Unitary Representation Theory

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Notation and Conventions

If $\mathcal{H}$ is a complex Hilbert space, then its scalar product is written $\langle\cdot,\cdot\rangle$. It is linear in the first and antilinear in the second argument

$$\lambda\langle v,w\rangle = \langle\lambda v,w\rangle = \langle v,\overline{\lambda}w\rangle,$$

and $\|v\| := \sqrt{\langle v,v\rangle}$ is the corresponding norm. A subset $E \subseteq \mathcal{H}$ is called *total* if $\operatorname{span} E$ is dense.

- $\mathbb{N} := \{1,2,3,\ldots\}$
- $\mathbb{R}_+ := \{x \in \mathbb{R} \colon x \geq 0\} = [0,\infty[$.
- $\mathbb{R}^\times := \mathbb{R} \setminus \{0\}$, $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$, $\mathbb{T} := \{z \in \mathbb{C} \colon |z| = 1\}$.

For two sets J and Y we write Y^J for the set of maps $f \colon J \to Y$. If J is a set and S an abelian semigroup with zero element 0, then we also write $S^{(J)} \subseteq S^J$ for the subset of finitely supported functions.

For Banach spaces X and Y we write

$$B(X,Y) := \{A \colon X \to Y \colon A \text{ linear}, \ \|A\| < \infty\}$$

for the Banach space of bounded linear operators from X to Y. For $X = Y$ we abbreviate $B(X) := B(X,X)$ and write $\mathrm{GL}(X)$ for the group of invertible elements in $B(X)$. If $\mathcal{H}$ is a complex Hilbert space, then we have an antilinear isometric map $B(\mathcal{H}) \to B(\mathcal{H}), A \mapsto A^*$, determined uniquely by

$$\langle Av,w\rangle = \langle v,A^*w\rangle \quad \text{for } v,w \in \mathcal{H}.$$

We write

$$\mathrm{U}(\mathcal{H}) := \{g \in \mathrm{GL}(\mathcal{H}) \colon g^{-1} = g^*\}$$

for the unitary group. For $\mathcal{H} = \mathbb{C}^n$, the corresponding matrix group is denoted

$$\mathrm{U}_n(\mathbb{C}) := \{g \in \mathrm{GL}_n(\mathbb{C}) \colon g^{-1} = g^*\}.$$

If G is a group, we write $\mathbf{1}$ for its neutral element and

$$\lambda_g(x) = gx, \quad \rho_g(x) = xg \quad \text{and} \quad c_g(x) = gxg^{-1}$$

for left multiplications, right multiplications, resp., conjugations.

For a metrix space (X,d), we write

$$B_r(x) := \{y \in X \colon d(x,y) < r\}$$

for the open balls.

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Introduction

A *unitary representation* of a group G is a homomorphism $\pi\colon G \to \mathrm{U}(\mathcal{H})$ to the unitary group

$$\mathrm{U}(\mathcal{H}) = \{g \in \mathrm{GL}(\mathcal{H})\colon g^* = g^{-1}\}$$

of a complex Hilbert space $\mathcal{H}$. Such a representation is said to be *irreducible* if $\{0\}$ and $\mathcal{H}$ are the only $\pi(G)$-invariant closed subspaces of $\mathcal{H}$. The two fundamental problems in representation theory are:

(FP1) To classify, resp., parametrize the irreducible representations of G, and

(FP2) to explain how a given unitary representation can be decomposed into irreducible ones. This is called the problem of *harmonic analysis* because it contains in particular the expansion of a periodic L^2-function as a Fourier series.

As formulated above, both problems are not well-posed. First, one has to specify the class of representations one is interested in, and this class may depend on the group G, resp., additional structure on this group. Only in very rare situations, one studies arbitrary unitary representations. If G is a *topological group*, i.e., if G carries a topology for which the group operations are continuous, one is only interested in unitary representations which are continuous in the sense that, for each $v \in \mathcal{H}$, the orbit map

$$\pi^v\colon G \to \mathcal{H}, \quad g \mapsto \pi(g)v$$

is continuous. If G is a *Lie group*, a concept refining that of a topological group, so that it makes sense to talk about smooth functions on G, then we consider only representations for which the subspace

$$\mathcal{H}^\infty := \{v \in \mathcal{H}\colon \pi^v\colon G \to \mathcal{H} \text{ is smooth}\}$$

of *smooth vectors* is dense in $\mathcal{H}$.

This means that there are three basic contexts for representation theory

- the discrete context (G is considered as a discrete group, no restrictions)
- the topological context (G is a topological group; continuity required)
- the Lie context (G is a Lie group; smoothness required).

In each of these contexts, the two fundamental problems mentioned above are of a completely different nature because they concern different classes of representations. For example one can show that the harmonic analysis problem has a good solution for the topological group $\mathrm{GL}_2(\mathbb{R})$, but not for the same group, considered as a discrete one. To make statements like this more precise is one of the fundamental tasks of representation theory.

To give a first impression of the major difficulties involved in this program, we discuss some examples.

Remark 1. If the group G is abelian, then one can show that all irreducible representations $(\pi, \mathcal{H})$ are one-dimensional, so that $\pi(g) = \chi(g)\mathbf{1}$ holds for a group homomorphism

$$\chi\colon G \to \mathbb{T} := \{z \in \mathbb{C}\colon |z| = 1\}$$

into the circle group. Such homomorphisms are called *characters*. For an abelian topological group G, we write

$$\widehat{G} := \mathrm{Hom}(G, \mathbb{T})$$

for the set of continuous characters. They form a group under pointwise multiplication, called the *character group of* G. Since all irreducible unitary representations of an abelian group G are one-dimensional, the group $\widehat{G}$ parametrizes the irreducible representations and the solution of (FP1) therefore consists in a description of the group $\widehat{G}$.

The second fundamental problem (FP2) is much harder to deal with. If $(\pi, \mathcal{H})$ is a unitary representation, then each irreducible subrepresentation is one-dimensional, hence generated by a G-eigenvector $v \in \mathcal{H}$ satisfying

$$\pi(g)v = \chi(g)v \quad \text{for} \quad g \in G$$

and some character $\chi \in \widehat{G}$. Now one would like to "decompose" $\mathcal{H}$ into the G-eigenspaces

$$\mathcal{H}_\chi := \{v \in \mathcal{H}\colon (\forall g \in G)\, \pi(g)v = \chi(g)v\}.$$

As the following two examples show, there are situations where this is possible, but this is not always the case.

Example 1. To solve (FP1) for the circle group $G := \mathbb{T}$, we first note that for each $n \in \mathbb{Z}$, $\chi_n(z) := z^n$ defines a continuous character of $\mathbb{T}$, and one can show that these are all continuous characters. Therefore $\chi_n \chi_m = \chi_{n+m}$ leads to

$$\widehat{\mathbb{T}} = \mathrm{Hom}(\mathbb{T}, \mathbb{T}) \cong \mathbb{Z}.$$

The group $\mathbb{T}$ has a continuous representation on the space $\mathcal{H} = L^2(\mathbb{T}, \mu)$, where μ is the probability measure on $\mathbb{T}$ specified by

$$\int_{\mathbb{T}} f(z)\, d\mu(z) := \frac{1}{2\pi} \int_0^{2\pi} f(e^{it})\, dt.$$

The *regular representation of* $\mathbb{T}$ is defined by

$$(\pi(t)f)(z) := f(tz).$$

Then the $\mathbb{T}$-eigenfunctions in $\mathcal{H}$ corresponding to χ_n are the functions χ_n themselves, and it is a basic result in the theory of Fourier series that any function $f \in \mathcal{H}$ can be expanded as a Fourier series

$$f = \sum_{n \in \mathbb{Z}} a_n \chi_n$$

converging in $\mathcal{H}$. In this sense $\mathcal{H}$ is a (topological) direct sum of the subspaces $\mathbb{C}\chi_n$, which means that the representation π decomposes nicely into irreducible pieces.

Example 2. For the group $G := \mathbb{R}$, the solution of (FP1) asserts that each continuous character is of the form

$$\chi_\lambda(x) := e^{i\lambda x}, \quad \lambda \in \mathbb{R},$$

so that $\chi_\lambda \chi_\mu = \chi_{\lambda+\mu}$ leads to the group isomorphism

$$\mathbb{R} \to \widehat{\mathbb{R}}, \quad \lambda \mapsto \chi_\lambda.$$

The group $\mathbb{R}$ has a continuous representation on the space $\mathcal{H} = L^2(\mathbb{R}, dx)$ given by

$$(\pi(x)f)(y) := f(x+y).$$

Then the $\mathbb{R}$-eigenfunctions in $\mathcal{H}$ corresponding to χ_λ solve the equation

$$f(x+y) = e^{i\lambda x} f(y)$$

for every x and, for a given x, for almost every $y \in \mathbb{R}$. This leads in particular to $|f(y+1)| = |f(y)|$ for almost every $y \in \mathbb{R}$, and thus to

$$\int_{\mathbb{R}} |f(y)|^2\, dy = \sum_{n\in\mathbb{Z}} \int_0^1 |f(y+n)|^2\, dy = \infty \cdot \int_0^1 |f(y)|^2\, dy = \infty,$$

whenever $\|f\|_2 \neq 0$. Therefore the representation $(\pi, \mathcal{H})$ contains no irreducible subspaces and we need refined methods to say what it means to decompose it into irreducible ones.

The problem of decomposing functions into simpler pieces with respect to the transformation behavior under a certain symmetry group arises in many situations, not only in mathematics, but also in the natural sciences. In mathematics, unitary representation theory has many applications in areas ranging from number theory, geometry, real and complex analysis to partial differential equations (see in particular [Ma78]).

One of the strongest motivations for the systematic development of the theory of unitary group representations that started in the 1940s, was its close connection to Quantum Mechanics. This connection is due to the fact that the *state space* of a quantum mechanical system is modeled by the set

$$\mathbb{P}(\mathcal{H}) := \{[v] = \mathbb{C}v \colon 0 \neq v \in \mathcal{H}\}$$

of one-dimensional subspaces of a complex Hilbert space, its *projective space*. This spaces carries several interesting structures. The most important one for physics is the function

$$\beta \colon \mathbb{P}(\mathcal{H}) \times \mathbb{P}(\mathcal{H}) \to [0,1], \quad \beta([v],[w]) := \frac{|\langle v, w\rangle|^2}{\|v\|^2\|w\|^2},$$

which is interpreted as a *transition probability* between the two states $[v]$ and $[w]$. A central feature of quantum physical models is that systems are often specified by their symmetries. More concretely, this means that each system has a symmetry group G. It acts on the corresponding set $\mathbb{P}(\mathcal{H})$ of states in such a way that it preserves the transition probabilities, i.e., we have a group action $G \times \mathbb{P}(\mathcal{H}) \to \mathbb{P}(\mathcal{H}), (g, [v]) \mapsto g[v]$, satisfying

$$\beta(g[v], g[w]) = \beta([v], [w]) \quad \text{for} \quad g \in G, 0 \neq v, w \in \mathcal{H}.$$

To link these structures to unitary representations, we have to quote Wigner's fundamental theorem that for each bijection φ of $\mathbb{P}(\mathcal{H})$ preserving β, there exists either a linear or an antilinear surjective isometry $\widetilde{\varphi}\colon \mathcal{H} \to \mathcal{H}$ such that $\varphi[v] = [\widetilde{\varphi}(v)]$ holds for each $0 \neq v \in \mathcal{H}$ (cf. [Ba64]; see also [FF00, Thm. 14.3.6] for Wigner's Theorem and various generalizations and [PF80] for the Lorentzian case). This leads to a surjective homomorphism

$$\Gamma\colon \mathrm{AU}(\mathcal{H}) \to \mathrm{Aut}(\mathbb{P}(\mathcal{H}), \beta), \quad \widetilde{\varphi} \mapsto \varphi,$$

where $\mathrm{AU}(\mathcal{H})$ denotes the set of *semilinear unitary operators*, where semilinear means either linear or antilinear. If $G \subseteq \mathrm{Aut}(\mathbb{P}(\mathcal{H}), \beta)$ is a quantum mechanical symmetry group, we thus obtain a subgroup $G^\sharp := \Gamma^{-1}(G) \subseteq \mathrm{AU}(\mathcal{H})$ with a semilinear unitary representation on $\mathcal{H}$, and the subgroup $G^\sharp_u := G^\sharp \cap \mathrm{U}(\mathcal{H})$ of index at most two is a unitary group. One subtlety that we observe here is that the homomorphism Γ is not injective. If $\dim \mathcal{H} > 1$, then its kernel consists of the circle group $\mathbb{T}\mathbf{1} = \{z\mathbf{1}\colon |z| = 1, z \in \mathbb{C}\}$, so that

$$G \cong G^\sharp/\mathbb{T}$$

and $G^\sharp$ is an *extension* of the group G by the circle group $\mathbb{T}$ which is central if $G^\sharp \subseteq \mathrm{U}(\mathcal{H})$, resp., $G \subseteq \Gamma(\mathrm{U}(\mathcal{H}))$.

It is this line of reasoning that leads from quantum mechanical symmetries to the problem of classifying irreducible unitary representation of a group G, resp., of its $\mathbb{T}$-extensions $G^\sharp$, because these representations correspond to systems with the same kind of symmetry. Similar questions lead in particular to the problem of classifying elementary particles in terms of representations of certain compact Lie groups (cf. [BH09, Va85, Ma78]).

Chapter 1

Continuous Unitary Representations

Throughout these notes we shall mainly be concerned with continuous representations of topological groups. Therefore Section 1.1 introduces topological groups and some important examples. In Section 1.2 we discuss continuity of unitary representations and provide some methods that can be used to verify continuity easily in many situations. We also introduce the strong topology on the unitary group $\mathrm{U}(\mathcal{H})$ for which a continuous unitary representation of G is the same as a continuous group homomorphism $\pi\colon G \to \mathrm{U}(\mathcal{H})$. As a first step in the decomposition theory of representations, we discuss in Section 1.3 direct sums of unitary representations and show that every representation is a direct sum of cyclic ones. Later we shall study cyclic representations in terms of positive definite functions on G.

1.1 Topological Groups

Definition 1.1.1. A *topological group* is a pair (G, τ) of a group G and a Hausdorff topology τ for which the group operations

$$m_G\colon G \times G \to G, \quad (x, y) \mapsto xy \quad \text{and} \quad \eta_G\colon G \to G, \quad x \mapsto x^{-1}$$

are continuous when $G \times G$ carries the product topology. Then we call τ a *group topology* on the group G.

Remark 1.1.2. The continuity of the group operations can also be translated into the following conditions which are more direct than referring to the product topology on G. The continuity of the multiplication m_G in $(x, y) \in G \times G$ means that for each neighborhood V of xy there exist neighborhoods U_x of x and U_y of y with $U_x U_y \subseteq V$. Similarly, the continuity of the inversion map η_G in x means that for each neighborhood V of x^{-1}, there exists a neighborhood U_x of x with $U_x^{-1} = \{y^{-1} \colon y \in U_x\} \subseteq V$.

Remark 1.1.3. For a group G with a topology τ, the continuity of m_G and η_G already follows from the continuity of the single map

$$\varphi\colon G \times G \to G, \quad (g,h) \mapsto gh^{-1}.$$

In fact, if φ is continuous, then the inversion $\eta_G(g) = g^{-1} = \varphi(\mathbf{1}, g)$ is the composition of φ and the continuous map $G \to G\times G, g \mapsto (\mathbf{1}, g)$. The continuity of η_G further implies that the product map

$$\mathrm{id}_G \times \eta_G \colon G \times G \to G \times G, \quad (g,h) \mapsto (g, h^{-1})$$

is continuous, and therefore $m_G = \varphi \circ (\mathrm{id}_G \times \eta_G)$ is continuous.

Remark 1.1.4. Every subgroup H of a topological group G is a topological group with respect to the subspace topology.

Examples 1.1.5. (a) $G = (\mathbb{R}^n, +)$ is an abelian topological group with respect to any metric defined by a norm.

More generally, the additive group $(X, +)$ of every normed space X is a topological group.

(b) $(\mathbb{C}^\times, \cdot)$ is a topological group and the circle group $\mathbb{T} := \{z \in \mathbb{C}^\times : |z| = 1\}$ is a compact subgroup.

(c) The group $\mathrm{GL}_n(\mathbb{R})$ of invertible $(n \times n)$-matrices is a topological group with respect to matrix multiplication. The continuity of the inversion follows from Cramer's Rule, which provides an explicit formula for the inverse in terms of determinants: For $g \in \mathrm{GL}_n(\mathbb{R})$, we define $b_{ij}(g) := \det(g_{mk})_{m\neq j, k\neq i}$. Then the inverse of g is given by

$$(g^{-1})_{ij} = \frac{(-1)^{i+j}}{\det g} b_{ij}(g)$$

(see Proposition 1.1.10 for a different argument).

(d) Any group G is a topological group with respect to the discrete topology.

We have already argued above that the group $\mathrm{GL}_n(\mathbb{R})$ carries a natural group topology. This group is the unit group of the algebra $M_n(\mathbb{R})$ of real $(n \times n)$-matrices. As we shall see now, there is a vast generalization of this construction.

Definition 1.1.6. A *Banach algebra* is a triple $(\mathcal{A}, m_{\mathcal{A}}, \|\cdot\|)$ of a Banach space $(\mathcal{A}, \|\cdot\|)$, together with an associative bilinear multiplication

$$m_{\mathcal{A}} \colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}, (a,b) \mapsto ab$$

for which the norm $\|\cdot\|$ is *submultiplicative*, i.e.,

$$\|ab\| \leq \|a\| \cdot \|b\| \quad \text{for} \quad a, b \in \mathcal{A}.$$

By abuse of notation, we shall mostly call $\mathcal{A}$ a Banach algebra, if the norm and the multiplication are clear from the context.

A *unital Banach algebra* is a pair $(\mathcal{A}, \mathbf{1})$ of a Banach algebra $\mathcal{A}$ and an element $\mathbf{1} \in \mathcal{A}$ satisfying $\mathbf{1}a = a\mathbf{1} = a$ for each $a \in \mathcal{A}$ and $\|\mathbf{1}\| = 1$. The subset

$$\mathcal{A}^\times := \{a \in \mathcal{A} \colon (\exists b \in \mathcal{A})\ ab = ba = \mathbf{1}\}$$

is called the *unit group of* $\mathcal{A}$ (cf. Exercise 1.1.15).

Example 1.1.7. (a) If $(X, \|\cdot\|)$ is a Banach space, then the space $B(X)$ of continuous linear operators $A \colon X \to X$ is a unital Banach algebra with respect to the *operator norm*

$$\|A\| := \sup\{\|Ax\| \colon x \in X, \|x\| \leq 1\}$$

and composition of maps. Note that the submultiplicativity of the operator norm, i.e.,

$$\|AB\| \leq \|A\| \cdot \|B\|,$$

is an immediate consequence of the estimate

$$\|ABx\| \leq \|A\| \cdot \|Bx\| \leq \|A\| \cdot \|B\| \cdot \|x\| \quad \text{for} \quad x \in X.$$

In this case the unit group is also denoted $\mathrm{GL}(X) := B(X)^\times$.

(b) If X is a compact space and $\mathcal{A}$ a Banach algebra, then the space $C(X, \mathcal{A})$ of $\mathcal{A}$-valued continuous functions on X is a Banach algebra with respect to pointwise multiplication $(fg)(x) := f(x)g(x)$ and the norm

$$\|f\| := \sup_{x \in X} \|f(x)\|$$

(Exercise 1.1.14)

(c) An important special case of (b) arises for $\mathcal{A} = M_n(\mathbb{C})$, where we obtain $C(X, M_n(\mathbb{C}))^\times = C(X, \mathrm{GL}_n(\mathbb{C})) = \mathrm{GL}_n(C(X, \mathbb{C}))$.

Example 1.1.8. For any norm $\|\cdot\|$ on $\mathbb{C}^n$, the choice of a basis yields an isomorphism of algebras $M_n(\mathbb{C}) \cong B(\mathbb{C}^n)$, so that $\mathrm{GL}_n(\mathbb{C}) \cong B(\mathbb{C}^n)^\times = \mathrm{GL}(\mathbb{C}^n)$.

Remark 1.1.9. In a Banach algebra $\mathcal{A}$, the multiplication is continuous because $a_n \to a$ and $b_n \to b$ implies $\|b_n\| \to \|b\|$ and therefore

$$\|a_n b_n - ab\| = \|a_n b_n - ab_n + ab_n - ab\| \leq \|a_n - a\| \cdot \|b_n\| + \|a\| \cdot \|b_n - b\| \to 0.$$

In particular, left and right multiplications

$$\lambda_a \colon \mathcal{A} \to \mathcal{A}, x \mapsto ax, \quad \text{and} \quad \rho_a \colon \mathcal{A} \to \mathcal{A}, x \mapsto xa,$$

are continuous with

$$\|\lambda_a\| \leq \|a\| \quad \text{and} \quad \|\rho_a\| \leq \|a\|. \tag{1.1}$$

Proposition 1.1.10. *The unit group $\mathcal{A}^\times$ of a unital Banach algebra is an open subset and a topological group with respect to the topology defined by the metric $d(a, b) := \|a - b\|$.*

Proof. The proof is based on the convergence of the Neumann series $\sum_{n=0}^\infty x^n$ for $\|x\| < 1$. For any such x we have

$$(\mathbf{1} - x)\sum_{n=0}^\infty x^n = \Big(\sum_{n=0}^\infty x^n\Big)(\mathbf{1} - x) = \mathbf{1},$$

so that $\mathbf{1} - x \in \mathcal{A}^\times$. We conclude that the open unit ball $B_1(\mathbf{1})$ is contained in $\mathcal{A}^\times$.

Next we note that left multiplications $\lambda_g \colon \mathcal{A} \to \mathcal{A}$ with elements $g \in \mathcal{A}^\times$ are continuous (Remark 1.1.9), hence homeomorphisms because $\lambda_g^{-1} = \lambda_{g^{-1}}$ is also continuous. Therefore $gB_1(\mathbf{1}) = \lambda_g B_1(\mathbf{1}) \subseteq \mathcal{A}^\times$ is an open subset, showing that g is an interior point of $\mathcal{A}^\times$. Hence $\mathcal{A}^\times$ is open.

The continuity of the multiplication of $\mathcal{A}^\times$ follows from the continuity of the multiplication on $\mathcal{A}$ by restriction and corestriction (Remark 1.1.9). The continuity of the inversion in $\mathbf{1}$ follows from the estimate

$$\|(\mathbf{1} - x)^{-1} - \mathbf{1}\| = \Big\|\sum_{n=1}^\infty x^n\Big\| \leq \sum_{n=1}^\infty \|x\|^n = \frac{1}{1 - \|x\|} - 1 = \frac{\|x\|}{1 - \|x\|},$$

which tends to 0 for $x \to 0$. The continuity of the inversion in $g \in \mathcal{A}^\times$ is now obtained as follows. If $g_n \to g$, then $g_n g^{-1} \to \mathbf{1}$ follows from the continity of the right multiplications, so that

$$g_n^{-1} = g^{-1}(g_n g^{-1})^{-1} \to g^{-1}$$

follows from the continuity of the inversion in $\mathbf{1}$ and the continuity of the left multiplications. This shows that $\mathcal{A}^\times$ is a topological group. □

As we shall see throughout these notes, dealing with unitary representations often leads us to Banach algebras with an extra structure given by an involution.

Definition 1.1.11. (a) An *involutive algebra* $\mathcal{A}$ is a pair $(\mathcal{A}, *)$ of a complex algebra $\mathcal{A}$ and a map $\mathcal{A} \to \mathcal{A}, a \mapsto a^*$, satisfying

(1) $(a^*)^* = a$ (Involutivity)

(2) $(\lambda a + \mu b)^* = \overline{\lambda} a^* + \overline{\mu} b^*$ (Antilinearity).

(3) $(ab)^* = b^* a^*$ ($*$ is an antiautomorphism of $\mathcal{A}$).

Then $*$ is called an *involution* on $\mathcal{A}$. A *Banach-$*$-algebra* is an involutive algebra $(\mathcal{A}, *)$, where $\mathcal{A}$ is a Banach algebra and $\|a^*\| = \|a\|$ holds for each $a \in \mathcal{A}$. If, in addition,

$$\|a^* a\| = \|a\|^2 \quad \text{for} \quad a \in \mathcal{A},$$

then $(\mathcal{A}, *)$ is called a *C^*-algebra*.

Example 1.1.12. (a) The algebra $B(\mathcal{H})$ of bounded operators on a complex Hilbert space $\mathcal{H}$ is a C^*-algebra. Here the main point is that, for each $A \in B(\mathcal{H})$, we have

$$\|A\| = \sup\{|\langle Av, w\rangle| \colon \|v\|, \|w\| \leq 1\},$$

which immediately implies that $\|A^*\| = \|A\|$. It also implies that

$$\|A^*A\| = \sup\{|\langle Av, Aw\rangle| \colon \|v\|, \|w\| \leq 1\} \geq \sup\{\|Av\|^2 \colon \|v\| \leq 1\} = \|A\|^2,$$

and since $\|A^*A\| \leq \|A^*\| \cdot \|A\| = \|A\|^2$ by Example 1.1.7, we see that $B(\mathcal{H})$ is a C^*-algebra.

(b) From (a) it immediately follows that every closed $*$-invariant subalgebra of $\mathcal{A} \subseteq B(\mathcal{H})$ also is a C^*-algebra.

(c) If X is a compact space, then the Banach space $C(X, \mathbb{C})$, endowed with

$$\|f\| := \sup_{x \in X} |f(x)|$$

is a C^*-algebra with respect to $f^*(x) := \overline{f(x)}$. In this case $\|f^*f\| = \||f|^2\| = \|f\|^2$ is trivial.

(d) If X is a locally compact space, then we say that a continuous function $f \colon X \to \mathbb{C}$ *vanishes at infinity* if, for each $\varepsilon > 0$, there exists a compact subset $K \subseteq X$ with $|f(x)| \leq \varepsilon$ for $x \notin K$. We write $C_0(X, \mathbb{C})$ for the set of all continuous functions vanishing at infinity and endow it with the norm

$$\|f\| := \sup_{x \in X} |f(x)|.$$

(cf. Exercise 1.1.16). Then $C_0(X, \mathbb{C})$ is a C^*-algebra with respect the involution $f^*(x) := \overline{f(x)}$.

Example 1.1.13. (a) If $\mathcal{H}$ is a (complex) Hilbert space, then its unitary group

$$\mathrm{U}(\mathcal{H}) := \{g \in \mathrm{GL}(\mathcal{H}) \colon g^* = g^{-1}\}$$

is a topological group with respect to the metric $d(g, h) := \|g - h\|$. It is a closed subgroup of the unit group $\mathrm{GL}(\mathcal{H}) = B(\mathcal{H})^\times$ of the C^*-algebra $B(\mathcal{H})$.

For $\mathcal{H} = \mathbb{C}^n$, endowed with the standard scalar product, we also write

$$\mathrm{U}_n(\mathbb{C}) := \{g \in \mathrm{GL}_n(\mathbb{C}) \colon g^* = g^{-1}\} \cong \mathrm{U}(\mathbb{C}^n),$$

and note that

$$\mathrm{U}_1(\mathbb{C}) = \{z \in \mathbb{C}^\times = \mathrm{GL}(\mathbb{C}) \colon |z| = 1\} \cong \mathbb{T}$$

is the circle group.

(b) If $\mathcal{A}$ is a unital C^*-algebra, then its *unitary group*

$$\mathrm{U}(\mathcal{A}) := \{g \in \mathcal{A} \colon gg^* = g^*g = \mathbf{1}\}$$

also is a topological group with respect to the norm topology.

Exercises for Section 1.1

Exercise 1.1.1. Show that, for each $n \in \mathbb{N}$, the unitary group

$$\mathrm{U}_n(\mathbb{C}) = \{g \in \mathrm{GL}_n(\mathbb{C}) \colon \mathbf{1} = g^*g = gg^*\}$$

is compact.

Exercise 1.1.2. Let $C_n := \{z \in \mathbb{C}^\times \colon z^n = 1\}$ denote the group of nth roots of unity and $(\pi, \mathcal{H})$ be a unitary representation of C_n. Show that:

(i) Every homomorphism $\chi \colon C_n \to \mathbb{T}$ is of the form $\chi_k(z) = z^k$ for some $k \in \mathbb{Z}$.

(ii) The set $\widehat{C_n} := \mathrm{Hom}(C_n, \mathbb{T})$ is a group with respect to pointwise multiplication which is isomorphic to C_n.

(iii) $P_k := \frac{1}{n}\sum_{z \in C_n} z^{-k}\pi(z)$ is an orthogonal projection onto the common eigenspace

$$\mathcal{H}_k := \{v \in \mathcal{H} \colon (\forall z \in C_n)\, \pi(z)v = z^k\}.$$

(iv) $P_k P_m = 0$ for $m - k \notin n\mathbb{Z}$.

(v) $P_0 + P_1 + \cdots + P_{k-1} = \mathrm{id}_{\mathcal{H}}$ and $\mathcal{H} = \mathcal{H}_0 \oplus \cdots \oplus \mathcal{H}_{k-1}$.

Exercise 1.1.3. Show that, for every finite abelian group A, the character group $\widehat{A} := \mathrm{Hom}(A, \mathbb{T})$ is isomorphic to A. Hint: Exercise 1.1.2(ii) and the primary decomposition of A.

Exercise 1.1.4. (Antilinear Isometries) Let $\mathcal{H}$ be a complex Hilbert space. Show that:

(a) There exists an antilinear isometric involution τ on $\mathcal{H}$. Such involutions are called *conjugations*. Hint: Use an orthonormal basis $(e_j)_{j \in J}$ of $\mathcal{H}$.

(b) A map $\varphi \colon \mathcal{H} \to \mathcal{H}$ is an antilinear isometry if and only if

$$\langle \varphi(v), \varphi(w) \rangle = \langle w, v \rangle \quad \text{for} \quad v, w \in \mathcal{H}.$$

(c) If σ is an antilinear isometric involution of $\mathcal{H}$, then there exists an orthonormal basis $(e_j)_{j \in J}$ fixed pointwise by σ. Hint: Show that $\mathcal{H}^\sigma := \{v \in \mathcal{H} \colon \sigma(v) = v\}$ is a real Hilbert space with $\mathcal{H}^\sigma \oplus i\mathcal{H}^\sigma = \mathcal{H}$ and pick an ONB in $\mathcal{H}^\sigma$.

(d) If $\dim \mathcal{H} > 1$, then no antilinear involution σ acts trivially on $\mathbb{P}(\mathcal{H})$, i.e., there exists an element $v \in \mathcal{H}$ with $\sigma(v) \notin \mathbb{C}v$.

(e) If $\dim \mathcal{H} = 1$, then every antilinear isometry $\sigma \colon \mathcal{H} \to \mathcal{H}$ is an involution.

Exercise 1.1.5. (Antilinear Isometries) Let $\mathcal{H}$ be a complex Hilbert space. Show that:

(a) In the group $\mathrm{AU}(\mathcal{H})$ of semilinear (=linear or antilinear) surjective isometries of $\mathcal{H}$, the unitary group $\mathrm{U}(\mathcal{H})$ is a normal subgroup of index 2 (cf. Exercise 1.1.4(a)).

(b) Each antilinear isometry φ of $\mathcal{H}$ induces a map $\overline{\varphi} \colon \mathbb{P}(\mathcal{H}) \to \mathbb{P}(\mathcal{H}), [v] \mapsto [\varphi(v)]$ preserving $\beta([v],[w]) = \frac{|\langle v, w\rangle|^2}{\|v\|^2\|w\|^2}$, i.e.,

$$\beta(\overline{\varphi}[v], \overline{\varphi}[w]) = \frac{|\langle \varphi(v), \varphi(w)\rangle|^2}{\|\varphi(v)\|^2\|\varphi(w)\|^2} = \beta([v],[w]).$$

(c) An element $g \in \mathrm{U}(\mathcal{H})$ induces the identity on $\mathbb{P}(\mathcal{H})$ if and only if $g \in \mathbb{T}\mathbf{1}$.

(d) If there exists an antilinear isometry inducing the identity on $\mathbb{P}(\mathcal{H})$, then $\dim \mathcal{H} = 1$.
Hint: Show first that $\sigma^2 = \mathbf{1}$ (Exercise 1.1.4(e)) and then use Exercise 1.1.4(c).

Exercise 1.1.6. Show that, for the one-dimensional Hilbert space $\mathcal{H} = \mathbb{C}$, the group $\mathrm{AU}(\mathcal{H})$ is isomorphic to the group $\mathrm{O}_2(\mathbb{R})$ of linear isometries of the euclidean plane.

Exercise 1.1.7. Let $\mathcal{H}$ be a complex Hilbert space. We endow its unit sphere

$$\mathbb{S}(\mathcal{H}) := \{v \in \mathcal{H} \colon \|v\| = 1\}$$

with the metric inherited from $\mathcal{H}$: $d(x, y) = \|x - y\|$ and consider the projective space $\mathbb{P}(\mathcal{H})$ as the set of $\mathbb{T}$-orbits in $\mathbb{P}(\mathcal{H})$. Show that the corresponding quotient metric on $\mathbb{P}(\mathcal{H})$ satisfies

$$d([x], [y]) := d(\mathbb{T}x, \mathbb{T}y) = \sqrt{2(1 - |\langle x, y\rangle|)} \in [0, \sqrt{2}].$$

Exercise 1.1.8. Let $(v_n)_{n\in\mathbb{N}}$ be a sequence in a Hilbert space $\mathcal{H}$ which converges weakly to v, i.e., $\langle v_n, w\rangle \to \langle v, w\rangle$ for every $w \in \mathcal{H}$, and assume that $\|v_n\| \to \|v\|$. Then we have $v_n \to v$.

Exercise 1.1.9. Let G be a topological group. Show that the following assertions hold:

(i) The left multiplication maps $\lambda_g \colon G \to G, x \mapsto gx$ are homeomorphisms.

(ii) The right multiplication maps $\rho_g \colon G \to G, x \mapsto xg$ are homeomorphisms.

(iii) The conjugation maps $c_g \colon G \to G, x \mapsto gxg^{-1}$ are homeomorphisms.

(iv) The inversion map $\eta_G \colon G \to G, x \mapsto x^{-1}$ is a homeomorphism.

Exercise 1.1.10. Let G be a group, endowed with a topology τ. Show that (G, τ) is a topological group if the following conditions are satisfied:

(i) The left multiplication maps $\lambda_g \colon G \to G, x \mapsto gx$ are continuous.

(ii) The inversion map $\eta_G \colon G \to G, x \mapsto x^{-1}$ is continuous.

(iii) The multiplication $m_G \colon G \times G \to G$ is continuous in $(\mathbf{1}, \mathbf{1})$.

Hint: Use (i) and (ii) to derive that all right multiplications and hence all conjugations are continuous.

Exercise 1.1.11. Let G be a group, endowed with a topology τ. Show that (G, τ) is a topological group if the following conditions are satisfied:

(i) The left multiplication maps $\lambda_g \colon G \to G, x \mapsto gx$ are continuous.

(ii) The right multiplication maps $\rho_g \colon G \to G, x \mapsto xg$ are continuous.

(iii) The inversion map $\eta_G \colon G \to G$ is continuous in $\mathbf{1}$.

(iv) The multiplication $m_G \colon G \times G \to G$ is continuous in $(\mathbf{1}, \mathbf{1})$.

Exercise 1.1.12. Show that if $(G_i)_{i\in I}$ is a family of topological groups, then the product group $G := \prod_{i\in I} G_i$ is a topological group with respect to the product topology.

Exercise 1.1.13. Let G and N be topological groups and suppose that the homomorphism $\alpha\colon G \to \mathrm{Aut}(N)$ defines a continuous map

$$G \times N \to N, \quad (g, n) \mapsto \alpha_g(n).$$

Then $N \times G$ is a group with respect to the multiplication

$$(n, g)(n', g') := (n\alpha_g(n'), gg'),$$

called the semidirect product of N and G with respect to α. It is denoted $N \rtimes_\alpha G$. Show that it is a topological group with respect to the product topology.

A typical example is the group

$$\mathrm{Mot}(\mathcal{H}) := \mathcal{H} \rtimes_\alpha \mathrm{U}(\mathcal{H})$$

of affine isometries of a complex Hilbert space $\mathcal{H}$; also called the *motion group*. In this case $\alpha_g(v) = gv$ and $\mathrm{Mot}(\mathcal{H})$ acts on $\mathcal{H}$ by $\alpha_{(b,g)}v := b + gv$ (hence the name). On $\mathrm{U}(\mathcal{H})$ we may either use the norm topology or the strong topology. For both we obtain group topologies on $\mathrm{Mot}(\mathcal{H})$ (verify this!) (cf. Exercise 1.2.3 below).

Exercise 1.1.14. Let X be a compact space and $\mathcal{A}$ be a Banach algebra. Show that:

(a) The space $C(X, \mathcal{A})$ of $\mathcal{A}$-valued continuous functions on X is a complex associative algebra with respect to pointwise multiplication $(fg)(x) := f(x)g(x)$.

(b) $\|f\| := \sup_{x\in X} \|f(x)\|$ is a submultiplicative norm on $C(X, \mathcal{A})$ for which $C(X, \mathcal{A})$ is complete, hence a Banach algebra. Hint: Continuous functions on compact spaces are bounded and uniform limits of sequences of continuous functions are continuous.

(c) $C(X, \mathcal{A})^\times = C(X, \mathcal{A}^\times)$.

(d) If $\mathcal{A}$ is a C^*-algebra, then $C(X, \mathcal{A})$ is also a C^*-algebra with respect to the involution $f^*(x) := f(x)^*$, $x \in X$.

Exercise 1.1.15. Let $\mathcal{A}$ be a Banach algebra over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. If $\mathcal{A}$ has no unit, we cannot directly associate a "unit group" to $\mathcal{A}$. However, there is a natural way to do that by considering on $\mathcal{A}$ the multiplication

$$x * y := x + y + xy.$$

Show that:

(a) The space $\mathcal{A}_+ := \mathcal{A} \times \mathbb{K}$ is a unital Banach algebra with respect to the multiplication
$$(a, t)(a', t') := (aa' + ta' + t'a, tt').$$

(b) The map $\eta\colon \mathcal{A} \to \mathcal{A}_+, x \mapsto (x, 1)$ is injective and satisfies $\eta(x * y) = \eta(x)\eta(y)$. Conclude in particular that $(\mathcal{A}, *, 0)$ is a monoid, i.e., a semigroup with neutral element 0.

(c) An element $a \in \mathcal{A}$ is said to be *quasi-invertible* if it is an invertible element in the monoid $(\mathcal{A}, *, 0)$. Show that the set $\mathcal{A}^\times$ of quasi-invertible elements of $\mathcal{A}$ is an open subset and that $(\mathcal{A}^\times, *, 0)$ is a topological group.

Exercise 1.1.16. Let X be a locally compact space and $\mathcal{A}$ be a Banach algebra. We say that a continuous function $f\colon X \to \mathcal{A}$ *vanishes at infinity* if, for each $\varepsilon > 0$, there exists a compact subset $K \subseteq X$ with $\|f(x)\| \leq \varepsilon$ for $x \notin K$. We write $C_0(X, \mathcal{A})$ for the set of all continuous $\mathcal{A}$-valued functions vanishing at infinity. Show that all assertions of Exercise 1.1.14 remain true in this more general context, where (c) has to be interpreted in the sense of quasi-invertible elements (Exercise 1.1.15).

1.2 Continuous Unitary Representations

For a topological group G, we only want to consider unitary representations which are continuous in some sense. Since we have already seen above that the unitary group $\mathrm{U}(\mathcal{H})$ of a Hilbert space is a topological group with respect to the metric induced by the operator norm, it seems natural to call a unitary representation $\pi\colon G \to \mathrm{U}(\mathcal{H})$ continuous if it is continuous with respect to the norm topology on $\mathrm{U}(\mathcal{H})$. However, the norm topology on $\mathrm{U}(\mathcal{H})$ is very fine, so that continuity with respect to this topology is a condition which is much too strong for many applications. We therefore need a suitable weaker topology on the unitary group.

We start by defining some topologies on the space $B(\mathcal{H})$ of all continuous operators which are weaker than the norm topology.

Definition 1.2.1. Let $\mathcal{H}$ be a Hilbert space. On $B(\mathcal{H})$ we define the *weak operator topology* τ_w as the coarsest topology for which all functions

$$f_{v,w}\colon B(\mathcal{H}) \to \mathbb{C}, \quad A \mapsto \langle Av, w\rangle, \quad v, w \in \mathcal{H},$$

are continuous. We define the *strong operator topology* τ_s as the coarsest topology for which all maps

$$B(\mathcal{H}) \to \mathcal{H}, \quad A \mapsto Av, \quad v \in \mathcal{H},$$

are continuous. This topology is also called the *topology of pointwise convergence.*

Remark 1.2.2. (a) Since

$$|f_{v,w}(A) - f_{v,w}(B)| = |\langle (A-B)v, w\rangle| \leq \|(A-B)v\| \cdot \|w\|$$

by the Cauchy–Schwarz Inequality, the functions $f_{v,w}$ are continuous on $B(\mathcal{H})$ with respect to the strong operator topology. Therefore the weak operator topology is weaker (=coarser) than the strong one.

(b) In the weak operator topology all left and right multiplications

$$\lambda_A\colon B(\mathcal{H}) \to B(\mathcal{H}), \quad X \mapsto AX \quad \text{and} \quad \rho_A\colon B(\mathcal{H}) \to B(\mathcal{H}), \quad X \mapsto XA$$

are continuous. Indeed, for $v, w \in \mathcal{H}$, we have

$$f_{v,w}(\lambda_A(X)) = \langle AXv, w\rangle = f_{v,A^*w}(X),$$

so that $f_{v,w} \circ \lambda_A$ is continuous, and this implies that λ_A is continuous. Similarly, we obtain $f_{v,w} \circ \rho_A = f_{Av,w}$, and hence the continuity of ρ_A.

Proposition 1.2.3. *On the unitary group* $\mathrm{U}(\mathcal{H})$*, the weak and the strong operator topology coincide and turn it into a topological group.*

We write $\mathrm{U}(\mathcal{H})_s$ for the topological group $(\mathrm{U}(\mathcal{H}), \tau_s)$.

Proof. For $v \in \mathcal{H}$ and $g, h \in \mathrm{U}(\mathcal{H})$, we have

$$\begin{aligned}\|hv - gv\|^2 &= \|hv\|^2 + \|gv\|^2 - 2\operatorname{Re}\langle hv, gv\rangle = 2\|v\|^2 - 2\operatorname{Re}\langle gv, hv\rangle \\ &= 2\|v\|^2 - 2\operatorname{Re} f_{v,hv}(g).\end{aligned}$$

Therefore the continuity of the function $f_{v,hv}$ implies that the orbit map $\mathrm{U}(\mathcal{H}) \to \mathcal{H}, g \mapsto gv$ is continuous in h with respect to the weak operator topology. We conclude that the weak operator topology on $\mathrm{U}(\mathcal{H})$ is finer than the strong one. Since it is also coarser by Remark 1.2.2, both topologies coincide on $\mathrm{U}(\mathcal{H})$.

The continuity of the multiplication in $\mathrm{U}(\mathcal{H})$ is most easily verified in the strong operator topology, where it follows from the estimate

$$\begin{aligned}\|g'h'v - ghv\| &= \|g'(h'-h)v + (g'-g)hv\| \leq \|g'(h'-h)v\| + \|(g'-g)hv\| \\ &= \|(h'-h)v\| + \|(g'-g)hv\|.\end{aligned}$$

This expression tends to zero for $g' \to g$ and $h' \to h$ in the strong operator topology.

The continuity of the inversion follows in the weak topology from the continuity of the functions

$$f_{v,w}(g^{-1}) = \langle g^{-1}v, w\rangle = \langle v, gw\rangle = \overline{\langle gw, v\rangle} = \overline{f_{w,v}(g)}$$

for $v, w \in \mathcal{H}$ and $g \in \mathrm{U}(\mathcal{H})$. ☐

Remark 1.2.4. (a) If $\dim \mathcal{H} < \infty$, then the norm topology and the strong operator topology coincide on $B(\mathcal{H})$, hence in particular on $\mathrm{U}(\mathcal{H})$. In fact, choosing an orthonormal basis $(e_1, \ldots, e_n)$ in $\mathcal{H}$, we represent $A \in B(\mathcal{H})$ by the matrix $A = (a_{ij}) \in M_n(\mathbb{C})$, where $a_{ij} = \langle Ae_j, e_i\rangle = f_{e_j,e_i}(A)$. If $E_{ij} \in M_n(\mathbb{C})$ denote the matrix units, we have $A = \sum_{i,j=1}^n a_{ij}E_{ij}$, so that

$$\|A\| \leq \sum_{i,j=1}^n |a_{ij}|\|E_{ij}\| = \sum_{i,j=1}^n |f_{e_j,e_i}(A)|\|E_{ij}\|,$$

which shows that convergence in the weak topology implies convergence in the norm topology.

(b) If $\dim \mathcal{H} = \infty$, then the strong operator topology on $\mathrm{U}(\mathcal{H})$ is strictly weaker than the norm topology. In fact, let $(e_i)_{i\in I}$ be an orthonormal basis of $\mathcal{H}$. Then I is infinite, so that we may w.l.o.g. assume that $\mathbb{N} \subseteq I$. For each n, we then define the unitary operator $g_n \in \mathrm{U}(\mathcal{H})$ by $g_ne_i := (-1)^{\delta_{in}}e_i$. For $n \neq m$, we then have

$$\|g_n - g_m\| \geq \|(g_n - g_m)e_n\| = \|-2e_n\| = 2,$$

and

$$\langle g_nv, w\rangle - \langle v, w\rangle = \langle g_nv - v, w\rangle = \langle -2\langle v, e_n\rangle e_n, w\rangle = -2\langle v, e_n\rangle\langle e_n, w\rangle \to 0$$

implies that $\lim_{n\to\infty} g_n = \mathbf{1}$ in the weak operator topology.

Definition 1.2.5. Let $\mathcal{H}$ be a complex Hilbert space and G a topological group. A continuous homomorphism

$$\pi \colon G \to \mathrm{U}(\mathcal{H})_s$$

is called a *(continuous) unitary representation* of G. We often denote unitary representations as pairs $(\pi, \mathcal{H})$. In view of Proposition 1.2.3, the continuity of π is equivalent to the continuity of all the *representative functions* (matrix coefficients)

$$\pi_{v,w} \colon G \to \mathbb{C}, \quad \pi_{v,w}(g) := \langle \pi(g)v, w\rangle.$$

A representation $(\pi, \mathcal{H})$ is called *norm continuous*, if it is continuous with respect to the operator norm on $\mathrm{U}(\mathcal{H})$. Clearly, this condition is stronger (Exercise 1.2.4).

Here is a convenient criterion for the continuity of a unitary representation:

Lemma 1.2.6. *A unitary representation $(\pi, \mathcal{H})$ of the topological group G is continuous if and only if there exists a total subset $E \subseteq \mathcal{H}$ (i.e.,* $\operatorname{span} E$ *is dense) such that the functions $\pi_{v,w}$ are continuous for $v, w \in E$.*

Proof. The condition is clearly necessary because we may take $E = \mathcal{H}$.

To see that it is also sufficient, we show that all functions $\pi_{v,w}$, $v, w \in \mathcal{H}$, are continuous (Proposition 1.2.3). If $F := \operatorname{span} E$, then all functions $\pi_{v,w}$, $v, w \in F$, are continuous because the space $C(G, \mathbb{C})$ of continuous functions on G is a vector space.

Let $v, w \in \mathcal{H}$ and $v_n \to v$, $w_n \to w$ with $v_n, w_n \in F$. We claim that the sequence π_{v_n,w_n} converges uniformly to $\pi_{v,w}$, which then implies its continuity. In fact, for each $g \in G$, we have

$$\begin{aligned}
|\pi_{v_n,w_n}(g) - \pi_{v,w}(g)| &= |\langle \pi(g)v_n, w_n\rangle - \langle \pi(g)v, w\rangle| \\
&= |\langle \pi(g)(v_n - v), w_n\rangle - \langle \pi(g)v, w - w_n\rangle| \\
&\leq \|\pi(g)(v_n - v)\|\|w_n\| + \|\pi(g)v\|\|w - w_n\| \\
&= \|v_n - v\|\|w_n\| + \|v\|\|w - w_n\| \to 0. \qquad \square
\end{aligned}$$

Example 1.2.7. If $(e_j)_{j\in J}$ is an orthonormal basis of $\mathcal{H}$, then $E := \{e_j : j \in J\}$ is a total subset. We associate to $A \in B(\mathcal{H})$ the matrix $(a_{jk})_{j,k\in J}$, defined by

$$a_{jk} := \langle Ae_k, e_j\rangle,$$

so that

$$A \sum_{k\in J} x_k e_k = \sum_{j\in J} \Big(\sum_{k\in J} a_{jk}x_k\Big) e_j.$$

Now Lemma 1.2.6 asserts that a unitary representation $(\pi, \mathcal{H})$ of G is continuous if and only if all matrix coefficients

$$\pi_{jk}(g) := \langle \pi(g)e_k, e_j\rangle = \pi_{e_k,e_j}(g)$$

are continuous. These functions are the entries of $\pi(g)$, considered as a $(J \times J)$-matrix.

For $\mathcal{H} = \mathbb{C}^n$, we have $\mathrm{U}(\mathcal{H}) \cong \mathrm{U}_n(\mathbb{C})$, and the preceding argument show that a unitary representation $\pi\colon G \to \mathrm{U}_n(\mathbb{C})$ is continuous if and only if all the functions $\pi_{ij}\colon G \to \mathbb{C}$ defined by the matrix entries $\pi(g) = (\pi_{ij}(g))_{1\leq i,j\leq n}$ are continuous.

To deal with unitary group representations, we shall frequently have to deal with representations of more general structures, called involutive semigroups.

Definition 1.2.8. (a) A pair $(S, *)$ of a semigroup S and an involutive antiautomorphism $s \mapsto s^*$ is called an *involutive semigroup*. Then we have $(st)^* = t^*s^*$ for $s, t \in S$ and $(s^*)^* = s$.

(b) If, in addition, S carries a topology τ for which involution and multiplication are continuous, then $(S, *, \tau)$ is called a *topological involutive semigroup*.

Example 1.2.9. (a) Any abelian (topological) semigroup S becomes an involutive (topological) semigroup with respect to $s^* := s$.

(b) If G is a (topological) group and $g^* := g^{-1}$, then $(G, *)$ is an involutive (topological) semigroup.

(c) An example of particular interest is the multiplicative semigroup $S = (B(\mathcal{H}), \cdot)$ of bounded operators on a complex Hilbert space $\mathcal{H}$ (Example 1.1.12(a)).

Definition 1.2.10. (a) A *representation* $(\pi, \mathcal{H})$ *of the involutive semigroup* $(S, *)$ is a homomorphism $\pi\colon S \to B(\mathcal{H})$ of semigroups satisfying $\pi(s^*) = \pi(s)^*$ for each $s \in S$.

(b) A representation $(\pi, \mathcal{H})$ of $(S, *)$ is called *non-degenerate*, if $\pi(S)\mathcal{H}$ spans a dense subspace of $\mathcal{H}$. This is in particular the case if $\mathbf{1} \in \pi(S)$.

(c) A representation $(\pi, \mathcal{H})$ is called *cyclic* if there exists a $v \in \mathcal{H}$ for which $\pi(S)v$ spans a dense subspace of $\mathcal{H}$.

(d) A representation $(\pi, \mathcal{H})$ is called *irreducible* if $\mathcal{H} \neq \{0\}$ and $\{0\}$ and $\mathcal{H}$ are the only closed $\pi(S)$-invariant subspaces of $\mathcal{H}$.

Example 1.2.11. If G is a group with $g^* = g^{-1}$, then the representations of the involutive semigroup $(G, *)$ mapping $\mathbf{1} \in G$ to $\mathbf{1} \in B(\mathcal{H})$, are precisely the unitary representations of G. All unitary representations of groups are non-degenerate since $\pi(\mathbf{1}) = \mathbf{1}$.

Exercises for Section 1.2

Exercise 1.2.1. Let $\mathcal{H}$ be a Hilbert space. Show that:

(a) The involution $*$ $B(\mathcal{H})$ is continuous with respect to the weak operator topology.

(b) On every bounded subset $K \subseteq B(\mathcal{H})$ the multiplication $(A, B) \mapsto AB$ is continuous with respect to the strong operator topology.

(c) On the unit sphere $\mathbb{S} := \{x \in H\colon \|x\| = 1\}$ the norm topology coincides with the weak topology.

Exercise 1.2.2. Suppose that $\dim \mathcal{H} = \infty$. Show that the unit sphere $\mathbb{S}(\mathcal{H})$ is dense in the closed unit ball $\mathcal{B} \subseteq \mathcal{H}$ with respect to the *weak topology*, which is the coarsest topology for which all functions $f_v = \langle \cdot, v\rangle \colon \mathcal{B} \to \mathbb{C}, v \in \mathcal{H}$ are continuous.

Exercise 1.2.3. Let $\mathcal{H}$ be a Hilbert space and $\mathrm{U}(\mathcal{H})_s$ its unitary group, endowed with the strong (=weak) operator topology. Show that the action map

$$\sigma \colon \mathrm{U}(\mathcal{H})_s \times \mathcal{H} \to \mathcal{H}, \quad (g, v) \mapsto gv$$

is continuous. Conclude that every continuous unitary representation $(\pi, \mathcal{H})$ of a topological group G defines a continuous action of G on $\mathcal{H}$ by $g.v := \pi(g)v$.

Exercise 1.2.4. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Show that we obtain a continuous unitary representation of $G = (\mathbb{R}, +)$ on $\mathcal{H} = \ell^2(\mathbb{N}, \mathbb{C})$ by

$$\pi(t)x = (e^{ita_1}x_1, e^{ita_2}x_2, \ldots).$$

Show further that, if the sequence (a_n) is unbounded, then π is not norm continuous. Is it norm continuous if the sequence (a_n) is bounded?

Exercise 1.2.5. Let $(\pi, \mathcal{H})$ be a representation of an involutive semigroup $(S, *)$. Show that:

(a) $(\pi, \mathcal{H})$ is non-degenerate if and only if $\pi(S)v \subseteq \{0\}$ implies $v = 0$.

(b) Show that $(\pi, \mathcal{H})$ is an orthogonal direct sum of a non-degenerate representation and a zero representation $(\zeta, \mathcal{K})$, i.e., $\zeta(S) = \{0\}$.

Exercise 1.2.6. Let $(\pi, \mathcal{H})$ be a representation of the involutive semigroup (G, η_G), where G is a group. Show that:

(a) $(\pi, \mathcal{H})$ is non-degenerate if and only if $\pi(\mathbf{1}) = \mathbf{1}$.

(b) $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, where $\mathcal{H}_j = \ker(\eta(\mathbf{1}) - j\mathbf{1})$, is an orthogonal direct sum.

Exercise 1.2.7. (A topology on the isometry group of a metric space) Let (X, d) be a metric space and $\mathrm{Aut}(X, d)$ be the group of automorphisms of (X, d), i.e., the group of bijective isometries. Show that the coarsest topology on $\mathrm{Aut}(X, d)$ for which all functions

$$f_x \colon \mathrm{Aut}(X, d) \to \mathbb{R}, \quad f_x(g) := d(gx, x)$$

are continuous turns $\mathrm{Aut}(X, d)$ into a topological group and that the action $\sigma \colon \mathrm{Aut}(X, d) \times X \to X, (g, x) \mapsto gx$ is continuous (cf. Exercise 1.2.3 and Proposition 1.2.3).

Exercise 1.2.8. Let $E \subseteq \mathcal{H}$ be a dense subspace. Show that on every bounded subset $\mathcal{B} \subseteq B(\mathcal{H})$ the weak operator topology is the coarsest topology for which all functions

$$f_{v,w} \colon \mathcal{B} \to \mathbb{C}, \quad v, w \in E,$$

are continuous.

1.3 Discrete Decomposition and Direct Sums

One major goal of the theory of unitary representations is to decompose a unitary representation into simpler pieces. The first basic observation is that, for any closed invariant subspace $\mathcal{K} \subseteq \mathcal{H}$, its orthogonal complement is also invariant, so that we obtain a decomposition into the two subrepresentations on $\mathcal{K}$ and $\mathcal{K}^\perp$. The next step is to iterate this process whenever either $\mathcal{K}$ of $\mathcal{K}^\perp$ is not irreducible. This method works well if $\mathcal{H}$ is finite dimensional, but in general it may not lead to a decomposition into irreducible pieces. However, we shall apply this strategy to show at least that every unitary representation is a direct sum of cyclic ones.

We start with the discussion of invariant subspaces.

Lemma 1.3.1. *Let $\mathcal{K} \subseteq \mathcal{H}$ be a closed subspace, $P \in B(\mathcal{H})$ be the orthogonal projection on $\mathcal{K}$ and $S \subseteq B(\mathcal{H})$ be a $*$-invariant subset. Then the following are equivalent*

(i) *$\mathcal{K}$ is S-invariant.*

(ii) *$\mathcal{K}^\perp$ is S-invariant.*

(iii) *P commutes with S.*

Proof. (i) $\Rightarrow$ (ii): If $w \in \mathcal{K}^\perp$ and $v \in \mathcal{K}$, we have for any $s \in S$ the relation $\langle sw, v\rangle = \langle w, s^*v\rangle = 0$ because $s^*v \in S\mathcal{K} \subseteq \mathcal{K}$.

(ii) $\Rightarrow$ (iii): First we observe that the same argument as above implies that the invariance of $\mathcal{K}^\perp$ entails the invariance of $\mathcal{K} = (\mathcal{K}^\perp)^\perp$.

We write $v = v_0 + v_1$, according to the decomposition $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}^\perp$. Then we have for any $s \in S$:

$$sPv = sv_0 = Psv_0 = P(sv_0 + sv_1) = Psv,$$

so that P commutes with S.

(iii) $\Rightarrow$ (i) follows from the fact that $\mathcal{K} = \ker(P - \mathbf{1})$ is an eigenspace of P, hence invariant under every operator commuting with P. □

We record an important consequence for $*$-representations:

Proposition 1.3.2. *If $(\pi, \mathcal{H})$ is a continuous representation of the topological involutive semigroup $(S, *)$ and $\mathcal{H}_1 \subseteq \mathcal{H}$ a closed invariant subspace, then $\mathcal{H}_2 := \mathcal{H}_1^\perp$ is also invariant.*

Writing elements of $B(\mathcal{H})$ according to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ as matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a \in B(\mathcal{H}_1)$, $b \in B(\mathcal{H}_2, \mathcal{H}_1)$, $c \in B(\mathcal{H}_1, \mathcal{H}_2)$ and $d \in B(\mathcal{H}_2)$ (Exercise 1.3.8)*, we then have*

$$\pi(s) = \begin{pmatrix} \pi_1(s) & \mathbf{0} \\ \mathbf{0} & \pi_2(s) \end{pmatrix},$$

where $(\pi_i, \mathcal{H}_i)$, $i = 1, 2$, are continuous representations.

Proof. The invariance of $\mathcal{H}_2$ follows from Lemma 1.3.1 because $\pi(S) \subseteq B(\mathcal{H})$ is $*$-invariant. Therefore $\pi_j(s) := \pi(s)|_{\mathcal{H}_j}^{\mathcal{H}_j}$ is a bounded operator for each $s \in S$ and we obtain two representations $(\pi_j, \mathcal{H}_j)$ of $(S, *)$. Their continuity follows immediately from the continuity of $(\pi, \mathcal{H})$. $\square$

Definition 1.3.3. (a) If $(\pi, \mathcal{H})$ is a representation of $(S, *)$ and $\mathcal{K} \subseteq \mathcal{H}$ a closed S-invariant subspace, then $\rho(s) := \pi(s)|_{\mathcal{K}}^{\mathcal{K}}$ defines a representation $(\rho, \mathcal{K})$ which is called a *subrepresentation of* $(\pi, \mathcal{H})$.

(b) If $(\pi, \mathcal{H})$ and $(\rho, \mathcal{K})$ are representations of $(S, *)$, then a bounded operator $A \colon \mathcal{K} \to \mathcal{H}$ satisfying

$$A \circ \rho(s) = \pi(s) \circ A \quad \text{for all} \quad s \in S$$

is called an *intertwining operator*. We write $B_S(\mathcal{K}, \mathcal{H})$ for the set of all intertwining operators. It is a closed subspace of the Banach space $B(\mathcal{K}, \mathcal{H})$ (Exercise 1.3.9).

(c) Two representations $(\pi, \mathcal{H})$ and $(\rho, \mathcal{K})$ of $(S, *)$ are said to be *equivalent* if there exists a *unitary* intertwining operator $A \colon \mathcal{K} \to \mathcal{H}$. It is easy to see that this defines indeed an equivalence relation on the class of all unitary representations. We write $[\pi]$ for the equivalence class of the representation $(\pi, \mathcal{H})$.

To understand the decomposition of representations into smaller pieces, we also need infinite "direct sums" of representations, hence the concept of a direct sum of Hilbert spaces which in turn requires the somewhat subtle concept of summability in Banach spaces.

Definition 1.3.4. Let I be a set and X a Banach space. Then a family $(x_i)_{i \in I}$ is called *summable to* $x \in X$ if, for every $\varepsilon > 0$, there exists a finite subset $I_\varepsilon \subseteq I$ with the property that, for every finite subset $F \supseteq I_\varepsilon$, we have

$$\Big\| \sum_{i \in F} x_i - x \Big\| < \varepsilon.$$

If $(x_i)_{i \in I}$ is summable to x, we write $x = \sum_{i \in I} x_i$.[1]

Remark 1.3.5. (a) Note that, for $I = \mathbb{N}$, the summability of a family $(x_n)_{n \in \mathbb{N}}$ in a Banach space X is stronger than the convergence of the series $\sum_{n=1}^\infty x_n$. In fact, if $x = \sum_{n \in \mathbb{N}} x_n$ holds in the sense of summability and $\mathbb{N}_\varepsilon \subseteq \mathbb{N}$ is a finite subset with the property that, for every finite subset $F \supseteq \mathbb{N}_\varepsilon$, we have $\| \sum_{n \in F} x_n - x \| < \varepsilon$, then we have for $N > \max \mathbb{N}_\varepsilon$ in particular

$$\Big\| \sum_{n=1}^N x_n - x \Big\| < \varepsilon,$$

[1] This can also be formulated in terms of convergence of nets. First we order the set $\mathcal{I} := \{F \subseteq I \colon |F| < \infty\}$ of finite subsets of I by set inclusion, so that $F \mapsto \sum_{i \in F} x_i$ is a net in X, called the *net of partial sums*. Then the summability of $(x_i)_{i \in I}$ in X is equivalent to the convergence of this net in X.

showing that the series $\sum_{n=1}^\infty x_n$ converges to x.

(b) If, conversely, the series $\sum_{n=1}^\infty x_n$ converges absolutely to some $x \in X$ and $\varepsilon > 0$, then there exists an $N \in \mathbb{N}$ with $\sum_{n=N}^\infty \|x_n\| < \varepsilon$. With $\mathbb{N}_\varepsilon := \{1, \ldots, N\}$ we then find for every finite superset $F \supseteq \mathbb{N}_\varepsilon$ that

$$\|x - \sum_{n\in F} x_n\| \leq \sum_{n\in\mathbb{N}\setminus F} \|x_n\| \leq \sum_{n>N} \|x_n\| < \varepsilon.$$

Therefore we also have $x = \sum_{n\in\mathbb{N}} x_n$ in the sense of summability.

(c) For $X = \mathbb{R}$ and $I = \mathbb{N}$, summability of $(x_n)_{n\in\mathbb{N}}$ implies in particular convergence of all reordered series $\sum_{n=1}^\infty x_{\sigma(n)}$, where $\sigma\colon \mathbb{N} \to \mathbb{N}$ is a bijection. Therefore Riemann's Reordering Theorem shows that summability implies absolute convergence.

(d) If $(x_i)_{i\in I}$ is a family in $\mathbb{R}_+ = [0, \infty[$, then the situation is much simpler. Here summability is easily seen to be equivalent to the existence of the supremum of the set $\mathcal{F} := \{\sum_{i\in F} x_i \colon F \subseteq I, |F| < \infty\}$ of all finite partial sums, and in this case $\sum_{i\in I} x_i = \sup \mathcal{F}$.

Lemma 1.3.6. *Let $(\mathcal{H}_j)_{j\in J}$ be a family of Hilbert spaces and*

$$\mathcal{H} := \Big\{(x_j)_{j\in J} \in \prod_{j\in J} \mathcal{H}_j \colon \sum_{j\in J} \|x_j\|^2 < \infty\Big\}.$$

Then $\mathcal{H}$ is a Hilbert space with respect to the scalar product

$$\langle (x_j)_{j\in J}, (y_j)_{j\in J}\rangle = \sum_{j\in J} \langle x_j, y_j\rangle.$$

Proof. First we show that $\mathcal{H}$ is a linear subspace of the complex vector space $\prod_{j\in J} \mathcal{H}_j$, in which we define addition and scalar multiplication componentwise. Clearly, $\mathcal{H}$ is invariant under multiplication with complex scalars. For $a, b \in \mathcal{H}_j$, the parallelogram identity

$$\|a + b\|^2 + \|a - b\|^2 = 2\|a\|^2 + 2\|b\|^2$$

(Exercise) implies that

$$\|a + b\|^2 \leq 2(\|a\|^2 + \|b\|^2).$$

For $x = (x_j)_{j\in J}, y = (y_j)_{j\in J} \in \mathcal{H}$, we therefore obtain

$$\sum_{j\in J} \|x_j + y_j\|^2 \leq 2\sum_{j\in J} \|x_j\|^2 + 2\sum_{j\in J} \|y_j\|^2 < \infty.$$

This shows that $x + y \in \mathcal{H}$, so that $\mathcal{H}$ is indeed a linear subspace.

For $x, y \in \mathcal{H}$, the polarization identity

$$\langle x, y\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \|x + i^k y\|^2$$

(Exercise 1.3.2(i)) and $x \pm y, x \pm iy \in \mathcal{H}$ imply that the sum

$$\langle x, y \rangle = \sum_{j \in J} \langle x_j, y_j \rangle$$

exists. For $0 \neq x$, some x_i is non-zero, so that $\langle x, x \rangle \geq \langle x_i, x_i \rangle > 0$. It is a trivial verification that $\langle \cdot, \cdot \rangle$ is a hermitian form. Therefore $\mathcal{H}$, endowed with $\langle \cdot, \cdot \rangle$, is a pre-Hilbert space.

It remains to show that it is complete. This is proved in the same way as the completeness of the space ℓ^2 of square-summable sequences, which is the special case $J = \mathbb{N}$ and $\mathcal{H}_j = \mathbb{C}$ for each $j \in J$. Let $(x^n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $\mathcal{H}$. Then $\|x_j^n - x_j^m\| \leq \|x^n - x^m\|$ holds for each $j \in J$, so that $(x_j^n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}_j$. Now the completeness of the spaces $\mathcal{H}_j$ imply the existence of elements $x_j \in \mathcal{H}_j$ with $x_j^n \to x_j$. For every finite subset $F \subseteq J$, we then have

$$\sum_{j \in F} \|x_j\|^2 = \lim_{n \to \infty} \sum_{j \in F} \|x_j^n\|^2 \leq \lim_{n \to \infty} \sum_{j \in J} \|x_j^n\|^2 = \lim_{n \to \infty} \|x^n\|^2,$$

which exists because $(x^n)_{n \in \mathbb{N}}$ is a Cauchy sequence. This implies that $x := (x_j)_{j \in J} \in \mathcal{H}$ with $\|x\|^2 \leq \lim_{n \to \infty} \|x^n\|^2$.

Finally, we show that $x^n \to x$ holds in $\mathcal{H}$. So let $\varepsilon > 0$ and $N_\varepsilon \in \mathbb{N}$ with $\|x^n - x^m\| \leq \varepsilon$ for $n, m \geq N_\varepsilon$. For a finite subset $F \subseteq J$, we then have

$$\sum_{j \in F} \|x_j - x_j^n\|^2 = \lim_{m \to \infty} \sum_{j \in F} \|x_j^m - x_j^n\|^2 \leq \lim_{m \to \infty} \|x^m - x^n\|^2 \leq \varepsilon^2$$

for $n \geq N_\varepsilon$. We therefore obtain

$$\|x - x^n\|^2 = \sup_{F \subseteq J, |F| < \infty} \sum_{j \in F} \|x_j - x_j^n\|^2 \leq \varepsilon^2.$$

This implies that $x^n \to x$ in $\mathcal{H}$, and thus $\mathcal{H}$ is complete. $\square$

Definition 1.3.7. For a family of $(\mathcal{H}_j)_{j \in J}$ of Hilbert spaces, we define

$$\widehat{\bigoplus_{j \in J}} \mathcal{H}_j := \Big\{ (x_j)_{j \in J} \in \prod_{j \in J} \mathcal{H}_j \colon \sum_{j \in J} \|x_j\|^2 < \infty \Big\}$$

with the scalar product from Lemma 1.3.6. We call this space the *Hilbert space direct sum* of the spaces $(\mathcal{H}_j)_{j \in J}$. This space is larger than the direct vector space sum of the $\mathcal{H}_j$, which is a dense subspace of $\widehat{\bigoplus}_{j \in J} \mathcal{H}_j$ (Exercise). In the following we always identify $\mathcal{H}_i$ with the subspace

$$\mathcal{H}_i \cong \{ (x_j)_{j \in J} \colon (\forall j \neq i)\ x_j = 0 \}.$$

Note that the requirement that $(\|x_j\|^2)_{j \in J}$ is summable implies in particular that, for each $x \in \mathcal{H}$, only countably many x_j are non-zero, even if J is uncountable (Exercise 1.3.4).

