Introduction to Categorical Homotopy Theory

Winter term 2023/24

Catherine Meusburger
Department Mathematik
Friedrich-Alexander-Universität Erlangen-Nürnberg

(Date: April 27, 2024)

To prepare this lecture I used the following textbooks that I recommend as reading:

- P. Goerss, J. Jardine, Simplicial homotopy theory, Springer, 2009.
- J. P. May, Simplicial objects in algebraic topology, University of Chicago Press, 1992
- B. Richter, From categories to homotopy theory, Cambridge University Press, 2020.
- E. Riehl, Category Theory in Context, Courier Dover Publications, 2017.
- E. Riehl, Categorical homotopy Theory, Cambridge University Press, 2014.
- C. Weibel, An introduction to homological algebra, Cambridge Studies in Advanced Mathematics 38.

I also used the following lecture notes and review articles

- G. Friedman, An elementary illustrated introduction to simplicial sets, arXiv:0809.4221v5 [math.AT]
- F. Loregian, Coend calculus. arXiv preprint arXiv:1501.02503.
- C. Rezk, Introduction to quasicategories. Lecture Notes for course at University of Illinois at Urbana-Champaign (2022).

Acknowledgements

I am grateful to the authors of the books and review articles listed above and also to Christoph Schweigert for sending me his lecture notes on simplicial sets.

Special thanks go to all students whose questions and comments helped me to improve these lecture notes and the exercises, in particular to Stefan Gebhart, Karin Hoffmann, Johannes Lindner, Maximilian Ludwig, Kenio Müller, Daniel Polster, Jana Sommerrock, David Wegmann, Fabian Weigelt.

Please send comments and remarks to

`catherine.meusburger@math.uni-erlangen.de`.

Contents

1 Background on categories **6**

1.1 Categories, functors and natural transformations 6

1.2 Universal properties and adjoint functors 12

1.3 Hom functors and the Yoneda lemma 20

2 Limits and Colimits **25**

2.1 (Co)limits: definitions and examples 25

2.2 Existence of (co)limits 36

2.3 (Co)limits in functor category and exchange of (co)limits 39

2.4 Transformations of (co)limits under functors 43

3 Kan extensions **51**

3.1 Kan extensions: definitions and examples 51

3.2 Formulas for Kan extensions 56

3.3 Preservation of Kan extensions 65

4 Ends and Coends **69**

4.1 Ends and coends: definition and properties 69

4.2 (Co)end calculus 74

5 Simplicial objects **79**

5.1 Simplexes and simplicial complexes 79

5.2 The simplex category and (co)simplicial objects 82

5.3 Geometric realisation 87

5.4 Simplicial nerve and homotopy category 91

5.5 Chain complexes and homologies 95

6 Homotopies **107**

6.1 Homotopies in Top 107

6.2 Simplicial homotopies 114

6.3 Simplicial homotopies and chain homotopies . . . 121

7 Kan complexes and quasicategories **127**

7.1 Kan complexes . . . 127

7.2 Simplicial homotopy groups . . . 136

7.3 Quasicategories . . . 146

8 Exercises **161**

8.1 Exercises for Chapter 1 . . . 161

8.2 Exercises for Chapter 2 . . . 162

8.3 Exercises for Chapter 3 . . . 164

8.4 Exercises for Chapter 4 . . . 166

8.5 Exercises for Chapter 5 . . . 167

8.6 Exercises for Chapter 6 . . . 170

8.7 Exercises for Chapter 7 . . . 172

Notation for categories

- Ab $= \mathbb{Z}$-Mod - category of abelian groups and group homomorphisms between them,
- $\mathrm{Alg}_{\mathbb{F}}$ - category of algebras over a field $\mathbb{F}$ and algebra homomorphisms,
- BG - category with a single object $\bullet$ and $\mathrm{Hom}_{BG}(\bullet, \bullet) = G$ for a group G,
- Cat - category of small categories and functors between them,
- $\mathrm{CAlg}_{\mathbb{F}}$ - category of commutative algebras over $\mathbb{F}$ and algebra homomorphisms,
- cHaus - category of compact Hausdorff spaces and continuous maps between them,
- CRing - category of commutative rings and ring homomorphisms,
- Field - category of fields and field homomorphisms,
- Grpd - category of groupoids and functors between them,
- G-Set $= \mathrm{Set}^{BG}$ - category of (left) G-sets and G-equivariant maps for a group G,
- Int - category of integral domains and injective ring homomorphisms between them,
- Kan - full subcategory of SSet with Kan complexes as objects,
- $\mathcal{O}(X)$ for a topological space X - poset category with open subsets $U \subset X$ as objects and the partial ordering relation $\preceq = \subset$,
- R-Mod - category of modules over a ring R and R-linear maps,
- Rel - category of sets and relations,
- $\mathrm{Rep}_{\mathbb{F}}(G) = \mathbb{F}[G]$-Mod $= \mathrm{Vect}_{\mathbb{F}}^{BG}$- category of representations of G over $\mathbb{F}$ and intertwiners,
- Set - category of sets and maps,
- SSet $= \mathrm{Set}^{\Delta^{op}}$ -category of simplicial sets and simplicial maps,
- sSet $= \mathrm{Set}^{\Delta_{\mathrm{inj}}^{op}}$ -category of semisimplicial sets and semisimplicial maps,
- Top - category of topological spaces and continuous maps,
- Top^* - category of pointed topological spaces and basepoint preserving continuous maps,
- Ring - category of unital rings and unital ring homomorphisms,
- $\mathrm{Vect}_{\mathbb{F}} = \mathbb{F}$-Mod - category of vector spaces over a field $\mathbb{F}$ and $\mathbb{F}$-linear maps
- $\mathcal{C}^{\mathcal{J}} = \mathrm{Fun}(\mathcal{J}, \mathcal{C})$ - category of functors $F : \mathcal{J} \to \mathcal{C}$ and natural transformations,
- Δ - simplex category
- Δ_{inj} - subcategory of Δ with the same objects and injective monotonic maps as morphisms

1 Background on categories

In this section we summarise the required background on categories, functors and natural transformations. We also recall important constructions with categories such as (co)products, adjoints and the Yoneda Lemma.

