathcal{H} \otimes \mathcal{K}) \times (\mathcal{H} \otimes \mathcal{K}) \to \mathbb{C}$$

mapping $(x \otimes y, x' \otimes y')$ to $\alpha_{x,y}(x' \otimes y')$.

so that the inclusion maps

$$\eta_j \colon \mathcal{K} \to \mathcal{H}\widehat{\otimes}\mathcal{K}, \quad v \mapsto e_j \otimes v$$

are isometric embeddings. This implies that

$$\mathcal{H}\widehat{\otimes}\mathcal{K} \cong \ell^2(J, \mathcal{K})$$

(cf. Example 3.6).

Definition 4.31. With a slight generalization, we can form tensor products of finitely many Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_n$ by

$$\widehat{\otimes}_{j=1}^n \mathcal{H}_j := (\widehat{\otimes}_{j=1}^{n-1} \mathcal{H}_j)\widehat{\otimes}\mathcal{H}_n,$$

so that an alternative construction is to apply the construction of twofold tensor products several times.

Pairs of linear operators define operators on the tensor product space:

Lemma 4.32. *Let $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$. Then there exists a unique bounded linear operator $A \otimes B$ on $\mathcal{H}\widehat{\otimes}\mathcal{K}$ with*

$$(A \otimes B)(v \otimes w) := (Av) \otimes (Bw) \quad \textit{for} \quad v \in \mathcal{H}, w \in \mathcal{K}. \tag{21}$$

It satisfies

$$\|A \otimes B\| \leq \|A\|\|B\| \quad \textit{and} \quad (A \otimes B)^* = A^* \otimes B^*.$$

Proof. Since the elements $v \otimes w$ span a dense subspace of $\mathcal{H} \otimes \mathcal{K}$, the operator $A \otimes B$ is uniquely determined by (21). It therefore remains to show its existence. To this end, we first consider the case $A = \mathbf{1}$.

Identifying $\mathcal{H}\widehat{\otimes}\mathcal{K}$ with $\ell^2(J, \mathcal{K})$ (Remark 4.30(b)), we see that B defines an operator $\widetilde{B}$ on $\ell^2(J, \mathcal{K}) \cong \widehat{\oplus}_{j\in J}\mathcal{K}$ by $\widetilde{B}(x_j) := (Bx_j)$, and $\|\widetilde{B}\| = \|B\|$ (Exercise 3.16). This proves the existence of $\mathbf{1} \otimes B$. We likewise obtain an operator $A \otimes \mathbf{1}$ with $\|A \otimes \mathbf{1}\| = \|A\|$, and we now put

$$A \otimes B := (A \otimes \mathbf{1})(\mathbf{1} \otimes B).$$

It satisfies

$$(A \otimes B)(v \otimes w) = (A \otimes \mathbf{1})(\mathbf{1} \otimes B)(v \otimes w) = (A \otimes \mathbf{1})(v \otimes Bw) = Av \otimes Bw$$

and

$$\|A \otimes B\| = \|(A \otimes \mathbf{1})(\mathbf{1} \otimes B)\| \leq \|A \otimes \mathbf{1}\|\|\mathbf{1} \otimes B\| = \|A\| \cdot \|B\|.$$

From

$$\begin{aligned}\langle (A \otimes B)(v \otimes w), v' \otimes w'\rangle &= \langle Av, v'\rangle\langle Bw, w'\rangle = \langle v, A^*v'\rangle\langle w, B^*w'\rangle \\ &= \langle v \otimes w, (A^* \otimes B^*)(v' \otimes w')\rangle\end{aligned}$$

we derive that $(A \otimes B)^* = A^* \otimes B^*$. $\square$

Lemma 4.33. *Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert spaces and $\mathcal{H}\widehat{\otimes}\mathcal{K}$ be their tensor product. In $B(\mathcal{H}\widehat{\otimes}\mathcal{K})$ we have*

$$(B(\mathcal{H}) \otimes \mathbf{1})' = \mathbf{1} \otimes B(\mathcal{K}) \quad \textit{and} \quad B(\mathcal{H}) \otimes \mathbf{1} = (\mathbf{1} \otimes B(\mathcal{K}))'.$$

Proof. Clearly, $B(\mathcal{H})\otimes\mathbf{1}$ commutes with $\mathbf{1}\otimes B(\mathcal{K})$, so that $B(\mathcal{H})\otimes\mathbf{1} \subseteq (\mathbf{1}\otimes B(\mathcal{K}))'$. We show that we actually have equality. Let $(e_j)_{j\in J}$ be an ONB in $\mathcal{H}$, so that $\mathcal{H}\widehat{\otimes}\mathcal{K} = \widehat{\bigoplus}_{j\in J} e_j \otimes \mathcal{K} \cong \widehat{\bigoplus}_{j\in J}\mathcal{K}$. We write

$$P_j \colon \mathcal{H}\widehat{\otimes}\mathcal{K} \to \mathcal{K}, \quad x\otimes v \mapsto \langle e_j, x\rangle v$$

for the corresponding "projections" which are in particular $B(\mathcal{K})$-intertwining operators. Let $A \in (\mathbf{1}\otimes B(\mathcal{K}))'$. Then, for $i,j\in J$, $A_{ij} := P_i A P_j^* \in B(\mathcal{K})\cap B(\mathcal{K})' = \mathbb{C}\mathbf{1}$. Let $a_{ij}\in\mathbb{C}$ with $A_{ij} = a_{ij}\mathbf{1}$ and $v,w\in\mathcal{K}$ be unit vectors. Then

$$\langle e_i\otimes v, A(e_j\otimes w)\rangle = \langle v, a_{ij}w\rangle = a_{ij}\langle v,w\rangle$$

implies that the closed subspaces $\mathcal{H}\otimes w \cong \mathcal{H}$ are A-invariant with

$$A(e_j\otimes w) = \Big(\sum_{i\in I} a_{ij}e_i\Big)\otimes w.$$

Therefore the matrix $(a_{ij})_{i,j\in J}$ defines a bounded operator $\widetilde{A}$ on $\mathcal{H}$ with $A = \widetilde{A}\otimes\mathbf{1}$ and thus $A\in B(\mathcal{H})\otimes\mathbf{1}$.

The relation $(B(\mathcal{H})\otimes\mathbf{1})' = \mathbf{1}\otimes B(\mathcal{K})$ follows by exchanging the roles of $\mathcal{H}$ and $\mathcal{K}$. $\square$

With the preceding lemma we can now describe the structure of type I factors:

Proposition 4.34. *A factor $\mathcal{M}\subseteq B(\mathcal{H})$ is of type I if and only if there exist Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ such that $\mathcal{H}\cong\mathcal{H}_1\widehat{\otimes}\mathcal{H}_2$ and $\mathcal{M}\cong B(\mathcal{H}_1)\otimes\mathbf{1}$.*

Proof. (a) If $\mathcal{M} = B(\mathcal{H}_1)\otimes\mathbf{1} \subseteq B(\mathcal{H}_1\otimes\mathcal{H}_2)$, then Lemma 4.33 shows that $\mathcal{M}' = \mathbf{1}\otimes B(\mathcal{H}_2)$. In particular, all subspaces $\Omega\otimes\mathcal{H}_2$, carry a representation of $\mathcal{M}'$ equivalent to the representation of $B(\mathcal{H}_2)$ on $\mathcal{H}_2$. This representation is irreducible and therefore $\mathcal{M}$ is of type I.

(b) Suppose, conversely, that $\mathcal{M}$ is of type I, so that $\mathcal{H}$ contains irreducible $\mathcal{M}'$-subrepresentations. To see that $\mathcal{H}$ is an orthogonal direct sum of irreducible $\mathcal{M}'$-representations, we use Zorn's Lemma to find a maximal set $\{\mathcal{K}_j \colon j\in J\}$ of mutually orthogonal irreducible $\mathcal{M}'$-invariant subspaces of $\mathcal{H}$. Set $\mathcal{K} := \overline{\sum_{j\in J}\mathcal{K}_j} \cong \widehat{\bigoplus}_{j\in J}\mathcal{K}_j$ (Exercise 3.5). Then $\mathcal{K}^\perp$ is also $\mathcal{M}'$-invariant. We write $p\colon\mathcal{H}\to\mathcal{K}^\perp$ for the orthogonal projection. Then p is surjective and intertwining for $\mathcal{M}'$. If $\mathcal{L}\subseteq\mathcal{H}$ is an $\mathcal{M}'$-irreducible subspace not contained in $\mathcal{K}$, then $p|_{\mathcal{L}}\colon\mathcal{L}\to\mathcal{K}^\perp$ is a non-zero intertwining operator, so that Corollary 3.15 implies the existence of an $\mathcal{M}'$-irreducible subspace of $\mathcal{K}^\perp$, contradicting the maximality of the family $(\mathcal{K}_j)_{j\in J}$. Therefore all $\mathcal{M}'$-irreducible subspaces are contained in $\mathcal{K}$.

For $A\in\mathcal{M}$ and $j\in J$, the restriction $A|_{\mathcal{K}_j}\colon\mathcal{K}_j\to\mathcal{H}$ commutes with $\mathcal{M}'$, so that Corollary 3.15 shows that either $A(\mathcal{K}_j) = \{0\}$ or $A(\mathcal{K}_j)$ is $\mathcal{M}'$-irreducible, and thus $A(\mathcal{K}_j)\subseteq\mathcal{K}$. This implies that the subspace $\mathcal{K}\subseteq\mathcal{H}$ is invariant under $\mathcal{M}$ and $\mathcal{M}'$. So the orthogonal projection onto $\mathcal{K}$ is contained in $\mathcal{M}\cap\mathcal{M}' = \mathbb{C}\mathbf{1}$, and, as $\mathcal{K}\neq\{0\}$ by assumption, $\mathcal{K} = \mathcal{H}$ (cf. Lemma 4.10(iii)).

The same argument shows that, for every $j_0\in J$, the subspace $\mathcal{M}\mathcal{K}_{j_0}$, which is invariant under $\mathcal{M}$ and $\mathcal{M}'$, is dense in $\mathcal{H}$. We thus find for every j an $A\in B(\mathcal{H})$ such that $A\mathcal{K}_{j_0}$ is not orthogonal to $\mathcal{K}_j$. Corollary 3.16 now shows that the $\mathcal{M}'$ representations on the $\mathcal{K}_j$ are mutually equivalent because they are equivalent to the representation of $\mathcal{K}_{j_0}$. Let us write $(\rho,\mathcal{H}_2)$ for the so-obtained irreducible representation of $\mathcal{M}'$. With $\mathcal{H}_1 := \ell^2(J,\mathbb{C})$, we then obtain an equivalence of $\mathcal{M}'$-representations:

$$\mathcal{H} = \widehat{\bigoplus}_{j\in J}\mathcal{K}_j \cong \ell^2(J,\mathcal{H}_2) \cong \mathcal{H}_1\widehat{\otimes}\mathcal{H}_2 \quad\text{with}\quad \mathcal{M}'\subseteq\mathbf{1}\otimes B(\mathcal{H}_2).$$

As $\mathcal{M}'$ acts irreducibly on $\mathcal{H}_2$, we obtain from Schur's Lemma and the fact that $\mathbf{1} \otimes B(\mathcal{H}_2)$ is a von Neumann algebra the relation $\mathcal{M}' = \mathcal{M}''' = \mathbf{1} \otimes B(\mathcal{H}_2)$ (Theorem 4.13), and finally $\mathcal{M} = \mathcal{M}'' = B(\mathcal{H}_1) \otimes \mathbf{1}$ with Lemma 4.33. □

Identifying $\mathcal{H}_1$ with a subspace of the form $\mathcal{H}_1 \otimes \Omega$, $\|\Omega\| = 1$, of $\mathcal{H} \cong \mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2$, we immediately obtain:

Corollary 4.35. *For every type I factor $\mathcal{M} \subseteq B(\mathcal{H})$, there exists a Hilbert space $\mathcal{H}_1$ and a $*$-isomorphism $\Phi\colon \mathcal{M} \to B(\mathcal{H}_1)$ that is continuous with respect to the weak operator topology on $\mathcal{H}$, resp., $\mathcal{H}_1$.*

Corollary 4.36. *Every finite dimensional factor is isomorphic to $M_n(\mathbb{C})$ for some $n \in \mathbb{N}$.*

Proof. Let $\mathcal{M} \subseteq B(\mathcal{H})$ be a finite dimensional factor. Since the lenght of any chain

$$p_1 < p_2 < \ldots < p_n$$

of projections in $\mathcal{M}$ is bounded by $\dim \mathcal{M}$, there exists a minimal (non-zero) projection. Therefore $\mathcal{M}$ is of type I and thus $\mathcal{M} \cong B(\mathcal{K})$ for a Hilbert space $\mathcal{K}$. As $\mathcal{M}$ is finite dimensional, we have $\mathcal{K} \cong \mathbb{C}^n$ for $n = \dim \mathcal{K} < \infty$, and therefore $\mathcal{M} \cong M_n(\mathbb{C})$. □

Corollary 4.37. *Every finite dimensional C^*-algebra $\mathcal{A}$ is an ℓ^∞ direct sum of matrix algebras $\mathcal{A} \cong \bigoplus_{j=1}^N M_{n_j}(\mathbb{C})$.*

Proof. Since every finite dimensional C^*-algebra is a von Neumann algebra (it is closed in the weak operator topology by Exercise 4.10), this follows by combining the preceding corollary on the structure of the finite dimensional factors with the direct sum decomposition into factors (Exercise 4.12). □

4.5 Tensor products and factor representations

Definition 4.38. A representation $(\pi, \mathcal{H})$ of a $*$-algebra $\mathcal{A}$ is called a *factor representation* if the von Neumann algebra $\pi(\mathcal{A})''$ generated by $\pi(\mathcal{A})$ is a factor.

Proposition 4.39. *Let $\mathcal{A}_j$, $j = 1, 2$, be two $*$-algebras and $(\pi_j, \mathcal{H})$ be $*$-representations on $\mathcal{H}$ such that $\pi_1(\mathcal{A}_1)$ commutes with $\pi_2(\mathcal{A}_2)$. Then $\pi(a_1 \otimes a_2) := \pi_1(a_1)\pi_2(a_2)$ defines a $*$-representation of $\mathcal{A} := \mathcal{A}_1 \otimes \mathcal{A}_2$ on $\mathcal{H}$.*

(a) *If $(\pi, \mathcal{H})$ is irreducible, then the representations π_1 and π_2 are factor representations.*

(b) *If $(\pi, \mathcal{H})$ is irreducible and π_1 is of type I, then there exists irreducible representations $(\rho_j, \mathcal{H}_j)$ of $\mathcal{A}_j$, $j = 1, 2$, such that $\pi \cong \rho_1 \otimes \rho_2$ in the sense that $\mathcal{H} \cong \mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2$ and $\pi(a_1 \otimes a_2) = \rho_1(a_1) \otimes \rho_2(a_2)$.*

Proof. (a) Let $\mathcal{M} := \pi_1(\mathcal{A}_1)''$. Then $\mathcal{M}' \supseteq \pi_2(\mathcal{A}_2)$, so that

$$\mathcal{M} \cap \mathcal{M}' \subseteq \pi_2(\mathcal{A}_2)' \cap \pi_1(\mathcal{A}_1)' = \pi(\mathcal{A})' = \mathbb{C}\mathbf{1}.$$

Therefore $\mathcal{M}$ is a factor. The same argument shows that $\pi_2(\mathcal{A}_2)''$ is a factor.

(b) If $\mathcal{M} = \pi_1(\mathcal{A}_1)''$ is a factor of type I, then there exist Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ with $\mathcal{H} \cong \mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2$ and $\mathcal{M} \cong B(\mathcal{H}_1) \otimes \mathbf{1}$. Then $\pi_2(\mathcal{A}_2) \subseteq \mathcal{M}' = \mathbf{1} \otimes B(\mathcal{H}_2)$, so that there exist representations ρ_j of $\mathcal{A}_j$ on $\mathcal{H}_j$ with

$$\pi(a_1 \otimes_2) = \pi_1(a_1)\pi_2(a_2) = (\rho_1(a_1) \otimes \mathbf{1})(\mathbf{1} \otimes \rho_2(a_2)) = \rho_1(a_1) \otimes \rho_2(a_2).$$ □

Remark 4.40. Note that a von Neumann algebra $\mathcal{M} \subseteq B(\mathcal{H})$ is a factor if and only the representation of $\mathcal{M} \otimes \mathcal{M}'$ on $\mathcal{H}$ is irreducible (Lemma 4.10(iii)). This is a "converse" to Proposition 4.39(a). It shows that a representation $(\pi_1, \mathcal{H})$ is a factor representation if and only if there exists a representation of some $*$-algebra $\mathcal{A}_2$ commuting with $\pi_1(\mathcal{A}_1)$ such that the corresponding representation of $\mathcal{A}_1 \otimes \mathcal{A}_2$ is irreducible (a natural choice is $\mathcal{A}_2 := \pi_1(\mathcal{A}_1)'$).

4.6 Type II$_1$ factors and ICC groups

Definition 4.41. A group G is called *ICC (infinite conjugacy classes)* if, for every $g \neq e$, the conjugacy class $C_g = \{hgh^{-1} \colon h \in G\}$ is infinite.

Example 4.42. Typical examples of ICC groups are: $\mathrm{PGL}_2(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R})/\mathbb{R}^\times \mathbf{1}$ or the group $S_{(X)}$ of finite permutations of an infinite set (cf. Exercise 4.20).

Proposition 4.43. (The regular representation of an ICC group is of type II$_1$) *Let $G \neq \{e\}$ be an ICC group. We consider the representation $(\pi, \mathcal{H})$ of $G \times G$ on the Hilbert space $\mathcal{H} = \ell^2(G, \mathbb{C}) \subseteq \mathbb{C}^G$ given by*

$$(\pi(g,h)f)(x) := f(g^{-1}xh).$$

This representation is irreducible and the restriction $\pi_\ell(g) := \pi(g,e)$ to the subgroup $G \times \{e\}$ is a factor representation of type II$_1$, i.e., $\pi_\ell(G)''$ is a factor of type II$_1$.

Proof. First we show that π is irreducible. We consider the ONB $(\delta_g)_{g \in G}$, consisting of δ-functions satisfying

$$\pi(g,h)\delta_x = \delta_{gxh^{-1}} \quad \text{for} \quad g,h,x \in G.$$

Clearly, the vector $\delta_{\mathbf{1}}$ is cyclic and invariant under the diagonal subgroup

$$K := \{(g,g) \colon g \in G\}.$$

The K-invariance means for an element $f \in \ell^2(G, \mathbb{C})$ that it is constant on conjugacy classes, and since all non-trivial conjugacy classes are infinite, we have

$$\ell^2(G, \mathbb{C})^K = \mathbb{C}\delta_{\mathbf{1}}.$$

As $\delta_{\mathbf{1}}$ is $G \times G$-cyclic, it is separating for the commutant $\mathcal{C} := \pi(G \times G)'$ (Exercise 4.3). Since $\mathcal{C}$ commutes with $\pi(K)$, it leaves the one-dimensional subspace $\ell^2(G, \mathbb{C})^K$ invariant, and this implies $\mathcal{C} = \mathbb{C}\mathbf{1}$, so that the irreducibility of π follows from Schur's Lemma.