Example 1.3.8. (a) If $\mathcal{H}_j = \mathbb{C}$ for each $j \in J$, we also write

$$\ell^2(J,\mathbb{C}) := \widehat{\bigoplus}_{j\in J}\mathbb{C} = \Big\{(x_j)_{j\in J} \in \mathbb{C}^J \colon \sum_{j\in J}|x_j|^2 < \infty\Big\}.$$

On this space we have

$$\langle x, y\rangle = \sum_{j\in J}\langle x_j, y_j\rangle \quad \text{and} \quad \|x\|^2 = \sum_{j\in J}|x_j|^2.$$

For $J = \{1,\ldots,n\}$, we obtain in particular the Hilbert space

$$\mathbb{C}^n \cong \ell^2(\{1,\ldots,n\},\mathbb{C}).$$

(b) If all Hilbert spaces $\mathcal{H}_j = \mathcal{K}$ are equal, we put

$$\ell^2(J,\mathcal{K}) := \widehat{\bigoplus}_{j\in J}\mathcal{K} = \Big\{(x_j)_{j\in J} \in \mathcal{K}^J \colon \sum_{j\in J}\|x_j\|^2 < \infty\Big\}.$$

On this space we also have

$$\langle x, y\rangle = \sum_{j\in J}\langle x_j, y_j\rangle \quad \text{and} \quad \|x\|^2 = \sum_{j\in J}\|x_j\|^2.$$

Proposition 1.3.9. *Let $(\pi_j, \mathcal{H}_j)_{j\in J}$ be a family of continuous unitary representation of G. Then*

$$\pi(g)(v_j)_{j\in J} := \big(\pi_j(g)v_j\big)_{j\in J}$$

defines on $\mathcal{H} := \widehat{\bigoplus}_{j\in J}\mathcal{H}_j$ a continuous unitary representation.

The representation $(\pi, \mathcal{H})$ is called the *direct sum of the representations* π_j, $j \in J$. It is also denoted $\pi = \sum_{j\in J}\pi_j$.

Proof. Since all operators $\pi_j(g)$ are unitary, we have

$$\sum_{j\in J}\|\pi_j(g)v_j\|^2 = \sum_{j\in J}\|v_j\|^2 < \infty \quad \text{for} \quad v = (v_j)_{j\in J} \in \mathcal{H}.$$

Therefore each $\pi(g)$ defines a unitary operator on $\mathcal{H}$ (cf. Exercise 1.3.1) and we thus obtain a unitary representation $(\pi, \mathcal{H})$ of G because each π_j is a unitary representation.

To see that it is continuous, we use Lemma 1.2.6, according to which it suffices to show that, for $v \in \mathcal{H}_i$ and $w \in \mathcal{H}_j$, the function

$$\pi_{v,w}(g) = \langle \pi(g)v, w\rangle = \delta_{ij}\langle \pi_j(g)v, w\rangle$$

is continuous, which immediately follows from the continuity of the representations π_j. $\square$

As we have seen in the introduction for the translation action of $\mathbb{R}$ on $L^2(\mathbb{R})$, we cannot expect in general that a unitary representation decomposes into irreducible ones, but the following proposition is often a useful replacement.

Proposition 1.3.10. *Every non-degenerate representation $(\pi, \mathcal{H})$ of a involutive semigroup $(S, *)$ is a direct sum of cyclic subrepresentations $(\pi_j, \mathcal{H}_j)_{j\in J}$.*

Proof. The proof is a typical application of Zorn's Lemma. We order the set $\mathcal{M}$ of all sets $(\mathcal{H}_j)_{j\in J}$ of mutually orthogonal closed S-invariant subspaces on which the representation is cyclic by set inclusion. Each chain $\mathcal{K}$ in this ordered space has an upper bound given by the union $\bigcup \mathcal{K} \in \mathcal{M}$. Now Zorn's Lemma yields a maximal element $(\mathcal{H}_j)_{j\in J}$ in $\mathcal{M}$.

Let $\mathcal{K} := \overline{\sum_{j\in J} \mathcal{H}_j}$. Since each $\mathcal{H}_j$ is S-invariant and each $\pi(s)$ is continuous, $\mathcal{K}$ is also S-invariant. In view of Proposition 1.3.2, the orthogonal complement $\mathcal{K}^\perp$ is also S-invariant. If $\mathcal{K}^\perp$ is non-zero, we pick $0 \neq v \in \mathcal{K}^\perp$. Then $\mathcal{C} := \overline{\operatorname{span} \pi(S)v}$ is a closed S-invariant subspace. We claim that $v \in \mathcal{C}$, which implies that the representation on $\mathcal{C}$ is cyclic. To this end, we write $v = v_0 + v_1$ with $v_1 \in \mathcal{C}$ and $v_0 \perp \mathcal{C}$. Since $\mathcal{C}^\perp$ is also S-invariant, we have

$$\mathcal{C} \ni \pi(s)v = \underbrace{\pi(s)v_0}_{\in \mathcal{C}^\perp} + \underbrace{\pi(s)v_1}_{\in \mathcal{C}},$$

so that $\pi(s)v_0 = 0$ for every $s \in S$. Since $(\pi, \mathcal{H})$ is non-degenerate, it follows that $v_0 = 0$ (Exercise 1.2.5). This shows that the representation on $\mathcal{C}$ is cyclic. Therefore $\mathcal{C}$, together with $(\mathcal{H}_j)_{j\in J}$ is an orthogonal family of S-cyclic subspaces. This contradicts the maximality of $(\mathcal{H}_j)_{j\in J}$. We thus obtain $\mathcal{K}^\perp = \{0\}$, which proves that $\mathcal{K} = \mathcal{H}$.

Finally, we note that the mutual orthogonality of the spaces $\mathcal{H}_j$ implies the existence of a map

$$\Phi \colon \widehat{\bigoplus}_{j\in J} \mathcal{H}_j \to \mathcal{H}, \quad \Phi(x) := \sum_{j\in J} x_j$$

which is easily seen to be isometric (Exercise 1.3.7). Since its range is dense and complete, it is also surjective. Moreover, $\pi(s)\Phi((x_j)) = \Phi((\pi_j(s)x_j))$ implies that Φ is an equivalence of representations. $\square$

Corollary 1.3.11. *Every non-degenerate continuous unitary representation $(\pi, \mathcal{H})$ of a topological group G is a direct sum of cyclic subrepresentations $(\pi_j, \mathcal{H}_j)_{j\in J}$.*

Proposition 1.3.12. *Every finite dimensional representation $(\pi, \mathcal{H})$ of an involutive semigroup $(S, *)$ is a direct sum of irreducible representations.*

Proof. This is proved easily by induction on $\dim \mathcal{H}$. If $\dim \mathcal{H} \leq 1$, there is nothing to show. Suppose that $\dim \mathcal{H} = d > 0$ and that the assertion is true for representations of dimension $< d$. Let $\mathcal{K} \subseteq \mathcal{H}$ be a minimal S-invariant subspace. Then the representation $\pi_{\mathcal{K}}$ of S on $\mathcal{K}$ is irreducible and $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}^\perp$ is an S-invariant decomposition (Proposition 1.3.2). Using the induction hypothesis on the representation on $\mathcal{K}^\perp$, the assertion follows. $\square$

Corollary 1.3.13. *If G is a finite group, then each unitary representation $(\pi, \mathcal{H})$ of G is a direct sum of irreducible representations $(\pi_j, \mathcal{H}_j)_{j \in J}$.*

Proof. First we use Proposition 1.3.10 to decompose π as a direct sum of cyclic representations π_j. Hence it suffices to show that each cyclic representation is a direct sum of irreducible ones. Since G is finite, each cyclic representation is finite dimensional, so that the assertion follows from Proposition 1.3.12. □

The preceding corollary remains true for representations of compact groups:

Theorem 1.3.14. (Fundamental Theorem on Unitary Representations of Compact Groups—Abstract Peter–Weyl Theorem) *If $(\pi, \mathcal{H})$ is a continuous unitary representation of the compact group G, then $(\pi, \mathcal{H})$ is a direct sum of irreducible representations and all irreducible representations of G are finite dimensional.*

In view of Corollary 1.3.11, the main point of the proof of this theorem is to show that every cyclic representation contains a finite dimensional invariant subspace. This can be derived from the existence of an invariant probability measure on a compact group (Haar measure), the theory of compact operators, and the Spectral Theorem for compact selfadjoint operators. We refer to Appendix A.4 for more details.

Exercises for Section 1.3

Exercise 1.3.1. Let $A\colon \mathcal{H}_1 \to \mathcal{H}_2$ be an isometric linear map between two Hilbert spaces. Show that A is unitary if $A(\mathcal{H}_1)$ is dense in $\mathcal{H}_2$. Hint: Subsets of complete metric spaces are complete if and only if they are closed.

Exercise 1.3.2. Let $b\colon V \times V \to \mathbb{C}$ be a sesquilinear form on the complex vector space V, i.e., b is linear in the first argument and antilinear in the second.

(i) Show that b satisfies the *polarization identity* which permits to recover all values of b from those on the diagonal:

$$b(x, y) = \frac{1}{4}\sum_{k=0}^{3} i^k b(x + i^k y, x + i^k y).$$

(ii) Show also that, if b is positive semidefinite, then it satisfies the *Cauchy–Schwarz inequality*:

$$|b(x, y)|^2 \leq b(x, x)b(y, y) \quad \text{for} \quad v, w \in V.$$

Exercise 1.3.3. Show that a unitary representation $(\pi, \mathcal{H})$ of a topological group G is continuous if and only if there exists a dense subspace $\mathcal{D} \subseteq \mathcal{H}$ such that all the functions

$$\pi_v(g) := \langle \pi(g)v, v\rangle, \quad v \in \mathcal{D}$$

are continuous. Hint: Apply the polarization identity to the map $(v, w) \mapsto \pi_{v,w}$.

Exercise 1.3.4. Show that, for each summable family $(x_j)_{j \in J}$ in the Banach space X, the set

$$J^\times = \{j \in J \colon x_j \neq 0\}$$

is countable, and that, if $J^\times = \{j_n \colon n \in \mathbb{N}\}$ is an enumeration of $J^\times$, then $\sum_{j \in J} x_j = \sum_{n=1}^\infty x_{j_n}$. Hint: Show that each set $J_n := \{j \in J \colon \|x_j\| > \frac{1}{n}\}$ is finite.

Exercise 1.3.5. Show that, for an orthogonal family $(x_j)_{j\in J}$ in the Hilbert space $\mathcal{H}$, the following are equivalent:

(i) $(x_j)_{j\in J}$ is summable.

(ii) $(\|x_j\|^2)_{j\in J}$ is summable in $\mathbb{R}$.

Show further that, if this is the case, then $\big\|\sum_{j\in J} x_j\big\|^2 = \sum_{j\in J}\|x_j\|^2$ and the set $\{j \in J : x_j \neq 0\}$ is countable.

Exercise 1.3.6. Show that for an orthonormal family $(x_j)_{j\in J}$ in the Hilbert space $\mathcal{H}$, the following assertions hold:

(i) $(\forall x \in \mathcal{H}) \sum_{j\in J} |\langle x_j, x\rangle|^2 \leq \|x\|^2$ (Bessel inequality).

(ii) $x = \sum_{j\in J}\langle x, x_j\rangle x_j$ holds if and only if $\sum_{j\in J} |\langle x, x_j\rangle|^2 = \|x\|^2$ (Parseval equality).

Exercise 1.3.7. Let $(\mathcal{H}_j)_{j\in J}$ be an orthogonal family of closed subspaces of the Hilbert space $\mathcal{H}$. Show that, for each $x = (x_j)_{j\in J} \in \widehat{\bigoplus}\mathcal{H}_j$, the sum $\Phi(x) := \sum_{j\in J} x_j$ converges in $\mathcal{H}$ and that $\Phi\colon \widehat{\bigoplus}_{j\in J}\mathcal{H}_j \to \mathcal{H}, (x_j)_{j\in J} \mapsto \sum_{j\in J} x_j$ defines an isometric embedding (cf. Exercise 1.3.5).

Exercise 1.3.8. Let V be a vector space which is the direct sum

$$V = V_1 \oplus \cdots \oplus V_n$$

of the subspaces V_i, $i = 1, \ldots, n$. Accordingly, we write $v \in V$ as a sum $v = v_1 + \cdots + v_n$ with $v_i \in V$. To each $\varphi \in \mathrm{End}(V)$ we associate the map $\varphi_{ij} \in \mathrm{Hom}(V_j, V_i)$, defined by $\varphi_{ij}(v) = \varphi(v)_i$ for $v \in V_j$. Show that

(a) $\varphi(v)_i = \sum_{j=1}^n \varphi_{ij}(v_j)$ for $v = \sum_{j=1}^n v_j \in V$.

(b) The map

$$\Gamma\colon \bigoplus_{i,j=1}^n \mathrm{Hom}(V_j, V_i) \to \mathrm{End}(V), \quad \Gamma((\psi_{ij}))(v) := \sum_{i,j=1}^n \psi_{ij}(v_j)$$

is a linear isomorphism. In this sense we may identify endomorphisms of V with $(n \times n)$-matrices with entries in $\mathrm{Hom}(V_j, V_i)$ in position (i, j).

(c) If V is a Banach space and each V_i is a closed subspace, then the map

$$S\colon V_1 \times \cdots \times V_n \to V, \quad (v_1, \ldots, v_n) \mapsto \sum_{i=1}^n v_i$$

is a homeomorphism. Moreover, a linear endomorphism $\varphi\colon V \to V$ is continuous if and only if each φ_{ij} is continuous. Hint: For the first assertion use the Open Mapping Theorem. Conclude that if $\iota_i\colon V_i \to V$ denotes the inclusion map and $p_j\colon V \to V_j$ the projection map, then both are continuous. Then use that $\varphi_{ij} = p_i \circ \varphi \circ \eta_j$.

Exercise 1.3.9. Let $(\pi, \mathcal{H})$ and $(\rho, \mathcal{K})$ be unitary representations of G. Show that the space $B_G(\mathcal{K}, \mathcal{H})$ of all intertwining operators is a closed subspace of the Banach space $B(\mathcal{K}, \mathcal{H})$

Exercise 1.3.10. Let G be a group. Show that:

(a) Each unitary representation $(\pi, \mathcal{H})$ of G is equivalent to a representation $(\rho, \ell^2(J, \mathbb{C}))$ for some set J. Therefore it makes sense to speak of the **set** of equivalence classes of representations with a fixed Hilbert dimension.

(b) Two unitary representations $\pi_j : G \to \mathrm{U}(\mathcal{H})$, $j = 1, 2$, are equivalent if and only if there exists a unitary operator $U \in \mathrm{U}(\mathcal{H})$ with

$$\pi_2(g) = U\pi_1(g)U^{-1} \quad \text{for each} \quad g \in G.$$

Therefore the set of equivalence classes of unitary representations of G on $\mathcal{H}$ is the set of orbits of the action of $\mathrm{U}(\mathcal{H})$ on the set $\mathrm{Hom}(G, \mathrm{U}(\mathcal{H}))$ for the action $(U * \pi)(g) := U\pi(g)U^{-1}$.

Exercise 1.3.11. Let V be a $\mathbb{K}$-vector space and $A \in \mathrm{End}(V)$. We write $V_\lambda(A) := \ker(A - \lambda\mathbf{1})$ for the *eigenspace of* A corresponding to the eigenvalue λ and $V^\lambda(A) := \bigcup_{n\in\mathbb{N}} \ker(A - \lambda\mathbf{1})^n$ for the *generalized eigenspace of* A corresponding to λ. Show that, if $A, B \in \mathrm{End}(V)$ commute, then

$$BV^\lambda(A) \subseteq V^\lambda(A) \quad \text{and} \quad BV_\lambda(A) \subseteq V_\lambda(A)$$

holds for each $\lambda \in \mathbb{K}$.

Chapter 2

The Commutant of a Representation

In this chapter we turn to finer information on unitary representations, resp., representations of involutive semigroups. We have already seen in Lemma 1.3.1 that for a representation $(\pi, \mathcal{H})$ of an involutive semigroup $(S, *)$, a closed subspace $\mathcal{K}$ of $\mathcal{H}$ is invariant if and only if the corresponding orthogonal projection $P_{\mathcal{K}}$ onto $\mathcal{K}$ belongs to the subalgebra

$$\pi(S)' := B_S(\mathcal{H}) = \{A \in B(\mathcal{H})\colon (\forall s \in S)\, A\pi(s) = \pi(s)A\}.$$

This algebra is called the *commutant of* $\pi(S)$ and since its hermitian projections are in one-to-one correspondence with the closed invariant subspaces of $\mathcal{H}$, it contains all information on how the representation $(\pi, \mathcal{H})$ decomposes.

A key result in this context is Schur's Lemma, asserting that $\pi(S)' = \mathbb{C}\mathbf{1}$ if and only if $(\pi, \mathcal{H})$ is irreducible. Its proof uses a result on C^*-algebras that can be found in Appendix A.3.

2.1 Commutants and von Neumann algebras

Definition 2.1.1. For a subset $S \subseteq B(\mathcal{H})$, we define the *commutant* by

$$S' := \{A \in B(\mathcal{H})\colon (\forall s \in S)\, sA = As\}.$$

If $(\pi, \mathcal{H})$ is a representation of an involutive semigroup S, then $\pi(S)' = B_S(\mathcal{H}, \mathcal{H})$ is called the *commutant of* $(\pi, \mathcal{H})$. It coincides with the space of self-intertwining operators of the representation $(\pi, \mathcal{H})$ with itself (cf. Definition 1.3.3).

Lemma 2.1.2. *For subsets $E, F \subseteq B(\mathcal{H})$, we have:*

(i) $E \subseteq F' \Leftrightarrow F \subseteq E'$.

(ii) *$E \subseteq E''$.*

(iii) *$E \subseteq F \Rightarrow F' \subseteq E'$.*

(iv) *$E' = E'''$.*

(v) *$E = E''$ if and only if $E = F'$ holds for some subset $F \subseteq B(\mathcal{H})$.*

Proof. (i) is trivial.

(ii) In view of (i), this is equivalent to $E' \subseteq E'$, hence trivial.

(iii) is also trivial.

(iv) From (ii) we get $E' \subseteq (E')'' = E'''$. Moreover, (ii) and (iii) imply $E''' \subseteq E'$.

(v) If $E = F'$, then $E'' = F''' = F' = E$ is a consequence of (iv). The converse is trivial, ☐

Lemma 2.1.3. *The commutant E' of a subset $E \subseteq B(\mathcal{H})$ has the following properties:*

(i) *If E is commutative, then so is E''.*

(ii) *E' is a subalgebra of $B(\mathcal{H})$ which is closed in the weak operator topology, hence in particular norm-closed.*

(iii) *If $E^* = E$, then E' is also $*$-invariant, hence in particular a C^*-subalgebra of $B(\mathcal{H})$.*

Proof. (i) That E is commutative is equivalent to $E \subseteq E'$, but this implies $E'' \subseteq E' = E'''$ (Lemma 2.1.2(iv)), which means that E'' is commutative.

(ii) Clearly E' is a linear subspace closed under products, hence a subalgebra of $B(\mathcal{H})$. To see that E' is closed in the weak operator topology, let $v, w \in \mathcal{H}$ and $B \in E$. For $A \in B(\mathcal{H})$ we then have

$$f_{v,w}(AB - BA) = \langle ABv, w\rangle - \langle BAv, w\rangle = (f_{Bv,w} - f_{v,B^*w})(A),$$

which leads to

$$E' = \bigcap_{v,w\in\mathcal{H},B\in E} \ker(f_{Bv,w} - f_{v,B^*w}),$$

which is subspace of $B(\mathcal{H})$ that is closed in the weak operator topology.

(iii) If $A \in E'$ and $B \in E$, then

$$A^*B - BA^* = (B^*A - AB^*)^* = 0$$

follows from $B^* \in E$. Therefore E' is $*$-invariant. Since it is in particular norm closed by (ii), E' is a C^*-subalgebra of $B(\mathcal{H})$. ☐

Definition 2.1.4. A unital $*$-subalgebra $\mathcal{A} \subseteq B(\mathcal{H})$ is called a *von Neumann algebra* if $\mathcal{A} = \mathcal{A}''$.

A von Neumann algebra $\mathcal{A}$ is called a *factor* if its center

$$Z(\mathcal{A}) := \{z \in \mathcal{A} \colon (\forall a \in \mathcal{A})\, az = za\}$$

is trivial, i.e., $Z(\mathcal{A}) = \mathbb{C}\mathbf{1}$. Factors are the building blocks of general von Neumann algebras, and there is a well-developed decomposition theory according to which any von Neumann algebra is a so-called "direct integral" of factors (cf. [Dix69]). The classification theory of factors is an important branch of noncommutative geometry (cf. [Co94]).

Remark 2.1.5. (a) In view of Lemma 2.1.3, any von Neumann algebra $\mathcal{A}$ is closed in the weak operator topology.

(b) For every $*$-invariant subset $E \subseteq B(\mathcal{H})$, the commutant E' is a von Neumann algebra because it is also $*$-invariant and $E''' = E'$ (Lemma 2.1.2). In particular, for any von Neumann algebra $\mathcal{A}$, the commutant $\mathcal{A}'$ is also a von Neumann algebra.

(c) Clearly, the center $Z(\mathcal{A})$ of a von Neumann algebra can also be written as

$$Z(\mathcal{A}) = \mathcal{A} \cap \mathcal{A}' = \mathcal{A}'' \cap \mathcal{A}' = Z(\mathcal{A}').$$

In particular, $\mathcal{A}$ is a factor if and only if its commutant $\mathcal{A}'$ is a factor. Note that $Z(\mathcal{A}) = (\mathcal{A} \cup \mathcal{A}')'$ also is a von Neumann algebra.

Example 2.1.6. The full algebra $\mathcal{A} = B(\mathcal{H})$ is a von Neumann algebra. In this case $\mathcal{A}' = \mathbb{C}\mathbf{1}$ (Exercise 2.2.1), which implies that $B(\mathcal{H})$ is a factor.

2.2 Schur's Lemma and some Consequences

The fact that commutants are always C^*-algebras is extremely useful in representation theory. We now use the results on commutative C^*-algebras explained in Appendix A.3.

Theorem 2.2.1. (Schur's Lemma) *A representation $(\pi, \mathcal{H})$ of an involutive semigroup is irreducible if and only if its commutant is trivial, i.e., $\pi(S)' = \mathbb{C}\mathbf{1}$.*

Proof. If $(\pi, \mathcal{H})$ is not irreducible and $\mathcal{K} \subseteq \mathcal{H}$ is a proper closed invariant subspace, then the orthogonal projection P onto $\mathcal{K}$ commutes with $\pi(S)$ (Lemma 1.3.1) and $P \notin \mathbb{C}\mathbf{1}$. Therefore $\pi(S)' \neq \mathbb{C}\mathbf{1}$ if π is not irreducible.

Suppose, conversely, that $\pi(S)' \neq \mathbb{C}\mathbf{1}$. Then Corollary A.3.3 applies to the C^*-algebra $\pi(S)'$ (Lemma 2.1.3), so that there exist non-zero commuting $A, B \in \pi(S)'$ with $AB = 0$. Then $\mathcal{K} := \overline{A(\mathcal{H})}$ is a non-zero closed subspace invariant under $\pi(S)$ and satisfying $B\mathcal{K} = \{0\}$. Therefore $(\pi, \mathcal{H})$ is not irreducible. □

For a version of Schur's Lemma for real Hilbert spaces, asserting that in this case the commutant is one of the skew fields $\mathbb{R}$, $\mathbb{C}$ of $\mathbb{H}$, we refer to [SV02].

Corollary 2.2.2. *Every irreducible representation $(\pi, \mathcal{H})$ of a commutative involutive semigroup $(S, *)$ is one-dimensional.*

Proof. If S is commutative, then $\pi(S) \subseteq \pi(S)'$. If $(\pi, \mathcal{H})$ is irreducible, then $\pi(S)' = \mathbb{C}\mathbf{1}$ by Schur's Lemma, and therefore $\pi(S) \subseteq \mathbb{C}\mathbf{1}$, so that the irreducibility further implies $\dim \mathcal{H} = 1$. □

Corollary 2.2.3. *Suppose that* $(\pi, \mathcal{H})$ *is an irreducible representation of an involutive semigroup and* $(\rho, \mathcal{K})$ *any representation of* $(S, *)$. *If* $B_S(\mathcal{H}, \mathcal{K}) \neq \{0\}$, *then* $(\pi, \mathcal{H})$ *is equivalent to a subrepresentation of* $(\rho, \mathcal{K})$. *In particular,* $B_S(\mathcal{H}, \mathcal{K}) = 0$ *if both representations are irreducible and non-equivalent.*

Proof. Let $A \in B_S(\mathcal{H}, \mathcal{K})$ be a non-zero intertwining operator. Then $A^*A \in B_S(\mathcal{H}) = \pi(S)' = \mathbb{C}\mathbf{1}$ by Schur's Lemma. If this operator is non-zero, then $\langle A^*Av, v\rangle = \|Av\|^2 \geq 0$ for $v \in \mathcal{H}$ implies that $A^*A = \lambda\mathbf{1}$ for some $\lambda > 0$. Then $B := \lambda^{-1/2}A$ is another intertwining operator with $B^*B = \mathbf{1}$. Hence $B \colon \mathcal{H} \to \mathcal{K}$ is an isometric embedding. In particular, its image $\mathcal{K}_0$ is a closed non-zero invariant subspace on which the representation induced by ρ is equivalent to $(\pi, \mathcal{H})$. □

Corollary 2.2.4. *If* $(\pi, \mathcal{H})$ *is a representation of an involutive semigroup and* $\mathcal{H}_1, \mathcal{H}_2 \subseteq \mathcal{H}$ *are non-equivalent irreducible subrepresentations, then* $\mathcal{H}_1 \perp \mathcal{H}_2$.

Proof. Let $P \colon \mathcal{H} \to \mathcal{H}_1$ denote the orthogonal projection onto $\mathcal{H}_1$. Since $\mathcal{H}_1$ is invariant under $\pi(S)$, Lemma 1.3.1 implies that $P \in B_S(\mathcal{H}, \mathcal{H}_1)$. Hence $P|_{\mathcal{H}_2} \in B_S(\mathcal{H}_2, \mathcal{H}_1) = \{0\}$ by Corollary 2.2.3. This means that $\mathcal{H}_1 \perp \mathcal{H}_2$. □

At this point it is natural to observe that any representation $(\pi, \mathcal{H})$ of an involutive semigroup S decomposes naturally into a *discrete part* $(\pi_d, \mathcal{H}_d)$ which is a direct sum of irreducible ones and a *continuous part* $(\pi_c, \mathcal{H}_c)$ which does not contain any irreducible subrepresentations.

Proposition 2.2.5. (Maximal discrete subrepresentation) *Let* $(\pi, \mathcal{H})$ *be a representation of the involutive semigroup* S *on* $\mathcal{H}$ *and* $\mathcal{H}_d \subseteq \mathcal{H}$ *be the closed subspace generated by all irreducible subrepresentations. Then the following assertions hold:*

(i) $\mathcal{H}_d$ *is* S*-invariant and the representation* $(\pi_d, \mathcal{H}_d)$ *of* S *is a direct sum of irreducible ones.*

(ii) *The orthogonal space* $\mathcal{H}_c := \mathcal{H}_d^\perp$ *carries a representation* $(\pi_c, \mathcal{H}_c)$ *of* S *which does not contain any irreducible subrepresentation.*

Proof. It is clear that the subspace $\mathcal{H}_d$ is S-invariant because it is generated by a family of S-invariant subspaces. To see that it is a direct sum of irreducible representations, we apply Zorn's Lemma. Let $\mathcal{H}_j$, $j \in J$, be a maximal set of S-invariant subspaces of $\mathcal{H}$ such that the corresponding representations $(\pi_j, \mathcal{H}_j)$ are irreducible and $\mathcal{H}_{j_1} \perp \mathcal{H}_{j_2}$ for $j_1 \neq j_2$. Set $\mathcal{H}_0 := \overline{\sum_{j \in J} \mathcal{H}_j} \subseteq \mathcal{H}_d$. Then $\mathcal{H}_1 := \mathcal{H}_0^\perp \cap \mathcal{H}_d$ is S-invariant. We write $p \colon \mathcal{H}_d \to \mathcal{H}_1$ for the orthogonal projection. Then p is surjective and if $\mathcal{H}_1 \neq \{0\}$, there exists an irreducible subspace $\mathcal{K} \subseteq \mathcal{H}_d$ with $p(\mathcal{K}) \neq \{0\}$. This means that $B_S(\mathcal{K}, \mathcal{H}_1) \neq \{0\}$ and hence, by Corollary 2.2.3, the representation on $\mathcal{K}$ is equivalent to an irreducible subrepresentation of $\mathcal{H}_1$. This contradicts the maximality of the family $(\mathcal{H}_j)_{j \in J}$. We conclude that $\mathcal{H}_1 = \{0\}$, and (i) follows from Exercise 1.3.7.

Assertion (ii) follows from the construction of $\mathcal{H}_d$. □

Example 2.2.6. On $\mathbb{R}$ we consider the Borel measure $\mu = \delta_0 + \lambda|_{[1,\infty[}$, where λ is the 1-dimensional Lebesgue measure and δ_0 is the Dirac measure in 0, i.e., $\delta_0(E) = 1$ if $0 \in E$ and $\delta_0(E) = 0$ otherwise. Then the decomposition

$$L^2(\mathbb{R}, \mu) = L^2(\mathbb{R}, \delta_0) \oplus L^2([1, \infty[) \cong \mathbb{C} \oplus L^2([1, \infty[)$$

is invariant under the unitary representation $(\pi(t)f)(x) = e^{itx}f(x)$ of $G = \mathbb{R}$. Clearly the function $\delta_0(x) = \delta_{0,x}$ is an eigenfunction, but this is the only eigenfunction of π because the equation $(e^{itx} - \lambda)f = 0$ implies that f vanishes on the complement of a countable set. We conclude that $\mathcal{H}_d = L^2(\mathbb{R}, \delta_0) = \mathbb{C}\delta_0$ is one-dimensional, and that $\mathcal{H}_c = L^2([1, \infty[)$ is the "continuous part" of the representation π (see also Exercise 2.2.7).

Definition 2.2.7. If $(\pi, \mathcal{H})$ is an irreducible representation of $(S, *)$, then we write $[\pi]$ for its (unitary) equivalence class. For a topological group G, we write $\widehat{G}$ for the set of equivalence classes of irreducible unitary representations (cf. Exercise 1.3.10). It is called the *unitary dual of* G.

Let $(\rho, \mathcal{H})$ be a continuous unitary representations of G. For $[\pi] \in \widehat{G}$, we write $\mathcal{H}_{[\pi]} \subseteq \mathcal{H}$ for the closed subspace generated by all irreducible subrepresentations of type $[\pi]$. From Corollaries 2.2.3 and 2.2.4 it follows that

$$\mathcal{H}_{[\pi]} \perp \mathcal{H}_{[\pi']} \quad \text{for} \quad [\pi] \neq [\pi'],$$

so that the discrete part of $(\rho, \mathcal{H})$ is an orthogonal direct sum

$$\mathcal{H}_d = \widehat{\bigoplus}_{[\pi] \in \widehat{G}} \mathcal{H}_{[\pi]}$$

(Exercise 1.3.7). The subspaces $\mathcal{H}_{[\pi]}$ are called the *isotypic components* of $\mathcal{H}$, resp., $(\pi, \mathcal{H})$.

Remark 2.2.8. (Reduction of commutants) Applying Corollary 2.2.3 to the decomposition

$$\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_d = \mathcal{H}_c \oplus \widehat{\bigoplus}_{[\pi] \in \widehat{G}} \mathcal{H}_{[\pi]},$$

we see that

$$B_G(\mathcal{H}_{[\pi]}, \mathcal{H}_c) = \{0\} \quad \text{and} \quad B_G(\mathcal{H}_{[\pi]}, \mathcal{H}_{[\pi']}) = \{0\} \quad \text{for} \quad [\pi] \neq [\pi'].$$

Therefore $B_G(\mathcal{H})$ preserves each $\mathcal{H}_{[\pi]}$, hence it also preserves $\mathcal{H}_c = \mathcal{H}_d^\perp$ because it is $*$-invariant.

From Exercise 2.2.3 we thus derive that

$$B_G(\mathcal{H}) = \Big\{(A_{[\pi]}) \in \prod_{[\pi] \in \widehat{G}} B_G(\mathcal{H}_{[\pi]}) \colon \sup_{[\pi] \in \widehat{G}} \|A_{[\pi]}\| < \infty\Big\} \oplus B_G(\mathcal{H}_c).$$

Using the concept of an ℓ^∞-direct sum of Banach spaces

$$\bigoplus_{j \in J}^{\infty} X_j := \Big\{(x_j)_{j \in J} \in \prod_{j \in J} X_j \colon \|x\| := \sup_{j \in J} \|x_j\| < \infty\Big\},$$

it follows that
$$B_G(\mathcal{H}) \cong \Big(\oplus_{[\pi]\in\widehat{G}}^{\infty} B_G(\mathcal{H}_{[\pi]})\Big) \oplus B_G(\mathcal{H}_c).$$
This reduces the determination of the commutant to discrete isotypical representations and to continuous ones.

Example 2.2.9. For an abelian topological group A, we have seen in Corollary 2.2.2 that all irreducible unitary representations are one-dimensional, hence given by continuous homomorphisms $\chi\colon A \to \mathbb{T}$. Two such homomorphisms define equivalent unitary representations if and only if they coincide. Therefore the unitary dual of A can be identified with the group

$$\widehat{A} := \mathrm{Hom}(A, \mathbb{T})$$

of continuous characters of A.

For any continuous unitary representation $(\pi, \mathcal{H})$ of A and $\chi \in \widehat{A}$, the isotypic subspace

$$\mathcal{H}_\chi := \mathcal{H}_{[\chi]} = \{v \in \mathcal{H}\colon (\forall a \in A)\, \pi(a)v = \chi(a)v\}$$

is the simultaneous eigenspace of A on $\mathcal{H}$ corresponding to the character χ. Taking all the simultaneous eigenspaces together, we obtain the subspace

$$\mathcal{H}_d = \widehat{\bigoplus}_{\chi\in\widehat{A}} \mathcal{H}_\chi$$

from Proposition 2.2.5.

Example 2.2.10. In general its orthogonal complement $\mathcal{H}_c$ is non-trivial, as the translation representation of $\mathbb{R}$ on $L^2(\mathbb{R})$ shows (cf. Example 2.2.6). To see this, we first observe that, in view of Exercise 2.2.4, every continuous character of $\mathbb{R}$ is of the form $\chi_\lambda(x) = e^{i\lambda x}$. Therefore any eigenfunction $f \in L^2(\mathbb{R})$ satisfies for each $x \in \mathbb{R}$ for almost every $y \in \mathbb{R}$ the relation

$$f(y + x) = e^{i\lambda x} f(y).$$

This implies in particular that the function $|f|$ is, as an element of $L^2(\mathbb{R}, dx)$, translation invariant. We thus obtain

$$\infty > \int_{\mathbb{R}} |f(x)|^2\, dx = \sum_{n\in\mathbb{Z}} \int_0^1 |f(x)|^2\, dx,$$

and therefore f vanishes almost everywhere. This proves that, for the translation action of $\mathbb{R}$ on $L^2(\mathbb{R}, dx)$, the discrete part is trivial, i.e., $L^2(\mathbb{R}, dx)_d = \{0\}$.

Example 2.2.11. We take a closer look at the circle group $A = \mathbb{T}$. To see how a unitary representation $(\pi, \mathcal{H})$ of $\mathbb{T}$ decomposes, we first recall from Exercise 2.2.4 that each character of $\mathbb{R}$ is of the form $\chi_\lambda(x) = e^{i\lambda x}$ for some $\lambda \in \mathbb{R}$. Since $q\colon \mathbb{R} \to \mathbb{T}, t \mapsto e^{ix}$ is continuous, any character $\chi \in \widehat{\mathbb{T}}$ satisfies

$$\chi(e^{ix}) = e^{i\lambda x}$$

for some $\lambda \in \mathbb{R}$ and any $x \in \mathbb{R}$. Then $\chi(1) = 1$ implies $\lambda \in \mathbb{Z}$, so that $\chi_n(z) = z^n$ for some $n \in \mathbb{Z}$ and all $z \in \mathbb{T}$. This proves that

$$\widehat{\mathbb{T}} = \{\chi_n \colon n \in \mathbb{Z}\} \cong \mathbb{Z}.$$

Let $(\pi, \mathcal{H})$ be a continuous unitary representation of $\mathbb{T}$. For $n \in \mathbb{Z}$ we write

$$\mathcal{H}_n := \{v \in \mathcal{H} \colon \pi(z)v = z^n v\}$$

for the corresponding eigenspace in $\mathcal{H}$. For $v \in \mathcal{H}$, we consider the $\mathcal{H}$-valued integral

$$P_n(v) := \frac{1}{2\pi} \int_0^{2\pi} e^{-int} \pi(e^{it}) v \, dt$$

whose existence follows from the continuity of the integrand (as a Riemann integral). It satisfies $P_n(v) = v$ for $v \in \mathcal{H}_n$ and $P_n(v) = 0$ for $v \in \mathcal{H}_m$, $m \neq n$ because $\int_0^{2\pi} e^{-int} e^{imt} \, dt = 0$. Since $\mathcal{H}$ is the orthogonal direct sum of the eigenspaces by compactness of $\mathbb{T}$ (Theorem 1.3.14), $P_n \colon \mathcal{H} \to \mathcal{H}_n$ is the orthogonal projection onto the eigenspace $\mathcal{H}_n$.

For $v \in \mathcal{H}$, we write $v_n := P_n(v) \in \mathcal{H}_n$, so that

$$v = \sum_{n \in \mathbb{Z}} v_n$$

holds in $\mathcal{H} = \widehat{\bigoplus}_{n \in \mathbb{Z}} \mathcal{H}_n$. This is an abstract form of the Fourier expansion of a periodic function, resp., a function on $\mathbb{T}$.

For $\mathcal{H} := L^2(\mathbb{T})$ with $(\pi(z)f)(t) = f(tz)$, we claim that the functions $\chi_n(t) = t^n$, $n \in \mathbb{Z}$, are the only eigenfunctions. In fact, any eigenfunction $f \in \mathcal{H}_n$ satisfies

$$\begin{aligned} f(e^{it}) = P_n(f)(e^{it}) &= \frac{1}{2\pi} \int_0^{2\pi} e^{-ins} f(e^{i(s+t)}) \, ds \\ &= \frac{1}{2\pi} \int_t^{2\pi + t} e^{-in(s-t)} f(e^{is}) \, ds \\ &= e^{int} \cdot \frac{1}{2\pi} \int_0^{2\pi} e^{-ins} f(e^{is}) \, ds. \end{aligned}$$

almost everywhere. Therefore $f \in \mathbb{C}\chi_n$ and therefore $\mathcal{H}_n = \mathbb{C}\chi_n$. Each $f \in L^2(\mathbb{T})$ has in $L^2(\mathbb{T})$ a convergent expansion

$$f = \sum_{n \in \mathbb{Z}} \widehat{f}(n) \chi_n \quad \text{with} \quad \widehat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} e^{-int} f(e^{it}) \, dt.$$

Identifying $\mathbb{T}$ with $\mathbb{R}/2\pi\mathbb{Z}$ and $L^2(\mathbb{T})$ with 2π-periodic functions, this takes the familiar form for Fourier series:

$$f(x) = \sum_{n \in \mathbb{Z}} \widehat{f}(n) e^{inx}, \qquad \widehat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} e^{-inx} f(x) \, dx,$$

where the series on the left convergence in $L^2([0, 2\pi])$.

Remark 2.2.12. (a) There exist abelian topological groups A with faithful continuous unitary representations for which all continuous characters are trivial, i.e., $\widehat{A} = \{\mathbf{1}\}$. For these groups, the two fundamental problems discussed in the introduction make no sense.

In [GN01] it is shown that the group

$$\mathcal{M}([0,1], \mathbb{T}) = \{f \in L^\infty([0,1], \mathbb{C}) \colon |f| = 1\}$$

of all measurable maps $f \colon [0,1] \to \mathbb{T}$, endowed with the weak operator topology obtained from the embedding

$$\lambda \colon L^\infty([0,1], \mathbb{C}) \hookrightarrow B(L^2([0,1], \mathbb{C})), \quad \lambda(f)h = f \cdot h,$$

(cf. Proposition 3.1.8) is an abelian topological group for which all continuous characters are trivial. [1]

(b) Another pathology that can occur for an (abelian) topological group G is that all its continuous unitary representations are trivial. Such topological groups are called *exotic*. In Chapter 2 of [Ba91] one finds various constructions of such group of the type $G = E/\Gamma$, where E is a Banach space and $\Gamma \subseteq E$ is a discrete subgroup. For any such exotic group G, all characters are trivial.

From Exercise 2.2.4, we immediately derive that all characters of $G = E/\Gamma$ are of the form

$$\chi(v + \Gamma) = e^{2\pi i \alpha(v)},$$

where $\alpha \in E'$ is a continuous linear functional satisfying $\alpha(\Gamma) \subseteq \mathbb{Z}$. If G is exotic, resp., $\widehat{G} = \{\mathbf{1}\}$, then the discrete subgroup Γ of E has the strange property that for any continuous linear functional $\alpha \in E'$ with $\alpha(\Gamma) \subseteq \mathbb{Z}$ we have $\alpha = 0$. It is not hard to see that this never happens if $\dim E < \infty$.

Exercises for Section 2

Exercise 2.2.1. Show that, for any Hilbert space $\mathcal{H}$,

$$Z(B(\mathcal{H})) = \{Z \in B(\mathcal{H}) \colon (\forall A \in B(\mathcal{H})\, AZ = ZA\} = \mathbb{C}\mathbf{1}.$$

Hint: Apply Exercise 1.3.11 with $A = \langle \cdot, v \rangle v$ to see that every $v \in \mathcal{H}$ is an eigenvector of Z.

Exercise 2.2.2. Let $(\pi, \mathcal{H})$ be an irreducible representation of the involutive semigroup $(S, *)$ and $\pi_n := \oplus_{j=1}^n \pi$ be the n-fold direct sum of π with itself on $\mathcal{H}^n = \oplus_{j=1}^n \mathcal{H}$. Show that

$$\pi_n(S)' \cong M_n(\mathbb{C}).$$

Hint: Write operators on $\mathcal{H}^n$ as matrices with entries in $B(\mathcal{H})$ (cf. Exercise 1.3.8) and evaluate the commuting condition.

---

[1] Since products of L^2-functions are L^1-functions, it is easy to see that the topology on $\mathcal{M}([0,1], \mathbb{C})$ is the coarsest topology for which all functions $F_h(g) := \int_0^1 h(t)g(t)\, dt$, $h \in L^1([0,1], \mathbb{C})$, are continuous. In functional analytic terms, it coincides with the weak-$*$-topology on the subset $\mathcal{M}([0,1], \mathbb{T})$ of $L^\infty([0,1], \mathbb{C}) \cong L^1([0,1], \mathbb{C})'$.

Exercise 2.2.3. Let $(\mathcal{H}_j)_{j\in J}$ be a family of Hilbert spaces and $A_j \in B(\mathcal{H}_j)$. Suppose that $\sup_{j\in J} \|A_j\| < \infty$. Then $A(x_j) := (A_j x_j)$ defines a bounded linear operator on $\widehat{\oplus}_{j\in J}\mathcal{H}_j$ with

$$\|A\| = \sup_{j\in J} \|A_j\|.$$

If, conversely, $\mathcal{H} = \widehat{\oplus}_{j\in J}\mathcal{H}_j$ is a Hilbert space direct sum and $A \in B(\mathcal{H})$ preserves each subspace $\mathcal{H}_j$, then the restrictions $A_j := A|_{\mathcal{H}_j}$ are bounded operators in $B(\mathcal{H}_j)$ satisfying $\|A\| = \sup_{j\in J} \|A_j\|$.

Exercise 2.2.4. Let V be a real topological vector space. Show that every continuous character $\chi\colon V \to \mathbb{T}$ is of the form $\chi(v) = e^{i\alpha(v)}$ for some continuous linear functional $\alpha \in V'$. Hint: Let $U \subseteq V$ be a circular 0-neighborhood (circular means that $\lambda U \subseteq U$ for $|\lambda| \leq 1$; such neighborhoods form a basis of 0-neighborhoods) with $\operatorname{Re}\chi(v) > 0$ for $v \in U + U$. Define a continuous (!) function

$$L\colon U \to]-\pi, \pi[\subseteq \mathbb{R} \quad \text{by} \quad e^{iL(u)} = \chi(u).$$

Observe that $L(x + y) = L(x) + L(y)$ for $x, y \in U$ and use this to see that

$$\alpha(x) := \lim_{n\to\infty} nL\Big(\frac{x}{n}\Big)$$

is an additive extension of L to V. Now it remains to observe that continuous additive maps $V \to \mathbb{R}$ are linear functionals (prove $\mathbb{Q}$-linearity first).

Exercise 2.2.5. Let A_1 and A_2 be abelian topological groups and $A := A_1 \times A_2$ be their topological direct product. Show that $\widehat{A} \cong \widehat{A_1} \times \widehat{A_2}$.

Exercise 2.2.6. Let $(A_j)_{j\in J}$ be a family of abelian topological groups and $A := \prod_{j\in J} A_j$ be the product group, endowed with the product topology. Show that the map

$$S\colon \bigoplus_{j\in J} \widehat{A_j} \to \widehat{A}, \quad S(\oplus_{j\in J}\chi_j)(a) := \prod_{j\in J} \chi_j(a_j)$$

is a group isomorphism. Hint: One has to reduce the problem to the case where J is finite. To this end, an important point is that the 1-neighborhood $\{z \in \mathbb{T}\colon \operatorname{Re} z > 0\}$ contains no non-trivial subgroup.

Exercise 2.2.7. Let μ be a Borel measure on $\mathbb{R}$ and $\mathcal{H} := L^2(\mathbb{R}, \mu)$. On this space we consider the unitary representation of $G = \mathbb{R}$, given by $(\pi(t)f)(x) = e^{itx}f(x)$. Describe the eigenspaces of $\pi(\mathbb{R})$ in terms of the measure.

Chapter 3

Representations on L^2-spaces

In the first two chapters we have seen how to deal with discrete decompositions of Hilbert spaces and unitary representations. We now turn to the continuous side. Here the simplest situations arise for Hilbert spaces of the type $L^2(X, \mu)$, where $(X, \mathfrak{S}, \mu)$ is a measure space. After discussing group representations on these spaces and twisting by cocycles for abstract measure spaces, we shall see how the context of Radon measures on locally compact spaces provides a natural context where these representations are continuous.

3.1 Representations by multiplication operators

In this section we introduce two types of operators on L^2-spaces: multiplications with functions and compositions with bijections of the underlying space. In general, the unitary representations constructed in this context come from a mixture of both.

Definition 3.1.1. A positive measure μ on $(X, \mathfrak{S})$ is said to be *σ-finite* if $X = \bigcup_{n\in\mathbb{N}} E_n$ with $E_n \in \mathfrak{S}$ and $\mu(E_n) < \infty$. This is an important assumption for many results in measure theory, such as Fubini's Theorem and the Radon–Nikodym Theorem.

A positive measure μ on $(X, \mathfrak{S})$ is called *semifinite* if, for each $E \in \mathfrak{S}$ with $\mu(E) = \infty$, there exists a measurable subset $F \subseteq E$ satisfying $0 < \mu(F) < \infty$.

Remark 3.1.2. (a) Any σ-finite measure is semifinite. If X is an uncountable set, then the counting measure

$$\mu\colon \mathbb{P}(X) \to \mathbb{N}_0 \cup \{\infty\}, \quad \mu(E) := |E|$$

is semifinite, but not σ-finite.

(b) If $(X_j, \mathfrak{S}_j, \mu_j)$, $j \in J$, are (semi-)finite measure spaces, and we put

$$X := \dot{\bigcup}_{j\in J} X_j, \quad \mathfrak{S} := \{E \subseteq X : (\forall j \in J)\, E \cap X_j \in \mathfrak{S}_j\}$$

and

$$\mu(E) := \sum_{j\in J} \mu_j(E \cap X_j),$$

then $\mathfrak{S}$ is a σ-algebra on X, μ is a measure, and $(X, \mathfrak{S}, \mu)$ is a semifinite measure space. In Exercise 3.3.1 we shall see a converse to this observation, namely that, up to sets of measure zero, any semifinite measure space is a direct sum of finite measure spaces.

The following lemma is an extremely important basic tool to deal with Hilbert spaces with continuous decompositions.

Lemma 3.1.3. (Multiplication operators) *Let $(X, \mathfrak{S}, \mu)$ be a measure space and $L^\infty(X, \mu)$ be the corresponding $*$-algebra of essentially bounded measurable functions. Then the following assertions hold:*

(i) *For each $f \in L^\infty(X, \mu)$, we obtain a bounded operator $M_f \in B(L^2(X, \mu))$ by $M_f(g) := fg$ satisfying $\|M_f\| \leq \|f\|_\infty$.*[1]

(ii) *The map $M : L^\infty(X, \mu) \to B(L^2(X, \mu)), f \mapsto M_f$ is a homomorphism of C^*-algebras.*

(iii) *If μ is semifinite , then λ is isometric, i.e., $\|M_f\| = \|f\|_\infty$ for each f.*

(iv) *If $(f_n)_{n\in\mathbb{N}}$ is a bounded sequence in $L^\infty(X, \mu)$ converging pointwise μ-almost everywhere to f, then $M_{f_n} \to M_f$ in the weak operator topology.*

In the following all measure spaces that we consider are semifinite, so that, in view of (iii), we may identify $L^\infty(X, \mu)$ with a subalgebra of $B(L^2(X, \mu))$.

Proof. (i) Since $|f(x)g(x)| \leq \|f\|_\infty |g(x)|$ holds μ-almost everywhere, M_f defines a bounded operator on $L^2(X, \mu)$ with $\|M_f\| \leq \|f\|_\infty$.

(ii) We clearly have $M_{f+g} = M_f + M_g$, $M_{fg} = M_f M_g$ and $M_f^* = M_{f^*}$, so that λ defines a homomorphism of C^*-algebras.

(iii) Now assume that $\|f\|_\infty > c \geq 0$. Then $F := \{|f| \geq c\}$ has positive measure, and since μ is semifinite, it contains a subset E of positive and finite measure. Then $\chi_E \in L^2(X, \mu)$ and

$$c\|\chi_E\|_2 \leq \|f\chi_E\|_2 \leq \|M_f\|\|\chi_E\|_2$$

lead to $\|M_f\| \geq c$. Since c was arbitrary, we obtain $\|f\|_\infty \leq \|M_f\|$.

[1] Although elements of $L^\infty(X, \mu)$ can be represented by bounded functions, they are equivalence classes of functions modulo functions h for which $h^{-1}(\mathbb{C}^\times)$ is a set of measure zero. Accordingly,

$$\|f\|_\infty = \inf\{c \in [0, \infty[: \mu(\{|f| > c\}) = 0\}$$

denotes the essential supremum of the function (cf. [Ru86]).

(iv) For $g, h \in L^2(X, \mu)$, the function $g\overline{h}$ is integrable and $|f_n g\overline{h}| \leq \|f_n\|_\infty |g\overline{h}|$, so that the Dominated Convergence Theorem implies that

$$\langle M_{f_n} g, h\rangle = \int_X f_n g\overline{h}\, d\mu \to \int_X f g\overline{h}\, d\mu = \langle M_f g, h\rangle. \qquad \square$$

Definition 3.1.4. For a measurable space $(X, \mathfrak{S})$ we write $\mathcal{M}(X, \mathbb{T})$ for the group of measurable $\mathbb{T}$-valued functions, where the group structure is defined by the pointwise product.

Proposition 3.1.5. *Let G be a topological group, $(X, \mathfrak{S}, \mu)$ be a measure space and $\gamma\colon G \to \mathcal{M}(X, \mathbb{T})$ a group homomorphism. Then*

$$(\pi(g)f)(x) = \gamma_g(x) f(x)$$

defines a unitary representation of G on $L^2(X, \mu)$ such that if $\lim_{n\to\infty} g_n = g$ in G always implies that $\gamma_{g_n} \to \gamma_g$ holds pointwise, then π is sequentially continuous. If, in addition, G is metrizable, then π is continuous.

Proof. Since $\gamma_{g^{-1}} = \overline{\gamma_g}$, Lemma 3.1.3(ii) implies that π is a unitary representation of G. If, in addition, $g_n \to g$ in G implies the pointwise convergence $\gamma(g_n) \to \gamma(g)$, then Lemma 3.1.3(iv) implies the sequential continuity of π. $\square$

Examples 3.1.6. (a) A typical example is the representation of $G = \mathbb{R}^n$ on $\mathcal{H} = L^2(\mathbb{R}^n, dx)$ obtained by

$$(\pi(g)f)(x) = e^{i\langle g, x\rangle} f(x).$$

The preceding proposition implies that this is a continuous unitary representation.

The same argument actually works for any Borel measure μ on $\mathbb{R}^n$ and the representation on $L^2(\mathbb{R}^n, \mu)$ given by the same formula.

(b) Here is a more abstract variant: If G is an abelian topological group, then we endow the group $\widehat{G} = \mathrm{Hom}(G, \mathbb{T})$ of continuous characters of G with the smallest σ-algebra $\widehat{\mathfrak{S}}$ for which all functions

$$\widehat{g}\colon \widehat{G} \to \mathbb{T}, \quad \widehat{g}(\chi) := \chi(g)$$

are measurable. For every measure μ on $(\widehat{G}, \widehat{\mathfrak{S}})$, we then obtain on $L^2(\widehat{G}, \mathfrak{S}, \mu)$ a sequentually continuous unitary representation of G by

$$(\pi(g)f)(\chi) := \chi(g) f(\chi), \qquad \pi(g)f = \widehat{g} \cdot f.$$

Here we use that $g_n \to g$ implies $\widehat{g_n}(\chi) = \chi(g_n) \to \chi(g) = \widehat{g}(\chi)$ for every $\chi \in \widehat{G}$.

(c) An important special case of (b) arises if $G = (V, +)$ is the additive group of a topological vector space V. If $V' = \mathrm{Hom}(V, \mathbb{R})$ denotes the linear space of continuous linear functionals on V, it follows from Exercise 2.2.4 that the character group

$$\widehat{V} = \{e^{i\alpha}\colon \alpha \in V'\} \cong V'$$

is isomorphic to the additive group of the dual space V'. From the pointwise convergence

$$x^*(\alpha) := \alpha(x) = \lim_{n\to\infty} -in(e^{i\alpha(x)/n} - 1)$$

it follows that the σ-algebra $\widehat{\mathfrak{S}}$ on V' coincides with the smallest σ-algebra for which all evaluation functionals x^*, $x \in V$, are measurable.

The following lemma shows that the two constructions from Examples 3.1.6(a),(b) are consistent:

Lemma 3.1.7. *Under the identification of $\mathbb{R}^n$ with $\widehat{\mathbb{R}^n} = \{e_y : y \in \mathbb{R}^n\}$, $e_y(x) = e^{i\langle x,y\rangle}$, the σ-algebra $\widehat{\mathfrak{S}}$ coincides with the Borel σ-algebra on $\mathbb{R}^n$.*

Proof. Since the functions e_y, $y \in \mathbb{R}^n$, are Borel measurable, we have $\widehat{\mathfrak{S}} \subseteq \mathfrak{B}(\mathbb{R}^n)$. The Borel σ-algebra is generated by the products of intervals, so that equality follows if we show that $\widehat{\mathfrak{S}}$ contains all products of intervals. This follows from Example 3.1.6 which implies that all coordinate functions $\mathbb{R}^n \to \mathbb{R}, x \mapsto x_j$, are $\widehat{\mathfrak{S}}$-measurable. □

To understand how the representations constructed above decompose, we have to compute their commutant. This is relatively easy for the representation of the full algebra $L^\infty(X, \mathfrak{S}, \mu)$, which is done in the following proposition that constitutes a basic tool in many more sophisticated constructions.

Proposition 3.1.8. *Let $(X, \mathfrak{S}, \mu)$ be a σ-finite measure space, $\mathcal{H} := L^2(X, \mu)$ the corresponding Hilbert space and*

$$\pi\colon L^\infty(X, \mu) \to B(L^2(X, \mu)), \quad \pi(f)g := fg.$$

be the homomorphism from Lemma 3.1.3*. Then the following assertions hold:*

(i) *If μ is finite, then $1 \in L^2(X, \mu)$ is a cyclic vector for π, i.e., not contained in a proper closed subspace invariant under $L^\infty(X, \mu)$.*

(ii) *$\pi(L^\infty(X, \mu)) = \pi(L^\infty(X, \mu))'$ is its own commutant, hence in particular a von Neumann algebra.*

Proof. (i) $\pi(L^\infty(X, \mu))1 = L^\infty(X, \mu)$ is dense in $L^2(X, \mu)$, because for each $f \in L^2(X, \mu)$, the sequence f_n, defined by

$$f_n(x) := \begin{cases} f(x) & \text{for } |f(x)| \leq n \\ 0 & \text{for } |f(x)| > n \end{cases}$$

converges to f because

$$\|f - f_n\|_2^2 = \int_{|f|>n} |f(x)|^2 \, d\mu(x) \to 0$$

follows from the Monotone Convergence Theorem. Here we use that

$$\{|f| = \infty\} = \bigcap_{n\in\mathbb{N}} \{x \in X \colon |f(x)| > n\}$$

is a set of measure zero.

(ii) Since $\pi(L^\infty(X,\mu))$ is commutative, it is contained in its own commutant. Suppose, conversely, that $B \in \pi(L^\infty(X,\mu))'$. Writing

$$X = \dot{\bigcup}_n X_n \quad \text{with} \quad \mu(X_n) < \infty,$$

all subspace $L^2(X_n,\mu) = \chi_{X_n} L^2(X,\mu)$ are invariant under B. Therefore it suffices to show that, on every such subspace, B is given by multiplication with a bounded function. We may therefore assume that μ is finite.

Then $h := B(1) \in L^2(X,\mu)$, and, for $f \in L^\infty(X,\mu) \subseteq L^2(X,\mu)$, we have

$$B(f) = B(f \cdot 1) = B(\pi(f)1) = \pi(f)B(1) = fh.$$

If $f = \chi_{E_n}$ is the characteristic function of the set

$$E_n := \{x \in X \colon n \leq |h(x)| \leq n+1\},$$

then $\|B(f)\|_2 = \|hf\|_2 \geq n\|f\|_2$, and since B is bounded, it follows that $\|f\|_2^2 = \mu(E_n) = 0$ if n is sufficiently large. This means that $h \in L^\infty(X,\mu)$. Now $\pi(h)$ and B coincide on the dense subspace $L^\infty(X,\mu)$, hence on all of $L^2(X,\mu)$. This proves that $B = \pi(h) \in \pi(L^\infty(X,\mu))$. □

Remark 3.1.9. (On the assumption of σ-finiteness) If $(X,\mathfrak{S},\mu)$ is not σ-finite (but semifinite), the determination of the commutant of $L^\infty(X,\mathfrak{S},\mu)$, acting by mutliplication operators, is connected to certain measure theoretic subtleties.

It is well known that for a σ-finite measure space $(X,\mathfrak{S},\mu)$, the natural map

$$\Phi \colon L^\infty(X,\mathfrak{S},\mu) \to L^1(X,\mathfrak{S},\mu)', \quad \Phi(f)(h) := \int_X fh\,d\mu$$

is an isometric isomorphism. Here $L^1(X,\mathfrak{S},\mu)'$ denote the dual Banach space with the operator norm and not a commutant. More generally, one can show easily that Φ is isometric if μ is semifinite. In general, Φ is not surjective, and the measures spaces for which Φ is a bijective isometry are called *localizable* (cf. [Fl81, 16.6.4], and [Fl81, 16.7] for an example of a measure space which is not localizable). One can show that localizability is equivalent to $(X,\mathfrak{S},\mu)$ being a direct sum (as in Remark 3.1.2(b)) of finite measure spaces (cf. [Sa71, Prop. 1.18.1]).

These issues are related to the commutant $\mathcal{A}$ of $L^\infty(X,\mathfrak{S},\mu)$ as follows. For $B \in \mathcal{A}$, we obtain with Proposition 3.1.8(ii) on every σ-finite subset $E \in \mathfrak{S}$ a unique element $h_E \in L^\infty(E,\mathfrak{S}|_E,\mu)$ with $Bf = h_E f$ for $f \in L^2(E,\mathfrak{S}|_E,\mu)$.

Since, for every $F \in L^1(X, \mathfrak{S}, \mu)$, the set $\{F \neq 0\}$ is σ-finite (Exercise 3.3.6), we thus obtain a natural map

$$\Psi\colon \mathcal{A} \to L^1(X, \mathfrak{S}, \mu)', \quad \Psi(B)F = \int_X h_E F\, d\mu \quad \text{if} \quad \{F \neq 0\} \subseteq E.$$

This map turns out to be a surjective isometry for every semifinite measure space. For the surjectivity, one only has to observe that, for every σ-finite subset $E \in \mathfrak{S}$, any continuous linear functional α on $L^1(X, \mathfrak{S}, \mu)$ satisfies $\alpha(\chi_E F) = \int_X h_E F\, d\mu$ for a unique $h_E \in L^\infty(E, \mathfrak{S}|_E, \mu)$, and that the corresponding multiplication operators fit together to a bounded operator on $L^2(X, \mathfrak{S}, \mu)$ commuting with $L^\infty(X, \mathfrak{S}, \mu)$.

This shows that we can always identify the commutant of $L^\infty(X, \mathfrak{S}, \mu)$ with the dual space of $L^1(X, \mathfrak{S}, \mu)$, but if the measure space is not localizable, then $L^\infty(X, \mathfrak{S}, \mu)$ is a proper subspace and this implies that it is not a von Neumann algebra. This justifies the assumption of σ-finiteness in Theorem 3.1.12 below.

Remark 3.1.10. (a) In the following we shall see many instances of representations on spaces $L^2(X, \mathfrak{S}, \mu)$, where $\pi(G)' \cong L^\infty(X, \mathfrak{S}, \mu)$. Since the hermitian projections in this algebra are in one-to-one correspondence with the G-invariant closed subspaces by Lemma 1.3.1, it is important to have a clear picture of the set

$$\mathcal{P} := \{p \in L^\infty(X, \mathfrak{S}, \mu)\colon p = p^* = p^2\}.$$

The condition $\overline{p} = p^* = p$ means that p is real-valued, and $p^2 = p$ means that $p(X) \subseteq \{0, 1\}$. Therefore

$$\mathcal{P} = \{\chi_E\colon E \in \mathfrak{S}\}.$$

For $E, F \in \mathfrak{S}$, the characteristic functions χ_E and χ_F coincide as elements of $L^\infty(X, \mathfrak{S}, \mu)$ if and only if $\mu(E \Delta F) = 0$. This defines an equivalence relation $\sim_\mu$ on $\mathfrak{S}$, and we obtain a bijection

$$\mathfrak{S}/\sim_\mu \to \mathcal{P}, \quad E \mapsto \chi_E.$$

(b) Passing from the σ-algebra $\mathfrak{S}$ to its *μ-completion*

$$\mathfrak{S}_\mu := \{E \subseteq X\colon (\exists F \in \mathfrak{S})\ \mu(E \Delta F) = 0\},$$

does neither change the corresponding space L^2 nor L^∞. Therefore we may pass from $\mathfrak{S}$ to $\mathfrak{S}_\mu$ whenever it is convenient. We say that σ is *μ-complete* if $\mathfrak{S}_\mu = \mathfrak{S}$, i.e., of all μ-zero sets are contained in $\mathfrak{S}$.

Example 3.1.11. We claim that, for the representation of $G := \mathcal{M}(X, \mathbb{T})$ on $L^2(X, \mu)$ (μ a finite measure) by $\pi(g)f = gf$, the commutant is $\pi(G)' = \pi(L^\infty(X, \mu))$.

In view of Proposition 3.1.8, it suffices to show that G spans $L^\infty(X, \mu)$, so that both have the same commutant. Since $L^\infty(X, \mu)$ is spanned by the real-valued functions f with $\|f\|_\infty \leq 1$, this follows from the fact that

$$f \pm i\sqrt{1 - f^2} \in \mathcal{M}(X, \mathbb{T}) \quad \text{for} \quad f = \overline{f}, \|f\|_\infty \leq 1.$$

In most concrete situations, one is not dealing with the full group $\mathcal{M}(X, \mathbb{T})$, but with much smaller groups or other subsemigroups of $\mathcal{M}(X, \mathbb{C})$. Therefore it is crucial to have good tools to calculate the commutants, resp., bicommutants in this case. The following theorem provides an effective tool to do so by giving a description of all von Neumann subalgebras of the von Neumann subalgebra $L^\infty(X, \mathfrak{S}, \mu) \subseteq B(L^2(X, \mathfrak{S}, \mu))$ if μ is σ-finite (cf. Proposition 3.1.8(ii)). Here we assume the σ-finiteness of the measure to ensure that the subspace $L^\infty(X, \mathfrak{S}, \mu) \subseteq B(L^2(X, \mathfrak{S}, \mu))$ is actually a von Neumann algebra (Remark 3.1.10).

Theorem 3.1.12. (The L^∞-Subalgebra Theorem) *Let $(X, \mathfrak{S}, \mu)$ be a σ-finite measure space and $\mathcal{A} \subseteq L^\infty(X, \mathfrak{S}, \mu) \subseteq B(L^2(X, \mathfrak{S}, \mu))$ be a von Neumann algebra. Then*

$$\mathfrak{A} := \{E \in \mathfrak{S} \colon \chi_E \in \mathcal{A}\}$$

is a σ-subalgebra of $\mathfrak{S}$ and

$$\mathcal{A} \cong L^\infty(X, \mathfrak{A}, \mu|_{\mathfrak{A}}).$$

Conversely, for every σ-subalgebra $\mathfrak{A} \subseteq \mathfrak{S}$, $L^\infty(X, \mathfrak{A}, \mu|_{\mathfrak{A}})$ is a von Neumann subalgebra of $L^\infty(X, \mathfrak{S}, \mu)$.

Proof. **Step 1:** First we show that $\mathfrak{A}$ is a σ-algebra. Clearly $0 \in \mathcal{A}$ implies $\emptyset \in \mathfrak{A}$, and since $\mathbf{1} \in \mathcal{A}'' = \mathcal{A}$, we also have $\chi_{E^c} = \mathbf{1} - \chi_E \in \mathcal{A}$ for each $E \in \mathfrak{A}$. From $\chi_E \cdot \chi_F = \chi_{E \cap F}$ we derive that $\mathfrak{A}$ is closed under finite intersections. Now let $(E_n)_{n\in\mathbb{N}}$ be a sequence of elements in $\mathfrak{A}$. It remains to show that $E := \bigcap_{n\in\mathbb{N}} E_n \in \mathfrak{A}$. Let $F_n := E_1 \cap \cdots \cap E_n$. Then $F_n \in \mathfrak{A}$ implies $\chi_{F_n} \in \mathcal{A}$. Moreover, $\chi_{F_n} \to \chi_F$ holds points, so that $\chi_{F_n} \to \chi_F$ in the weak operator topology by Lemma 3.1.3(iv). As $\mathcal{A}$ is closed in this topology by Lemma 2.1.3, we obtain $\chi_F \in \mathcal{A}$ and thus $F \in \mathfrak{A}$. This proves that $\mathfrak{A}$ is a σ-algebra.