This implies that the left regular representation $(\pi_\ell, \mathcal{H})$ of G, resp., the linear extension to the $*$-algebra $\mathbb{C}[G]$ on $\ell^2(G, \mathbb{C})$, is a factor representation (Proposition 4.39). We claim that the factor $\mathcal{M} := \pi_\ell(G)''$ is of type II$_1$. In view of Theorem 4.25 it suffices to show that $\mathcal{M}$ admits a trace and that it is not of type I$_n$ for some $n \in \mathbb{N}$.

We consider the state

$$\omega \colon \mathcal{M} \to \mathbb{C}, \quad \omega(A) := \langle \delta_{\mathbf{1}}, A\delta_{\mathbf{1}} \rangle = (A\delta_{\mathbf{1}})(\mathbf{1}).$$

We claim that it is a trace, i.e.,

$$\omega(AB) = \omega(BA) \quad \text{for} \quad A, B \in \pi_r(G)'.$$

For each $A \in \mathcal{M} \subseteq \pi_r(G)'$, the function $a \colon = A\delta_{\mathbf{1}}$ satisfies

$$A\delta_x = A\pi_r(x)^{-1}\delta_{\mathbf{1}} = \pi_r(x)^{-1}A\delta_{\mathbf{1}} = \pi_r(x)^{-1}a, \quad \text{i.e.,} \quad (A\delta_x)(y) = a(yx^{-1}).$$

This further leads to

$$(Af)(\mathbf{1}) = \sum_{x\in G} f(x)(A\delta_x)(\mathbf{1}) = \sum_{x\in G} f(x)a(x^{-1}).$$

For $b := B\delta_{\mathbf{1}}$, this leads to

$$\omega(AB) = A(B\delta_{\mathbf{1}})(\mathbf{1}) = \sum_{x\in G} a(x^{-1})b(x) = \sum_{x\in G} b(x^{-1})a(x) = \ldots = \omega(BA).$$

Therefore ω is a trace, so that $\mathcal{M}$ is of type I_n for some finite n or of type II_1 (cf. Exercise 4.22). If $\mathcal{M}$ is of type I_n, then $\mathcal{M} \cong M_n(\mathbb{C})$ is finite dimensional, so that $\dim(\operatorname{span}\pi_\ell(G)) < \infty$. This in turn implies that $\pi_\ell(G)\delta_{\mathbf{1}} = \{\delta_g \colon g \in G\}$ is finite-dimensional, which implies that G is finite and ICC, hence trivial. This contradicts our initial hypothesis. $\square$

4.7 Tensor products of von Neumann algebras

Definition 4.44. For two von Neumann algebra $\mathcal{M}_j \subseteq B(\mathcal{H}_j)$, $j = 1, 2$, we define their *tensor product* by

$$\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2 := (\mathcal{M}_1 \otimes \mathcal{M}_2)'' \subseteq B(\mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2).$$

Lemma 4.45. *If $\mathcal{M}_j \subseteq B(\mathcal{H}_j)$, $j = 1, 2$, are factors, then*

$$\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2 \subseteq B(\mathcal{H}_1 \otimes \mathcal{H}_2)$$

is a factor.

Proof. The commutant of the center $Z(\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2)$ contains $\mathcal{M}_1 \otimes \mathcal{M}_2$ and $\mathcal{M}_1' \otimes \mathcal{M}_2'$. As $\mathcal{M}_1 \otimes \mathbf{1}$ and $\mathcal{M}_1' \otimes \mathbf{1}$ generate $B(\mathcal{H}_1) \otimes \mathbf{1}$ (Lemma 4.10) and likewise $\mathbf{1} \otimes \mathcal{M}_2$ and $\mathbf{1} \otimes \mathcal{M}_2'$ generate $\mathbf{1} \otimes B(\mathcal{H}_2)$ (Lemma 4.10(iii)), it follows that

$$Z(\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2) \subseteq (B(\mathcal{H}_1) \otimes \mathbf{1})' \cap (\mathbf{1} \otimes B(\mathcal{H}_2))' = (\mathbf{1} \otimes B(\mathcal{H}_2)) \cap (\mathbf{1} \otimes B(\mathcal{H}_2))' = \mathbb{C}(\mathbf{1} \otimes \mathbf{1}). \ \square$$

More generally, one even has the general *Commutation Theorem* for von Neumann algebras, asserting that

$$(\mathcal{M} \overline{\otimes} \mathcal{N})' = \mathcal{M}' \overline{\otimes} \mathcal{N}',$$

but this is a deep theorem beyond the scope of this lecture (cf. [Bl06]). Some special cases are easy. For instance

$$(\mathcal{M} \overline{\otimes} B(\mathcal{H}_2))' = (\mathcal{M} \otimes \mathbf{1})' \cap (\mathbf{1} \otimes B(\mathcal{H}_2))' = (\mathcal{M} \otimes \mathbf{1})' \cap (B(\mathcal{H}_1) \otimes \mathbf{1}) = \mathcal{M}' \otimes \mathbf{1},$$

which in turn implies

$$(\mathcal{M} \otimes \mathbf{1})' = (\mathcal{M}'' \otimes \mathbf{1})' = \mathcal{M}' \overline{\otimes} B(\mathcal{H}_2).$$

Remark 4.46. (On the classification of type II factors) (a) One can shows that any type II_∞-factor is of the form

$$\mathcal{M} = \mathcal{M}_1 \overline{\otimes} B(\ell^2),$$

where $\mathcal{M}_1$ is a type II_1-factor that can be obtained as $\mathcal{M}_1 = p\mathcal{M}p$ for a projection $p \in P(\mathcal{M})$ which is *finite* in the sense that, $q \leq p$ and $q \sim p$ imply $p = q$ (p is not equivalent to a proper subprojection; this is equivalent to $p \not\sim \mathbf{1}$).

(b) Conversely, for every type II_1 factor $\mathcal{N}$, the factor $\mathcal{M} := \mathcal{N} \overline{\otimes} B(\ell^2)$ is of type II_∞.

(c) As we have seen in Theorem 4.25, the types of finite non-zero projections in a factor $\mathcal{M}$ can be labeled by positive real numbers $\tau([p])$. In view of (a), it is a natural question when two factors $p\mathcal{M}p$ and $q\mathcal{M}q$ are isomorphic. It turns out that the set of numbers $\frac{\tau([p])}{\tau([q])} \in \mathbb{R}^\times$ for which this is the case is a subgroup $\Gamma \subseteq (\mathbb{R}^\times, \cdot)$ called the *fundamental group of* $\mathcal{M}$. It is an important invariant of type II_∞-factors that plays a key role in their classification theory.

4.8 Direct limits and infinite tensor products

Definition 4.47. (Direct limits of Hilbert spaces) Let $(\mathcal{H}_n)_{n\in\mathbb{N}}$ be a sequence of Hilbert spaces such that $\mathcal{H}_n \subseteq \mathcal{H}_{n+1}$ for every $n \in \mathbb{N}$. Then the union $\bigcup_{n\in\mathbb{N}} \mathcal{H}_n$ (more formally it is a direct limit) carries a natural scalar product defined by

$$\langle x, y\rangle := \langle x, y\rangle_{\mathcal{H}_n} \quad \text{if} \quad x, y \in \mathcal{H}_n.$$

The completion of this space is denoted

$$\mathcal{H} := \varinjlim \mathcal{H}_n$$

and called the *direct limit Hilbert space*. It is a Hilbert space in which the $\mathcal{H}_n$ are naturally embedded as subspaces and the union $\bigcup_n \mathcal{H}_n$ of these subspaces is dense.

Example 4.48. (a) If $J = \bigcup_n J_n$, where the J_n form an increasing sequence of subsets of J, then $\ell^2(J) \cong \varinjlim \ell^2(J_n)$.

(b) In $\mathcal{H} = L^2([0,1], \mathbb{C})$ we consider the subspace $\mathcal{H}_n$ of functions constant on the intervals of the form $[k2^{-n}, (k+1)2^{-n})$, $0 \leq k < 2^n$. Then $\dim \mathcal{H}_n = 2^n$, $\mathcal{H}_n \subseteq \mathcal{H}_{n+1}$ and the union of the $\mathcal{H}_n$ is dense in $\mathcal{H}$. Therefore $\mathcal{H} \cong \varinjlim \mathcal{H}_n$.

Definition 4.49. (Infinite tensor products of Hilbert spaces) Let $(\mathcal{H}_n, v_n)_{n\in\mathbb{N}}$ be a sequence of Hilbert spaces $\mathcal{H}_n$ and $v_n \in \mathcal{H}_n$ be a unit vector. Then we obtain isometric embeddings

$$\alpha_n \colon \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n \to \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{n+1}, \quad x_1 \otimes \cdots \otimes x_n \mapsto x_1 \otimes \cdots \otimes x_n \otimes v_{n+1}.$$

We write

$$\mathcal{H} := \widehat{\bigotimes}_{n\in\mathbb{N}} (\mathcal{H}_n, v_n) := \varinjlim \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$$

for the corresponding direct limit Hilbert space. We think of it as an *incomplete infinite tensor product* because it only contains very specific infinite tensor products. Usually the image of an element $x_1 \otimes \cdots \otimes x_n \in \otimes_{k=1}^n \mathcal{H}_k$ in $\mathcal{H}$ is denoted

$$x_1 \otimes \cdots \otimes x_n \otimes v_{n+1} \otimes v_{n+2} \otimes \cdots .$$

In this sense $\mathcal{H}$ is generated by the infinite tensor products whose tail is determined by the sequence $(v_n)_{n\in\mathbb{N}}$ of unit vectors.

Definition 4.50. (Infinite tensor products of von Neumann algebras) Now let $(\mathcal{A}_n, \omega_n)_{n\in\mathbb{N}}$ be a sequence of C^*-algebras and $\omega_n \in \mathfrak{S}(\mathcal{A}_n)$ be states. We consider the corresponding GNS representation $(\pi_n, \mathcal{H}_n)$ of $\mathcal{A}_n$ with cyclic unit vector Ω_n. Recall that this representation is determined up to equivalence by the requirement that

$$\omega_n(A) = \langle \Omega_n, \pi_n(A)\Omega_n\rangle \quad \text{for} \quad A \in \mathcal{A}_n$$

(GNS Theorem 3.9).

We now form the infinite tensor product Hilbert space

$$\mathcal{H} := \widehat{\bigotimes}_{n\in\mathbb{N}}(\mathcal{H}_n, \Omega_n).$$

On this space we have a natural representation of each algebra $\mathcal{A}_n$ determined by

$$\rho_n(A)(x_1 \otimes \cdots \otimes x_n \otimes x_{n+1} \otimes \cdots) = x_1 \otimes \cdots \otimes x_{n-1} \otimes Ax_n \otimes x_{n+1} \otimes \cdots,$$

resp.,

$$\rho_n(A) = \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{n-1 \text{ times}} \otimes A \otimes \mathbf{1} \otimes \mathbf{1} \otimes \cdots.$$

Then

$$\mathcal{M} := \{\rho_n(\mathcal{A}_n) \colon n \in \mathbb{N}\}'' \subseteq B(\mathcal{H})$$

is a von Neumann algebra that can be considered as an infinite tensor product of von Neumann algebras:

$$\mathcal{M} \cong \overline{\bigotimes}_{n\in\mathbb{N}} \pi_n(\mathcal{A}_n)''.$$

The unit vector

$$\Omega := \Omega_1 \otimes \Omega_2 \otimes \Omega_3 \otimes \cdots \otimes \cdots \in \mathcal{H}$$

is a cyclic vector for $\mathcal{M}$ and the corresponding state is given by

$$\omega(\rho_1(A_1) \cdots \rho_n(A_n)) = \langle \Omega, \rho_1(A_1) \cdots \rho_n(A_n)\Omega \rangle = \prod_{j=1}^n \omega_j(A_j).$$

Accordingly, ω is called an *infinite product state*.

With these tools we can now describe how type III factors can be obtained rather easily. This construction is due to R. T. Powers who found with the factors $(\mathcal{R}_\lambda)_{0<\lambda<1}$ the first uncountable family of mutually non-equivalent type III factors ([Po67]). The classification of type III factors was carried out (respectively reduced to other problems) later by A. Connes (cf. [Co73, Co94]) who received the Fields Medal for these results.

Theorem 4.51. (Powers) *For $0 \leq \lambda \leq 1$, we consider on $\mathcal{A}_n = M_2(\mathbb{C})$ the state*

$$\omega_n(A) := \frac{a_{11} + \lambda a_{22}}{1 + \lambda}.$$

Then the von Neumann algebras

$$\mathcal{R}_\lambda := \overline{\bigotimes}_{n\in\mathbb{N}} \pi_n(\mathcal{A}_n)'' \cong \overline{\bigotimes}_{n\in\mathbb{N}} M_2(\mathbb{C})$$

on the corresponding infinite tensor product is a factor. Moreover,

(i) *$\mathcal{R}_0$ is of type I.*

(ii) *$\mathcal{R}_1$ is of type II_1*

(iii) *For $0 < \lambda < 1$, the factors $\mathcal{R}_\lambda$ are of type III and mutually non-isomorphic.*

Proof. We refer to [BR02] for the proof. Here are a few comments.

(i) For $\lambda = 0$ one shows that $\mathcal{R}_0 = B(\mathcal{H})$, i.e., that the representation on $\mathcal{H}$ is irreducible. This can be derived from the fact that, for every n, we have $\mathcal{H}_n \cong \mathbb{C}^2$ with the canonical representation of $M_2(\mathbb{C})$ and $\Omega_n = e_1$ (the first basis vector). This implies that the representation of

$$\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n \cong \bigotimes_{k=1}^{n} M_2(\mathbb{C}) \cong M_2(\mathbb{C}^{2^k})$$

is the canonical representation on

$$\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n \cong \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 \cong \mathbb{C}^{2^k}.$$

Then one uses the general result that a "direct limit" of irreducible representations are irreducible (Exercise 4.27).

(ii) For $\lambda = 1$, the states ω_n are the normalized trace $\omega_n(A) = \frac{1}{2}\operatorname{tr}(A)$ on $M_2(\mathbb{C})$. This implies that ω also is a normalized trace on the factor $\mathcal{R}_1$. As it is not finite dimensional, it must be of type II_1 (Exercise 4.22 and Theorem 4.25). $\square$

Definition 4.52. A von Neumann algebra $\mathcal{M}$ is called *hyperfinite* if there exists an increasing sequence $(\mathcal{M}_n)_{n\in\mathbb{N}}$ of finite dimensional $*$-subalgebras $\mathcal{M}_n$ such that the subalgebra $\bigcup_n \mathcal{M}_n$ is dense in $\mathcal{M}$ with respect to the weak operator topology, resp., generates $\mathcal{M}$ (see von Neumann's Bicommutant Theorem 4.13).

Remark 4.53. (On the classification of hyperfinite factors) The Powers factors $\mathcal{R}_\lambda$ are hyperfinite by construction. A classification of hyperfinite factors is known and much less complicated than the general case. Clearly all type I factors $B(\mathcal{H})$, $\mathcal{H}$ at most separable, are hyperfinite. According to a classical result of Murray and von Neumann all hyperfinite type II_1 factors are isomorphic, hence isomorphic to $\mathcal{R}_1$. Further, A. Connes showed that there is only one hyperfinite factor of type II_∞, namely $\mathcal{R}_1\overline{\otimes}B(\ell^2)$ (the "infinite matrix algebra" with entries in $\mathcal{R}_1$) [Co73, Co75]. Type III factors can be labeled by a parameter $\lambda \in [0,1]$, so that, for $0 < \lambda < 1$, the Powers factor is of type III_λ and Connes' results imply that it is the only hyperfinite factor of this type. Hyperfinite factors of type III_0 have been classified by W. Krieger (the *Krieger factors*) [Kr76] and the uniqueness of hyperfinite type III_1-factors is due to U. Haagerup [Ha87].

Exercises for Section 4

Exercise 4.1. Let $(p_n)_{n\in\mathbb{N}}$ be a sequence of projections in a von Neumann algebra $\mathcal{M} \subseteq B(\mathcal{H})$. Suppose that it either is increasing ($p_n \leq p_{n+1}$) or decreasing ($p_n \geq p_{n+1}$). Then $p := \lim_{n\to\infty} p_n$ exists in $\mathcal{M}$ with respect to the weak operator topology and $p \in P(\mathcal{M})$.

Exercise 4.2. Let $(L, \leq)$ be a partially ordered set. Then L is called a *lattice* if the following axioms are satisfied:

(L1) L has a maximal element $\mathbf{1}$ and a minimal element $\mathbf{0}$.

(L2) For two elements $x, y \in L$ supremum and infimum

$$x \vee y := \min\{z \in L \colon x \leq z, y \leq z\}, \quad \text{resp.,} \quad x \wedge y := \max\{z \in L \colon z \leq x, z \leq y\}$$

exist.

We write $a < b$ for $a \leq b$ and $a \neq b$. If $a \in \mathcal{L}$, then an element a' is said to be a *complement for* a if $a \wedge a' = \mathbf{0}$ and $a \vee a' = \mathbf{1}$. A lattice $\mathcal{L}$ is said to be a *Boolean algebra* if every element has a complement and $\mathcal{L}$ is distributive, i.e., for any three elements a, b, $c \in \mathcal{L}$ the identities

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \quad \text{and} \quad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

are satisfied. A *Boolean σ-algebra* is a Boolean algebra in which countable sups and infs exist.

Show that, for every commutative von Neumann algebra $\mathcal{M}$, the ordered set $(P(\mathcal{M}), \leq)$ is a Boolean σ-algebra. Hint: Exercise 4.1.

Exercise 4.3. Let $S \subseteq \mathcal{H}$ be a subset and $\mathcal{M} \subseteq B(\mathcal{H})$ be a unital $*$-subalgebra. Then the following are equivalent:

(i) S is $\mathcal{M}$-generating, i.e., $[\![\mathcal{M}S]\!] = \mathcal{H}$.

(ii) S is $\mathcal{M}'$-separating, i.e., if $M \in \mathcal{M}'$ satisfies $MS = \{0\}$, then $M = 0$.

Hint: Consider the projection P onto $[\![\mathcal{M}S]\!]$.

Exercise 4.4. Let $(v_n)_{n\in\mathbb{N}}$ be a sequence in a Hilbert space $\mathcal{H}$ which converges weakly to v, i.e., $\langle v_n, w\rangle \to \langle v, w\rangle$ for every $w \in \mathcal{H}$, and assume that $\|v_n\| \to \|v\|$. Then $v_n \to v$ in the norm topology.

Exercise 4.5. Show that, on the unit sphere $\mathbb{S} := \{x \in H : \|x\| = 1\}$ of a Hilbert space $\mathcal{H}$, the norm topology coincides with the weak topology.

Exercise 4.6. Suppose that $\dim \mathcal{H} = \infty$. Show that the unit sphere $\mathbb{S}(\mathcal{H})$ is dense in the closed unit ball $\mathcal{B} \subseteq \mathcal{H}$ with respect to the *weak topology*, which is the coarsest topology for which all functions $f_v = \langle \cdot, v\rangle : \mathcal{B} \to \mathbb{C}, v \in \mathcal{H}$ are continuous.