Step 2: That $\mathcal{A} \supseteq L^\infty(X, \mathfrak{A}, \mu|_{\mathfrak{A}})$ follows directly from the fact that $\mathcal{A}$ contains all finite linear combinations $\sum_j c_j \chi_{E_j}$, $E_j \in \mathfrak{A}$, the norm-closedness of $\mathcal{A}$ and the fact that every element $f \in L^\infty(X, \mathfrak{A}, \mu|_{\mathfrak{A}})$ is a norm-limit of a sequence of step functions f_n. To verify the latter fact, it suffices to verify this claim for bounded real-valued function f, and this can be done by defining

$$f_n(x) := \frac{k}{n} \quad \text{for} \quad x \in \Big\{\frac{k}{n} \leq f(x) < \frac{k+1}{n}\Big\}, \quad k \in \mathbb{Z},$$

because this implies that $\|f - f_n\|_\infty \leq \frac{1}{n}$.

Step 3: Finally we show that $\mathcal{A} \subseteq L^\infty(X, \mathfrak{A}, \mu|_{\mathfrak{A}})$, i.e., that all elements of $\mathcal{A}$ are $\mathfrak{A}$-measurable (if possibly modified on sets of measure zero).

We recall that Lemma 3.1.3(iv) implies that $\mathcal{A}$ is closed under bounded pointwise limits. Let $(p_n)_{n\in\mathbb{N}}$ be the sequence of polynomials from Lemma A.2.3 converging on $[0,1]$ uniformly to the square root function. For $0 \neq f \in \mathcal{A}$, we consider the functions $p_n\big(\frac{|f|^2}{\|f\|_\infty^2}\big)$, which also belong to $\mathcal{A}$. Since they converge pointwise to $\frac{|f|}{\|f\|_\infty}$, we see that $|f| \in \mathcal{A}$. For real-valued elements $f, g \in \mathcal{A}$, this further implies that

$$\max(f,g) = \frac{1}{2}(f + g + |f - g|) \in \mathcal{A}.$$

For any $c \in \mathbb{R}$, it now follows that $\max(f,c) \in \mathcal{A}$. The sequence $e^{-n(\max(f,c)-c)} \in \mathcal{A}$ is bounded and converges pointwise to the characteristic function $\chi_{\{f \leq c\}}$ of the set

$$\{f \leq c\} := \{x \in X \colon f(x) \leq c\}.$$

We thus obtain that $\chi_{\{f \leq c\}} \in \mathcal{A}$. We conclude that the set $\{f \leq c\}$ is contained in the μ-completion $\mathfrak{A}_\mu$ of $\mathfrak{A}$, and this finally shows that $f \in L^\infty(X, \mathfrak{A}_\mu, \mu) = L^\infty(X, \mathfrak{A}, \mu)$ (Remark 3.1.10(b)). $\square$

Corollary 3.1.13. *If $(X, \mathfrak{S}, \mu)$ is a σ-finite measure space and*

$$\mathcal{F} \subseteq L^\infty(X, \mathfrak{S}, \mu)$$

is a subset with the property that $\mathfrak{S}$ is the smallest σ-algebra for which all elements of $\mathcal{F}$ are measurable, then $\mathcal{F}'' = L^\infty(X, \mathfrak{S}, \mu)$, i.e., $\mathcal{F}$ generates $L^\infty(X, \mathfrak{S}, \mu)$ as a von Neumann algebra.

Proof. We have seen in Theorem 3.1.12 that $\mathcal{F}'' = L^\infty(X, \mathfrak{A}, \mu|_{\mathfrak{A}})$ holds for a σ-subalgebra $\mathfrak{A} \subseteq \mathfrak{S}$. Then all elements of $\mathcal{F}$ are measurable with respect to the μ-completion $\mathfrak{A}_\mu$ of $\mathfrak{A}$, so that $\mathfrak{S} \subseteq \mathfrak{A}_\mu$. This implies that

$$\mathcal{F}'' = L^\infty(X, \mathfrak{A}, \mu|_{\mathfrak{A}}) = L^\infty(X, \mathfrak{A}_\mu, \mu|_{\mathfrak{A}}) \supseteq L^\infty(X, \mathfrak{S}, \mu). \qquad \square$$

Example 3.1.14. (a) Recall Example 3.1.6(b), where G is an abelian topological group and $\widehat{\mathfrak{S}}$ is the smallest σ-algebra on $\widehat{G}$ for which all evaluation functions $\widehat{g} \colon \widehat{G} \to \mathbb{T}$ are measurable. Then Corollary 3.1.13 implies that $\pi(G)'' = L^\infty(\widehat{G}, \widehat{\mathfrak{S}}, \mu)$.

If, in addition, μ is σ-finite, it further follows from Proposition 3.1.8 that

$$\pi(G)' = \pi(G)''' = L^\infty(\widehat{G}, \widehat{\mathfrak{S}}, \mu)' = L^\infty(\widehat{G}, \widehat{\mathfrak{S}}, \mu).$$

It follows in particular, that, for every closed $\pi(G)$-invariant closed subspace $\mathcal{K} \subseteq \mathcal{H} := L^2(\widehat{G}, \widehat{\mathfrak{S}}, \mu)$, there exists an $E \in \mathfrak{S}$ with $\mathcal{K} = \chi_E \mathcal{H} = \{f \in \mathcal{H} \colon f|_{E^c} = 0\}$.

(b) For the special case where $G = (V, +)$ for a topological vector space V, we obtain with Example 3.1.6(c) that the same assertion holds for the smallest σ-algebra $\widehat{\mathfrak{S}}$ on $\widehat{G} \cong V'$ for which all the linear evaluation functionals $x^*(\alpha) := \alpha(X), x \in V$, are measurable.

Remark 3.1.15. (Refinements for locally compact groups) If G is a locally compact abelian group, then its character group $\widehat{G}$ carries a natural group topology, called the *compact open topology* or the *topology of uniform convergence on compact subsets of* G. This topology turns $\widehat{G}$ into a locally compact abelian group all of whose characters are of the form $\widehat{g}$, $g \in G$, so that $G \cong \widehat{\widehat{G}}$ as topological groups (cf. [HM98]). All these functions are measurable with respect to the σ-algebra of Borel subsets of $\widehat{G}$, so that Example 3.1.6(b) applies to any Borel measure on $\widehat{G}$.

3.2 Group actions preserving measure classes

Definition 3.2.1. Let $(X,\mathfrak{S})$ be a measurable space and λ,μ be positive measures on $(X,\mathfrak{S})$. We call λ and μ *equivalent*, and write $\lambda \sim \mu$, if there exists a measurable function

$$f\colon X \to]0,\infty[\quad \text{with} \quad \lambda(E) = \int_E f(x)\,d\mu(x) \quad \text{for every} \quad E \in \mathfrak{S}. \tag{3.1}$$

It is easy to see that $\sim$ defines an equivalence relation on the set of positive measures on $(X,\mathfrak{S})$. The corresponding equivalence classes $[\lambda]$ are called *measure classes*.

For simplicity we also write (3.1) as

$$\lambda = f\cdot\mu.$$

The function

$$\frac{d\lambda}{d\mu} := f$$

is called the *Radon–Nikodym derivative of* λ *with respect to* μ.

The Theorem of Radon–Nikodym ([Ru86, Thm. 6.10]) is a central result in abstract measure theory. It asserts that, for σ-finite measures, $\lambda\sim\mu$ if and only if μ and λ have the same zero sets. The latter condition is always necessary for $\lambda\sim\mu$.

Remark 3.2.2. (a) It is natural to restrict our considerations to σ-finite measures because we shall mostly deal with finite measures μ and with measures of the form $f\mu$ for an integrable function $f\in L^1(X,\mu)$.

However, it is instructive to observe that, if μ is σ-finite and $\lambda\sim\mu$, then λ is also σ-finite. In fact, we have

$$\lambda(X) = \int_X f(x)\,d\mu(x) = \lim_{n\to\infty}\int_{\{0\le f\le n\}} f(x)\,d\mu(x),$$

and if $X=\bigcup_{n\in\mathbb{N}} X_n$ with $\mu(X_n)<\infty$, then the subsets $X_n\cap\{0\le f\le m\}$ have finite λ-measure.

(b) If λ and μ are σ-finite, then the function f is uniquely determined up to its values on a set of measure zero. If $\lambda = f_1\mu = f_2\mu$, then $h := f_1 - f_2$ satisfies $\int_E h\,d\mu = 0$ for every $E\in\mathfrak{S}$ with $\lambda(E)<\infty$ and hence for every $E\in\mathfrak{S}$ because λ is σ-finite. Applying this to $E=\{h>0\}$ and $E=\{h<0\}$, we see that h vanishes μ almost everywhere.

We are interested in unitary group representations on $L^2(X,\mu)$ for a measure space $(X,\mathfrak{S},\mu)$. A natural group to be considered in this context is

$$\mathrm{Aut}(X,\mathfrak{S}) := \{\varphi\colon X\to X : \varphi \text{ meas.},\ \exists\psi\colon X\to X \text{ meas.}, \psi\circ\varphi = \varphi\circ\psi = \mathrm{id}_X\}$$

of *automorphism of the measurable space* $(X,\mathfrak{S})$.

Recall that, for a measurable map $\varphi\colon (X,\mathfrak{S}) \to (X',\mathfrak{S}')$ and a measure μ on $(X,\mathfrak{S})$, we obtain a measure $\varphi_*\mu$ on $(X',\mathfrak{S}')$ by

$$(\varphi_*\mu)(E) := \mu(\varphi^{-1}(E)) \quad \text{for} \quad E \in \mathfrak{S}'.$$

It is called the *push-forward of μ by φ*. The corresponding transformation formula for integrals reads

$$\int_{X'} f(x)\,d(\varphi_*\mu)(x) = \int_X f(\varphi(x))\,d\mu(x). \tag{3.2}$$

For a given σ-finite positive measure μ on $(X,\mathfrak{S})$, we thus obtain two subgroups of $\mathrm{Aut}(X,\mathfrak{S})$:

$$\mathrm{Aut}(X,\mu) := \{\varphi \in \mathrm{Aut}(X,\mathfrak{S})\colon \varphi_*\mu = \mu\}$$

and the larger group

$$\mathrm{Aut}(X,[\mu]) := \{\varphi \in \mathrm{Aut}(X,\mathfrak{S})\colon \varphi_*\mu \sim \mu\}.$$

If $\varphi_*\mu \sim \mu$, then μ is said to be *quasi-invariant under φ*. Clearly,

$$\mathrm{Aut}(X,\mu) \subseteq \mathrm{Aut}(X,[\mu]).$$

For $\varphi \in \mathrm{Aut}(X,[\mu])$, we define

$$\delta(\varphi) := \delta_\mu(\varphi) := \frac{d(\varphi_*\mu)}{d\mu}$$

and note that, with the notation $\varphi_* f := f \circ \varphi^{-1}$, we have the cocycle property

$$\delta(\varphi\psi) = \delta(\varphi)\cdot\varphi_*(\delta(\psi)) \tag{3.3}$$

μ-almost everywhere because

$$(\varphi\psi)_*\mu = \varphi_*\psi_*\mu = \varphi_*(\delta(\psi)\mu) = \varphi_*(\delta(\psi))\cdot\varphi_*\mu = \varphi_*(\delta(\psi))\delta(\varphi)\mu$$

and the Radon–Nikodym derivatives are unique almost everywhere.

Proposition 3.2.3. *Suppose that μ is σ-finite. For $f \in L^2(X,\mu)$ and $\varphi \in \mathrm{Aut}(X,[\mu])$, we put*

$$(\pi(\varphi)f)(x) := \sqrt{\delta(\varphi)(x)}\,f(\varphi^{-1}(x)).$$

Then $(\pi, L^2(X,\mu))$ is a unitary representation of the group $\mathrm{Aut}(X,[\mu])$.

Proof. Clearly, $\pi(\varphi)f$ is measurable, and we also find

$$\begin{aligned}\|\pi(\varphi)f\|_2^2 &= \int_X \delta(\varphi)(x)|f(\varphi^{-1}(x))|^2\,d\mu(x) = \int_X |f(\varphi^{-1}(x))|^2\,d(\varphi_*\mu)(x)\\ &= \int_X |f(x)|^2\,d\mu(x) = \|f\|^2,\end{aligned}$$

so that $\pi(\varphi)$ defines an isometry of $L^2(X,\mu)$. We also observe that, for $\varphi, \psi \in \mathrm{Aut}(X, [\mu])$, we obtain with (3.3):

$$\begin{aligned}\pi(\varphi\psi)f &= \sqrt{\delta(\varphi\psi)}\varphi_*(\psi_* f) = \sqrt{\delta(\varphi)}\varphi_*\sqrt{\delta(\psi)}\varphi_*(\psi_* f)\\ &= \sqrt{\delta(\varphi)}\varphi_*\big(\sqrt{\delta(\psi)}(\psi_* f)\big) = \pi(\varphi)\pi(\psi)f.\end{aligned}$$

In particular, we see that each isometry $\pi(\varphi)$ is surjective with $\pi(\varphi^{-1}) = \pi(\varphi)^{-1}$. □

3.2.1 Representation defined by cocycles]

The following corollary consitutes the basic tool to construct representations on L^2-spaces.

Corollary 3.2.4. *Let $(X, \mathfrak{S}, \mu)$ be a σ-finite measure space and*

$$\sigma \colon G \times X \to X, \quad (g, x) \mapsto g.x$$

a group action by measurable maps such that each map $\sigma_g(x) := \sigma(g,x)$ preserves the measure class $[\mu]$. Let $\gamma_g \colon X \to \mathbb{T}$ be measurable functions satisfying the cocycle condition, i.e.,

$$\gamma_{gh} = \gamma_g \cdot g_*\gamma_h \quad \textit{for} \quad g, h \in G.$$

Then

$$(\pi_\gamma(g)f)(x) := \gamma_g(x)\sqrt{\delta(\sigma_g)(x)}f(\sigma_g^{-1}(x)), \quad \pi_\gamma(g)f := \gamma_g \cdot \sqrt{\delta(\sigma_g)}(\sigma_g)_* f$$

defines a unitary representation of G on $L^2(X,\mu)$.

Proof. It follows from Proposition 3.2.3 that

$$(\pi(g)f)(x) := \sqrt{\delta(\sigma_g)(x)}f(\sigma_g^{-1}(x))$$

defines a unitary representation $(\pi, L^2(X,\mu))$ of G, and since γ is a cocycle, we also have $\pi_\gamma(gh) = \pi_\gamma(g)\pi_\gamma(h)$ for $g, h \in G$. □

Example 3.2.5. If the measure class $[\mu]$ is G-invariant, we obtain in particular a natural class of cocycles by

$$\gamma(g) := \delta(\sigma_g)^{is} = e^{is \log \delta(\sigma_g)}, \quad s \in \mathbb{R}.$$

This leads to a family of unitary representations on $L^2(X,\mu)$, parametrized by $s \in \mathbb{R}$:

$$(\pi_s(g)f) := \delta(\sigma_g)^{\frac{1}{2}+is}(\sigma_g)_* f.$$

Example 3.2.6. Let $U \subseteq \mathbb{R}^n$ be an open subset and λ be Lebesgue measure, restricted to U. For each C^1-diffeomorphism $\varphi\colon U \to U$, we then have the transformation formulas

$$\int_U f(x)\, d(\varphi_*\lambda)(x) = \int_U f(\varphi(x))\, d\lambda(x)$$

and

$$\int_U f(\varphi(x))|\det(\mathtt{d}\varphi(x))|\, dx = \int_U f(x)\, dx.$$

Comparing these two implies that

$$\begin{aligned}\int_U f(x)\, d(\varphi_*\lambda)(x) &= \int_U f(\varphi(x))|\det(\mathtt{d}\varphi(x))||\det(\mathtt{d}\varphi(x))|^{-1}\, d\lambda(x)\\ &= \int_U f(x)|\det(\mathtt{d}\varphi(\varphi^{-1}(x)))|^{-1}\, d\lambda(x)\\ &= \int_U f(x)|\det(\mathtt{d}\varphi^{-1}(x))|\, d\lambda(x),\end{aligned}$$

and therefore

$$\delta(\varphi)(x) := \frac{d\varphi_*\lambda}{d\lambda}(x) = |\det(\mathtt{d}\varphi^{-1}(x))|.$$

Thus

$$(\pi(\varphi)f)(x) := \sqrt{|\det(\mathtt{d}\varphi^{-1}(x))|}f(\varphi^{-1}(x))$$

defines a unitary representation of the group $\mathrm{Diff}^1(U)$ of C^1-diffeomorphisms of U on $L^2(U,\lambda)$.

For $U = \mathbb{R}^n$ and the subgroup

$$\mathrm{Aff}_n(\mathbb{R}) := \{\varphi_{A,b}(x) = Ax + b\colon A \in \mathrm{GL}_n(\mathbb{R}), b \in \mathbb{R}^n\}$$

we obtain in particular

$$\delta(\varphi_{A,b})(x) = |\det A|^{-1},$$

so that

$$(\pi(\varphi_{A,b})f)(x) := \sqrt{|\det A|}^{-1} f(\varphi_{A,b}^{-1}(x))$$

defines a unitary representation of the affine group $\mathrm{Aff}_n(\mathbb{R})$ on $L^2(\mathbb{R}^n, \lambda)$.

Remark 3.2.7. One can show that every σ-compact n-dimensional smooth manifold M carries a measure μ which, in every chart (φ, U) has a smooth positive density with respect to Lebesgue measure on $\varphi(U)$ ([HiNe12]). Then $\mathrm{Diff}(M) \subseteq \mathrm{Aut}(M, [\mu])$, so that we obtain a unitary representation of the full diffeomorphism group $\mathrm{Diff}(M)$ on $L^2(M,\mu)$.

The following examples describes a continuous irreducible unitary representation of the affine group of the real line. One can actually show that, up to equivalence, there is only one such representation.

Example 3.2.8. (An irreducible representation of the affine group) We consider the affine group $G := \mathrm{Aff}_1(\mathbb{R})$ consisting of all affine automorphisms $\varphi_{(a,b)}(x) = ax + b$ of $\mathbb{R}$. It is also called the $ax+b$-group. Below we construct an irreducible unitary representation of this group and one can show that, up to equivalence, this is the only one.

For simplicity, we write the elements of this groups as pairs (b, a), so that the multiplication (corresponding to composition of affine functions) is given by

$$(b, a)(b', a') = (b + ab', aa'), \quad a, a' \in \mathbb{R}^\times, b, b' \in \mathbb{R}.$$

We define a unitary representation of this group $G = \mathbb{R} \rtimes \mathbb{R}^\times$ on $L^2(\mathbb{R})$ by

$$(\pi(b, a)f)(x) := e^{ibx}\sqrt{|a|}f(ax).$$

It is easy to verify that this defines indeed a unitary representation (cf. Corollary 3.2.4). Here we use that $\sigma_{(a,b)}x = a^{-1}x$ implies $(\sigma_{(a,b)})_*dx = |a|dx$.

To calculate its commutant, we first recall from Example 3.1.6 that

$$\pi(\mathbb{R} \times \{1\})' = L^\infty(\mathbb{R}).$$

For $a \in \mathbb{R}^\times$, $h \in L^\infty(\mathbb{R})$ and $f \in L^2(\mathbb{R})$, we have

$$\pi(0, a)(hf)(x) = h(ax)(\pi(0, a)f)(x).$$

Therefore $h \in L^\infty(\mathbb{R})$ defines an element in $\pi(G)'$ if and only if $h_a(x) = h(ax)$ satisfies $h_a = h$ almost everywhere for every $a \in \mathbb{R}^\times$.

Let $0 \leq f \in C_c(\mathbb{R}^\times)$ be a continuous function with compact support and $\int_{\mathbb{R}^\times} f(a)\,\frac{da}{|a|} = 1$. Then

$$F(x) := \int_{\mathbb{R}^\times} f(a)h(ax)\,\frac{da}{|a|} = \int_{\mathbb{R}^\times} f(ax^{-1})h(a)\,\frac{da}{|a|}$$

is a continuous function on $\mathbb{R}^\times$ (Exercise; an easy consequence of the Dominated Convergence Theorem). For each measurable subset $E \subseteq \mathbb{R}^\times$, we further have

$$\begin{aligned}\int_E F(x)\,\frac{dx}{|x|} &= \int_E \int_{\mathbb{R}^\times} f(a)h(ax)\,\frac{da}{|a|}\,\frac{dx}{|x|} = \int_{\mathbb{R}^\times}\int_E f(a)h(ax)\,\frac{dx}{|x|}\,\frac{da}{|a|} \\ &= \int_{\mathbb{R}^\times}\int_E f(a)h(x)\,\frac{dx}{|x|}\,\frac{da}{|a|} = \int_E h(x)\,\frac{dx}{|x|}\int_{\mathbb{R}^\times} f(a)\,\frac{da}{|a|} = \int_E h(x)\,\frac{dx}{|x|},\end{aligned}$$

so that $h = F$ almost everywhere (cf. Remark 3.2.2). Since $F(ax) = F(x)$ for almost every x, the continuity of F implies that $F(ax) = F(x)$ for every $a, x \in \mathbb{R}^\times$, and therefore F is constant. We conclude that

$$\pi(G)' = L^\infty(\mathbb{R}) \cap \pi(\{0\} \times \mathbb{R}^\times)' = \mathbb{C}\mathbf{1},$$

so that π is irreducible.

Remark 3.2.9. The mollifying techniqe encountered in the preceding example can be used to show that, for every homogeneous space G/H of a locally compact group G, the subspace $L^\infty(G/H)^G$ of G-invariant elements coincides with the constant functions. Again it is based on the convolution product

$$C_c(G) \times L^\infty(G/H) \to C(G/H).$$

Here is a general setting generalizing Example 3.2.8. We shall see more concrete examples of this type below.

Example 3.2.10. Let A be an abelian topological group and $\alpha\colon G \to \mathrm{Aut}(A)$ be a continuous action of the group G (presently we consider no topology on G) by topological automorphisms of A. Accordingly, we obtain an action $\widehat{\alpha}$ of G on the character group $\widehat{A}$ by $\widehat{\alpha}_g(\chi) := \chi \circ \alpha_g^{-1}$. It is easy to verify that every $\widehat{\alpha}_g$ is an automorphism of the measurable space $(\widehat{A}, \widehat{\mathfrak{S}})$ (cf. Example 3.1.6). Let $[\mu]$ be a G-invariant σ-finite measure class on $(\widehat{A}, \widehat{\mathfrak{S}})$. Then we obtain a unitary representation of the group $A \rtimes_\alpha G$ on $L^2(\widehat{A}, \widehat{\mathfrak{S}}, \mu)$ by

$$\pi(a,g)f := \widehat{a}\delta(g)\sqrt{\delta_\mu(g)}(\widehat{\alpha}_g)_* f, \qquad (\pi(a,g)f)(\chi) := \chi(a)\sqrt{\delta_\mu(g)(\chi)}f(\chi \circ \alpha_g).$$

Then $\pi(A \times \{\mathbf{1}\})' = L^\infty(\widehat{A}, \widehat{\mathfrak{S}}, \mu)$ by Example 3.1.14(a), so that

$$\pi(A \rtimes_\alpha G)' = L^\infty(\widehat{A}, \widehat{\mathfrak{S}}, \mu)^G.$$

If the measure μ is G-ergodic in the sense that $L^\infty(\widehat{A}, \widehat{\mathfrak{S}}, \mu)^G = \mathbb{C}\mathbf{1}$, then Schur's Lemma implies that the representation π is irreducible.

Specializing this construction to the group $A = \mathbb{R}$, $G = \mathbb{R}^\times$, $\alpha_a(x) = ax$ and the Lebesgue measure μ on $\widehat{A} \cong \mathbb{R}$, we obtain Example 3.2.8.

3.2.2 Translations of gaussian measures

In this subsection we discuss a class of examples which are of central importance in quantum field theory ([Si74]) and probability ([Hid80]).

We start with the *Gaussian probability measure*

$$d\gamma(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx$$

on $\mathbb{R}$ with expectation value 0 and variance 1.

For every $n \in \mathbb{N}$, we then obtain on $\mathbb{R}^n$ a product probability measure

$$\gamma^n = \gamma^{\otimes n} \quad \text{with} \quad d\gamma^n(x) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{\|x\|^2}{2}}\, dx.$$

This measure on $\mathbb{R}^n$ is equivalent to Lebesgue measure, hence quasiinvariant under the translations $\tau_y(x) := x + y$. The corresponding Radon–Nikodym derivatives are given by

$$\delta(y)(x) = \frac{d((\tau_y)_*\gamma)}{d\gamma}(x) = \frac{e^{-\frac{\|x-y\|^2}{2}}}{e^{-\frac{\|x\|^2}{2}}} = e^{-\frac{\|y\|^2}{2} + \langle x,y\rangle}. \tag{3.4}$$

We thus obtain on $L^2(\mathbb{R}^n, \gamma^n)$ the unitary representation of the translation group $\mathbb{R}^n$ by

$$(\pi_\gamma(y)f)(x) := \sqrt{\delta(y)(x)}f(x-y) = e^{-\frac{\|y\|^2}{4}+\frac{\langle x,y\rangle}{2}}f(x-y).$$

This representation is equivalent to the translation representation of $\mathbb{R}^n$ on $L^2(\mathbb{R}^n, dx)$ (Exercise 3.3.12).

The situation becomes more interesting on infinite dimensional spaces. For a set J we endow the product set $X := \mathbb{R}^J$ of all real-valued functions on $\mathbb{R}$ with the smallest σ-algebra $\mathfrak{S} = \mathfrak{B}(\mathbb{R})^{\otimes J}$ for which all projections $p_k((x_j)_{j\in J}) = x_k$ are Borel measurable maps. Then Kolmogorov's Product Theorem implies the existence of a unique probability measure γ^J on $(X, \mathfrak{S})$, such that, for every finite subset $F \subseteq J$, we have $(p_F)_*\gamma^J = \gamma^F$ for the projection map $p_F(x) = (x_j)_{j\in F}$ (cf. [Ba78, Satz 33.2]).

Definition 3.2.11. The measure γ^J on $(\mathbb{R}^J, \mathfrak{B}(\mathbb{R})^{\otimes J})$ is called the *canonical Gaussian measure on* $\mathbb{R}^J$.

Remark 3.2.12. There are many other Gaussian product measures on $\mathbb{R}^J$. For $m \in \mathbb{R}$ and $\sigma > 0$, the Gaussian measure with expectation value m and variance σ^2 on $\mathbb{R}$ is given by

$$d\gamma_{m,\sigma^2}(x) := \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-m)^2}{2\sigma^2}}\,dx.$$

Accordingly, we obtain arbitrary product measures $\otimes_{j\in J}\gamma_{m_j,\sigma_j^2}$ on $\mathbb{R}^J$, and this leads already to uncountably many measure classes on $\mathbb{R}^{\mathbb{N}}$.

Proposition 3.2.13. (Translational quasi-invariance of Gaussian measure) *The standard Gaussian measure γ^J on $(\mathbb{R}^J, \mathfrak{B}(\mathbb{R})^{\otimes J})$ is quasi-invariant under all translations $\tau_y(x) = x + y$ for $y \in \ell^2(J, \mathbb{R})$.*

Proof. If

$$y \in \mathbb{R}^{(J)} := \{x \in \mathbb{R}^J : |\{j \in J : x_j \neq 0\}| < \infty,$$

then $F := \{j \in J : y_j \neq 0\}$ is finite. Now the decomposition $J = F\dot{\cup}F^c$ leads to a factorization $\gamma^J = \gamma^F \otimes \gamma^{F^c}$ on $\mathbb{R}^J \cong \mathbb{R}^F \times \mathbb{R}^{F^c}$. Therefore the quasi-invariance of γ under τ_y follows from the quasi-invariance of γ^F on $\mathbb{R}^F \cong \mathbb{R}^{|F|}$ and the fact that $\tau_y(x_F, x_{F^c}) = (x_F + y, x_{F^c})$ with respect to the factorization $\mathbb{R}^J \cong \mathbb{R}^F \times \mathbb{R}^{F^c}$.

To complete the proof, we can argue similarly in the case where F is infinite, hence countable (cf. Exercise 1.3.4). This reduces the problem to the case where $J = \mathbb{N}$. Here the problem is to show that the function $\varphi(y)(x) := e^{\sum_{n=1}^\infty y_n x_n}$ defines an element of $L^1(\mathbb{R}^{\mathbb{N}}, \gamma^{\mathbb{N}})$ for every $y \in \ell^2(\mathbb{N}, \mathbb{R})$. This can be done as follows. We consider the sequence

$$\varphi_n(y)(x) := e^{\sum_{j=1}^n y_j x_j}$$

of functions in $L^2(\mathbb{R}^{\mathbb{N}}, \gamma^{\mathbb{N}})$ and want to show that it is a Cauchy sequence.

From the above formula for $\delta(y)$, it follows that, for $y \in \mathbb{R}^{(\mathbb{N})}$, the function $\varphi(y)$ on $\mathbb{R}^{\mathbb{N}}$ is $\gamma^{\mathbb{N}}$-integrable with

$$\int_{\mathbb{R}^{\mathbb{N}}} \varphi(y)\, d\gamma^{\mathbb{N}} = \int_{\mathbb{R}^{\mathbb{N}}} \varphi(y)(x)\, d\gamma^{\mathbb{N}}(x) = e^{\frac{\|y\|^2}{2}}.$$

This further leads to

$$\int_{\mathbb{R}^{\mathbb{N}}} \varphi(y)\varphi(z)\, d\gamma^{\mathbb{N}} = \int_{\mathbb{R}^{\mathbb{N}}} \varphi(y+z)\, d\gamma^{\mathbb{N}} = e^{\frac{\|y+z\|^2}{2}},$$

and hence to

$$\|\varphi(y) - \varphi(z)\|_2^2 = e^{2\|y\|^2} + e^{2\|z\|^2} - 2e^{\frac{\|y+z\|^2}{2}} =: D(y,z).$$

The function D extends to a continuous function on $\ell^2(\mathbb{N},\mathbb{R})$, vanishing on the diagonal. For $y \in \ell^2(\mathbb{N},\mathbb{R})$, we write $y^n := (y_1, \ldots, y_n, 0, \cdots) \in \mathbb{R}^{(\mathbb{N})}$. Then $\varphi_n(y) = \varphi(y^n)$ leads to

$$\|\varphi_n(y) - \varphi_m(y)\|_2^2 = D(y^n, y^m) \to D(y,y) = 0$$

for $n, m \to \infty$. This implis that $(\varphi_n(y))_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\mathbb{R}^{\mathbb{N}}, \gamma^{\mathbb{N}}) \subseteq L^1(\mathbb{R}^{\mathbb{N}}, \gamma^{\mathbb{N}})$. We write $\varphi_\infty(y)$ for its limit.

If $F \subseteq \mathbb{N}$ is finite and $E_1 \subseteq \mathbb{R}^F$ a Borel set, we write $E := E_1 \times \mathbb{R}^{F^c} \subseteq \mathbb{R}^{\mathbb{N}}$ for the corresponding cylinder set. Then

$$\begin{aligned}((\tau_y)_*\gamma^{\mathbb{N}})(E) = ((\tau_{y^n})_*\gamma^n)(E_1) = e^{-\frac{\|y^n\|_2^2}{2}} \int_{E_1} \varphi_n(y)\, d\gamma^n = e^{-\frac{\|y^n\|_2^2}{2}} \int_E \varphi_n(y)\, d\gamma^{\mathbb{N}} \\ \to e^{-\frac{\|y\|_2^2}{2}} \int_E \varphi_\infty(y)\, d\gamma^{\mathbb{N}}.\end{aligned}$$

This implies that

$$(\tau_y)_*\gamma^{\mathbb{N}} = e^{-\frac{\|y\|_2^2}{2}} \varphi_\infty(y) \cdot \gamma^{\mathbb{N}}$$

for every $y \in \ell^2(\mathbb{N},\mathbb{R})$, and this completes our proof. □

Although we won't need it in the following, we add a theorem clarifying possible "supports" for the standard Gaussian measure $\gamma^{\mathbb{N}}$. It follows in particular that it can be realized on Hilbert spaces on functions which are much smaller than the space $\mathbb{R}^{\mathbb{N}}$ of all sequences. Such phenomena are studied systematically in the context of stochastic processes.

Theorem 3.2.14. *For $a = (a_n)_{n\in\mathbb{N}} \in]0,\infty[^{\mathbb{N}}$, the set*

$$\ell^2(a) := \Big\{ x \in \mathbb{R}^{\mathbb{N}} \colon \|x\|_a^2 := \sum_{n=1}^{\infty} a_n x_n^2 < \infty \Big\}$$

satisfies

$$\gamma^{\mathbb{N}}(\ell^2(a)) = \begin{cases} 1 & \text{if } \sum_n a_n < \infty \\ 0 & \text{if } \sum_n a_n = \infty. \end{cases}$$

In particular $\gamma^{\mathbb{N}}(\ell^2) = \{0\}$.

Proof. (cf. [Dr03, Thm. 5.2]) We consider the functions

$$q\colon \mathbb{R}^{\mathbb{N}} \to [0,\infty], \quad q(x) := \sum_n a_n x_n^2$$

and put $q_N(x) := \sum_{n=1}^N a_n x_n^2$. Then $\lim_{\varepsilon\to 0+} e^{-\varepsilon q/2} = \chi_{\ell^2(a)}$ and $0 \leq e^{-\varepsilon q/2} \leq 1$ lead to

$$\gamma^{\mathbb{N}}(\ell^2(a)) = \lim_{\varepsilon\to 0+} \int_{\mathbb{R}^{\mathbb{N}}} e^{-\varepsilon q/2}\, d\gamma^{\mathbb{N}}$$

by the Monotone Convergence Theorem. For any $\varepsilon > 0$, we derive from the Monotone Convergence Theorem

$$\begin{aligned}
\int_{\mathbb{R}^{\mathbb{N}}} e^{-\varepsilon q/2}\, d\gamma^{\mathbb{N}} &= \int_{\mathbb{R}^{\mathbb{N}}} \lim_{N\to\infty} e^{-\varepsilon q_N/2}\, d\gamma^{\mathbb{N}} = \lim_{N\to\infty} \int_{\mathbb{R}^{\mathbb{N}}} e^{-\varepsilon q_N/2}\, d\gamma^{\mathbb{N}} \\
&= \lim_{N\to\infty} \int_{\mathbb{R}^N} e^{-\frac{\varepsilon}{2}\sum_{n=1}^N a_n x_n^2}\, d\gamma^N(x_1,\ldots,x_N) \\
&= \lim_{N\to\infty} \prod_{n=1}^N \int_{\mathbb{R}} e^{-\frac{\varepsilon}{2} a_n x_n^2}\, d\gamma(x_n) = \lim_{N\to\infty} \prod_{n=1}^N \frac{\sqrt{\frac{2\pi}{\varepsilon a_n + 1}}}{\sqrt{2\pi}} \\
&= \prod_{n=1}^\infty (1+\varepsilon a_n)^{-\frac{1}{2}} = \Big(\prod_{n=1}^\infty (1+\varepsilon a_n)\Big)^{-\frac{1}{2}}.
\end{aligned}$$

Taking logarithms and then passing to the limit $\varepsilon \to 0+$ thus leads to

$$-\log \gamma^{\mathbb{N}}(\ell^2(a)) = \lim_{\varepsilon\to 0} \frac{1}{2}\sum_{n=1}^\infty \log(1+\varepsilon a_n) = \begin{cases} 0 & \text{if } \sum_n a_n < \infty \\ \infty & \text{if } \sum_n a_n = \infty. \end{cases}$$

To verify the last equality, we first observe that, if $\sum_n a_n < \infty$, this follows from the Monotone Convergence Theorem applied to the counting measure on $\mathbb{N}$. If $\sum_n a_n = \infty$, then $\sum_{n=1}^\infty \log(1+\varepsilon a_n) = \infty$ for every $\varepsilon > 0$. This proves the assertion. □

Remark 3.2.15. (a) One remarkable consequence of the preceding theorem is that the Hilbert space ℓ^2 is a zero set for the Gaussian measure on $\mathbb{R}^{\mathbb{N}}$. Therefore it is necessary to enlarge this space to some $\ell^2(a)$, $0 < a \in \ell^1$, to obtain a Hilbert space on which the measure $\gamma^{\mathbb{N}}$ can be realized.

(b) Another interesting point is that

$$\bigcap_{0<a\in\ell^1} \ell^2(a) = \Big\{x \in \mathbb{R}^{\mathbb{N}}\colon (\forall a \in \ell^1)\ \sum_n |a_n| x_n^2 < \infty\Big\} = \ell^\infty.$$

All sets $\ell^2(a)$ have full $\gamma^{\mathbb{N}}$-measure, but the subspace ℓ^∞ is a zero set because $\gamma([-r,r]) < 1$ for every $r > 0$ leads to

$$\gamma^{\mathbb{N}}([-r,r]^{\mathbb{N}}) = \lim_{N\to\infty} \gamma([-r,r])^N = 0.$$

3.3 L^2-spaces on locally compact spaces

In this section we briefly discuss some specific issues related to L^2-spaces on locally compact spaces. Here the main point is the concept of a Radon measure, i.e., a Borel measure which is closely linked with the topology on X. In particular, for a Radon measure μ, the subspace $C_c(X)$ of compactly supported continuous functions is dense in $L^2(X,\mu)$, as the following proposition shows.

Definition 3.3.1. A measure μ on a locally compact space X is called *a Radon measure* if

(i) $\mu(K) < \infty$ for each compact subset K of X.

(ii) (Outer regularity) For each Borel subset $E \subseteq X$, we have

$$\mu(E) = \inf\{\mu(U) \colon E \subseteq U, U \text{open}\}.$$

(iii) If $E \subseteq X$ is open or E is a Borel set with $\mu(E) < \infty$, then

$$\mu(E) = \sup\{\mu(K) \colon K \subseteq E, K \text{ compact}\}.$$

The measure μ is called *regular* if (ii) and (iii) are satisfied.

Proposition 3.3.2. *If μ is a Radon measure on a locally compact space X, then $C_c(X)$ is dense in $L^2(X,\mu)$.*

Proof. Since the step functions form a dense subspace of $L^2(X,\mu)$, it suffices to show that any characteristic function χ_E with $\mu(E) < \infty$ can be approximated by elements of $C_c(X)$ in the L^2-norm. Since every such Borel set is inner regular, we may w.l.o.g. assume that E is compact. Then the outer regularity implies for each $\varepsilon > 0$ the existence of an open subset $U \subseteq X$ with $\mu(U \setminus E) < \varepsilon$. Next we use Urysohn's Theorem A.1.6 to find a continuous function $f \in C_c(X)$ with $0 \leq f \leq 1$, $f|_E = 1$, and $\mathrm{supp}(f) \subseteq U$. Then

$$\|f - \chi_E\|_2^2 = \int_X |f(x) - \chi_E(x)|^2 \, d\mu(x) = \int_{U\setminus E} |f(x)|^2 \, d\mu(x) \leq \mu(U \setminus E) < \varepsilon,$$

and this completes the proof. □

Remark 3.3.3. In many cases the regularity of a Borel measure μ on a locally compact space X for which all compact subspaces have finite measure is automatic.

In [Ru86, Thm. 2.18] one finds the convenient criterion that this is the case whenever every open subset $O \subseteq X$ is a countable union of compact subsets.

This is in particular the case for $\mathbb{R}^n$, because we can write

$$O = \bigcup_{n\in\mathbb{N}} O_n \quad \text{with} \quad O_n := \Big\{x \in O \colon \mathrm{dist}(x, O^c) \geq \frac{1}{n}, \|x\| \leq n\Big\}.$$

Proposition 3.3.4. *Let G be a topological group and $\sigma\colon G \times X \to X$ be a continuous action of G on the locally compact space X. Further, let μ be a Radon measure on X whose measure class is G-invariant and for which the Radon–Nikodym derivative $\delta(g) := \frac{d(\sigma_g)_*\mu}{d\mu}$ can be realized by a continuous function*

$$\widetilde{\delta}\colon G \times X \to \mathbb{R}^\times, \quad \widetilde{\delta}(g,x) := \delta(g)(x).$$

Then the unitary representation $(\pi, L^2(X,\mu))$, defined by

$$(\pi(g)f)(x) := \sqrt{\delta(g)(x)}f(g^{-1}.x)$$

is continuous.

Proof. In Proposition 3.3.2 we have seen that $C_c(X)$ is dense in $L^2(X,\mu)$. In view of Lemma 1.2.6, it therefore suffices to show that, for $f, h \in C_c(X)$, the function

$$\begin{aligned}\pi_{f,h}\colon G \to \mathbb{C}, \quad & g \mapsto \langle \pi(g)f, h\rangle \\ &= \int_X \sqrt{\delta(g)(x)}f(g^{-1}.x)\overline{h(x)}\, d\mu(x) = \int_{\mathrm{supp}(h)} \sqrt{\delta(g)(x)}f(g^{-1}.x)\overline{h(x)}\, d\mu(x)\end{aligned}$$

is continuous. This is an integral of the form

$$F(g) := \int_K H(x,g)\, d\mu(x),$$

where $K := \mathrm{supp}(h) \subseteq X$ is compact and $H\colon K \times G \to \mathbb{C}$ is continuous. The map

$$\widetilde{H}\colon G \to C(K), \quad \widetilde{H}(g)(x) := H^g(x) = H(x,g)$$

is continuous with respect to $\|\cdot\|_\infty$ on $C(K)$.[2] Therefore the continuity of the function F follows from the continuity of the linear functional $C(K) \to \mathbb{C}, h \mapsto \int_K h\, d\mu$ with respect to $\|\cdot\|_\infty$ on $C(K)$ (cf. Exercise 3.3.2). $\square$

Corollary 3.3.5. *Let G be a topological group, $\sigma\colon G \times X \to X$ be a continuous action of G on the locally compact space X and μ be a G-invariant Radon measure on X. Then the unitary representation $(\pi, L^2(X,\mu))$, defined by*

$$\pi(g)f := f \circ \sigma_g^{-1}$$

is continuous.

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[2]For $g_0 \in G$ and $\varepsilon > 0$, the set

$$M := \{(g,x) \in G \times K \colon |H(g,x) - H(g_0,x)| < \varepsilon\}$$

is open and contains $\{g_0\} \times K$, hence also a set of the form $U \times K$, where U is a neighborhood of g_0. This means that for $g \in U$, we $\|H^g - H^{g_0}\|_\infty \leq \varepsilon$.

Example 3.3.6. (a) The translation representation of $G = \mathbb{R}^n$ on $L^2(\mathbb{R}^n, dx)$, given by

$$(\pi(x)f)(y) := f(x+y)$$

is continuous.

(b) On the circle group $G = \mathbb{T}$, we consider the Radon measure $\mu_{\mathbb{T}}$, given by

$$\int_{\mathbb{T}} f(z)\, d\mu_{\mathbb{T}}(z) := \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\, dt.$$

Then the unitary representation of $\mathbb{T}$ on $L^2(\mathbb{T}, \mu_{\mathbb{T}})$, given by

$$(\pi(t)f)(z) := f(tz)$$

is continuous.

Exercises for Chapter 3

Exercise 3.3.1. Let $(X, \mathfrak{S}, \mu)$ be a semifinite measure space. Show that there exist measurable subsets $X_j \subseteq X$, $j \in J$, of finite measure such that

$$L^2(X, \mu) \cong \widehat{\bigoplus}_{j\in J} L^2(X_j, \mu|_{X_j}).$$

Hint: Use Zorn's Lemma to find a maximal family $(X_j)_{j\in J}$ of measurable subsets of X for which $\mu(X_j \cap X_k) = 0$ for $j \neq k$. Conclude that the corresponding subspaces $L^2(X_j, \mu|_{X_j})$ of $L^2(X, \mu)$ are mutually orthogonal and that the intersection of their orthogonal complements is trivial.

Exercise 3.3.2. Let μ be a Radon measure on the locally compact space X and $K \subseteq X$ be a compact subset. Show that the integral

$$I \colon C(K) \to \mathbb{C}, \quad f \mapsto \int_K f(x)\, d\mu(x)$$

satisfies

$$|I(f)| \leq \|f\|_\infty \mu(K).$$

In particular, I is continuous.

Exercise 3.3.3. Show that each σ-finite measure μ on a measurable space $(X, \mathfrak{S})$ is equivalent to a finite measure.

Exercise 3.3.4. Let μ and λ be equivalent σ-finite measures on $(X, \mathfrak{S})$ and $h := \frac{d\mu}{d\lambda}$. Show that

$$\Phi \colon L^2(X, \mu) \to L^2(X, \lambda), \quad f \mapsto \sqrt{h} f$$

defines a unitary map.

Exercise 3.3.5. Consider the Gaussian measure

$$d\gamma^n(x) := \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}\|x\|^2} dx$$

on $\mathbb{R}^n$. Its measure class is invariant under the action of the affine group $\mathrm{Aff}_n(\mathbb{R})$. Find a formula for the unitary representation of this group on $L^2(\mathbb{R}^n, \gamma^n)$.

Exercise 3.3.6. Let $(X, \mathfrak{S}, \mu)$ be a measure space. Show that:

(a) If $f \in L^p(X, \mu)$, $1 \leq p < \infty$, then the measurable subset $\{f \neq 0\}$ of X is σ-finite.

(b) If $\mathcal{H} \subseteq L^2(X, \mu)$ is a separable subspace, then there exists a σ-finite measurable subset $X_0 \subseteq X$ with the property that each $f \in \mathcal{H}$ vanishes μ-almost everywhere on $X_0^c = X \setminus X_0$.

Exercise 3.3.7. (Limitations of the Radon–Nikodym Theorem) Let $(X, \mathfrak{S}, \mu)$ be a finite measure space. Define $\nu \colon \mathfrak{S} \to \mathbb{R}_+$ by

$$\nu(E) := \begin{cases} 0 & \text{for } \mu(E) = 0 \\ \infty & \text{otherwise.} \end{cases}$$

Show that ν is a measure with the same zero sets as μ, but there exists no measurable function $f \colon X \to \mathbb{R}$ with $\nu = f\mu$. However, the constant function $f = \infty$ satisfies $\nu(E) = \int_E f(x)\, d\mu(x)$ for each $E \in \mathfrak{S}$.

Exercise 3.3.8. Let $(X, \mathfrak{S}, \mu)$ be a measure and $\mathfrak{S}_{\mathrm{fin}} := \{E \in \mathfrak{S} \colon \mu(E) < \infty\}$. Verify the following assertions:

(a) On $\mathfrak{S}_{\mathrm{fin}}$ we obtain by $d(E, F) := \mu(E \Delta F)$ a semimetric.

(b) Let
$$[E] := \{F \in \mathfrak{S} \colon \mu(E \Delta F) = 0\}$$
denote the corresponding equivalence class of $E \in \mathfrak{S}$ and $\overline{\mathfrak{S}} := \mathfrak{S}/\sim$ denote the set of equivalence classes. Then $d([E], [F]) := \mu(E \Delta F)$ defines a metric on $\overline{\mathfrak{S}}_{\mathrm{fin}}$.

(c) The map $\gamma \colon \mathfrak{S}_{\mathrm{fin}} \to L^1(X, \mathfrak{S}, \mu), E \mapsto \chi_E$ is an isometry, i.e.,
$$d(E, F) = \mu(E \Delta F) = \|\chi_E - \chi_F\|_1 = \|\chi_E - \chi_F\|_2^2.$$

Exercise 3.3.9. Let $(X, \mathfrak{S}, \mu)$ be a finite measure space and $G := \mathrm{Aut}(X, \mu)$ be its automorphism group. Verify the following assertions:

(a) $N := \{g \in G \colon (\forall E \in \mathfrak{S}_{\mathrm{fin}})\, \mu(gE \Delta E) = 0\}$ is a normal subgroup of G, and if $\pi \colon G \to \mathrm{U}(L^2(X, \mu)), \pi(g)f := f \circ g^{-1}$ is the canonical unitary representation of G on $L^2(X, \mu)$, then $\ker \mu = N$.

(b) On the quotient group $\overline{G} := G/N$, we consider the coarsest topology for which all functions
$$f_E \colon \overline{G} \to \mathbb{R}, \quad g \mapsto \mu(gE \Delta E)$$
are continuous. Show that $\overline{G}$ is a topological group and that π factors through a topological embedding $\overline{\pi} \colon \overline{G} \to \mathrm{U}(L^2(X, \mu))_s$. Hint: Exercise 1.2.7 and Lemma 1.2.6.

Exercise 3.3.10. We consider the group $G := \mathrm{GL}_2(\mathbb{R})$ and the *real projective line*

$$\mathbb{P}_1(\mathbb{R}) = \{[v] := \mathbb{R}v \colon 0 \neq v \in \mathbb{R}^2\}$$

of 1-dimensional linear subspaces of $\mathbb{R}^2$. We write $[x : y]$ for the line $\mathbb{R}\begin{pmatrix} x \\ y \end{pmatrix}$. Show that:

(a) We endow $\mathbb{P}_1(\mathbb{R})$ with the quotient topology with respect to the map $q \colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{P}_1(\mathbb{R}), v \mapsto [v]$. Show that $\mathbb{P}_1(\mathbb{R})$ is homeomorphic to $\mathbb{S}^1$. Hint: Consider the squaring map on $\mathbb{T} \subseteq \mathbb{C}$.

(b) The map $\mathbb{R} \to \mathbb{P}_1(\mathbb{R}), x \mapsto [x : 1]$ is injective and its complement consists of the single point $\infty := [1 : 0]$ (the horizontal line). We thus identify $\mathbb{P}_1(\mathbb{R})$ with the one-point compactification of $\mathbb{R}$. These are the so-called *homogeneous coordinates* on $\mathbb{P}_1(\mathbb{R})$.

(c) The natural action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{P}_1(\mathbb{R})$ by $g.[v] := [gv]$ is given in the coordinates of (b) by

$$g.x = \sigma_g(x) := \frac{ax+b}{cx+d} \quad \text{for} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(d) There exists a unique Radon measure μ with total mass π on $\mathbb{P}_1(\mathbb{R})$ which is invariant under the group $\mathrm{O}_2(\mathbb{R})$. Hint: Identify $\mathbb{P}_1(\mathbb{R})$ with the compact group $\mathrm{SO}_2(\mathbb{R})/\{\pm\mathbf{1}\} \cong \mathbb{T}$.

(e) Show that, in homogeneous coordinates, we have $d\mu(x) = \frac{dx}{1+x^2}$.
Hint: $\begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}.0 = -\tan x$, and the image of Lebesgue measure on $]-\pi/2, \pi/2[$ under tan is $\frac{dx}{1+x^2}$.

(f) Show that the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{P}_1(\mathbb{R})$ preserves the measure class of μ. Hint: Show that $\sigma_g(x) := \frac{ax+b}{cx+d}$ satisfies $\sigma_g'(x) = \frac{1}{(cx+d)^2}$ and derive the formula

$$\delta(\sigma_g)(x) = \frac{d((\sigma_g)_*\mu)}{d\mu}(x) = \frac{1+x^2}{(a-cx)^2+(b-dx)^2}, \quad \delta(\sigma_g)(\infty) = \frac{1}{c^2+d^2}.$$

(g) The density function also has the following metric interpretation with respect to the euclidean norm on $\mathbb{R}^2$:

$$\delta(\sigma_g)([v]) = \frac{\|g^{-1}v\|^2}{\|v\|^2}.$$

The corresponding unitary representations of $\mathrm{SL}_2(\mathbb{R})$ on $L^2(\mathbb{P}_1(\mathbb{R}), \mu)$ defined by

$$\pi_s(g)f := \delta(\sigma_g)^{\frac{1}{2}+is}(\sigma_g)_*f$$

(cf. Example 3.2.5) form the so-called *spherical principal series*.

Exercise 3.3.11. Let $(\pi, \mathcal{H})$ be a non-degenerate representation of the involutive semigroup $(S, *)$ and $v \in \mathcal{H}$. Show that the following assertions are equivalent:

(a) v is a cyclic vector for $\pi(S)$.

(b) v is a cyclic vector for the von Neumann algebra $\pi(S)''$.

(c) v is separating for the von Neumann algebra $\pi(S)'$, i.e., the map $\pi(S)' \to \mathcal{H}, A \mapsto Av$ is injective.

Hint: To see that (c) implies (a), consider the projection P onto $(\pi(S)v)^\perp$, which is an element of $\pi(S)'$.

Exercise 3.3.12. Let γ^n be the centered Gaussian measure on $\mathbb{R}^n$ with variance $\mathbf{1}$. Show that the unitary representation

$$(\pi_\gamma(y)f)(x) := \sqrt{\delta(y)(x)}f(x-y) = e^{-\frac{\|y\|^2}{4}+\frac{\langle x,y\rangle}{2}}f(x-y)$$

of $\mathbb{R}^n$ on $L^2(\mathbb{R}^n, \gamma^n)$ is equivalent to the translation representation of $\mathbb{R}^n$ on $L^2(\mathbb{R}^n, dx)$. Hint: Exercise 3.3.4.

Exercise 3.3.13. Show that, for a measure space $(X, \mathfrak{S}, \mu)$, we have $L^2(X, \mathfrak{S}, \mu) = \{0\}$ if and only if, $\mu(\mathfrak{S}) \subseteq \{0, \infty\}$.

Exercise 3.3.14. Let $(X, \mathfrak{S}, \mu)$ be a measure space and $\mathfrak{S}_f \subseteq \mathfrak{S}$ be the set of those elements $E \in \mathfrak{S}$ for which either $\mu|_{\mathfrak{S} \cap E}$ or $\mu|_{\mathfrak{S} \cap E^c}$ is σ-finite. Show that

(a) $\mathfrak{S}_f$ is a σ-subalgebra of $\mathfrak{S}$.

(b) $\mathfrak{S}_f$ is generated by the μ-finite subsets of $\mathfrak{S}$.

(c) $L^2(X, \mathfrak{S}, \mu) = L^2(X, \mathfrak{S}_f, \mu|_{\mathfrak{S}_f})$.

Exercise 3.3.15. (Cyclic elements in $L^2(X, \mathfrak{S}, \mu)$) Let $(X, \mathfrak{S}, \mu)$ be a semifinite measure space and $\mathcal{A} := L^\infty(X, \mathfrak{S}, \mu) \subseteq B(L^2(X, \mathfrak{S}, \mu))$ acting by the multiplication operators $M_f(h) = fh$ (Proposition 3.1.8). Show that a function $h \in L^2(X, \mathfrak{S}, \mu)$ is $\mathcal{A}$-cyclic if and only if $X_0 := \{h = 0\}$ contains no subsets of positive finite measure, i.e., $L^2(X_0, \mathfrak{S}|_{X_0}, \mu) = \{0\}$. Hint: If $X_0 = \emptyset$ and $L^2(X, \mathfrak{S}, \mu) \ni f \perp \mathcal{A}h$, then every subset $E := \{n < |f| \leq n+1\}$ is of finite measure and $\mathcal{A}\chi_E = L^2(E, \mathfrak{S}|_E, \mu) \perp h$. Conclude that $\mu(E) = 0$ and hence that $f = 0$.

Exercise 3.3.16. (σ-finiteness and cyclicity) Let $(X, \mathfrak{S}, \mu)$ be a measure space for which $L^2(X, \mathfrak{S}, \mu)$ contains a cyclic element for $L^\infty(X, \mathfrak{S}, \mu)$. Show that there exists a σ-finite subset $X_1 \in \mathfrak{S}$ with $L^2(X, \mathfrak{S}, \mu) = L^2(X_1, \mathfrak{S} \cap X_1, \mu)$.

Chapter 4

Reproducing Kernel Spaces

In Chapter 3 we have seen how Hilbert spaces and continuous unitary representations can be constructed on L^2-spaces. An L^2-space of a measure space $(X, \mathfrak{S}, \mu)$ has the serious disadvantage that its elements are not functions on X, they are only equivalence classes of functions modulo those vanishing on μ-zero sets. However, many important classes of unitary representations can be realized in spaces of continuous functions. In particular for infinite dimensional Lie groups, this is the preferred point of view because measure theory on infinite dimensional spaces has serious defects that one can avoid by using other methods.

In this chapter we introduce the concept of a reproducing kernel Hilbert space. These are Hilbert spaces $\mathcal{H}$ of *functions* on a set X for which all point evaluations $\mathcal{H} \to \mathbb{C}, f \mapsto f(x)$, are continuous linear functionals. Representing these functions according to the Fréchet–Riesz Theorem by an element $K_x \in \mathcal{H}$, we obtain a function

$$K \colon X \times X \to \mathbb{C}, \quad K(x, y) := K_y(x)$$

called the *reproducing kernel of* $\mathcal{H}$. Typical questions arising in this context are: Which functions on $X \times X$ are reproducing kernels and, if we have a group action on X, how can we construct unitary representations on reproducing kernel spaces.

Throughout this chapter $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$.

4.1 Hilbert Spaces with Continuous Point Evaluations

Definition 4.1.1. Let X be a set.

(a) Consider a Hilbert space $\mathcal{H}$ which is contained in the space $\mathbb{K}^X$ of $\mathbb{K}$-valued functions on X. We say that $\mathcal{H}$ has *continuous point evaluations* if, for each $x \in X$, the linear functional

$$\mathrm{ev}_x \colon \mathcal{H} \to \mathbb{K}, \quad f \mapsto f(x)$$

is continuous. In view of the Fréchet–Riesz Theorem, this implies the existence of some $K_x \in \mathcal{H}$ with

$$f(x) = \langle f, K_x \rangle \quad \text{for} \quad f \in \mathcal{H}, x \in X.$$

The corresponding function

$$K \colon X \times X \to \mathbb{K}, \quad K(x,y) := K_y(x)$$

is called the *reproducing kernel of* $\mathcal{H}$. As we shall see below, $\mathcal{H}$ is uniquely determined by K, so that we shall denote it by $\mathcal{H}_K$ to emphasize this fact and call it the *reproducing kernel Hilbert space (RKHS)* associated to K.

(b) A function $K \colon X \times X \to \mathbb{K}$ is called a *positive definite kernel* if, for each finite subset $\{x_1, \ldots, x_n\} \subseteq X$, the matrix $\big(K(x_i, x_j)\big)_{i,j=1,\ldots,n}$ is positive semidefinite. For a function $K \colon X \times X \to \mathbb{K}$ we write $K^*(x,y) := \overline{K(y,x)}$ and say that K is *hermitian* (or *symmetric* for $\mathbb{K} = \mathbb{R}$) if $K^* = K$.

We write $\mathcal{P}(X, \mathbb{K})$ for the set of positive definite kernels on the set X.

Remark 4.1.2. (a) Over $\mathbb{K} = \mathbb{C}$, the positive definiteness of a kernel K already follows from the requirement that for all choices $x_1, \ldots, x_n \in X$ and $c_1, \ldots, c_n \in \mathbb{C}$ we have

$$\sum_{j,k=1}^{n} c_j \overline{c_k} K(x_j, x_k) \geq 0$$

because this implies that K is hermitian (Exercise 4.1.1).

For $\mathbb{K} = \mathbb{R}$, the requirement of the kernel to be hermitian is not redundant. Indeed, the matrix

$$(K_{ij})_{i,j=1,2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

considered as a kernel on the two element set $X = \{1, 2\}$, satisfies

$$\sum_{j,k=1}^{2} c_j c_k K(x_j, x_k) = 0$$

for $x_1, \ldots, x_n \in X$ and $c_1, \ldots, c_n \in \mathbb{R}$, but K is not hermitian.

(b) For any positive definite kernel $K \in \mathcal{P}(X)$ and $x, y \in \mathcal{P}(X)$, the positive definiteness of the hermitian matrix

$$\begin{pmatrix} K(x,x) & K(x,y) \\ K(y,x) & K(y,y) \end{pmatrix}$$

implies in particular that

$$|K(x,y)|^2 \leq K(x,x) K(y,y) \tag{4.1}$$

In the following we call a subset S of a Hilbert space $\mathcal{H}$ *total* if it spans a dense subspace.

Theorem 4.1.3. (Characterization Theorem) *The following assertions hold for a function $K\colon X \times X \to \mathbb{K}$:*

(a) *If K is the reproducing kernel of the Hilbert space $\mathcal{H} \subseteq \mathbb{K}^X$ with continuous point evaluations, then the following assertions hold:*

  (1) *K is positive definite.*

  (2) *$\{K_x \colon x \in X\}$ is total in $\mathcal{H}$.*

  (3) *$K(x,y) = \sum_{j\in J} e_j(x)\overline{e_j(y)}$ for any orthonormal basis $(e_j)_{j\in J}$ of $\mathcal{H}$.*

(b) *If K is positive definite, then $\mathcal{H}_K^0 := \operatorname{span}\{K_x \colon x \in X\} \subseteq \mathbb{K}^X$ carries a unique positive definite hermitian form satisfying*

$$\langle K_y, K_x\rangle = K(x,y) \quad \text{for} \quad x,y \in X. \tag{4.2}$$

The completion $\mathcal{H}_K$ of $\mathcal{H}_K^0$ permits an injection

$$\iota\colon \mathcal{H}_K \to \mathbb{K}^X, \quad \iota(v)(x) := \langle v, K_x\rangle$$

whose image is a Hilbert space with reproducing kernel K that we identify with $\mathcal{H}_K$.

(c) *K is positive definite if and only if there exists a Hilbert space $\mathcal{H} \subseteq \mathbb{K}^X$ with reproducing kernel K.*

Proof. (a)(1) That K is hermitian follows from

$$K(y,x) = K_x(y) = \langle K_x, K_y\rangle = \overline{\langle K_y, K_x\rangle} = \overline{K(x,y)}.$$

For $c \in \mathbb{K}^n$ we further have

$$\sum_{j,k} \overline{c_j} c_k K(x_j,x_k) = \sum_{j,k} \overline{c_j} c_k \langle K_{x_k}, K_{x_j}\rangle = \Big\| \sum_k c_k K_{x_k} \Big\|^2 \geq 0.$$

This proves (1).

(2) If $f \in \mathcal{H}$ is orthogonal to each K_x, then $f(x) = 0$ for each $x \in X$ implies $f = 0$. Therefore $\{K_x \colon x \in X\}$ spans a dense subspace.

(3) If $(e_j)_{j\in J}$ is an ONB of $\mathcal{H}$, then we have for each $y \in X$ the relation

$$K_y = \sum_{j\in J} \langle K_y, e_j\rangle e_j = \sum_{j\in J} \overline{e_j(y)} e_j,$$

and therefore

$$K(x,y) = K_y(x) = \sum_{j\in J} \overline{e_j(y)} e_j(x).$$

(b) We want to put

$$\Big\langle \sum_j c_j K_{x_j}, \sum_k d_k K_{x_k} \Big\rangle := \sum_{j,k} c_j \overline{d_k} K(x_k, x_j), \tag{4.3}$$

so that we have to show that this is well-defined.

So let $f = \sum_j c_j K_{x_j}$ and $h = \sum_k d_k K_{x_k} \in \mathcal{H}_K^0$. Then we obtain for the right hand side

$$\sum_{j,k} c_j \overline{d_k} K(x_k, x_j) = \sum_{j,k} c_j \overline{d_k} K_{x_j}(x_k) = \sum_k \overline{d_k} f(x_k). \tag{4.4}$$

This expression does not depend on the representation of f as a linear combination of the K_{x_j}. Similarly, we see that the right hand side does not depend on the representation of h as a linear combination of the K_{x_k}. Therefore

$$\langle f, h \rangle := \sum_{j,k} c_j \overline{d_k} K(x_k, x_j)$$

is well-defined. Since K is positive definite, we thus obtain a positive semidefinite hermitian form on $\mathcal{H}_K^0$. From (4.4) we obtain for $h = K_x$ the relation

$$\langle f, K_x \rangle = f(x) \quad \text{for} \quad x \in X, f \in \mathcal{H}_K^0.$$

If $\langle f, f \rangle = 0$, then the Cauchy–Schwarz inequality yields

$$|f(x)|^2 = |\langle f, K_x \rangle|^2 \leq K(x,x) \langle f, f \rangle = 0,$$

so that $f = 0$. Therefore $\mathcal{H}_K^0$ is a pre-Hilbert space.

Now let $\mathcal{H}_K$ be the completion of $\mathcal{H}_K^0$. Then

$$\iota \colon \mathcal{H}_K \to \mathbb{K}^X, \quad \iota(v)(x) := \langle v, K_x \rangle$$

is an injective linear map because the set $\{K_x \colon x \in X\}$ is total in $\mathcal{H}_K^0$, hence also in $\mathcal{H}_K$. The subspace $\mathcal{H}_K \cong \iota(\mathcal{H}_K) \subseteq \mathbb{K}^X$ is a Hilbert space with continuous point evaluations and reproducing kernel K.

(c) follows (a) and (b). □

Lemma 4.1.4. (Uniqueness Lemma for Reproducing Kernel Spaces) *If $\mathcal{H} \subseteq \mathbb{K}^X$ is a Hilbert space with continuous point evaluations and reproducing kernel K, then $\mathcal{H} = \mathcal{H}_K$.*

Proof. Since K is the reproducing kernel of $\mathcal{H}$, it contains the subspace $\mathcal{H}_K^0 := \operatorname{span}\{K_x \colon x \in X\}$ of $\mathcal{H}_K$, and the inclusion $\eta \colon \mathcal{H}_K^0 \to \mathcal{H}$ is isometric because the scalar products coincide on the pairs (K_x, K_y). Now η extends to an isometric embedding $\widehat{\eta} \colon \mathcal{H}_K \to \mathcal{H}$, and since $\mathcal{H}_K^0$ is also dense in $\mathcal{H}$, we see that $\widehat{\eta}$ is surjective. For $f \in \mathcal{H}_K$ we now have

$$\widehat{\eta}(f)(x) = \langle \widehat{\eta}(f), K_x \rangle_{\mathcal{H}} = \langle f, K_x \rangle_{\mathcal{H}_K} = f(x),$$

so that $\widehat{\eta}(f) = f$, and we conclude that $\mathcal{H}_K = \mathcal{H}$. □

Definition 4.1.5. The preceding lemma justifies the notation $\mathcal{H}_K$ for the unique Hilbert subspace of $\mathbb{K}^X$ with continuous point evaluations and reproducing kernel K. We call it the reproducing kernel Hilbert space defined by K.

Lemma 4.1.6. *If $\mathcal{H}_K \subseteq \mathbb{K}^X$ is a reproducing kernel space and $S \subseteq X$ a subset with*

$$K(x,x) \leq C \quad \textit{for} \quad x \in S,$$

then

$$|f(x)| \leq \sqrt{C}\|f\| \quad \textit{for} \quad x \in S, f \in \mathcal{H}_K.$$

In particular, convergence in $\mathcal{H}_K$ implies uniform convergence on S.

Proof. For $f \in \mathcal{H}_K$ and $x \in S$, we have

$$|f(x)| = |\langle f, K_x\rangle| \leq \|f\| \cdot \|K_x\| = \|f\|\sqrt{\langle K_x, K_x\rangle} = \|f\|\sqrt{K(x,x)} \leq \sqrt{C}\|f\|.$$

□

Proposition 4.1.7. *Let X be a topological space and K a positive definite kernel. Then the following assertions hold:*

(a) *The map $\gamma\colon X \to \mathcal{H}_K, \gamma(x) = K_x$ is continuous if and only if K is continuous.*

(b) *$\mathcal{H}_K \subseteq C(X)$, i.e., $\mathcal{H}_K$ consists of continuous functions.*

Proof. (a) If γ is continuous, then $K(x,y) = \langle \gamma(y), \gamma(x)\rangle$ is obviously continuous. If, conversely, K is continuous, then the continuity of γ follows from the continuity of

$$\|K_x - K_y\|^2 = K(x,x) + K(y,y) - K(x,y) - K(y,x).$$

(b) Since the scalar product is a continuous function $\mathcal{H} \times \mathcal{H} \to \mathbb{K}$, the continuity of each $f \in \mathcal{H}_K$ now follows from $f(x) = \langle f, K_x\rangle = \langle f, \gamma(x)\rangle$ and the continuity of γ. □

Exercises for Section 4.1

Exercise 4.1.1. Show that, if $A \in M_n(\mathbb{C})$ satisfies $x^*Ax \geq 0$ for every $x \in \mathbb{C}^n$, then $A^* = A$.

Exercise 4.1.2. Let X be a non-empty set and $T \subseteq X \times X$ be a subset containing the diagonal. Then the characteristic function χ_T of T is a positive definite kernel if and only if T is an equivalence relation.

Exercise 4.1.3. Show that if K is a positive definite kernel and $c > 0$, then $\mathcal{H}_{cK} = \mathcal{H}_K$ as subspaces of $\mathbb{K}^X$. Explain how their scalar products are related.

Exercise 4.1.4. Let $X = \{1, \ldots, n\}$ and $A = (a_{ij}) \in M_n(\mathbb{K})$ be positive semidefinite. We identify $\mathbb{K}^X$ canonically with $\mathbb{K}^n$. Show that, if we consider A as a positive definite kernel on X, then $\mathcal{H}_A \subseteq \mathbb{K}^n$ coincides with the column space of A. In particular

$$\dim \mathcal{H}_A = \operatorname{rank} A.$$

Exercise 4.1.5. Show that a hermitian kernel $K\colon X \times X \to \mathbb{K}$ is positive definite if and only if, for every finite sequence $x_1, \ldots, x_n \in X$, we have

$$\det(K(x_i, x_j))_{1 \leq i,j \leq n} \geq 0.$$

4.2 Basic Properties of Positive Definite Kernels

The key advantage of Hilbert spaces with continuous point evaluations is that they can be completely encoded in the function K, which is a much less complex object than an infinite dimensional Hilbert space. Before we discuss some important examples of positive definite kernels, we take a closer look at the closure properties of the set $\mathcal{P}(X)$ of all positive definite kernels under several operations.

Proposition 4.2.1. (Permanence properties of positive definite kernels) *The set $\mathcal{P}(X)$ of positive definite kernels on $X \times X$ has the following properties:*

(a) *$\mathcal{P}(X)$ is a convex cone in $\mathbb{K}^{X \times X}$, i.e., $K, Q \in \mathcal{P}(X)$ and $\lambda \in \mathbb{R}_+$ imply*

$$K + Q \in \mathcal{P}(X) \quad \text{and} \quad \lambda K \in \mathcal{P}(X).$$

(b) *The cone $\mathcal{P}(X)$ is closed under pointwise limits. In particular, if $(K_j)_{j \in J}$ is a family of positive definite kernels on X and all sums $K(x,y) := \sum_{j \in J} K_j(x,y)$ exist, then K is also positive definite.*

(c) *If μ is a positive measure on $(J, \mathfrak{S})$ and $(K_j)_{j \in J}$ is a family of positive definite kernels such that for $x, y \in X$ the functions $j \mapsto K_j(x,y)$ are measurable and the functions $j \mapsto K_j(x,x)$ are integrable, then*

$$K(x,y) := \int_J K_j(x,y)\, d\mu(j)$$

is also positive definite.

(d) (Schur) *$\mathcal{P}(X)$ is closed under pointwise multiplication: If $K, Q \in \mathcal{P}(X)$, then the kernel*

$$(KQ)(x,y) := K(x,y)Q(x,y)$$

is also positive definite.

(e) *If $K \in \mathcal{P}(X)$, then $\overline{K}$ and $\operatorname{Re} K \in \mathcal{P}(X)$.*

Proof. A hermitian kernel K is positive definite if

$$S(K) := \sum_{j,k=1}^{n} K(x_i, x_j) c_i \overline{c_j} \geq 0$$

holds for $x_1, \ldots, x_n \in X$ and $c_1, \ldots, c_n \in \mathbb{K}$.

(a) follows from $S(K+Q) = S(K) + S(Q)$ and $S(\lambda K) = \lambda S(K)$.

(b) follows from $S(K_j) \to S(K)$ if $K_j \to K$ holds pointwise on $X \times X$ and the fact that the set of positive semidefinite $(n \times n)$-matrices is closed. For $K = \sum_j K_j$ we likewise have $S(K) = \sum_{j \in J} S(K_j) \geq 0$.

(c) To see that the functions $j \mapsto K_j(x,y)$ are integrable, we first observe that the positive definiteness of the kernels K_j implies that

$$|K_j(x,y)| \leq \sqrt{K_j(x,x)}\sqrt{K_j(y,y)}$$

(Remark 4.1.2) and since the functions $j \mapsto \sqrt{K_j(x,x)}$ are square integrable by assumption, the product $\sqrt{K_j(x,x)}\sqrt{K_j(y,y)}$ is integrable. Now the assertion follows from $S(K) = \int_J S(K_j)\, d\mu(j) \geq 0$, because μ is a positive measure.

(d) We have to show that the pointwise product $C = (a_{jk}b_{jk})$ of two positive semidefinite matrices A and B is positive semidefinite.

On the $\mathbb{K}$-Hilbert space $\mathcal{H} := \mathbb{K}^n$, the operator defined by B is orthogonally diagonalizable with non-negative eigenvalues. Let $f_1, \ldots, f_n$ be an ONB of eigenvectors for B and $\lambda_1, \ldots, \lambda_n$ be the corresponding eigenvalues. Then

$$Bv = \sum_{j=1}^n \langle v, f_j \rangle B f_j = \sum_{j=1}^n \lambda_j \langle v, f_j \rangle f_j = \sum_{j=1}^n \lambda_j f_j^* v \cdot f_j = \sum_{j=1}^n \lambda_j f_j f_j^* v$$

(where we use matrix products) implies $B = \sum_j \lambda_j f_j f_j^*$, and since the λ_j are non-negative, it suffices to prove the assertion for the special case $B = vv^*$ for some $v \in \mathbb{K}^n$, i.e., $b_{jk} = v_j \overline{v_k}$. Then we obtain for $d \in \mathbb{K}^n$

$$\sum_{j,k} d_j \overline{d_k} c_{jk} = \sum_{j,k} d_j \overline{d_k} v_j \overline{v}_k a_{jk} = \sum_{j,k} (d_j v_j) \overline{d_k v_k} a_{jk} \geq 0,$$

and thus C is positive semidefinite.

(e) Since K is hermitian, we have $\overline{K}(x,y) = K(y,x)$, and this kernel is positive definite. In view of (a), this implies that $\operatorname{Re} K = \frac{1}{2}(K + \overline{K})$ is also positive definite. □

Corollary 4.2.2. *If $f(z) := \sum_{n=0}^\infty a_n z^n$ is a power series with $a_n \geq 0$ converging for $|z| < r$ and $K \in \mathcal{P}(X)$ is a positive definite kernel with $|K(x,y)| < r$ for $x, y \in X$, then the kernel*

$$(f \circ K)(x,y) := f(K(x,y)) = \sum_{n=0}^\infty a_n K(x,y)^n$$

is positive definite.

Proof. This follows from Proposition 4.2.1(b) because Proposition 4.2.1(d) implies that the kernels $K(x,y)^n$ are positive definite. □

4.3 Realization as Reproducing Kernel Spaces

At this point we know how to get new positive definite kernels from given ones, but we should also have a more effective means to recognize positive definite kernels quickly.

Remark 4.3.1. For any map $\gamma\colon X \to \mathcal{H}$ of a set X into a Hilbert space $\mathcal{H}$, the kernel $K_\gamma(x,y) := \langle \gamma(y), \gamma(x)\rangle$ is positive definite because it clearly is hermitian, and for $x_1, \ldots, x_n \in X$ and $c_1, \ldots, c_n \in \mathbb{K}$, we have

$$\sum_{i,j=1}^n c_j \overline{c_k} K_\gamma(x_j, x_k) = \sum_{i,j=1}^n c_j \overline{c_k} \langle \gamma(x_k), \gamma(x_j)\rangle = \Big\| \sum_{i=1}^n \overline{c_k}\gamma(x_k)\Big\|^2 \geq 0.$$

Definition 4.3.2. Let $\mathcal{H}$ be a Hilbert space. A triple $(X, \gamma, \mathcal{H})$ consisting of a set X and a map $\gamma\colon X \to \mathcal{H}$ is called a *realization triple* if $\gamma(X)$ spans a dense subspace of $\mathcal{H}$. Then $K(x,y) := \langle \gamma(y), \gamma(x)\rangle$ is called the corresponding positive definite kernel.

Theorem 4.3.3. (Realization Theorem) *For every positive definite kernel K on X the following assertions hold:*

(a) *If $(X, \gamma, \mathcal{H})$ is a realization triple, then the map*

$$\Phi_\gamma\colon \mathcal{H} \to \mathcal{H}_K, \quad \Phi_\gamma(v)(x) := \langle v, \gamma(x)\rangle$$

is a unitary operator with $\Phi_\gamma(\gamma(x)) = K_x$ for every $x \in X$.

(b) *The map $\gamma\colon X \to \mathcal{H}_K, x \mapsto K_x$ defines a realization triple $(X, \gamma, \mathcal{H}_K)$.*

(c) *For two realization triples $(X, \gamma_1, \mathcal{H}_1)$ and $(X, \gamma_2, \mathcal{H}_2)$ there exists a unique unitary operator $\varphi\colon \mathcal{H}_1 \to \mathcal{H}_2$ with $\varphi \circ \gamma_1 = \gamma_2$.*

Proof. (a) Clearly, Φ_γ defines a map $\mathcal{H} \to \mathbb{K}^X$, and for $y \in Y$ we have

$$\Phi_\gamma(\gamma(y))(x) = \langle \gamma(y), \gamma(x)\rangle = K(x,y) = K_y(x), \qquad \text{hence} \qquad \Phi_\gamma(\gamma(y)) = K_y.$$

We conclude that Φ_γ maps the dense subspace $\mathcal{H}^0 := \operatorname{span}\gamma(X)$ onto $\mathcal{H}_K^0$. The relation $\langle \gamma(y), \gamma(x)\rangle = K(x,y) = \langle K_y, K_x\rangle$ further implies that $\Phi_\gamma|_{\mathcal{H}^0}$ is isometric. Since its image is dense in $\mathcal{H}_K$, the map $\Phi_\gamma|_{\mathcal{H}^0}$ extends to a unitary operator $\widehat{\Phi}_\gamma\colon \mathcal{H} \to \mathcal{H}_K$, For $v \in \mathcal{H}$, we then have

$$\widehat{\Phi}_\gamma(v)(x) = \langle \widehat{\Phi}_\gamma(v), K_x\rangle = \langle v, \widehat{\Phi}_\gamma^*(K_x)\rangle = \langle v, \widehat{\Phi}_\gamma^{-1}(K_x)\rangle = \langle v, \gamma(x)\rangle,$$

so that $\widehat{\Phi}_\gamma = \Phi_\gamma$.

(b) follows from the density of $\mathcal{H}_K^0$ in $\mathcal{H}_K$ and the relation $\langle K_y, K_x\rangle = K(x,y)$ for $x, y \in X$.

(c) Let $\Phi_j\colon \mathcal{H}_j \to \mathcal{H}_K$, $j = 1, 2$, be the two unitary operators obtained from (a) which satisfy $\Phi_j(\gamma_j(x)) = K_x$ for $x \in X$. Then $U := \Phi_2^{-1} \circ \Phi_1\colon \mathcal{H}_1 \to \mathcal{H}_2$ is unitary with $U \circ \gamma_1 = \gamma_2$. ☐

Definition 4.3.4. (a) For a positive definite kernel $K\colon X \times X \to \mathbb{C}$, the realization triple $(X, \gamma, \mathcal{H}_K)$ with $\gamma(x) = K_x$, used in the previous proof, is called the *canonical realization triple*.

(b) For a realization triple $(X, \gamma, \mathcal{H})$, the corresponding unitary map

$$\Phi_\gamma\colon \mathcal{H} \to \mathcal{H}_K \quad \text{with} \quad \Phi_\gamma(\gamma(x)) = K_x \quad \text{for} \quad x \in X,$$

is called a *realization of $\mathcal{H}$ as a reproducing kernel space.*

Examples 4.3.5. (a) If $\mathcal{H}$ is a Hilbert space, then the kernel

$$K(x, y) := \langle y, x\rangle$$

on $\mathcal{H}$ is positive definite (Remark 4.3.1). A corresponding realization is given by the map $\gamma = \mathrm{id}_{\mathcal{H}}$. In particular, $\mathcal{H} \cong \mathcal{H}_K \subseteq \mathbb{K}^{\mathcal{H}}$.

(b) The kernel $K(x, y) := \langle x, y\rangle = \overline{\langle y, x\rangle}$ is also positive definite (Proposition 4.2.1(e)). To identify the corresponding Hilbert space, we consider the dual space $\mathcal{H}'$ of continuous linear functionals on $\mathcal{H}$. According to the Fréchet–Riesz Theorem, every element of $\mathcal{H}'$ has the form $\gamma_v(x) := \langle x, v\rangle$ for a uniquely determined $v \in \mathcal{H}$, and the map

$$\gamma\colon \mathcal{H} \to \mathcal{H}', \quad v \mapsto \gamma_v$$

is an antilinear isometry. In particular, $\mathcal{H}'$ also is a Hilbert space, and the scalar product on $\mathcal{H}'$ (which is determined uniquely by the norm via polarization) is given by

$$\langle \gamma_y, \gamma_x\rangle := \langle x, y\rangle = K(x, y).$$

Therefore $\gamma\colon \mathcal{H} \to \mathcal{H}'$ yields a realization of the kernel K, which leads to $\mathcal{H}_K \cong \mathcal{H}'$.

(c) If $(e_j)_{j\in J}$ is an orthonormal basis in $\mathcal{H}$, then the map

$$\gamma\colon J \to \mathcal{H}, \quad j \mapsto e_j$$

has total range, and $K(i, j) := \delta_{ij} = \langle e_i, e_j\rangle$ is the corresponding positive definite kernel on J. The element $v \in \mathcal{H}$ then corresponds to the function

$$\Phi_\gamma(v)\colon J \to \mathbb{K}, \quad j \mapsto \langle v, e_j\rangle$$

of its coefficients in the expansion $v = \sum_{j\in J} \langle v, e_j\rangle e_j$, and the map

$$\Phi_\gamma\colon \mathcal{H} \to \ell^2(J, \mathbb{K}), \quad v \mapsto (\langle v, e_j\rangle)_{j\in J}$$

is an isomorphism of Hilbert spaces. We conclude that $\mathcal{H}_K \cong \ell^2(J, \mathbb{K}) \subseteq \mathbb{K}^J$ is the corresponding reproducing kernel space.

(d) Let $(X, \mathfrak{S}, \mu)$ be a measure space, $\mathfrak{S}_{\mathrm{fin}} := \{E \in \mathfrak{S}\colon \mu(E) < \infty\}$ and $\mathcal{H} = L^2(X, \mu)$. Then the map

$$\gamma\colon \mathfrak{S}_{\mathrm{fin}} \to L^2(X, \mu), \quad E \mapsto \chi_E$$

has total range because the step functions form a dense subspace of $L^2(X,\mu)$. We thus obtain a realization

$$\Phi_\gamma \colon L^2(X,\mu) \to \mathcal{H}_K \subseteq \mathbb{C}^{\mathfrak{S}_{\rm fin}}, \quad \Phi_\gamma(f)(E) = \langle f, \chi_E\rangle = \int_E f\,d\mu,$$

of $L^2(X,\mu)$ as a reproducing kernel space on $\mathfrak{S}_{\rm fin}$ whose kernel is given by

$$K(E,F) = \langle \chi_F, \chi_E\rangle = \mu(E\cap F).$$

(e) If $\mathcal{H}$ is a complex Hilbert space, then the kernel $K(z,w) := e^{\langle z,w\rangle}$ is also positive definite (Corollary 4.2.2). The corresponding Hilbert space $\mathcal{H}_K \subseteq \mathbb{C}^{\mathcal{H}}$ is called the *(symmetric) Fock space $\mathcal{F}(\mathcal{H})$ of $\mathcal{H}$.* As we shall see below, it plays an important role in representations theory, and in particular in Quantum Field Theory (cf. Section 4.5).

We also note that the same argument shows that for each $\lambda \geq 0$, the kernel $e^{\lambda\langle z,w\rangle}$ is positive definite.

(f) Let $\mathcal{H}$ be a Hilbert space and $\mathcal{D} := \{z \in \mathcal{H} \colon \|z\| < 1\}$ be the open unit ball. For each $s \geq 0$, we find with Corollary 4.2.2 that the kernel

$$\begin{aligned} K(z,w) := (1-\langle z,w\rangle)^{-s} &= \sum_{n=0}^\infty \binom{-s}{n}(-1)^n\langle z,w\rangle^n \\ &= \sum_{n=0}^\infty \frac{(-s)(-s-1)\cdots(-s-n+1)}{n!}(-1)^n\langle z,w\rangle^n \\ &= \sum_{n=0}^\infty \frac{s(s+1)\cdots(s+n-1)}{n!}\langle z,w\rangle^n \end{aligned}$$

is positive definite.

We shall see below how these kernels can be used to obtain interesting unitary representations of various Lie groups.

Exercises for Section 4.3

Exercise 4.3.1. Let $\mathcal{H}_K \subseteq \mathbb{K}^X$ be a reproducing kernel Hilbert space and $\mathcal{H}_K = \widehat{\bigoplus}_{j\in J}\mathcal{H}_j$ be a direct Hilbert space sum. Show that there exist positive definite kernels $K^j \in \mathcal{P}(X)$ with $K = \sum_{j\in J} K^j$ and $\mathcal{H}_j = \mathcal{H}_{K^j}$ for $j \in J$. Hint: Consider $\mathcal{H}_j$ as a Hilbert space with continuous point evaluations and let K^j be its reproducing kernel.

Exercise 4.3.2. Let $X = [a,b]$ be a compact interval in $\mathbb{R}$ and $K \colon [a,b]^2 \to \mathbb{C}$ be a continuous function. Then K is positive definite if and only if

$$\int_a^b\int_a^b c(x)\overline{c(y)}K(x,y)\,dx\,dy \geq 0 \quad \text{ for each } \quad c \in C([a,b],\mathbb{C}).$$

Exercise 4.3.3. Show that for $a \in \mathbb{C}$ with $\operatorname{Re} a > 0$ and $z \in \mathbb{C}$, the following integral exists and verify the formula:

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{xz - \frac{ax^2}{2}}\,dx = \frac{1}{\sqrt{a}}e^{\frac{z^2}{2a}},$$

where $\sqrt{a}$ refers to the canonical branch of the square root on the right half plane with $\sqrt{1} = 1$. Hint: Assume first that $a, z \in \mathbb{R}$. Then use a dominated convergence argument to verify that the integral depends holomorphically on z and a.

Exercise 4.3.4. Fix $a > 0$ and define $\gamma\colon \mathbb{C} \to L^2(\mathbb{R}, dx)$ by $\gamma(z)(x) := e^{xz - \frac{ax^2}{2}}$. Show that

$$\langle \gamma(z), \gamma(w)\rangle = \sqrt{\frac{\pi}{a}} e^{\frac{(z+\overline{w})^2}{4a}}$$

and that $\gamma(\mathbb{C})$ is total in $L^2(\mathbb{R})$. Use this to derive an isomorphism Φ_γ of $L^2(\mathbb{R}, dx)$ with a reproducing kernel space of holomorphic functions on $\mathbb{C}$.

Exercise 4.3.5. Define

$$\gamma\colon \mathbb{C}_+ := \{z \in \mathbb{C}\colon \operatorname{Im} z > 0\} \to L^2(\mathbb{R}_+, dx), \quad \gamma(z)(x) = e^{-zx}.$$

Show that

$$\langle \gamma(z), \gamma(w)\rangle = \frac{1}{z + \overline{w}} =: K(z, w)$$

and that $\gamma(\mathbb{C}_+)$ is total in $L^2(\mathbb{R})$. Use this to derive an isomorphism Φ_γ of $L^2(\mathbb{R}_+, dx)$ with the reproducing kernel space $\mathcal{H}_K$ of holomorphic functions on $\mathbb{C}_+$. This is the *Hardy space of the upper half plane*. Hint: The totality of $\gamma(\mathbb{C}_+)$ follows easily once we have that $\gamma(\mathbb{C}_+)'' = L^\infty(\mathbb{R}_+)$ (cf. Corollary 3.1.13).

Exercise 4.3.6. Let $(X, \mathfrak{S}, \mu)$ be a probability space. Show that on $X = \mathfrak{S}$, the kernel

$$K(E, F) := \mu(E \cap F) - \mu(E)\mu(F)$$

is positive definite. Hint: Consider the hyperplane $\{f \in L^2(X, \mu)\colon \int_X f\, d\mu = 0\}$.

Exercise 4.3.7. Show that on $X := [0, 1]$, the kernel $K(x, y) := \min(x, y) - xy$ is positive definite.

Exercise 4.3.8. On the interval $[0, 1] \subseteq \mathbb{R}$, we consider $\mathcal{H} = L^2([0, 1], dx)$ and the map

$$\gamma\colon [0, 1] \to \mathcal{H}, \quad \gamma(x) := \chi_{[0,x]}.$$

Show that:

(a) $K(x, y) := \langle \gamma(y), \gamma(x)\rangle = \min(x, y)$.

(b) $\operatorname{im}(\gamma)$ is total in $\mathcal{H}$. Hint: The subspace spanned by $\operatorname{im}(\gamma)$ contains all Riemannian step functions (those corresponding to finite partitions of $[0, 1]$ into subintervals). From this one derives that its closure contains all continuous functions and then use the density of continuous functions in L^2.

(c) The reproducing kernel space $\mathcal{H}_K$ consists of continuous functions and we have a unitary map

$$\Phi_\gamma\colon L^2([0, 1]) \to \mathcal{H}_K, \quad \Phi_\gamma(f)(x) := \int_0^x f(t)\, dt.$$

The space $\mathcal{H}_K$ is also denoted $H^1_*([0, 1])$. It is the Sobolev space of all continuous functions on $[0, 1]$, vanishing in 0 whose derivatives are L^2-functions.

Exercise 4.3.9. Show that on $X :=]0, \infty[$ the kernel $K(x, y) := \frac{1}{x+y}$ is positive definite. Hint: Consider the elements $e_\lambda(x) := e^{-\lambda x}$ in $L^2(\mathbb{R}_+, dx)$.

Exercise 4.3.10. Let X be a topological space and $Y \subseteq X$ be a dense subspace. Show that, if $K: X \times X \to \mathbb{C}$ is a continuous positive definite kernel, then the restriction map

$$r: \mathcal{H}_K \to \mathbb{C}^Y, \quad f \mapsto f|_Y$$

induces a unitary isomorphism onto the reproducing kernel space $\mathcal{H}_Q$ with $Q := K|_{Y \times Y}$.

Exercise 4.3.11. Let X be a set and $K \in \mathcal{P}(X, \mathbb{C})$ be a positive definite kernel. Show that

(a) $\mathcal{H}_{\overline{K}} = \overline{\mathcal{H}_K}$ and that the map $\sigma: \mathcal{H}_K \to \mathcal{H}_{\overline{K}}, f \mapsto \overline{f}$ is anti-unitary.

(b) The map $\sigma(f) = \overline{f}$ preserves $\overline{\mathcal{H}_K} = \mathcal{H}_K$ and acts isometrically on this space if and only if K is real-valued.

Exercise 4.3.12. Show that every Hilbert space $\mathcal{H}$ can be realized as a closed subspace of some space $L^2(X, \mathfrak{S}, \mu)$, where $(X, \mathfrak{S}, \mu)$ is a probability space. Hint: Let $(e_j)_{j \in J}$ be an ONB and consider the Gaussian measure space $(X, \mathfrak{S}, \mu) = (\mathbb{R}^J, \mathfrak{B}(\mathbb{R})^{\otimes J}, \gamma^J)$. Now map e_j to the coordinate function x_j.

Exercise 4.3.13. (Kolmogoroff's Theorem) Let $K: X \times X \to \mathbb{R}$ be a positive definite kernel. Show that:

(a) There exists a probability space $(\Omega, \mathfrak{S}, \mu)$ and real-valued random variables F_x, $x \in X$, such that the common distribution of every finite subset $F_{x_1}, \ldots, F_{x_n}$ is Gaussian with expectation value 0, and the covariance satisfies

$$\int_\Omega F_x F_y \, d\mu = K(x, y) \quad \text{for} \quad x, y \in X.$$

(b) Consider the measurable map

$$\Psi: \Omega \to \mathbb{R}^X, \quad \Psi(q)(x) := F_x(q)$$

and show that the measure $\nu := \Psi_* \mu$ on $\mathbb{R}^X$ satisfies (a) with respect to the coordinate functions on $\mathbb{R}^X$.

4.4 Representations on Reproducing Kernel Spaces

Next we explain how group actions on a space X lead to unitary representations on reproducing kernel spaces on X and discuss a variety of examples in Section 4.5 below. A key advantage of this general setup is that it specializes to many interesting settings. In Section 4.6 below, we shall see in particular how cyclic continuous unitary representations are encoded in positive definite functions (GNS Theorem).

Remark 4.4.1. (a) For $K \in \mathcal{P}(X)$ and $f: X \to \mathbb{C}$, the kernel

$$Q(x, y) := f(x) K(x, y) \overline{f(y)}$$

is also positive definite. In fact, it is the product of K with the kernel $f(x)\overline{f(y)}$ whose positive definiteness follows from Remark 4.3.1, applied to the function

$$\gamma = f\colon X \to \mathbb{C} = \mathcal{H}$$

and Proposition 4.2.1(d).

(b) If $K \in \mathcal{P}(X)$ and $\varphi\colon Y \to X$ is a function, then the kernel

$$\varphi^*K\colon Y \times Y \to \mathbb{C}, \quad (x, y) \mapsto K(\varphi(x), \varphi(y))$$

is also positive definite. This is a direct consequence of the definitions.

Lemma 4.4.2. *For $\varphi \in S_X$, the group of bijections of X and the function $\theta\colon X \to \mathbb{K}^\times$, we consider the linear operator*

$$(Uf)(x) := \theta(x)f(\varphi^{-1}(x))$$

on $\mathbb{K}^X$. Let $K \in \mathcal{P}(X, \mathbb{K})$ be a positive definite kernel. Then the corresponding reproducing kernel Hilbert space $\mathcal{H}_K$ is invariant under U and $U_K := U|_{\mathcal{H}_K}$ is a unitary operator if and only if K satisfies the invariance condition

$$K(\varphi(x), \varphi(y)) = \theta(\varphi(x))K(x, y)\overline{\theta(\varphi(y))} \quad \text{for} \quad x, y \in X, \tag{4.5}$$

which is equivalent to

$$UK_x = \overline{\theta(\varphi(x))}^{-1} K_{\varphi(x)} \quad \text{for} \quad x \in X. \tag{4.6}$$

Proof. First we observe that the condition

$$(UK_x)(y) = \theta(y)K(\varphi^{-1}(y), x) = \overline{\theta(\varphi(x))}^{-1} K(y, \varphi(x))$$

for all $x, y \in X$ holds if and only if

$$\theta(\varphi(y))K(y, x) = \overline{\theta(\varphi(x))}^{-1} K(\varphi(y), \varphi(x)) \quad \text{for} \quad x, y \in X.$$

This is equivalent to (4.5).

Suppose first that $\mathcal{H}_K$ is U-invariant and that we thus obtain a unitary operator U_K on $\mathcal{H}_K$. For $f \in \mathcal{H}_K$ and $x \in X$ we then have

$$\theta(x)\langle f, K_{\varphi^{-1}(x)}\rangle = \theta(x)f(\varphi^{-1}(x)) = (U_K f)(x) = \langle U_K f, K_x\rangle = \langle f, U^{-1}K_x\rangle,$$

which leads to

$$U^{-1}K_x = \overline{\theta(x)}K_{\varphi^{-1}(x)}.$$

Replacing x by $\varphi(x)$, we obtain (4.6).

Suppose, conversely, that (4.6) holds. Then U preserves the pre-Hilbert space $\mathcal{H}_K$ and

$$\langle UK_x, UK_y\rangle = \overline{\theta(\varphi(x))}^{-1}\theta(\varphi(y))^{-1}\langle K_{\varphi(x)}, K_{\varphi(y)}\rangle = K(y, x) = \langle K_x, K_y\rangle.$$

Therefore $U|_{\mathcal{H}_K^0}$ is unitary, hence extends to a unitary operator U_K on $\mathcal{H}_K$. For $f \in \mathcal{H}_K$ we then have

$$(U_K f)(x) = \langle U_K f, K_x\rangle = \langle f, U^{-1}K_x\rangle = \theta(x)\langle f, K_{\varphi^{-1}(x)}\rangle = \theta(x)f(\varphi^{-1}(x)).$$

This shows that $U_K = U|_{\mathcal{H}_K}$. □

Definition 4.4.3. Let $\sigma\colon G \times X \to X, (g,x) \mapsto \sigma_g(x) = g.x$ be an action of G on the set X. We call a function $J\colon G \to (\mathbb{K}^\times)^X$ a *cocycle* if

$$J_{gh} = J_g \cdot g_* J_h \quad \text{for} \quad g,h \in G,$$

where $(g_* f)(x) = f(g^{-1}.x)$.

Clearly, the group G acts by automorphisms on the group $(\mathbb{K}^\times)^X$ via $\alpha_\varphi f := f \circ \sigma^{-1} = (\sigma_g) * f$. We may therefore form the semidirect product

$$(\mathbb{K}^\times)^X \rtimes_\alpha G \quad \text{with} \quad (f,g)(f',g') := (f \cdot (\sigma_g)_* f', gg').$$

In this context the cocycle condition is equivalent to the map

$$G \to (\mathbb{K}^\times)^X \rtimes_\alpha G, \qquad g \mapsto (J_g, g)$$

being a group homomorphism.

Remark 4.4.4. The cocycle condition implies in particular that $J_1 = J_1^2$, so that $J_1 = \mathbf{1}$. This in turn implies that

$$J_g^{-1} = g_* J_{g^{-1}}, \qquad J_g(x)^{-1} = J_{g^{-1}}(g^{-1}.x) \quad \text{for} \quad g \in G, x \in X. \tag{4.7}$$

Proposition 4.4.5. *Let $\sigma\colon G \times X \to X$ be a group action and $J\colon G \to (\mathbb{K}^\times)^X$ be a cocycle, i.e.,*

$$J_{gh} = J_g \cdot g_* J_h \quad \textit{for} \quad g,h \in G.$$

Then

$$(\pi(g)f)(x) := J_g(x) f(g^{-1}.x), \quad \pi(g)f = J_g \cdot g_* f, \tag{4.8}$$

defines a representation of G on the space $\mathbb{K}^X$ of all $\mathbb{K}$-valued functions on X.

In addition, let $K \in \mathcal{P}(X,\mathbb{K})$ be a positive definite kernel and $\mathcal{H}_K \subseteq \mathbb{K}^X$ be the corresponding reproducing kernel Hilbert space. Then $\mathcal{H}_K$ is invariant under $\pi(G)$ and $\pi_K(g) := \pi(g)|_{\mathcal{H}_K}$ defines a unitary representation of G on $\mathcal{H}_K$ if and only if K satisfies the invariance condition

$$K(g.x, g.y) = J_g(g.x) K(x,y) \overline{J_g(g.y)} \quad \textit{for} \quad g \in G, x,y \in X, \tag{4.9}$$

which is equivalent to

$$\pi(g) K_x = \overline{J_{g^{-1}}(x)} K_{g.x} \quad \textit{for} \quad g \in G, x \in X. \tag{4.10}$$

If these conditions are satisfied, we further have:

(a) *If X is a topological space, G a topological group, and σ, K is continuous, and the maps $G \to \mathbb{K}^\times, g \mapsto J_g(x)$, $x \in X$, are continuous, then the representation $(\pi_K, \mathcal{H}_K)$ of G is continuous.*

(b) *Any G-invariant closed subspace $\mathcal{K} \subseteq \mathcal{H}_K$ is a reproducing kernel space $\mathcal{H}_Q$ whose kernel Q satisfies*

$$Q(g.x, g.y) = J_g(g.x) Q(x,y) \overline{J_g(g.y)} \quad \textit{for} \quad g \in G, x,y \in X. \tag{4.11}$$

Proof. The first part follows immediately from Lemma 4.4.2 and (4.6).

(a) We apply Lemma 1.2.6 to the total subset $E := \{K_x \colon x \in X\}$. For $x, y \in X$ we have

$$\langle \pi(g)K_y, K_x\rangle = (\pi(g)K_y)(x) = \overline{J_{g^{-1}}(y)}K_{g.y}(x) = \overline{J_{g^{-1}}(y)}K(x, g.y),$$

which depends continuously on g. Therefore the representation $(\pi_K, \mathcal{H}_K)$ is continuous.

(b) Since the inclusion $\mathcal{K} \to \mathcal{H}_K$ is continuous, $\mathcal{K}$ has continuous point evaluations, hence is a reproducing kernel space $\mathcal{H}_Q$ (Lemma 4.1.4). By assumption, $\mathcal{H}_Q = \mathcal{K}$ is invariant under the unitary G-action defined by

$$(\pi_K(g)f)(x) = J_g(x)f(g^{-1}.x),$$

so that (4.11) follows from the first part of the proof. $\square$

Definition 4.4.6. If (4.9) is satisfied, the cocycle J is called a *multiplier for the kernel* K.

Remark 4.4.7. The preceding proposition applies in particular if the kernel K is G*-invariant*, i.e.,

$$K(g.x, g.y) = K(x, y) \quad \text{for} \quad g \in G, x, y \in X.$$

Then we may use the cocycle $J = 1$ and obtain a unitary representation of G on $\mathcal{H}_K$ by

$$(\pi(g)f)(x) := f(g^{-1}.x), \quad f \in \mathcal{H}_K, x \in X, g \in G.$$

Exercises for Section 4.4

Exercise 4.4.1. Let $K \colon X \times X \to \mathbb{C}$ be a positive definite kernel and $\theta \colon X \to \mathbb{C}^\times$ a function. Determine necessary and sufficient conditions on θ such that

$$\theta(x)K(x, y)\overline{\theta(y)} = K(x, y) \quad \text{for} \quad x, y \in X.$$

Hint: Consider the subset $X_1 := \{x \in X \colon K(x, x) > 0\}$ and its complement X_0 separately.

Exercise 4.4.2. Let $K, Q \in \mathcal{P}(X, \mathbb{C})$ be positive definite kernels on X and $\theta \colon X \to \mathbb{C}^\times$. Show that

$$m_\theta \colon \mathcal{H}_K \to \mathcal{H}_Q, \quad f \mapsto \theta f$$

defines a unitary map if and only if

$$Q(x, y) = \theta(x)K(x, y)\overline{\theta(y)} \quad \text{for} \quad x, y \in X.$$

Exercise 4.4.3. Let $(V, \|\cdot\|)$ be a normed space,

$$\mathbb{P}(V) := \{[v] := \mathbb{R}v \colon 0 \neq v \in V\}$$

be the space of one-dimensional subspace of V (the projective space). Show that

(a) $g.[v] := [gv]$ defines an action of $\mathrm{GL}(V)$ on $\mathbb{P}(V)$.

(b) $J \colon \mathrm{GL}(V) \times \mathbb{P}(V) \to \mathbb{R}^\times, J_g([v]) := \frac{\|g^{-1}v\|}{\|v\|}$ is a cocycle with respect to this action.

4.5 Some Examples

Let $\mathcal{H}$ be a complex Hilbert space. We consider the kernel $K(z,w) := e^{\langle z,w\rangle}$ corresponding to the Fock space $\mathcal{F}(\mathcal{H}) := \mathcal{H}_K \subseteq \mathbb{C}^{\mathcal{H}}$. Fock spaces play a central role in operator theory and mathematical physics, in particular in Quantum Field Theory (QFT). In this section we discuss several interesting unitary representations of groups on $\mathcal{F}(\mathcal{H})$.

Since our approach is based on reproducing kernels, we start with groups acting on $\mathcal{H}$, and then discuss the cocycles that are needed to make the kernel invariant under the group action.

4.5.1 The Schrödinger Representation of the Heisenberg Group

The simplest group acting on $\mathcal{H}$ is the group of translations. For $v \in \mathcal{H}$, we write $\tau_v(x) := x + v$ for the corresponding translation. We want to associate to τ_v a unitary operator on the Hilbert space $\mathcal{H}_K$. Since the kernel $K(z,w) = e^{\langle z,w\rangle}$ is not translation invariant, this requires a multiplier $\theta_v \colon \mathcal{H} \to \mathbb{C}^\times$ for the kernel K, i.e.,

$$K(z+v, w+v) = \theta_v(z+v)K(z,w)\overline{\theta_v(w+v)}, \quad z,v,w \in \mathcal{H}. \tag{4.12}$$

To find this multiplier, we observe that

$$\begin{aligned} K(z+v,w+v) &= e^{\langle z+v,w+v\rangle} = e^{\langle z,v\rangle}e^{\langle z,w\rangle}e^{\langle v,w\rangle}e^{\langle v,v\rangle} \\ &= e^{\langle z,v\rangle+\frac{1}{2}\langle v,v\rangle}K(z,w)e^{\langle v,w\rangle+\frac{1}{2}\langle v,v\rangle}. \end{aligned}$$

Therefore

$$\theta_v(z) := e^{\langle z-v,v\rangle+\frac{1}{2}\langle v,v\rangle} = e^{\langle z,v\rangle-\frac{1}{2}\langle v,v\rangle}$$

satisfies (4.12). Hence

$$(\pi(v)f)(z) := \theta_v(z)f(z-v) = e^{\langle z,v\rangle-\frac{1}{2}\langle v,v\rangle}f(z-v)$$

defines a unitary operator on $\mathcal{H}_K$ (cf. Lemma 4.4.2).

However, this assignment does not define a unitary representation $(\mathcal{H},+) \to \mathrm{U}(\mathcal{H}_K)$ because we have in the group $(\mathbb{C}^\times)^X \rtimes S_X$ the relation

$$(\theta_v, \tau_v)(\theta_w, \tau_w) = (\theta_v \cdot (\tau_v)_*\theta_w, \tau_{v+w}) \neq (\theta_{v+w}, \tau_{v+w})$$

because

$$\begin{aligned} (\theta_v \cdot (\tau_v)_*\theta_w)(z) &= e^{\langle z,v\rangle-\frac{1}{2}\langle v,v\rangle}e^{\langle z-v,w\rangle-\frac{1}{2}\langle w,w\rangle} \\ &= e^{\langle z,v+w\rangle-\frac{1}{2}\langle v+w,v+w\rangle}e^{-\langle v,w\rangle+\frac{1}{2}(\langle v,w\rangle+\langle w,v\rangle)} \\ &= \theta_{v+w}(z)e^{\frac{1}{2}(\langle w,v\rangle-\langle v,w\rangle)} = \theta_{v+w}(z)e^{-i\,\mathrm{Im}\langle v,w\rangle}. \end{aligned}$$

This leads us to the *Heisenberg group of* $\mathcal{H}$, which is given by

$$\mathrm{Heis}(\mathcal{H}) := \mathbb{R} \times \mathcal{H} \quad \text{and} \quad (t,v)(s,w) := \Big(t + s - \mathrm{Im}\langle v, w\rangle, v + w\Big).$$

It is easy to verify that this defines a group structure on $\mathbb{R} \times \mathcal{H}$ with

$$(0,v)(0,w) = \Big(- \mathrm{Im}\langle v, w\rangle, v + w\Big).$$

Proposition 4.5.1. *The group* $\mathrm{Heis}(\mathcal{H})$ *is a topological group with respect to the product topology on* $\mathbb{R} \times \mathcal{H}$*, where* $\mathcal{H}$ *is endowed with the norm topology,* $\sigma(t,v)(z) := z + v$ *defines a continuous action of* $\mathrm{Heis}(\mathcal{H})$ *on* $\mathcal{H}$ *and*

$$J_{(t,v)}(z) := e^{it}\theta_v(z) = e^{it + \langle z, v\rangle - \frac{1}{2}\langle v, v\rangle}$$

is a continuous cocycle. Further,

$$(\pi(t,v)f)(z) := e^{it}\theta_v(z)f(z - v) = e^{it + \langle z, v\rangle - \frac{1}{2}\langle v, v\rangle} f(z - v)$$

defines a continuous unitary representation of $\mathrm{Heis}(\mathcal{H})$ *on* $\mathcal{F}(\mathcal{H})$*.*

Proof. The continuity of the group operations on $\mathrm{Heis}(\mathcal{H})$ is clear and the continuity of the action on $\mathcal{H}$ is trivial.

From the preceding calculations we know that the map

$$\mathrm{Heis}(\mathcal{H}) \to (\mathbb{C}^\times)^X \rtimes S_X, \quad (t,v) \mapsto (e^{it}\theta_v, \tau_v)$$

is a homomorphism, and this implies that J is a cocycle. Its continuity is clear, and therefore Proposition 4.4.5 implies that π defines a continuous unitary representation of $\mathrm{Heis}(\mathcal{H})$ on $\mathcal{H}_K = \mathcal{F}(\mathcal{H})$. $\square$

Remark 4.5.2. (a) If $\sigma\colon \mathcal{H} \to \mathcal{H}$ is an antilinear isometric involution, then

$$\mathcal{H}^\sigma := \{v \in \mathcal{H} \colon \sigma(v) = v\}$$

is a *real form of* $\mathcal{H}$, i.e., a closed real subspace for which

$$\mathcal{H} = \mathcal{H}^\sigma \oplus i\mathcal{H}^\sigma$$

is orthogonal with respect to the real scalar product $(x,y) := \mathrm{Re}\langle x, y\rangle$. To verify this claim, we recall the relation

$$\langle z, w\rangle = \langle \sigma(w), \sigma(z)\rangle$$

from Exercise 1.1.4. For $z, w \in \mathcal{H}^\sigma$ it implies that $\langle z, w\rangle \in \mathbb{R}$, and for $z \in \mathcal{H}^\sigma, w \in i\mathcal{H}^\sigma = \mathcal{H}^{-\sigma}$ we obtain $\langle z, w\rangle \in i\mathbb{R}$, so that $(z,w) = 0$.

This observation has the interesting consequence that $\{0\} \times \mathcal{H}^\sigma$ is a subgroup of the Heisenberg group $\mathrm{Heis}(\mathcal{H})$ and that

$$(\pi(v)f)(x) = \theta_v(x)f(x - v) = e^{\langle v, x\rangle - \frac{1}{2}\langle v, v\rangle} f(x - v)$$

defines a unitary representation of $\mathcal{H}^\sigma$ on the Fock space $\mathcal{F}(\mathcal{H})$. We shall see later that $\mathcal{F}(\mathcal{H})$ is isomorphic to the reproducing kernel space on $\mathcal{H}^\sigma$ defined by the real-valued kernel $K(z,w) = e^{(z,w)}$.

(b) For a real Hilbert space $\mathcal{H}$, the situation is simpler. Then we have the relation $\theta_{v+w} = \theta_v \cdot (\tau_v)_*\theta_w$, so that

$$(\pi(v)f)(x) = \theta_v(x)f(x-v) = e^{\langle v,x\rangle - \frac{1}{2}\langle v,v\rangle} f(x-v).$$

defines a unitary representation of the additive group $(\mathcal{H},+)$ on the reproducing kernel space $\mathcal{H}_K$ with kernel $K(z,w) = e^{(z,w)}$.

4.5.2 The Fock Representation of the Unitary Group

Proposition 4.5.3. *Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{F}(\mathcal{H}) := \mathcal{H}_K \subseteq \mathbb{C}^{\mathcal{H}}$ be the Fock space on $\mathcal{H}$ with the reproducing kernel $K(z,w) = e^{\langle z,w\rangle}$. Further, let $\mathcal{F}_m(\mathcal{H}) \subseteq \mathcal{F}(\mathcal{H})$ denote the subspace of those functions in $\mathcal{F}(\mathcal{H})$ which are homogeneous of degree m, i.e., $f(\lambda z) = \lambda^m f(z)$ for $\lambda \in \mathbb{C}$, $z \in \mathcal{H}$. Then the following assertions hold:*

(i) *The action $\big(\pi(g)f\big)(v) := f(g^{-1}v)$ defines a continuous unitary representation of $\mathrm{U}(\mathcal{H})_s$ on $\mathcal{F}(\mathcal{H})$. The closed subspaces $\mathcal{F}_m(\mathcal{H})$ are invariant under this action and their reproducing kernel is given by $K^m(z,w) = \frac{1}{m!}\langle z,w\rangle^m$.*

(ii) *Let $(e_j)_{j\in J}$ be an orthonormal basis of $\mathcal{H}$. Then the functions*

$$p_{\mathbf{m}}(z) = z^{\mathbf{m}} := \prod_{j\in J} z_j^{m_j} \quad \textit{for} \quad z_j := \langle z, e_j\rangle, \mathbf{m} \in \mathbb{N}_0^{(J)},$$

form a complete orthogonal system in $\mathcal{F}(\mathcal{H})$ and $\|p_{\mathbf{m}}\|^2 = \mathbf{m}! := \prod_{j\in J} m_j!$. Here $\mathbb{N}_0^{(J)} \subseteq \mathbb{N}_0^J$ denotes the subset of finitely supported tuples.

Proof. (i) Since the action of $\mathrm{U}(\mathcal{H})_s$ on $\mathcal{H}$ given by $(g,v) \mapsto gv$ is continuous (Exercise 1.2.3), it follows from the invariance of K under this action that $\big(\pi(g)f\big)(v) = f(g^{-1}v)$ defines a continuous unitary action of $\mathrm{U}(\mathcal{H})_s$ on $\mathcal{F}(\mathcal{H})$ (Proposition 4.4.5). It is clear that the subspaces $\mathcal{F}_m(\mathcal{H})$ are invariant under this action.

Next we consider the action of the subgroup $T := \mathbb{T}\mathbf{1} \subseteq \mathrm{U}(\mathcal{H})$ on $\mathcal{F}(\mathcal{H})$. For $m \in \mathbb{Z}$, let

$$\mathcal{F}(\mathcal{H})_m := \{f \in \mathcal{F}(\mathcal{H})\colon (\forall t \in \mathbb{T})(\forall z \in \mathcal{H})\, f(tz) = t^m f(z)\}$$

be the common eigenspace corresponding to the character $t\mathbf{1} \mapsto t^{-m}$ of T (cf. Example 2.2.11). According to the discussion in Example 2.2.11 and Theorem 1.3.14, we have an orthogonal decomposition

$$\mathcal{F}(\mathcal{H}) = \widehat{\bigoplus}_{m\in\mathbb{Z}} \mathcal{F}(\mathcal{H})_m.$$

In view of Exercise 4.3.1, we have a corresponding decomposition $K = \sum_{m\in\mathbb{Z}} K^m$ of the reproducing kernel Then $K^m_x \in \mathcal{F}(\mathcal{H})_m$ is the projection of K_x to the subspace $\mathcal{F}(\mathcal{H})_m$, which leads with the discussion in Example 2.2.11 to

$$\begin{aligned} K^m_x(y) &= \frac{1}{2\pi}\int_0^{2\pi} e^{-imt} K_x(e^{it}y)\,dt = \frac{1}{2\pi}\int_0^{2\pi}\sum_{n=0}^\infty \frac{1}{n!}e^{-imt}\langle e^{it}y, x\rangle^n\,dt \\ &= \sum_{n=0}^\infty \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{n!}e^{it(n-m)}\langle y,x\rangle^n\,dt = \begin{cases} \frac{1}{m!}\langle y,x\rangle^m, & \text{for } m\in\mathbb{N}_0\\ 0, & \text{for } m<0.\end{cases}\end{aligned}$$

We conclude that $K^m(z,w) = \frac{1}{m!}\langle z,w\rangle^m$ for $m\in\mathbb{N}_0$ and that $\mathcal{F}(\mathcal{H})_m = 0$ for $m<0$. We also see that $\mathcal{F}(\mathcal{H})_m \subseteq \mathcal{F}_m(\mathcal{H})$ for all $m\in\mathbb{N}_0$ and therefore obtain $\mathcal{F}_m(\mathcal{H}) = \mathcal{F}(\mathcal{H})_m$ for each $m\in\mathbb{N}$ because the inclusion $\mathcal{F}_m(\mathcal{H})\subseteq\mathcal{F}(\mathcal{H})_m$ is trivial.

(ii) We consider the topological product group $T := \mathbb{T}^J$ and note that Tychonov's Theorem implies that this group is compact with respect to the product topology. Next we observe that the natural homomorphism

$$\alpha\colon T\to \mathrm{U}(\mathcal{H}),\quad \alpha\big((t_j)_{j\in J}\big)\sum_{j\in J} z_je_j := \sum_{j\in J} t_jz_je_j$$

(the action of T by diagonal matrices) defines a continuous unitary representation. In view of the continuity criterion Lemma 1.2.6, this follows from the fact that the functions $T\to\mathbb{C}, t=(t_j)_{j\in J}\mapsto \langle\alpha(t)e_k,e_\ell\rangle = \delta_{\ell k}t_k$ are continuous for all $k,\ell\in J$. Next we use (i) to conclude that $\pi\circ\alpha$ is a continuous unitary representation of T on $\mathcal{F}(\mathcal{H})$. Since T is compact abelian, the Fundamental Theorem on Unitary Representations of Compact Groups (Theorem 1.3.14) shows that the eigenfunctions of T form a total subset of $\mathcal{F}(\mathcal{H})$. So we have to determine these eigenfunctions.

Each continuous character $\chi\colon T\to\mathbb{T}$ is of the form $\chi_{\mathbf{m}}(z) = \prod_{j\in J} z_j^{m_j}$ for $\mathbf{m}\in\mathbb{Z}^{(J)}$, where $\mathbb{Z}^{(J)}\subseteq\mathbb{Z}^J$ denotes the subset of all functions with finite support, i.e., the free abelian group on J (Exercise 4.5.2). Accordingly, we have

$$\mathcal{F}(\mathcal{H}) = \widehat{\bigoplus}_{\mathbf{m}\in\mathbb{Z}^{(J)}}\mathcal{F}(\mathcal{H})_{\mathbf{m}},$$

where

$$\mathcal{F}(\mathcal{H})_{\mathbf{m}} = \{f\in\mathcal{F}(\mathcal{H})\colon (\forall t\in T)\pi(\alpha(t))f = f\circ\alpha(t)^{-1} = \chi_{-\mathbf{m}}(t)f\}$$

(cf. Example 2.2.11). Then we have a corresponding decomposition

$$K = \sum_{\mathbf{m}\in\mathbb{Z}^{(J)}} K^{\mathbf{m}}$$

of the reproducing kernel (Exercise 4.3.1). To determine the kernels $K^{\mathbf{m}}$, we first observe that, in view of (i), $\mathcal{F}(\mathcal{H})_{\mathbf{m}}\subseteq\mathcal{F}(\mathcal{H})_m$ holds for $\sum_{j\in J} m_j = m$.

In a similar fashion as for the circle group $\mathbb{T}$ (cf. Example 2.2.11, Exercise 2.2.6), we obtain an orthogonal projection

$$P_{\mathbf{m}}\colon \mathcal{F}(\mathcal{H})_m \to \mathcal{F}(\mathcal{H})_{\mathbf{m}}, \quad P_{\mathbf{m}}(f)(z) = \int_T \chi_{\mathbf{m}}(t) f(\alpha(t)^{-1}z)\, d\mu_T(t).$$

In particular, we obtain

$$\begin{aligned} K^{\mathbf{m}}_w(z) = P_{\mathbf{m}}(K^m_w)(z) &= \int_T \chi_{\mathbf{m}}(t) K^m_w(\alpha(t)^{-1}z)\, d\mu_T(t) \\ &= \frac{1}{m!}\int_T \chi_{\mathbf{m}}(t^{-1})\langle \alpha(t)z, w\rangle^m\, d\mu_T(t). \end{aligned}$$

To evaluate this expression, we recall the multinomial formula

$$(x_1 + \ldots + x_n)^k = \sum_{|\alpha|=k} \binom{k}{\alpha} x^\alpha, \quad x^\alpha := x_1^{\alpha_1}\cdots x_n^{\alpha_n}, \quad \binom{k}{\alpha} := \frac{k!}{\alpha_1!\cdots\alpha_n!},$$

where $|\alpha| := \alpha_1 + \cdots + \alpha_n$. We thus obtain for $z, w \in \mathcal{H}$ with the Cauchy Product Formula

$$\langle \alpha(t)z, w\rangle^m = \Big(\sum_{j\in J} t_j z_j \overline{w_j}\Big)^m = \sum_{\mathbf{m}\in\mathbb{N}_0^{(J)}, |\mathbf{m}|=m} \binom{m}{\mathbf{m}} t^{\mathbf{m}} z^{\mathbf{m}} \overline{w}^{\mathbf{m}}$$

with uniform convergence in $t \in T$. This leads to $K^{\mathbf{m}}_w(z) = 0$ for $\mathbf{m} \not\in \mathbb{N}_0^{(J)}$, and for $\mathbf{m} \in \mathbb{N}_0^{(J)}$ we get

$$K^{\mathbf{m}}_w(z) = \frac{1}{m!}\binom{m}{\mathbf{m}} z^{\mathbf{m}}\overline{w}^{\mathbf{m}} = \frac{1}{\mathbf{m}!} z^{\mathbf{m}}\overline{w}^{\mathbf{m}} = \frac{1}{\mathbf{m}!} p_{\mathbf{m}}(z)\overline{p_{\mathbf{m}}(w)}.$$

This shows that $\mathcal{F}(\mathcal{H})_{\mathbf{m}} = \mathbb{C}p_{\mathbf{m}}$ and Theorem 4.1.3(3) implies that $\|p_{\mathbf{m}}\|^2 = \mathbf{m}!$. □

Remark 4.5.4. At this point we have unitary representations of the Heisenberg group $\mathrm{Heis}(\mathcal{H})$ and the unitary group $\mathrm{U}(\mathcal{H})$ on the Fock space $\mathcal{F}(\mathcal{H})$. These two representations are compatible in the following sense.

For each $g \in \mathrm{U}(\mathcal{H})$, we obtain a topological automorphism of $\mathrm{Heis}(\mathcal{H})$ by $\alpha(g)(t, v) := (t, gv)$, and we thus obtain a homomorphism

$$\alpha\colon\ \mathrm{U}(\mathcal{H}) \to \mathrm{Aut}(\mathrm{Heis}(\mathcal{H})),$$

defining a continuous action of $\mathrm{U}(\mathcal{H})_s$ on $\mathrm{Heis}(\mathcal{H})$ (cf. Exercise 1.2.3). Therefore we obtain a topological semidirect product group

$$\mathrm{Heis}(\mathcal{H}) \rtimes_\alpha \mathrm{U}(\mathcal{H}).$$

In view of the relation

$$\begin{aligned} (\pi(g)\pi(t, v)f)(z) &= e^{it + \langle g^{-1}z, v\rangle - \frac{1}{2}\langle v, v\rangle} f(g^{-1}z - v) \\ &= e^{it + \langle z, gv\rangle - \frac{1}{2}\langle gv, gv\rangle} f(g^{-1}(z - gv)) = (\pi(t, gv)\pi(g)f)(z), \end{aligned}$$

we have

$$\pi(g)\pi(t,v)\pi(g)^{-1} = \pi(t,gv),$$

so that the representations of $\mathrm{Heis}(\mathcal{H})$ and $\mathrm{U}(\mathcal{H})$ on the Fock space combine to a continuous unitary representation $\pi(t,v,g) := \pi(t,v)\pi(g)$ of the semidirect product group.

4.5.3 Hilbert Spaces on the Unit Disc

Example 4.5.5. Let $\mathcal{D} := \{z \in \mathbb{C} \colon |z| < 1\}$ denote the unit disc, and consider for a real $m > 0$ the reproducing kernel Hilbert space $\mathcal{H}_m := \mathcal{H}_{K^m}$ with kernel

$$K^m(z,w) := (1 - z\overline{w})^{-m}$$

(cf. Example 4.3.5). Since all functions $K^m_w = K^m(\cdot, w)$ are holomorphic, the dense subspace $\mathcal{H}^0_{K^m}$ consists of holomorphic functions. Further, the function $z \mapsto K^m(z,z)$ is bounded on every compact subset of $\mathcal{D}$, so that Lemma 4.1.6 implies that convergence in $\mathcal{H}_m$ implies uniform convergence on every compact subset of $\mathcal{D}$. This shows that $\mathcal{H}_m \subseteq \mathcal{O}(\mathcal{D})$. For $m = 2$, the Hilbert space $\mathcal{H}_m$ is called the *Bergman space of* $\mathcal{D}$ and, for $m = 1$, the *Hardy space of* $\mathcal{D}$ (cf. Example 4.5.6 below).

Since the kernel $\mathcal{H}_m$ is invariant under the action of $\mathbb{T}$ by scalar multiplication,

$$K^m(tz, tw) = K^m(z,w),$$

we obtain a continuous unitary representation of $\mathbb{T}$ on $\mathcal{H}_m$, given by $(\pi(t)f)(w) := f(tw)$ (Proposition 4.4.5). From the Fundamental Theorem on Unitary Representations of Compact Groups (Theorem 1.3.14) we now derive that $\mathcal{H}_m$ is an orthogonal direct sum of the $\mathbb{T}$-eigenspaces $\mathcal{H}_{m,n}$, corresponding to the characters $\chi_n(t) := t^n$, and in Example 2.2.11 we have seen that the orthogonal projection on $\mathcal{H}_{m,n}$ is given by

$$P_n(f)(z) := \int_{\mathbb{T}} t^{-n} f(tz)\, dt,$$

where dt refers to the invariant probability measure on $\mathbb{T}$. Applying this to the functions K^m_w leads to

$$\begin{aligned} P_n(K^m_w)(z) &= \int_{\mathbb{T}} t^{-n}(1 - tz\overline{w})^{-m}\, dt = \sum_{k=0}^{\infty} \binom{-m}{k}(-1)^k \int_{\mathbb{T}} t^{k-n} z^k \overline{w}^k\, dt \\ &= \binom{-m}{n}(-1)^n z^n \overline{w}^n = \frac{m(m+1)\cdots(m+n-1)}{n!} z^n \overline{w}^n. \end{aligned}$$

We conclude that $\mathcal{H}_{m,n} = \mathbb{C}p_n(z)$ for $p_n(z) = z^n$ and $n \geq 0$, and $\mathcal{H}_{m,n} = \{0\}$ otherwise. Further, Exercise 4.3.1 implies that

$$K^{m,n}(z,w) := \frac{m(m+1)\cdots(m+n-1)}{n!} z^n \overline{w}^n$$

is the reproducing kernel of $\mathcal{H}_{m,n}$, so that

$$\|p_n\|^2 = \frac{n!}{m(m+1)\cdots(m+n-1)}.$$

Example 4.5.6. (The Hardy space) On $\mathcal{O}(\mathcal{D})$ we consider

$$\|f\|^2 := \lim_{\substack{r\to 1\\ r<1}} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^2 \, dt \in [0,\infty].$$

To evaluate this expression, let $f(z) = \sum_{n=0}^\infty a_n z^n$ denote the Taylor series of f in 0 which converges uniformly on each compact subset of $\mathcal{D}$. Hence we can interchange integration and summation and obtain

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^2 \, dt = \sum_{n,m=0}^\infty a_n\overline{a_m}\frac{1}{2\pi}\int_0^{2\pi} r^{n+m}e^{it(n-m)} \, dt = \sum_{n=0}^\infty |a_n|^2 r^{2n}.$$

Applying the Monotone Convergence Theorem to the sequences $(|a_n|^2 r^{2n})_{n\in\mathbb{N}} \in \ell^1(\mathbb{N}_0)$, we see that $(|a_n|^2)_{n\in\mathbb{N}} \in \ell^1(\mathbb{N}_0)$ if and only if $\|f\| < \infty$, and that in this case $\|f\|^2 = \sum_{n=0}^\infty |a_n|^2$. Therefore

$$\mathcal{H}_1 := \{f \in \mathcal{O}(\mathcal{D})\colon \|f\| < \infty\} \cong \ell^2(\mathbb{N}_0, \mathbb{C})$$

is a Hilbert space and the polynomials form a dense subspace of $\mathcal{H}_1$. Moreover, the monomials $p_n(z) = z^n$ form an orthonormal basis of $\mathcal{H}_1$. We put

$$K^1(z,w) := \sum_{n=0}^\infty p_n(z)\overline{p_n(w)} = \sum_{n=0}^\infty z^n\overline{w}^n = \frac{1}{1-z\overline{w}}$$

(cf. Theorem 4.1.3(a)). Then, for $w \in \mathcal{D}$, the functions $K^1_w(z) = \frac{1}{1-z\overline{w}} = \sum_{n=0}^\infty \overline{w}^n z^n$ are contained in $\mathcal{H}_1$, and for $f(z) = \sum_{n=0}^\infty a_n z^n$, we get

$$\langle f, K^1_w\rangle = \sum_{n=0}^\infty a_n\langle p_n, \overline{w}^n p_n\rangle = \sum_{n=0}^\infty a_n w^n = f(w).$$

This proves that $\mathcal{H}_1$ has continuous point evaluations and that its reproducing kernel is given by K^1 (cf. Exercise 4.5.5 above).

The space $\mathcal{H}_1$ is called the *Hardy space of* $\mathcal{D}$ and K^1 is called the *Cauchy kernel*. This is justified by the following observation. For each holomorphic function f on $\mathcal{D}$ extending continuously to the boundary, we obtain the simpler formula for the norm:

$$\|f\|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|^2 \, dt.$$

We see in particular that such a function is contained in $\mathcal{H}_1$ and thus

$$\begin{aligned} f(z) = \langle f, K^1_z\rangle &= \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\overline{K^1_z(e^{it})} \, dt = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(e^{it})}{1-ze^{-it}} \, dt \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(e^{it})}{e^{it}-z} \, e^{it} i dt = \frac{1}{2\pi i}\oint_{|\zeta|=1} \frac{f(\zeta)}{\zeta - z} \, d\zeta, \end{aligned}$$

where the latter integral denotes a complex line integral. This means that the fact that K^1 is the reproducing kernel for $\mathcal{H}_1$ is equivalent to *Cauchy's integral formula*

$$f(z) = \frac{1}{2\pi i} \oint_{|\zeta|=1} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Now we turn to the representations of the group $\mathrm{SU}_{1,1}(\mathbb{C})$ on the spaces $\mathcal{H}_m \subseteq \mathcal{O}(\mathcal{D})$. We have already seen how the spaces $\mathcal{H}_m$ decompose under the unitary representation of the group $\mathbb{T}$, but the spaces $\mathcal{H}_m$ carry for $m \in \mathbb{N}$ a unitary representation of the larger group

$$G := \mathrm{SU}_{1,1}(\mathbb{C}) := \Big\{ g = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})\colon a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1 \Big\}.$$

We claim that

$$\sigma_g(z) := g.z := (az + b)(\overline{b}z + \overline{a})^{-1}$$

defines a continuous action of G on $\mathcal{D}$ by biholomorphic maps. Note that this expression is always defined because $|z| < 1$ and $|b| < |a|$ implies that $\overline{b}z + \overline{a} \neq 0$. That $\sigma_g(z) \in \mathcal{D}$ for $z \in \mathcal{D}$ follows from

$$|az+b|^2 = |a|^2|z|^2 + (a\overline{b}z + \overline{a}b\overline{z}) + |b|^2 < |b|^2|z|^2 + (a\overline{b}z + \overline{a}b\overline{z}) + |a|^2 = |\overline{b}z + \overline{a}|^2.$$

The relations $\sigma_{\mathbf{1}}(z) = z$ and $\sigma_{gg'} = \sigma_g \sigma_{g'}$ are easily verified (see Exercise 4.5.4). To see that this action is transitive, we note that for $|z| < 1$,

$$g := \frac{1}{\sqrt{1 - |z|^2}} \begin{pmatrix} 1 & z \\ \overline{z} & 1 \end{pmatrix} \in \mathrm{SU}_{1,1}(\mathbb{C})$$

satisfies $g.0 = z$.

To obtain a unitary action of G on $\mathcal{H}_m$, we have to see how the corresponding kernel K^m transforms under the action of G. For the kernel $Q(z,w) = 1 - z\overline{w}$ an easy calculation shows that

$$\begin{aligned} Q(g.z, g.w) &= 1 - \frac{(az+b)}{(\overline{b}z + \overline{a})} \frac{(\overline{a}\overline{w} + \overline{b})}{(b\overline{w} + a)} = \frac{(\overline{b}z + \overline{a})(a + b\overline{w}) - (az+b)(\overline{a}\overline{w} + \overline{b})}{(\overline{b}z + \overline{a})(a + b\overline{w})} \\ &= \frac{(|a|^2 - |b|^2)(1 - z\overline{w})}{(\overline{b}z + \overline{a})(a + b\overline{w})} = \frac{Q(z,w)}{(\overline{b}z + \overline{a})(a + b\overline{w})}. \end{aligned}$$

Finally, we note that,

$$J_g(z) := a - \overline{b}z$$

defines a cocycle for the action of G on $\mathcal{D}$, which can be verified by direct calculation, and

$$J_g(g.z) = a - \overline{b}\frac{az+b}{\overline{b}z + \overline{a}} = \frac{a\overline{b}z + |a|^2 - \overline{b}az - |b|^2}{\overline{b}z + \overline{a}} = \frac{1}{\overline{b}z + \overline{a}},$$

so that we obtain for

$$J^m(g,z) := J_g(z)^{-m}$$

the relation

$$K^m(g.z, g.w) = J^m(g, g.z)K^m(z, w)\overline{J^m(g, g.w)},$$

and Proposition 4.4.5 show that, for $m \in \mathbb{N}$,

$$(\pi_m(g)f)(z) = J^m(g, z)f(g^{-1}.z) = (a - \overline{b}z)^{-m} f\left(\frac{\overline{a}z - b}{a - \overline{b}z}\right)$$

defines a continuous unitary representation of $G = \mathrm{SU}_{1,1}(\mathbb{C})$ on $\mathcal{H}^m$.

Remark 4.5.7. From Example 4.3.5 we recall the positive definite kernels

$$K^s(z, w) := (1 - z\overline{w})^{-s}, \quad s > 0,$$

on the open unit disc $\mathcal{D} \subseteq \mathbb{C}$. We have seen in Example 4.5.6 that, for $s \in \mathbb{N}$, we have a unitary representation of $\mathrm{SU}_{1,1}(\mathbb{C})$ on the corresponding Hilbert space. The reason for restricting to integral values of s is that otherwise we don't have a corresponding cocycle. However, for $Q(z, w) = 1 - z\overline{w}$, we have

$$Q(g.z, g.w) = \frac{Q(z, w)}{(\overline{b}z + \overline{a})(a + b\overline{w})} = \frac{Q(z, w)}{|a|^2(1 + (\overline{b}/\overline{a})z)(1 + (b/a)\overline{w})}Q(z, w),$$

and therefore

$$K^s(g.z, g.w) = \theta_g(z)K^s(z, w)\overline{\theta_g(w)}$$

for

$$\theta_g(z) := |a|^s(1 + (\overline{b}/a)z)^s,$$

where, in view of $|b| < |a|$, the right hand side can be defined by a power series converging in $\mathcal{D}$.

One can show that these considerations lead to a projective unitary representation of $\mathrm{SU}_{1,1}(\mathbb{C})$ on $\mathcal{H}_s$ by

$$(\pi_s(g)f)(z) := \theta_g(g^{-1}.z)f(g^{-1}.z).$$

Exercises for Section 4.5

Exercise 4.5.1. Let V be a real vector space and $\omega\colon V \times V \to \mathbb{R}$ be a bilinear map.

(a) Show that on $\mathbb{R} \times V$ we obtain a group structure by

$$(t, v)(s, w) := (t + s + \omega(v, w), v + w).$$

This group is called the *Heisenberg group* $\mathrm{Heis}(V, \omega)$.

More generally, we obtain for any two abelian groups V and Z and any biadditive map $\omega\colon V \times V \to Z$ a group structure on $Z \times V$ by

$$(t, v)(s, w) := (t + s + \omega(v, w), v + w).$$

(b) Let $\mathcal{H}$ be a complex Hilbert space. How do we have to choose V and ω to obtain an isomorphism $\mathrm{Heis}(V, \omega) \cong \mathrm{Heis}(\mathcal{H})$?

(c) Verify that $Z(\mathrm{Heis}(V,\omega)) = \mathbb{R} \times \mathrm{rad}(\omega_s)$, where

$$\omega_s(v,w) := \omega(v,w) - \omega(w,v) \quad \text{and} \quad \mathrm{rad}(\omega_s) := \{v \in V : \omega_s(v,V) = \{0\}\}.$$

(d) Show that for $V = \mathbb{R}^2$ with $\omega(x,y) = x_1 y_2$, the Heisenberg group $H(V,\omega)$ is isomorphic to the matrix group

$$H := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x,y,z \in \mathbb{R} \right\}.$$

Exercise 4.5.2. Let $G = \prod_{j \in J} G_j$ be a product of abelian topological groups and $p_j \colon G \to G_j$ be the projection maps. Show that the map

$$S \colon \bigoplus_{j \in J} \widehat{G_j} \to \widehat{G}, \quad (\chi_j)_{j \in J} \mapsto \prod_{j \in J} (\chi_j \circ p_j)$$

is an isomorphism of abelian groups.

Exercise 4.5.3. On $\mathbb{R}^n$ we consider the vector space $\mathcal{P}_k$ of all homogeneous polynomials of degree k:

$$p(x) = \sum_{|\alpha|=k} c_\alpha x^\alpha, \quad c_\alpha \in \mathbb{R}, x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n.$$

We associate to such a polynomial p a differential operator by

$$p(\partial) := \sum_{|\alpha|=k} c_\alpha \partial^\alpha, \quad \partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}, \quad \partial_i := \frac{\partial}{\partial x_i}.$$

Show that the *Fischer inner product*

$$\langle p, q \rangle := (p(\partial) q)(0)$$

defines on $\mathcal{P}_k$ the structure of a real Hilbert space with continuous point evaluations. Show further that its kernel is given by

$$K(x,y) = \frac{1}{k!} \langle x, y \rangle^k = \frac{1}{k!} \Big(\sum_{j=1}^n x_j y_j \Big)^k.$$

Hint: Show that the monomials $p_\alpha(x) = x^\alpha$ form an orthogonal subset with $\langle p_\alpha, p_\alpha \rangle = \alpha!$ and conclude with Theorem 4.1.3 that $K(x,y) = \sum_{|\alpha|=m} \frac{x^\alpha y^\alpha}{\alpha!}$.