Exercise 4.7. (A neighborhood basis for the weak operator topology) Let $\mathcal{H}$ be a Hilbert space and $A \in B(\mathcal{H})$. Show that, a basis for the neighborhoods of A in the weak operator topology is given by the subsets of the form

$$U := \{B \in B(\mathcal{H}) : (\forall j = 1, \ldots, n)\ |\langle v_j, (B - A)w_j\rangle| < \varepsilon\}, \quad n \in \mathbb{N}, v_j, w_j \in \mathcal{H}, \varepsilon > 0.$$

Exercise 4.8. Let $\mathcal{H}$ be a Hilbert space. Show that:

(a) The involution $*$ on $B(\mathcal{H})$ is continuous with respect to the weak operator topology.

(b) On every bounded subset $K \subseteq B(\mathcal{H})$ the multiplication $(A, B) \mapsto AB$ is continuous with respect to the strong operator topology.

(c) The left and right multiplications $L_A(B) := AB$ and $R_A(B) := BA$ on $B(\mathcal{H})$ are continuous in the weak and the strong operator topology.

Exercise 4.9. Let $E \subseteq \mathcal{H}$ be a dense subspace. Show that, on every bounded subset $\mathcal{B} \subseteq B(\mathcal{H})$, the weak operator topology is the coarsest topology for which all functions

$$f_{v,w} : \mathcal{B} \to \mathbb{C}, \quad v, w \in E,$$

are continuous.

Exercise 4.10. Show that every finite dimensional subspace $F \subseteq B(\mathcal{H})$ is closed in the weak operator topology.
Hint: Let $A \in \overline{F}$ and consider the subspace $E := F + \mathbb{C}A$. It suffices to show that F is closed in E. To this end, observe that the linear functionals $f_{v,w}|_E$ separate the points on E and derive that every linear functional on E is continuous with respect to the weak operator topology. Conclude that F is closed in E.

Exercise 4.11. Let $\mathcal{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra and let $p = p^* = p^2 \in Z(\mathcal{M})$ be a projection. Then the following assertions hold:

(a) $\mathcal{M}_p := p\mathcal{M} = \mathcal{M}p$ is an ideal of $\mathcal{M}$.

(b) $\mathcal{M}_p = \{M \in \mathcal{M} \colon (\mathbf{1} - p)M = 0\}$.

(c) $\mathcal{M}_p$ is a von Neumann algebra in $B(p\mathcal{H})$. Hint: Exercise 4.8.

(d) $\mathcal{M} = \mathcal{M}_p \oplus \mathcal{M}_{(\mathbf{1}-p)}$, where both summands are ideals.

(e) $Z(\mathcal{M}) = Z(\mathcal{M}_p) \oplus Z(\mathcal{M}_{\mathbf{1}-p})$.

If $Z(\mathcal{M}) \neq \mathbb{C}$, i.e., if $\mathcal{M}$ is not a factor, then $Z(\mathcal{M})$ contains a projection $p \neq 0, \mathbf{1}$ (Proposition 4.18), so that $\mathcal{M}$ is not simple.

Exercise 4.12. Let $\mathcal{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra and suppose that $\mathcal{Z} := Z(\mathcal{M}) \cong \mathbb{C}^n$ as a $*$-algebra (Example 1.7(b)). Then there exist central projections $p_1, \ldots, p_n$ with

$$\mathbf{1} = p_1 + \cdots + p_n \quad \text{and} \quad p_i p_j = \delta_{ij} p_j.$$

We then have

$$\mathcal{M} \cong \bigoplus_{j=1}^n \mathcal{M}_{p_j},$$

where each ideal $\mathcal{M}_{p_j} \subseteq B(p_j\mathcal{H})$ is a factor.

Exercise 4.13. Let $\mathcal{R}$ be a unital ring an $M_n(\mathcal{R})$ the corresponding matrix ring. Show that, for a unital subring $\mathcal{S} \subseteq \mathcal{R}$, the commutants in $M_n(\mathcal{R})$ satisfy

$$(\mathcal{S}\mathbf{1})' = M_n(\mathcal{S}') \quad \text{and} \quad M_n(\mathcal{S})' = \mathcal{S}'\mathbf{1}.$$

Exercise 4.14. Let $\mathcal{S} \subseteq B(\mathcal{H})$ be a $*$-invariant subset. Show that $\mathcal{S}$ and the von Neumann algebra $\mathcal{S}''$ leave the same closed subspaces of $\mathcal{H}$ invariant.

Exercise 4.15. Show that, for any Hilbert space $\mathcal{H}$,

$$Z(B(\mathcal{H})) = \{Z \in B(\mathcal{H}) \colon (\forall A \in B(\mathcal{H})\, AZ = ZA\} = \mathbb{C}\mathbf{1}.$$

Hint: Apply Exercise 1.7 with $A = \langle v, \cdot \rangle v$ to see that every $v \in \mathcal{H}$ is an eigenvector of Z.

Exercise 4.16. Let $\mathcal{H}$ be a Hilbert space and $\beta \colon \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ be a continuous sesquilinear from. Then there exists a uniquely determined bounded operator $A \in B(\mathcal{H})$ with

$$\beta(v, w) = \langle v, Aw \rangle \quad \text{for all} \quad v, w \in \mathcal{H}.$$

Show further that β is hermitian (positive semidefinite) if and only if A is hermitian (positive).

Exercise 4.17. Show that the weak operator topology on $B(\mathcal{H})$ coincides with the coarsest topology for which all functionals $f_v(A) := \langle v, Av \rangle$, $v \in \mathcal{H}$, are continuous. Hint: Polarization identity, Exercise 3.1.

Exercise 4.18. Let $\mathcal{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra and $p \in P(\mathcal{M})$ be a projection. Show that $p\mathcal{M}p \subseteq B(p\mathcal{H})$ coincides with the commutant of $\mathcal{M}'|_{p\mathcal{H}}$. In particular, $p\mathcal{M}p$ is a von Neumann algebra. Hint: If $A \in B(p\mathcal{H})$ commutes with $\mathcal{M}'$ on $p\mathcal{H}$, then the trivial extension by 0 on $(p\mathcal{H})^\perp$ also commutes with $\mathcal{M}'$, and therefore $A \in \mathcal{M}''$ satisfies $A = pAp$ and therefore $A \in p\mathcal{M}p$.

Exercise 4.19. (Direct sums of von Neumann algebras) Let $\mathcal{M}_j \subseteq B(\mathcal{H}_j)$ be a family of von Neumann algebras, $\mathcal{H} := \widehat{\bigoplus}_{j\in J} \mathcal{H}_j$ the Hilbert space direct sum of the $\mathcal{H}_j$ and

$$\mathcal{M} := \overline{\bigoplus}_{j\in J} \mathcal{M}_j := \Big\{(M_j)_{j\in J} \in \prod_{j\in J} \mathcal{M}_j \colon \sup_{j\in J} \|M_j\| < \infty\Big\}$$

the ℓ^∞-direct sum of the von Neumann algebras $\mathcal{M}_j$. Show that $\mathcal{M}$ can be realized in a natural way as a von Neumann algebra on $\mathcal{H}$.

Exercise 4.20. Let X be an infinite set and $S_{(X)}$ be the group of all those permutations φ of X moving only finitely many points, i.e.,

$$|\{x \in X \colon \varphi(x) \neq x\}| < \infty.$$

Show that, for each element $\varphi \neq \mathrm{id}_X$ in $S_{(X)}$, the conjugacy class

$$C_\varphi := \{\psi\varphi\psi^{-1} \colon \psi \in S_{(X)}\}$$

is infinite, i.e., $S_{(X)}$ is ICC.
Hin: For each $\varphi \in S_{(X)}$, consider its fixed point set $\mathrm{Fix}(\varphi)$. How does it behave under conjugation?

Exercise 4.21. Consider the groups $\mathrm{PGL}_n(\mathbb{C}) := \mathrm{GL}_n(\mathbb{C})/\mathbb{C}^\times \mathbf{1}$ and $\mathrm{PU}_n(\mathbb{C}) := \mathrm{U}_n(\mathbb{C})/\mathbb{T}\mathbf{1}$. Show that in both groups the conjugacy class of every element $g \neq \mathbf{1}$ is infinite.

Exercise 4.22. Let $\mathcal{H}$ be an infinite-dimensional Hilbert space. Show that every trace functional (Definition 4.26) $T \colon B(\mathcal{H}) \to \mathbb{C}$ vanishes, i.e., $T \geq 0$ and

$$T(AB) = T(BA) \quad \text{for} \quad A, B \in B(\mathcal{H})$$

implies $T = 0$. Here are some steps to follow:

(a) T is conjugation invariant, i.e., $T(gAg^{-1}) = T(A)$ for $g \in \mathrm{GL}(\mathcal{H})$ and $A \in B(\mathcal{H})$.

(b) If P and Q are two orthogonal projections in $B(\mathcal{H})$ for which there are unitary isomorphisms $P(\mathcal{H}) \to Q(\mathcal{H})$ and $P(\mathcal{H})^\perp \to Q(\mathcal{H})^\perp$, then $T(P) = T(Q)$.

(c) For each $n \in \mathbb{N}$, there exists a unitary isomorphism $u_n \colon \mathcal{H} \to \mathcal{H}^n$, i.e.,

$$\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n \quad \text{with} \quad \mathcal{H}_j \cong \mathcal{H}.$$

Let $P_j^{(n)}$ denote the orthogonal projection onto $\mathcal{H}_j$.

(d) Show that $T(P_j^{(n)}) = \frac{1}{n}T(\mathbf{1})$ and use (b) to derive $T(P_1^{(2)}) = T(P_1^{(3)})$. Conclude that $T(\mathbf{1}) = 0$.

Exercise 4.23. (Jauch's Theorem) Let $\mathcal{A} \subseteq B(\mathcal{H})$ be a von Neumann algebra. Show that the following assertions are equivalent [16]

(a) The commutant $\mathcal{A}'$ is commutative (the representation of $\mathcal{A}$ on $\mathcal{H}$ is then called *multiplicity free*).

(b) Any maximal commutative von Neumann subalgebra $\mathcal{B} \subseteq \mathcal{A}$ satisfies $\mathcal{B}' = \mathcal{B}$, i.e., it is maximal commutative in $B(\mathcal{H})$.

Hint: If $\mathcal{A}'$ is commutative, then $\mathcal{A}' = Z(\mathcal{A}') = Z(\mathcal{A})$, and if $\mathcal{B} \subseteq \mathcal{A}$ is maximal commutative, then $\mathcal{A}' = Z(\mathcal{A}) \subseteq \mathcal{B}$, which in turn leads to $\mathcal{B}' \subseteq \mathcal{A}$. If, conversely, $\mathcal{B} \subseteq \mathcal{A}$ is commutative with $\mathcal{B}' = \mathcal{B}$, then $\mathcal{A}' \subseteq \mathcal{B}' \subseteq \mathcal{A}$.

Exercise 4.24. Let $(\pi, \mathcal{H})$ be an irreducible representation of the $*$-algebra $\mathcal{A}$ and $\pi_n := \sum_{j=1}^n \pi$ be the n-fold direct sum of π with itself on $\mathcal{H}^n = \bigoplus_{j=1}^n \mathcal{H}$. Show that

$$\pi_n(\mathcal{A})' \cong M_n(\mathbb{C}).$$

Hint: Write operators on $\mathcal{H}^n$ as matrices with entries in $B(\mathcal{H})$ (cf. Exercise 1.8) and evaluate the commuting condition.

[16] This exercise illustrates the hypothesis of Commutative Superselection Rules in Quantum Field Theory (cf. [Wi95, p. 759]). Here $\mathcal{A}$ plays the role of the von Neumann algebra generated by the observables of a quantum system, so that its commutant corresponds to the superselection rules. A commutative von Neumann algebra $\mathcal{B}$ which is maximal in $B(\mathcal{H})$ is called a *complete set of commuting observables*. The result discussed in the exercise (Jauch's Theorem) then states that the commutativity of the superselection rules is equivalent to the existence of a complete commuting set of observables.

Exercise 4.25. Let $\mathcal{A}$ be a $*$-algebra and $P(\mathcal{A})$ be the set of projections (hermitian idempotents) in $\mathcal{A}$. We define a relation $\sim$ on $P(\mathcal{A})$ by $p \sim q$ if there exists a $u \in \mathcal{A}$ with $p = u^*u$ and $q = uu^*$. Show that $\sim$ defines an equivalence relation on $P(\mathcal{A})$.

Exercise 4.26. (Partial isometries) An operator $U \in B(\mathcal{H})$ is called a *partial isometry* if $U|_{\ker(U)^\perp}$ is an isometry. Show that the following are equivalent:

(i) U is a partial isometry.

(ii) $P := U^*U$ is a projection (the *domain projection of* U).

(iii) U^* is a partial isometry.

(iv) $Q := UU^*$ is a projection (the *range projection of* U).

Exercise 4.27. Let $\mathcal{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra generated by the subalgebras $\mathcal{M}_n$, $n \in \mathbb{N}$. Suppose further, that $(\mathcal{H}_n)_{n\in\mathbb{N}}$ is an increasing sequence of subspaces whose union is dense in $\mathcal{H}$. Suppose that, for every $n \in \mathbb{N}$, the subspace $\mathcal{H}_n$ is $\mathcal{M}_n$-invariant and the representation of $\mathcal{M}_n$ on $\mathcal{H}_n$ is irreducible. Then the representation of $\mathcal{M}$ on $\mathcal{H}$ is irreducible.

5 Fock spaces and second quantization

In this section we use tensor powers to construct the symmetric (bosonic) Fock space

$$\mathcal{F}_+(\mathcal{H}) = S(\mathcal{H}) = \widehat{\bigoplus}_{n=0}^{\infty} S^n(\mathcal{H})$$

and the skew-symmetric (fermionic) Fock space

$$\mathcal{F}_-(\mathcal{H}) = \Lambda(\mathcal{H}) = \widehat{\bigoplus}_{n=0}^{\infty} \Lambda^n(\mathcal{H})$$

of a complex Hilbert space. In Subsection 5.1 we introduce the basic ingredients, the symmetric and alternating powers $S^n(\mathcal{H})$ and $\Lambda^n(\mathcal{H})$, which are subspaces of the n-fold tensor product $\mathcal{H}^{\otimes n}$. Then we construct the Weyl operators $W(v)$, $v \in \mathcal{H}$, on the symmetric Fock space in Subsection 5.2 and show that they define a unitary representation of the Heisenberg group $\mathrm{Heis}(\mathcal{H})$. For every real subspace $V \subseteq \mathcal{H}$, we thus obtain a von Neumann algebra $\mathcal{R}(V) := W(V)''$ on $S(\mathcal{H})$ (Subsection 5.3) and observe that this leads in many cases to factors. In Quantum Field Theory one uses this construction to construct so-called free field from a unitary representation of the Poincaré group ([Ar99]).

5.1 Symmetric and exterior powers

In this section we discuss two important constructions of new Hilbert spaces from old ones: symmetric and alternating powers. The symmetric group S_n has a unitary representation on $\mathcal{H}^{\otimes n}$ defined by permutation of the factors

$$\rho(\sigma)(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}.$$

The two most important subspaces of $\mathcal{H}^{\otimes n}$ are the two eigenspaces of S_n for the two characters of this group. For the trivial character we obtain the subspace

$$S^n(\mathcal{H}) := (\mathcal{H}^{\otimes n})^{S_n}$$

of S_n-invariant vectors. It is called the *nth symmetric power of* $\mathcal{H}$, and for the signature character $\mathrm{sgn}\colon S_n \to \{\pm 1\}$, we obtain the subspace

$$\Lambda^n(\mathcal{H}) := (\mathcal{H}^{\otimes n})^{S_n,\mathrm{sgn}} = \{v \in \mathcal{H}^{\otimes n}\colon (\forall \sigma \in S_n)\ \rho(\sigma)v = \mathrm{sgn}(\sigma)v\}.$$

It is called the *nth exterior power of* $\mathcal{H}$.

It is easy to write down projections onto these subspaces using the invariant probability measures on the finite group S_n:

$$P_+ := \frac{1}{n!}\sum_{\sigma\in S_n} \rho(\sigma)$$

is the projection onto $S^n(\mathcal{H})$ and

$$P_- := \frac{1}{n!}\sum_{\sigma\in S_n} \mathrm{sgn}(\sigma)\rho(\sigma)$$

is the projection onto $\Lambda^n(\mathcal{H})$ (Exercise 5.1).