Exercise 4.5.4. We consider the group $G := \mathrm{GL}_2(\mathbb{C})$ and the *complex projective line* (the *Riemann sphere*)

$$\mathbb{P}_1(\mathbb{C}) = \{[v] := \mathbb{C}v \colon 0 \neq v \in \mathbb{C}^2\}$$

of 1-dimensional linear subspaces of $\mathbb{C}^2$. We write $[x : y]$ for the line $\mathbb{C}\begin{pmatrix} x \\ y \end{pmatrix}$. Show that:

(a) The map $\mathbb{C} \to \mathbb{P}_1(\mathbb{C}), z \mapsto [z : 1]$ is injective and its complement consists of the single point $\infty := [1 : 0]$ (the horizontal line). We thus identify $\mathbb{P}_1(\mathbb{C})$ with the one-point compactification $\widehat{\mathbb{C}}$ of $\mathbb{C}$. These are the so-called *homogeneous coordinates* on $\mathbb{P}_1(\mathbb{C})$.

(b) The natural action of $\mathrm{GL}_2(\mathbb{C})$ on $\mathbb{P}_1(\mathbb{C})$ by $g.[v] := [gv]$ is given in the coordinates of (b) by

$$g.z = \sigma_g(z) := \frac{az+b}{cz+d} \quad \text{for} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(c) On $\mathbb{C}^2$ we consider the indefinite hermitian form

$$\beta(z,w) := z_1\overline{w_1} - z_2\overline{w_2} = w^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} z.$$

We define

$$\mathrm{U}_{1,1}(\mathbb{C}) := \{g \in \mathrm{GL}_2(\mathbb{C})\colon (\forall z,w \in \mathbb{C}^2)\, \beta(gz,gw) = \beta(z,w)\}.$$

Show that $g \in \mathrm{U}_{1,1}(\mathbb{C})$ is equivalent to

$$g^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Show further that the above relation is equivalent to

$$\det g \in \mathbb{T}, \quad d = \overline{a}\det g \quad \text{and} \quad c = \overline{b}\det g.$$

In particular, we obtain $|a|^2 - |b|^2 = 1$.

(d) The hermitian form β is negative definite on the subspace $[z_1 : z_2]$ if and only if $|z_1| < |z_2|$, i.e., $[z_1 : z_2] = [z : 1]$ for $|z| < 1$. Conclude that $g.z := \frac{az+b}{cz+d}$ defines an action of $\mathrm{U}_{1,1}(\mathbb{C})$ on the open unit disc $\mathcal{D}$ in $\mathbb{C}$.

4.6 Positive Definite Functions

It was one of our first observations in Section 1.3 that any unitary representation decomposes as a direct sum of cyclic ones. The main point of this section is to describe the bridge between cyclic representations and positive definite kernels on the group G which are invariant under right translations. Such kernels are determined by the function $\varphi := K_{\mathbf{1}}$ via $K(g,h) = \varphi(gh^{-1})$, and φ is said to be positive definite if K has this property. We shall see that any cyclic representation of G is equivalent to one in a reproducing kernel subspace $\mathcal{H}_\varphi \subseteq C(G)$ corresponding to a continuous positive definite function φ, and characterize the irreducible ones geometrically by the condition that φ is an extreme points in the set $\mathcal{S}(G)$ of states of G, the set of normalized positive definite functions on G.

Definition 4.6.1. (a) A function $\varphi\colon S \to \mathbb{C}$ on an involutive semigroup $(S,*)$ is called *positive definite* if the kernel

$$K_\varphi\colon S \times S \to \mathbb{C}, \quad K_\varphi(s,t) := \varphi(st^*)$$

is positive definite. We then write $\mathcal{H}_\varphi := \mathcal{H}_{K_\varphi} \subseteq \mathbb{C}^S$ for the corresponding reproducing kernel Hilbert space.

(b) If G is a group, then (G, η_G) is an involutive semigroup, so that a function $\varphi\colon G \to \mathbb{C}$ is positive definite if the kernel

$$K_\varphi\colon G \times G \to \mathbb{C}, \quad K_\varphi(s,t) := \varphi(st^{-1})$$

is positive definite.

A kernel $K\colon G \times G \to \mathbb{C}$ is called *right invariant* if

$$K(xg, yg) = K(x,y) \quad \text{holds for} \quad g,x,y \in G.$$

For any such kernel K, the function $\varphi := K_{\mathbf{1}}$ satisfies

$$K(x,y) = K(xy^{-1}, \mathbf{1}) = \varphi(xy^{-1}).$$

Conversely, for every function $\varphi\colon G \to \mathbb{C}$, the kernel $K(x,y) := \varphi(xy^{-1})$ is right invariant. Therefore the right invariant positive definite kernels on G correspond to positive definite functions.

(c) For a topological group G, we write $\mathcal{P}(G)$ for the set of continuous positive definite functions on G. The subset

$$\mathcal{S}(G) := \{\varphi \in \mathcal{P}(G)\colon \varphi(\mathbf{1}) = 1\}$$

is called the *set of states of* G.

Clearly, $\mathcal{P}(G)$ is a convex cone and $\mathcal{S}(G) \subseteq \mathcal{P}(G)$ is a convex subset with $\mathcal{P}(G) = \mathbb{R}_+\mathcal{S}(G)$. The extreme points of $\mathcal{S}(G)$ are called *pure states of* G.

Recall from Proposition 4.1.7 that $\mathcal{H}_\varphi \subseteq C(G)$ for each continuous positive definite function $\varphi \in \mathcal{P}(G)$.

(d) If $\mathcal{A}$ is an involutive algebra, then a linear functional $f\colon \mathcal{A} \to \mathbb{C}$ is said to be *positive* if it is a positive definite function on the involutive semigroup $((\mathcal{A}, \cdot), *)$, i.e., if the sesquilinear kernel

$$K(a,b) := f(ab^*)$$

is positive definite, resp., a positive semidefinite hermitian form. Clearly, this is equivalent to $f(aa^*) \geq 0$ for every $a \in \mathcal{A}$.

Remark 4.6.2. If $(\pi, \mathcal{H})$ is a unitary representation of the involutive semigroup $(S, *)$ and $v \in \mathcal{H}$, then the function

$$\pi^v\colon S \to \mathbb{C}, \quad s \mapsto \langle \pi(s)v, v\rangle$$

is positive definite because

$$K(s,t) := \pi^v(st^*) = \langle \pi(s)\pi(t^*)v, v\rangle = \langle \pi(t^*)v, \pi(s^*)v\rangle,$$

and the positive definiteness of this kernel follows from Remark 4.3.1. The corresponding realization map is

$$\gamma\colon S \to \mathcal{H}, \quad \gamma(s) = \pi(s^*)v$$

(Theorem 4.3.3). If v is a cyclic vector, then this map has total range, so that

$$\Phi_\gamma \colon \mathcal{H} \to \mathcal{H}_K, \quad \Phi_\gamma(w)(s) := \langle w, \pi(s^*)v\rangle = \langle \pi(s)w, v\rangle$$

is an isomorphism of Hilbert spaces. In view of

$$\Phi_\gamma(\pi(t)w)(s) = \langle \pi(st)w, v\rangle = \Phi_\gamma(w)(st),$$

this map intertwines the representation π of S on $\mathcal{H}$ with the representation of S on $\mathcal{H}_K$ by $(s.f)(x) := f(xs)$.

Definition 4.6.3. In the following we write a cyclic unitary representation $(\pi, \mathcal{H})$ with cyclic vector v as a triple $(\pi, \mathcal{H}, v)$.

Proposition 4.6.4. (GNS (Gelfand–Naimark–Segal) Theorem) *Let G be a topological group.*

(a) *For every continuous unitary representation $(\pi, \mathcal{H})$ of G and $v \in \mathcal{H}$,*

$$\pi^v(g) := \langle \pi(g)v, v\rangle$$

is a continuous positive definite function.

(b) *Conversely, for every continuous positive definite function $\varphi \colon G \to \mathbb{C}$, the reproducing kernel space $\mathcal{H}_\varphi \subseteq C(G, \mathbb{C})$ with the kernel $K(g, h) := \varphi(gh^{-1})$ carries a continuous unitary representation of G, given by*

$$(\pi_\varphi(g)f)(x) := f(xg),$$

satisfying

$$\varphi(g) = \langle \pi_\varphi(g)\varphi, \varphi\rangle \quad \text{for} \quad g \in G.$$

(c) *A continuous unitary representation $(\pi, \mathcal{H})$ of G is cyclic if and only if it is equivalent to some $(\pi_\varphi, \mathcal{H}_\varphi)$ with $\varphi \in \mathcal{P}(G)$.*

(d) *For two cyclic unitary representations $(\pi_j, \mathcal{H}_j, v_j)$, $j = 1, 2$ of G, there exists a unitary intertwining operator $\Gamma \colon \mathcal{H}_1 \to \mathcal{H}_2$ with $\Gamma(v_1) = v_2$ if and only if $\pi_1^{v_1} = \pi_2^{v_2}$.*

Proof. (a) follows immediately from Remark 4.6.2.

(b) We first observe that the kernel K is invariant under right multiplications:

$$K(xg, yg) = \varphi(xg(yg)^{-1}) = K(x, y), \quad x, y, g \in G,$$

so that we obtain a continuous unitary representation $(\pi_\varphi, \mathcal{H}_\varphi)$ of G (Proposition 4.4.5). Finally, we note that $K(x, g) = \varphi(xg^{-1})$ leads to $K_g = \varphi \circ \rho_{g^{-1}}$, so that

$$\langle \pi_\varphi(g)\varphi, \varphi\rangle = \langle \pi_\varphi(g)\varphi, K_{\mathbf{1}}\rangle = (\pi_\varphi(g)\varphi)(\mathbf{1}) = \varphi(g).$$

(c) To see that $(\pi_\varphi, \mathcal{H}_\varphi)$ is cyclic, we show that φ is a cyclic vector. In fact, if $f \in \mathcal{H}_\varphi$ is orthogonal to $\pi_\varphi(G)\varphi$, then we have

$$f(g) = (\pi(g)f)(\mathbf{1}) = \langle \pi(g)f, \varphi\rangle = \langle f, \pi(g)^{-1}\varphi\rangle = 0,$$

for each $g \in G$, and therefore $f = 0$.

If, conversely, $(\pi, \mathcal{H}, v)$ is a cyclic continuous unitary representation of G and $v \in \mathcal{H}$ a cyclic vector, then $\varphi := \pi^v \in \mathcal{P}(G)$ by (a), and Remark 4.6.2 implies that the map

$$\Phi_\gamma \colon \mathcal{H} \to \mathcal{H}_\varphi, \quad \Phi_\gamma(w)(g) = \langle \pi(g)w, v\rangle$$

is a unitary intertwining operator. We conclude that $(\pi, \mathcal{H}, v) \cong (\pi_\varphi, \mathcal{H}_\varphi, \varphi)$.

(d) If $\Gamma \colon \mathcal{H}_1 \to \mathcal{H}_2$ is a unitary intertwining operator with $\Gamma(v_1) = v_2$, then we have for each $g \in G$ the relation

$$\pi_2^{v_2}(g) = \langle \pi_2(g)v_2, v_2\rangle = \langle \Gamma(\pi_1(g)v_1), \Gamma(v_1)\rangle = \langle \pi_1(g)v_1, v_1\rangle = \pi_1^{v_1}(g).$$

Suppose, conversely, that $\varphi := \pi_1^{v_1} = \pi_2^{v_2}$. Then we obtain with Remark 4.6.2 unitary intertwining operators

$$\Gamma_j \colon \mathcal{H}_j \to \mathcal{H}_\varphi \quad \text{with} \quad \Gamma_j(v_j) = \varphi.$$

Then $\Gamma_2^{-1} \circ \Gamma_1 \colon \mathcal{H}_1 \to \mathcal{H}_2$ is a unitary intertwining operator mapping v_1 to v_2. □

Proposition 4.6.5. *Let $(\pi, \mathcal{H}, v)$ be a cyclic representation, where v is a unit vector. Then π is irreducible if and only if π^v is an extreme point of the convex set $\mathcal{S}(G)$, i.e., a pure state.*

Proof. (a) If $\pi = \pi_1 \oplus \pi_2$ is a proper direct sum decomposition and $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, accordingly, then the cyclicity of $v = v_1 + v_2$ implies that $v_j \neq 0$ for $j = 1, 2$. Then

$$\pi^v = \pi^{v_1} + \pi^{v_2} = \|v_1\|^2 \frac{\pi^{v_1}}{\|v_1\|^2} + \|v_2\|^2 \frac{\pi^{v_2}}{\|v_2\|^2},$$

and since $1 = \|v\|^2 = \|v_1\|^2 + \|v_2\|^2$, the function π^v is not an extreme point in $\mathcal{S}(G)$. This shows that π is irreducible if π^v is an extreme point of $\mathcal{S}(G)$.

(b) Suppose, conversely, that

$$\pi^v = \lambda_1\varphi_1 + \lambda_2\varphi_2, \quad 0 < \lambda_j, \lambda_1 + \lambda_2 = 1, \quad \varphi_j \in \mathcal{S}(G).$$

Let $(\pi_j, \mathcal{H}_j, v_j)$, $j = 1, 2$, be cyclic representations of G with $\lambda_j\varphi_j = \pi_j^{v_j}$. Then the unit vector $w := (v_1, v_2) \in \mathcal{H}_1 \oplus \mathcal{H}_2$ satisfies $\pi^w = \pi^v$, so that the cyclic representation $(\pi, \mathcal{H}, v)$ is equivalent to the cyclic subrepresentation of $\mathcal{K} := \mathcal{H}_1 \oplus \mathcal{H}_2$ generated by w (Proposition 4.6.4). We may therefore assume that $v = w$ and $\mathcal{H} \subseteq \mathcal{K}$.

If π is irreducible, then the fact that the projections $P_j \colon \mathcal{H} \to \mathcal{H}_j$ are non-zero intertwining operators implies that $0 \neq P_j^*P_j \in \mathbb{C}\mathbf{1}$ by Schur's Lemma. In view of

$$\lambda_j = \|v_j\|^2 = \|P_j w\|^2 = \langle P_j^*P_j w, w\rangle,$$

we obtain $P_j^*P_j = \lambda_j\mathbf{1}$. Then $\Phi_j := \lambda_j^{-\frac{1}{2}} P_j \colon \mathcal{H} \to \mathcal{H}_j$ is unitary and maps w to the unit vector $w_j := \lambda_j^{-\frac{1}{2}} v_j$. This implies that

$$\pi^w = \pi^{w_j} = \lambda_j^{-1}\pi^{v_j} = \varphi_j$$

for $j = 1, 2$, and therefore $\pi^w = \pi^v$ is an extreme point of $\mathcal{S}(G)$. □

Examples 4.6.6. On reproducing kernel spaces, cyclic representations arise naturally from transitive actions of G on X. So let us assume that G acts transitively on the set X and pick a point $x_0 \in X$.

For a positive definite kernel K on X we then study unitary representations of G on $\mathcal{H}_K$ of the form

$$(\pi(g)f)(x) = J_g(x)f(g^{-1}.x)$$

for which we have

$$\pi(g)K_x = \overline{J_{g^{-1}}(x)}K_{g.x}$$

(cf. Proposition 4.4.5). The latter relation implies in particular that $\pi(g)K_x \in \mathbb{C}K_{g.x}$. As $\{K_x \colon x \in X\}$ is total in $\mathcal{H}_K$, it follows that K_{x_0} is a cyclic vector for $(\pi, \mathcal{H}_K)$.

The corresponding positive definite function is given by

$$\varphi(g) := \langle \pi(g)K_{x_0}, K_{x_0}\rangle = \overline{J_{g^{-1}}(x_0)}K_{g.x_0}(x_0) = \overline{J_{g^{-1}}(x_0)}K(x_0, g.x_0).$$

We now evaluate this expression for several classes of cyclic unitary representations of this type:

(a) For $G = \mathrm{Heis}(\mathcal{H})$ and $K(x,y) = e^{\langle x,y\rangle}$ on $\mathcal{H}$ (Subsection 4.5.1), we put $x_0 := 0$. Now $J_{(t,v)}(x) = e^{it}e^{\langle x,v\rangle - \frac{1}{2}\|v\|^2}$ leads to

$$\varphi(t,v) = \overline{J_{(-t,-v)}(0)}K(0, g.0) = e^{-it}e^{-\frac{1}{2}\|v\|^2}.$$

(b) For the transitive action of $G = \mathrm{U}(\mathcal{H})$ on $X = \{x \in \mathcal{H} \colon \|x\| = 1\}$ (cf. Exercise 4.6.2), we consider the invariant kernel $K(x,y) = \langle x,y\rangle^n$ (Proposition 4.5.3). Then $J_g = 1$ for every g leads to

$$\varphi(g) = K(x_0, g.x_0) = \langle x_0, g.x_0\rangle^n = \overline{\langle x_0, g.x_0\rangle^n}.$$

If $x_0 = e_1$ is the first element in an orthonormal bases $e_1, e_2, \ldots$, then $\varphi(g) = \overline{g_{11}}^n$ is obtained from the left upper entry of the matrix representing g.

(c) For the transitive action of $G = \mathrm{SU}_{1,1}(\mathbb{C})$ on $X = \mathcal{D}$ and the kernel $K(z,w) = (1 - z\overline{w})^{-m}$, $m \in \mathbb{N}_0$ (Subsection 4.5.3), we obtain for $x_0 = 0$ and the cocycle $J^m_g(z) := (a - \overline{b}z)^{-m}$ the corresponding positive definite function

$$\varphi(g) = \overline{J^m_{g^{-1}}(0)}K(0, g.0) = \overline{a}^{-m} = \overline{g_{11}}^{\ -m}.$$

Note that this is similar to (b), but that the sign of m is different.

Exercises for Section 4.6

Exercise 4.6.1. Let $C \subseteq V$ be a convex cone in the real vector space V and $\alpha \in V^*$ with $\alpha(c) > 0$ for $0 \neq c \in C$. Show that

$$S := \{c \in C \colon \alpha(c) = 1\}$$

satisfies:

(a) $C = \mathbb{R}_+ S$.

(b) $x \in S$ is an extreme point of S if and only if $\mathbb{R}_+ x$ is an extremal ray of C.

Exercise 4.6.2. Show that for a euclidean space V, the group $\mathrm{O}(V)$ of linear surjective isometries acts transitively on the sphere

$$\mathbb{S}(V) = \{v \in V : \|v\| = 1\}.$$

Hint: For a unit vector $v \in \mathbb{S}(V)$, consider the map

$$\sigma_v(x) := x - 2\langle x, v\rangle v.$$

Show that $\sigma_v \in \mathrm{O}(V)$ and that for $x, y \in \mathbb{S}(V)$ there exists a $v \in \mathbb{S}$ with $\sigma_v(x) = y$.

Exercise 4.6.3. Let $\sigma \colon G \times X \to X, (g, x) \mapsto g.x$ be a transitive continuous action of the topological group G on the topological space X. Fix $x_0 \in X$ and let $K := \{g \in G \colon g.x_0 = x_0\}$ be the stabilizer subgroup of x_0. Show that:

(a) We obtain a continuous bijective map $\eta \colon G/K \to X, gK \mapsto g.x_0$.

(b) Suppose that η has a continuous local section, i.e., x_0 has a neighborhood U for which there exists a continuous map $\tau \colon U \to G$ with $\tau(y).x_0 = y$ for $y \in U$. Then η is open, hence a homeomorphism.

(c) Let $G := \mathbb{R}_d$ be the group $(\mathbb{R}, +)$, endowed with the discrete topology and $X := \mathbb{R}$, endowed with the canonical topology. Then $\sigma(x, y) := x + y$ defines a continuous transitive action of G on X for which the orbit map η is continuous and bijective but not open.

Chapter 5

Spectral Measures and Integrals

We have already seen in Chapter 1 that forming direct sums of Hilbert spaces and decomposing a given Hilbert space as an orthogonal direct sum of closed subspaces is an important technique in representation theory. However, this technique only leads to a complete understanding of those representations which are direct sums of irreducible ones, i.e., $\mathcal{H} = \mathcal{H}_d$ in the notation of Section 2.2. In this chapter we develop the concept of a projection valued measure, which provides a continuous analog of direct sum decompositions of Hilbert spaces. In particular, it can be used to study the structure of representations without irreducible subrepresentations. The general idea is that a representation may be composed from irreducible ones in the same way as a measure space is composed from its points, which need not have positive measure.

5.1 Spectral Measures

Definition 5.1.1. Let $\mathcal{H}$ be a Hilbert space and

$$\mathcal{P}_{\mathcal{H}} := \{P \in B(\mathcal{H}) \colon P = P^2 = P^*\}$$

be the set of all orthogonal projections on $\mathcal{H}$. Further, let $(X, \mathfrak{S})$ be a measurable space. A map $P \colon \mathfrak{S} \to \mathcal{P}_{\mathcal{H}}$ is called a *spectral measure* or a *projection valued measure* if

(SM1) $P(X) = \mathbf{1}$ and $P(\emptyset) = \mathbf{0}$,

(SM2) If $(E_j)_{j\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{S}$, then

$$P\Big(\cup_{j=1}^{\infty} E_j\Big)v = \sum_{j=1}^{\infty} P(E_j)v \quad \text{for each} \quad v \in \mathcal{H}.$$

In this sense we have

$$P\Big(\bigcup_{j=1}^{\infty} E_j\Big) = \sum_{j=1}^{\infty} P(E_j)$$

in the strong operator topology, and

(SM3) $P(E \cap F) = P(E)P(F)$ for $E, F \in \mathfrak{S}$.

Remark 5.1.2. On $P_{\mathcal{H}}$ we define an order structure by

$$P \leq Q \quad \text{if} \quad P = PQ.$$

Note that this implies that $P = P^* = Q^*P^* = QP$, so that P commutes with Q. The relation $P \leq Q$ is equivalent to $\operatorname{im}(P) \subseteq \operatorname{im}(Q)$.

For $P \leq Q$ and $v \in \mathcal{H}$, we have

$$\begin{aligned} \|Pv - Qv\|^2 &= \langle Pv, Pv\rangle - 2\operatorname{Re}\langle Pv, Qv\rangle + \langle Qv, Qv\rangle \\ &= \langle Pv, v\rangle - 2\langle Pv, v\rangle + \langle Qv, v\rangle \\ &= \langle Qv, v\rangle - \langle Pv, v\rangle. \end{aligned} \tag{5.1}$$

We conclude that, if (P_n) is a monotone sequence of projections, then $P_n \to P$ in the strong operator topology if equivalent to $\langle P_n v, v\rangle \to \langle Pv, v\rangle$ for every $v \in \mathcal{H}$.

Since $P_{\mathcal{H}} \subseteq B(\mathcal{H})$ is bounded, it even suffices that this relation holds for all elements v in a dense subspace (cf. Exercise 1.2.8).

Remark 5.1.3. (a) If $(X, \mathfrak{S})$ is a measurable space, then $\mathfrak{S}$ is an abelian involutive semigroup with respect to the operations

$$A \cdot B := A \cap B \quad \text{and} \quad A^* := A.$$

The set X is a neutral element. Condition (SM3) implies that every spectral measure is in particular a representation of this involutive semigroup $(\mathfrak{S}, *)$ with identity.

(b) The axiomatics in our definition of a spectral measure are not free of redundancy. One can show that (SM3) follows from (SM1) and (SM2) and that the requirement $P(\emptyset) = \mathbf{0}$ in (SM1) can also be omitted (cf. [Ne09]).

(c) For each spectral measure P on $(X, \mathfrak{S})$ and each $v \in \mathcal{H}$,

$$P^v(E) := \langle P(E)v, v\rangle = \|P(E)v\|^2$$

defines a positive measure on $(X, \mathfrak{S})$ with total mass $\|v\|^2$. In particular, it is a probability measure if v is a unit vector.

(d) In practice, the verification of (SM2) can be simplified as follows. Suppose that we know already that (SM2) holds for finite sums. Then $\sum_{n=1}^\infty P(E_n) = \lim_{n\to\infty} P\Big(\bigcup_{k=1}^n E_k\Big)$, so that we are dealing with an increasing sequence of projections. In view of Remark 5.1.2, it therefore suffices to show that $P^v(\bigcup_{j=1}^\infty E_j) = \sum_{j=1}^\infty P^v(E_j)$ for every v in a dense subspace.

The following lemma describes a typical situation where spectral measures arise.

Lemma 5.1.4. (a) *Let $(X, \mathfrak{S}, \mu)$ be a measure space and $\mathcal{H} := L^2(X, \mu)$. For $E \in \mathfrak{S}$ we define an operator on $\mathcal{H}$ by $P(E)f := \chi_E f$. Then P defines a projection valued measure on $\mathcal{H}$.*

If, in addition, μ is finite, then $v = 1$ is a cyclic vector for P and $P^v = \mu$.

(b) *For $v \in \mathcal{H}$, the induced spectral measure on the cyclic subspace $\mathcal{H}_v := \overline{\operatorname{span} P(\mathfrak{S})v} \subseteq \mathcal{H}$ is unitarily equivalent to the canonical spectral measure on $L^2(X, P^v)$, given by multiplication with characteristic functions.*

Proof. (a) From Lemma 3.1.3 we recall the homomorphism of C^*-algebras

$$\rho\colon L^\infty(X, \mu) \to B(L^2(X, \mu)), \quad \rho(h)f = hf.$$

For each $E \in \mathfrak{S}$, the characteristic function χ_E satisfies $\chi_E = \chi_E^* = \chi_E^2$, so that $P(E) = \rho(\chi_E) \in \mathcal{P}_{\mathcal{H}}$. We also obtain for finitely many pairwise disjoint sets $E_1, \ldots, E_n$ the relation

$$\chi_{E_1 \cup \cdots \cup E_n} = \chi_{E_1} + \cdots + \chi_{E_n},$$

so that P is finitely additive.

Clearly, $P(X) = \mathbf{1}$ and $P(\emptyset) = \mathbf{0}$. Now let $(E_j)_{j \in \mathbb{N}}$ be a disjoint sequence in X and $f \in \mathcal{H}$. We put $F_k := \bigcup_{j=1}^k E_j$ and $F := \bigcup_{j=1}^\infty E_j$. Then $\chi_{F_k} \to \chi_F$ pointwise, so that the finite additivity of P together with Lemma 3.1.3(iv) imply that $P(F_k) = M_{\chi_{F_k}} \to M_{\chi_F} = P(F)$. This proves (SM2). Finally (SM3) follows from

$$P(E \cap F) = \rho(\chi_{E \cap F}) = \rho(\chi_E \chi_F) = \rho(\chi_E)\rho(\chi_F) = P(E)P(F).$$

If μ is finite, then 1 is cyclic for $L^\infty(X, \mu)$, and since $\operatorname{span}\{\chi_E \colon E \in \mathfrak{S}\}$ is dense in $L^\infty(X, \mu)$, the function 1 is also cyclic for P.

(b) For $E, F \in \mathfrak{S}$, we have

$$\begin{aligned} \langle P(E)v, P(F)v \rangle &= \langle P(F)P(E)v, v \rangle = \langle P(E \cap F)v, v \rangle \\ &= P^v(E \cap F) = \langle \chi_E, \chi_F \rangle_{L^2(X, P^v)}. \end{aligned}$$

This means that the two maps $\gamma_1(E) := P(E)v$ and $\gamma_2(E) := \chi_E$ define two realization triples $(\mathcal{S}, \gamma_1, \mathcal{H}_v)$ and $(\mathcal{S}, \gamma_2, L^2(X, P^v))$ of the same positive definite kernel $K(E, F) = P^v(E \cap F)$. The Realization Theorem 4.3.3 therefore implies the existence of a unique unitary operator

$$\Gamma\colon \mathcal{H}_v \to L^2(X, P^v) \quad \text{with} \quad \Gamma(P(E)v) = \chi_E \quad \text{for} \quad E \in \mathfrak{S}.$$

Then

$$\Gamma \circ P(F)(P(E)v) = \Gamma(P(E \cap F)v) = \chi_{F \cap E} = \chi_F \chi_E = \chi_F \Gamma(P(E)v)$$

implies that

$$\Gamma \circ P(F) = \chi_F \cdot \Gamma \quad \text{for} \quad F \in \mathfrak{S}.$$

This means that Γ intertwines the spectral measure P on $\mathcal{H}_v$ with the canonical spectral measure on $L^2(X, P^v)$. $\square$

Lemma 5.1.5. (Direct sum of spectral measures) *Let $\mathcal{H} \cong \widehat{\oplus}_{j\in J}\mathcal{H}_j$ be a direct sum of Hilbert spaces, $(X, \mathfrak{S})$ be a measurable space and $P_j \colon \mathfrak{S} \to P_{\mathcal{H}_j}$, $j \in J$, be spectral measures. Then*

$$P(E)v := \big(P_j(E)v_j\big) \quad \textit{for} \quad v = (v_j)_{j\in J} \in \mathcal{H}$$

defines a spectral measure $P \colon \mathfrak{S} \to P_{\mathcal{H}}$.

Proof. Clearly, $P(E)^* = P(E) = P(E)^2$, so that each $P(E)$ is indeed an orthogonal projection. Further, $P(X) = \mathbf{1}$, $P(\emptyset) = \mathbf{0}$, and $P(E \cap F) = P(E)P(F)$. For any disjoint sequence $(E_n)_{n\in\mathbb{N}}$ of Borel subsets of X, we thus obtain $P(E_n)P(E_m) = 0$ for $n \neq m$. Clearly, (SM2) holds for any $v \in \mathcal{H}_j$, $j \in J$, and, in view of Remark 5.1.3(d) this implies (SM2) for P. $\square$

Remark 5.1.6. (Decomposition into cyclic spectral measures) If P is a spectral measure on $(X, \mathfrak{S})$, then we can always decompose $\mathcal{H}$ as an orthogonal direct sum $\mathcal{H} = \widehat{\oplus}_{j\in J}\mathcal{H}_j$ of P-invariant subspaces on which the corresponding spectral measure $P_j(E) := P(E)|_{\mathcal{H}_j} \in P_{\mathcal{H}_j}$ is cyclic with cyclic vector v_j (Proposition 1.3.10). In view of Lemma 5.1.4(b), we then have $\mathcal{H}_j \cong L^2(P^{v_j}, X)$ with $P_j(E)f = \chi(E)f$. In this sense every spectral measure is a direct sum of canonical spectral measures on L^2-spaces of finite measure spaces.

Conversely, Lemma 5.1.5 shows that, for any family $(\mu_j)_{j\in J}$ of finite measures on $(X, \mathfrak{S})$, we obtain on $\widehat{\bigoplus}_{j\in J} L^2(X, \mu_j)$ a canonical spectral measure.

5.2 Spectral Integrals for Measurable Functions

Proposition 5.2.1. *Let $P \colon \mathfrak{S} \to \mathcal{P}_{\mathcal{H}}$ be a spectral measure on $(X, \mathfrak{S})$. Then there exists a unique continuous linear map*

$$\widehat{P} \colon L^\infty(X, \mathfrak{S}) \to B(\mathcal{H})$$

with $\widehat{P}(\chi_E) = P(E)$ for $E \in \mathfrak{S}$. This map is called the spectral integral *and we also write*

$$\widehat{P}(f) = \int_X f(x)\, dP(x). \tag{5.2}$$

This map satisfies

(i) *$\widehat{P}(f)^* = \widehat{P}(\overline{f})$, $\widehat{P}(fg) = \widehat{P}(f)\widehat{P}(g)$ and $\|\widehat{P}(f)\| \leq \|f\|_\infty$ for $f, g \in L^\infty(X, \mathfrak{S})$. In particular, $(\widehat{P}, \mathcal{H})$ is a representation of the commutative C^*-algebra $L^\infty(X, \mathfrak{S})$ of all bounded measurable functions on $(X, \mathfrak{S})$.*

(ii) *If* (f_n) *is a bounded sequence in* $L^\infty(X,\mathfrak{S})$ *for which* $f_n \to f$ *holds pointwise on the complement of a subset* $N \in \mathfrak{S}$ *with* $P(N) = \mathbf{0}$*, then* $\widehat{P}(f_n) \to \widehat{P}(f)$ *in the strong operator topology.*

Proof. (i) Decomposing $\mathcal{H}$ as a direct sum of cyclic subspaces (Remark 5.1.6), we may w.l.o.g. assume that $\mathcal{H} = L^2(X,\mu)$ for a finite measure μ and $P(E)f = \chi_E f$ (Lemma 5.1.5). In this case (i) follows from Lemma 3.1.3.

The uniqueness of the spectral integral follows from the density of the subspace $\operatorname{span}\{\chi_E \colon E \in \mathfrak{S}\}$ of step functions in $L^\infty(X,\mathfrak{S})$.

(ii) If the spectral measure is cyclic, hence equivalent to the canonical one on some $L^2(X,P^v)$, this follows from Lemma 3.1.3(iv). Since every spectral measure is a direct sum of cyclic ones $(P_j,\mathcal{H}_j)_{j\in J}$, it follows that $\widehat{P}(f_n)v \to \widehat{P}(f)v$ on the dense subspace $\sum_{j\in J}\mathcal{H}_j$. Now the boundedness of $(P(f_n))_{n\in\mathbb{N}}$ implies that this relation holds for every $v \in \mathcal{H}$. □

Remark 5.2.2. If P is a spectral measure on $(X,\mathfrak{S})$, then we obtain for each $v \in \mathcal{H}$ a measure P^v on $(X,\mathfrak{S})$. For each measurable function $f \in L^\infty(X,\mathfrak{S})$, we then obtain the relations

$$\langle \widehat{P}(f)v, v\rangle = \int_X f(x)\, dP^v(x) \tag{5.3}$$

$$\|\widehat{P}(f)v\|^2 = \int_X |f(x)|^2\, dP^v(x). \tag{5.4}$$

between usual integrals with respect to the measure P^v and the spectral integrals with respect to P. Note that (5.3) justifies the notation (5.2).

If $f = \chi_E$ is a characteristic function, then (5.3) reproduces simply the definition of the measure P^v, which implies the first relation for step functions because both sides of (5.3) are linear in f. Since both sides define continuous linear functionals on $L^\infty(X,\mathfrak{S})$

$$\Big|\int_X f(x)\, dP^v(x)\Big| \leq \int_X |f(x)|\, dP^v(x) \leq \|f\|_\infty P^v(X) = \|f\|_\infty \|v\|^2$$

(cf. Proposition 5.2.1), and step functions form a dense subspace, both sides of (5.3) coincide on all of $L^\infty(X,\mathfrak{S})$. The second relation now follows from (5.3):

$$\|\widehat{P}(f)v\|^2 = \langle \widehat{P}(f)^*\widehat{P}(f)v, v\rangle = \langle \widehat{P}(|f|^2)v, v\rangle = \int_X |f(x)|^2\, dP^v(x).$$

Remark 5.2.3. An important but subtle point of the theory of non-discrete spectral measures is the measurement of multiplicities. For a discrete spectral measure $P \colon \mathfrak{S} = 2^X \to \mathcal{P}_\mathcal{H}$, i.e., $P(E) = \sum_{x\in X} P(\{x\})$ for $E \in \mathfrak{S}$, the multiplicity of $x \in X$ can simply be measured by $\dim \mathcal{H}_x$ for $\mathcal{H}_x := P(\{x\})\mathcal{H}$ and P leads to an orthogonal decomposition

$$\mathcal{H} = \widehat{\oplus}_{x\in X}\mathcal{H}_x.$$

For non-discrete spectral measures the situation is more complicated. One way to deal with this problem is to decompose $\mathcal{H}$ into cyclic subspaces

$$\mathcal{H}_v := \overline{\mathrm{span}\{P(E)v \colon E \in \mathfrak{S}\}} \cong L^2(X, P^v).$$

On this space the representation of the involutive semigroup $(\mathfrak{S}, \mathrm{id}_{\mathfrak{S}})$ is multiplicity free because $(P(\mathfrak{S})|_{\mathcal{H}_v})' \cong L^\infty(X, P^v)$ (Proposition 3.1.8) is commutative. Now $\mathcal{H}$ is a direct sum of such spaces $\mathcal{H}_v$, so that one may count multiplicities by comparing the measures P^v. This problem is studied systematically in Halmos' nice book [Ha57] which is still one of the best reference for these issues. [Nel69] also contains an excellent exposition of separable spectral multiplicity theory and its applications).

5.3 Existence of Spectral Measures

The main result on the existence of spectral measures is the following theorem. The first part is an immediate consequence of Lemma 3.1.3. The main point is the existence in part (ii).

We recall from Appendix A.3 that for every commutative Banach $*$-algebra $\mathcal{A}$, the set

$$\widehat{\mathcal{A}} := \mathrm{Hom}(\mathcal{A}, \mathbb{C}) \setminus \{0\}$$

of non-zero continuous $*$-homomorphisms $\chi \colon \mathcal{A} \to \mathbb{C}$, is a locally compact space with respect to the topology of pointwise convergence on $\mathcal{A}$.

Definition 5.3.1. If X is a locally compact space, then we call a Borel spectral measure P on X *regular* if all the measures P^v are regular.

Theorem 5.3.2. (Spectral Theorem for commutative Banach-$*$-algebras) *Let $\mathcal{A}$ be a commutative Banach-$*$-algebra. Then the following assertions hold:*

(i) *If $P \colon \mathfrak{B}(\widehat{\mathcal{A}}) \to P_{\mathcal{H}}$ is a regular Borel spectral measure on the locally compact space $\widehat{\mathcal{A}}$ and $\widehat{a}(\chi) := \chi(a)$ for $a \in \mathcal{A}$, $\chi \in \widehat{\mathcal{A}}$, then*

$$\pi_P(a) := \widehat{P}(\widehat{a}) = \int_{\widehat{\mathcal{A}}} \chi(a)\, dP(\chi)$$

defines a non-degenerate representation of $\mathcal{A}$ on $\mathcal{H}$.

(ii) *If $(\pi, \mathcal{H})$ is a non-degenerate representation of $\mathcal{A}$, then there exists a unique regular spectral measure P on $\widehat{\mathcal{A}}$ with $\pi = \pi_P$.*

Proof. (i) Since the Gelfand transform $\mathcal{G} \colon \mathcal{A} \to C_0(\widehat{\mathcal{A}}), a \mapsto \widehat{a}$ is a (contractive) homomorphism of Banach-$*$-algebras (Appendix A.3) and the same holds for the spectral integral $\widehat{P} \colon L^\infty(\widehat{\mathcal{A}}) \to B(\mathcal{H})$ (Proposition 5.2.1), the composition $\pi_P := \widehat{P} \circ \mathcal{G} \colon \mathcal{A} \to B(\mathcal{H})$ is a representation of the Banach-$*$-algebra $\mathcal{A}$.

To see that the representation $(\pi_P, \mathcal{H})$ is non-degenerate, we may w.l.o.g. assume that the spectral measure is cyclic (Remark 5.1.6), which implies that

$\mathcal{H} \cong L^2(\widehat{\mathcal{A}}, \mu)$ for some finite regular measure μ on $\widehat{\mathcal{A}}$ and $P(E)f = \chi_E f$ (Lemma 5.1.4). Since $\mathcal{G}(\mathcal{A})$ is dense in $C_0(\widehat{\mathcal{A}})$ by the Stone–Weierstraß Theorem (Remark A.3.2) and $C_c(\widehat{\mathcal{A}})$ is dense in $L^2(\widehat{\mathcal{A}}, \mu)$ by Proposition 3.3.2, the relation $\pi_P(a)1 = \widehat{a}$ implies that 1 is a cyclic vector for $(\pi_P, L^2(\widehat{\mathcal{A}}, \mu))$. Hence π_P is non-degenerate.

(ii)[1] First we show that we may assume that $\mathcal{A} = C_0(X)$ for some locally compact space X. So let $\mathcal{B} := \overline{\pi(\mathcal{A})}$. Then $\mathcal{B}$ is a commutative C^*-algebra and $\pi\colon \mathcal{A} \to \mathcal{B}$ is a morphism of Banach-$*$-algebras with dense range. In view of the Gelfand Representation Theorem (Theorem A.3.1), $\mathcal{B} \cong C_0(Y)$ for some locally compact space Y. For $y \in Y$ and $\delta_y(f) := f(y)$, we have $\pi^*(\delta_y) := \delta_y \circ \pi \in \widehat{\mathcal{A}}$ because $\pi^*(\delta_y) \neq 0$ follows from the fact that π has dense range. The so obtained map $\pi^*\colon Y \to \widehat{\mathcal{A}}$ is continuous because for each $a \in \mathcal{A}$, the function $y \mapsto \pi^*(\delta_y)(a) = \pi(a)(y)$ is continuous. Moreover, it extends to an injective continuous map

$$\pi^*\colon Y \cup \{0\} = \mathrm{Hom}(\mathcal{B}, \mathbb{C}) \to \widehat{\mathcal{A}} \cup \{0\} = \mathrm{Hom}(\mathcal{A}, \mathbb{C})$$

of compact spaces which therefore is a topological embedding. This implies that $\pi^*(Y) \cup \{0\}$ is a compact subset of $\widehat{\mathcal{A}} \cup \{0\}$ so that $Y \cong \pi^*(Y)$ is a closed subset of $\widehat{\mathcal{A}}$. We may therefore assume that Y is a closed subset of $\widehat{\mathcal{A}}$, so that π obtains the simple form $\pi(a) = \widehat{a}|_Y$. If P_B is a regular Borel spectral measure on Y with $\widehat{P}_B(f) = f$ for $f \in \mathcal{B} \subseteq B(\mathcal{H})$, then $P_A(E) := P_B(E \cap Y)$ is a regular Borel spectral measure on $\widehat{\mathcal{A}}$ (cf. Exercise A.1.6(d)), and for $a \in \mathcal{A}$ we have

$$\widehat{P}_A(\widehat{a}) = \widehat{P}_B(\widehat{a}|_Y) = \pi(a).$$

Replacing $\mathcal{A}$ by $\mathcal{B}$, we may thus assume that $\mathcal{A} = C_0(X)$ holds for some locally compact space X.

Uniqueness: Next we show the uniqueness of the spectral measure. Let P and Q be regular spectral measures with the desired properties. For $v \in \mathcal{H}$ we then obtain two positive measures P^v and Q^v on $\widehat{\mathcal{A}}$ with

$$\int_{\widehat{\mathcal{A}}} \widehat{a}(\chi)\, dP^v(\chi) = \langle \widehat{P}(\widehat{a})v, v\rangle = \langle \pi(a)v, v\rangle = \int_{\widehat{\mathcal{A}}} \widehat{a}(\chi)\, dQ^v(\chi)$$

for $a \in \mathcal{A}$. In view of the Riesz Representation Theorem, the regularity assumption implies $P^v = Q^v$. Since each $P(E)$ is uniquely determined by the numbers $P^v(E) = \langle P(E)v, v\rangle$, $v \in \mathcal{H}$, the uniqueness of P follows.

Existence: Now we prove the existence. To this end, we decompose the representation $(\pi, \mathcal{H})$ into cyclic representations $(\pi_j, \mathcal{H}_j)$, $j \in J$ (Proposition 1.3.10). If we have for each $j \in J$ a spectral measure P^j with values on $B(\mathcal{H}_j)$ and $\widehat{P}_j \circ \mathcal{G} = \pi_j$, then Lemma 5.1.5 implies that

$$P(E)v := \big(P_j(E)v_j\big) \quad \text{for} \quad v = (v_j)_{j \in J} \in \mathcal{H}$$

[1]This part of the proof draws heavily from the theory of commutative Banach $*$-algebras developed in Appendix A.3. Readers which are not familiar with this theory should consider it as a sketch of a proof which gives a good impression of the main ideas behind this existence result.

defines a spectral measure. We may thus assume that the representation of $\mathcal{A} = C_0(X)$ is cyclic. Let $v \in \mathcal{H}$ be a cyclic vector, so that $\pi(\mathcal{A})v$ is dense in $\mathcal{H}$. Then

$$\pi^v : C_0(X) \to \mathbb{C}, \quad f \mapsto \langle \pi(f)v, v\rangle$$

is a positive functional, and the Riesz Representation Theorem provides a unique regular Borel measure P^v on X with

$$\pi^v(f) = \int_X f(\chi)\, dP^v(\chi) \quad \text{for} \quad f \in C_0(X).$$

Next we show that $\mathcal{H} \cong L^2(X, P^v)$. To this end, we consider the map $\widetilde{\Phi} : C_0(X) \to \mathcal{H}, a \mapsto \pi(a)v$. Then

$$\begin{aligned}\langle \widetilde{\Phi}(a), \widetilde{\Phi}(b)\rangle &= \langle \pi(a)v, \pi(b)v\rangle = \langle \pi(ab^*)v, v\rangle \\ &= \pi^v(ab^*) = \int_X a(x)\overline{b(x)}\, dP^v(x) = \langle a, b\rangle_{L^2(X,P^v)}.\end{aligned}$$

Hence the map $L^2(X, P^v) \ni a \to \pi(a)v \in \mathcal{H}$ is well defined and, since $C_0(X)$ is dense in $L^2(X, P^v)$ (Proposition 3.3.2), it extends to an isometric embedding

$$\Phi : L^2(X, P^v) \to \mathcal{H},$$

which is surjective because $\pi(\mathcal{A})v$ is dense. For each $a \in \mathcal{A}$ we have

$$\langle \pi(a)v, v\rangle = \langle a, 1\rangle_{L^2(X,P^v)} = \langle \Phi(a), \Phi(1)\rangle = \langle \pi(a)v, \Phi(1)\rangle,$$

so that $\Phi(1) = v$.

Let $\rho : L^\infty(X) \to B\big(L^2(X, P^v)\big)$ denote the multiplication representation from Lemma 3.1.3. For $a, b \in \mathcal{A}$ we then have

$$\pi(a)\Phi(b) = \pi(a)\pi(b)v = \pi(ab)v = \Phi(ab) = \Phi(\rho(a)b).$$

In view of the density of $C_0(X)$ in $L^2(X, P^v)$, Φ is an intertwining operator for the representations ρ and π of $\mathcal{A}$. We may thus assume that $\mathcal{H} = L^2(X, P^v)$.

Finally, let $P(E)f = \chi_E f$ denote the spectral measure on $L^2(X, P^v)$ from Lemma 5.1.4. For $a \in \mathcal{A} \cong C_0(X)$, we now have $\widehat{P}(a) = \rho(a)$. It remains to show that, for $f \in L^2(X, P^v)$, the measures

$$E \mapsto P^f(E) = \langle P(E)f, f\rangle = \langle \chi_E f, f\rangle = \int_E |f(x)|^2\, dP^v(x)$$

are regular, but this is a consequence of the following Lemma 5.3.3. $\square$

Lemma 5.3.3. *If μ is a regular Borel measure on the locally compact space X and $f \in \mathcal{L}^2(X, \mu)$, then the finite measure $\mu_f(E) := \int_E |f(x)|^2\, d\mu(x)$ is also regular.*

Proof. Let $E \subseteq X$ be a Borel set. We have to show that E is outer regular. We may assume $\mu(E) < \infty$ because otherwise there is nothing to show. Let $\varepsilon > 0$. For $n \in \mathbb{N}$ we consider the sets $F_n := \{x \in X : |f(x)| \geq n\}$. Then $\mu_f(X \setminus F_n) \to \mu_f(X) = \|f\|_2^2$ implies that $\mu_f(F_n) \leq \varepsilon$ for $n \geq N_\varepsilon$. If $V \supseteq E$ is an open subset with $\mu(V) \leq \mu(E) + \frac{\varepsilon}{N_\varepsilon^2}$, then we obtain for $n = N_\varepsilon$:

$$\begin{aligned}\mu_f(V \setminus E) &= \mu_f\big((V \cap F_n) \setminus E\big) + \mu_f\big((V \setminus F_n) \setminus E\big) \\ &\leq \mu_f(F_n) + \mu_f\big((V \setminus F_n) \setminus E\big) \leq \varepsilon + \frac{\varepsilon}{n^2} n^2 = 2\varepsilon.\end{aligned}$$

This proves the outer regularity of E.

To see that each open subset $U \subseteq X$ is inner regular, we argue similarly. □

The proof of the preceding theorem directly implies the following:

Corollary 5.3.4. *A representation $(\pi, \mathcal{H})$ of a commutative Banach-$*$-algebra $\mathcal{A}$ is cyclic with cyclic vector v if and only if there exists a finite Radon measure μ on the locally compact space $\widehat{\mathcal{A}}$ such that $(\pi, \mathcal{H}, v)$ is unitarily equivalent to the representation $\big(\pi_\mu, L^2(\widehat{\mathcal{A}}, \mu), 1\big)$ with $\pi_\mu(a)f = \widehat{a} \cdot f$.*

Lemma 5.3.5. *Let $(\pi, \mathcal{H})$ be a non-degenerate representation of the commutative Banach-$*$-algebra $\mathcal{A}$ and $P \colon \mathfrak{S} = \mathfrak{B}(\widehat{\mathcal{A}}) \to P_{\mathcal{H}}$ the corresponding regular Borel spectral measure with $\widehat{P}(\widehat{a}) = \pi(a)$ for $a \in \mathcal{A}$. Then*

$$\pi(\mathcal{A})'' = P(\mathfrak{S})''.$$

Proof. Clearly $P(\mathfrak{S})''$ is a von Neumann algebra in $B(\mathcal{H})$, hence in particular norm closed. Since it contains all operators $P(E)$, $E \in \mathfrak{S}$, it contains the operators $\widehat{P}(f)$ for all measurable step functions $f \colon \widehat{\mathcal{A}} \to \mathbb{C}$. As these form a dense subspace of $L^\infty(\widehat{\mathcal{A}})$, we see that

$$\pi(\mathcal{A}) = \widehat{P}(\{\widehat{a} \colon a \in \mathcal{A}\}) \subseteq \widehat{P}(C_0(\widehat{\mathcal{A}})) \subseteq \widehat{P}(L^\infty(\widehat{\mathcal{A}})) \subseteq P(\mathfrak{S})''.$$

To prove the converse inclusion, we have to show that each $P(E)$ is contained in $\pi(\mathcal{A})''$, i.e., that it commutes with $\pi(\mathcal{A})'$. In view of Exercise A.3.5, the unital C^*-algebra $\pi(\mathcal{A})'$ is spanned by its unitary elements, so that it suffices to show that $P(E)$ commutes with all unitary elements $u \in \pi(\mathcal{A})'$. For any such unitary element

$$P_u(E) := uP(E)u^{-1}$$

defines a regular spectral measure with the property that, for $a \in \mathcal{A}$, we have

$$\widehat{P}_u(\widehat{a}) = u\widehat{P}(\widehat{a})u^{-1} = u\pi(a)u^{-1} = \pi(a).$$

Therefore the uniqueness of the spectral measure representing π implies that $P_u = P$, i.e., that each $P(E)$ commutes with u. □

5.4 Applications to locally compact abelian groups

Any locally compact group G carries a left invariant Radon measure μ_G, called the *Haar measure* of G. It is unique up to positive multiples. For $G = \mathbb{R}^n$, this is (any positive multiple of) Lebesgue measure and if G is discrete, the Haar measure is simply the counting measure on G. With this measure, we can define a Banach-$*$-algebra $L^1(G) := L^1(G, \mu_G)$ by the convolution product

$$(f * g)(x) := \int_G f(y)g(y^{-1}x)\, d\mu_G(y) = \int_G f(xy)g(y^{-1})\, d\mu_G(y). \tag{5.5}$$

If G is abelian, the involution on $L^1(G)$ is simply given by

$$f^*(x) := \overline{f(x^{-1})}. \tag{5.6}$$

Suppose that G is locally compact and abelian. Then each character $\chi \in \widehat{G}$ defines a homomorphism

$$\Gamma_\chi \colon L^1(G) \to \mathbb{C}, \quad \Gamma_\chi(f) := \int_G f(g)\chi(g)\, d\mu_G(g),$$

and it turns out that we thus obtain a bijection

$$\Gamma \colon \widehat{G} \to \widehat{L^1(G)}$$

(cf. Appendix A.6.6). The Spectral Theorem for commutative Banach-$*$-algebras now applies to $L^1(G)$, and this leads to a one-to-one correspondence between continuous unitary representations $(\pi, \mathcal{H})$ of G and regular spectral measures on the locally compact space $\widehat{G}$. In particular, every unitary representation of G is given in terms of such a spectral measure P as

$$\pi_P(g) = \widehat{P}(\widehat{g}).$$

Combining this discussion with Theorem A.6.25 from the appendix, resp., with Corollary 5.3.4, we obtain:

Corollary 5.4.1. (Cyclic representations of locally compact abelian groups) *A representation $(\pi, \mathcal{H})$ of an abelian locally compact group G is cyclic if and only if there exists a finite Radon measure μ on $\widehat{G}$ such that $(\pi, \mathcal{H})$ is equivalent to the cyclic representation $\big(\pi_\mu, L^2(\widehat{G}, \mu), 1\big)$, given by $\pi_\mu(g)f = \widehat{g} \cdot f$. In particular, all these representations are continuous and cyclic.*

Theorem 5.4.2. (Bochner's Theorem) *A continuous function φ on the locally compact abelian group G is positive definite if and only if there exists a finite Radon measure μ on $\widehat{G}$ with $\varphi = \widehat{\mu}$, where*

$$\widehat{\mu}(g) := \int_{\widehat{G}} \chi(g)\, d\mu(\chi)$$

is the Fourier transform of the measure μ. Then μ is uniquely determined by φ.

Proof. We have already seen in Proposition 4.6.4 that φ is positive definite if and only if $\varphi = \pi^v$ holds for a continuous cyclic unitary representation $(\pi, \mathcal{H}, v)$. In view of Corollary 5.4.1, any such representation is equivalent to a representation of the form $(\pi_\mu, \widehat{G}, 1)$, where μ is a regular Borel measure on $\widehat{G}$ and $\pi_\mu(g)f = \widehat{g}f$. Now the assertion follows from

$$\langle \pi_\mu(g)1, 1\rangle = \langle \widehat{g}, 1\rangle = \int_G \widehat{g}(\chi)\, d\mu(\chi) = \widehat{\mu}(g).$$

To see that μ is unique, we note that, for $f \in C_c(G)$, we obtain with Fubini's Theorem the relation

$$\begin{aligned}
\int_G f(g)\varphi(g)\, d\mu_G(g) &= \int_G \int_{\widehat{G}} f(g)\chi(g)\, d\mu(\chi)\, d\mu_G(g) \\
&= \int_{\widehat{G}} \int_G f(g)\chi(g)\, d\mu_G(g)\, d\mu(\chi) \\
&= \int_{\widehat{G}} \Gamma_\chi(f)\, d\mu(\chi) = \int_{\widehat{G}} \mathcal{G}(f)\, d\mu.
\end{aligned}$$

Therefore the function φ determines the μ-integral of all functions in $\mathcal{G}(C_c(G))$, hence of all functions in $C_0(\widehat{G})$, and this determines the Radon measure μ uniquely. $\square$

The following theorem shows how spectral measures lead to unitary representations of the group $(\mathbb{R}, +)$ and vice versa. It may be considered as a classification of unitary one-parameter groups in terms of spectral measures on $\mathbb{R}$ which provides important structural information.

Theorem 5.4.3. *Let $P\colon \mathfrak{B}(\mathbb{R}) \to \mathcal{P}_\mathcal{H}$ be a spectral measure. Then*

$$\pi(t) := \widehat{P}(e^{it\,\mathrm{id}_\mathbb{R}}) = \int_X e^{itx}\, dP(x).$$

defines a continuous unitary representation $\pi\colon \mathbb{R} \to \mathrm{U}(\mathcal{H})$. Conversely, every continuous unitary representation of $\mathbb{R}$ is of this form.

Proof. Since $\mathbb{R} \to \mathrm{U}(L^\infty(X, \mathbb{C})), t \mapsto e^{it\,\mathrm{id}_\mathbb{R}}$ is a homomorphism into the unitary group of the C^*-algebra $L^\infty(\mathbb{R}, \mathbb{C})$ and $t_n \to t$ implies $e^{it_n x} \to e^{itx}$ pointwise, it follows from Proposition 5.2.1 that $\pi(t) := \widehat{P}(e^{it\,\mathrm{id}_\mathbb{R}})$ defines a continuous unitary representation of $\mathbb{R}$.

If, conversely, $(\pi, \mathcal{H})$ is a continuous unitary representation of $\mathbb{R}$, then Theorem A.6.25 implies the existence of a spectral measure P on $\widehat{\mathbb{R}}$ with $\pi(t) = \widehat{P}(\widehat{t})$ for $t \in \mathbb{R}$. Identifying the locally compact character group $\widehat{\mathbb{R}}$ with $\mathbb{R}$ in such a way that $\widehat{t}(x) = e^{itx}$ (Example A.6.24), the assertion follows. $\square$

Exercises for Chapter 5

Exercise 5.4.1. Let P and Q be two commuting projections in $P_\mathcal{H}$. Show that PQ is the orthogonal projection onto the closed subspace $\mathrm{im}(P) \cap \mathrm{im}(Q)$.

Exercise 5.4.2. (One-parameter groups of $\mathrm{U}(\mathcal{H})$)

(1) Let $A = A^* \in B(\mathcal{H})$ be a bounded hermitian operator. Then $\gamma_A(t) := e^{itA}$ defines a norm-continuous unitary representation of $(\mathbb{R}, +)$.

(2) Let $P\colon (X, \mathfrak{S}) \to B(\mathcal{H})$ be a spectral measure and $f\colon X \to \mathbb{R}$ a measurable function. Then $\gamma_f(t) := \widehat{P}(e^{itf}) = \int_{\mathbb{R}} e^{itf(x)}\, dP(x)$ is a continuous unitary representation of $(\mathbb{R}, +)$. Show that γ_f is norm-continuous if and only if f is essentially bounded.

Exercise 5.4.3. Two representations $(\pi_j, \mathcal{H}_J)$, $j = 1, 2$, of an involutive semigroup $(S, *)$ are called *disjoint* if $B_S(\mathcal{H}_1, \mathcal{H}_2) = \{0\}$.

Show that a family $(\pi_j, \mathcal{H}_j)_{j\in J}$ of representations of $(S, *)$ is pairwise disjoint and multiplicity free if and only if their direct sum $\pi := \oplus_{j\in J}\pi_j$ is multiplicity free.

Exercise 5.4.4. Let $(\pi, \mathcal{H})$ be a non-degenerate representation of the involutive semigroup $(S, *)$ on the separable Hilbert space $\mathcal{H}$ which is *multiplicity free* in the sene that $\pi(S)'$ is commutative. Show that $(\pi, \mathcal{H})$ is cyclic. Hint: Write $\mathcal{H}$ as a direct sum of at most countably many cyclic representations $(\pi_j, \mathcal{H}_j, v_j)$ with cyclic unit vectors $(v_j)_{j\in J}$ and find $c_j > 0$ such that $v := \sum_{j\in J} c_j v_j$ converges in $\mathcal{H}$. Now show that v is a separating vector for $\pi(S)'$ and use Exercise 3.3.11. Note that the orthogonal projections P_j onto $\mathcal{H}_j$ are contained in $\pi(S)'$.

Exercise 5.4.5. Let $\mathcal{A} \subseteq B(\mathcal{H})$ be a commutative von Neumann algebra, where $\mathcal{H}$ is separable. Show that the following assertions are equivalent

(a) $\mathcal{A}$ is maximal commutative, i.e., $\mathcal{A}' = \mathcal{A}$.

(b) The representation of $\mathcal{A}$ on $\mathcal{H}$ is multiplicity free, i.e., $\mathcal{A}'$ is commutative.

(c) The representation of $\mathcal{A}$ on $\mathcal{H}$ is cyclic.

Hint: Use Exercise 5.4.4 for (b) $\Rightarrow$ (c) and for (b) $\Rightarrow$ (a) observe that $\mathcal{A}'$ is commutative if and only if $\mathcal{A}' \subseteq \mathcal{A}'' = \mathcal{A}$. For (c) $\Rightarrow$ (b) use Corollary 5.3.4 to identify the cyclic representations as some $L^2(\widehat{\mathcal{A}}, \mu)$, and then Lemma 5.3.5 to see that in this case the commutant is the commutative algebra $L^\infty(\widehat{\mathcal{A}}, \mu)' = L^\infty(\widehat{\mathcal{A}}, \mu) \subseteq B(L^2(\widehat{\mathcal{A}}, \mu))$ (Proposition 3.1.8).

Exercise 5.4.6. Let $X = \{x_1, \ldots, x_n\}$ be a finite set and $\mathfrak{S} = 2^X$ be the σ-algebra of all subsets of X. Show that:

(i) Spectral measures $P\colon \mathfrak{S} \to P_{\mathcal{H}}$ are in one-to-one correspondence with n-tuples $(\mathcal{H}_1, \ldots, \mathcal{H}_n)$ of closed subspaces of $\mathcal{H}$ for which $\mathcal{H} = \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_n$ is an orthogonal direct sum. For the corresponding orthogonal projections $P_j\colon \mathcal{H} \to \mathcal{H}_j \subseteq \mathcal{H}$ this means that

$$\mathbf{1} = \sum_{j=1}^{n} P_j \quad \text{and} \quad P_j P_k = \delta_{jk} P_j,$$

i.e., the P_j form a *resolution of the identity*.

(ii) Two spectral measures $P\colon \mathfrak{S} \to P_{\mathcal{H}}$ and $Q\colon \mathfrak{S} \to P_{\mathcal{K}}$ are unitarily equivalent if and only if the Hilbert spaces $P(\{x_j\})$ and $Q(\{x_j\})$ are isomorphic (=have the same Hilbert dimension) for $j = 1, \ldots, n$.

Exercise 5.4.7. Let $\mathcal{H} = \mathbb{C}^n$ and $A \in B(\mathcal{H})$ be a normal operator.

(i) Describe the unique spectral measure $P\colon 2^{\mathrm{Spec}\,A} \to P_{\mathcal{H}}$ with $A = \int_{\mathrm{Spec}\,A} \lambda\, dP(\lambda)$ (the spectral itegral) in terms of the eigenspaces of A.

(ii) Describe the spectral integral in terms of the decomposition of $\mathcal{H}$ into A-eigenspaces.

(iii) Which property of A corresponds to the multiplicity freeness of the spectral measure?

(iv) When are two normal operators $A, B \in B(\mathcal{H})$ unitarily equivalent?

Exercise 5.4.8. (A glimpse of spectral multiplicity theory) Let $(X, \mathfrak{S})$ be a measurable space and $\mu_1, \mu_2 \colon \mathfrak{S} \to \mathbb{R}_+$ be finite measures. We put $\mu := \mu_1 + \mu_2$ and want to compare the L^2-spaces of μ_1, μ_2 and μ. Show that:

(i) We have an isometric embedding

$$\eta\colon L^2(X,\mu) \to L^2(X,\mu_1) \oplus L^2(X,\mu_2), \quad f \mapsto (f,f).$$

(ii) Suppose that $\mu_j = \rho_j \mu$ with density functions ρ_j (their existence follows from the Radon–Nikodym Theorem). Then the adjoint map $\eta^*\colon L^2(X,\mu_1) \oplus L^2(X,\mu_2) \to L^2(X,\mu)$ has the form $\eta^*(f_1, f_2) := \rho_1 f_1 + \rho_2 f_2$. Conclude that $L^2(X,\mu) = \rho_1 L^2(X,\mu_1) + \rho_2 L^2(X,\mu_2)$.

(iii) The summation map η^* has non-trivial kernel if and only if there exists an $E \in \mathfrak{S}$ with $\mu_1(E) > 0$ and $\mu_2(E) > 0$. If this is not the case, then we call the two measures *orthogonal*: $\mu_1 \bot \mu_2$. Hint: For any such E verify $(-\rho_2\chi_E, \rho_1\chi_E) \in \ker \eta^*$ and if, conversely, $(f_1, f_2) \in \ker \eta^*$ with $f_1 \neq 0$, consider $E := \{f_1 \neq 0\}$.

(iv) If $\mu_1 \bot \mu_2$, then the canonical spectral measure on $L^2(X,\mu_1) \oplus L^2(X,\mu_2)$ is multiplicity free. Hint: η is a unitary equivalence with $L^2(X,\mu)$.

(v) If μ_1 and μ_2 are equivalent, then $P(\mathfrak{S})' \cong L^\infty(X,\mu) \otimes M_2(\mathbb{C})$, i.e., we have a representation of “multiplicity 2”.

One can show that in general one has a mixture of the situation under (iv) and (v).

Chapter 6

Stone's Theorem

In this chapter we discuss the connection between a strongly continuous unitary one-parameter group $(U_t)_{t\in\mathbb{R}}$, i.e., a continuous unitary representation of $\mathbb{R}$ and its infinitesimal generator defined by

$$Av := -i\frac{d}{dt}\Big|_{t=0}U_t v$$

whenever the right hand side exists for $v \in \mathcal{H}$. Although this is only the case on a dense subspace $\mathcal{D} \subseteq \mathcal{H}$, the operator $A\colon \mathcal{D} \to \mathcal{H}$ determines the one-parameter group $(U_t)_{t\in\mathbb{R}}$ uniquely, hence deserves to be called its *infinitesimal generator*. We shall see that it is self-adjoint in a sense that has to be made precise for operators not defined on all of $\mathcal{H}$.

6.1 Unbounded operators

Before we can study the infinitesimal generator of a unitary one-parameter group, we have to develop some concepts related to "unbounded" operators, i.e., operators not defined on all of $\mathcal{H}$.

Definition 6.1.1. (a) Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. An *(unbounded) operator* from $\mathcal{H}_1$ to $\mathcal{H}_2$ is a linear map A from a subspace $\mathcal{D}(A) \subseteq \mathcal{H}_1$, called the *domain of* A, to $\mathcal{H}_2$.

(b) The operator A is called *densely defined* if $\mathcal{D}(A)$ is a dense subspace of $\mathcal{H}_1$. It is called *closed* if its graph $\Gamma(A) := \{(x, Ax)\colon x \in \mathcal{D}(A)\}$ is a closed subset of $\mathcal{H}_1 \times \mathcal{H}_2$. We write

$$\mathcal{N}(A) := \ker(A) \subseteq \mathcal{D}(A) \quad \text{and} \quad \mathcal{R}(A) := \mathrm{im}(A) \subseteq \mathcal{H}_2$$

for kernel and range of A.

(c) If A and B are operators from $\mathcal{H}_1$ to $\mathcal{H}_2$, then B is called an *extension of* A if $\mathcal{D}(A) \subseteq \mathcal{D}(B)$ and $B|_{\mathcal{D}(A)} = A$. We write $A \subseteq B$ if B is an extension of A. The operator A is called *closable* if it has a closed extension, i.e., if the closure $\overline{\Gamma(A)}$ of the graph of A is the graph of a linear operator which we then call $\overline{A}$.

(d) Suppose that A is a densely defined operator from $\mathcal{H}_1$ to $\mathcal{H}_2$. We define the *adjoint operator* A^* from $\mathcal{H}_2$ to $\mathcal{H}_1$ as follows. We put

$$\mathcal{D}(A^*) = \{w \in \mathcal{H}_2 \colon (\exists u \in \mathcal{H}_1)(\forall v \in \mathcal{D}(A))\, \langle Av, w\rangle = \langle v, u\rangle\}.$$

Then $A^*w := u \in \mathcal{H}_1$ satisfies $\langle Av, w\rangle = \langle v, A^*w\rangle$ for all $v \in \mathcal{D}(A)$. Since $\mathcal{D}(A)$ is dense, this relation determines A^*w uniquely. We thus obtain a linear operator $A^* \colon \mathcal{D}(A^*) \to \mathcal{H}_1$ whose domain consists of those elements $w \in \mathcal{H}_2$ for which the linear functional

$$v \mapsto \langle Av, w\rangle, \quad \mathcal{D}(A) \to \mathbb{C}$$

is continuous.

Note that $A \subseteq B$ trivially implies $B^* \subseteq A^*$.

(e) An operator A is called *symmetric* if it is densely defined with $A \subseteq A^*$, and *selfadjoint* if $A^* = A$. We say that A is *essentially selfadjoint* if it is closable and $\overline{A}$ is selfadjoint.

Proposition 6.1.2. *For a densely defined operator A from $\mathcal{H}_1$ to $\mathcal{H}_2$ the following assertions hold:*

(i) *For the unitary operator $V \colon \mathcal{H}_2 \oplus \mathcal{H}_1 \to \mathcal{H}_1 \oplus \mathcal{H}_2, V(x,y) = (-y,x)$, we have*
$$\Gamma(A^*) = V(\Gamma(A)^\perp) = V(\Gamma(A))^\perp.$$

(ii) *A^* is closed.*

(iii) *A^* is densely defined if and only if A is closable, and in this case*
$$A^* = \overline{A}^* \quad \text{and} \quad \overline{A} = A^{**}.$$

(iv) *If $\mathcal{D}(A) = \mathcal{H}_1$, then the following are equivalent:*

   (1) *A is bounded.*

   (2) *A is closed.*

   (3) *$\mathcal{D}(A^*) \subseteq \mathcal{H}_2$ is dense.*

Proof. (i) We endow $\mathcal{H}_1 \times \mathcal{H}_2$ with its natural Hilbert space structure given by

$$\langle (a,b), (a',b')\rangle := \langle a, a'\rangle + \langle b, b'\rangle$$

for $a, a' \in \mathcal{H}_1$ and $b, b' \in \mathcal{H}_2$. Then $(y,z) \in \Gamma(A^*)$ if and only if $\langle Av, y\rangle = \langle v, z\rangle$ for all $v \in \mathcal{D}(A)$, i.e. if $(-z, y) \in \Gamma(A)^\perp$.

(ii) follows immediately from (i) because orthogonal subspace are closed.

(iii) From (i) we know that

$$\overline{\Gamma(A)} = \Gamma(A)^{\perp\perp} = V^{-1}(\Gamma(A^*))^\perp.$$

Therefore an element of the form $(0, w)$ is contained in $\overline{\Gamma(A)}$ if and only if $(w, 0) \perp \Gamma(A^*)$, which is equivalent to $w \in \mathcal{D}(A^*)^\perp$. That any such element w is 0 means that A is closable, so that $\mathcal{D}(A^*)$ is dense if and only if A is closable.

Assume that this is the case. Then $\Gamma(\overline{A}) = \overline{\Gamma(A)}$ has the same orthogonal space as $\Gamma(A)$, so that

$$\Gamma(A^*) = V(\Gamma(A)^\perp) = V(\Gamma(\overline{A})^\perp) = \Gamma(\overline{A}^*),$$

where the last equality follows from (i). This proves that $\overline{A}^* = A^*$.

We also obtain

$$\Gamma(\overline{A}) = \overline{\Gamma(A)} = \Gamma(A)^{\perp\perp} = V^{-1}(\Gamma(A^*))^\perp = \Gamma(A^{**}),$$

where the last equality follows by applying (i) to the operator A^* from $\mathcal{H}_2$ to $\mathcal{H}_1$ and $V^{-1}(v, w) = (w, -v)$. This proves $A^{**} = \overline{A}$.

(iv) The equivalence between (1) and (2) is the Closed Graph Theorem ([Ru73, Thm. 2.15]). It is also clear that (1) implies (3). So it remains to show that (3) implies (2). In view of (iii), we see that A has a closed extension $\overline{A}$. But A is defined on $\mathcal{H}_1$, hence $A = \overline{A}$, so that the graph of A is closed. $\square$

Examples 6.1.3. (a) Let $(X, \mathfrak{S}, \mu)$ be a measure space and $f\colon X \to \mathbb{C}$ be a measurable function. On the Hilbert space $\mathcal{H} = L^2(X, \mu)$ we consider the operator M_f with

$$\mathcal{D}(M_f) := \{h \in L^2(X, \mu)\colon fh \in L^2(X, \mu)\} \quad \text{and} \quad M_f h = fh.$$

To see that $\mathcal{D}(M_f)$ is dense, let $X_n := \{x \in X\colon |f(x)| \le n\}$. Then $\chi_{X_n} L^2(X, \mu) \subseteq \mathcal{D}(M_f)$, and for $h \in L^2(X, \mu)$ we have

$$\|h - \chi_{E_n} h\|_2^2 = \int_{X \setminus X_n} |h(x)|^2 \, d\mu(x) \to 0.$$

Therefore $\mathcal{D}(M_f)$ is dense. For $g, h \in \mathcal{D}(M_f)$ we have

$$\langle M_f g, h\rangle = \int_X f g \overline{h} \, d\mu = \langle g, M_{\overline{f}} h\rangle.$$

For $g \in \mathcal{H}$ the map

$$\mathcal{D}(M_f) \to \mathbb{C}, \quad h \mapsto \langle M_f h, g\rangle = \int_X f h \overline{g} \, d\mu$$

is continuous if and only if $f\overline{g} \in L^2(X, \mu)$. Therefore $\mathcal{D}(M_f^*) = \mathcal{D}(M_f)$ implies that $M_f^* = M_{\overline{f}}$. In particular, M_f is selfadjoint if $f(X) \subseteq \mathbb{R}$.

(b) Let $\mathcal{H} = \ell^2(\mathbb{N}, \mathbb{C})$ with the canonical ONB $(e_n)_{n\in\mathbb{N}}$. For a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of complex numbers we define an operator T on $\mathcal{H}$ by

$$\mathcal{D}(T) := \operatorname{span}\{e_n\colon n \in \mathbb{N}\} \quad \text{and} \quad Tv = (\lambda_n v_n).$$

Then T is densely defined. We can also calculate its adjoint.

The condition that the map

$$\mathcal{D}(T) \to \mathbb{C}, \quad y \mapsto \langle Ty, v\rangle = \sum_{n\in\mathbb{N}} \lambda_n y_n v_n$$

is continuous, is equivalent to $\sum_{n\in\mathbb{N}} |\lambda_n|^2 \cdot |v_n|^2 < \infty$. Therefore

$$\mathcal{D}(T^*) = \Big\{v = (v_n) \in \mathcal{H} \colon \sum_{n\in\mathbb{N}} |\lambda_n|^2 |v_n|^2 < \infty\Big\} \quad \text{and} \quad T^*(v_n) = (\overline{\lambda_n} v_n).$$

This shows that T is symmetric if and only if all numbers λ_n are real. Applying the same argument to T^* (which is also densely defined), we see that $\overline{T} = T^{**}$ is given by the same formula as T:

$$T^{**}(v_n) = (\lambda_n v_n).$$

(c) An important special case arises for the operator on $\mathcal{H} = L^2(\mathbb{R})$ given by

$$(Qf)(x) = xf(x) \quad \text{and} \quad \mathcal{D}(Q) = \Big\{f \in L^2(\mathbb{R}) \colon \int_{\mathbb{R}} x^2 |f(x)|^2\, dx < \infty\Big\}.$$

(d) We consider the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$ and the operator P with $\mathcal{D}(P) := C^1_c(\mathbb{R})$ (continuously differentiable functions with compact support) and $Pf = if'$. Then P is densely defined (Exercise) and, for $f, h \in \mathcal{D}(P)$, we have

$$\langle Pf, h\rangle = \int_{\mathbb{R}} if'(x)\overline{h(x)}\; dx = \int_{\mathbb{R}} f(x)\overline{ih'(x)}\; dx = \langle f, Ph\rangle,$$

according to the Product Rule, because, for sufficiently large R, we have

$$\int_{\mathbb{R}} (if\overline{h})'(x)\; dx = \int_{-R}^{R} (if\overline{h})'(x)\; dx = if(R)\overline{h(R)} - if(-R)\overline{h(-R)} = 0.$$

Therefore P is symmetric. In this case it is slightly harder to calculate the domain of its closure. Again, one can show that $P^* = P^{**}$, so that P^* is a selfadjoint extension of P.

Proposition 6.1.4. *For any symmetric operator T on $\mathcal{H}$ with domain $\mathcal{D}(T)$, the following assertions hold:*

(i) $\|Tx + ix\|^2 = \|Tx\|^2 + \|x\|^2$ *for* $x \in \mathcal{D}(T)$.

(ii) *T is closed if and only if $\mathcal{R}(T + i\mathbf{1})$ is closed.*

(iii) *$T + i\mathbf{1}$ is injective.*

(iv) *If $\mathcal{R}(T + i\mathbf{1}) = \mathcal{H}$, then T has no proper symmetric extension.*

(v) *Statements* (i)-(iv) *remain correct if we replace i by $-i$.*

Proof. (i) follows from

$$\|Tx + ix\|^2 = \|Tx\|^2 + \|x\|^2 - i\langle Tx, x\rangle + i\langle x, Tx\rangle = \|Tx\|^2 + \|x\|^2.$$

(ii) In view of (i), the map $\mathcal{R}(T + i\mathbf{1}) \to \Gamma(T), (T + i\mathbf{1})x \mapsto (x, Tx)$ is isometric. Therefore $\mathcal{R}(T + i\mathbf{1})$ is closed, resp., complete if and only if this holds for the graph of T. This implies (ii).
(iii) follows from (i).
(iv) Let T_1 be a symmetric extension of T. Then $T_1 + i\mathbf{1}$ is an extension of $T + i\mathbf{1}$. In view of the bijectivity of $T + i\mathbf{1}$, we obtain $T_1 = T$.
(v) is clear. □

Lemma 6.1.5. (Range-Kernel Lemma) *For a densely defined operator T on $\mathcal{H}$ we have*

$$\mathcal{N}(T^*) = \mathcal{R}(T)^\perp.$$

In particular,

$$\mathcal{N}(T^* + \overline{\lambda}\mathbf{1}) = \mathcal{R}(T + \lambda\mathbf{1})^\perp \quad \textit{for} \quad \lambda \in \mathbb{C}.$$

Proof. In view of $(T + \lambda\mathbf{1})^* = T^* + \overline{\lambda}\mathbf{1}$ (verify!), the second part follows from the first one.

An element $y \in \mathcal{H}$ is contained in $\mathcal{R}(T)^\perp$ if and only if, for all $v \in \mathcal{D}(T)$, we have $\langle Tv, y\rangle = 0$. This is equivalent to $y \in \mathcal{D}(T^*)$ and $T^*y = 0$. □

Proposition 6.1.6. *For a symmetric operator T on $\mathcal{H}$, the following are equivalent:*

(i) *T is selfadjoint.*

(ii) *T is closed and $T^* \pm i\mathbf{1}$ are both injective.*

(iii) *T is closed and $T \pm i\mathbf{1}$ both have dense range.*

(iv) *$\mathcal{R}(T \pm i\mathbf{1}) = \mathcal{H}$.*

Proof. (i) ⇒ (ii): According to Proposition 6.1.2(ii), the closedness of T follows from $T = T^*$. Since T is symmetric, the injectivity of $T^* \pm i\mathbf{1} = T \pm i\mathbf{1}$ is a consequence of Proposition 6.1.4(iii).
(ii) ⇒ (iii) follows from Lemma 6.1.5.
(iii) ⇒ (iv) follows from Proposition 6.1.4(ii),(v).
(iv) ⇒ (i): In view of $T \subseteq T^*$, we only have to show that $\mathcal{D}(T^*) \subseteq \mathcal{D}(T)$. Let $y \in \mathcal{D}(T^*)$. With (iv) we find $x \in \mathcal{D}(T)$ with $(T^* + i\mathbf{1})y = (T + i\mathbf{1})x$. Then $T \subseteq T^*$ implies $(T^* + i\mathbf{1})y = (T^* + i\mathbf{1})x$, hence $y = x \in \mathcal{D}(T)$, because $T^* + i\mathbf{1}$ is injective which in turn follows from the density of $\mathcal{R}(T - i\mathbf{1})$ and the Range-Kernel Lemma 6.1.5. □

Corollary 6.1.7. *For a symmetric operator T on $\mathcal{H}$, the follows are equivalent:*

(i) *T is essentially selfadjoint.*

(ii) $T^* \pm i\mathbf{1}$ *are injective.*

(iii) $T \pm i\mathbf{1}$ *have dense range.*

Proof. (i) $\Rightarrow$ (ii): If T is essentially selfadjoint, then its closure $\overline{T}$ is selfadjoint, so that $T^* = \overline{T}^* = \overline{T}$ (Proposition 6.1.2(iii)). Hence (ii) follows from Proposition 6.1.6(ii).

(ii) $\Rightarrow$ (iii) follows from the Range-Kernel Lemma 6.1.5.

(iii) $\Rightarrow$ (i): Our assumption implies in particular that the range of $\overline{T} \pm i\mathbf{1}$ is dense. Moreover, $T \subseteq T^*$ and the closedness of T^* imply $\overline{T} \subseteq T^* = \overline{T}^*$ (Proposition 6.1.2), so that $\overline{T}$ is also symmetric. Now Proposition 6.1.6 implies that $\overline{T}$ is selfadjoint. $\square$

$\boxed{\text{The Spectral Integral for unbounded measurable functions}}$

Proposition 6.1.8. *Let P be a spectral measure on $(X, \mathfrak{S})$ and $f: X \to \mathbb{C}$ a measurable function. Then the following assertions hold for*

$$\mathcal{D}(f) := \{v \in \mathcal{H} : f \in L^2(X, P^v)\}.$$

(i) *$\mathcal{D}(f)$ is a dense subspace of $\mathcal{H}$, and there exists a unique linear operator*

$$\widehat{P}(f): \mathcal{D}(f) \to \mathcal{H} \quad \textit{with} \quad \langle \widehat{P}(f)v, v\rangle = \int_X f(x)\, dP^v(x)$$

for $v \in \mathcal{D}(f)$. If f is bounded, then $\mathcal{D}(f) = \mathcal{H}$.

(ii) *$\mathcal{D}(f) = \mathcal{D}(\overline{f})$ and $P(f)^* = P(\overline{f})$.*

(iii) *If $f(X) \subseteq \mathbb{T}$, then $P(f)$ is unitary, and if $f(X) \subseteq \mathbb{R}$, then $P(f)$ is selfadjoint.*

Proof. Decomposing into cyclic subspaces for P, we see with Lemma 5.1.4(b) that $\mathcal{H} \cong \widehat{\bigoplus}_{j\in J} L^2(X, \mu_j)$ with finite measures μ_j on $(X, \mathfrak{S})$ and $P(E)(f_j)_{j\in J} = (\chi_E f_j)_{j\in J}$ for $(f_j) \in \mathcal{H}$.

(i), (ii) Write $v = (v_j)$ with $v_j \in L^2(X, \mu_j)$. Then

$$P^v(E) = \sum_{j\in J} P^{v_j}(E) = \sum_{j\in J} \int_E |v_j(x)|^2\, d\mu_j(x)$$

implies that

$$\int_X |f(x)|^2\, dP^v(x) = \sum_{j\in J} \int_X |f(x)|^2 |v_j(x)|^2\, d\mu_j(x).$$

Therefore $v \in \mathcal{D}(f)$ is equivalent to $fv_j \in L^2(X, \mu_j)$ for every j and $\|fv\|^2 = \sum_{j\in J} \|fv_j\|_2^2 < \infty$. Hence $\widehat{P}(f): \mathcal{D}(f) \to \mathcal{H}$ is the direct sum of the corresponding multiplication operators $M_f(h) = fh$ on the subspaces $L^2(X, \mu_j)$ (cf.

Examples 6.1.3(a)). Now Exercise 6.3.1 and Examples 6.1.3 imply that $\widehat{P}(f)$ is densely defined and closed with $\widehat{P}(f)^* = \widehat{P}(\overline{f})$.

In view of the Polarization Identity, the operator $\widehat{P}(f)$ is uniquely determined by the numbers

$$\langle \widehat{P}(f)v, v\rangle = \sum_{j\in J}\int_X f(x)|v_j(x)|^2\,d\mu_j(x) = \int_X f(x)\,dP^v(x), \quad v \in \mathcal{D}(f).$$

(iii) If $f(X) \subseteq \mathbb{T}$, then all multiplication operators λ_f on $L^2(X,\mu_j)$ are unitary, so that $\widehat{P}(f)$ is also unitary. If $f(X) \subseteq \mathbb{R}$, then $f = \overline{f}$, (i) and (ii) imply that $\widehat{P}(f)$ is selfadjoint. □

Theorem 6.1.9. (Spectral Theorem for Selfadjoint Operators) *If $P\colon \mathfrak{B}(\mathbb{R}) \to \mathcal{H}$ is a spectral measure on $\mathbb{R}$, then $P(\mathrm{id}_{\mathbb{R}})$ is a selfadjoint operator, and for each selfadjoint operator A on $\mathcal{H}$ there exists a unique regular Borel spectral measure P on $\mathbb{R}$ such that $A = P(\mathrm{id}_{\mathbb{R}})$.*

Proof. (Sketch) The first part follows from Proposition 6.1.8(iii). The second part is more difficult. The main idea is to use the Cayley transform to transform A into a unitary operator $C(A) := (A - i\mathbf{1})(A + i\mathbf{1})^{-1}$ and then use the spectral measure of $C(A)$ on the circle $\mathbb{T}$ to obtain a spectral measure on $\mathbb{R}$ by the map $c\colon \mathbb{R} \to \mathbb{T}, c(t) = (t - i)(t + i)^{-1}$. For the detailed proof we refer to [Ru73, Thm. 13.30] (see also [Ne09]). □

6.2 Infinitesimal generators of unitary one-parameter groups

In this section we address the problem we prove Stone's Theorem about unitary one-parameter groups and their infinitesimal generators, which are (up to multiplication with i) the, possibly unbounded, selfadjoint operators.

Definition 6.2.1. Let $\mathcal{H}$ be a Hilbert space and $(U_t)_{t\in\mathbb{R}}$ be a strongly continuous unitary one-parameter group, i.e., a continuous unitary representation $U\colon \mathbb{R} \to \mathrm{U}(\mathcal{H})$. We define an unbounded operator $A\colon \mathcal{D}(A) \to \mathcal{H}$ by

$$\mathcal{D}(A) := \left\{ v \in \mathcal{H}\colon \lim_{t\to 0}\frac{1}{t}\big(U_t v - v\big) \text{ exists} \right\} \quad \text{and} \quad Av := \lim_{t\to 0}\frac{1}{it}\big(U_t v - v\big).$$

This operator is called the *infinitesimal generator of U* and the elements of the space $\mathcal{D}(A)$ are called *differentiable vectors for U*.

Lemma 6.2.2. *For a continuous unitary one-parameter group $(U_t)_{t\in\mathbb{R}}$, the following assertions hold:*

(a) *The operator A is hermitian in the sense that*

$$\langle Av, w\rangle = \langle v, Aw\rangle \quad \textit{for} \quad v, w \in \mathcal{D}(A).$$

(b) *For every $t \in \mathbb{R}$ and $v \in \mathcal{D}(A)$, we have $U_t v \in \mathcal{D}(A)$ and $U_t A v = A U_t v$.*

(c) *For $v \in \mathcal{D}(A)$, the curve $\gamma(t) := U_t v$ is the unique solution of the initial value problem*

$$\gamma(0) = v \quad \text{and} \quad \gamma'(t) = iA\gamma(t), \quad t \in \mathbb{R}. \tag{6.1}$$

Proof. (a) For $v, w \in \mathcal{D}(A)$ we have

$$\langle Av, w \rangle = \frac{d}{dt}\Big|_{t=0} - i\langle U_t v, w \rangle = \frac{d}{dt}\Big|_{t=0} \langle v, iU_{-t} w \rangle = \langle v, Aw \rangle.$$

(b) For $v \in \mathcal{D}(A)$, we have

$$\lim_{h \to 0} \frac{1}{h}(U_{t+h}v - U_t v) = \lim_{h \to 0} \frac{1}{h}(U_t U_h v - U_t v) = U_t \lim_{h \to 0} \frac{1}{h}(U_h v - v) = U_t(iAv).$$

In view of $U_{t+h} = U_h U_t$, the existence of this limit implies that $U_t v \in \mathcal{D}(A)$ with $AU_t v = U_t A v$, so that the curve $\gamma(t) := U_t v$ solves the initial value problem (6.1).

(c) Suppose that $\gamma \colon \mathbb{R} \to \mathcal{H}$ satisfies the initial value problem (6.1), which means implicitly that $\gamma(\mathbb{R}) \subseteq \mathcal{D}(A)$, so that $A\gamma(t)$ makes sense. For $w \in \mathcal{D}(A)$ and $\beta(t) := U_{-t}\gamma(t)$ we then have

$$\begin{aligned} &\frac{1}{h}(\beta(t+h) - \beta(t)) \\ &= U_{-t-h}\Big(\frac{1}{h}(\gamma(t+h) - \gamma(t) - iA\gamma(t)\Big) + iU_{-t-h}A\gamma(t) + \frac{1}{h}(U_{-t-h} - U_{-t})\gamma(t). \end{aligned}$$

Since $(U_s)_{s \in \mathbb{R}}$ is a bounded family of operators, we obtain

$$\lim_{h \to 0} U_{-t-h}\Big(\frac{1}{h}(\gamma(t+h) - \gamma(t) - iA\gamma(t)\Big) = 0,$$

and therefore

$$\lim_{h \to 0} \frac{1}{h}(\beta(t+h) - \beta(t)) = iU_{-t}A\gamma(t) + -iAU_{-t}\gamma(t) = iU_{-t}A\gamma(t) + -iU_{-t}A\gamma(t) = 0.$$

Therefore the curve β is differentiable with $\beta' = 0$, and this implies that β is constant. We thus obtain for every $t \in \mathbb{R}$ from $\beta(t) = \beta(0) = v$ the relation $\gamma(t) = U_t v$. $\square$

Remark 6.2.3. (a) If A is a bounded hermitian operator, then $U_t := e^{itA}$ defines a norm-continuous unitary one-parameter group with infinitesimal generator A. In fact, the estimate

$$\|e^{itA} - \mathbf{1} - itA\| \leq \sum_{n=2}^{\infty} \frac{1}{n!}|t|^n \|A\|^n = e^{|t|\|A\|} - 1 - |t|\|A\|$$

implies that

$$\lim_{t\to 0} \frac{U_t - \mathbf{1}}{it} = A$$

holds in the norm topology.

(b) Given a symmetric operator $A\colon \mathcal{D}(A) \to \mathcal{H}$, there is no guarantee that a solution to the initial value problem (6.1) exists for every $v \in \mathcal{D}(A)$. As we shall see below, this requires extra conditions on A.

(c) If $\mathcal{D}(A) = \mathcal{H}$, then the symmetry implies that $A \subseteq A^*$, so that Proposition 6.1.2(iv) implies that A is bounded. The curves $\gamma(t) := e^{itA}v$ are the unique solutions of the initial value problem

$$\gamma(0) = v \quad \text{ and } \quad \gamma'(t) = iA\gamma(t), \quad t \in \mathbb{R},$$

so that Lemma 6.2.2(c) implies that $U_t = e^{itA}$ for $t \in \mathbb{R}$.

The preceding remark shows that A is unbounded if and only if there exists non-differentiable vectors in $\mathcal{H}$, i.e., $\mathcal{D}(A) \neq \mathcal{H}$. Here is a typical example where this happens.

Example 6.2.4. On $\mathcal{H} = L^2(\mathbb{R})$ we consider the continuous one-parameter group given by $(U_t f)(x) = f(x+t)$ (Example 3.3.6). Then, for every $f \in C_c^1(\mathbb{R})$, the limit

$$\lim_{t\to 0} \frac{1}{t}(U_t f - f) = f'$$

exists uniformly on $\mathbb{R}$ and since the support of f is compact, the limit exists in particular in $L^2(\mathbb{R})$. Therefore $f \in \mathcal{D}(A)$ and $Af = -if'$. That A is unbounded follows immediately by applying it to functions $f_n \in C_c^1(\mathbb{R})$ with $f_n(x) = e^{inx}$ for $x \in [0,1]$

Theorem 6.2.5. (Stone's Theorem for One-Parameter Groups, 1932) *Let $(U_t)_{t\in\mathbb{R}}$ be a strongly continuous unitary one-parameter group and $A\colon \mathcal{D} := \mathcal{D}(A) \to \mathcal{H}$ be its infinitesimal generator. Then the following assertions hold:*

(i) *The space $\mathcal{D}$ of differentiable vectors is dense in $\mathcal{H}$.*

(ii) *If $\mathcal{D}_0 \subseteq \mathcal{D}$ is dense and U-invariant, then $A|_{\mathcal{D}_0}$ is essentially selfadjoint and its closure coincides with A.*

(iii) *A is selfadjoint.*

(iv) *If $(V_t)_{t\in\mathbb{R}}$ is another strongly continuous unitary one-parameter group with the same generator A, then $U_t = V_t$ for every $t \in \mathbb{R}$.*

Proof. (i) This is done by a mollifying argument. Let $v \in \mathcal{H}$ and $T > 0$. We consider the element $v_T := \int_0^T U_t v\, dt$. The integral exists because $(U_t)_{t\in\mathbb{R}}$ is strongly is strongly continuous, which further implies that

$$\left\| \frac{1}{T} v_T - v \right\| = \left\| \frac{1}{T} \int_0^T U_t v - v\, dt \right\| \leq \sup_{0\leq t\leq T} \|U_t v - v\| \to 0 \quad \text{ for } \quad T \to 0.$$

Therefore it suffices to show that $v_T \in \mathcal{D}$. The relation

$$\begin{aligned}
U_t v_T - v_T &= U_t \int_0^T U_s v\, ds - \int_0^T U_s v\, ds = \int_0^T U_s U_t v\, ds - \int_0^T U_s v\, ds \\
&= \int_t^{T+t} U_s v\, ds - \int_0^T U_s v\, ds \\
&= \int_T^{T+t} U_s v\, ds - \int_T^t U_s v\, ds - \int_0^T U_s v\, ds \\
&= U_T \int_0^t U_s v\, ds - \int_0^t U_s v\, ds = U_T v_t - v_t = (U_T - \mathbf{1}) v_t
\end{aligned}$$

implies that, for $T > 0$, we have

$$\lim_{t \to 0} \frac{1}{t}(U_t v_T - v_T) = (U_T - \mathbf{1}) \lim_{t \to 0} \frac{1}{t} v_t = (U_T - \mathbf{1}) v.$$

(ii) We have already seen that A is symmetric (Lemma 6.2.2), and therefore $A_0 := A|_{\mathcal{D}_0}$ is also symmetric. In view of the criterior for essential selfadjointness (Corollary 6.1.7), we have to show that the operators $A_0 \pm i\mathbf{1}$ have dense range. If the range of $A_0 + i\mathbf{1}$ is not dense, there exists a vector $0 \neq v \in \mathcal{R}(A_0 + i\mathbf{1})^\perp$. As $\mathcal{D}_0$ is dense, there exists a $w \in \mathcal{D}_0$ with $\langle v, w \rangle \neq 0$. Then the differentiable function $f(t) := \langle U_t w, v \rangle$ satisfies

$$f'(t) = i\langle A U_t w, v \rangle = i\langle (A + i\mathbf{1}) U_t w, v \rangle + \langle U_t w, v \rangle = \langle U_t w, v \rangle = f(t),$$

which leads to $f(t) = f(0)e^t$, contradicting $|f(t)| \leq \|v\|\|w\|$. We likewise see that $\mathcal{R}(A_0 - i\mathbf{1})$ is dense. This shows that $\overline{A_0}$ is a selfadjoint operator.

We postpone the proof of the relation $A = \overline{A_0}$ until we have proved (iii).

(iii) From (ii) it follows that A is essentially selfadjoint, so it remains to show that A is closed. In Lemma 6.2.2 we have seen that, for $v \in \mathcal{D}(A)$, the curve $\gamma(t) := U_t v$ is differentiable with derivative $\gamma'(t) = U_t i A v$, which is continuous. The Fundamental Theorem of Calculus thus leads to the relation

$$U_t v - v = \int_0^t U_s i A v\, ds.$$

To see that A is closed, let $(v_n) \in \mathcal{D}(A)$ be a sequence for which $(v_n, A v_n) \to (v, w)$ in $\mathcal{H} \oplus \mathcal{H}$. We then obtain for each $t \in \mathbb{R}$ the relation

$$U_t v - v = \lim_{n \to \infty} U_t v_n - v_n = \lim_{n \to \infty} \int_0^t U_s i A v_n\, ds = \int_0^t U_s i w\, ds$$

and thus $iAv = \frac{d}{dt}\big|_{t=0} U_t v = iw$. We conclude that $v \in \mathcal{D}(A)$ with $Av = w$.

(ii) (continued) Now we show that $\overline{A_0} = A$. We know from above that $\overline{A_0}$ is selfadjoint and that A is selfadjoint, hence in particular closed. Thus $A_0 \subseteq A$ implies $\overline{A_0} \subseteq A$, so that

$$A = A^* \subseteq \overline{A_0}^* = \overline{A_0} \subseteq A$$

(cf. Definition 6.1.1(d)).

(iv) For $v \in \mathcal{D}(A)$, the curve $\gamma(t) = V_t v$ satisfies $\gamma(0) = v$ and $\gamma'(t) = iA\gamma(t)$, so that Lemma 6.2.2(c) implies $V_t v = U_t v$. As V_t and U_t are bounded operators and $\mathcal{D}(A)$ is dense, it follows that $V_t = U_t$ for every $t \in \mathbb{R}$. $\square$

Lemma 6.2.6. *Let $(U_t)_{t\in\mathbb{R}}$ be a continuous unitary one-parameter group and A be its infinitesimal generator. Suppose that $\mathcal{H} = \oplus_{j\in J}\mathcal{H}_j$ is an orthogonal decomposition into U-invariant subspaces and that $U_t^j := U_t|_{\mathcal{H}_j}$. Then $A = \oplus_{j\in J}A_j$ is the direct sum of the infinitesimal generators A_j of $(U_t^j)_{t\in\mathbb{R}}$, i.e.,*

$$\mathcal{D}(A) = \Big\{(v_j) \in \mathcal{H}\colon (\forall j)\, v_j \in \mathcal{D}(A_j),\ \sum_{j\in J} \|A_j v_j\|_2^2 < \infty\Big\} \to \mathcal{H}, \quad A(v_j) := (A_j v_j).$$

Proof. Let $v = (v_j)_{j\in J} \in \mathcal{H}$. Then $v \in \mathcal{D}(A)$ clearly implies that $v_j \in \mathcal{D}(A_j)$ for every $j \in J$ and that $Av = (A_j v_j)$. This means that that $A \subseteq B := \oplus_{j\in J}A_j$. Since the operators A_j are selfadjoint by Stone's Theorem, Exercise 6.3.1 shows that B is selfadjoint. We therefore have $B = B^* \subseteq A^* = A$, and thus $A = B$. $\square$

Example 6.2.7. (a) Let $(X, \mathfrak{S}, \mu)$ be a measure space and consider the unitary one-parameter group given by $U_t h = e^{itf}h$ for some measurable function $f\colon X \to \mathbb{R}$. We claim that its infinitesimal generator A coincides with the multiplication operator M_f from Example 6.1.3. To this end, let $X_n := \{|f| \leq n\}$ and consider the subspace

$$\mathcal{D}_0 := \{h \in L^2(X, \mu)\colon (\exists n \in \mathbb{N}) f|_{X\setminus X_n} = 0\}.$$

Then, for each $h \in \mathcal{D}_0$, the curve $t \mapsto U_t h$ is analytic, given by the convergent exponential series, and we obtain $Ah = fh = M_f h$. Since $\mathcal{D}_0$ is invariant under U, Theorem 6.2.5 implies that $A = \overline{A_0}$ for $A_0 := M_f|_{\mathcal{D}_0}$. As M_f is closed, this leads to $A \subseteq M_f$, and since it is selfadjoint, we obtain $M_f = M_f^* \subseteq A^* = A$, which proves equality.

(b) Now let $P\colon \mathfrak{S} \to P_{\mathcal{H}}$ be a spectral measure on $(X, \mathfrak{S})$ and $f\colon X \to \mathbb{R}$ be a measurable function. Then $U_t := \widehat{P}(e^{itf})$ is a continuous unitary one-parameter group by Proposition 5.2.1(ii).

We write $\mathcal{H} = \widehat{\oplus}_{j\in J}L^2(X, \mu_j)$ as a direct sum of L^2-spaces on which $P(E)(v_j) = (\chi_E v_j)$ (Remark 5.1.6). Then $U_t = (U_t^j)$, where $U_t^j h = e^{itf}h$ for $h \in L^2(X, \mu_j)$. Now Lemma 6.2.6 implies that the infinitesimal generator of U is $\oplus_{j\in J}M_f = \widehat{P}(f)$.

Corollary 6.2.8. *For every seladjoint operator A on a Hilbert space $\mathcal{H}$, there exists a unique continuous unitary one-parameter group $(U_t)_{t\in\mathbb{R}}$ with infinitesimal generator A.*

Proof. We know already from Stone's Theorem that $(U_t)_{t\in\mathbb{R}}$ is determined by A, so it only remains to show existence.

From the Spectral Theorem for selfadjoint operators (Theorem 6.1.9), we first obtain a regular spectral measure on $(\mathbb{R}, \mathfrak{B}(\mathbb{R}))$ with $A = \widehat{P}(\mathrm{id}_{\mathbb{R}})$. We now put

$$U_t := \widehat{P}(e^{it\,\mathrm{id}_{\mathbb{R}}}).$$

Proposition 5.2.1(ii) then implies that U is a continuous unitary one-parameter group and Example 6.2.7(b) shows that $A = \widehat{P}(\mathrm{id}_{\mathbb{R}})$ is its infinitesimal generator. $\square$

6.3 Invariant subspaces and analytic vectors

Remark 6.3.1. (a) If A is bounded, then a closed subspace $\mathcal{K}$ of the Hilbert space $\mathcal{H}$ is A-invariant if and only if it is invariant under the corresponding one-parameter group $(U_t)_{t\in\mathbb{R}}$, defined by $U_t = e^{itA}$.

In fact, $A\mathcal{K} \subseteq \mathcal{K}$ implies $A^n\mathcal{K} \subseteq \mathcal{K}$ for every $n \in \mathbb{N}$, and therefore

$$U_t v = \sum_{n=0}^{\infty} \frac{1}{n!}(it)^n A^n v \in \mathcal{K}$$

for every $v \in \mathcal{K}$ and $t \in \mathbb{R}$. Conversely, for $v \in \mathcal{K}$, we have

$$Av = -i \lim_{t\to 0} \frac{1}{t}(U_t v - v) \in \mathcal{K}.$$

(b) If A is unbounded, the situation is more subtle. What remains true is that the invariance of $\mathcal{K}$ under (U_t) implies that

$$A(\mathcal{D}(A) \cap \mathcal{K}) \subseteq \mathcal{K}. \tag{6.2}$$

However, this condition does not imply that $\mathcal{K}$ is invariant under the operators U_t.

A prototypical example is obtained by $\mathcal{H} = L^2(\mathbb{R})$, $(U_t f)(x) = f(x-t)$ with $Af = if'$ for $f \in C^1_c(\mathbb{R})$, and $\mathcal{K} := L^2([0,1])$. For every $f \in \mathcal{K} \cap \mathcal{D}(A)$ and $g \in L^2(\mathbb{R})$ vanishing on $[-\varepsilon, 1+\varepsilon]$ for some $\varepsilon > 0$ we obtain

$$\langle U_t f, g\rangle = 0 \quad \text{for} \quad |t| < \varepsilon$$

and therefore $\langle Af, g\rangle = 0$. This implies that $Af \in L^2([0,1])$. On the other hand $U_t L^2([0,1]) = L^2([t, 1+t])$ shows that $\mathcal{K}$ is not invariant under the operators U_t.

The problem of deciding whether a closed subspace $\mathcal{K} \subseteq \mathcal{H}$ is invariant under (U_t) is a serious problem in many applications and it obviously is of central importance in representation theory. The preceding remark shows that the necessary invariance condition (6.2) under $A|_{\mathcal{D}(A)\cap\mathcal{K}}$ is not sufficient.

Definition 6.3.2. We have already seen that $\mathcal{D}(A)$ is the space of those vectors v for which the curve $U_t v$ is continuously differentiable. For $k \in \mathbb{N} \cup \{\infty, \omega\}$ we write $\mathcal{D}^k(A)$ for the set of all elements $v \in \mathcal{H}$ for which the curve

$$\gamma_v(t) \colon \mathbb{R} \to \mathcal{H}, \quad \gamma_v(t) := U_t v$$

is of class C^k. The elements of $\mathcal{D}^k(A)$ are called *C^k-vectors*. For $k = \infty$ they are called *smooth vectors* and for $k = \omega$ *analytic vectors*.

Definition 6.3.3. If T and S are unbounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ and $a \in \mathbb{C}$, then we define

(1) aT by $\mathcal{D}(aT) := \mathcal{D}(T)$ and $(aT)v = aTv$ if $a \neq 0$, and $0T := \mathbf{0}$.

(2) $\mathcal{D}(T+S) := \mathcal{D}(T) \cap \mathcal{D}(S)$ with $(T+S)v := Tv + Sv$ for $v \in \mathcal{D}(T+S)$.

(3) If S is an unbounded Operator from $\mathcal{H}_2$ to $\mathcal{H}_3$, then we define

$$\mathcal{D}(ST) := T^{-1}\big(\mathcal{D}(S)\big) \quad \text{and} \quad (ST)v := S\big(Tv\big).$$

Remark 6.3.4. For $k \in \mathbb{N}$, any $v \in \mathcal{D}^k(A)$ is contained in $\mathcal{D}(A)$, so that $\gamma_v'(t) = iAU_tv = U_t(iAv)$. We conclude that $Av \in \mathcal{D}^{k-1}(A)$. Iterating this procedure, we see inductively that $\mathcal{D}^k(A) \subseteq \mathcal{D}(A^k)$. Conversely, every $v \in \mathcal{D}(A^k)$ is a C^k-vector. In fact, for $k = 1$ this is clear, and for $k > 1$, the relation $\gamma_v'(t) = U_t(iAv)$ implies inductively that γ_v' is C^{k-1}, so that γ_v is C^k. Therefore

$$\mathcal{D}^k(A) = \mathcal{D}(A^k) \quad \text{for} \quad k \in \mathbb{N}.$$

Since this holds for any $k \in \mathbb{N}$, we also obtain

$$\mathcal{D}^\infty(A) = \bigcap_{k=1}^\infty \mathcal{D}(A^k).$$

Clearly, $\mathcal{D}^\omega(A) \subseteq \mathcal{D}^\infty(A)$, and for $v \in \mathcal{D}^\omega(A)$ Taylor's Theorem implies that, for some $\varepsilon > 0$, we have

$$U_tv = \gamma_v(t) = \sum_{n=0}^\infty \frac{t^n}{n!}\gamma_v^{(n)}(0) = \sum_{n=0}^\infty \frac{(it)^n}{n!}A^nv \quad \text{for} \quad |t| < \varepsilon.$$

Conversely, if $v \in \mathcal{D}^\infty(A)$ satisfies

$$\sum_{n=0}^\infty \frac{\varepsilon^n}{n!}\|A^nv\| < \infty \quad \text{for some} \quad \varepsilon > 0,$$

then the Taylor series of γ_v in 0 converges for $|t| < \varepsilon$ to an analytic curve η solving the initial value problem

$$\eta(0) = v, \qquad \eta'(t) = iA\eta(t),$$

so that Lemma 6.2.2(c) implies that $\eta(t) = \gamma_v(t)$. This implies that γ_v is analytic in a 0-neighborhood. From $\gamma_v(t+h) = U_t\gamma_v(h)$, it now follows that γ_v is analytic. This leads to the characterization

$$\mathcal{D}^\omega(A) = \Big\{v \in \mathcal{H} \colon (\exists\varepsilon > 0) \sum_{n=0}^\infty \frac{\varepsilon^n}{n!}\|A^nv\| < \infty\Big\}.$$

In particular, this space is determined completely in terms of the unbounded operator A.

The following proposition shows that the problem of passing from "A-invariance" of a subspace to the invariance under the corresponding unitary one-parameter group is due to the difference between smoothness and analyticity which is absolutely crucial in the context of unbounded operators.

Proposition 6.3.5. *Let $(U_t)_{t\in\mathbb{R}}$ be a strongly continuous unitary one-parameter group with infinitesimal generator A. Let $\mathcal{K} \subseteq \mathcal{H}$ be a closed subspace for which $\mathcal{K} \cap \mathcal{D}^\omega(A)$ is dense in $\mathcal{K}$ and $Av \in \mathcal{K}$ for every $v \in \mathcal{K} \cap \mathcal{D}^\omega(A)$. Then $\mathcal{K}$ is invariant under $(U_t)_{t\in\mathbb{R}}$.*

Proof. Let $v \in \mathcal{K} \cap \mathcal{D}^\omega(A)$ and $\varepsilon > 0$ be such that

$$\sum_{n=0}^\infty \frac{\varepsilon^n}{n!}\|A^n v\| < \infty.$$

We have seen in Remark 6.3.4 that, for $|t| < \varepsilon$, we have

$$U_t v = \sum_{n=0}^\infty \frac{(it)^n}{n!} A^n v.$$

As $\mathcal{K}$ is closed and $A^n v \in \mathcal{K}$ for every $n \in \mathbb{N}$, it follows that $U_t v \in \mathcal{K}$. Now we consider the closed subset

$$J := \{t \in \mathbb{R} \colon U_t v \in \mathcal{K}\}.$$

The preceding argument shows that J is a neighborhood of 0. Applying the same argument to the vectors $U_s v \in \mathcal{D}^\omega(A)$, $s \in J$, we see that J is open. Therefore the connectedness of $\mathbb{R}$ implies that $J = \mathbb{R}$, so that $U_t v \in \mathcal{K}$ for every $t \in \mathbb{R}$. As $\mathcal{K} \cap \mathcal{D}^\omega(A)$ is dense in $\mathcal{K}$ and each U_t is continuous, we obtained $U_t\mathcal{K} \subseteq \mathcal{K}$ for $t \in \mathbb{R}$. □

Exercises for Chapter 6

Exercise 6.3.1. Let $A_j \colon \mathcal{D}(A_j) \to \mathcal{H}_j$ be unbounded operators on the Hilbert spaces $\mathcal{H}_j$ and $\mathcal{H} := \widehat{\bigoplus}_{j\in J}\mathcal{H}_j$. We define the unbounded operator $A := \oplus_{j\in J} A_j$ on $\mathcal{H}$ by

$$\mathcal{D}(A) := \Big\{(v_j) \in \mathcal{H} \colon (\forall j)\, v_j \in \mathcal{D}(A_j),\ \sum_{j\in J}\|A_j v_j\|_2^2 < \infty\Big\} \to \mathcal{H}, \quad A(v_j) := (A_j v_j).$$

Then the following assertions hold:

(i) A is closed if and only if each A_j is closed.

(ii) A is densely defined if and only if each A_j is densely defined.

(iii) $A^* = \oplus_{j\in J} A_j^*$.

(iv) A is selfadjoint if and only if each A_j is selfadjoint.

Exercise 6.3.2. Let $A \colon \mathcal{D}(A) \to \mathcal{H}_2$ be a densely defined unbounded operator from $\mathcal{H}_1$ to $\mathcal{H}_2$ and $B \colon \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded operator. Show that

$$(A + B)^* = A^* + B^*.$$

Exercise 6.3.3. On $\mathcal{H} = L^2(\mathbb{T})$ we consider the unitary one-parameter group given by

$$(U_t f)(e^{is}) = f(e^{i(s+t)x}).$$

Show that the domain of its infinitesimal generator A can be described in terms of Fourier series as

$$\mathcal{D}(A) = \Big\{ \sum_{n\in\mathbb{Z}} a_n e_n \colon \sum_n n^2 |a_n|^2 < \infty \Big\}, \quad \text{where} \quad e_n(t) = e^{int}.$$

Hint: Consider the dense U_t-invariant subspace $\mathcal{D}_0 := \operatorname{span}\{e_n \colon n \in \mathbb{Z}\}$ and use Exercise 6.1.3.

Exercise 6.3.4. (A symmetric operator which is not essentially selfadjoint) In $\mathcal{H} = L^2(\mathbb{T})$ we consider the subspace

$$\mathcal{D}_0 := \operatorname{span}\{e_n(z) = z^n \colon n \in \mathbb{Z}\} \quad \text{and} \quad \mathcal{D}_1 := \{f \in \mathcal{D}_0 \colon f(1) = 0\}.$$

On $\mathcal{D}_0$ we consider the operators $A_0 f := if'$ and $A_1 := A_0|_{\mathcal{D}_1}$. Show that:

(i) A_0 is essentially selfadjoint.

(ii) A_1 is symmetric.

(iii) The function $f(e^{it}) := t$, for $0 \leq t < 2\pi$, is contained in the domain of A_1^*. Hint: Verify that $(-i, f) \in \Gamma(A_1)^\perp$.

(iv) For $n \neq 0$, the Fourier coefficients of f are of the form $a_n = \frac{c}{n}$. Conclude that $f \notin \mathcal{D}(A_0^*)$, so that $A_1^* \supseteq A_0^*$ is a proper extension and A_1 is not essentially selfadjoint.

Exercise 6.3.5. Consider on $\mathcal{H} := \ell^2 = \ell^2(\mathbb{N})$ the selfadjoint operator

$$A(x_n) := (nx_n), \quad \mathcal{D}(A) := \Big\{ x \in \ell^2 \colon \sum_n n^2 |x_n|^2 < \infty \Big\}.$$

Describe the subspaces $\mathcal{D}^k(A)$ of C^k-vectors of A, resp., the corresponding unitary one-parameter group $U_t = e^{itA}$ for $k \in \mathbb{N} \cup \{\infty, \omega\}$.

Exercise 6.3.6. Let $\mathcal{H}$ be a Hilbert space, $a < b$ real numbers, and $\gamma \colon [a, b] \to \mathcal{H}$ be a function. Show the following assertions by reducing them to the case $\mathcal{H} = \mathbb{R}$:

(i) If γ is differentiable and $\gamma' = 0$, then γ is constant.

(ii) If $\alpha \colon [a, b] \to \mathcal{H}$ is continuous, then $\gamma(t) := \int_a^t \alpha(s)\, ds$ is differentiable with $\gamma' = \alpha$.

(iii) If γ is continuously differentiable, then $\gamma(t) = \gamma(a) + \int_a^t \gamma'(s)\, ds$.

Chapter 7

Representations in Spaces of Polynomials

In this section we take a closer look at representation of unitary and orthogonal groups in spaces of polynomials. We start in with an elementary but useful criterion for irreducibility in Section 7.1. This criterion implies easily that the representation of the unitary group $\mathrm{U}(\mathcal{H})$ in the space $\mathcal{F}_d(\mathcal{H})$ of polynomials of degree d is irreducible. As a consequence, $\mathcal{F}(\mathcal{H}) = \widehat{\bigoplus}_{d=0}^{\infty}\mathcal{F}_d(\mathcal{H})$ describes the decomposition of the Fock space into irreducible representations of $\mathrm{U}(\mathcal{H})$.

If $\mathcal{H} = \mathbb{C}^n$ is finite dimensional, then $\mathcal{F}_d(\mathbb{C}^n)$ coincides with the space $\mathcal{P}_d$ of homogeneous polynomials of degree d, so that our argument shows that $\mathrm{U}_n(\mathbb{C})$ acts irreducibly on this space. The situation becomes more complicated for the action of the orthogonal group $\mathrm{O}_n(\mathbb{R})$ on the space of polynomials of a fixed degree d on $\mathbb{R}^n$. In Section 7.2 we show how the theory of harmonic polynomials can be used to understand how the space $\mathcal{P}_d(\mathbb{R}^n)$ decomposes under $\mathrm{SO}_n(\mathbb{R})$. This decomposition further leads to the decomposition of the representation of $\mathrm{SO}_n(\mathbb{R})$ on $L^2(\mathbb{S}^{n-1})$.

7.1 A Criterion for Irreducibility

The following criterion for irreducibility is often quite useful.

Proposition 7.1.1. (Subgroup criterion for irreducibility) *Let G be a group, $K \subseteq G$ be a subgroup and suppose that the unitary representation $(\pi, \mathcal{H})$ of G is generated by a subspace $\mathcal{F}$ on which K acts irreducibly. Let $(\rho, \mathcal{F})$ denote the corresponding representation of K. If the multiplicity of ρ in $\pi|_K$ is 1, i.e., $\dim B_K(\mathcal{F}, \mathcal{H}) = 1$, then π is irreducible.*

Proof. Our assumption $\dim B_K(\mathcal{F}, \mathcal{H}) = 1$ implies that $B_K(\mathcal{F}, \mathcal{F}^\perp) = \{0\}$, so that the decomposition $\mathcal{H} = \mathcal{F} \oplus \mathcal{F}^\perp$ is invariant under the commutant $B_K(\mathcal{H}) = \pi(K)'$.

Let $\mathcal{E} \subseteq \mathcal{H}$ be a closed non-zero G-invariant subspace and $P\colon \mathcal{H} \to \mathcal{E}$ be the orthogonal projection. Since P commutes with $\pi(K)$, we have $P(\mathcal{F}) \subseteq \mathcal{F}$. Since $\mathcal{H}$ is generated, as a unitary G-representation, by $\mathcal{F}$, the representation of G on $\mathcal{E}$ is generated by $P(\mathcal{F})$, which implies that $P(\mathcal{F}) \neq \{0\}$. As $\mathcal{F}$ is irreducible under K, this leads to $P(\mathcal{F}) = \mathcal{F}$. We conclude that $\mathcal{F} \subseteq \mathcal{E}$, so that $\mathcal{H} = \mathcal{E}$ follows from the fact that $\mathcal{F}$ generates $\mathcal{H}$. This proves that the representation $(\pi, \mathcal{H})$ is irreducible. $\square$

Since one-dimensional representations are irreducible, we obtain in particular the following specialization.

Corollary 7.1.2. (Irreducibility criterion for cyclic representations) *Let $(\pi, \mathcal{H})$ be a unitary representation of the group G, $K \subseteq G$ a subgroup and $\chi\colon K \to \mathbb{T}$ be a character. If the G-representation on $\mathcal{H}$ is generated by the subspace*

$$\mathcal{H}^{K,\chi} := \{v \in \mathcal{H}\colon (\forall k \in K)\, \pi(k)v = \chi(k)v\}$$

and $\dim \mathcal{H}^{K,\chi} = 1$, then $(\pi, \mathcal{H})$ is irreducible.

Example 7.1.3. Let $\mathcal{H}$ be a complex Hilbert space. We have already seen in Proposition 4.5.3 that the unitary group $\mathrm{U}(\mathcal{H})$ has a continuous unitary representation $\pi(g)f := f \circ g^{-1}$ on the subspace $\mathcal{F}_d(\mathcal{H})$ consisting of homogeneous functions of degree d in the Fock space $\mathcal{F}(\mathcal{H})$. Recall that $\mathcal{F}_d(\mathcal{H})$ is a reproducing kernel Hilbert space with kernel

$$K^d(z, w) = \frac{1}{d!}\langle z, w\rangle^d.$$

Fix an ONB $(e_j)_{j\in J}$ of $\mathcal{H}$ and write

$$T := \{U \in \mathrm{U}(\mathcal{H})\colon (\forall j \in J) U e_j \in \mathbb{T} e_j\} \cong \mathbb{T}^J$$

for the corresponding "diagonal" subgroup. According to Proposition 4.5.3(ii), the functions

$$p_{\mathbf{m}}(z) = z^{\mathbf{m}} = \prod_{j\in J} z_j^{m_j}, \qquad z_j := \langle z, e_j\rangle, \quad |\mathbf{m}| = d,$$

form an orthogonal basis of T-eigenvectors corresponding to different characters.

Since the kernel K^d is invariant under $\mathrm{U}(\mathcal{H})$, we have $UK^d_v = K^d_{Uv}$ for $v \in \mathcal{H}$. As $\mathrm{U}(\mathcal{H})$ acts transitively on the unit sphere $\mathbb{S}(\mathcal{H})$ (cf. Exercise 4.6.2) and $K^d_{\lambda v} = \overline{\lambda}^d K^d_v$, for any $0 \neq v \in \mathcal{H}$, the element $K^d_v \in \mathcal{F}_d(\mathcal{H})$ is a $\mathrm{U}(\mathcal{H})$-cyclic vector.

Therefore Corollary 7.1.2, applied with $K = T$ and $\mathcal{F} = \mathbb{C}K^d_{e_j}$ for some $j \in J$, shows that the representation of $\mathrm{U}(\mathcal{H})$ on $\mathcal{F}_m(\mathcal{H})$ is irreducible.

Remark 7.1.4. (a) The subgroup $\mathbb{T}\mathbf{1}$ acts on $\mathcal{F}_d(\mathcal{H})$ by multiples of the identity $\pi(t\mathbf{1}) = t^{-d}\mathbf{1}$.

(b) If $\dim \mathcal{H} = n$ is finite, then $\mathrm{U}_n(\mathbb{C}) = \mathbb{T}\,\mathrm{SU}_n(\mathbb{C})$ and (a) imply that the representation of $\mathrm{SU}_n(\mathbb{C})$ on $\mathcal{F}_d(\mathbb{C}^n)$ is irreducible.

For $n = 2$, we thus obtain a sequence π_d, $d \in \mathbb{N}_0$, of irreducible unitary representations on the spaces $\mathcal{F}_d(\mathbb{C}^2)$ of dimension $d+1$ (cf. Proposition 4.5.3(ii)), and one can show that, up to equivalence, these are all the irreducible representations of the group $\mathrm{SU}_2(\mathbb{C})$.

7.2 The Fock space $\mathcal{F}(\mathbb{R}^n)$

On $\mathbb{R}^n$ we consider the kernel $K(x, y) = e^{\langle x,y\rangle}$. Since this is the restriction of the corresponding kernel on the complex Hilbert space $\mathbb{C}^n$, the kernel K is also positive definite (cf. Example 4.3.5(e)) and we write $\mathcal{F}(\mathbb{R}^n) := \mathcal{H}_K \subseteq \mathbb{C}^{\mathbb{R}^n}$ for the corresponding reproducing kernel Hilbert space, called the *Fock space on* $\mathbb{R}^n$.

Lemma 7.2.1. *The restriction map*

$$R\colon \mathcal{F}(\mathbb{C}^n) \to \mathcal{F}(\mathbb{R}^n), \quad f \mapsto f|_{\mathbb{R}^n}$$

is unitary.

Proof. We write $\widetilde{K}(z, w) = e^{\langle z,w\rangle}$ for the corresponding kernel on $\mathbb{C}^n$. We consider the map

$$\gamma\colon \mathbb{R}^d \to \mathcal{F}(\mathbb{C}^n), \quad \gamma(x) := \widetilde{K}_x.$$

We claim that $\gamma(\mathbb{R}^n)$ is total in $\mathcal{F}(\mathbb{C}^n)$, i.e., that every function $f \in \mathcal{F}(\mathbb{C}^n)$ vanishing on $\mathbb{R}^n$ vanishes on $\mathbb{C}^n$.

Let $(e_1, \ldots, e_n)$ be the canonical ONB of $\mathbb{R}^n$, resp., $\mathbb{C}^n$. Then Proposition 4.5.3(ii) implies that, for every $f \in \mathcal{F}(\mathbb{C}^n)$, the expansion of f with respect to the orthogonal basis $p_{\mathbf{m}}$ converges in $\mathcal{F}(\mathbb{C}^n)$ and takes the form

$$f(z) = \sum_{\mathbf{m}} c_{\mathbf{m}} z^{\mathbf{m}},$$

so that it corresponds to the Taylor series of f in 0. In particular, all functions in $\mathcal{F}(\mathbb{C}^n)$ are analytic. If f vanishes on $\mathbb{R}^n$, then all its Taylor coefficients vanish (they can be obtained by real partial derivatives), and this implies that every $c_{\mathbf{m}}$ vanishes, i.e., $f = 0$.

We conclude that $\gamma(\mathbb{R}^n)$ is total in $\mathcal{F}(\mathbb{C}^n)$, so that $(\mathbb{R}^n, \gamma, \mathcal{F}(\mathbb{C}^n))$ is a realization triple for the Fock kernel K, and thus

$$R\colon \mathcal{F}(\mathbb{C}^n) \to \mathcal{F}(\mathbb{R}^n), \quad (Rf)(x) = \langle f, \gamma(x)\rangle = f(x)$$

is unitary by the Realization Theorem 4.3.3. □

Clearly, the Fock kernel K is invariant under the action of the orthogonal group $G := \mathrm{O}_n(\mathbb{R})$, so that we obtain a unitary representation

$$\big(\pi(g)f\big)(x) = f(g^{-1}x)$$

of $\mathrm{O}_n(\mathbb{R})$ on $\mathcal{F}(\mathbb{R}^n)$ (Proposition 4.4.5). It leaves the subspaces $\mathcal{F}_d(\mathbb{R}^n)$ of functions homogeneous of degree d invariant. We have already seen in Example 7.1.3 that the representations of $\mathrm{U}_n(\mathbb{C})$ on the subspaces $\mathcal{F}_d(\mathbb{C}^n) \cong \mathcal{F}_d(\mathbb{R}^n)$ are irreducible. The restriction of these representations to $\mathrm{SO}_n(\mathbb{R})$, resp., $\mathrm{O}_n(\mathbb{R})$ are in general not irreducible. The main goal of this section is to understand how these finite dimensional representations decompose into irreducible ones.

In the following we write $\mathcal{P}_\ell \cong \mathcal{F}_\ell(\mathbb{R}^n)$ for the subspace of homogeneous polynomials of degree ℓ.

Lemma 7.2.2. *For $\ell \geq 1$, let*

$$\partial_j := \frac{\partial}{\partial x_j} \colon \mathcal{P}_\ell \to \mathcal{P}_{\ell-1}$$

be the partial derivatives and

$$M_j = M_{x_j} \colon \mathcal{P}_{\ell-1} \to \mathcal{P}_\ell, \quad (M_j f)(x) = x_j f(x_j)$$

be the multiplication operators. Then

$$M_j^* = \partial_j \quad \textit{and} \quad \partial_j^* = M_j.$$

Proof. Let $K^\ell(x,y) = \frac{1}{\ell!}\langle x,y\rangle^\ell$ be the reproducing kernel of $\mathcal{P}_\ell$. Then

$$\begin{aligned}\langle \partial_j K_x^\ell, K_y^{\ell-1}\rangle &= (\partial_j K_x^\ell)(y) = \frac{\ell}{\ell!}\langle y,x\rangle^{\ell-1} x_j = \frac{1}{(\ell-1)!}\overline{\langle y,x\rangle^{\ell-1} x_j} \\ &= \overline{M_j K_y^{\ell-1}(x)} = \overline{\langle M_j K_y^{\ell-1}, K_x^\ell\rangle} = \langle K_x^\ell, M_j K_y^{\ell-1}\rangle.\end{aligned}$$

This implies the assertion because the subset $\{K_x^\ell \colon x \in \mathbb{R}^n\}$ is total in $\mathcal{P}_\ell$. □

From Lemma 7.2.2 we immediately obtain:

Proposition 7.2.3. *Let $\Delta = \sum_{j=1}^n \partial_j^2$ be the Laplacian and*

$$M_{r^2} = \sum_{j=1}^n M_j^2$$

be the multiplication with $r^2 := \sum_{j=1}^n x_j^2$. Then the operators

$$\Delta \colon \mathcal{P}_{\ell+2} \to \mathcal{P}_\ell \quad \textit{and} \quad M_{r^2} \colon \mathcal{P}_\ell \to \mathcal{P}_{\ell+2}$$

are mutual adjoints for $\ell \in \mathbb{N}_0$.

Lemma 7.2.4. *Let π_ℓ be the representation of $\mathrm{O}_n(\mathbb{R})$ on $\mathcal{P}_\ell$. Then*

$$\pi_\ell(g) \circ \Delta = \Delta \circ \pi_{\ell+2}(g) \quad \textit{for} \quad g \in \mathrm{O}_n(\mathbb{R}).$$

Proof. From

$$\begin{aligned}(\pi(g) M_{r^2} f)(x) &= (M_{r^2} f)(g^{-1}x) = \|g^{-1}x\|^2 f(g^{-1}x) \\ &= \|x\|^2 f(g^{-1}x) = \Big(M_{r^2}\big(\pi(g)f\big)\Big)(x)\end{aligned}$$

we derive $\pi_{\ell+2}(g) \circ M_{r^2} = M_{r^2} \circ \pi_\ell(g)$ for $g \in \mathrm{O}_n(\mathbb{R})$. With Proposition 7.2.3 we now get

$$\Delta \circ \pi_{\ell+2}(g)^{-1} = M_{r^2}^* \circ \pi_{\ell+2}(g)^* = \pi_\ell(g)^* \circ M_{r^2}^* = \pi_\ell(g)^{-1} \circ \Delta. \qquad \square$$

7.2.1 Harmonic Polynomials and Spherical Harmonics

We consider the space $\mathcal{P} = \bigoplus_{\ell=0}^{\infty} \mathcal{P}_\ell = \mathcal{P}(\mathbb{R}^n) \subseteq \mathcal{F}(\mathbb{R}^n)$ of complex-valued polynomial functions on $\mathbb{R}^n$. We write

$$\mathcal{H}_\ell := \{f \in \mathcal{P}_\ell \colon \Delta f = 0\}$$

for the subspace of *harmonic polynomials* of degree ℓ and note that $\mathcal{H}_0 = \mathcal{P}_0$ and $\mathcal{H}_1 = \mathcal{P}_1$.

Lemma 7.2.5. *For $\ell \geq 2$, we have*

$$\dim \mathcal{P}_\ell = \binom{\ell+n-1}{n-1} \quad \textit{and} \quad \dim \mathcal{H}_\ell = \binom{\ell+n-2}{n-2} + \binom{\ell+n-3}{n-2}.$$

Proof. In $\mathcal{P}_\ell$ the monomials

$$x^{\mathbf{m}} = x_1^{m_1} \cdots x_n^{m_n}, \qquad \mathbf{m} = (m_1, \ldots, m_n), \quad |\mathbf{m}| = m_1 + \ldots + m_n = \ell$$

form a basis. The number of possibilities to write ℓ as a sum of n elements of $\mathbb{N}_0$ coincides with the number of possibilities to delete $n-1$ elements from a set with $\ell + n - 1$ elements. This implies the first formula.

Now let $f \in \mathcal{P}_\ell$. We write

$$f(x) = \sum_{k=0}^{\ell} \frac{x_n^k}{k!} f_k(x_1, \ldots, x_{n-1}), \tag{7.1}$$

where every f_k is homogeneous of degree $\ell - k$. Then

$$\begin{aligned} \Delta f &= \sum_{k=2}^{\ell} \frac{x_n^{k-2}}{(k-2)!} f_k(x_1, \ldots, x_{n-1}) + \sum_{k=0}^{\ell} \frac{x_n^k}{k!} (\Delta f_k)(x_1, \ldots, x_{n-1}) \\ &= \sum_{k=0}^{\ell-2} \frac{x_n^k}{k!} f_{k+2}(x_1, \ldots, x_{n-1}) + \sum_{k=0}^{\ell-2} \frac{x_n^k}{k!} (\Delta f_k)(x_1, \ldots, x_{n-1}). \end{aligned} \tag{7.2}$$

Therefore $\Delta f = 0$ is equivalent to

$$f_{k+2} = -\Delta f_k \quad \text{for} \quad k = 0, \ldots, \ell - 2. \tag{7.3}$$

If, conversely, $f_0 \in \mathcal{P}_\ell(\mathbb{R}^{n-1})$ and $f_1 \in \mathcal{P}_{\ell-1}(\mathbb{R}^{n-1})$ and we definef_k for $k = 2, \ldots, \ell$ inductively by (7.3), then (7.1) defines a harmonic polynomial f of degree ℓ. This proves the second formula. □

Lemma 7.2.6. *The space $\mathcal{H}_\ell$ is invariant under the action of* $\mathrm{O}_n(\mathbb{R})$ *on $\mathcal{P}_\ell$.*

Proof. In view of Lemma 7.2.4, the Laplacian Δ intertwines the representation of $\mathrm{O}_n(\mathbb{R})$ on $\mathcal{P}_\ell$ with the representation on $\mathcal{P}_{\ell-2}$. In particular, the kernel $\mathcal{H}_\ell$ of Δ on $\mathcal{P}_\ell$ is invariant. □

Proposition 7.2.7. *For $n \geq 3$, the representation of* $\mathrm{SO}_n(\mathbb{R})$ *on* $\mathcal{H}_\ell$ *is irreducible.*

Proof. Let $G = \mathrm{SO}_n(\mathbb{R})$ and $e_n \in \mathbb{R}^n$ be the last basis vector. We consider the subgroup

$$H := \{h \in G \colon he_n = e_n\}$$

and identify H with the group $\mathrm{SO}_{n-1}(\mathbb{R})$ by considering the restriction of $h \in H$ to the $(n-1)$-dimensional subspace $e_n^\perp \cong \mathbb{R}^{n-1}$. We now verify the assumptions of Corollary 7.1.2 for the trivial character $\chi = 1$.

Let $f \in \mathcal{H}_\ell$ be fixed by H. We write

$$f = \sum_{j=1}^{\ell} \frac{1}{j!} x_n^j f_j(x_1, \ldots, x_{n-1}) \quad \text{with} \quad f_j \in \mathcal{P}_{\ell-j}(\mathbb{R}^{n-1}).$$

Since this expansion is unique and compatible with the action of H, we see with $he_n = e_n$ for $h \in H$ that H fixes $f_0, \ldots, f_\ell$. In view of the assumption $n \geq 3$, we have $n - 1 \geq 2$ and the group $H = \mathrm{SO}_{n-1}(\mathbb{R})$ acts transitively on the unit sphere $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^{n-1} \colon \|x\| = 1\}$. Therefore f_j is constant on $\mathbb{S}^{n-1}$, so that its homogeneity implies that $f_j(x) = c\|x\|^{\ell-j}$ for some $c \in \mathbb{C}$.

If ℓ is odd, then $f_0 = 0$ follows from the smoothness of f_0.[1] The proof of Lemma 7.2.5 shows that $f_{k+2} = -\Delta f_k$. Therefore $f_{2j} = 0$ for every j and f_{2j+1} is determined by f_1. Since f_1 is a multiple of $\|x\|^{\ell-1}$, it follows that the dimension of the space $\mathcal{H}_\ell^H$ of H-fixed vectors in $\mathcal{H}_\ell$ is bounded by 1. If ℓ is even, we argue similarly.

Now let $f = K^\ell_{e_n}$, where $K^\ell_x(y) = \frac{1}{\ell!}\langle y, x\rangle^\ell$. Then $f(y) = \frac{1}{\ell!} y_n^\ell \in \mathcal{P}_\ell$ is H-invariant and this carries over to $P(f)$, where $P \colon \mathcal{P}_\ell \to \mathcal{H}_\ell$ is the orthogonal projection. On the other hand $\pi(g)f = K^\ell_{ge_n}$ and thus

$$\pi(G)f = \{K^\ell_x \colon \|x\| = 1\}.$$

In view of $K^\ell_{\lambda x} = \lambda^d K^\ell_x$, this set is total in $\mathcal{P}_\ell$. Consequently $P(f)$ is G-cyclic in $\mathcal{H}_\ell$. Now the assertion follows from Corollary 7.1.2. $\square$

Remark 7.2.8. For the special case $n = 3$, we get

$$\dim \mathcal{H}_\ell = \binom{\ell+1}{1} + \binom{\ell}{1} = 2\ell + 1.$$

Therefore $\mathcal{H}_\ell$ carries a $2\ell + 1$-dimensional irreducible unitary representation of $\mathrm{SO}_3(\mathbb{R})$. One can show that every irreducible unitary representation of $\mathrm{SO}_3(\mathbb{R})$ is equivalent to one of these.

Remark 7.2.9. We consider the case $n = 2$ in the situation of Proposition 7.2.7. For $n = 2$ we obtain for $\ell \geq 2$ that $\dim \mathcal{H}_\ell = 2$ (Lemma 7.2.5). This also holds for $\ell = 1$. A basis for this space is obtained by

$$f(x, y) = (x + iy)^\ell \quad \text{and} \quad \overline{f(x, y)} = (x - iy)^\ell.$$

[1] It suffices to consider the restriction to a one-dimensional line, where $h(x) := |x|^\ell$ satisfies $h^{(\ell)}(x) = \ell! \,\mathrm{sgn}(x)$ for $0 \neq x$.

One readily verifies that these functions are eigenfunctions of $\mathrm{SO}_2(\mathbb{R})$. They are exchanged by the reflection in the x-axis. This shows that the representation of $\mathrm{O}_2(\mathbb{R})$ on $\mathcal{H}_\ell$ is irreducible, whereas the restriction to $\mathrm{SO}_2(\mathbb{R})$ decomposes into two summands. In particular, Proposition 7.2.7 does not extend to $n = 2$.

We have already seen some irreducible subrepresentations for $\mathrm{SO}_n(\mathbb{R})$ on $\mathcal{P}(\mathbb{R}^n)$. We now use them to obtain the complete decomposition.

Lemma 7.2.10. *Let*

$$D := M_{r^2}\Delta\colon \mathcal{P}_\ell \to \mathcal{P}_\ell.$$

Then $\mathcal{P}_\ell = \mathcal{H}_\ell \oplus \operatorname{im} D$ is an orthogonal $\mathrm{O}_n(\mathbb{R})$-invariant decomposition.