For $v_1,\ldots,v_n \in \mathcal{H}$, we define the *symmetric product*

$$v_1\cdots v_n := v_1 \vee \cdots \vee v_n := \sqrt{n!}P_+(v_1\otimes\cdots\otimes v_n)$$

and the *exterior (=alternating) product* by

$$v_1 \wedge \cdots \wedge v_n := \sqrt{n!}P_-(v_1\otimes\cdots\otimes v_n).$$

These products define continuous complex n-linear maps $\mathcal{H}^n \to S^n(\mathcal{H})$ and $\mathcal{H}^n \to \Lambda^n(\mathcal{H})$. It follows directly from the definition that the $\vee$-product is symmetric and the wedge product $\wedge$ is alternating, i.e.,

$$v_{\sigma(1)} \vee \cdots \vee v_{\sigma(n)} = v_1 \vee \cdots \vee v_n \quad \text{for} \quad \sigma \in S_n$$

and

$$v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(n)} = \mathrm{sgn}(\sigma) v_1 \vee \cdots \vee v_n. \quad \text{for} \quad \sigma \in S_n$$

The inner products of such elements are given by

$$\begin{aligned}\langle v_1 \vee \cdots \vee v_n, w_1 \vee \cdots \vee w_n\rangle &= \sqrt{n!}\langle v_1 \vee \cdots \vee v_n, w_1 \otimes \cdots \otimes w_n\rangle \\ &= \sum_{\sigma\in S_n} \langle v_{\sigma(1)}, w_1\rangle \cdots \langle v_{\sigma(n)}, w_m\rangle \end{aligned} \tag{22}$$

and likewise

$$\begin{aligned} &\langle v_1 \wedge \cdots \wedge v_n, w_1 \wedge \cdots \wedge w_n\rangle = \sqrt{n!}\langle v_1 \wedge \cdots \wedge v_n, w_1 \otimes \cdots \otimes w_n\rangle \\ = &\sum_{\sigma\in S_n} \mathrm{sgn}(\sigma)\langle v_{\sigma(1)}, w_1\rangle \cdots \langle v_{\sigma(n)}, w_m\rangle = \det(\langle v_i, w_j\rangle)_{1\le i,j\le n}. \end{aligned} \tag{23}$$

We also note that, for $v \in \mathcal{H}$ and $v^n = \underbrace{v \vee \cdots \vee v}_{n-\text{times}}$, we have

$$\langle v^n, w^n\rangle = n!\langle v, w\rangle^n \quad \text{and} \quad \|v^n\| = \sqrt{n!}\|v\|^n. \tag{24}$$

Lemma 5.1. *Let $(e_j)_{j\in J}$ be an ONB in $\mathcal{H}$.*

(i) *For $\mathbf{m} \in \mathbb{N}_0^{(J)}$, the set of finitely supported functions $J \to \mathbb{N}_0, j \mapsto m_j$, considered as multi-indices, we put $|\mathbf{m}| := \sum_{j\in J} m_j$. Then the elements*

$$e^{\mathbf{m}} := \bigvee_{j\in J} e_j^{m_j} = e_{j_1}^{m_{j_1}} \vee \cdots \vee e_{j_k}^{m_{j_k}} \quad \textit{for} \quad \{j \in J \colon m_j \neq 0\} = \{j_1, \ldots, j_k\}$$

with $|\mathbf{m}| = n$ form an orthogonal basis of $S^n(\mathcal{H})$ satisfying

$$\|e^{\mathbf{m}}\|^2 = \mathbf{m}! \quad \textit{for} \quad \mathbf{m}! := \prod_{j\in J} m_j!.$$

(ii) *Suppose that $\leq$ is a linear order on J and, for $F = \{f_1, \ldots, f_n\} \subseteq J$ with $f_1 < \ldots < f_n$ put $e_F := e_{f_1} \wedge \cdots \wedge e_{f_n}$. Then $\{e_F \colon F \subseteq J, |F| = n\}$ is an orthonormal basis of $\Lambda^n(\mathcal{H})$.*

Proof. (i) Expanding wedge products $v_1 \vee \cdots \vee v_n \in S^n(\mathcal{H})$ with respect to the orthonormal basis $(e_j)_{j\in J}$, we see that the

$$e_{j_1} \vee \cdots \vee e_{j_n}, \quad j_1, \ldots, j_n \in J,$$

form a total subset of $S^n(\mathcal{H})$. This proves that the $e^{\mathbf{m}}$, $|\mathbf{m}| = n$, form a total subset of $S^n(\mathcal{H})$. Next we observe that, for

$$e^{\mathbf{m}} = e_{j_1}^{m_{j_1}} \vee \cdots \vee e_{j_k}^{m_{j_k}} \quad \text{and} \quad e^{\mathbf{n}} = e_{j_1}^{n_{j_1}} \vee \cdots \vee e_{j_\ell}^{n_{j_\ell}}$$

to have a non-zero scalar product we need that $m_j = n_j$ for every $j \in J$. Therefore the system $(e^{\mathbf{m}})_{|\mathbf{m}|=n}$ is orthogonal in $S^n(\mathcal{H})$. From (22) we further obtain that

$$\langle e^{\mathbf{m}}, e^{\mathbf{m}} \rangle = \prod_{j\in J} m_j! = \mathbf{m}!.$$

(ii) Expanding wedge products $v_1 \wedge \cdots \wedge v_n$ with respect to the orthonormal basis $(e_j)_{j\in J}$, we see that the

$$e_{j_1} \wedge \cdots \wedge e_{j_n}, \quad j_1, \ldots, j_n \in J,$$

form a total subset of $\Lambda^n(\mathcal{H})$. If $|\{j_1, \ldots, j_n\}| < n$, then $e_{j_1}, \ldots, e_{j_n}$ are linearly dependent, so that $e_{j_1} \wedge \cdots \wedge e_{j_n} = 0$. If $|\{j_1, \ldots, j_n\}| = n$, we put $F := \{j_1, \ldots, j_n\}$ and note that for $\sigma \in S_n$ we have

$$e_{j_1} \wedge \cdots \wedge e_{j_n} = \operatorname{sgn}(\sigma) e_{j_{\sigma(1)}} \wedge \cdots \wedge e_{j_{\sigma(n)}}.$$

This proves that the e_F, $F \subseteq J$ an n-element subset, form a total subset of $\Lambda^n(\mathcal{H})$. Since

$$\langle e_F, e_{F'} \rangle = \delta_{F,F'},$$

the e_F form an orthonormal basis of $\Lambda^n(\mathcal{H})$. □

Definition 5.2. The direct sum Hilbert space

$$S(\mathcal{H}) := \mathcal{F}_+(\mathcal{H}) := \widehat{\bigoplus}_{n=0}^{\infty} S^n(\mathcal{H})$$

is called the *symmetric (bosonic) Fock space* of $\mathcal{H}$. The direct sum Hilbert space

$$\Lambda(\mathcal{H}) := \mathcal{F}_-(\mathcal{H}) := \widehat{\bigoplus}_{n=0}^{\infty} \Lambda^n(\mathcal{H})$$

is called the *antisymmetric (fermionic) Fock space* of $\mathcal{H}$.

Remark 5.3. The terms bosonic and fermionic are due to the interpretation of these spaces in physics. There are two types of elementary particles: bosons and fermions. These are related to symmetric, resp., alternating tensors as follows.

(a) Suppose that the Hilbert space $\mathcal{H}_1$ describes the states of a single (quantum mechanical) particle P_1 in the sense explained in the introduction. Suppose further that $\mathcal{H}_2$ describes the states of a second particle P_2. The tensor product Hilbert space $\mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2$ corresponds to the composition of the systems corresponding to $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Therefore it describes the states of all those particles composed of one particle of type P_1 and one particle of type P_2.

(b) If P is a single particle described by the Hilbert space $\mathcal{H}$, then particles composed of n particles of this type are described by the Hilbert space $\mathcal{H}^{\otimes n}$. However, it turns out that this Hilbert space is (in general) far too large and one only needs a suitable subspace. If P is a *boson*, then the n-particle states are described by the subspace $S^n(\mathcal{H})$ of symmetric tensors and if P is a *fermion*, then the n-particle states are described by the subspace $\Lambda^n(\mathcal{H})$ of alternating tensors. This means that n bosons of the same type form symmetric states in which they cannot be distinguished. In particular, all bosons contributing to an n-particle state may be in the same state, and this corresponds to the states $[v^n] \in \mathbb{P}(S^n(\mathcal{H}))$.

Fermions behave very differently: in an n-fermion state all n particles have to be in different states. This has the interesting consequence that, if $\mathcal{H}$ is of finite dimension n (interpreted as n different fermion states), then $\Lambda^k(\mathcal{H}) = \{0\}$ for $k > n$ means that there are no k-fermion state for $k > n$ that can be formed with particles corresponding to $\mathcal{H}$.

(c) The bosonic Fock space $\mathcal{F}_+(\mathcal{H}) = S(\mathcal{H})$ is the Hilbert spaces describing all finite particle states of a single boson and the fermionic Fock space $\mathcal{F}_-(\mathcal{H}) = \Lambda(\mathcal{H})$ is the Hilbert spaces describing all finite particle states of a single fermion. In this context the space $\mathcal{H}$, considered as a subspace of $\mathcal{F}_\pm(\mathcal{H})$ is called the *one-particle space*. The passage from one-particle systems to multi-particle systems, resp., fields, is often called *second quantization*. Here the idea is that a "first quantization" leads from a mechanical system to a quantum system corresponding to $\mathcal{H}$ and the "second quantization" establishes the passage from single particles to fields. The one-dimensional space $\mathcal{F}^0_\pm(\mathcal{H})$ corresponds to the zero-particle states and the unit vectors in this space, denoted Ω, are called *vacuum vectors*.

5.2 Weyl operators on the symmetric Fock space

In this subsection we consider the bosonic Fock space $S(\mathcal{H})$ of the complex Hilbert space $\mathcal{H}$. We want to define natural unitary operators on this space, called the *Weyl operators*. They will form a unitary representation of the *Heisenberg group* $\mathrm{Heis}(\mathcal{H})$.

We start by observing that, for every $v \in \mathcal{H}$, the series

$$\mathrm{Exp}(v) := \sum_{n=0}^\infty \frac{1}{n!} v^n ,$$

defines an element in $S(\mathcal{H})$ and that by (24) the scalar product of two such elements is given by

$$\langle \mathrm{Exp}(v), \mathrm{Exp}(w) \rangle = \sum_{n=0}^\infty \frac{n!}{(n!)^2} \langle v, w \rangle^n = e^{\langle v, w \rangle}.$$

Lemma 5.4. $\mathrm{Exp}(\mathcal{H})$ *is total in* $S(\mathcal{H})$, *i.e., it spans a dense subspace.*

Proof. Let $\mathcal{K} \subseteq S(\mathcal{H})$ be the closed subspace generated by $\mathrm{Exp}(\mathcal{H})$. We consider the unitary representation of the circle group $\mathbb{T} \subseteq \mathbb{C}^\times$ on $S(\mathcal{H})$ by

$$U_z(v_1 \vee \cdots \vee v_n) := z^n(v_1 \vee \cdots \vee v_n) \quad \text{for} \quad n \in \mathbb{N}_0, v_j \in \mathcal{H}.$$

The decomposition $S(\mathcal{H}) = \widehat{\bigoplus}_{n=0}^\infty S^n(\mathcal{H})$ is the eigenspace decomposition with respect to the operators U_z and it is easy to see that the action of $\mathbb{T}$ on $S(\mathcal{H})$ has continuous orbit maps (Exercise 5.6).

For $\xi \in S(\mathcal{H})$ with $\xi = \sum_{n=0}^\infty \xi_n$ and $\xi_n \in S^n(\mathcal{H})$, we have $U_z\xi = \sum_n z^n \xi_n$, so that

$$\xi_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-2\pi i n t} U_{e^{it}}\xi\, dt$$

(observe the analogy with Fourier coefficients). It follows that, for $\xi \in \mathcal{K}$, the existence of the above Riemann integral in the closed subspace $\mathcal{K}$ implies $\xi_n \in \mathcal{K}$. We conclude that $v^n \in \mathcal{K}$ for $v \in \mathcal{H}$ and $n \in \mathbb{N}_0$. Therefore it suffices to observe that the subset $\{v^n \colon v \in \mathcal{H}\}$ is total in $S^n(\mathcal{H})$ (Exercise 5.4). $\square$

For $v, x \in \mathcal{H}$ we have

$$\langle \mathrm{Exp}(v+x), \mathrm{Exp}(w+x)\rangle = e^{\langle v+x, w+x\rangle} = e^{\langle v,w\rangle} e^{\langle x,w\rangle + \frac{\|x\|^2}{2}} e^{\langle v,x\rangle + \frac{\|x\|^2}{2}},$$

so that there exists a well-defined and uniquely determined unitary operator U_x on $S(\mathcal{H})$ satisfying

$$U_x \,\mathrm{Exp}(v) = e^{-\langle x,v\rangle - \frac{\|x\|^2}{2}} \,\mathrm{Exp}(v+x) \quad \text{for} \quad x, v \in \mathcal{H} \tag{25}$$

(Exercise 5.5; the surjectivity of U_x follows from the totality of $\mathrm{Exp}(\mathcal{H})$). A direct calculation then shows that

$$U_x U_y = e^{-i \,\mathrm{Im}\langle x,y\rangle} U_{x+y} \quad \text{for} \quad x, y \in \mathcal{H}. \tag{26}$$

In fact, for $v \in \mathcal{H}$, we have

$$\begin{aligned}
U_x U_y \,\mathrm{Exp}(v) &= U_x e^{-\langle y,v\rangle - \frac{\|y\|^2}{2}} \,\mathrm{Exp}(v+y) \\
&= e^{-\langle y,v\rangle - \frac{\|y\|^2}{2}} e^{-\langle x, v+y\rangle - \frac{\|x\|^2}{2}} \,\mathrm{Exp}(v+y+x) \\
&= e^{-\langle x+y, v\rangle} e^{-\frac{\|y\|^2}{2} - \frac{\|x\|^2}{2} - \langle x,y\rangle} \,\mathrm{Exp}(v+y+x)
\end{aligned}$$

and

$$\begin{aligned}
U_{x+y} \,\mathrm{Exp}(v) &= e^{-\langle x+y, v\rangle - \frac{\|x+y\|^2}{2}} \,\mathrm{Exp}(v+y+x) \\
&= e^{-\langle x+y, v\rangle - \frac{\|x\|^2}{2} - \frac{\|y\|^2}{2} - \mathrm{Re}\langle x,y\rangle} \,\mathrm{Exp}(v+y+x)
\end{aligned}$$

The relation (26) shows that the map $U \colon (\mathcal{H}, +) \to \mathrm{U}(S(\mathcal{H}))$ is not a group homomorphism. Instead, we have to replace the additive group of $\mathcal{H}$ by the *Heisenberg group*

$$\mathrm{Heis}(\mathcal{H}) := \mathbb{T} \times \mathcal{H} \quad \text{with} \quad (z,v)(z',v') := (zz' e^{-i\,\mathrm{Im}\langle v,v'\rangle}, v+v').$$

For this group, we obtain a unitary representation

$$\widehat{U} \colon \mathrm{Heis}(\mathcal{H}) \to \mathrm{U}(S(\mathcal{H})) \quad \text{by} \quad \widehat{U}_{(z,v)} := zU_v.$$

The operators

$$W(v) := U_{iv/\sqrt{2}}, \qquad v \in \mathcal{H}$$

are called *Weyl operators.* They satisfying the *Weyl relations*

$$W(v)W(w) = e^{-i\operatorname{Im}\langle v,w\rangle/2}W(v+w), \qquad v, w \in \mathcal{H}. \tag{27}$$

The *Weyl algebra*

$$W(\mathcal{H}) := C^*(\{W(v) \colon v \in \mathcal{H}\}) \subseteq B(S(\mathcal{H}))$$

is the C^*-subalgebra of $B(S(\mathcal{H}))$ generated by the Weyl operators. The Weyl algebra plays an important role in Quantum (Statistical) Mechanics and Quantum Field Theory. This is partly due to the fact that it is a simple C^*-algebra (all ideals are trivial), which implies that all its representations are faithful. Closely related is its universal property: If $\mathcal{A}$ is a unitary C^*-algebra and $\varphi\colon \mathcal{H} \to \mathrm{U}(\mathcal{A})$ a map satisfying the Weyl relations in the form

$$\varphi(v)\varphi(w) = e^{-i\operatorname{Im}\langle v,w\rangle/2}\varphi(v+w), \qquad v, w \in \mathcal{H}, \tag{28}$$

then there exists a unique homomorphism $\Phi\colon W(\mathcal{H}) \to \mathcal{A}$ of unital C^*-algebras with $\Phi \circ W = \varphi$. An excellent discussion of the Weyl algebra and its properties can be found in the monograph [BR96] which also describes the physical applications in great detail.

5.3 From real subspaces to von Neumann algebras

In this subsection we describe a mechanism that associates to real subspaces of a Hilbert space $\mathcal{H}$ von Neumann algebras on the symmetric Fock space $S(\mathcal{H})$. This constructions plays an important role in recent developments of Algebraic Quantum Field Theory (AQFT) because it provides natural links between the geometric structure of spacetime and operator algebras (see in particular [Ar99, Lo08, Le15]). It has also been of great interest for the classification of factors because it provides very controlled constructions of factors whose type can be determined in some detail ([AW63, AW68]).

We write

$$\gamma(v, w) := \operatorname{Im}\langle v, w\rangle \quad \text{for} \quad v, w \in \mathcal{H}$$

and observe that γ is skew-symmetric and non-degenerate, so that the underlying real Hilbert space $\mathcal{H}^{\mathbb{R}}$ carries the structure of a *symplectic vector space* $(\mathcal{H}^{\mathbb{R}}, \gamma)$. Accordingly, we write

$$V' := \{w \in \mathcal{H} \colon (\forall v \in V)\ \gamma(v, w) = 0\}$$

for the *symplectic orthogonal space* of V. It is easy to see that $V' = iV^{\perp_{\mathbb{R}}}$, where $V^{\perp_{\mathbb{R}}}$ is the real orthogonal space of V with respect to the real-valued scalar product $\operatorname{Re}\langle v, w\rangle$. Note that $V'' = \overline{V}$ follows from the Hahn–Banach Extension Theorem.

Using the Weyl operators, we associate to every real linear subspace $V \subseteq \mathcal{H}$ a von Neumann subalgebra

$$\mathcal{R}(V) := W(V)'' = \{W(v) \colon v \in V\}'' \subseteq B(S(\mathcal{H})).$$

This leads to a variety of interesting von Neumann algebras and also to factors of various types.

Lemma 5.5. *We have*

(i) $\mathcal{R}(\mathcal{H}) = B(\mathcal{F}_+(\mathcal{H}))$, *resp., the representation of* $\mathrm{Heis}(\mathcal{H})$ *on* $\mathcal{F}_+(\mathcal{H})$ *is irreducible.*

(ii) $\mathcal{R}(V) = \mathcal{R}(\overline{V})$.

(iii) $\Omega = \mathrm{Exp}(0) \in \mathcal{F}_+(\mathcal{H})$ *is cyclic for* $\mathcal{R}(V)$ *if and only if* $V + iV$ *is dense in* $\mathcal{H}$.

(iv) $\Omega = \mathrm{Exp}(0) \in \mathcal{F}_+(\mathcal{H})$ *is separating for* $\mathcal{R}(V)$ *if and only if* $\overline{V} \cap i\overline{V} = \{0\}$.

(v) $\mathcal{R}(V) \subseteq \mathcal{R}(W)'$ *if and only if* $V \subseteq W'$.

(vi) $\mathcal{R}(V)$ *is commutative if and only if* $V \subseteq V'$.

Proof. (i) follows from [BR96, Prop. 5.2.4(3)].

(ii) follows from the fact that $\mathcal{H} \to B(\mathcal{F}_+(\mathcal{H})), v \mapsto W_v$ is strongly continuous and $\mathcal{R}(V)$ is closed in the weak operator topology.

(iii) Let $\mathcal{K} := \overline{V + iV}$. Then $\mathcal{R}(V)\Omega \subseteq \mathcal{F}_+(\mathcal{K})$, so that Ω cannot be cyclic if $\mathcal{K} \neq \mathcal{H}$.

Suppose, conversely, that $\mathcal{K} = \mathcal{H}$ and that $f \in (\mathcal{R}(V)\Omega)^\perp$. Then the holomorphic function $\widehat{f}(v) := \langle f, \mathrm{Exp}(v)\rangle$ on $\mathcal{H}$ vanishes on iV, hence also on $V + iV$, and since this subspace is dense in $\mathcal{H}$, we obtain $f = 0$ because $\mathrm{Exp}(\mathcal{H})$ is total in $\mathcal{F}_+(\mathcal{H})$.

(iv) In view of (ii), we may assume that V is closed. Let $0 \neq w \in \mathcal{K} := V \cap iV$. To see that Ω is not separating for $\mathcal{R}(V)$, it suffices to show that, for the one-dimensional Hilbert space $\mathcal{H}_0 := \mathbb{C}w$, the vector Ω is not separating for $\mathcal{R}(\mathbb{C}w) = B(\mathcal{F}_+(\mathbb{C}w))$ (see (i)). This is obviously the case because $\dim \mathcal{F}_+(\mathbb{C}w) > 1$.

Suppose that $\mathcal{K} = \{0\}$. As $\mathcal{K} = V'' \cap (iV'') = (V' + iV')'$, it follows that $V' + iV'$ is dense in $\mathcal{H}$. By (iii), Ω is cyclic for $\mathcal{R}(V')$ which commutes with $\mathcal{R}(V)$. Therefore Ω is separating for $\mathcal{R}(V)$.

(v) follows directly from the Weyl relations (27).