Proof. In view of Proposition 7.2.3, the operator $D := M_{r^2}\Delta$ is selfadjoint. We thus have an orthogonal decomposition

$$\mathcal{P}_\ell = \ker D \oplus \operatorname{im} D.$$

Further, D commutes with the action of $\mathrm{O}_n(\mathbb{R})$ (Corollary **??**). Therefore the subspaces $\ker D$ and $\operatorname{im} D$ are $\mathrm{O}_n(\mathbb{R})$-invariant. As the operator M_{r^2} is injective, we obtain $\ker D = \mathcal{H}_\ell$. □

Lemma 7.2.11. *For $\ell \geq 2$, the map $\Delta\colon \operatorname{im} D \to \mathcal{P}_{\ell-2}$ is bijective and commutes with $\mathrm{O}_n(\mathbb{R})$.*

Proof. That Δ commutes with the actions of $\mathrm{O}_n(\mathbb{R})$ follows from Lemma **??**. As $(\ker\Delta)\cap\mathcal{P}_\ell = \mathcal{H}_\ell$, the intersection of $\ker\Delta$ with $\operatorname{im} D$ is trivial (Lemma 7.2.10). Further,

$$\begin{aligned}\dim(\operatorname{im} D) &= \dim\mathcal{P}_\ell - \dim\mathcal{H}_\ell\\ &= \binom{\ell+n-1}{n-1} - \binom{\ell+n-2}{n-2} - \binom{\ell+n-3}{n-2}\\ &= \binom{\ell+n-2}{n-1} - \binom{\ell+n-3}{n-2}\\ &= \binom{\ell+n-3}{n-1} = \dim\mathcal{P}_{\ell-2}.\end{aligned}$$

Counting dimensions, it follows that the map $\Delta\colon \operatorname{im} D \to \mathcal{P}_{\ell-2}$ is also bijective. □

We now combine the results obtained so far:

Theorem 7.2.12. *Let $n \geq 3$. Under $\mathrm{SO}_n(\mathbb{R})$, the space $\mathcal{P}_\ell$ decomposes as*

$$\mathcal{P}_\ell = \bigoplus_{j=0}^{[\ell/2]} r^{2j}\mathcal{H}_{\ell-2j} \cong \mathcal{H}_\ell \oplus \mathcal{H}_{\ell-2} \oplus \ldots \oplus \begin{cases}\mathcal{P}_0 = \mathcal{H}_0, & \text{for } \ell \in 2\mathbb{Z}\\ \mathcal{P}_1 = \mathcal{H}_1, & \text{for } \ell \in 2\mathbb{Z}+1.\end{cases}$$

Proof. Combining Lemmas 7.2.10 and 7.2.11, we see that

$$\mathcal{P}_\ell = \mathcal{H}_\ell \oplus r^2\mathcal{P}_{\ell-2} \cong \mathcal{H}_\ell \oplus \mathcal{P}_{\ell-2}.$$

Now the assertion follows by induction. □

Remark 7.2.13. Considering the space $\mathcal{P} = \mathcal{P}(\mathbb{R}^n)$ of all polynomials and $\mathcal{H} = \oplus_{\ell=0}^\infty \mathcal{H}_\ell$ the subspace of harmonic polynomians, the decomposition in the preceding theorem can also be formulated as the assertion that the multiplication map

$$\mathbb{C}[r^2] \otimes \mathcal{H} \to \mathcal{P}, \quad f \otimes h \mapsto f \cdot h$$

is a linear bijection.

7.2.2 The representation of $\mathrm{SO}_n(\mathbb{R})$ on $L^2(\mathbb{S}^{n-1})$

The following elementary lemma provides a very direct way to describe the surface measure on the sphere in terms of Lebesgue measure.

Lemma 7.2.14. (The invariant measure on $\mathbb{S}^{n-1}$) *Let $\mathbb{B}^n \subseteq \mathbb{R}^n$ be the closed unit ball. Then*

$$\int_{\mathbb{S}^{n-1}} f(x)\ d\mu(x) = \int_{\mathbb{B}^n} \|x\| f\left(\frac{x}{\|x\|}\right)\ dx$$

defines an $\mathrm{O}_n(\mathbb{R})$-invariant measure on $\mathbb{S}^{n-1}$.

With Proposition 3.3.4 we now obtain a continuous unitary representation of $\mathrm{O}_n(\mathbb{R})$ on $L^2(\mathbb{S}^{n-1}) := L^2(\mathbb{S}^{n-1}, \mu)$. Using the results of the preceding subsection, we can now describe how this representation decomposes into irreducible ones. To this end, we write $\mathcal{Y}_\ell \subseteq C(\mathbb{S}^{n-1})$ for the space of restrictions of the space $\mathcal{H}_\ell$ of harmonic polynomials of degree ℓ to the sphere. The elements of this space are called *spherical harmonics of degree ℓ*.

Theorem 7.2.15. *For $n \geq 3$, the representation of $\mathrm{SO}_n(\mathbb{R})$ on $L^2(\mathbb{S}^{n-1})$ decomposes as follows into irreducible subrepresentations:*

$$L^2(\mathbb{S}^{n-1}) = \widehat{\bigoplus}_{\ell\in\mathbb{N}_0} \mathcal{Y}_\ell.$$

Proof. Since the restriction map $C(\mathbb{R}^n) \to C(\mathbb{S}^{n-1})$ commutes with the action of $\mathrm{SO}_n(\mathbb{R})$, the subspaces $\mathcal{Y}_\ell$ of $L^2(\mathbb{S}^{n-1})$ are irreducible under $\mathrm{SO}_n(\mathbb{R})$ (Theorem 7.2.12). With Lemma 7.2.5, we see that $\dim \mathcal{H}_{\ell+1} > \dim \mathcal{H}_\ell$. Therefore the subrepresentation on $\mathcal{Y}_\ell$ are pairwise inequivalent and therefore orthogonal (Corollary 2.2.4).

Since the algebra $\mathcal{P}$ of polynomials separates the points of $\mathbb{S}^{n-1}$ and is invariant under conjugation, the Stone–Weierstraß Theorem implies that the restriction map $R\colon \mathcal{P} \to C(\mathbb{S}^{n-1})$ has dense range. This implies that its range is also dense in $L^2(\mathbb{S}^{n-1})$ (cf. Proposition 3.3.2).

From Theorem 7.2.12 and the following remarks, we recall that

$$\mathcal{P} = \bigoplus_{j=0}^{\infty} \bigoplus_{\ell=0}^{\infty} r^{2j} \mathcal{H}_\ell.$$

As the functions r^{2j} restrict to constant functions on the sphere, we derive that

$$R(\mathcal{P}) = \sum_{\ell=0}^{\infty} \mathcal{Y}_\ell,$$

which shows that $\sum_{\ell=0}^{\infty} \mathcal{Y}_\ell$ is dense in $L^2(\mathbb{S}^{n-1})$. This completes the proof. □

Remark 7.2.16. For $n = 2$, the group $\mathrm{SO}_2(\mathbb{R}) \cong \mathbb{T}$ acts on $\mathbb{S}^1$ by rotations. Therefore the decomposition of $L^2(\mathbb{S}^1) \cong L^2(\mathbb{T})$ is given by expansion in Fourier series (cf. Example 2.2.11).

For a more detailed discussion of spherical harmonics on $\mathbb{S}^2$ and the connection to Legendre polynomials and functions, we refer to [BtD85, §II.10].

Exercises for Chapter 7

Exercise 7.2.1. Let $V_n = \mathcal{P}_n(\mathbb{C}^2)$ be the space of homogeneous polynomials of degree n on $\mathbb{C}^2$ with reproducing kernel $K(z,w) = \frac{\langle z,w\rangle^n}{n!}$ and recall the unitary representation of $\mathrm{SU}_2(\mathbb{C})$ on this space by $\big(\pi(g)f\big)(x) = f(g^{-1}.x)$. The same formula defines a representation of $\mathrm{SL}_2(\mathbb{C})$ which is not unitary. We want to calculate the complex linear extension $(\mathtt{d}\pi)_{\mathbb{C}}\colon \mathfrak{sl}_2(\mathbb{C}) = \mathfrak{su}_2(\mathbb{C})_{\mathbb{C}} \to \mathfrak{gl}(V_n)$ of the derived representation. Show that:

(a) For $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, we have $(\mathtt{d}\pi)_{\mathbb{C}}(X) = -z_2 \frac{\partial}{\partial z_1}$.

(b) For $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, we have $(\mathtt{d}\pi)_{\mathbb{C}}(Y) = -z_1 \frac{\partial}{\partial z_2}$.

(c) For $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ we have $(\mathtt{d}\pi)_{\mathbb{C}}(H) = -z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}$.

(d) $(\mathtt{d}\pi)_{\mathbb{C}}(Z)^* = (\mathtt{d}\pi)_{\mathbb{C}}(Z^*)$ for $Z \in \mathfrak{sl}_2(\mathbb{C})$.

(e) Find the matrices for these operators in the basis consisting of monomials.

Exercise 7.2.2. We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, so that the scalar product on $\mathbb{C}^n$ takes the form

$$\langle (p,q),(p',q')\rangle = \langle p,p'\rangle + \langle q,q'\rangle + i(\langle q,p'\rangle - \langle p,q'\rangle).$$

Recall the unitary representation of the Heisenberg group

$$\mathrm{Heis}(\mathbb{R}^{2n}) := \mathbb{R} \times \mathbb{R}^{2n}, \qquad (t,q,p)(t',q',p') := \Big(t + t' - \langle p,q'\rangle - \langle p',q\rangle, q + q', p + p'\Big)$$

on $\mathcal{F}(\mathbb{C}^n)$ by

$$(\pi(t,q,p)f)(z) = e^{it + \langle z, q+ip\rangle - \frac{1}{2}(\|p\|^2 + \|q\|^2)} f(z - q - ip).$$

We want to calculate the derived representation on functions in $\mathcal{H}^\infty$. Show that:

(a) For $Z := (1, 0, 0)$, we have $\mathtt{d}\pi(Z) = i\mathbf{1}$.

(b) For $Q_j = (0, e_j, 0)$, we have $\mathtt{d}\pi(Q_j) = z_j - \frac{\partial}{\partial z_j}$.

(c) For $P_j = (0, 0, e_j)$, we have $\mathtt{d}\pi(P_j) = -iz_j - i\frac{\partial}{\partial z_j}$.

(d) Verify the commutator relations:

$$[Q_j, P_k] = 2\delta_{jk}Z, \quad [Q_j, Q_k] = [P_j, P_k] = 0, \quad [Q_j, Z] = [P_j, Z] = 0.$$

Chapter 8

Unitary Representations of Lie Groups

In this final chapter we briefly discuss some specific aspects of unitary representations of Lie groups. We shall develop the concepts in such a way that everything remains true for infinite dimensional Lie groups.

8.1 Infinite dimensional Lie groups

The main difference between finite and infinite dimensional Lie groups are the corresponding categories of smooth manifolds. In principle, one could develop a theory of smooth manifolds for which the model spaces are general (real) topological vector spaces, but in this general context one runs into difficulties, such as the non-validity of the Fundamental Theorem of Calculus. It turns out that a sufficiently general context is provided by model spaces which are *locally convex* in the sense that 0 has a neighborhood basis consisting of convex sets.

Definition 8.1.1. (a) Let E and F be locally convex spaces, $U \subseteq E$ open and $f\colon U \to F$ a map. Then the *derivative of f at x in the direction h* is defined as

$$\mathtt{d}f(x)(h) := (\partial_h f)(x) := \frac{d}{dt}\Big|_{t=0} f(x+th) = \lim_{t\to 0} \frac{1}{t}(f(x+th) - f(x))$$

whenever it exists. The function f is called *differentiable at x* if $\mathtt{d}f(x)(h)$ exists for all $h \in E$. It is called *continuously differentiable*, if it is differentiable at all points of U and

$$\mathtt{d}f\colon U \times E \to F, \quad (x,h) \mapsto \mathtt{d}f(x)(h)$$

is a continuous map. This implies that the maps $\mathtt{d}f(x)$ are linear (cf. [GN, Lemma 2.2.14]). The map f is called a *$\mathcal{C}^k$-map*, $k \in \mathbb{N} \cup \{\infty\}$, if it is continuous, the iterated directional derivatives

$$\mathtt{d}^j f(x)(h_1, \ldots, h_j) := (\partial_{h_j} \cdots \partial_{h_1} f)(x)$$

exist for all integers $1 \leq j \leq k$, $x \in U$ and $h_1, \ldots, h_j \in E$, and all maps $\mathtt{d}^j f : U \times E^j \to F$ are continuous. As usual, $\mathcal{C}^\infty$-maps are called *smooth*.

(b) If E and F are complex locally convex spaces, then f is called *complex analytic* if it is continuous and for each $x \in U$ there exists a 0-neighborhood V with $x + V \subseteq U$ and continuous homogeneous polynomials $\beta_k : E \to F$ of degree k such that, for each $h \in V$, we have

$$f(x+h) = \sum_{k=0}^{\infty} \beta_k(h),$$

as a pointwise limit. The map f is called *holomorphic* if it is $\mathcal{C}^1$ and, for each $x \in U$, the map $\mathtt{d}f(x) : E \to F$ is complex linear. If F is sequentially complete, then f is holomorphic if and only if it is complex analytic ([BS71, Ths. 3.1, 6.4]).

(c) If E and F are real locally convex spaces, then we call a map $f : U \to F$, $U \subseteq E$ open, *real analytic* or a $\mathcal{C}^\omega$-map, if for each point $x \in U$ there exists an open neighborhood $V \subseteq E_{\mathbb{C}}$ and a holomorphic map $f_{\mathbb{C}} : V \to F_{\mathbb{C}}$ with $f_{\mathbb{C}}|_{U \cap V} = f|_{U \cap V}$. The advantage of this definition, which differs from the one in [BS71], is that it also works nicely for non-complete spaces. Any analytic map is smooth, and the corresponding chain rule holds without any condition on the underlying spaces, which is the key to the definition of analytic manifolds (see [Gl02] for details).

Once the concept of a smooth map between open subsets of locally convex spaces and the Chain Rule are established (cf. [Ne06], [GN]), it is clear how to define a locally convex smooth manifold. A *(locally convex) Lie group* G is a group equipped with a smooth manifold structure modeled on a locally convex space for which the group multiplication and the inversion are smooth maps. We write $\mathbf{1} \in G$ for the identity element. Then each $x \in T_{\mathbf{1}}(G)$ corresponds to a unique left invariant vector field x_l with $x_l(\mathbf{1}) = x$. The space of left invariant vector fields is closed under the Lie bracket of vector fields, hence inherits a Lie algebra structure. In this sense we obtain on $\mathfrak{g} := T_{\mathbf{1}}(G)$ a continuous Lie bracket which is uniquely determined by $[x, y] = [x_l, y_l](\mathbf{1})$ for $x, y \in \mathfrak{g}$. We shall also use the functorial notation $\mathbf{L}(G) := (\mathfrak{g}, [\cdot, \cdot])$ for the Lie algebra of G. The adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ of G on $\mathfrak{g}$ is defined by $\mathrm{Ad}(g) := T_{\mathbf{1}}(c_g)$, where $c_g(x) = gxg^{-1}$ is the conjugation map. The adjoint action is smooth and each $\mathrm{Ad}(g)$ is a topological isomorphism of $\mathfrak{g}$. If $\mathfrak{g}$ is a Fréchet, resp., a Banach space, then G is called a *Fréchet-*, resp., a *Banach–Lie group*.

A smooth map $\exp_G : \mathfrak{g} \to G$ is called an *exponential function* if each curve $\gamma_x(t) := \exp_G(tx)$ is a one-parameter group with $\gamma_x'(0) = x$. The Lie group G is said to be *locally exponential* if it has an exponential function for which there is an open 0-neighborhood U in $\mathfrak{g}$ mapped diffeomorphically by $\exp_G$ onto an open subset of G. If, in addition, G is an analytic Lie group (an analytic manifold with analytic group operations) and the exponential function is an analytic diffeomorphism in a 0-neighborhood, then G is called a *BCH–Lie group* (for Baker–Campbell–Hausdorff). The class of BCH–Lie groups contains in particular all Banach–Lie groups, which includes the class of finite dimensional Lie groups ([Ne06, Prop. IV.1.2]).

Not every Lie group G has an exponential function and it is still an open problem to show the existence of an exponential function if the model space, resp., the Lie algebra $\mathfrak{g}$, is a complete space.

We refer to [Ne06] and [GN] for more details on the following classes of examples of infinite dimensional Lie groups.

Example 8.1.2. (a) For a unital Banach algebra $\mathcal{A}$, the unit group $\mathcal{A}^\times$ is a Banach–Lie group with Lie algebra $(\mathcal{A}, [\cdot,\cdot])$ and exponential function

$$\exp x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

If $\mathcal{A}$ is a C^*-algebra, then its unitary group $\mathrm{U}(\mathcal{A})$ is a Banach–Lie group with Lie algebra

$$\mathfrak{u}(\mathcal{A}) = \{x \in \mathcal{A} \colon x^* = -x\}.$$

(b) If K is a Lie group and M a compact smooth manifold, then $G := C^\infty(M,K)$ is a Fréchet–Lie group with Lie algebra $\mathfrak{g} := C^\infty(M,\mathfrak{k})$. If $\exp_K \colon \mathfrak{k} \to K$ is an exponential function for K, then

$$\exp_G \colon \mathfrak{g} \to G, \quad \xi \mapsto \exp_K \circ \xi$$

is an exponential function of G.

(c) If M is a compact smooth manifold, then $G := \mathrm{Diff}(M)$ is a Fréchet–Lie group with Lie algebra $\mathcal{V}(M)$, the Lie algebra of smooth vector fields on M and exponential function

$$\exp \colon \mathcal{V}(M) \to \mathrm{Diff}(M), \quad \exp(X) = \Phi_1^X \text{ (time-1-flow)}.$$

If $\dim M > 0$, then this Lie group is not locally exponential, so that one cannot obtain charts from the exponential function. One can also show that it is not analytic.

8.2 The derived representation

Let G be a Lie group with Lie algebra $\mathfrak{g} = \mathbf{L}(G)$ and exponential function $\exp \colon \mathfrak{g} \to G$. To study unitary representations of G, we have to specify smoothness properties of the representation to make the passage between group and Lie algebra work.

Definition 8.2.1. Let $(\pi, \mathcal{H})$ be a unitary representation of G. We call an element $v \in \mathcal{H}$ a *smooth vector* if the orbit map

$$\pi^v \colon G \to \mathcal{H}, \quad g \mapsto \pi(g)v$$

is a smooth map. The space of smooth vectors is denoted $\mathcal{H}^\infty$.

If G is an analytic Lie group, we call v an *analytic vector* if the orbit map

$$\pi^v \colon G \to \mathcal{H}, \quad g \mapsto \pi(g)v$$

is analytic. For BCH–Lie groups G, this is equivalent to the analyticity of the map $\pi^v \circ \exp \colon \mathfrak{g} \to \mathcal{H}$. Note that the latter condition also makes sense if G is not analytic.

The representation $(\pi, \mathcal{H})$ is said to be *smooth* if $\mathcal{H}^\infty$ is dense and it is called *analytic* if the smaller subspace $\mathcal{H}^\omega$ is dense.

Definition 8.2.2. (The derived representation) For $v \in \mathcal{H}^\infty$ and $x \in \mathfrak{g}$ we put

$$\mathtt{d}\pi(x)v := \frac{d}{dt}\Big|_{t=0}\pi(\exp tx)v$$

and consider

$$\mathtt{d}\pi(x) \colon \mathcal{H}^\infty \to \mathcal{H}$$

as an unbounded operator on $\mathcal{H}$.

Proposition 8.2.3. (The derived representation)

(i) *The subspace $\mathcal{H}^\infty$ is invariant under $\pi(G)$ and*

$$\mathtt{d}\pi(\mathrm{Ad}(g)x) = \pi(g)\mathtt{d}\pi(x)\pi(g)^{-1} \quad \textit{for} \quad g \in G, x \in \mathfrak{g}.$$

(ii) $\mathtt{d}\pi(x)\mathcal{H}^\infty \subseteq \mathcal{H}^\infty$.

(iii) $\mathtt{d}\pi \colon \mathfrak{g} \to \mathrm{End}(\mathcal{H}^\infty)$ *is a representation of* $\mathfrak{g}$ *on* $\mathcal{H}^\infty$.

(iv) *If π is smooth, i.e., $\mathcal{H}^\infty$ is dense, then the operators $i\mathtt{d}\pi(x)$ are essentially selfadjoint.*

(v) *If G is locally exponential and connected, then any smooth unitary representation of G is uniquely determined by its derived representation.*

The representation $(\mathtt{d}\pi, \mathcal{H}^\infty)$ is called the derived representation of $(\pi, \mathcal{H})$.

Proof. (i) For $v \in \mathcal{H}^\infty$ and $g \in G$, the function $\pi^v \circ \rho_g = \pi^{\pi(g)v}$, $\rho_g(h) = hg$, is also smooth. Therefore $\pi(g)v$ is also a smooth vector. We further have

$$\pi(g)\pi(\exp tx)\pi(g^{-1})v = \pi(g(\exp tx)g^{-1})v = \pi(\exp(\mathrm{Ad}(g)tx))v,$$

so that (i) follows by taking derivatives in $t = 0$.

(ii) and (iii): We consider the embedding

$$\eta \colon \mathcal{H}^\infty \to C^\infty(G, \mathcal{H}), \quad \eta(v) := \pi^v, \quad \pi^v(g) = \pi(g)v.$$

Then we obtain for the left invariant vector field $x_l(g) = gx$ on G that

$$(x_l\pi^v)(g) = \frac{d}{dt}\Big|_{t=0}\pi(g)\pi(\exp tx)v = \pi(g)\mathtt{d}\pi(x)v.$$

Since the function $x_l \pi^v$ is also smooth, $\mathtt{d}\pi(x)v$ is a smooth vector by (ii). We also obtain the relation

$$x_l \pi^v = \pi^{\mathtt{d}\pi(x)v},$$

which leads to

$$\pi^{\mathtt{d}\pi([x,y])v} = [x,y]_l \pi^v = x_l y_l \pi^v - y_l x_l \pi^v = \pi^{\mathtt{d}\pi(x)\mathtt{d}\pi(y)v - \mathtt{d}\pi(x)\mathtt{d}\pi(y)v}.$$

Evaluating in $g = \mathbf{1}$, we obtain

$$\mathtt{d}\pi([x,y])v = [\mathtt{d}\pi(x), \mathtt{d}\pi(y)]v,$$

which proves (iii).

(iv) Since $-i\mathtt{d}\pi(x)$ is the restriction of the infinitesimal generator of the unitary one-parameter group $\pi(\exp tx)$ to the dense subspace $\mathcal{H}^\infty$ and this subspace is invariant under $\pi(\exp tx)$ by (i), this follows from Stone's Theorem 6.2.5(ii).

(v) follows from (iv) and the uniqueness assertion in Stone's Theorem which ensures that the restriction of π to each one-parameter group $\exp(\mathbb{R}x)$ is uniquely determined by the derived representation $\mathtt{d}\pi$. The assertion now follows from $G = \langle \exp \mathfrak{g} \rangle$, which holds for any locally exponential connected Lie group G because the subgroup $\langle \exp \mathfrak{g} \rangle$ generated by the image of the exponential function is open. □

Definition 8.2.4. Let $\mathfrak{g}$ be a real Lie algebra and $\mathcal{D}$ be a complex pre-Hilbert space. Then a homomorphism $\rho \colon \mathfrak{g} \to \mathfrak{gl}(\mathcal{D})$ of Lie algebras is called a *unitary representation* if the operators $\rho(x)$, $x \in \mathfrak{g}$, are skew-symmetric, i.e.,

$$\langle \rho(x)v, w \rangle = -\langle v, \rho(x)w \rangle \quad \text{for} \quad x \in \mathfrak{g}, v, w \in \mathcal{D}.$$

Although the preceding proposition shows that (for connected locally exponential Lie groups) smooth unitary representations can be recovered from their derived representation, it is a difficult problem to determine which unitary representation $\rho \colon \mathfrak{g} \to \mathrm{End}(\mathcal{D})$ can actually be *integrated* in the sense that it is obtained by restricting the derived representation of a smooth group representation $(\pi, \mathcal{H})$ on the completion $\mathcal{H} = \overline{\mathcal{D}}$ to the dense subspace $\mathcal{D} \subseteq \mathcal{H}^\infty$. For the group $G = \mathbb{R}$, the corresponding problem is to determine which symmetric operator $A \colon \mathcal{D} \to \mathcal{D} \subseteq \mathcal{H}$ is essentially self-adjoint, because this means that its closure $\overline{A}$ generates a unitary one-parameter group in the sense of Stone's Theorem 6.2.5.

Theorem 8.2.5. (Nelson's Criterion) *Let G be a finite dimensional simply connected Lie group. If $\rho \colon \mathfrak{g} \to \mathrm{End}(\mathcal{D})$ is a representation of $\mathfrak{g}$ on the pre-Hilbert space $\mathcal{D}$ by skew-symmetric operators and $\mathcal{D}$ consists of analytic vectors for each operator $\rho(x)$, then there exists a unique unitary representation $(\pi, \mathcal{H})$ of G on $\mathcal{H}$, for which $\mathcal{D} \subseteq \mathcal{H}^\omega$ and $\rho(x) = \mathtt{d}\pi(x)|_{\mathcal{D}}$ for every $x \in \mathfrak{g}$.*

Nelson's Criterion is also valid for Banach–Lie groups (cf. [Ne11, Thm. 6.8]). Applying Nelson's criterion to $G = \mathbb{R}$, it provides the following criterion for essential selfadjointness:

Corollary 8.2.6. *If $A\colon \mathcal{D} \to \mathcal{H}$ is a symmetric operator for which the subspace $\mathcal{D}^\omega(A)$ of analytic vectors is dense, then A is essentially selfadjoint.*

The following criterion ([Mer11]) is sometimes more useful because it does not require the a priori existence of analytic vectors:

Theorem 8.2.7. (Merigon's Criterion) *Let G be a simply connected Banach–Lie group. If $\rho\colon \mathfrak{g} \to \mathrm{End}(\mathcal{D})$ is a representation of $\mathfrak{g}$ on the pre-Hilbert space $\mathcal{D}$ by operators for which the following conditions are satisfied:*

(i) *$i\rho(x)$ is essentially selfadjoint for every $x \in \mathfrak{g}$. In particular, we obtain a unitary one-parameter group $U_t^x := e^{t\rho(x)}$ by Stone's Theorem.*

(ii) *$\mathcal{D}$ is invariant under every U_t^x, $x \in \mathfrak{g}$, $t \in \mathbb{R}$.*

(iii) *$U_1^x \rho(y)(U_1^x)^* = \rho(e^{\mathrm{ad}\, x} y)$ for $x, y \in \mathfrak{g}$.*

Then there exists a smooth unitary representation $(\pi, \mathcal{H})$ of G on $\mathcal{H}$ for which $\mathcal{D} \subseteq \mathcal{H}^\infty$ and $\rho(x) = \mathtt{d}\pi(x)|_{\mathcal{D}}$ for every $x \in \mathfrak{g}$.

Theorem 8.2.8. (Nelson, 1959) *Every continuous unitary representation of a finite dimensional Lie group G is analytic, i.e., the subspace $\mathcal{H}^\omega$ of analytic vectors is dense.*

Proof. (Sketch) The idea is to use a mollifying technique. One can show that the fundamental solutions $(\rho_t)_{t>0}$ of a suitable variant of the heat equation on G are L^1-functions with respect to Haar measure.[1] Moreover, they are analytic and the map $G \to L^1(G), g \mapsto \rho_t \circ \lambda_g$ is analytic. Therefore the operators

$$\pi(\rho_t)v := \int_G \rho_t(g)\pi(g)v\, dg$$

are well-defined. Now the relation

$$\pi(h)\pi(\rho_t)v = \int_G \rho_t(g)\pi(hg)v\, dg = \int_G \rho_t(h^{-1}g)\pi(g)v\, dg = \pi(\rho_t \circ \lambda_{h^{-1}})v$$

implies that all vectors in the range of $\pi(\rho_t)$ are analytic. The density of $\mathcal{H}^\omega$ now follows from

$$\lim_{t\to 0} \pi(\rho_t)v = v. \qquad \square$$

It is much easier to show that the space $\mathcal{H}^\infty$ of smooth vectors is dense (Gårding's Theorem). One simply follows the argument in the preceding proof, where $\rho_n \in C_c^\infty(G)$ is a sequence of compactly supported smooth functions with $\int_G \rho_n(g)\, dg = 1$ and $\mathrm{supp}(\rho_n) \to \mathbf{1}$ in the sense that, for every $\mathbf{1}$-neighborhood U of $\mathbf{1}$ there exists an $N \in \mathbb{N}$ with $\mathrm{supp}(\rho_n) \subseteq U$ for $n \geq N$. Then $\pi(\rho_n)v \to v$ for every $v \in \mathcal{H}$ and $\pi(\rho_n)v \in \mathcal{H}^\infty$.

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[1] For $G = \mathbb{R}^n$, one obtains functions of the form $\rho_t = \frac{c}{\sqrt{t}} e^{-\frac{\|x\|^2}{4t}}$, where c is a constant ensuring that $\int_{\mathbb{R}^n} \rho_t(x)\, dx = 1$.

8.3 Bounded representations

The Lie theoretic tools work particularly well for unitary representations that behave like finite dimensional ones. These representations are called bounded:

Definition 8.3.1. A unitary representation $(\pi, \mathcal{H})$ of a topological group G is called *bounded* or *norm continuous* if $\pi\colon G \to \mathrm{U}(\mathcal{H})$ is continuous with respect to the operator norm

The boundedness of a unitary representation has particularly strong consequences for Lie groups:

Proposition 8.3.2. *For a unitary representation $(\pi, \mathcal{H})$ of a locally exponential Lie group G, the following are equivalent:*

(i) *π is bounded.*

(ii) *$\pi\colon G \to \mathrm{U}(\mathcal{H})$ is a smooth homomorphism of Lie groups, where $\mathrm{U}(\mathcal{H})$ carries the Lie group structure defined by the norm topology.*

(iii) *$\mathcal{H}^\infty = \mathcal{H}$ and the derived representation $\mathtt{d}\pi\colon \mathfrak{g} \to B(\mathcal{H})$ is a continuous representation by bounded operators.*

Proof. (i) $\Rightarrow$ (ii): For $x \in \mathfrak{g}$, let $\pi_x(t) := \pi(\exp tx)$ denote the corresponding one-parameter group. Since it is norm-continuous, combining Exercise 5.4.2 with the Spectral Theorem for selfadjoint operators (Theorem 6.1.9), it follows that the operator $\mathbf{L}(\pi)x := \frac{d}{dt}\big|_{t=0}\pi(\exp tx)$ is bounded and satisfies $\pi(\exp x) = e^{\mathbf{L}(\pi)x}$. With the same arguments as for finite dimensional Lie groups (Trotter Product Formula and Commutator Formula) one then shows that $\mathbf{L}(\pi)\colon \mathfrak{g} \to \mathfrak{u}(\mathcal{H})$ is a homomorphism of Lie algebras. Since $\exp_G$ and $\exp_{\mathrm{U}(\mathcal{H})}$ are local homeomorphisms in 0, the relation $\pi \circ \exp_G = \exp_{\mathrm{U}(\mathcal{H})} \circ \mathbf{L}(\pi)$ implies that $\mathbf{L}(\pi)$ is continuous in 0, hence a continuous linear map. As continuous linear maps are smooth, we use that $\exp_G$ and $\exp_{\mathrm{U}(\mathcal{H})}$ are local diffeomorphisms in 0 to see that the same relation shows that π is smooth in an identity neighborhood. But this implies the smoothness of π.

(ii) $\Rightarrow$ (iii) follows from the fact that $\mathbf{L}(\pi) = T_{\mathbf{1}}(\pi)\colon \mathfrak{g} \to \mathfrak{u}(\mathcal{H})$ is a continuous homomorphism of Lie algebras and $\mathbf{L}(\pi)x = \mathtt{d}\pi(x)$ for $x \in \mathfrak{g}$.

(iii) $\Rightarrow$ (ii): From the uniqueness assertion in Stone's Theorem it follows that $e^{\mathtt{d}\pi(x)} = \pi(\exp x)$ for every $x \in \mathfrak{g}$. Now the same argument as above implies that π is smooth.

(ii) $\Rightarrow$ (i) is trivial. $\square$

For bounded representations of Lie algebras, their integrability is simply a covering issue, as for finite dimensional representations.

Theorem 8.3.3. *Let $\rho\colon \mathfrak{g} \to B(\mathcal{H})$ be a representation of $\mathfrak{g}$ by bounded skew-hermitian operators and G a 1-connected locally exponential Lie group with Lie algebra $\mathfrak{g}$. Then the following assertions hold:*

(a) *There exists a unique bounded representation* $(\pi, \mathcal{H})$ *of* G *on* $\mathcal{H}$ *with* $\rho = \mathtt{d}\pi$.

(b) *We have the equality* $\rho(\mathfrak{g})' = \pi(G)'$ *of commutants.*

(c) *A closed subspace of* $\mathcal{H}$ *is invariant under* $\rho(\mathfrak{g})$ *if and only if it is invariant under* $\pi(G)$.

Proof. (a) The existence of π follows from basic Lie theory. The representation π is uniquely determined by the relation $\pi(\exp x) = e^{\rho(x)}$ for $x \in \mathfrak{g}$.

(b) Since G is generated by $\exp \mathfrak{g}$, $\pi(G) \subseteq \rho(\mathfrak{g})''$ follows from

$$\pi(\exp x) = e^{\rho(x)} = \sum_{n=0}^{\infty} \frac{1}{n!}\rho(x)^n \in \rho(\mathfrak{g})'' \quad \text{for} \quad x \in \mathfrak{g}.$$

The relation $\rho(\mathfrak{g}) \subseteq \pi(G)''$ follows from the existence of the norm limit

$$\rho(x) = \lim_{t\to 0} \frac{1}{t}(\pi(\exp tx) - \mathbf{1}) \in \pi(G)''.$$

This shows that $\rho(\mathfrak{g})'' = \pi(G)''$, and hence also that $\rho(\mathfrak{g})' = \pi(G)'$.

(c) follows immediately from (b) because the closed invariant subspace are in one-to-one correspondence with the hermitian projections in the commutant (Lemma 1.3.1). □

Among the finite dimensional connected Lie groups, the compact ones are the only groups for which all irreducible unitary representations are bounded. They are actually finite dimensional, hence can be dealt with completely in the realm of finite dimensional Lie theory.

Remark 8.3.4. If G is a connected finite dimensional Lie group and $(\pi, \mathcal{H})$ a faithful bounded representation, then one can show that $G \cong K \times \mathbb{R}^n$ is a direct product group, where K is compact.[2] Therefore the existence of bounded unitary representations has strong structural consequences for finite dimensional Lie groups. In particular, they only occur for compact and abelian groups (and their products). This is drastically different for infinite dimensional Lie groups such as the Banach–Lie group $\mathrm{U}(\mathcal{H})$, where the underlying topology is defined by the operator norm.

8.4 Compact Lie groups—Weyl's Unitary Trick

The preceding theorem implies in particular, that, for a simply connected compact Lie group G, the irreducible unitary representations correspond precisely

[2] Roughly the argument proceeds as follows. First we observe that $\|\mathtt{d}\pi(x)\|$ defines an $\mathrm{Ad}(G)$-invariant norm on $\mathfrak{g}$, and this in turn implies that the closure of $\mathrm{Ad}(G)$ in $\mathrm{GL}(\mathfrak{g})$ is compact, i.e., $\mathfrak{g}$ is a compact Lie algebra ([HiNe12, Sect. 12.1]). Now the assertion follows from the Structure Theorem for Groups with Compact Lie Algebra ([HiNe12, Thm. 12.1.18]).

to the homomorphisms $\rho\colon \mathfrak{g} \to \mathfrak{u}_n(\mathbb{C})$. To classify such representations, one passes to complex Lie algebras by observing that

$$\rho_{\mathbb{C}}\colon \mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} \to \mathfrak{u}_n(\mathbb{C})_{\mathbb{C}} \cong \mathfrak{gl}_n(\mathbb{C})$$

is a complex linear representation of $\mathfrak{g}_{\mathbb{C}}$ (which actually is a semisimple complex Lie algebra).

If, conversely, $\rho_{\mathbb{C}}\colon \mathfrak{g}_{\mathbb{C}} \to \mathfrak{gl}_n(\mathbb{C})$ is a complex linear representation of $\mathfrak{g}_{\mathbb{C}}$, then $\rho := \rho_{\mathbb{C}}|_{\mathfrak{g}}\colon \mathfrak{g} \to \mathfrak{gl}_n(\mathbb{C})$ is a representation of $\mathfrak{g}$, but it need not be unitary. However, it integrates to a representation $\pi\colon G \to \mathrm{GL}_n(\mathbb{C})$ of the compact simply connected Lie group G. This representation can be made unitary by replacing the canonical scalar product $\langle \cdot, \cdot \rangle_{\mathbb{C}^n}$ on $\mathbb{C}^n$ by

$$\langle v, w \rangle := \int_G \langle \pi(g)v, \pi(g)w \rangle_{\mathbb{C}^n}\, dg, \tag{8.1}$$

where dg stands for the invariant probability measure of G, the Haar measure on G. The right invariance of dg implies that the scalar product (8.1) is invariant, so that we obtain a unitary representation. This procedure is called *Weyl's unitary trick*. It leads to the following theorem:

Theorem 8.4.1. (Weyl's Unitary Trick) *Let G be a* 1*-connected compact Lie group. If $(\pi, \mathcal{H})$ is a finite dimensional unitary representation of G, then π is bounded and $(\mathtt{d}\pi)_{\mathbb{C}}\colon \mathfrak{g}_{\mathbb{C}} \to \mathfrak{gl}(\mathcal{H})$ is a complex linear representation of the semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$.*

If, conversely, (ρ, V) is a finite dimensional representation of $\mathfrak{g}_{\mathbb{C}}$, then there exists a positive definite hermitian form on V for which the operators $\rho(x)$, $x \in \mathfrak{g}$, are skew-symmetric and $\rho|_{\mathfrak{g}}$ integrates to a unitary representation (π, V) of G.

At this point, the classification of unitary representations of G has been translated into the completely algebraic finite dimensional representation theory of the semisimple complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$. This is solved by the Cartan–Weyl Theorem, parametrizing the irreducible representations of the semisimple complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$ by their "highest weights."

For non-compact Lie groups, the theory of unitary representations goes much beyond a completely algebraic theory. We have seen in Chapter 6 how it looks for the one-dimensional Lie group $G = \mathbb{R}$. In this case the problem is completely analytical and solved by the theory of spectral measures and selfadjoint operators. If G is non-commutative, then we have to study unitary representations of non-abelian Lie algebras by unbounded operators, and this is still an important challenge, in particular for infinite dimensional Lie groups.

End of Lecture

Appendix A

Complementary material

A.1 Locally Compact Spaces

Proposition A.1.1. *For a topological space X, the following are equivalent:*

(i) *X is quasicompact, i.e., every open cover has a finite subcover.*

(ii) *For each family $(A_i)_{i\in I}$ of closed subsets of X with $\bigcap_{i\in I} A_i = \emptyset$, there exists a finite subset $F \subseteq I$ with $\bigcap_{i\in F} A_i = \emptyset$.*

Proof. (i) $\Leftrightarrow$ (ii) follows by taking complements: The condition $\bigcap_{i\in I} A_i = \emptyset$ means that the family $(A_i^c)_{i\in I}$ of complements is an open covering of X because $X = \emptyset^c = \bigcup_{i\in I} A_i^c$. Similarly, $\bigcap_{i\in F} A_i = \emptyset$ means that $(A_i^c)_{i\in F}$ is a finite subcovering. □

Definition A.1.2. A separated topological space X is called *locally compact* if each point $x \in X$ has a compact neighborhood.

Lemma A.1.3. *If X is locally compact and $x \in X$, then each neighborhood U of x contains a compact neighborhood of x.*

Proof. Let K be a compact neighborhood of $x \in X$. Since it suffices to show that $U \cap K$ contains a compact neighborhood of x, we may w.l.o.g. assume that X is compact. Replacing U by its interior, we may further assume that U is open, so that its complement U^c is compact.

We argue by contradiction and assume that U does not contain any compact neighborhood of x. Then the family $\mathcal{F}$ of all intersections $C \cap U^c$, where C is a compact neighborhood of x, contains only non-empty sets and is stable under finite intersections. We thus obtain a family of closed subsets of the compact space U^c for which all finite intersections are non-empty, and therefore Proposition A.1.1 implies that its intersection $\bigcap_C (C \cap U^c)$ contains a point y. Then $y \in U^c$ implies $x \neq y$, and since X is separated, there exist open neighborhoods U_x of x and U_y of y with $U_x \cap U_y = \emptyset$. Then U_y^c is a compact neighborhood of x, which leads to the contradiction $y \in U_y^c \cap U^c$ to $y \in U_y$. □

Definition A.1.4. A subset A of a topological space X is said to be *relatively compact* if $\overline{A}$ is compact.

Lemma A.1.5. *Let X be locally compact, $K \subseteq X$ compact and $U \supseteq K$ open. Then there exists a compact subset $V \subseteq X$ with*

$$K \subseteq V^0 \subseteq V \subseteq U.$$

Proof. For each $x \in K$ we choose a compact neighborhood $V_x \subseteq U$ (Lemma A.1.3). Then there exist finitely many $x_1, \ldots, x_n$ with $K \subseteq \bigcup_{i=1}^n V_{x_i}^0$ and we put $V := \bigcup_{i=1}^n V_{x_i} \subseteq U$. $\square$

Proposition A.1.6. (Urysohn's Theorem) *Let X be locally compact, $K \subseteq X$ compact and $U \supseteq K$ be an open subset. Then there exists a continuous function $h \colon X \to \mathbb{R}$ with*

$$h|_K = 1 \quad \textit{and} \quad h|_{X \setminus U} = 0.$$

Proof. We put $U(1) := U$. With Lemma A.1.5, we find an open, relatively compact subset $U(0)$ with $K \subseteq U(0) \subseteq \overline{U(0)} \subseteq U(1)$. Iterating this procedure leads to a subset $U(\frac{1}{2})$ with

$$\overline{U(0)} \subseteq U\Big(\frac{1}{2}\Big) \subseteq \overline{U\Big(\frac{1}{2}\Big)} \subseteq U(1).$$

Continuing like this, we find for each dyadic number $\frac{k}{2^n} \in [0,1]$ an open, relatively compact subset $U(\frac{k}{2^n})$ with

$$\overline{U\Big(\frac{k}{2^n}\Big)} \subseteq U\Big(\frac{k+1}{2^n}\Big) \quad \text{for} \quad k = 0, \ldots, 2^n - 1.$$

Let $\mathbb{D} := \{\frac{k}{2^n} \colon k = 0, \ldots, 2^n, n \in \mathbb{N}\}$ for the set of dyadic numbers in $[0,1]$. For $r \in [0,1]$, we put

$$U(r) := \bigcup_{s \leq r, s \in \mathbb{D}} U(s).$$

For $r = \frac{k}{2^n}$ this is consistent with the previous definition. For $t < t'$ we now find $r = \frac{k}{2^n} < r' = \frac{k+1}{2^n}$ in $\mathbb{D}$ with $t < r < r' < t'$, so that we obtain

$$\overline{U(t)} \subseteq \overline{U(r)} \subseteq U(r') \subseteq U(t').$$

We also put $U(t) = \emptyset$ for $t < 0$ and $U(t) = X$ for $t > 1$. Finally, we define

$$f(x) := \inf\{t \in \mathbb{R} : x \in U(t)\}.$$

Then $f(K) \subseteq \{0\}$ and $f(X \setminus U) \subseteq \{1\}$.

We claim that f is continuous. So let $x_0 \in X$, $f(x_0) = t_0$ and $\varepsilon > 0$. We put $V := U(t_0 + \varepsilon) \setminus \overline{U(t_0 - \varepsilon)}$ and note that this is a neighborhood of x_0. From $x \in V \subseteq \underline{U(t_0 + \varepsilon)}$ we derive $f(x) \leq t_0 + \varepsilon$. If $f(x) < t_0 - \varepsilon$, then also $x \in U(t_0 - \varepsilon) \subseteq \overline{U(t_0 - \varepsilon)}$, which is a contradiction. Therefore $|f(x) - f(x_0)| \leq \varepsilon$ holds on V, and this implies that f is continuous. Finally, we put $h := 1 - f$. $\square$

Exercises for Section A.1

Exercise A.1.1. (One point compactification) Let X be a locally compact space. Show that:

(i) There exists a compact topology on the set $X_\omega := X \cup \{\omega\}$, where ω is a symbol of a point not contained in X. Hint: A subset $O \subseteq X_\omega$ is open if it either is an open subset of X or $\omega \in O$ and $X \setminus O$ is compact.

(ii) The inclusion map $\eta_X \colon X \to X_\omega$ is a homeomorphism onto an open subset of X_ω.

(iii) If Y is a compact space and $f \colon X \to Y$ a continuous map which is a homeomorphism onto the complement of a point in Y, then there exists a homeomorphism $F \colon X_\omega \to Y$ with $F \circ \eta_X = f$.

The space X_ω is called the *Alexandroff compactification* or the *one point compactification* of X.[1]

Exercise A.1.2. (Stereographic projection) We consider the n-dimensional sphere

$$\mathbb{S}^n := \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \colon x_0^2 + x_1^2 + \cdots + x_n^2 = 1\}.$$

We call the unit vector $e_0 := (1, 0, \ldots, 0)$ the *north pole* of the sphere and $-e_0$ the *south pole*. We then have the corresponding *stereographic projection maps*

$$\varphi_+ \colon U_+ := \mathbb{S}^n \setminus \{e_0\} \to \mathbb{R}^n, \quad (y_0, y) \mapsto \frac{1}{1 - y_0} y$$

and

$$\varphi_- \colon U_- := \mathbb{S}^n \setminus \{-e_0\} \to \mathbb{R}^n, \quad (y_0, y) \mapsto \frac{1}{1 + y_0} y.$$

Show that these maps are homeomorphisms with inverse maps

$$\varphi_\pm^{-1}(x) = \Big(\pm \frac{\|x\|_2^2 - 1}{\|x\|_2^2 + 1}, \frac{2x}{1 + \|x\|_2^2} \Big).$$

Exercise A.1.3. Show that the one-point compactification of $\mathbb{R}^n$ is homeomorphic to the n-dimensional sphere $\mathbb{S}^n$. Hint: Exercise A.1.2.

Exercise A.1.4. Show that the one-point compactification of an open interval $]a, b[\subseteq \mathbb{R}$ is homeomorphic to $\mathbb{S}^1$.

Exercise A.1.5. Let X be a locally compact space and $Y \subseteq X$ be a subset. Show that Y is locally compact with respect to the subspace topology if and only if there exists an open subset $O \subseteq X$ and a closed subset A with $Y = O \cap A$. Hint: If Y is locally compact, write it as a union of compact subsets of the form $O_i \cap Y$, O_i open in X, where $O_i \cap Y$ has compact closure, contained in Y. Then put $O := \bigcup_{i \in I} O_i$ and $A := \overline{Y \cap O}$.

Exercise A.1.6. Let $f \colon X \to Y$ be a continuous proper map between locally compact spaces, i.e., inverse image of compact subsets are compact. Show that

(a) f is a closed map, i.e., maps closed subsets to closed subsets.

(b) If f is injective, then it is a topological embedding onto a closed subset.

[1] Alexandroff, Pavel (1896–1982)

(c) There is a well-defined homomorphism $f^*\colon C_0(Y) \to C_0(X)$ of C^*-algebras, defined by $f^*h := h \circ f$.

(d) For each regular Borel measure μ on X, the push-forward measure $f_*\mu$ on Y, defined by $(f_*\mu)(E) := \mu(f^{-1}(E))$ is regular. Hint: To verify outer regularity, pick an open $O \supseteq f^{-1}(E)$ with $\mu(O \setminus f^{-1}(E)) < \varepsilon$. Then $U := f(O^c)^c$ is an open subset of Y containing E and $\widetilde{O} := f^{-1}(U)$ satisfies $f^{-1}(E) \subseteq \widetilde{O} \subseteq O$, which leads to $(f_*\mu)(U \setminus E) < \varepsilon$.

A.2 The Stone–Weierstraß Theorem

Definition A.2.1. (a) Let M be a set and $\mathcal{A} \subseteq \mathbb{K}^M$ be a set of functions $M \to \mathbb{K}$. We say that $\mathcal{A}$ *separates the points of* M if for two points $x \neq y$ in X there exists some $f \in \mathcal{A}$ with $f(x) \neq f(y)$.

(b) A linear subspace $\mathcal{A} \subseteq \mathbb{K}^M$ is called an *algebra* if it is closed under pointwise multiplication.

Theorem A.2.2. (Dini's Theorem) [2] *Let X be a compact space and $(f_n)_{n\in\mathbb{N}}$ be a monotone sequence of functions in $C(X,\mathbb{R})$. If $(f_n)_{n\in\mathbb{N}}$ converges pointwise to some $f \in C(X,\mathbb{R})$, then the convergence is uniform, i.e., $\|f_n - f\|_\infty \to 0$.*

Proof. Idea: First we find for each $x \in X$ and each $\varepsilon > 0$ a neighborhood U_x and an $n_x \in \mathbb{N}$ with $|f(x) - f_n(y)| < \varepsilon$ for $y \in U_x$ and $n \geq n_x$. Then X is covered by finitely many such U_x and the monotony is used.

Here are the details: Replacing f_n by $f - f_n$ or $f_n - f$, we may w.l.o.g. assume that $f = 0$ and $f_n \geq f_{n+1} \geq 0$ for $n \in \mathbb{N}$. For $\varepsilon > 0$ and $x \in X$ we now find an $n_x \in \mathbb{N}$ with

$$(\forall n \geq n_x) \quad 0 \leq f_m(x) \leq \frac{\varepsilon}{3}.$$

The continuity of f and f_{n_x} yields a neighborhood U_x of x with

$$(\forall y \in U_x) \quad |f_{n_x}(x) - f_{n_x}(y)| \leq \frac{\varepsilon}{3}.$$

We thus obtain

$$(\forall y \in U_x) \quad 0 \leq f_{n_x}(y) \leq \varepsilon.$$

Now we choose $x_1, \ldots, x_k \in X$ such that the U_{x_j} cover X and put $n_0 := \max\{n_{x_1}, \ldots, n_{x_k}\}$. Then, by monotony of the sequence,

$$0 \leq f_{n_0}(x) \leq f_{n_{x_j}}(x) \leq \varepsilon \quad \text{for} \quad x \in U_{x_j}$$

and thus

$$(\forall n \geq n_0)(\forall x \in X) \quad 0 \leq f_n(x) \leq f_{n_0}(x) \leq \varepsilon.$$

This completes the proof. □

Lemma A.2.3. *There exists an increasing sequence of real polynomials p_n which converges in $[0,1]$ uniformly to the square root function $x \mapsto \sqrt{x}$.*

[2] Dini, Ulisse (1845–1918)

Proof. Idea: We start with $p_1 := 0$ and construct p_n inductively by the rule

$$p_{n+1}(x) := p_n(x) + \frac{1}{2}(x - p_n(x)^2). \tag{A.1}$$

Then we show that this sequence is monotone and bounded. The iteration procedure produces an equation for the limit which turns out to be $\sqrt{x}$. Then we apply Dini's Theorem.

Details: We prove by induction that that

$$(\forall n \in \mathbb{N})(\forall x \in [0,1]) \quad 0 \leq p_n(x) \leq \sqrt{x} \leq 1.$$

In fact,

$$\begin{aligned}
\sqrt{x} - p_{n+1}(x) &= \sqrt{x} - p_n(x) - \frac{1}{2}\big(x - p_n(x)^2\big) \\
&= (\sqrt{x} - p_n(x))\Big(1 - \frac{1}{2}\big(\sqrt{x} + p_n(x)\big)\Big)
\end{aligned}$$

and $p_n(x) \leq \sqrt{x}$ yields

$$(\forall x \in [0,1]) \qquad 0 \leq \frac{1}{2}(\sqrt{x} + p_n(x)) \leq \sqrt{x} \leq 1.$$

Therefore the definition of p_{n+1} yields $p_n \leq p_{n+1}$ on $[0,1]$, so that our claim follows by induction. Therefore the sequence $(p_n)_{n\in\mathbb{N}}$ is increasing on $[0,1]$ and bounded, hence converges pointwise to some function $f\colon [0,1] \to [0,1]$. Passing in (A.1) to the limit on both sides, we obtain the relation $f(x)^2 = x$, which proves that $f(x) = \sqrt{x}$. Now Dini's Theorem A.2.2 implies that the convergence $p_n \to f$ is uniform. □

Theorem A.2.4. (Stone–Weierstraß) [3] [4] *Let X be a compact space and $\mathcal{A} \subseteq C(X,\mathbb{R})$ be a point separating subalgebra containing the constant functions. Then $\mathcal{A}$ is dense in $C(X,\mathbb{R})$ w.r.t. $\|\cdot\|_\infty$.*

Proof. Let $\mathcal{B} := \overline{\mathcal{A}}$ denote the closure of $\mathcal{A}$ in the Banach space

$$(C(X,\mathbb{R}), \|\cdot\|_\infty).$$

Then $\mathcal{B}$ also contains the constant functions, separates the points and is a subalgebra (Exercise A.2.1). We have to show that $\mathcal{B} = C(X,\mathbb{R})$.

Here is the idea of the proof. First we use Lemma A.2.3 to see that for $f, g \in \mathcal{B}$, also $|f|$, $\min(f,g)$ and $\max(f,g)$ are contained in $\mathcal{B}$. Then we use the point separation property to approximate general continuous functions locally by elements of $\mathcal{B}$. Now the compactness of X permits to complete the proof.

[3] Stone, Marshall (1903–1989)
[4] Weierstraß, Karl (1815–1897)

Here are the details: Let $(p_n)_{n\in\mathbb{N}}$ be the sequence of polynomials from Lemma A.2.3. For $f \in \mathcal{B}$, we consider the functions $p_n\big(\frac{f^2}{\|f\|_\infty^2}\big)$, which also belong to $\mathcal{B}$. In view of Lemma A.2.3, they converge uniformly to $\sqrt{\frac{f^2}{\|f\|_\infty^2}} = \frac{|f|}{\|f\|_\infty}$, so that $|f| \in \mathcal{B}$.

Now let $f, g \in \mathcal{B}$. Then $\mathcal{B}$ contains the functions

$$\min(f,g) = \frac{1}{2}(f + g - |f-g|) \quad \text{and} \quad \max(f,g) = \frac{1}{2}(f+g+|f-g|).$$

Next let $x \neq y$ in X and $r, s \in \mathbb{R}$. According to our assumption, there exists a function $g \in \mathcal{B}$ with $g(x) \neq g(y)$. For

$$h := r + (s-r)\frac{g - g(x)}{g(y)-g(x)} \in \mathcal{B}$$

we then have $h(x) = r$ and $h(y) = s$.

Claim: For $f \in C(X,\mathbb{R})$, $x \in X$ and $\varepsilon > 0$, there exists a function $g_x \in \mathcal{B}$ with

$$f(x) = g_x(x) \quad \text{and} \quad (\forall y \in X) \quad g_x(y) \leq f(y) + \varepsilon.$$

To verify this claim, pick for each $z \in X$ a function $h_z \in \mathcal{B}$ with $h_z(x) = f(x)$ and $h_z(z) \leq f(z) + \frac{\varepsilon}{2}$. Then there exists a neighborhood U_z of z with

$$(\forall y \in U_z) \quad h_z(y) \leq f(y) + \varepsilon.$$

Since X is compact, it is covered by finitely many $U_{z_1}, \ldots, U_{z_k}$ of these neighborhoods. Then $g_x := \min\{h_{z_1}, \ldots, h_{z_k}\}$ is the desired function.

Now we complete the proof by showing that $\mathcal{B} = C(X,\mathbb{R})$. So let $f \in C(X,\mathbb{R})$ and $\varepsilon > 0$. For each $x \in X$, pick $g_x \in \mathcal{B}$ with

$$(\forall y \in X) \quad f(x) = g_x(x) \quad \text{and} \quad g_x(y) \leq f(y) + \varepsilon.$$

Then the continuity of f and g_x yield neighborhoods U_x of x with

$$\forall y \in U_x \ : \quad g_x(y) \geq f(y) - \varepsilon.$$

Now the compactness of X implies the existence of finitely many points $x_1, \ldots, x_k$ such that $X \subseteq U_{x_1} \cup \cdots \cup U_{x_k}$. We now put $\varphi_\varepsilon := \max\{g_{x_1}, \ldots, g_{x_k}\} \in \mathcal{B}$. Then

$$\forall y \in X \ : \quad f(y) - \varepsilon \leq \varphi_\varepsilon(y) \leq f(y) + \varepsilon.$$

This implies that $\|f - \varphi_\varepsilon\|_\infty \leq \varepsilon$ and since ε was arbitrary, $f \in \mathcal{B}$. □

Corollary A.2.5. *Let X be a compact space and $\mathcal{A} \subseteq C(X,\mathbb{C})$ be a point separating subalgebra containing the constant functions which is invariant under complex conjugation, i.e., $f \in \mathcal{A}$ implies $\overline{f} \in \mathcal{A}$. Then $\mathcal{A}$ is dense in $C(X,\mathbb{C})$ w.r.t. $\|\cdot\|_\infty$.*

Proof. Let $\mathcal{A}_\mathbb{R} := \mathcal{A} \cap C(X,\mathbb{R})$. Since $\mathcal{A}$ is conjugation invariant, we have $\mathcal{A} = \mathcal{A}_\mathbb{R} \oplus i\mathcal{A}_\mathbb{R}$. This implies that $\mathcal{A}_\mathbb{R}$ contains the real constants and separates the points of X. Now Theorem A.2.4 implies that $\mathcal{A}_\mathbb{R}$ is dense in $C(X,\mathbb{R})$, and therefore $\mathcal{A}$ is dense in $C(X,\mathbb{C}) = C(X,\mathbb{R}) + iC(X,\mathbb{R})$. □

Exercises for Section A.2

Exercise A.2.1. If X is a compact topological space and $\mathcal{A} \subseteq C(X, \mathbb{R})$ is a subalgebra, then its closure also is a subalgebra. Hint: If $f_n \to f$ and $g_n \to g$ uniformly, then also $f_n + g_n \to f + g, \lambda f_n \to \lambda f$ and $f_n g_n \to fg$ uniformly.

Exercise A.2.2. Let $[a, b] \subseteq \mathbb{R}$ be a compact interval. Show that the space

$$\mathcal{A} := \Big\{ f|_{[a,b]} \colon (\exists a_0, \ldots, a_n \in \mathbb{R}, n \in \mathbb{N})\ f(x) = \sum_{i=0}^{n} a_i x^i \Big\}$$

of polynomial functions on $[a, b]$ is dense in $C([a, b], \mathbb{R})$ with respect to $\| \cdot \|_\infty$.

Exercise A.2.3. Let $K \subseteq \mathbb{R}^n$ be a compact subset. Show that the space $\mathcal{A}$ consisting of all restrictions of polynomial functions

$$f(x) = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha x^\alpha, \quad a_\alpha \in \mathbb{R}, \quad x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

to K is dense in $C(K, \mathbb{R})$ with respect to $\| \cdot \|_\infty$.

Exercise A.2.4. Let $\mathbb{S}^1 = \{ z \in \mathbb{C} \colon |z| = 1 \}$ and

$$\mathcal{A} := \Big\{ f|_{\mathbb{S}^1} \colon (\exists a_0, \ldots, a_n \in \mathbb{C}, n \in \mathbb{N})\ f(z) = \sum_{j=0}^{n} a_n z^n \Big\}.$$

Show that $\mathcal{A}$ is not dense in $C(\mathbb{S}^1, \mathbb{C})$. Hint: Consider the function $f(z) := z^{-1}$ on $\mathbb{S}^1$ and try to approximate it by elements f_n of $\mathcal{A}$; then consider the complex path integrals $\oint_{|z|=1} f_n(z)\, dz$. Why does the Stone–Weierstraß Theorem not apply?

Exercise A.2.5. For a locally compact space X, we consider the Banach space $C_0(X)$ of all continuous functions $f \colon X \to \mathbb{C}$ vanishing at infinity, i.e., with the property that for each $\varepsilon > 0$ there exists a compact subset $C_\varepsilon \subseteq X$ with $|f(x)| \le \varepsilon$ for $x \notin C_\varepsilon$. Suppose that $\mathcal{A} \subseteq C_0(X)$ is a complex subalgebra satisfying

(a) $\mathcal{A}$ is invariant under conjugation.

(b) $\mathcal{A}$ has no zeros, i.e., for each $x \in X$ there exists an $f \in \mathcal{A}$ with $f(x) \neq 0$.

(c) $\mathcal{A}$ separates the points of X.

Show that $\mathcal{A}$ is dense in $C_0(X)$ with respect to $\| \cdot \|_\infty$. Hint: Let X_ω be the one-point compactification of X. Then each function $f \in C_0(X)$ extends to a continuous function $\widetilde{f}$ on X_ω by $\widetilde{f}(\omega) := 0$, and this leads to bijection

$$C_*(X_\omega) := \{ f \in C(X_\omega) \colon f(\omega) = 0 \} \to C_0(X), \quad f \mapsto f|_X.$$

Use the Stone–Weierstraß Theorem to show that the algebra

$$\widetilde{\mathcal{A}} := \mathbb{C}\mathbf{1} + \{ \widetilde{a} \colon a \in \mathcal{A} \}$$

is dense in $C(X_\omega)$ and show that if $\widetilde{f_n} + \lambda \mathbf{1} \to \widetilde{f}$ for $\lambda_n \in \mathbb{C}, f \in C_0(X), f_n \in \mathcal{A}$, then $\lambda_n \to 0$ and $f_n \to f$.

A.3 Commutative C^*-algebras

Let $\mathcal{A}$ be a commutative Banach-$*$-algebra. We write

$$\widehat{\mathcal{A}} := \mathrm{Hom}(\mathcal{A}, \mathbb{C}) \setminus \{0\},$$

where $\mathrm{Hom}(\mathcal{A}, \mathbb{C})$ denotes the set of all morphisms of Banach-$*$-algebras, i.e., continuous linear functionals $\chi \colon \mathcal{A} \to \mathbb{C}$ with the additional property that

$$\chi(ab) = \chi(a)\chi(b) \quad \text{and} \quad \chi(a^*) = \overline{\chi(a)} \quad \text{for} \quad a, b \in \mathcal{A}. \tag{A.2}$$

Thinking of $\mathbb{C}$ as a one-dimensional Hilbert space, we have $\mathbb{C} \cong B(\mathbb{C})$, so that $\mathrm{Hom}(\mathcal{A}, \mathbb{C})$ can also be considered as the set of one-dimensional (involutive) representations of the Banach-$*$-algebra $\mathcal{A}$.

Since the set $\mathrm{Hom}(\mathcal{A}, \mathbb{C})$ is defined by the equations (A.2), it is a weak-$*$-closed subset of the topological dual space $\mathcal{A}'$. One can also show that $\|\chi\| \leq 1$ for any $\chi \in \mathrm{Hom}(\mathcal{A}, \mathbb{C})$ (Exercise A.3.1), so that $\mathrm{Hom}(\mathcal{A}, \mathbb{C})$ is also bounded, hence weak-$*$-compact by the Banach–Alaoglu Theorem (cf. [Ru73]). Therefore $\widehat{\mathcal{A}}$ is a locally compact space.

Since $\widehat{\mathcal{A}} \subseteq \mathbb{C}^{\mathcal{A}}$ carries the weak-$*$-topology, i.e., the topology of pointwise convergence, each element $a \in \mathcal{A}$ defines a continuous function

$$\widehat{a} \colon \widehat{\mathcal{A}} \to \mathbb{C}, \quad \widehat{a}(\chi) := \chi(a).$$

Since $\widehat{a}$ extends to a continuous function on the compact space $\mathrm{Hom}(\mathcal{A}, \mathbb{C})$ vanishing in the 0-functional, we have $\widehat{a} \in C_0(\widehat{\mathcal{A}})$ (Exercise A.3.2), with

$$|\widehat{a}(\chi)| = |\chi(a)| \leq \|\chi\|\|a\| \leq \|a\|$$

(cf. Exercise A.3.1). We thus obtain a map

$$\mathcal{G} \colon \mathcal{A} \to C_0(\widehat{\mathcal{A}}), \quad a \mapsto \widehat{a},$$

called the *Gelfand transform*. For $a, b \in \mathcal{A}$ and $\chi \in \widehat{\mathcal{A}}$ we have

$$\mathcal{G}(ab)(\chi) = \chi(ab) = \chi(a)\chi(b) = \mathcal{G}(a)(\chi)\mathcal{G}(b)(\chi)$$

and

$$\mathcal{G}(a^*)(\chi) = \chi(a^*) = \overline{\chi(a)} = \mathcal{G}(a)^*(\chi),$$

so that $\mathcal{G}$ is a morphism of Banach-$*$-algebras, i.e., a continuous homomorphism compatible with the involution.

Theorem A.3.1. (Gelfand Representation Theorem) *If $\mathcal{A}$ is a commutative C^*-algebra, then the Gelfand transform*

$$\mathcal{G} \colon \mathcal{A} \to C_0(\widehat{\mathcal{A}})$$

is an isometric isomorphism.

For a proof we refer to [Ru73, Thm. 11.18].

Remark A.3.2. (a) If $\mathcal{A}$ is already of the form $\mathcal{A} = C_0(X)$ for a locally compact space, then one can show that the natural map

$$\eta\colon X \to \widehat{\mathcal{A}}, \quad \eta(x)(f) := f(x)$$

is a homeomorphism, so that we can recover the space X as $\widehat{\mathcal{A}}$.

(b) The image $\mathcal{G}(\mathcal{A})$ of the Gelfand transform is a $*$-subalgebra of $C_0(\widehat{\mathcal{A}})$ separating the points of $\mathcal{A}$ and for each $\chi \in \mathcal{A}$, there exists an element $a \in \mathcal{A}$ with $\widehat{a}(\chi) \neq 0$. Therefore the Stone–Weierstraß Theorem for locally compact spaces (Exercise A.2.5) implies that $\mathcal{G}(\mathcal{A})$ is dense in $C_0(\widehat{\mathcal{A}})$.

Corollary A.3.3. *If $\mathcal{A}$ is a unital C^*-algebra and $\dim \mathcal{A} > 1$, then there exist non-zero commuting elements $a, b \in \mathcal{A}$ with $ab = 0$.*

Proof. Since $\mathcal{A} \neq \mathbb{C}\mathbf{1}$, there exists an element $x \in \mathcal{A} \setminus \mathbb{C}\mathbf{1}$. Writing $x = y + iz$ with $y^* = y$ and $z^* = z$, it follows immediately that $\mathcal{A}$ contains a hermitian element $a \in \mathcal{A} \setminus \mathbb{C}\mathbf{1}$. Let $\mathcal{B} \subseteq \mathcal{A}$ be the closed unital subalgebra of $\mathcal{A}$ generated by a. Then $\mathcal{B}$ is commutative and larger than $\mathbb{C}\mathbf{1}$, hence isomorphic to $C_0(X)$ for some locally compact space X (Theorem A.3.1). Then X contains at least two points $x \neq y$, and Urysohn's Theorem A.1.6 implies the existence of non-zero elements $a, b \in C_c(X) \subseteq C_0(X) \cong \mathcal{B}$ with $ab = 0$. $\square$

Exercises for Section A.3

Exercise A.3.1. c Let $\mathcal{A}$ be a Banach algebra and $\chi\colon \mathcal{A} \to \mathbb{C}$ be an algebra homomorphism. Show that:

(a) χ extends to the unital Banach algebra $\mathcal{A}_+ := \mathcal{A} \times \mathbb{C}$ with the multiplication

$$(a, t)(a', t') := (aa' + ta' + t'a, tt')$$

(cf. Exercise 1.1.15).

(b) If $\mathcal{A}$ is unital and $\chi \neq 0$, then

$$\chi(\mathbf{1}) = 1 \quad \text{and} \quad \chi(\mathcal{A}^\times) \subseteq \mathbb{C}^\times.$$

Conclude further that $\chi(B_1(\mathbf{1})) \subseteq \mathbb{C}^\times$ and derive that χ is continuous with $\|\chi\| \leq 1$.

Exercise A.3.2. Suppose that Y is a compact space $y_0 \in Y$ and $X := Y \setminus \{y_0\}$. Show that the restriction map yields an isometric isomorphism of C^*-algebras:

$$r\colon C_*(Y, \mathbb{C}) := \{f \in C(Y, \mathbb{C})\colon f(y_0) = 0\} \to C_0(X, \mathbb{C}).$$

Exercise A.3.3. Let $\mathcal{A}$ be a C^*-algebra. Show that:

(i) If $a = a^* \in \mathcal{A}$ is a hermitian element, then $\|a^n\| = \|a\|^n$ holds for each $n \in \mathbb{N}$. Hint: Consider the commutative C^*-subalgebra generated by a.