(vi) follows from (v). □

Theorem 5.6. ([Ar63]) (Araki's Theorem) *For closed real subspaces* V, W, V_j *of* $\mathcal{H}$, *the following assertions hold:*

(i) $\mathcal{R}(V) \subseteq \mathcal{R}(W)$ *if and only if* $V \subseteq W$.

(ii) $\mathcal{R}\big(\bigcap_{j\in J} V_j\big) = \bigcap_{j\in J} \mathcal{R}(V_j)$.

(iii) $\mathcal{R}(V)' = \mathcal{R}(V')$ *(Duality).*

(iv) $Z(\mathcal{R}(V)) = \mathcal{R}(V \cap V')$. *In particular,* $\mathcal{R}(V)$ *is a factor if and only if* $V \cap V' = \{0\}$.

Proof. We only comment on some of these statements:

(i) That $V \subseteq W$ implies $\mathcal{R}(V) \subseteq \mathcal{R}(W)$ is clear, but the converse is non-trivial. It can be derived from the duality propery (iv), which is a deep result, basically the main result of [Ar63].

(ii) here "$\subseteq$" is easy.

(iii) is a deep theorem.

(iv) follows from (ii) and (iii). □

The preceding theorem asserts in particular that $\mathcal{R}(V)$ is a factor if and only if $V \cap V' = \{0\}$. Subspaces with this property are very easy to construct. In [Ar64b] many results on the types of the so-obtained factors have been derived. In particular, it is shown that factors of type II do not arise from this construction and [Ar64] provides an explicit criterion for $\mathcal{R}(V)$ to be of type I. "Generically", the so-obtained factors are of type III.

5.4 The canonical commutation relations (CCR)

On the dense subspace $S(\mathcal{H})_0 = \sum_{n=0}^\infty S^n(\mathcal{H})$ of the symmetric Fock space $S(\mathcal{H})$, we have for each $f \in \mathcal{H}$ the *creation operator*

$$a^*(f)(f_1 \vee \cdots \vee f_n) := f \vee f_1 \vee \cdots \vee f_n.$$

This operator has an adjoint $a(f)$ on $S(\mathcal{H})_0$, given by

$$a(f)\Omega = 0, \quad a(f)(f_1 \vee \cdots \vee f_n) = \sum_{j=1}^n \langle f, f_j\rangle f_1 \vee \cdots \vee \widehat{f_j} \vee \cdots \vee f_n,$$

where $\widehat{f_j}$ means omitting the factor f_j. Note that $a(f)$ defines a derivation on the commutative algebra $S(\mathcal{H})_0$. One easily verifies that these operators satisfy the *canonical commutation relations (CCR)*:

$$[a(f), a(g)] = 0 \quad [a(f), a^*(g)] = \langle f, g\rangle \mathbf{1}. \tag{29}$$

Lemma 5.7. $\frac{d}{dt}\big|_{t=0} U(tx) = a^*(x) - a(x)$ *on* $S_0(\mathcal{H})$.

Proof. We associate to every $F \in S(\mathcal{H})$ the function on $\mathcal{H}$ defined by

$$\Theta(F)(v) := \langle F, \mathrm{Exp}(v)\rangle$$

and note that

$$\Theta(v_1 \vee \cdots \vee v_n)(v) = \langle v_1 \vee \cdots \vee v_n, \mathrm{Exp}(v)\rangle = \frac{1}{n!}\langle v_1 \vee \cdots \vee v_n, v^n\rangle = \prod_{j=1}^n \langle v_j, v\rangle = \prod_{j=1}^n \Theta(v_j)(v).$$

This implies in particular that

$$\Theta|_{S(\mathcal{H})_0} \colon S(\mathcal{H})_0 \to \mathrm{Pol}(\mathcal{H}) \tag{30}$$

is a unital algebra homomorphism with respect to the natural commutative algebra structure on $S(\mathcal{H})_0$ defined by

$$(v_1 \vee \cdots \vee v_k) \cdot (w_1 \vee \cdots \vee w_m) := v_1 \vee \cdots \vee v_k \vee w_1 \vee \cdots \vee w_m.$$

It follows in particular that

$$\Phi \circ a^*(v) = m_{\Theta(v)} \circ \Phi \quad \text{and} \quad \Phi \circ a(v) = \partial_v \circ \Phi,$$

where $m_g(h) = g \cdot h$ and

$$(\partial_v F)(x) = \frac{d}{dt}\Big|_{t=0} F(x + tv).$$

Next we observe that

$$\Theta(U_x F)(v) = \langle F, U_{-x}\,\mathrm{Exp}(v)\rangle = e^{\langle x, v\rangle - \frac{\|x\|^2}{2}}\langle F, \mathrm{Exp}(v - x)\rangle = e^{\langle x, v\rangle - \frac{\|x\|^2}{2}}\Theta(F)(v - x).$$

Accordingly, we define the operators $\widehat{U}_x$ on functions $F \colon \mathcal{H} \to \mathbb{C}$ by

$$(\widehat{U}_x F)(v) := e^{\langle x, v\rangle - \frac{\|x\|^2}{2}} F(v - x).$$

If F is partially differentiable, this leads to

$$\frac{d}{dt}\Big|_{t=0}(\widehat{U}_{tx}F)(v) = \langle x, v\rangle F(v) - (\partial_x F)(v) = \Theta(x)(v)F(v) - (\partial_x F)(v),$$

i.e.,

$$\Theta \circ \frac{d}{dt}\Big|_{t=0} U_{tx} = \frac{d}{dt}\Big|_{t=0} \widehat{U}_{tx} \circ \Theta = (m_{\Theta(x)} - \partial_x) \circ \Theta = \Theta \circ (a^*(x) - a(x))$$

by (30). This implies the assertion because Θ is injective. $\square$

Remark 5.8. In the context of the theory of unbounded operators, one can show that the skew-hermitian operators $a^*(v) - a(v)$ on $S(\mathcal{H})_0$ have a skew-adjoint closure, so that Stone's Theorem leads to unitary operators $e^{\overline{a^*(v)-a(v)}}$, and the preceding calculations then imply that

$$U_v = e^{\overline{a^*(v)-a(v)}} \quad \text{for} \quad v \in \mathcal{H}.$$

Accordingly, we obtain

$$W(v) = U_{iv/\sqrt{2}} = e^{\frac{i}{\sqrt{2}}\overline{a^*(v)+a(v)}} \quad \text{for} \quad v \in \mathcal{H}.$$

Remark 5.9. As the Weyl algebra $W(\mathcal{H}) \subseteq B(S(\mathcal{H}))$ is generated by the Weyl operators, the *Fock state* corresponding to the vacuum vector $\Omega = \mathrm{Exp}(0) \in S(\mathcal{H})_0$ is uniquely determined by

$$\langle \Omega, W(f)\Omega\rangle = \langle \mathrm{Exp}(0), U_{if/\sqrt{2}}\,\mathrm{Exp}(0)\rangle = \langle \mathrm{Exp}(0), e^{-\|f\|^2/4}\,\mathrm{Exp}(if/\sqrt{2})\rangle = e^{-\frac{1}{4}\|f\|^2}. \quad (31)$$

Exercises for Section 5

Exercise 5.1. Let G be a finite group, $(\pi, \mathcal{H})$ be a unitary representation and $\chi\colon G \to \mathbb{C}^\times$ be a homomorphism (a character of G). Show that:

(i) $P_\chi := \frac{1}{|G|}\sum_{g\in G}\chi(g)^{-1}\pi(g)$ is the orthogonal projection onto the χ-eigenspace

$$\mathcal{H}_\chi := \{v \in \mathcal{H}\colon (\forall g \in G)\, \pi(g)v = \chi(g)v\}.$$

(ii) $P_\chi P_\eta = 0$ for $\chi \neq \eta$.

(iii) $P := \sum_{\chi\in\mathrm{Hom}(G,\mathbb{T})} P_\chi$ is the projection onto the closed subspace of $\mathcal{H}$ spanned by the G-eigenvectors.

Exercise 5.2. (Second quantization as a functor) Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces and $A\colon \mathcal{H} \to \mathcal{K}$ be a contraction. Show that:

(a) There are unique contractions

$$\Gamma_\pm(A)\colon \mathcal{F}_\pm(\mathcal{H}) \to \mathcal{F}_\pm(\mathcal{K})$$

defined by

$$\Gamma_+(A)(v_1 \vee \cdots \vee v_n) := Av_1 \vee \cdots \vee Av_n, \quad \text{resp.,} \quad \Gamma_-(A)(v_1 \wedge \cdots \wedge v_n) := Av_1 \wedge \cdots \wedge Av_n.$$

(b) $\Gamma_\pm(\mathbf{1}_\mathcal{H}) = \mathbf{1}_{\mathcal{F}_\pm(\mathcal{H})}$, $\Gamma_\pm(AB) = \Gamma_\pm(A)\Gamma_\pm(B)$ and $\Gamma_\pm(A^*) = \Gamma_\pm(A)^*$.

Exercise 5.3. Let V and W be $\mathbb{K}$-vector spaces and $\beta\colon V^n \to W$ be a symmetric n-linear map. Show that β is completely determined by the values on the diagonal $\beta(v,\ldots,v)$, $v \in V$.
Hint: Consider

$$\beta((t_1v_1+\ldots+t_nv_n)^n) = \sum_{m_1+\ldots+m_n=n} \frac{n!}{m_1!\cdots m_n!} t_1^{m_1}\cdots t_n^{m_n}\beta(v_1^{m_1},\ldots,v_n^{m_n})$$

and recover $\beta(v_1,\ldots,v_n)$ as a suitable partial derivative.

Exercise 5.4. Let V be $\mathbb{K}$-vector space and $S^n(V) := (V^{\otimes})^{S_n}$ be the nth symmetric power of V. Show that

$$S^n(V) = \operatorname{span}\{v^{\otimes n}\colon v \in V\}.$$

Hint: Use the same technique as in Exercise 5.3.

Exercise 5.5. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert space, X be a set and $\gamma_j\colon X \to \mathcal{H}_j$, $j = 1, 2$, be maps with total range. Then the following are equivalent:

(a) There exists a unitary operator $U\colon \mathcal{H}_1 \to \mathcal{H}_2$ with $U \circ \gamma_1 = \gamma_2$.

(b) $\langle \gamma_2(x), \gamma_2(y)\rangle = \langle \gamma_1(x), \gamma_1(y)\rangle$ for all $x, y \in X$.

Exercise 5.6. Let X be a topological space, $\mathcal{H}$ be a Hilbert space and $\gamma\colon X \to \mathcal{H}$ be a map. Show that γ is continuous if and only if the corresponding kernel function

$$K\colon X \times X \to \mathbb{C}, \quad K(x,y) := \langle \gamma(x), \gamma(y)\rangle$$

is continuous.

A Complementary material

A.1 Initial and Final Topologies

If $f\colon X \to Y$ is a map and τ_X is a topology on X, then the topology

$$f_*\tau_X := \{A \subseteq Y\colon f^{-1}(A) \in \tau_X\}$$

is called the *push-forward of τ_X by f*. Similarly, we obtain for each topology τ_Y on Y a topology

$$f^*\tau_Y := f^{-1}\tau_Y := \langle f^{-1}(O)\colon O \in \tau_Y\rangle_{\mathrm{top}}$$

on X, called the *pull-back of τ_Y by f*. The main point of initial and final topologies is to extend these concepts to families of maps.

Definition A.1. Let X be a set and $(Y_i, \tau_i)_{i\in I}$ be topological spaces.

(a) Let $f_i\colon X \to Y_i$ be maps. Then the topology

$$\tau := \langle f_i^{-1}(\tau_i), i \in I\rangle_{top}$$

generated by all inverse images $f_i^{-1}(O_i)$, $O_i \in \tau_i$, is called the *initial topology* defined by the family $(f_i, Y_i)_{i\in I}$.

(b) Let $f_i\colon Y_i \to X$ be maps. Then the topology

$$\tau := \{U \subseteq X\colon (\forall i \in I) f_i^{-1}(U) \in \tau_i\} = \bigcap_{i\in I} f_{i,*}\tau_i$$

is called the *final topology on X* defined by the family $(f_i, Y_i)_{i\in I}$. That τ is indeed a topology is due to the fact that the assignment $U \mapsto f_i^{-1}(U)$ preserves arbitrary intersections and unions.

Lemma A.2. *If $f : (X, \tau_X) \to (Y, \tau_Y)$ is a map between topological spaces and $\mathcal{B}$ a sub-basis of τ_Y, then f is continuous if and only if for each $B \in \mathcal{B}$, the inverse image $f^{-1}(B)$ is open.*

Proof. The set

$$f_*\tau_X := \{A \subseteq Y : f^{-1}(A) \in \tau_X\}$$

is easily seen to be a topology on Y. Now f is continuous if and only if $f_*\tau_X \supseteq \tau_Y$, and since $\mathcal{B}$ generates τ_Y, this happens if and only if $\mathcal{B} \subseteq f_*\tau_X$. □

Lemma A.3. *The initial topology τ defined by the family $f_i : X \to Y_i$, $i \in I$, of maps is the coarsest topology for which all maps f_i are continuous. It has the following universal property: If Z is a topological space, then a map $h : Z \to X$ is continuous if and only if all maps $f_i \circ h : Z \to Y_i$ are continuous.*

Proof. Apply Lemma A.2 to the sub-basis $\{f_i^{-1}(O_i) : O_i \subseteq Y_i \text{ open}\}$ of τ. □

Lemma A.4. *The final topology defined by the family $f_i : Y_i \to X$, $i \in I$, is the finest topology for which all maps f_i are continuous. It has the following universal property: If Z is a topological space, then a map $h : X \to Z$ is continuous if and only if all maps $h \circ f_i$, $i \in I$, are continuous.*

Proof. For an open subset $O \subseteq Z$, the inverse image $h^{-1}(O) \subseteq X$ is open if and only if for each $i \in I$, the set $f_i^{-1}(h^{-1}(O)) = (h \circ f_i)^{-1}(O)$ is open in Y_i. Therefore h is continuous if and only if each map $h \circ f_i$ is continuous. □

Example A.5. Let (X, τ) be a topological space and $Y \subseteq X$ be a subset. We write $\iota_Y : Y \to X$ for the canonical embedding, mapping each $y \in Y$ to itself. Then the initial topology on Y with respect to ι_Y coincides with the subspace topology

$$\tau_Y = \{\iota_Y^{-1}(U) = U \cap Y : U \in \tau\}.$$

Definition A.6. (Quotient topology) (a) Let $\sim$ be an equivalence relation on the topological space X, $[X] := X/\sim = \{[x] : x \in X\}$ be the set of equivalence classes, and $q : X \to [X], x \mapsto [x]$ the quotient map. Then the final topology on $[X]$ defined by $q : X \to [X]$ is called the *quotient topology*.

(b) According to Definition A.1, a subset $U \subseteq [X]$ is open if and only its inverse image $q^{-1}(U)$ is an open subset of X and Lemma A.4 implies that a map $h : [X] \to Z$ to a topological space Z is continuous if and only if $h \circ q\ X \to Z$ is continuous.

(c) An important special cases arises if $S \subseteq X$ is a subset and we define the equivalence relation $\sim$ in such a way that $S = [x]$ for each $x \in S$ and $[y] = \{y\}$ for each $y \in S^c$. Then the quotient space is also denoted $X/S := X/\sim$. It is obtained by collapsing the subset S to a point.

Definition A.7. (Product topology) Let $(X_i)_{i\in I}$ be a family of topological spaces and $X := \prod_{i\in I} X_i$ be their product set. We think of its elements as all tuples $(x_i)_{i\in I}$ with $x_i \in X_i$, or, equivalently, as the set of all maps $x : I \to \bigcup_{i\in I} X_i$ with $x_i := x(i) \in X_i$ for each $i \in I$.

We have for each $i \in I$ a projection map

$$p_i : X \to X_i,\ (x_j)_{j\in I} \mapsto x_i.$$

The initial topology on X with respect to this family $p_i : X \to X_i$ is called the *product topology* and X, endowed with this topology, it called the *topological product space*.

Example A.8. Typical examples of product spaces are $\mathbb{R}^n$ and $\mathbb{C}^n$. More generally, one can shows that for any finite collection $(X_1, d_1), \ldots, (X_n, d_n)$ of metric spaces, the metrics

$$d_1(x, y) := \sum_{j=1}^{n} d_j(x_j, y_j) \quad \text{and} \quad d_\infty(x, y) := \max\{d_j(x_j, y_j) \colon j = 1, \ldots, n\}$$

define the product topology on $X := \prod_{j=1}^{n} X_j$.

From Lemma A.4, we immediately obtain:

Proposition A.9. *A map $f = (f_i) : Y \to \prod_{i \in I} X_i$ to a product space is continuous if and only if all component maps $f_i = p_i \circ f : Y \to X_i$ are continuous.*

Example A.10. (The topology of pointwise convergence) Let X be a set and Y be a topological space. We identify the set $\mathcal{F}(X, Y)$ of all maps $X \to Y$ with the product space $Y^X = \prod_{x \in X} Y$. Then the product topology on Y^X yields a topology on $\mathcal{F}(X, Y)$, called the *topology of pointwise convergence*. We shall see later, when we discuss convergence in topological spaces, why this makes sense.

It is the coarsest topology on $\mathcal{F}(X, Y)$ for which all evaluation maps

$$\mathrm{ev}_x \colon \mathcal{F}(X, Y) \to Y, \quad f \mapsto f(x)$$

are continuous because these maps correspond to the projections $Y^X \to Y$.

A.2 Locally Compact Spaces

Proposition A.11. *For a topological space X, the following are equivalent:*

(i) *X is quasi-compact, i.e., every open cover has a finite subcover.*

(ii) *For each family $(A_i)_{i \in I}$ of closed subsets of X with $\bigcap_{i \in I} A_i = \emptyset$, there exists a finite subset $F \subseteq I$ with $\bigcap_{i \in F} A_i = \emptyset$.*

Proof. (i) $\Leftrightarrow$ (ii) follows by taking complements: The condition $\bigcap_{i \in I} A_i = \emptyset$ means that the family $(A_i^c)_{i \in I}$ of complements is an open covering of X because $X = \emptyset^c = \bigcup_{i \in I} A_i^c$. Similarly, $\bigcap_{i \in F} A_i = \emptyset$ means that $(A_i^c)_{i \in F}$ is a finite subcovering. $\square$

Definition A.12. A separated topological space X is called *locally compact* if each point $x \in X$ has a compact neighborhood.

Lemma A.13. *If X is locally compact and $x \in X$, then each neighborhood U of x contains a compact neighborhood of x.*

Proof. Let K be a compact neighborhood of $x \in X$. Since it suffices to show that $U \cap K$ contains a compact neighborhood of x, we may w.l.o.g. assume that X is compact. Replacing U by its interior, we may further assume that U is open, so that its complement U^c is compact.