(ii) If $\mathcal{B}$ is a Banach-$*$-algebra and $\alpha\colon \mathcal{B} \to \mathcal{A}$ a continuous morphism of Banach-$*$-algebras, then $\|\alpha\| \leq 1$. Hint: Let $C := \|\alpha\|$ and derive with (i) for $b \in \mathcal{B}$ the relation

$$\|\alpha(b)\|^{2n} = \|\alpha(bb^*)\|^n = \|\alpha((bb^*)^n)\| \leq C\|(bb^*)^n\| \leq C\|b\|^{2n}.$$

Finally, use that $C^{1/n} \to 1$.

Exercise A.3.4. Let $\mathcal{A}$ be a C^*-algebra. We call a hermitian element $a = a^* \in \mathcal{A}$ *positive* if $a = b^2$ for some hermitian element $b = b^* \in \mathcal{A}$. Show that:

(a) Every positive Element $a \in \mathcal{A}$ has a positive square root. Hint: Consider the commutative C^*-subalgebra $\mathcal{B}$ generated by b and recall that $\mathcal{B} \cong C_0(X)$ for some locally compact space.

(b) If $C_0(X)$, X a locally compact space, is generated as a C^*-algebra by some $f \geq 0$, then it is also generated by f^2. Hint: Use the Stone–Weierstraß Theorem.

(c) If b is a positive square root of a, then there exists a commutative C^*-subalgebra of $\mathcal{A}$ containing a and b in which b is positive. Hint: Write $b = c^2$ and consider the C^*-algebra generated by c.

(d) Every positive Element $a \in \mathcal{A}$ has a unique positive square root. Hint: Use (b) and (c) to see that any positive square root of a is contained in the C^*-algebra generated by a; then consider the special case $\mathcal{A} = C_0(X)$.

Exercise A.3.5. Let $\mathcal{A}$ be a unital C^*-algebra and $a = a^* \in \mathcal{A}$ with $\|a\| < 1$. Show that

$$b := \sqrt{1 - a^2} := \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-1)^n a^{2n}$$

is hermitian and satisfies $b^2 = 1 - a^2$. Show further that

$$u := a + i\sqrt{\mathbf{1} - a^2} \in \mathrm{U}(\mathcal{A})$$

and conclude that $\mathcal{A} = \operatorname{span} \mathrm{U}(\mathcal{A})$. Hint: To verify $b^2 = \mathbf{1} - a^2$, it suffices to consider the commutative C^*-algebra generated by a.

A.4 Discrete Decomposability for Compact Groups

Let $(\pi, \mathcal{H})$ be a continuous unitary representation of the compact group G and μ_G be a normalized Haar measure on G. We assume that $\mathcal{H} \neq \{0\}$ and want to show that $\mathcal{H} = \mathcal{H}_d$, i.e., that $\mathcal{H}$ decomposes as a direct sum of irreducible representations. This will follow, as soon as we can show that $\mathcal{H}$ contains a non-zero finite dimensional G-invariant subspace because every finite dimensional representation is a direct sum of irreducible ones (Proposition 1.3.12).

If $0 \neq A = A^* \in B_G(\mathcal{H})$ is a non-zero compact intertwining operator, then the Spectral Theorem for compact hermitian operators implies that $\mathcal{H} = \widehat{\bigoplus}_{\lambda \in \mathbb{R}} \mathcal{H}_\lambda(A)$ is the orthogonal direct sum of the eigenspaces

$$\mathcal{H}_\lambda(A) := \ker(A - \lambda\mathbf{1})$$

and if $\lambda \neq 0$, then $\dim \mathcal{H}_\lambda(A) < \infty$. Since A is non-zero, it has a non-zero eigenvalue λ, and therefore $\mathcal{H}_\lambda(A)$ is a finite dimensional subspace of $\mathcal{H}$ which is G-invariant (Exercise 1.3.11). It therefore remains to construct a non-zero hermitian compact element of $B_G(\mathcal{H})$.

Proposition A.4.1. *For each $A \in B(\mathcal{H})$, there exists a unique operator $A_G \in B_G(\mathcal{H})$ with the property that*

$$\langle A_G v, w\rangle = \int_G \langle \pi(g)A\pi(g)^{-1}v, w\rangle \, d\mu_G \quad \text{for} \quad v, w \in \mathcal{H}.$$

Moreover, $(A_G)^* = (A^*)_G$ and if A is compact, then A_G is also compact.

We also write this operator as an operator-valued integral

$$\int_G \pi(g)A\pi(g)^{-1} \, d\mu_G := A_G.$$

Proof. On $\mathcal{H}$ we consider the sesquilinear form defined by

$$F(v, w) := \int_G \langle \pi(g)A\pi(g)^{-1}v, w\rangle \, d\mu_G(g).$$

Then

$$\begin{aligned} |F(v, w)| &\leq \int_G \|\pi(g)A\pi(g)^{-1}\|\|v\|\|w\| \, d\mu_G(g) \\ &= \int_G \|A\|\|v\|\|w\| \, d\mu_G(g) = \|A\|\|v\|\|w\|, \end{aligned}$$

and we conclude the existence of a unique bounded operator $A_G \in B(\mathcal{H})$ with

$$F(v, w) = \langle A_G v, w\rangle \quad \text{for} \quad v, w \in \mathcal{H}$$

(Exercise in Functional Analysis). To see that A_G commutes with each $\pi(g)$, we calculate

$$\begin{aligned} \langle \pi(g)A_G\pi(g)^{-1}v, w\rangle &= \int_G \langle \pi(g)\pi(h)A\pi(h)^{-1}\pi(g)^{-1}v, w\rangle \, d\mu_G(h) \\ &= \int_G \langle \pi(gh)A\pi(gh)^{-1}v, w\rangle \, d\mu_G(h) \\ &= \int_G \langle \pi(h)A\pi(h)^{-1}v, w\rangle \, d\mu_G(h) = \langle A_G v, w\rangle. \end{aligned}$$

Next we note that

$$\begin{aligned} \langle (A_G)^* v, w\rangle = \langle v, A_G w\rangle &= \int_G \langle v, \pi(g)A\pi(g)^{-1}w\rangle \, d\mu_G(g) \\ &= \int_G \langle \pi(g)A^*\pi(g)^{-1}v, w\rangle \, d\mu_G(g) = \langle (A^*)_G v, w\rangle. \end{aligned}$$

Finally, we assume that A is compact, i.e., the image of the closed unit ball $B \subseteq \mathcal{H}$ is relatively compact. We have to show that the same holds for A_G.

Since G is compact, the set $B' := \overline{\pi(G)AB} \subseteq \pi(G)\overline{AB}$ is also compact because the action $G \times \mathcal{H} \to \mathcal{H}, (g,v) \mapsto \pi(g)v$ is continuous (Exercise 1.2.3). The closed convex hull $K := \overline{\mathrm{conv}(B')}$ is also compact (Exercise A.4.1).

We claim that $A_G(B) \subseteq K$, and this will imply that A_G is compact. So let $v \in \mathcal{H}$ and $c \in \mathbb{R}$ with $\mathrm{Re}\langle v, x\rangle \leq c$ for all $x \in K$. Then we obtain for $w \in B$:

$$\begin{aligned}\mathrm{Re}\langle v, A_G w\rangle &= \mathrm{Re}\int_G \langle v, \pi(g)A\pi(g)^{-1}w\rangle\, d\mu_G(g)\\ &= \int_G \mathrm{Re}\langle v, \pi(g)A\pi(g)^{-1}w\rangle\, d\mu_G(g) \leq \int_G c\, d\mu_G(g) = c.\end{aligned}$$

Since

$$K = \{x \in \mathcal{H} \colon (\forall v \in \mathcal{H})\, \mathrm{Re}\langle v, x\rangle \leq \sup \mathrm{Re}\langle v, K\rangle\}$$

by the Hahn–Banach Separation Theorem, it follows that $A_G w \in K$, and thus $A_G B \subseteq K$. □

Combining the preceding proposition with the discussion above, we obtain:

Proposition A.4.2. *If $(\pi, \mathcal{H})$ is a non-zero continuous unitary representation of the compact group G, then $\mathcal{H}$ contains a non-zero finite dimensional G-invariant subspace.*

Proof. We have to show the existence of a non-zero compact hermitian intertwining operator. So let $v_0 \in \mathcal{H}$ be a unit vector and consider the orthogonal projection $P(v) := \langle v, v_0\rangle v_0$ onto $\mathbb{C}v_0$. Then $\dim(\mathrm{im}(P)) = 1$ implies that P is compact, and since it is an orthogonal projection, we also have $P^* = P$. Therefore

$$P_G(v) := \int_G (\pi(g)P\pi(g)^{-1})v\, d\mu_G(g) = \int_G \langle \pi(g)^{-1}v, v_0\rangle \pi(g)v_0\, d\mu_G(g)$$

from Proposition A.4.1 is a compact hermitian operator. To see that it is non-zero, we simply observe that

$$\begin{aligned}\langle P_G v_0, v_0\rangle &= \int_G \langle \pi(g)^{-1}v_0, v_0\rangle \langle \pi(g)v_0, v_0\rangle\, d\mu_G(g)\\ &= \int_G |\langle \pi(g)v_0, v_0\rangle|^2\, d\mu_G(g) > 0\end{aligned}$$

follows from $\langle \pi(\mathbf{1})v_0, v_0\rangle > 0$ and the defining property of the Haar measure μ_G. □

Theorem A.4.3. (Fundamental Theorem on Unitary Representations of Compact Groups—Abstract Peter–Weyl Theorem) *If $(\pi, \mathcal{H})$ is a continuous unitary representation of the compact group G, then $(\pi, \mathcal{H})$ is a direct sum of irreducible representations and all irreducible representations of G are finite dimensional.*

Proof. Writing $\mathcal{H} = \mathcal{H}_d \oplus \mathcal{H}_c$ for the decomposition into discrete and continuous part (Proposition 2.2.5), we use Proposition A.4.2 to see that if $\mathcal{H}_c \neq \{0\}$, then it contains a finite dimensional invariant subspace, contradicting the definition of $\mathcal{H}_c$ (Proposition 1.3.12). Therefore $\mathcal{H}_c = \{0\}$ and thus $\mathcal{H} = \mathcal{H}_d$, so that the first part follows from Proposition 2.2.5. Applying Proposition A.4.2 to an irreducible representation $(\pi, \mathcal{H})$ of G, we thus get $\dim \mathcal{H} < \infty$. □

Exercises for Section A.4

Exercise A.4.1. Show that if B is a compact subset of a Banach space E, then its closed convex hull $K := \overline{\mathrm{conv}(B)}$ is also compact. Hint: Since we are dealing with metric spaces, it suffices to show precompactness, i.e., that for each $\varepsilon > 0$, there exists a finite subset $F \subseteq K$ with $K \subseteq B_\varepsilon(F) := F + B_\varepsilon(0)$. Since B is compact, there exists a finite subset $F_B \subseteq B$ with $B \subseteq B_\varepsilon(F_B)$. Then $\mathrm{conv}(B) \subseteq \mathrm{conv}(F_B) + B_\varepsilon(0)$, and since $\mathrm{conv}(F_B)$ is compact (why?), $\mathrm{conv}(F_B) \subseteq B_\varepsilon(F)$ for a finite subset $F \subseteq \mathrm{conv}(F_B)$. This leads to $\mathrm{conv}(B) \subseteq F + B_{2\varepsilon}(0)$, which implies implies $K \subseteq F + B_{\leq 2\varepsilon}(0)$.

Exercise A.4.2. Let $\mathcal{H}$ be a complex Hilbert space and $G \subseteq \mathrm{U}(\mathcal{H})_s$ be a closed subgroup. Show that G is compact if and only if $\mathcal{H}$ can be written as an orthogonal direct sum $\mathcal{H} = \widehat{\bigoplus}_{j\in J} \mathcal{H}_j$ of finite dimensional G-invariant subspaces. Hint: Use Tychonov's Theorem and Exercise 1.1.1 to see that for any family of finite dimensional Hilbert spaces $(\mathcal{H}_j)_{j\in J}$, the topological group $\prod_{j\in J} \mathrm{U}(\mathcal{H}_j)_s$ is compact.

A.5 The Fourier transform on $\mathbb{R}^n$

For $f \in L^1(\mathbb{R}^n)$, we define its *Fourier transform* by

$$\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{i\langle \xi, x\rangle}\, dx.$$

The Dominated Convergence Theorem immediately implies that $\widehat{f} \colon \mathbb{R}^n \to \mathbb{R}$ is a continuous function satisfying

$$\|\widehat{f}\|_\infty \leq \|f\|_1.$$

The Banach space $L^1(\mathbb{R}^n)$ is a Banach-$*$-algebra with respect to the involution

$$f^*(x) := \overline{f(-x)}, \quad x \in \mathbb{R}^n$$

and the *convolution product*

$$(f_1 * f_2)(x) := \int_{\mathbb{R}^n} f_1(y) f_2(x-y)\, dy$$

which satisfies

$$\|f_1 * f_2\|_1 \leq \|f_1\| \cdot \|f_2\|$$

(an easy application of Fubini's Theorem).

With Fubini's Theorem, we obtain for $f_1, f_2 \in L^1(\mathbb{R}^n)$:

$$
\begin{aligned}
(f_1 * f_2)\widehat{}(\xi) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f_1(y) f_2(x-y) e^{i\langle \xi, x\rangle}\, dy\, dx \\
&= \int_{\mathbb{R}^n} f_1(y) e^{i\langle \xi, y\rangle} \int_{\mathbb{R}^n} f_2(x-y) e^{i\langle \xi, x-y\rangle}\, dx\, dy \\
&= \int_{\mathbb{R}^n} f_1(y) e^{i\langle \xi, y\rangle} \int_{\mathbb{R}^n} f_2(x) e^{i\langle \xi, x\rangle}\, dx\, dy \\
&= \int_{\mathbb{R}^n} f_1(y) e^{i\langle \xi, y\rangle} \widehat{f_2}(\xi)\, dy \\
&= \widehat{f_1}(\xi)\widehat{f_2}(\xi)
\end{aligned}
$$

and we also note that

$$\overline{\widehat{f}} = \widehat{f^*} \quad \text{for} \quad f \in L^1(\mathbb{R}^n).$$

Therefore the Fourier transform

$$\mathcal{F}\colon L^1(\mathbb{R}^n) \to (C_b(\mathbb{R}^n), \|\cdot\|_\infty), \quad f \mapsto \widehat{f}$$

is a morphism of Banach-$*$-algebras.

Proposition A.5.1. (Riemann–Lebesgue Lemma) *For each $f \in L^1(\mathbb{R}^n)$ the Fourier transform $\widehat{f}$ vanishes at infinity, i.e., $\widehat{f} \in C_0(\mathbb{R}^n)$.*

Proof. For each $0 \neq x \in \mathbb{R}^n$ we obtain with $e^{-i\pi} = -1$ and the translation invariance of Lebesgue measure the relation

$$
\begin{aligned}
\widehat{f}(x) &= 2\frac{1}{2}\int_{\mathbb{R}^n} e^{i\langle x, y\rangle} f(y)\, dy \\
&= \frac{1}{2}\int_{\mathbb{R}^n} e^{i\langle x, y\rangle} f(y)\, dy - \frac{1}{2}\int_{\mathbb{R}^n} e^{i\langle x, y - \frac{\pi}{\|x\|^2}x\rangle} f(y)\, dy \\
&= \frac{1}{2}\int_{\mathbb{R}^n} e^{i\langle x, y\rangle} f(y)\, dy - \frac{1}{2}\int_{\mathbb{R}^n} e^{i\langle x, y\rangle} f\Big(y + \frac{\pi}{\|x\|^2}x\Big)\, dy \\
&= \frac{1}{2}\int_{\mathbb{R}^n} e^{i\langle x, y\rangle} \Big[f(y) - f\Big(y + \frac{\pi}{\|x\|^2}x\Big)\Big]\, dy.
\end{aligned}
$$

This implies that

$$|\widehat{f}(x)| \leq \frac{1}{2}\int_{\mathbb{R}^n} \Big| f(y) - f\Big(y + \frac{\pi}{\|x\|^2}x\Big)\Big|\, dy.$$

Now the assertion follows from the continuity of the map

$$\mathbb{R}^n \to L^1(\mathbb{R}^n), \quad x \mapsto \lambda_x f$$

in 0 (Exercise) and $\lim_{x\to\infty} \frac{x}{\|x\|^2} = 0$. $\square$

Proposition A.5.2. *$\mathcal{F}(L^1(\mathbb{R}^n))$ is dense in $C_0(\mathbb{R}^n)$ bzgl. $\|\cdot\|_\infty$.*

Proof. We know already that $\mathcal{A} := \mathcal{F}(L^1(\mathbb{R}^n))$ is a conjugation invariant subalgebra of $C_0(\mathbb{R}^n)$. According to the Stone–Weierstraß Theorem for non-compact spaces (Exercise A.2.5), we have to see that $\mathcal{A}$ has no common zeros and that it separates the points of $\mathbb{R}^n$. This is verified in (a) and (b) below.

(a) Let $x_0 \in \mathbb{R}^n$ and B be the ball of radius 1 around x_0. Then $f(x) := \chi_B(x)e^{-i\langle x_0,x\rangle}$ is an element of $L^1(\mathbb{R}^n)$ and

$$\widehat{f}(x_0) = \int_B dx = \mathrm{vol}(B) > 0.$$

Therefore $\mathcal{A}$ has no common zeros.

(b) For $x_0 \neq y_0 \in \mathbb{R}^n$, we pick $\alpha \in \mathbb{R}$ such that $z_0 := \alpha(x_0 - y_0)$ satisfies $e^{i\langle x_0-y_0,z_0\rangle} \neq 1$. Then there exists a ball B with center z_0 such that $e^{i\langle x_0-y_0,z\rangle} \neq 1$ for every $z \in B$. Then the L^1-function

$$f(x) := \chi_B(x)(e^{-i\langle x_0,x\rangle} - e^{-i\langle y_0,x\rangle})$$

has a Fourier transform satisfying

$$\begin{aligned}\widehat{f}(x_0) - \widehat{f}(y_0) &= \int_B (e^{-i\langle x_0,x\rangle} - e^{-i\langle y_0,x\rangle})(e^{i\langle x_0,x\rangle} - e^{i\langle y_0,x\rangle})\,dx \\ &= \int_B |e^{i\langle x_0,x\rangle} - e^{i\langle y_0,x\rangle}|^2\,dx > 0.\end{aligned}$$

□

A.6 The Group Algebra of a Locally Compact Group

A.6.1 Haar measure on locally compact groups

In this section G always denotes a locally compact group.

Definition A.6.1. (a) A positive Radon measure μ on G is called *left invariant* if

$$\int_G f(gx)\ d\mu(x) = \int_G f(x)\ d\mu(x) \quad \text{for} \quad f \in C_c(G), g \in G.$$

We likewise define right invariance by

$$\int_G f(xg)\ d\mu(x) = \int_G f(x)\ d\mu(x) \quad \text{for} \quad f \in C_c(G), g \in G.$$

(b) A positive left invariant Radon measure μ on G is called a *(left) Haar integral*, resp., a *(left) Haar measure*, if $0 \neq f \geq 0$ for $f \in C_c(G)$ implies

$$\int_G f(x)\,d\mu(x) > 0.$$

In the following we shall denote Haar measures on G by μ_G.

Remark A.6.2. One can show that every locally compact group G possesses a Haar measure and that for two Haar measures μ and μ' there exists a $\lambda > 0$ with $\mu' = \lambda\mu$ ([Neu90], [HiNe91]).

If G is compact and μ a Haar measure on G, then $\mu(G)$ is finite positive, so that we obtain a unique Haar probability measure on G. We call this Haar measure *normalized.*

Example A.6.3. (a) If G is a discrete group, then $C_c(G)$ is the space of finitely supported functions on G, and the counting measure

$$\int_G f\,d\mu := \sum_{g\in G} f(g)$$

is a Haar measure on G. If, in addition, G is finite, then

$$\int_G f\,d\mu := \frac{1}{|G|}\sum_{g\in G} f(g)$$

is a normalized Haar measure.

(b) For $G = \mathbb{Z}$ we obtain in particular a Haar measure by

$$\int_{\mathbb{Z}} f\,d\mu_{\mathbb{Z}} := \sum_{n\in\mathbb{Z}} f(n).$$

(c) On $G = \mathbb{R}^n$, the Riemann, resp., Lebesgue integral defines a Haar measure by

$$\int_{\mathbb{R}^n} f\,d\mu_G := \int_{\mathbb{R}^n} f(x)\ dx.$$

(d) On the circle group $G = \mathbb{T}$,

$$\int_{\mathbb{T}} f\,d\mu_{\mathbb{T}} := \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\ dt$$

is a Haar measure.

(e) If $G = (\mathbb{R}^\times, \cdot)$ is the multiplicative group of real numbers, then

$$\int_{\mathbb{R}^\times} f d\mu := \int_{\mathbb{R}^\times} \frac{f(x)}{|x|}\ dx$$

is a Haar measure on G. Note that a continuous function with compact support on $\mathbb{R}^\times$ vanishes in a neighborhood of 0, so that the integral is defined.

Lemma A.6.4. *If μ_G is a Haar measure on G and $h \in C(G)$ with $\int_G fh\,d\mu_G = 0$ for all $f \in C_c(G)$, then $h = 0$.*

Proof. Let $g \in G$. Then there exists a function $f \in C_c(G)$ with $0 \leq f$ and $f(g) > 0$ (Urysohn's Theorem). Now $f\overline{h} \in C_c(G)$ satisfies $f\overline{h} \cdot h = f|h|^2 \geq 0$, so that $\int_G f|h|^2\,d\mu_G = 0$ implies $f|h|^2 = 0$, and therefore $h(g) = 0$. □

Proposition A.6.5. *Let μ_G be a Haar measure on G. Then there exists a continuous homomorphism*

$$\Delta_G\colon G \to (\mathbb{R}^\times_+, \cdot) \quad \text{with} \quad (\rho_g)_*\mu_G = \Delta_G(g)^{-1}\mu_G \quad \text{for} \quad g \in G.$$

Proof. Since left and right multiplications on G commute, the Radon measure $(\rho_g)_*\mu_G$ is also left invariant and satisfies

$$\int_G f\, d\big((\rho_g)_*\mu_G\big) = \int_G f \circ \rho_g \, d\mu_G > 0$$

for $0 \neq f \geq 0$ (cf. (3.2)). Therefore $(\rho_g)_*\mu_G$ is a left Haar measure, and hence there exists $\Delta_G(g) \in]0, \infty[$ with $(\rho_g)_*\mu_G = \Delta_G(g)^{-1}\mu_G$ (Remark A.6.2).

Let $0 \neq f \in C_c(G)$ with $f \geq 0$. To see that

$$\Delta_G(g)^{-1} = \frac{1}{\int_G f d\mu_G} \int_G (f \circ \rho_g)\, d\mu_G$$

depends continuously on g, we note that for a fixed $g \in G$, we actually integrate only over $\mathrm{supp}(f)g^{-1}$. For any compact neighborhood K of g_0, the subset $\mathrm{supp}(f)K^{-1}$ of G is compact (it is the image of the compact product set $\mathrm{supp}(f) \times K$ under the continuous map $(x, y) \mapsto xy^{-1}$), and for any $g \in K$ we have

$$\Delta_G(g)^{-1} = \frac{1}{\int_G f d\mu_G} \int_{\mathrm{supp}(f)K^{-1}} (f \circ \rho_g)\, d\mu_G,$$

so that the continuity in g_0 follows as in the proof of Proposition 3.3.4. That Δ_G is a homomorphism is an immediate consequence of the definition:

$$\begin{aligned}(\rho_{gh})_*\mu_G &= (\rho_h\rho_g)_*\mu_G = (\rho_h)_*(\rho_g)_*\mu_G \\ &= \Delta_G(g)^{-1}(\rho_h)_*\mu_G = \Delta_G(g)^{-1}\Delta_G(h)^{-1}\mu_G.\end{aligned}$$

□

Definition A.6.6. The function Δ_G is called the *modular factor of* G. Clearly, it does not depend on the choice of the Haar measure μ_G. A locally compact group G is called *unimodular* if $\Delta_G = 1$, i.e., each left invariant Haar measure is also right invariant, hence biinvariant.

Proposition A.6.7. *A locally compact group G is unimodular if it satisfies one of the following conditions:*

(a) *G is compact.*

(b) *G is abelian.*

(c) *Its commutator group (G, G) is dense.*

Proof. (a) In this case $\Delta_G(G)$ is a compact subgroup of $\mathbb{R}^\times_+$, hence equal to $\{1\}$.

(b) Follows from the fact that $\rho_g = \lambda_g$ for any $g \in G$.

(c) Since $\mathbb{R}^\times_+$ is abelian, $(G, G) \subseteq \ker \Delta_G$. If (G, G) is dense, the continuity of Δ_G implies that $\Delta_G = 1$. □

Lemma A.6.8. *Let G be a locally compact group, μ_G a Haar measure and Δ_G be the modular factor. Then we have for $f \in L^1(G, \mu_G)$ the following formulas:*

(a) $\int_G f(xg)\ d\mu_G(x) = \Delta_G(g)^{-1} \int_G f(x)\ d\mu_G(x)$.

(b) $\Delta_G^{-1} \cdot \mu_G$ *is a right invariant measure on* G.

(c) $\int_G f(x^{-1})\ d\mu_G(x) = \int_G f(x)\Delta_G(x)^{-1}\ d\mu_G(x)$.

Proof. (a) is the definition of the modular factor.

(b) Using (a), we obtain $(\rho_g)_*\mu_G = \Delta_G(g)^{-1}\mu_G$. Since we also have $(\rho_g)_*\Delta_G = \Delta_G(g)^{-1}\Delta_G$, (b) follows.

(c) Let $I(f) := \int_G f(x^{-1})\Delta_G(x)^{-1}\ d\mu_G(x)$. Then (b) implies that

$$\begin{aligned} I(f \circ \lambda_{g^{-1}}) &= \int_G f(g^{-1}x^{-1})\Delta_G(x)^{-1}\ d\mu_G(x) = \int_G f((xg)^{-1})\Delta_G(x)^{-1}\ d\mu_G(x) \\ &= \int_G f(x^{-1})\Delta_G(x)^{-1}\ d\mu_G(x) = I(f), \end{aligned}$$

so that I is left invariant. For $0 \leq f \neq 0$ in $C_c(G)$ we also have $I(f) > 0$, so that I is a Haar integral. In view of the Uniqueness of Haar measure, there exists a $C > 0$ with

$$I(f) = \int_G f(x^{-1})\Delta_G(x)^{-1}\ d\mu_G(x) = C \int_G f(x)\ d\mu_G(x) \quad \text{for} \quad f \in C_c(G).$$

It remains to show that $C = 1$. We apply the preceding relation to the compactly supported function $\widetilde{f}(x) := f(x^{-1})\Delta_G(x)^{-1}$ to find

$$C \int_G \widetilde{f}(x)\, d\mu_G(x) = \int_G f(x)\Delta_G(x)\Delta_G(x)^{-1}\ d\mu_G(x) = \int_G f(x)\ d\mu_G(x),$$

which leads to $C = 1/C$, and hence to $C = 1$. □

Proposition A.6.9. *Let G be a locally compact group and μ_G a (left) Haar measure on G. On $L^2(G) := L^2(G, \mu_G)$ we have two continuous unitary representations of G. The* left regular representation

$$\pi_l(g)f := f \circ \lambda_g^{-1}$$

and the right regular representation

$$\pi_r(g)f := \sqrt{\Delta_G(g)} f \circ \rho_g.$$

Proof. The continuity of the left regular representation follows from Corollary 3.3.5. For the right regular representation we apply Proposition 3.3.4 to the left action of G on G defined by $\sigma_g(x) := xg^{-1} = \rho_{g^{-1}}x$. Then

$$\widetilde{\delta}(g, x) := \frac{d\big((\sigma_g)_*\mu_G\big)}{d\mu_G}(x) = \frac{d\big((\rho_g^{-1})_*\mu_G\big)}{d\mu_G}(x) = \Delta_G(g)$$

is a continuous function on $G \times G$, which implies the continuity of π_r. □

Corollary A.6.10. *For a locally compact group G, the left regular representation is injective. In particular, G has a faithful continuous unitary representation.*

Proof. For $g \neq \mathbf{1}$, pick disjoint open neighborhoods U of $\mathbf{1}$ and V of g with $gU \subseteq V$. Let $0 \leq f \in C_c(G)$ be non-zero with $\mathrm{supp}(f) \subseteq U$. Then

$$\langle \pi_l(g)f, f\rangle = \int_G f(g^{-1}x)\overline{f(x)}\, d\mu_G(x) = 0$$

because $\mathrm{supp}(\pi_l(g)f) = g\,\mathrm{supp}(f) \subseteq gU \subseteq V$ intersects U trivially. On the other hand the definition of Haar measure implies $\|f\|_2 > 0$, so that $\pi_l(g) \neq \mathbf{1}$. $\square$

A.6.2 The Convolution Product

Let G be a locally compact group and μ_G be a left Haar measure on G. For $f, g \in C_c(G)$ we define we define the *convolution product*

$$(f * g)(x) := \int_G f(y)g(y^{-1}x)\, d\mu_G(y) = \int_G f(xy)g(y^{-1})\, d\mu_G(y). \tag{A.3}$$

This integral is defined because the first integrand is supported by the compact set $\mathrm{supp}(f)$. Using the modular factor Δ_G, we define an involution on $C_c(G)$ by

$$f^*(x) := \Delta_G(x)^{-1}\overline{f(x^{-1})}. \tag{A.4}$$

Lemma A.6.11. *For $f, g \in C_c(G)$, we have*

(i) $f * g \in C_c(G)$ *with* $\mathrm{supp}(f * g) \subseteq \mathrm{supp}(f)\,\mathrm{supp}(g)$ *and convolution is associative.*

(ii) $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.

(iii) $\|f^*\|_1 = \|f\|_1$.

(iv) $(f * g)^* = g^* * f^*$.

(v) *For $x \in G$ and $f \in C_c(G)$ we put $\lambda_x f := f \circ \lambda_x^{-1}$ and $\rho_x f := f \circ \rho_x$. Then*

   (a) $\lambda_x(f * g) = (\lambda_x f) * g$, $\rho_x(f * g) = f * \rho_x g$,

   (b) $\rho_x(f) * g = f * \Delta_G(x)^{-1}(\lambda_{x^{-1}} g)$.

   (c) $(\lambda_x f)^* = \Delta_G(x)\rho_x f^*$ *and* $(\rho_x f)^* = \Delta_G(x)^{-1}\lambda_x f^*$.

   (d) $\|\lambda_x f\|_1 = \|f\|_1$ *and* $\|\rho_x f\|_1 = \Delta_G(x)^{-1}\|f\|_1$.

(vi) *For $f \in C_c(G)$, the map $G \to L^1(G, \mu_G), x \mapsto \lambda_x(f)$ is continuous.*

Proof. (i) The continuity of $f*g$ follows from the continuity of the integrand and the fact that we actually integrate over a compact subset of G. If $(f*g)(x) \neq 0$, then there exists a $y \in G$ with $f(y)g(y^{-1}x) \neq 0$. Then $y \in \mathrm{supp}(f)$ and

$y^{-1}x \in \operatorname{supp}(g)$, so that $x \in \operatorname{supp}(f)\operatorname{supp}(g)$. In particular, $f * g$ has compact support if f and g have.

For the associativity of the convolution product, we calculate

$$\begin{aligned}((f * g) * h)(x) &= \int_G \int_G f(z)g(z^{-1}y)h(y^{-1}x)\, d\mu_G(y)\, d\mu_G(z) \\ &= \int_G f(z) \int_G g(y)h(y^{-1}z^{-1}x)\, d\mu_G(y)\, d\mu_G(z) \\ &= \int_G f(z)(g * h)(z^{-1}z)\, d\mu_G(z) = (f * (g * h))(x).\end{aligned}$$

(ii) We have

$$|f * g(x)| \leq \int_G |f(y)||g(y^{-1}x)|\ d\mu_G(y)$$

and therefore

$$\begin{aligned}\|f * g\|_1 &\leq \int_G \int_G |f(y)||g(y^{-1}x)|\ d\mu_G(y)\ d\mu_G(x) \\ &\overset{\text{Fubini}}{=} \int_G |f(y)| \int_G |g(y^{-1}x)|\ d\mu_G(x)\ d\mu_G(y) \\ &= \int_G |f(y)| \int_G |g(x)|\ d\mu_G(x)\ d\mu_G(y) = \|g\|_1 \int_G |f(y)|\ d\mu_G(y) \\ &= \|g\|_1\|f\|_1.\end{aligned}$$

Here the application of Fubini's Theorem is justified by the fact that both integrals extend over sets of finite measure, so that the assumption of σ-finiteness is satisfied for the corresponding restricted measures.

(iii) In view of Lemma A.6.8, we have

$$\|f^*\|_1 = \int_G \Delta_G(x^{-1})|f(x^{-1})|\ d\mu_G(x) = \int_G |f(x)|\ d\mu_G(x) = \|f\|_1.$$

(iv) As in (iii), we get

$$\begin{aligned}(f * g)^*(x) &= \Delta_G(x)^{-1} \int_G \overline{f(y)g(y^{-1}x^{-1})}\ d\mu_G(y) \\ &= \Delta_G(x)^{-1} \int_G \overline{f(x^{-1}y)g(y^{-1})}\ d\mu_G(y) \\ &= \int_G g^*(y)\Delta_G(x^{-1}y)\overline{f(x^{-1}y)}\ d\mu_G(y) \\ &= \int_G g^*(y)f^*(y^{-1}x)\ d\mu_G(y) = (g^* * f^*)(x).\end{aligned}$$

(v) (a) follows immediately from (A.3).

(b) follows from

$$((\rho_x f) * g)(y) = \int_G f(zx)g(z^{-1}y)\, d\mu_G(z) = \Delta_G(x)^{-1} \int_G f(z)g(xz^{-1}y)\, d\mu_G(z)$$
$$= \Delta_G(x)^{-1}(f * \lambda_{x^{-1}}g)(y).$$

(c) follows from

$$(\lambda_x f)^*(y) = \Delta_G(y)^{-1}\overline{f(x^{-1}y^{-1})} = \Delta_G(x)\Delta_G(yx)^{-1}\overline{f(x^{-1}y^{-1})}$$
$$= \Delta_G(x)f^*(yx) = \Delta_G(x)(\rho_x f^*)(y),$$

which in turn implies

$$(\rho_x f)^* = \Delta_G(x)^{-1}\lambda_x f^*.$$

(d) follows from the left invariance of μ_G.

(vi) Let $f \in C_c(G)$. In view of

$$\|\lambda_x f - \lambda_{x'} f\|_1 = \int_G |f(x^{-1}g) - f(x'^{-1}g)|\; d\mu_G(g),$$

the assertion follows from the continuity of the integrand and the fact that we actually integrate over a compact subset of G. □

Let $L^1(G) := L^1(G, \mu_G)$ denote the completion of $C_c(G)$ with respect to $\|\cdot\|_1$. The preceding lemma implies that the convolution product and the involution extend to continuous maps on $L^1(G)$, turning it into a Banach-$*$-algebra. We also extend the left translations λ_g and the maps $\Delta_G(g)\rho_g$ to isometries of $L^1(G)$.

Proposition A.6.12. *Let $(\pi, \mathcal{H})$ be a continuous unitary representation of the locally compact group G. For $f \in L^1(G)$ and $v, w \in \mathcal{H}$, we define by*

$$\langle \pi(f)v, w\rangle := \int_G f(g)\langle \pi(g)v, w\rangle\; d\mu_G(g) \tag{A.5}$$

an operator $\pi(f) \in B(\mathcal{H})$, which we also denote symbolically by

$$\pi(f) = \int_G f(g)\pi(g)\; d\mu_G(g).$$

It has the following properties:

(i) $\|\pi(f)\| \leq \|f\|_1$.

(ii) *The continuous linear extension $\pi\colon L^1(G) \to B(\mathcal{H})$ defines a representation of the Banach-$*$-algebra $L^1(G)$, i.e., it is a homomorphism of algebras satisfying $\pi(f)^* = \pi(f^*)$ for $f \in L^1(G)$.*

(iii) *For $x \in G$ and $f \in L^1(G)$ we have $\pi(x)\pi(f) = \pi(\lambda_x f)$ and $\pi(f)\pi(x) = \Delta_G(x)\pi(\rho_x f)$.*

(iv) *The representation π of $L^1(G)$ on $\mathcal{H}$ is non-degenerate.*

(v) $\pi(L^1(G))' = \pi(G)'$ *and, in particular,* $\pi(G) \subseteq \pi(L^1(G))''$ *and* $\pi(L^1(G)) \subseteq \pi(G)''$.

(vi) *The representations of G and $L^1(G)$ have the same closed invariant subspaces. In particular, one is irreducible if and only if the other has this property.*

Proof. (i) First we observe that the sesquilinear map

$$(v, w) \mapsto \int_G f(g)\langle \pi(g)v, w\rangle \, d\mu_G(g)$$

is continuous:

$$\Big| \int_G f(g)\langle \pi(g)v, w\rangle \, d\mu_G(g)\Big| \leq \int_G |f(g)| \|\pi(g)v\| \|w\| \, d\mu_G(g) = \|f\|_1 \|v\| \|w\|.$$

Hence there exists a unique operator $\pi(f) \in B(\mathcal{H})$ satisfying (i) and (A.5).

(ii) In view of (i), π defines a continuous linear map $L^1(G) \to B(\mathcal{H})$. It remains to verify

$$\pi(f * g) = \pi(f)\pi(g) \quad \text{and} \quad \pi(f)^* = \pi(f^*).$$

Since $C_c(G)$ is dense in $L^1(G)$, it suffices to verify these relations for $f, g \in C_c(G)$. For $v, w \in \mathcal{H}$, we have

$$\begin{aligned}
\langle \pi(f * g)v, w\rangle &= \int_G \int_G f(y) g(y^{-1}x) \langle \pi(x)v, w\rangle \, d\mu_G(y) \, d\mu_G(x) \\
&\overset{\text{Fubini}}{=} \int_G f(y) \int_G g(y^{-1}x) \langle \pi(x)v, w\rangle \, d\mu_G(x) \, d\mu_G(y) \\
&= \int_G f(y) \int_G g(x) \langle \pi(yx)v, w\rangle \, d\mu_G(x) \, d\mu_G(y) \\
&= \int_G f(y) \int_G g(x) \langle \pi(x)v, \pi(y^{-1})w\rangle \, d\mu_G(x) \, d\mu_G(y) \\
&= \int_G f(y) \langle \pi(g)v, \pi(y^{-1})w\rangle \, d\mu_G(y) \\
&= \int_G f(y) \langle \pi(y)\pi(g)v, w\rangle \, d\mu_G(y) \\
&= \langle \pi(f)\pi(g)v, w\rangle.
\end{aligned}$$

This proves that $\pi(f * g) = \pi(f)\pi(g)$. We further have

$$\begin{aligned}
\langle v, \pi(f^*)w\rangle &= \int_G \Delta_G(x) f(x^{-1}) \langle v, \pi(x)w\rangle \, d\mu_G(x) \\
&= \int_G \Delta_G(x) f(x^{-1}) \langle \pi(x^{-1})v, w\rangle \, d\mu_G(x) \\
&= \int_G f(x) \langle \pi(x)v, w\rangle \, d\mu_G(x) \\
&= \langle \pi(f)v, w\rangle,
\end{aligned}$$

which implies $\pi(f)^* = \pi(f^*)$.

(iii) Since λ_x defines an isometry of $L^1(G)$, it suffices to assume that $f \in C_c(G)$. For $v, w \in \mathcal{H}$, we have

$$\begin{aligned}\langle \pi(x)\pi(f)v, w\rangle &= \langle \pi(f)v, \pi(x^{-1})w\rangle = \int_G f(y)\langle \pi(y)v, \pi(x^{-1})w\rangle\ d\mu_G(y)\\ &= \int_G f(y)\langle \pi(xy)v, w\rangle\ d\mu_G(y) = \int_G f(x^{-1}y)\langle \pi(y)v, w\rangle\ d\mu_G(y)\\ &= \langle \pi(\lambda_x f)v, w\rangle.\end{aligned}$$

From this relation and (ii) we further derive

$$(\pi(f)\pi(x))^* = \pi(x^{-1})\pi(f^*) = \pi(\lambda_{x^{-1}} f^*) = \pi(\Delta_G(x)(\rho_x f)^*) = \Delta_G(x)\pi(\rho_x f)^*,$$

and this proves (iii).

(iv) To see that the representation of $L^1(G)$ is non-degenerate, we show that for every $0 \neq v \in \mathcal{H}$ there exists an $f \in C_c(G)$ with $\|\pi(f)v - v\| < \varepsilon$. To find such an f, let U be a $\mathbf{1}$-neighborhood in G with $\|\pi(g)v - v\| < \varepsilon$ for $g \in U$. Urysohn's Lemma implies the existence of $0 \neq f \in C_c(G)$ with $0 \leq f$ and $\operatorname{supp}(f) \subseteq U$. Then $\int_G f(g)\, d\mu_G(g) > 0$, and after multiplication with a suitable scalar, we may w.l.o.g. assume that $\int_G f(g)\, d\mu_G(g) = 1$. Then

$$\begin{aligned}\|\pi(f)v - v\| &= \Big\| \int_G f(g)\pi(g)v\, d\mu_G(g) - \int_G f(g)v\, d\mu_G(g)\Big\|\\ &= \Big\| \int_G f(g)(\pi(g)v - v)\, d\mu_G(g)\Big\|\\ &\leq \int_G |f(g)|\|\pi(g)v - v\|\, d\mu_G(g) \leq \varepsilon \int_G f(g)\, d\mu_G(g) = \varepsilon.\end{aligned}$$

(v) First we show that $\pi(L^1(G)) \subseteq \pi(G)''$. So let $A \in \pi(G)'$. For $f \in L^1(G)$ and $v, w \in \mathcal{H}$ we then have

$$\begin{aligned}&\langle A\pi(f)v, w\rangle = \langle \pi(f)v, A^*w\rangle = \int_G f(g)\langle \pi(g)v, A^*w\rangle\, d\mu_G(g)\\ &= \int_G f(g)\langle A\pi(g)v, w\rangle\, d\mu_G(g) = \int_G f(g)\langle \pi(g)Av, w\rangle\, d\mu_G(g) = \langle \pi(f)Av, w\rangle,\end{aligned}$$

which implies that $A\pi(f) = \pi(f)A$.

Next we show that $\pi(G) \subseteq \pi(L^1(G))''$. If $A \in \pi(L^1(G))'$, then

$$\pi(g)A\pi(f) = \pi(g)\pi(f)A = \pi(\lambda_g f)A = A\pi(\lambda_g f) = A\pi(g)\pi(f)$$

for each $f \in L^1(G)$, and since the representation of $L^1(G)$ on $\mathcal{H}$ is non-degenerate, it follows that $\pi(g)A = A\pi(g)$.

From $\pi(L^1(G)) \subseteq \pi(G)''$, we now get $\pi(G)''' = \pi(G)' \subseteq \pi(L^1(G))'$, and likewise we derive from $\pi(G) \subseteq \pi(L^1(G))''$ that $\pi(L^1(G))' \subseteq \pi(G)'$, so that we have equality.

(vi) Since the closed invariant subspaces correspond to the orthogonal projections in the commutant (Lemma 1.3.1), this follows from (v). $\square$

We have just seen how to "integrate" a continuous unitary representation of G to a representation of the Banach-$*$-algebra $L^1(G)$. Thinking of $\pi(f) = \int_G f(x)\pi(x)\,d\mu_G(x)$, this means a "smearing" of the unitary operators $\pi(x)$. If the group G is not discrete, then G is not contained in $L^1(G)$ (as δ-functions), so that it is not obvious how to recover the unitary representation of G from the corresponding representation of $L^1(G)$. However, since the representation of $L^1(G)$ is non-degenerate, the operators $\pi(x)$, $x \in G$, are uniquely determined by the relation $\pi(x)\pi(f) = \pi(\lambda_x f)$ for $f \in L^1(G)$. To make systematic use of such relations, we now introduce the concept of a multiplier of an involutive semigroup.

A.6.3 Unitary Multiplier Actions on Semigroups

Definition A.6.13. Let $(S, *)$ be an involutive semigroup. A *multiplier* of S is a pair (λ, ρ) of maps $\lambda, \rho \colon S \to S$ satisfying the following conditions:

$$a\lambda(b) = \rho(a)b, \qquad \lambda(ab) = \lambda(a)b, \qquad \text{and} \qquad \rho(ab) = a\rho(b).$$

We write $M(S)$ for the set of all multipliers of S and turn it into an involutive semigroup by

$$(\lambda, \rho)(\lambda', \rho') := (\lambda \circ \lambda', \rho' \circ \rho) \quad \text{and} \quad (\lambda, \rho)^* := (\rho^*, \lambda^*),$$

where $\lambda^*(a) := \lambda(a^*)^*$ and $\rho^*(a) = \rho(a^*)^*$. We write

$$\mathrm{U}(M(S)) := \{(\lambda, \rho) \in M(S) \colon (\lambda, \rho)(\lambda, \rho)^* = (\lambda, \rho)^*(\lambda, \rho) = \mathbf{1}\}$$

for the *unitary group* of $M(S)$.

Remark A.6.14. (a) The assignment $\eta_S \colon S \to M(S), a \mapsto (\lambda_a, \rho_a)$ defines a morphism of involutive semigroups which is surjective if and only if S has an identity. Its image is an involutive semigroup ideal in $M(S)$, i.e.,

$$M(S)\eta_S(S) \subseteq \eta_S(S) \quad \text{and} \quad \eta_S(S)M(S) \subseteq \eta_S(S).$$

(b) The map

$$M(S) \times S \to S, \quad ((\lambda, \rho), s) \mapsto \lambda(s)$$

defines a left action of the semigroup $M(S)$ on S, and

$$S \times M(S) \to S, \quad (s, (\lambda, \rho)) \mapsto \rho(s)$$

defines a right action of $M(S)$ on S.

Example A.6.15. (a) The C^*-algebra $(C^b(X), \|\cdot\|_\infty)$ of bounded continuous functions on a locally compact space acts via the multipliers

$$\lambda(f)h = \rho(f)h = fh$$

on the commutative C^*-algebra $C_0(X)$.

(b) Let $\mathcal{H}$ be a complex Hilbert space and $K(\mathcal{H})$ be the C^*-algebra of compact operators on $\mathcal{H}$. Then we obtain for each $A \in B(\mathcal{H})$ a multiplier (λ_A, ρ_A) on $K(\mathcal{H})$.

Lemma A.6.16. *Let G be a locally compact group and $(L^1(G), *)$ be its convolution algebra. Then, for each $g \in G$, the pair*

$$m(g) := \big(\lambda_g, \Delta_G(g)^{-1}\rho_{g^{-1}}\big)$$

is a unitary multiplier of $L^1(G)$ and $m\colon G \to \mathrm{U}(M(L^1(G)))$ is a group homomorphism.

Proof. That each $m(g)$ is a multiplier of the involutive semigroup $L^1(G)$ follows from Lemma A.6.11(v)(a),(b). We further obtain from Lemma A.6.11(v)(c) that

$$\lambda_g^* = \Delta_G(g)\rho_g,$$

so that

$$m(g)^* = (\Delta_G(g^{-1})\rho_{g^{-1}}^*, \lambda_g^*) = (\lambda_{g^{-1}}, \Delta_G(g)\rho_g) = m(g^{-1}) = m(g)^{-1},$$

which shows that $m(g)$ is unitary. That m is multiplicative is an immediate consequence of the definitions. □

Proposition A.6.17. *For each non-degenerate representation $(\pi, \mathcal{H})$ of S there exists a unique unitary representation $(\widetilde{\pi}, \mathcal{H})$ of $\mathrm{U}(M(S))$, determined by*

$$\widetilde{\pi}(g)\pi(s) = \pi(gs) \quad \textit{for} \quad g \in \mathrm{U}(M(S)), s \in S. \tag{A.6}$$

Proof. Every non-degenerate representation of S is a direct sum of cyclic ones (Proposition 1.3.10), which in turn are (up to unitary equivalence) of the form $(\pi_\varphi, \mathcal{H}_\varphi)$ (Remark 4.6.2). We therefore may assume that $(\pi, \mathcal{H}) = (\pi_\varphi, \mathcal{H}_\varphi)$. The reproducing kernel K of $\mathcal{H}_\varphi$ is $K(s,t) := \varphi(st^*)$, and it is invariant under the right action of any $g = (\lambda_g, \rho_g) \in \mathrm{U}(M(S))$:

$$\begin{aligned} K(\rho_g(s), \rho_g(t)) &= \varphi(\rho_g(s)\rho_g(t)^*) = \varphi(\rho_g(s)\rho_g^*(t^*)) = \varphi(\rho_g(s)\lambda_g^{-1}(t^*)) \\ &= \varphi(s\lambda_g\lambda_g^{-1}(t^*)) = \varphi(st^*) = K(s,t). \end{aligned}$$

Hence $\widetilde{\pi}_\varphi(g)(f) := f \circ \rho_g$ defines a unitary representation $(\widetilde{\pi}_\varphi, \mathcal{H}_\varphi)$ of $\mathrm{U}(M(S))$ satisfying (A.6). That this condition determines $\widetilde{\pi}_\varphi$ uniquely follows from the non-degeneracy of the cyclic representation $(\pi_\varphi, \mathcal{H}_\varphi)$ (Proposition 1.3.10). □

A.6.4 Intermezzo on Banach Space-Valued Integrals

Let X be a compact space, μ a Radon probability measure on X, E a Banach space and $f\colon X \to E$ a continuous function. We want to define the E-valued integral $\int_X f(x)\,d\mu(x)$.

Lemma A.6.18. *There exists at most one element $I \in E$ with*

$$\lambda(I) = \int_X \lambda(f(x))\,d\mu(x) \quad \textit{for each} \quad \lambda \in E'.$$

Proof. This is an immediate consequence of the fact that E' separates the points of E. $\square$

We define a linear functional

$$\widetilde{I}\colon E' \to \mathbb{C}, \quad \widetilde{I}(\lambda) := \int_X \lambda(f(x))\, d\mu(x)$$

and observe that the integral exists because the integrand is a continuous function on X. We also observe that

$$|\widetilde{I}(\lambda)| \leq \int_X |\lambda(f(x))|\, d\mu(x) \leq \int_X \|\lambda\| \cdot \|f(x)\|\, d\mu(x) = \|\lambda\| \cdot \int_X \|f(x)\|\, d\mu(x),$$

so that $\widetilde{I} \in E''$ with

$$\|\widetilde{I}\| \leq \int_X \|f(x)\|\, d\mu(x).$$

We recall the isometric embedding

$$\eta_E\colon E \to E'', \quad \eta(v)(\lambda) = \lambda(v).$$

A Banach space E is said to be *reflexive* if η_E is surjective, but this is not the case for each Banach space. A typical examples is c_0 with $c_0'' = (\ell^1)' = \ell^\infty$. However, we want to show that $\widetilde{I} = \eta(I)$ for some $I \in E$, which means that I satisfies the condition in Lemma A.6.18.

Let $K := \overline{\mathrm{conv}(f(X))}$ and recall from Exercise A.4.1 that K is a compact subset of E with respect to the norm topology. Write E_w for the space E, endowed with the weak topology, i.e., the coarsest topology for which all elements $\lambda \in E'$ are continuous. Then the identity $E \to E_w$ is continuous and E_w is Hausdorff, which implies that K is also compact with respect to the weak topology. The embedding $\eta\colon E_w \to E''$ is clearly continuous with respect to the weak-$*$-topology on E'' (with respect to E') and the weak topology on E, because for each $\lambda \in E'$ the map $E_w \to \mathbb{C}, v \mapsto \eta(v)(\lambda) = \lambda(v)$ is continuous. Therefore the image $\eta(K) \subseteq E''$ is weak-$*$-compact.

Finally we show that $\widetilde{I} \in \eta(K)$. In fact, for each $\lambda \in E'$ we have

$$\widetilde{I}(\lambda) = \int_X \lambda(f(x))\, d\mu(x) \leq \big(\sup \lambda(K)\big)\mu(X) = \sup \lambda(K) = \sup \eta(K)(\lambda),$$

so that the Hahn–Banach Separation Theorem and the weak-$*$-closedness of $\eta(K)$ imply that

$$\widetilde{I} \in \eta(K).$$

This proves the following theorem:

Theorem A.6.19. *Let X be a compact space, μ a Radon measure on X, E a Banach space and $f\colon X \to E$ a continuous function. Then there exists a unique element $I \in E$ with*

$$\lambda(I) = \int_X \lambda(f(x))\, d\mu(x) \quad \textit{for} \quad \lambda \in E'.$$

Proof. It only remains to argue that the requirement that $\mu(X) = 1$ can be dropped. If $\mu(X) = 0$, we anyway have $I = 0$, and if $\mu(X) > 0$, then we simply replace μ by $\frac{1}{\mu(X)}\mu$ and f by $\mu(X)f$, and apply the preceding arguments. $\square$

We denote the element I also by

$$\int_X f(x)\, d\mu(x).$$

We have already seen that

$$\Big\| \int_X f(x)\, d\mu(x) \Big\| \leq \int_X \|f(x)\|\, d\mu(x). \tag{A.7}$$

Remark A.6.20. If $A\colon E \to F$ is a continuous linear map between Banach spaces, then

$$A \int_X f(x)\, d\mu(x) = \int_X Af(x)\, d\mu(x).$$

For each $\lambda \in F'$ we have

$$\begin{aligned}\lambda\Big(A \int_X f(x)\, d\mu(x)\Big) &= (\lambda \circ A)\Big(\int_X f(x)\, d\mu(x)\Big) = \int_X (\lambda \circ A)(f(x))\, d\mu(x)\\ &= \lambda\Big(\int_X Af(x)\, d\mu(x)\Big),\end{aligned}$$

so that the assertion follows from Lemma A.6.18.

A.6.5 Recovering the Representation of G

Proposition A.6.21. *For $f, h \in C_c(G)$ we have*

$$f * h = \int_G f(x)\, \lambda_x(h)\, d\mu_G(x)$$

as an $L^1(G)$-valued integral.

Proof. Let $K \subseteq G$ be a compact subset containing

$$\operatorname{supp}(f) \cdot \operatorname{supp}(h) \supseteq \operatorname{supp}(f * h).$$

Since $\operatorname{supp}(f)$ is compact and the map

$$G \to C(K), \quad x \mapsto f(x)\lambda_x(h)|_K$$

is continuous (see the proof of Proposition 3.3.4) Theorem A.6.19 implies the existence of a $C(K)$-valued integral

$$I := \int_G f(x)\, \lambda_x(h)|_K\, d\mu_G(x).$$

If $A\colon C(K) \to L^1(G)$ is the canonical inclusion, defined by extending a function $f\colon K \to \mathbb{C}$ by 0 on $G \setminus K$, then

$$AI = \int_G f(x)\,\lambda_x(h)\,d\mu_G(x)$$

follows from the fact that $\lambda_x(h)$ vanishes outside of K. Since point evaluations on $C(K)$ are continuous, we have for each $y \in K$:

$$I(y) = \int_G f(x)\,\lambda_x(h)(y)\,d\mu_G(x) = (f * h)(y),$$

hence $I = (f * h)|_K$, and finally $AI = f * h$ follows from $\mathrm{supp}(f * h) \subseteq K$. $\square$

Theorem A.6.22. *Let G be a locally compact group. Then there exists for each non-degenerate representation $(\pi, \mathcal{H})$ of the Banach-$*$-algebra $L^1(G)$ a unique unitary representation $(\pi_G, \mathcal{H})$ with the property that*

$$\pi_G(g)\pi(f) = \pi(\lambda_g f) \quad \text{for} \quad g \in G, f \in L^1(G).$$

The representation $(\pi_G, \mathcal{H})$ is continuous, and for $f \in L^1(G)$ we have

$$\pi(f) = \int_G f(x)\pi_G(x)\,d\mu_G(x),$$

so that π coincides with the representation of $L^1(G)$ defined by π_G.

Proof. Since we have the homomorphism

$$m\colon G \to \mathrm{U}(M(L^1(G))), \quad g \mapsto (\lambda_g, \Delta_G(g)^{-1}\rho_{g^{-1}})$$

from Example A.6.15, the existence of π_G follows from Proposition A.6.17.

To see that π_G is continuous, let $v \in \mathcal{H}$ and $f \in C_c(G)$. Then the map

$$G \to \mathcal{H}, g \mapsto \pi_G(g)\pi(f)v = \pi(\lambda_g f)v$$

is continuous because the map $G \to L^1(G), g \mapsto \lambda_g f$ is continuous (Lemma A.6.11(vi)). Since the elements of the form $\pi(f)v$ span a dense subspace, the continuity of π_G follows from Lemma 1.2.6.

To see that integration of π_G yields the given representation π, it suffices to show that for $f, h \in C_c(G)$ and $v \in \mathcal{H}$ we have

$$\pi_G(f)\pi(h)v = \pi(f)\pi(h)v$$

because the elements of the form $\pi(h)v$, $h \in C_c(G)$, $v \in \mathcal{H}$, form a dense subset of $\mathcal{H}$. For $v, w \in \mathcal{H}$ we obtain a continuous linear functional

$$\omega\colon L^1(G) \to \mathbb{C}, \quad f \mapsto \langle \pi(f)v, w\rangle.$$

Applying, Proposition A.6.21 to this functional, we get with Proposition A.6.12(iii):

$$\begin{aligned}
&\langle \pi(f)\pi(h)v, w\rangle = \langle \pi(f * h)v, w\rangle = \omega(f * h) = \int_G f(x)\omega(\lambda_x h)\, d\mu_G(x) \\
&= \int_G f(x)\langle \pi(\lambda_x h)v, w\rangle\, d\mu_G(x) = \int_G f(x)\langle \pi_G(x)\pi(h)v, w\rangle\, d\mu_G(x) \\
&= \langle \pi_G(f)\pi(h)v, w\rangle.
\end{aligned}$$

This proves that $\pi(f)\pi(h) = \pi_G(f)\pi(h)$. □

A.6.6 Representations of Abelian Locally Compact Groups

Proposition A.6.23. *For an abelian locally compact group G, the following assertions hold:*

(a) *$L^1(G)$ is a commutative Banach-$*$-algebra.*

(b) *The map*

$$\eta\colon \widehat{G} \to L^1(G)', \quad \eta(\chi)(f) := \int_G f(x)\chi(x)\, d\mu_G(x)$$

maps the character group $\widehat{G}$ bijectively onto $\widehat{L^1(G)}$.

Proof. (a) It suffices to show that the convolution product is commutative on the dense subalgebra $C_c(G)$. Since the modular factor Δ_G of G is trivial (Proposition A.6.7), we have

$$\begin{aligned}
(f * h)(y) &= \int_G f(x)h(x^{-1}y)\, d\mu_G(x) = \int_G f(x^{-1})h(yx)\, d\mu_G(x) \\
&= \int_G f(x^{-1}y)h(x)\, d\mu_G(x) = (h * f)(y).
\end{aligned}$$

(b) Since each character $\chi \in \widehat{G}$ is a bounded measurable function, it defines an element in $L^1(G)'$. If $\pi_\chi(g) = \chi(g)\mathbf{1}$ is the one-dimensional irreducible representation of G, defined by the character χ, then the corresponding integrated representation of $L^1(G)$ is given by

$$\pi_\chi(f) = \int_G f(x)\chi(x)\mathbf{1}\, d\mu_G(x) = \eta(\chi)(f)\mathbf{1},$$

so that $\eta(\chi)\colon L^1(G) \to \mathbb{C}$ defines a non-zero algebra homomorphism because it is a non-degenerate representation.

If, conversely, $\gamma\colon L^1(G) \to \mathbb{C}$ is a non-zero continuous homomorphism of Banach-$*$-algebras, then $\pi(f) := \gamma(f)\mathbf{1}$ defines a one dimensional non-degenerate representation of $L^1(G)$, and the corresponding representation of G is given by a continuous character χ with $\pi_\chi = \pi$. This implies that $\gamma = \eta(\chi)$. □

In the following we endow the character group $\widehat{G}$ always with the locally compact topology for which η is a homeomorphism. This is the coarsest topology for which all functions

$$\widehat{f}\colon \widehat{G} \to \mathbb{C}, \quad \chi \mapsto \int_G f(x)\chi(x)\,d\mu_G(x)$$

are continuous, and, by definition, all these functions vanish at infinity, i.e., $\widehat{f} \in C_0(\widehat{G})$. The function $\widehat{f}$ is called the *Fourier transform* of f.

Example A.6.24. For $G = \mathbb{R}^n$ we have already seen that each element of $\widehat{G}$ is of the form $\chi_x(y) = e^{i\langle x,y\rangle}$ for some $x \in \mathbb{R}^n$. Therefore the Fourier transform can be written as

$$\widehat{f}(y) = \int_{\mathbb{R}^n} f(x)e^{i\langle x,y\rangle}\,dx.$$

From Lebesgue's Dominated Convergence Theorem it follows immediately that all the functions $\widehat{f}$ are continuous with respect to the standard topology of $\mathbb{R}^n$. Therefore the bijection

$$\iota\colon \mathbb{R}^n \to \widehat{\mathbb{R}}^n, \quad x \mapsto \chi_x$$

is continuous. Further, the Riemann–Lebesgue Lemma (Proposition A.5.1) implies that all functions $\widehat{f}$ vanish at infinity, and this implies that ι extends to a continuous map

$$\iota_\omega\colon (\mathbb{R}^n)_\omega \to (\widehat{\mathbb{R}}^n)_\omega \cong \mathrm{Hom}(L^1(\mathbb{R}^n), \mathbb{C})$$

of the one-point compactifications. As ι_ω is a bijection and $\mathbb{R}^n_\omega$ is compact, it follows that ι_ω, and hence also ι, is a homeomorphism.

Theorem A.6.25. (Spectral Theorem for locally compact abelian groups) *Let G be a locally compact abelian group and $\widehat{G} \cong L^1(G)\widehat{\ }$ be its character group. Then, for each regular spectral measure P on the locally compact space $\widehat{G}$, the unitary representation*

$$\pi_P\colon G \to \mathrm{U}(\mathcal{H}), \quad \pi_P(g) := P(\widehat{g}), \quad \widehat{g}(\chi) = \chi(g),$$

is continuous. If, conversely, $(\pi, \mathcal{H})$ is a continuous unitary representation of G on $\mathcal{H}$, then there exists a unique regular spectral measure P with $\pi = \pi_P$.

Proof. First we use the spectral measure P to define a non-degenerate representation π_P of $L^1(G)$ by $\pi_P(f) := P(\widehat{f})$ (Theorem 5.3.2). Then we clearly have

$$\pi_P(g)\pi_P(f) = P(\widehat{g})P(\widehat{f}) = P(\widehat{g}\widehat{f}), \quad g \in G, f \in L^1(G).$$

Next we observe that for each character $\chi \in \widehat{G}$ we have

$$\begin{aligned}(\lambda_g f)\widehat{\ }(\chi) &= \int_G (\lambda_g f)(x)\chi(x)\,d\mu_G(x) = \int_G f(g^{-1}x)\chi(x)\,d\mu_G(x)\\ &= \int_G f(x)\chi(gx)\,d\mu_G(x) = \chi(g)\int_G f(x)\chi(x)\,d\mu_G(x) = \widehat{g}(\chi)\widehat{f}(\chi).\end{aligned}$$

We thus obtain

$$\pi_P(g)\pi_P(f) = P(\widehat{g}\widehat{f}) = P((\lambda_g f)\widehat{\ }) = \pi_P(\lambda_g f),$$

so that $\pi_P \colon G \to \mathrm{U}(\mathcal{H})$ is the unique continuous unitary representation of G on $\mathcal{H}$ corresponding to the representation of $L^1(G)$ (Theorem A.6.22). In particular, π_P is continuous. It follows in particular that π_P is continuous.

If, conversely, $(\pi, \mathcal{H})$ is a continuous unitary representation of G and $\pi \colon L^1(G) \to B(\mathcal{H})$ the corresponding non-degenerate representation of $L^1(G)$, then we use Theorem 5.3.2 to obtain a regular spectral measure P on $\widehat{G} \cong L^1(G)\widehat{\ }$ with $\pi(f) = P(\widehat{f})$, $f \in L^1(G)$. Then

$$\pi(g)\pi(f) = \pi(\lambda_g f) = P(\widehat{g}\widehat{f}) = P(\widehat{g})P(\widehat{f}) = P(\widehat{g})\pi(f)$$

implies that $P(\widehat{g}) = \pi(g)$ holds for each $g \in G$ (Theorem A.6.22). $\square$

Definition A.6.26. Let P be a regular spectral measure on $\widehat{G}$. If $(U_i)_{i\in I}$ are open subsets of $\widehat{G}$ with $P(U_i) = \mathbf{0}$, then the same holds for $U := \bigcup_{i\in I} U_i$. In fact, since P is inner regular, it suffices to observe that for each compact subset $C \subseteq U$ we have $P(C) = \mathbf{0}$, but this follows from the fact that C is covered by finitely many U_i. We conclude that there exists a maximal open subset $U \subseteq \widehat{G}$ with $P(U) = \mathbf{0}$, and its complement

$$\mathrm{supp}(P) := U^c$$

is called the *support of* P, resp., the support of the corresponding representation. It is the smallest closed subset A of $\widehat{G}$ with $P(A) = \mathbf{1}$.

Exercises for Section A.6

Exercise A.6.1. Let $\lambda = dX$ denote Lebesgue measure on the space $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ of real $(n \times n)$-matrices. Show that a Haar measure on $\mathrm{GL}_n(\mathbb{R})$ is given by

$$d\mu_{\mathrm{GL}_n(\mathbb{R})}(g) = \frac{1}{|\det(g)|^n} d\lambda(g).$$

Hint: Calculate the determinant of the linear maps $\lambda_g \colon M_n(\mathbb{R}) \to M_n(\mathbb{R}), x \mapsto gx$.

Exercise A.6.2. Let $G = \mathrm{Aff}_1(\mathbb{R}) \cong \mathbb{R} \ltimes \mathbb{R}^\times$ denote the affine group of $\mathbb{R}$, where (b, a) corresponds to the affine map $\varphi_{b,a}(x) := ax + b$. This group is sometimes called the *$ax + b$-group*. Show that a Haar measure on this group is obtained by

$$\int_G f(b,a)\, d\mu_G(b,a) := \int_{\mathbb{R}} \int_{\mathbb{R}^\times} f(b,a)\, db \frac{da}{|a|^2}.$$

Show further that $\Delta_G(b,a) = |a|^{-1}$, which implies that G is not unimodular.

Exercise A.6.3. Let X be a locally compact space, μ a positive Radon measure on X, $\mathcal{H}$ a Hilbert space and $f \in C_c(X, \mathcal{H})$ be a compactly supported continuous function.

(a) Prove the existence of the $\mathcal{H}$-valued integral

$$I := \int_X f(x)\, d\mu(x),$$

i.e., the existence of an element $I \in \mathcal{H}$ with

$$\langle v, I \rangle = \int_X \langle v, f(x) \rangle\, d\mu(x) \quad \text{for} \quad v \in \mathcal{H}.$$

Hint: Verify that the right hand side of the above expression is defined and show that it defines a continuous linear functional on $\mathcal{H}$.

(b) Show that, if μ is a probability measure, then

$$I \in \overline{\mathrm{conv}(f(X))}.$$

Hint: Use the Hahn–Banach Separation Theorem.

Exercise A.6.4. Let G be a locally compact group. Show that the convolution product on $C_c(G)$ satisfies

$$\|f * h\|_\infty \leq \|f\|_1 \cdot \|h\|_\infty.$$

Conclude that convolution extends to a continuous bilinear map

$$L^1(G, \mu_G) \times C_0(G) \to C_0(G).$$

Conclude that for $f \in L^1(G, \mu_G)$ and $h \in C_c(G)$, the convolution product $f * h$ can be represented by a continuous function in $C_0(G)$.

Exercise A.6.5. Let G be a compact group. Show that every left or right invariant closed subspace of $L^2(G)$ consists of continuous functions. Hint: Use Exercise A.6.4 and express the integrated representation of $L^1(G)$ on $L^2(G)$ in terms of the convolution product.

Exercise A.6.6. Let $\pi\colon G \to \mathrm{U}(\mathcal{H})$ be a unitary representation of the locally compact group G on $\mathcal{H}$ which is norm-continuous, i.e., continuous with respect to the norm topology on $\mathrm{U}(\mathcal{H})$. Show that there exists an $f \in C_c(G)$ for which the operator $\pi(f)$ is invertible.

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