We argue by contradiction and assume that U does not contain any compact neighborhood of x. Then the family $\mathcal{F}$ of all intersections $C \cap U^c$, where C is a compact neighborhood of x, contains only non-empty sets and is stable under finite intersections. We thus obtain a family of closed subsets of the compact space U^c for which all finite intersections are non-empty, and therefore Proposition A.11 implies that its intersection $\bigcap_C (C \cap U^c)$ contains a point y. Then $y \in U^c$ implies $x \neq y$, and since X is separated, there exist open neighborhoods U_x of x and U_y of y with $U_x \cap U_y = \emptyset$. Then U_y^c is a compact neighborhood of x, which leads to the contradiction $y \in U_y^c \cap U^c$ to $y \in U_y$. $\square$

Definition A.14. A subset A of a topological space X is said to be *relatively compact* if $\overline{A}$ is compact.

Lemma A.15. *Let X be locally compact, $K \subseteq X$ compact and $U \supseteq K$ open. Then there exists a compact subset $V \subseteq X$ with*

$$K \subseteq V^0 \subseteq V \subseteq U.$$

Proof. For each $x \in K$ we choose a compact neighborhood $V_x \subseteq U$ (Lemma A.13). Then there exist finitely many $x_1, \ldots, x_n$ with $K \subseteq \bigcup_{i=1}^n V_{x_i}^0$ and we put $V := \bigcup_{i=1}^n V_{x_i} \subseteq U$. ☐

Proposition A.16. (Urysohn's Theorem) *Let X be locally compact, $K \subseteq X$ compact and $U \supseteq K$ be an open subset. Then there exists a continuous function $h\colon X \to \mathbb{R}$ with*

$$h|_K = 1 \quad \textit{and} \quad h|_{X \setminus U} = 0.$$

Proof. We put $U(1) := U$. With Lemma A.15, we find an open, relatively compact subset $U(0)$ with $K \subseteq U(0) \subseteq \overline{U(0)} \subseteq U(1)$. Iterating this procedure leads to a subset $U(\frac{1}{2})$ with

$$\overline{U(0)} \subseteq U\Big(\frac{1}{2}\Big) \subseteq \overline{U\Big(\frac{1}{2}\Big)} \subseteq U(1).$$

Continuing like this, we find for each dyadic number $\frac{k}{2^n} \in [0,1]$ an open, relatively compact subset $U(\frac{k}{2^n})$ with

$$\overline{U\Big(\frac{k}{2^n}\Big)} \subseteq U\Big(\frac{k+1}{2^n}\Big) \quad \text{for} \quad k = 0, \ldots, 2^n - 1.$$

Let $\mathbb{D} := \{\frac{k}{2^n} : k = 0, \ldots, 2^n, n \in \mathbb{N}\}$ for the set of dyadic numbers in $[0,1]$. For $r \in [0,1]$, we put

$$U(r) := \bigcup_{s \le r, s \in \mathbb{D}} U(s).$$

For $r = \frac{k}{2^n}$ this is consistent with the previous definition. For $t < t'$ we now find $r = \frac{k}{2^n} < r' = \frac{k+1}{2^n}$ in $\mathbb{D}$ with $t < r < r' < t'$, so that we obtain

$$\overline{U(t)} \subseteq \overline{U(r)} \subseteq U(r') \subseteq U(t').$$

We also put $U(t) = \emptyset$ for $t < 0$ and $U(t) = X$ for $t > 1$. Finally, we define

$$f(x) := \inf\{t \in \mathbb{R} : x \in U(t)\}.$$

Then $f(K) \subseteq \{0\}$ and $f(X \setminus U) \subseteq \{1\}$.

We claim that f is continuous. So let $x_0 \in X$, $f(x_0) = t_0$ and $\varepsilon > 0$. We put $V := U(t_0 + \varepsilon) \setminus \overline{U(t_0 - \varepsilon)}$ and note that this is a neighborhood of x_0. From $x \in V \subseteq U(t_0 + \varepsilon)$ we derive $f(x) \le t_0 + \varepsilon$. If $f(x) < t_0 - \varepsilon$, then also $x \in U(t_0 - \varepsilon) \subseteq \overline{U(t_0 - \varepsilon)}$, which is a contradiction. Therefore $|f(x) - f(x_0)| \le \varepsilon$ holds on V, and this implies that f is continuous. Finally, we put $h := 1 - f$. ☐

Exercises for Section A.2

Exercise A.1. (One point compactification) Let X be a locally compact space. Show that:

(i) There exists a compact topology on the set $X_\omega := X \cup \{\omega\}$, where ω is a symbol of a point not contained in X. Hint: A subset $O \subseteq X_\omega$ is open if it either is an open subset of X or $\omega \in O$ and $X \setminus O$ is compact. Note that the local compactness of X is needed for the Hausdorff property of X_ω.

(ii) The inclusion map $\eta_X : X \to X_\omega$ is a homeomorphism onto an open subset of X_ω.

(iii) If Y is a compact space and $f : X \to Y$ a continuous map which is a homeomorphism onto the complement of a point in Y, then there exists a homeomorphism $F : X_\omega \to Y$ with $F \circ \eta_X = f$.

The space X_ω is called the *Alexandroff compactification* or the *one point compactification* of X.[17]

Exercise A.2. (Stereographic projection) We consider the n-dimensional sphere

$$\mathbb{S}^n := \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : x_0^2 + x_1^2 + \cdots + x_n^2 = 1\}.$$

We call the unit vector $e_0 := (1, 0, \ldots, 0)$ the *north pole* of the sphere and $-e_0$ the *south pole*. We then have the corresponding *stereographic projection maps*

$$\varphi_\pm : U_\pm := \mathbb{S}^n \setminus \{\pm e_0\} \to \mathbb{R}^n, \quad (y_0, y) \mapsto \frac{1}{1 \mp y_0} y.$$

Show that these maps are homeomorphisms with inverse maps

$$\varphi_\pm^{-1}(x) = \Big(\pm \frac{\|x\|_2^2 - 1}{\|x\|_2^2 + 1}, \frac{2x}{1 + \|x\|_2^2}\Big).$$

Exercise A.3. Show that the one-point compactification of $\mathbb{R}^n$ is homeomorphic to the n-dimensional sphere $\mathbb{S}^n$. Hint: Exercise A.2.

Exercise A.4. Show that the one-point compactification of an open interval $]a, b[\subseteq \mathbb{R}$ is homeomorphic to $\mathbb{S}^1$.

Exercise A.5. Show that the one-point compactification of $\mathbb{N}$ is homeomorphic to $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$.

Exercise A.6. Let X be a locally compact space and $Y \subseteq X$ be a subset. Show that Y is locally compact with respect to the subspace topology if and only if there exists an open subset $O \subseteq X$ and a closed subset A with $Y = O \cap A$. Hint: If Y is locally compact, write it as a union of compact subsets of the form $O_i \cap Y$, O_i open in X, where $O_i \cap Y$ has compact closure, contained in Y. Then put $O := \bigcup_{i \in I} O_i$ and $A := \overline{Y \cap O}$.

Exercise A.7. Let $f : X \to Y$ be a continuous proper map between locally compact spaces, i.e., inverse image of compact subsets are compact. Show that

(a) f is a closed map, i.e., maps closed subsets to closed subsets.

(b) If f is injective, then it is a topological embedding onto a closed subset.

(c) There is a well-defined homomorphism $f^* : C_0(Y) \to C_0(X)$ of C^*-algebras, defined by $f^*h := h \circ f$.

(d) For each regular Borel measure μ on X, the push-forward measure $f_*\mu$ on Y, defined by $(f_*\mu)(E) := \mu(f^{-1}(E))$ is regular. Hint: To verify outer regularity, pick an open $O \supseteq f^{-1}(E)$ with $\mu(O \setminus f^{-1}(E)) < \varepsilon$. Then $U := f(O^c)^c$ is an open subset of Y containing E and $\widetilde{O} := f^{-1}(U)$ satisfies $f^{-1}(E) \subseteq \widetilde{O} \subseteq O$, which leads to $(f_*\mu)(U \setminus E) < \varepsilon$.

[17] Alexandroff, Pavel (1896–1982)

A.3 Hilbert–Schmidt and Trace Class Operators

In subsection we collect some facts on Hilbert–Schmidt and trace class operators.

A.3.1 Hilbert–Schmidt Operators

Lemma A.17. *Let $\mathcal{H}$, $\mathcal{K}$ be Hilbert spaces, $(e_j)_{j\in J}$ an orthonormal basis in $\mathcal{H}$, and $(f_k)_{k\in K}$ an orthonormal basis in $\mathcal{K}$. For $A \in B(\mathcal{H},\mathcal{K})$, we then have*

$$\sum_{j\in J} \|Ae_j\|^2 = \sum_{k\in K} \|A^* f_k\|^2.$$

Proof. $\sum_j \|Ae_j\|^2 = \sum_{j,k} |\langle Ae_j, f_k\rangle|^2 = \sum_{j,k} |\langle e_j, A^* f_k\rangle|^2 = \sum_k \|A^* f_k\|^2$. □

Definition A.18. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and $(e_j)_{j\in J}$ an orthonormal basis. An operator $A \in B(\mathcal{H},\mathcal{K})$ is called a *Hilbert–Schmidt operator* if

$$\|A\|_2 := \Big(\sum_{j\in J} \|Ae_j\|^2\Big)^{\frac{1}{2}} < \infty.$$

In view of Lemma A.17, the preceding expression does not depend on the choice of the orthonormal basis in $\mathcal{H}$. We write $B_2(\mathcal{H},\mathcal{K})$ for the space of Hilbert–Schmidt operators in $B(\mathcal{H},\mathcal{K})$, $B_2(\mathcal{H}) := B_2(\mathcal{H},\mathcal{H})$ for the Hilbert–Schmidt operators on $\mathcal{H}$, and $B_{\rm fin}(\mathcal{H})$ for the space of continuous finite rank operators on $\mathcal{H}$.

Proposition A.19. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces.*

(i) *For $A \in B_2(\mathcal{H},\mathcal{K})$, we have $\|A\| \leq \|A\|_2 = \|A^*\|_2$.*

(ii) *If $A, B \in B_2(\mathcal{H},\mathcal{K})$ and $(e_j)_{j\in J}$ is an orthonormal basis of $\mathcal{H}$, then*

$$\langle A, B\rangle := \sum_j \langle B^* Ae_j, e_j\rangle = \sum_j \langle Ae_j, Be_j\rangle \tag{32}$$

converges and defines the structure of a complex Hilbert space on $B_2(\mathcal{H},\mathcal{K})$ with the norm

$$\|A\|_{HS} := \|A\|_2 := \sqrt{\langle A, A\rangle}.$$

(iii) *$\langle A, B\rangle$ as in (32) does not depend on the chosen orthonormal basis.*

(iv) *$\langle A, B\rangle = \langle B^*, A^*\rangle$ for $A, B \in B_2(\mathcal{H},\mathcal{K})$.*

(v) *If $A \in B(\mathcal{K})$ and $B, C \in B_2(\mathcal{H},\mathcal{K})$, then $AB \in B_2(\mathcal{H},\mathcal{K})$ with*

$$\|AB\|_2 \leq \|A\| \cdot \|B\|_2 \quad \text{and} \quad \langle AB, C\rangle = \langle B, A^* C\rangle.$$

(vi) *Hilbert–Schmidt operators are compact, i.e., $B_2(\mathcal{H},\mathcal{K}) \subseteq K(\mathcal{H},\mathcal{K})$.*

Proof. (i) The relation $\|A\|_2 = \|A^*\|_2$ is immediate from the proof of Lemma A.17. To prove that $\|A\| \leq \|A\|_2$, let $\varepsilon > 0$ and $(e_j)_{j\in J}$ be an orthonormal basis of $\mathcal{H}$ such that $\|Ae_{j_0}\| \geq \|A\| - \varepsilon$ for an element $j_0 \in J$. Then

$$\|A\|_2^2 = \sum_{j\in J} \|Ae_j\|^2 \geq (\|A\| - \varepsilon)^2.$$

Since $\varepsilon > 0$ was arbitrary, we obtain $\|A\|_2 \geq \|A\|$.

(ii) It is clear from the definition that

$$\Phi \colon B_2(\mathcal{H}, \mathcal{K}) \to \ell^2(J, \mathcal{K}), \quad A \mapsto (Ae_j)_{j \in J}$$

is an isometric embedding. That Φ is surjective follows from the observation that, for $v \in \ell^2(J, \mathbb{K})$ and $x \in \mathcal{H}$, the prescription $Ax := \sum_{j \in J} \langle e_j, x \rangle v_j$ defines a bounded operator $\mathcal{H} \to \mathcal{K}$ with

$$\|Ax\|^2 \leq \sum_{j \in J} |\langle x, e_j \rangle|^2 \sum_j \|v_j\|^2 = \|x\|^2 \|v\|^2$$

(Cauchy–Schwarz inequality). Hence $A \in B_2(\mathcal{H}, \mathcal{K})$ with $\Phi(A) = v$. This shows that Φ is an isometric bijection, and therefore that $B_2(\mathcal{H}, \mathcal{K})$ is a Hilbert space with scalar product given by $\langle A, B \rangle = \sum_{j \in J} \langle Ae_j, Be_j \rangle = \sum_{j \in J} \langle B^* A e_j, e_j \rangle$.

(iii) This follows from the fact that the scalar product on the Hilbert space $B_2(\mathcal{H}, \mathcal{K})$ is uniquely determined by the norm via the polarization identity.

$$\langle A, B \rangle = \frac{1}{4} \sum_{k=0}^{3} i^{-k} \langle A + i^k B, A + i^k B \rangle.$$

(iv) We know already that $A \mapsto A^*$ is an isometry of $B_2(\mathcal{H}, \mathcal{K})$ onto $B_2(\mathcal{K}, \mathcal{H})$. Hence both sides in (iv) are hermitian forms on $B_2(\mathcal{H}, \mathcal{K})$ which define the same norm. Since the scalar product is uniquely determined by the norm via the polarization identity, the assertion follows.

(v) The first part follows from

$$\|AB\|_2^2 = \sum_{j \in J} \|ABe_j\|^2 \leq \sum_{j \in J} \|A\|^2 \|Be_j\|^2 = \|A\|^2 \|B\|_2^2.$$

For the second part, we calculate

$$\langle AB, C \rangle = \sum_{j \in J} \langle C^* ABe_j, e_j \rangle = \sum_{j \in J} \langle (A^* C)^* Be_j, e_j \rangle = \langle B, A^* C \rangle.$$

(vi) If $A \in B_2(\mathcal{H}, \mathcal{K})$ and $(e_j)_{j \in J}$ is an orthonormal basis of $\mathcal{H}$, then the finiteness of $\sum_j \|Ae_j\|^2 < \infty$ implies that $J_A := \{j \in J \colon Ae_j \neq 0\}$ is countable. If J_A is finite, then A is a finite rank operator, hence in particular compact. If J_A is infinite, then it can be written as $J_A = \{j_n \colon n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, we we consider the operator A_n defined by

$$A_n e_j := \begin{cases} Ae_j & \text{for } j \in \{j_1, \ldots, j_n\} \\ 0 & \text{otherwise }. \end{cases}$$

Then the sequence $(A_n)_{n \in \mathbb{N}}$ converges to A in $B_2(\mathcal{H}, \mathcal{K})$ and hence, in particular, with respect to the operator norm. Therefore A is a limit of finite rank operators and therefore compact. □

A.3.2 Trace Class Operators

Definition A.20. We say that an operator $A \in B(\mathcal{H})$ is of *trace class* if $A \in B_2(\mathcal{H})$ and

$$\|A\|_1 := \sup\{|\langle A, B \rangle| \colon B \in B_2(\mathcal{H}), \|B\| \leq 1\}$$

is finite. We write $B_1(\mathcal{H}) \subseteq B_2(\mathcal{H})$ for the subspace of trace class operators. It follows easily from the definition that $\| \cdot \|_1$ defines a norm on $B_1(\mathcal{H})$.

Proposition A.21. *The following assertions hold:*

(i) *If $A \in B(\mathcal{H})$ and $B \in B_1(\mathcal{H})$, then $AB \in B_1(\mathcal{H})$ with $\|AB\|_1 \leq \|A\| \cdot \|B\|_1$.*

(ii) *$\|A\|_2 \leq \|A\|_1$ for $A \in B_1(\mathcal{H})$.*

(iii) *$B_1(\mathcal{H})$ is invariant under taking adjoints, and $\|A^*\|_1 = \|A\|_1$.*

(iv) *$B_2(\mathcal{H})B_2(\mathcal{H}) \subseteq B_1(\mathcal{H})$.*

(v) *For $x, y \in \mathcal{H}$ we define the operator*

$$P_{x,y} := |x\rangle\langle y| \quad \text{by} \quad P_{x,y}(z) = \langle y, z\rangle x.$$

Then $\|P_{x,y}\| = \|P_{x,y}\|_1 = \|x\| \cdot \|y\|$.

Proof. (i) This follows from $|\langle AB, X\rangle| = |\langle B, A^*X\rangle| \leq \|B\|_1\|A\| \cdot \|X\|$ for $X \in B_2(\mathcal{H})$ with $\|X\| \leq 1$.

(ii) From $\|X\| \leq \|X\|_2$ for $X \in B_2(\mathcal{H})$ (Proposition A.19(i)), it follows that

$$\{X \in B_2(\mathcal{H}) \colon \|X\| \leq 1\} \supseteq \{X \in B_2(\mathcal{H}) \colon \|X\|_2 \leq 1\}.$$

Hence the assertion follows from $\|A\|_2 = \sup\{|\langle A, X\rangle| \colon \|X\|_2 \leq 1\}$.

(iii) From $|\langle A^*, X\rangle| = |\langle X^*, A\rangle| = |\langle A, X^*\rangle|$ and the fact that $X \mapsto X^*$ is an isometry of $B(\mathcal{H})$ and $B_2(\mathcal{H})$, we see that $A^* \in B_1(\mathcal{H})$ with $\|A^*\|_1 = \|A\|_1$.

(iv) If $A = BC$ with $B, C \in B_2(\mathcal{H})$, then we have for $X \in B_2(\mathcal{H})$ the estimate

$$|\langle A, X\rangle| = |\langle BC, X\rangle| = |\langle C, B^*X\rangle| \leq \|C\|_2\|B^*X\|_2 \leq \|C\|_2\|B\|_2\|X\|.$$

Hence $A \in B_1(\mathcal{H})$ with $\|A\|_1 \leq \|B\|_2\|C\|_2$.

(v) For $A \in B_2(\mathcal{H})$, we have

$$\langle A, P_{x,y}\rangle = \sum_j \langle Ae_j, P_{x,y}e_j\rangle = \sum_j \langle Ae_j, x\rangle\langle y, e_j\rangle = \sum_j \langle e_j, A^*x\rangle\langle y, e_j\rangle = \langle y, A^*x\rangle = \langle Ay, x\rangle, \tag{33}$$

hence $\|P_{x,y}\| \leq \|P_{x,y}\|_1 \leq \|x\| \cdot \|y\|$. As $\|P_{x,y}(y)\| = \|y\|^2\|x\|$ shows that $\|P_{x,y}\| \geq \|x\| \cdot \|y\|$, the assertion follows. □

Proposition A.22. *Let $(e_j)_{j\in J}$ be an orthonormal basis and $A \in B_1(\mathcal{H})$. Then the sum*

$$\operatorname{tr} A := \sum_{j\in J} \langle Ae_j, e_j\rangle$$

converges absolutely and has the following properties:

(i) *$|\operatorname{tr} A| \leq \|A\|_1$, i.e., tr is a continuous linear functional on $B_1(\mathcal{H})$ and it is independent of the chosen orthonormal basis.*

(ii) *$\langle A, B\rangle = \operatorname{tr}(AB^*)$ and $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for $A, B \in B_2(\mathcal{H})$.*

(iii) *For $A \in B_1(\mathcal{H})$, the function $X \mapsto \operatorname{tr}(XA)$ on $B(\mathcal{H})$ is continuous and extends the linear functional $X \mapsto \langle A^*, X\rangle$ on $B_2(\mathcal{H})$. Moreover, $\operatorname{tr}(AX) = \operatorname{tr}(XA)$.*

(iv) *Each* $A \in B_1(\mathcal{H})$ *can be written as* $A = \sum_{n=1}^\infty P_{v_n,w_n}$*, where* $(v_n)_{n\in\mathbb{N}}$ *and* $(w_n)_{n\in\mathbb{N}}$ *are orthogonal sequences. Then* $\|A\|_1 = \sum_{n=1}^\infty \|v_n\| \cdot \|w_n\|$ *and* $\operatorname{tr} A = \sum_{n=1}^\infty \langle w_n, v_n\rangle$.

(v) $B_{\text{fin}}(\mathcal{H})$ *is dense in* $B_1(\mathcal{H})$.

(vi) $B_1(\mathcal{H}) \cong K(\mathcal{H})'$ *and* $B_1(\mathcal{H})' \cong B(\mathcal{H})$*, where the pairings are given by the bilinear form* $(A, B) \mapsto \operatorname{tr}(AB)$.

(vii) $\big(B_1(\mathcal{H}), \|\cdot\|_1\big)$ *is a Banach space.*

(viii) *If* $(e_j)_{j\in J}$ *is an orthonormal basis, then* $\|A\|_1 \leq \sum_{i,j\in J} |\langle Ae_i, e_j\rangle|$ *holds for all* $A \in B(\mathcal{H})$.

Proof. Let $J_0 \subseteq J$ be a finite subset and $(\lambda_j)_{j\in J_0}$ be complex numbers with $|\lambda_j| = 1$ and $\lambda_j\langle Ae_j, e_j\rangle = |\langle Ae_j, e_j\rangle|$. Then (33) shows that

$$\sum_{j\in J_0} |\langle Ae_j, e_j\rangle| = \sum_{j\in J_0} \lambda_j\langle Ae_j, e_j\rangle = \Big\langle A, \sum_{j\in J_0} \lambda_j P_{e_j}\Big\rangle \leq \|A\|_1 \Big\|\sum_{j\in J_0} \lambda_j P_{e_j}\Big\| \leq \|A\|_1.$$

This proves the estimate under (i) and the absolute convergence of the series. To see that $\operatorname{tr}(A)$ does not depend on the chosen basis, let $(f_k)_{k\in K}$ be another basis and calculate

$$\begin{aligned}\sum_k \langle Af_k, f_k\rangle &= \sum_{k,j} \langle Af_k, e_j\rangle\langle e_j, f_k\rangle = \sum_{k,j} \langle f_k, A^*e_j\rangle\langle e_j, f_k\rangle \\ &= \sum_j \langle e_j, A^*e_j\rangle = \sum_j \langle Ae_j, e_j\rangle.\end{aligned}$$

(ii) The first part is precisely Proposition A.19(ii). The second follows from Proposition A.19(iv):

$$\operatorname{tr}(AB) = \langle A, B^*\rangle = \langle B, A^*\rangle = \operatorname{tr}(BA).$$

(iii) For $A \in B_1(\mathcal{H})$ and $B \in B(\mathcal{H})$, we use Proposition A.21(i) to see that $AB \in B_1(\mathcal{H})$ with $|\operatorname{tr}(XA)| \leq \|XA\|_1 \leq \|X\| \cdot \|A\|_1$. This proves the first part of (iii).

To show that $\operatorname{tr}(AX) = \operatorname{tr}(XA)$ holds for $A \in B_1(\mathcal{H})$ and $X \in B(\mathcal{H})$, we note that, since both sides define complex bilinear forms, we may assume that A is hermitian. Then the fact that A is compact (Proposition A.19(vi)) shows that there exists an orthogonal basis $(e_j)_{j\in J}$ with $Ae_j = \lambda_j e_j$. We may thus assume that $Ae_j = \lambda_j e_j$ and note that this implies $A^*e_j = \overline{\lambda_j} e_j$. Then

$$\operatorname{tr}(AX) = \sum_j \langle AXe_j, e_j\rangle = \sum_j \langle Xe_j, A^*e_j\rangle = \sum_j \overline{\lambda_j}\langle Xe_j, e_j\rangle = \sum_j \langle XAe_j, e_j\rangle = \operatorname{tr}(XA).$$

(iv) Since A is compact, it can be written as $A = \sum_{n=1}^\infty P_{v_n,w_n}$, as required (cf. [We76, Satz 7.6]). Now Proposition A.21 yields $\|A\|_1 \leq \sum_{n=1}^\infty \|v_n\| \cdot \|w_n\|$.

To obtain the converse estimate, we consider the operator $X_n = \sum_{j=1}^n c_j P_{v_j,w_j}$, where $c_j = \frac{1}{\|v_j\|\cdot\|w_j\|}$ if $P_{v_j,w_j} \neq 0$. Then $\|X\| \leq 1$ follows from Exercise A.8. Moreover, we have

$$\|A\|_1 \geq \langle A, X_n\rangle = \sum_{j=1}^n \langle v_j, X_n.w_j\rangle = \sum_{j=1}^n c_j\|v_j\|^2\|w_j\|^2 = \sum_{j=1}^n \|v_j\| \cdot \|w_j\|.$$

Since n was arbitrary, we obtain $\|A\|_1 \geq \sum_{j=1}^{\infty} \|v_j\| \cdot \|w_j\|$ and therefore equality. It follows, in particular, that $A = \lim_{m\to\infty} A_m$ with $A_m = \sum_{n=1}^m P_{v_n,w_n}$ because $\|A - A_m\| = \sum_{j>m} \|v_j\| \cdot \|w_j\|$. Therefore

$$\operatorname{tr} A = \lim_{m\to\infty} \operatorname{tr} A_m = \lim_{m\to\infty} \sum_{j=1}^m \langle v_j, w_j\rangle = \sum_{j=1}^{\infty} \langle v_j, w_j\rangle.$$

(v) Since $B_1(\mathcal{H})$ is invariant under taking adjoints, it suffices to show that each symmetric element A in $B_1(\mathcal{H})$ can be approximated by finite rank operators with respect to $\|\cdot\|_1$. We write $A = \sum_{n=1}^{\infty} \lambda_n P_{v_n}$, where $(v_n)_{n\in\mathbb{N}}$ is an orthonormal system and conclude as in (iv) that $A_n \to A$.

(vi) The continuity of the pairing $B_1(\mathcal{H}) \times B(\mathcal{H}) \to \mathbb{C}, (A, B) \mapsto \operatorname{tr}(AB)$ follows from $|\operatorname{tr}(AB)| \leq \|AB\|_1 \leq \|A\|_1 \|B\|$. First we show that this pairing yields an isomorphism of $B_1(\mathcal{H})$ with $K(\mathcal{H})'$. So let $f \in K(\mathcal{H})'$. Then $f|_{B_2(\mathcal{H})}$ is a linear functional with $|f(X)| \leq \|f\| \cdot \|X\| \leq \|f\| \cdot \|X\|_2$ (Proposition A.19(i)), hence can be represented by an element $Y \in B_2(\mathcal{H})$. Then $f(X) = \langle X, Y\rangle = \operatorname{tr}(XY^*)$ holds for all $X \in B_2(\mathcal{H})$, and with $|f(X)| \leq \|f\| \cdot \|X\|$ we obtain $Y \in B_1(\mathcal{H})$ with $\|Y\|_1 = \|Y^*\|_1 \leq \|f\|$. The converse follows from the density of $B_{\mathrm{fin}}(\mathcal{H}) \subseteq B_2(\mathcal{H})$ in $K(\mathcal{H})$.

Next we show that $B_1(\mathcal{H})' \cong B(\mathcal{H})$. So we have to represent each continuous linear functional f on $B_1(\mathcal{H})$ by a bounded linear operator on $\mathcal{H}$. From Proposition A.21 we recall that $\|P_{v,w}\|_1 = \|v\| \cdot \|w\|$. Therefore, for each $w \in \mathcal{H}$, the mapping $v \mapsto f(P_{v,w})$ is continuous and linear, hence can be represented by a vector a_w in the sense that $f(P_{v,w}) = \langle v, a_w\rangle$ holds for all $v \in \mathcal{H}$. Moreover, the above calculation shows that $\|a_w\| \leq \|f\| \cdot \|w\|$. Since the assignment $w \mapsto a_w$ is linear, we find a bounded operator A on $\mathcal{H}$ with $Aw = a_w$ for all $w \in \mathcal{H}$ and $\|A\| \leq \|f\|$. Now $f(P_{v,w}) = \langle v, Aw\rangle = \langle P_{v,w}, A\rangle$ holds for $v, w \in \mathcal{H}$. From that we obtain $f(X) = \operatorname{tr}(XA^*)$ for $X \in B_{\mathrm{fin}}(\mathcal{H})$ and since, in view of (v), $B_{\mathrm{fin}}(\mathcal{H})$ is dense in $B_1(\mathcal{H})$, we obtain $f(X) = \operatorname{tr}(XA^*)$ for all $X \in B_1(\mathcal{H})$. This proves (vi).

(vii) Since $B_1(\mathcal{H}) \cong K(\mathcal{H})'$ follows from (vi), the completeness of $B_1(\mathcal{H})$ follows from the fact that dual spaces of normed spaces are Banach spaces.

(viii) Let $(v_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ be sequences in $\mathcal{H}$ satisfying $\sum_{n=1}^{\infty} \|v_n\|^2 < \infty$ and $\sum_{n=1}^{\infty} \|w_n\|^2 < \infty$. Then $\|P_{v_n,w_n}\|_1 = \|v_n\| \cdot \|w_n\|$ (Proposition A.21(v)) implies that the series $A := \sum_{n=1}^{\infty} P_{v_n,w_n}$ converges absolutely in $B_1(\mathcal{H})$. Therefore

$$\sum_{n=1}^{\infty} \langle Xv_n, w_n\rangle = \sum_{n=1}^{\infty} \operatorname{tr}(XP_{v_n,w_n}) = \operatorname{tr}(XA).$$

This proves that each ultraweakly continuous linear functional on $B(\mathcal{H})$ is also weak-$*$-continuous.

If, conversely, $f(X) = \operatorname{tr}(AX)$ with $A \in B_1(\mathcal{H})$, then we want to show that f is ultraweakly continuous. Writing $A = B_+ - B_- + i(C_+ - C_-)$, where B_+, B_-, C_+ and C_- are positive trace class operators, we may assume that A is positive. Then $A = \sum_{n=1}^{\infty} P_{u_n}$, where $(u_n)_{n\in\mathbb{N}}$ is an orthogonal sequence consisting of eigenvectors of eigenvalue $\|u_n\|^2$, and, in view of (iv), $\|A\|_1 = \sum_{n=1}^{\infty} \|u_n\|^2 < \infty$. We conclude that $f(X) = \operatorname{tr}(XA) = \sum_{n=1}^{\infty} \langle X.u_n, u_n\rangle$ with $\sum_{n=1}^{\infty} \|u_n\|^2 = \|A\|_1 < \infty$. Hence f is ultraweakly continuous. Now the assertion follows from the fact that the weak-$*$-topology and the ultraweak topology are the coarsest topology for which the same set of linear functionals is continuous.

(ix) We may assume that the sum $\sum_{i,j} |\langle Ae_i, e_j\rangle|$ exists. In view of $|\langle Ae_i, e_j\rangle| \leq \|A\|$, this implies that $\sum_{i,j} |\langle Ae_i, e_j\rangle|^2 < \infty$, i.e., $A \in B_2(\mathcal{H})$.

Let $B = \sum_{i,j\in J} b_{i,j} P_{e_j,e_i}$ be a finite sum of the operators P_{e_j,e_i}. Then $B \in B_2(\mathcal{H})$, $|b_{i,j}| \leq \|B\|$ for all i, j, and

$$|\operatorname{tr}(AB)| \leq \sum_{i,j} |b_{i,j} \operatorname{tr}(AP_{e_i,e_j})| \leq \sum_{i,j} |b_{i,j}| \cdot |\langle Ae_i, e_j\rangle| \leq \|B\| \sum_{i,j} |\langle Ae_i, e_j\rangle|.$$

This prove that whenever the sum $\sum_{i,j} |\langle Ae_i, e_j\rangle|$ exists, then $A \in B_1(\mathcal{H})$ with $\|A\|_1 \leq \sum_{i,j} |\langle Ae_i, e_j\rangle|$. ☐

Exercise A.8. Show that, it $A = \sum_{j=1}^n \lambda_j P_{v_j,w_j}$, where the finite sequences $v_1, \dots, v_n$, $w_1, \dots, w_n$ are orthonormal, then $\|A\| = \max\{|\lambda_j| \colon j = 1, \dots, n\}$.
Hint: $Aw_j = \lambda_j v_j$ for every j.

A.4 The Stone–Weierstraß Theorem

Definition A.23. (a) Let M be a set and $\mathcal{A} \subseteq \mathbb{K}^M$ be a set of functions $M \to \mathbb{K}$. We say that $\mathcal{A}$ *separates the points of* M if for two points $x \neq y$ in X there exists some $f \in \mathcal{A}$ with $f(x) \neq f(y)$.

(b) A linear subspace $\mathcal{A} \subseteq \mathbb{K}^M$ is called an *algebra* if it is closed under pointwise multiplication.

Lemma A.24. *There exists an increasing sequence of real polynomials p_n which converges in $[0,1]$ uniformly to the square root function $x \mapsto \sqrt{x}$.*

Proof. We consider the function $f(t) := t^{-1/2}$ on $(0,1]$. Then

$$f^{(k)}(t) = (-1)^k \frac{1 \cdot 3 \cdots (2k-1)}{2^k} t^{-1/2-k}.$$

Therefore the Taylor polynomial q_k of order k in $t = 1$ of f satisfy the relation

$$q_{k+1}(t) = q_k(t) + \frac{1 \cdot 3 \cdots (2k+1)}{2^{k+1}(k+1)!}(1-t)^{k+1}.$$

We consider the polynomials $p_k(t) := tq_k(t)$. Then $(p_n)_{n\in\mathbb{N}_0}$ is an increasing sequence of non-negative polynomials on $[0,1]$ whose pointwise limit is the function $t \mapsto tt^{-1/2} = \sqrt{t}$. For every $\varepsilon \in (0,1)$, the convergence is uniform on the interval $[\varepsilon, 1]$ because the Taylor series of f converges uniformly in this interval. On the interval $[0, \varepsilon]$, we have

$$\sqrt{t} - p_k(t) \leq \sqrt{t} \leq \sqrt{\varepsilon}.$$

Hence the convergence is uniform on $[0,1]$. ☐

Theorem A.25. (Stone–Weierstraß) [18] [19] *Let X be a compact space and $\mathcal{A} \subseteq C(X,\mathbb{R})$ be a point separating subalgebra containing the constant functions. Then $\mathcal{A}$ is dense in $C(X,\mathbb{R})$ w.r.t. $\|\cdot\|_\infty$.*

[18] Stone, Marshall (1903–1989)
[19] Weierstraß, Karl (1815–1897)

Proof. Let $\mathcal{B} := \overline{\mathcal{A}}$ denote the closure of $\mathcal{A}$ in the Banach space

$$(C(X,\mathbb{R}), \|\cdot\|_\infty).$$

Then $\mathcal{B}$ also contains the constant functions, separates the points and is a subalgebra (Exercise A.9). We have to show that $\mathcal{B} = C(X,\mathbb{R})$.

Here is the idea of the proof. First we use Lemma A.24 to see that for $f, g \in \mathcal{B}$, also $|f|$, $\min(f,g)$ and $\max(f,g)$ are contained in $\mathcal{B}$. Then we use the point separation property to approximate general continuous functions locally by elements of $\mathcal{B}$. Now the compactness of X permits to complete the proof.

Here are the details: Let $(p_n)_{n\in\mathbb{N}}$ be the sequence of polynomials from Lemma A.24. For $f \in \mathcal{B}$, we consider the functions $p_n\big(\frac{f^2}{\|f\|_\infty^2}\big)$, which also belong to $\mathcal{B}$. In view of Lemma A.24, they converge uniformly to $\sqrt{\frac{f^2}{\|f\|_\infty^2}} = \frac{|f|}{\|f\|_\infty}$, so that $|f| \in \mathcal{B}$.

Now let $f, g \in \mathcal{B}$. Then $\mathcal{B}$ contains the functions

$$\min(f,g) = \frac{1}{2}(f + g - |f-g|) \quad \text{and} \quad \max(f,g) = \frac{1}{2}(f+g+|f-g|).$$

Next let $x \neq y$ in X and $r, s \in \mathbb{R}$. According to our assumption, there exists a function $g \in \mathcal{B}$ with $g(x) \neq g(y)$. For

$$h := r + (s-r)\frac{g - g(x)}{g(y) - g(x)} \in \mathcal{B}$$

we then have $h(x) = r$ and $h(y) = s$.

Claim: For $f \in C(X,\mathbb{R})$, $x \in X$ and $\varepsilon > 0$, there exists a function $g_x \in \mathcal{B}$ with

$$f(x) = g_x(x) \quad \text{and} \quad (\forall y \in X) \quad g_x(y) \leq f(y) + \varepsilon.$$

To verify this claim, pick for each $z \in X$ a function $h_z \in \mathcal{B}$ with $h_z(x) = f(x)$ and $h_z(z) \leq f(z) + \frac{\varepsilon}{2}$. Then there exists a neighborhood U_z of z with

$$(\forall y \in U_z) \quad h_z(y) \leq f(y) + \varepsilon.$$

Since X is compact, it is covered by finitely many $U_{z_1}, \ldots, U_{z_k}$ of these neighborhoods. Then $g_x := \min\{h_{z_1}, \ldots, h_{z_k}\}$ is the desired function.

Now we complete the proof by showing that $\mathcal{B} = C(X,\mathbb{R})$. So let $f \in C(X,\mathbb{R})$ and $\varepsilon > 0$. For each $x \in X$, pick $g_x \in \mathcal{B}$ with

$$(\forall y \in X) \quad f(x) = g_x(x) \quad \text{and} \quad g_x(y) \leq f(y) + \varepsilon.$$

Then the continuity of f and g_x yield neighborhoods U_x of x with

$$\forall y \in U_x \; : \quad g_x(y) \geq f(y) - \varepsilon.$$

Now the compactness of X implies the existence of finitely many points $x_1, \ldots, x_k$ such that $X \subseteq U_{x_1} \cup \cdots \cup U_{x_k}$. We now put $\varphi_\varepsilon := \max\{g_{x_1}, \ldots, g_{x_k}\} \in \mathcal{B}$. Then

$$\forall y \in X \; : \quad f(y) - \varepsilon \leq \varphi_\varepsilon(y) \leq f(y) + \varepsilon.$$

This implies that $\|f - \varphi_\varepsilon\|_\infty \leq \varepsilon$ and since ε was arbitrary, $f \in \mathcal{B}$. $\square$

Corollary A.26. *Let X be a compact space and $\mathcal{A} \subseteq C(X, \mathbb{C})$ be a point separating subalgebra containing the constant functions which is invariant under complex conjugation, i.e., $f \in \mathcal{A}$ implies $\overline{f} \in \mathcal{A}$. Then $\mathcal{A}$ is dense in $C(X, \mathbb{C})$ w.r.t. $\| \cdot \|_\infty$.*

Proof. Let $\mathcal{A}_\mathbb{R} := \mathcal{A} \cap C(X, \mathbb{R})$. Since $\mathcal{A}$ is conjugation invariant, we have $\mathcal{A} = \mathcal{A}_\mathbb{R} \oplus i\mathcal{A}_\mathbb{R}$. This implies that $\mathcal{A}_\mathbb{R}$ contains the real constants and separates the points of X. Now Theorem A.25 implies that $\mathcal{A}_\mathbb{R}$ is dense in $C(X, \mathbb{R})$, and therefore $\mathcal{A}$ is dense in the complex Banach space $C(X, \mathbb{C}) = C(X, \mathbb{R}) + iC(X, \mathbb{R})$. □

Exercises for Section A.4

Exercise A.9. If X is a compact topological space and $\mathcal{A} \subseteq C(X, \mathbb{R})$ is a subalgebra, then its closure also is a subalgebra. Hint: If $f_n \to f$ and $g_n \to g$ uniformly, then also $f_n + g_n \to f + g, \lambda f_n \to \lambda f$ and $f_n g_n \to fg$ uniformly.

Exercise A.10. Let $[a, b] \subseteq \mathbb{R}$ be a compact interval. Show that the space

$$\mathcal{A} := \Big\{ f|_{[a,b]} \colon (\exists a_0, \ldots, a_n \in \mathbb{R}, n \in \mathbb{N})\ f(x) = \sum_{i=0}^n a_i x^i \Big\}$$

of polynomial functions on $[a, b]$ is dense in $C([a, b], \mathbb{R})$ with respect to $\| \cdot \|_\infty$.

Exercise A.11. Let $K \subseteq \mathbb{R}^n$ be a compact subset. Show that the space $\mathcal{A}$ consisting of all restrictions of polynomial functions

$$f(x) = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha x^\alpha, \quad a_\alpha \in \mathbb{R}, \quad x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

to K is dense in $C(K, \mathbb{R})$ with respect to $\| \cdot \|_\infty$.

Exercise A.12. Let $\mathbb{S}^1 = \{z \in \mathbb{C} \colon |z| = 1\}$ and

$$\mathcal{A} := \Big\{ f|_{\mathbb{S}^1} \colon (\exists a_0, \ldots, a_n \in \mathbb{C}, n \in \mathbb{N})\ f(z) = \sum_{j=0}^n a_n z^n \Big\}.$$

Show that $\mathcal{A}$ is not dense in $C(\mathbb{S}^1, \mathbb{C})$. Hint: Consider the function $f(z) := z^{-1}$ on $\mathbb{S}^1$ and try to approximate it by elements f_n of $\mathcal{A}$; then consider the complex path integrals $\oint_{|z|=1} f_n(z)\, dz$. Why does the Stone–Weierstraß Theorem not apply?

Exercise A.13. For a locally compact space X, we consider the Banach space $C_0(X)$ of all continuous functions $f \colon X \to \mathbb{C}$ vanishing at infinity, i.e., with the property that for each $\varepsilon > 0$ there exists a compact subset $C_\varepsilon \subseteq X$ with $|f(x)| \leq \varepsilon$ for $x \notin C_\varepsilon$. Suppose that $\mathcal{A} \subseteq C_0(X)$ is a complex subalgebra satisfying

(a) $\mathcal{A}$ is invariant under conjugation.
(b) $\mathcal{A}$ has no zeros, i.e., for each $x \in X$ there exists an $f \in \mathcal{A}$ with $f(x) \neq 0$.
(c) $\mathcal{A}$ separates the points of X.

Show that $\mathcal{A}$ is dense in $C_0(X)$ with respect to $\| \cdot \|_\infty$. Hint: Let X_ω be the one-point compactification of X. Then each function $f \in C_0(X)$ extends to a continuous function $\widetilde{f}$ on X_ω by $\widetilde{f}(\omega) := 0$, and this leads to bijection

$$C_*(X_\omega) := \{f \in C(X_\omega) \colon f(\omega) = 0\} \to C_0(X), \quad f \mapsto f|_X.$$

Use the Stone–Weierstraß Theorem to show that the algebra

$$\widetilde{\mathcal{A}} := \mathbb{C}\mathbf{1} + \{\widetilde{a} \colon a \in \mathcal{A}\}$$

is dense in $C(X_\omega)$ and show that if $\widetilde{f}_n + \lambda \mathbf{1} \to \widetilde{f}$ for $\lambda_n \in \mathbb{C}, f \in C_0(X), f_n \in \mathcal{A}$, then $\lambda_n \to 0$ and $f_n \to f$.

A.5 Summability in Banach spaces

Definition A.27. Let I be a set and X a Banach space. Then a family $(x_i)_{i\in I}$ is called *summable to* $x \in X$ if, for every $\varepsilon > 0$, there exists a finite subset $I_\varepsilon \subseteq I$ with the property that, for every finite subset $F \supseteq I_\varepsilon$, we have

$$\Big\| \sum_{i\in F} x_i - x \Big\| < \varepsilon.$$

If $(x_i)_{i\in I}$ is summable to x, we write $x = \sum_{i\in I} x_i$.[20]

Remark A.28. (a) Note that, for $I = \mathbb{N}$, the summability of a family $(x_n)_{n\in\mathbb{N}}$ in a Banach space X is stronger than the convergence of the series $\sum_{n=1}^\infty x_n$. In fact, if $x = \sum_{n\in\mathbb{N}} x_n$ holds in the sense of summability and $\mathbb{N}_\varepsilon \subseteq \mathbb{N}$ is a finite subset with the property that, for every finite subset $F \supseteq \mathbb{N}_\varepsilon$, we have $\| \sum_{n\in F} x_n - x\| < \varepsilon$, then we have for $N > \max \mathbb{N}_\varepsilon$ in particular

$$\Big\| \sum_{n=1}^N x_n - x \Big\| < \varepsilon,$$

showing that the series $\sum_{n=1}^\infty x_n$ converges to x.

(b) If, conversely, the series $\sum_{n=1}^\infty x_n$ converges absolutely to some $x \in X$ and $\varepsilon > 0$, then there exists an $N \in \mathbb{N}$ with $\sum_{n=N}^\infty \|x_n\| < \varepsilon$. With $\mathbb{N}_\varepsilon := \{1, \ldots, N\}$ we then find for every finite superset $F \supseteq \mathbb{N}_\varepsilon$ that

$$\|x - \sum_{n\in F} x_n\| \leq \sum_{n\in\mathbb{N}\setminus F} \|x_n\| \leq \sum_{n>N} \|x_n\| < \varepsilon.$$

Therefore we also have $x = \sum_{n\in\mathbb{N}} x_n$ in the sense of summability.

(c) For $X = \mathbb{R}$ and $I = \mathbb{N}$, summability of $(x_n)_{n\in\mathbb{N}}$ implies in particular convergence of all reordered series $\sum_{n=1}^\infty x_{\sigma(n)}$, where $\sigma\colon \mathbb{N} \to \mathbb{N}$ is a bijection. Therefore Riemann's Reordering Theorem shows that summability implies absolute convergence.

(d) If $(x_i)_{i\in I}$ is a family in $\mathbb{R}_+ = [0, \infty)$, then the situation is much simpler. Here summability is easily seen to be equivalent to the existence of the supremum of the set $\mathcal{F} := \{\sum_{i\in F} x_i \colon F \subseteq I, |F| < \infty\}$ of all finite partial sums, and in this case $\sum_{i\in I} x_i = \sup \mathcal{F}$.

Lemma A.29. *Let $(\mathcal{H}_j)_{j\in J}$ be a family of Hilbert spaces and*

$$\mathcal{H} := \Big\{ (x_j)_{j\in J} \in \prod_{j\in J} \mathcal{H}_j \colon \sum_{j\in J} \|x_j\|^2 < \infty \Big\}.$$

Then $\mathcal{H}$ is a Hilbert space with respect to the scalar product

$$\langle (x_j)_{j\in J}, (y_j)_{j\in J} \rangle = \sum_{j\in J} \langle x_j, y_j \rangle.$$

Proof. First we show that $\mathcal{H}$ is a linear subspace of the complex vector space $\prod_{j\in J} \mathcal{H}_j$, in which we define addition and scalar multiplication componentwise. Clearly, $\mathcal{H}$ is invariant under multiplication with complex scalars. For $a, b \in \mathcal{H}_j$, the parallelogram identity

$$\|a + b\|^2 + \|a - b\|^2 = 2\|a\|^2 + 2\|b\|^2$$

[20] This can also be formulated in terms of convergence of nets. First we order the set $\mathcal{I} := \{F \subseteq I \colon |F| < \infty\}$ of finite subsets of I by set inclusion, so that $F \mapsto \sum_{i\in F} x_i$ is a net in X, called the *net of partial sums*. Then the summability of $(x_i)_{i\in I}$ in X is equivalent to the convergence of this net in X.

(verify!) implies that

$$\|a+b\|^2 \leq 2(\|a\|^2 + \|b\|^2).$$

For $x = (x_j)_{j\in J}, y = (y_j)_{j\in J} \in \mathcal{H}$, we therefore obtain

$$\sum_{j\in J} \|x_j + y_j\|^2 \leq 2\sum_{j\in J} \|x_j\|^2 + 2\sum_{j\in J} \|y_j\|^2 < \infty.$$

This shows that $x + y \in \mathcal{H}$, so that $\mathcal{H}$ is indeed a linear subspace.

For $x, y \in \mathcal{H}$, the polarization identity

$$\langle x, y\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \|x + i^k y\|^2$$

(Exercise 3.1(i)) and $x \pm y, x \pm iy \in \mathcal{H}$ imply that the sum

$$\langle x, y\rangle = \sum_{j\in J} \langle x_j, y_j\rangle$$

exists. For $0 \neq x$, some x_i is non-zero, so that $\langle x, x\rangle \geq \langle x_i, x_i\rangle > 0$. It is a trivial verification that $\langle\cdot,\cdot\rangle$ is a hermitian form. Therefore $\mathcal{H}$, endowed with $\langle\cdot,\cdot\rangle$, is a pre-Hilbert space.

It remains to show that it is complete. This is proved in the same way as the completeness of the space ℓ^2 of square-summable sequences, which is the special case $J = \mathbb{N}$ and $\mathcal{H}_j = \mathbb{C}$ for each $j \in J$. Let $(x^n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{H}$. Then $\|x_j^n - x_j^m\| \leq \|x^n - x^m\|$ holds for each $j \in J$, so that $(x_j^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}_j$. Now the completeness of the spaces $\mathcal{H}_j$ imply the existence of elements $x_j \in \mathcal{H}_j$ with $x_j^n \to x_j$. For every finite subset $F \subseteq J$, we then have

$$\sum_{j\in F} \|x_j\|^2 = \lim_{n\to\infty} \sum_{j\in F} \|x_j^n\|^2 \leq \lim_{n\to\infty} \sum_{j\in J} \|x_j^n\|^2 = \lim_{n\to\infty} \|x^n\|^2,$$

which exists because $(x^n)_{n\in\mathbb{N}}$ is a Cauchy sequence. This implies that $x := (x_j)_{j\in J} \in \mathcal{H}$ with $\|x\|^2 \leq \lim_{n\to\infty} \|x^n\|^2$.

Finally, we show that $x^n \to x$ holds in $\mathcal{H}$. So let $\varepsilon > 0$ and $N_\varepsilon \in \mathbb{N}$ with $\|x^n - x^m\| \leq \varepsilon$ for $n, m \geq N_\varepsilon$. For a finite subset $F \subseteq J$, we then have

$$\sum_{j\in F} \|x_j - x_j^n\|^2 = \lim_{m\to\infty} \sum_{j\in F} \|x_j^m - x_j^n\|^2 \leq \lim_{m\to\infty} \|x^m - x^n\|^2 \leq \varepsilon^2$$

for $n \geq N_\varepsilon$. We therefore obtain

$$\|x - x^n\|^2 = \sup_{F\subseteq J, |F|<\infty} \sum_{j\in F} \|x_j - x_j^n\|^2 \leq \varepsilon^2.$$

This implies that $x^n \to x$ in $\mathcal{H}$, and thus $\mathcal{H}$ is complete. □

A.6 The Fourier transform on $\mathbb{R}^n$

For $f \in L^1(\mathbb{R}^n)$, we define its *Fourier transform* by

$$\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{i\langle\xi, x\rangle}\, dx.$$

The Dominated Convergence Theorem immediately implies that $\widehat{f}\colon \mathbb{R}^n \to \mathbb{R}$ is a continuous function satisfying

$$\|\widehat{f}\|_\infty \leq \|f\|_1.$$

The Banach space $L^1(\mathbb{R}^n)$ is a Banach-$*$-algebra with respect to the involution

$$f^*(x) := \overline{f(-x)}, \quad x \in \mathbb{R}^n$$

and the *convolution product*

$$(f_1 * f_2)(x) := \int_{\mathbb{R}^n} f_1(y) f_2(x-y)\, dy$$

which satisfies

$$\|f_1 * f_2\|_1 \leq \|f_1\| \cdot \|f_2\|$$

(an easy application of Fubini's Theorem).

With Fubini's Theorem, we obtain for $f_1, f_2 \in L^1(\mathbb{R}^n)$:

$$\begin{aligned}
(f_1 * f_2)\widehat{\ }(\xi) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f_1(y) f_2(x-y) e^{i\langle \xi, x\rangle}\, dy\, dx \\
&= \int_{\mathbb{R}^n} f_1(y) e^{i\langle \xi, y\rangle} \int_{\mathbb{R}^n} f_2(x-y) e^{i\langle \xi, x-y\rangle}\, dx\, dy \\
&= \int_{\mathbb{R}^n} f_1(y) e^{i\langle \xi, y\rangle} \int_{\mathbb{R}^n} f_2(x) e^{i\langle \xi, x\rangle}\, dx\, dy \\
&= \int_{\mathbb{R}^n} f_1(y) e^{i\langle \xi, y\rangle} \widehat{f_2}(\xi)\, dy \\
&= \widehat{f_1}(\xi)\widehat{f_2}(\xi)
\end{aligned}$$

and we also note that

$$\overline{\widehat{f}} = \widehat{f^*} \quad \text{for} \quad f \in L^1(\mathbb{R}^n).$$

Therefore the Fourier transform

$$\mathcal{F}\colon L^1(\mathbb{R}^n) \to (C_b(\mathbb{R}^n), \|\cdot\|_\infty), \quad f \mapsto \widehat{f}$$

is a morphism of Banach-$*$-algebras.

Proposition A.30. (Riemann–Lebesgue Lemma) *For each $f \in L^1(\mathbb{R}^n)$ the Fourier transform $\widehat{f}$ vanishes at infinity, i.e., $\widehat{f} \in C_0(\mathbb{R}^n)$.*

Proof. For each $0 \neq x \in \mathbb{R}^n$ we obtain with $e^{-i\pi} = -1$ and the translation invariance of Lebesgue measure the relation

$$\begin{aligned}
\widehat{f}(x) &= 2\frac{1}{2}\int_{\mathbb{R}^n} e^{i\langle x, y\rangle} f(y)\, dy \\
&= \frac{1}{2}\int_{\mathbb{R}^n} e^{i\langle x, y\rangle} f(y)\, dy - \frac{1}{2}\int_{\mathbb{R}^n} e^{i\langle x, y - \frac{\pi}{\|x\|^2}x\rangle} f(y)\, dy \\
&= \frac{1}{2}\int_{\mathbb{R}^n} e^{i\langle x, y\rangle} f(y)\, dy - \frac{1}{2}\int_{\mathbb{R}^n} e^{i\langle x, y\rangle} f\Big(y + \frac{\pi}{\|x\|^2}x\Big)\, dy \\
&= \frac{1}{2}\int_{\mathbb{R}^n} e^{i\langle x, y\rangle} \Big[f(y) - f\Big(y + \frac{\pi}{\|x\|^2}x\Big)\Big]\, dy.
\end{aligned}$$

This implies that

$$|\widehat{f}(x)| \leq \frac{1}{2}\int_{\mathbb{R}^n} \Big| f(y) - f\Big(y + \frac{\pi}{\|x\|^2}x\Big)\Big|\, dy.$$

Now the assertion follows from the continuity of the map

$$\mathbb{R}^n \to L^1(\mathbb{R}^n), \quad x \mapsto \lambda_x f$$

in 0 (Exercise) and $\lim_{x\to\infty} \frac{x}{\|x\|^2} = 0$. $\square$

Proposition A.31. *$\mathcal{F}(L^1(\mathbb{R}^n))$ is dense in $C_0(\mathbb{R}^n)$ w.r.t. $\|\cdot\|_\infty$.*

Proof. We know already that $\mathcal{A} := \mathcal{F}(L^1(\mathbb{R}^n))$ is a conjugation invariant subalgebra of $C_0(\mathbb{R}^n)$. According to the Stone–Weierstraß Theorem for non-compact spaces (Exercise A.13), we have to see that $\mathcal{A}$ has no common zeros and that it separates the points of $\mathbb{R}^n$. This is verified in (a) and (b) below.

(a) Let $x_0 \in \mathbb{R}^n$ and B be the ball of radius 1 around x_0. Then $f(x) := \chi_B(x)e^{-i\langle x_0, x\rangle}$ is an element of $L^1(\mathbb{R}^n)$ and

$$\widehat{f}(x_0) = \int_B dx = \mathrm{vol}(B) > 0.$$

Therefore $\mathcal{A}$ has no common zeros.

(b) For $x_0 \neq y_0 \in \mathbb{R}^n$, we pick $\alpha \in \mathbb{R}$ such that $z_0 := \alpha(x_0 - y_0)$ satisfies $e^{i\langle x_0 - y_0, z_0\rangle} \neq 1$. Then there exists a ball B with center z_0 such that $e^{i\langle x_0 - y_0, z\rangle} \neq 1$ for every $z \in B$. Then the L^1-function

$$f(x) := \chi_B(x)(e^{-i\langle x_0, x\rangle} - e^{-i\langle y_0, x\rangle})$$

has a Fourier transform satisfying

$$\begin{aligned}\widehat{f}(x_0) - \widehat{f}(y_0) &= \int_B (e^{-i\langle x_0, x\rangle} - e^{-i\langle y_0, x\rangle})(e^{i\langle x_0, x\rangle} - e^{i\langle y_0, x\rangle})\, dx \\ &= \int_B |e^{i\langle x_0, x\rangle} - e^{i\langle y_0, x\rangle}|^2\, dx > 0.\end{aligned}$$

$\square$

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