1.1 Categories, functors and natural transformations

The concept of a category encodes many examples of mathematical structures and structure preserving maps between them, but it goes beyond them and replaces structure preserving maps by the more abstract notion of a morphism. The crucial features are that morphisms have a fixed source and target, can be composed and can be identity morphisms. This generalises the notions of domain and codomain of structure preserving maps, of their composition and of the structure preserving identity maps.

Definition 1.1.1: A **category** $\mathcal{C}$ consists of:

- a class $\mathrm{Ob}\,\mathcal{C}$ of **objects**,
- for each pair of objects $X, Y \in \mathrm{Ob}\,\mathcal{C}$ a class $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of **morphisms**,
- for each triple of objects X, Y, Z a **composition map**

$$\circ : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z),$$

such that the following axioms are satisfied:

(C1) The classes $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of morphisms are pairwise disjoint,

(C2) The composition is associative: $f \circ (g \circ h) = (f \circ g) \circ h$ for all morphisms $h \in \mathrm{Hom}_{\mathcal{C}}(W, X)$, $g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, $f \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$,

(C3) For every object X there is a morphism $1_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$, the **identity morphism** on X, with $1_X \circ f = f$ and $g \circ 1_X = g$ for all $f \in \mathrm{Hom}_{\mathcal{C}}(W, X)$, $g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

Instead of $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, we also write $f : X \to Y$. The object X is called the **source** of f, and the object Y the **target** of f. A morphism $f : X \to X$ is called an **endomorphism**.

A morphism $f : X \to Y$ is called an **isomorphism**, if there is a morphism $g : Y \to X$ with $g \circ f = 1_X$ and $f \circ g = 1_Y$. In this case, we call the objects X and Y **isomorphic**.

Often, one requires that the *morphisms* between fixed objects form not only a class, but a set. This is the case in essentially all familiar categories from algebra and topology. Nevertheless, it is sometimes necessary to relax this condition. In contrast, requiring that the *objects* of a category form a set is very restrictive and excludes many familiar and important categories.

Definition 1.1.2: A category $\mathcal{C}$ is called

- **locally small**, if $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a set for all objects $X, Y \in \mathrm{Ob}\mathcal{C}$,
- **small**, if it is locally small and $\mathrm{Ob}\mathcal{C}$ is a set.

The following examples of categories are all locally small, but none of them is small.

Example 1.1.3:

1. The category Set: objects are sets, morphisms are maps. The composition is the composition of maps, and identity morphisms are identity maps. Isomorphisms are bijective maps.

2. The category Top of topological spaces: objects are topological spaces, morphisms are continuous maps, isomorphisms are homeomorphisms.

3. The category Top* of **pointed topological spaces**: Objects are pairs (X, x) of a topological space X and a point $x \in X$, morphisms $f : (X, x) \to (Y, y)$ are continuous maps $f : X \to Y$ with $f(x) = y$.

4. The category Top(2) of **pairs of topological spaces**: Objects are pairs (X, A) of a topological space X and a subspace $A \subset X$, morphisms $f : (X, A) \to (Y, B)$ are continuous maps $f : X \to Y$ with $f(A) \subset B$. Isomorphisms are homeomorphisms $f : X \to Y$ with $f(A) = B$.

5. Many examples of categories we will use in the following are categories of algebraic structures. This includes the following:

   - the category Vect$_{\mathbb{F}}$ of vector spaces over a field $\mathbb{F}$:
     objects: vector spaces over $\mathbb{F}$, morphisms: $\mathbb{F}$-linear maps,

   - the category Vect$_{\mathbb{F}}^{fin}$ of finite dimensional vector spaces over a field $\mathbb{F}$:
     objects: finite-dimensional vector spaces over $\mathbb{F}$, morphisms: $\mathbb{F}$-linear maps,

   - the category Grp of groups:
     objects: groups, morphisms: group homomorphisms,

   - the category Ab of abelian groups:
     objects: abelian groups, morphisms: group homomorphisms,

   - the category Ring of unital rings:
     objects: unital rings, morphisms: unital ring homomorphisms,

   - the category Field of fields:
     objects: fields, morphisms: field homomorphisms,

   - the category Alg$_{\mathbb{F}}$ of algebras over a field $\mathbb{F}$:
     objects: algebras over $\mathbb{F}$, morphisms: algebra homomorphisms,

   - the categories R-Mod and Mod-R of left and right modules over a ring R:
     objects: R-left or right modules, morphisms: R-left or right module homomorphisms.

   - the category R-Mod-S of (R, S)-bimodules:
     objects: (R, S)-bimodules, morphisms: (R, S)-bimodule homomorphisms.

In all of the categories in Example 1.1.3 the morphisms are *maps*. A category for which this is the case is called a **concrete category**. A category that is not concrete is the category of sets and relations in Exercise 1. Further examples of non-concrete categories arise from some of the basic categorical concepts and constructions in the next example.

Example 1.1.4:

1. A small category $\mathcal{C}$ in which all morphisms are isomorphisms is called a **groupoid**.
2. A category with a single object is a **monoid** and a groupoid with a single object a **group**.

   Group elements are identified with endomorphisms of the object, and the composition of morphisms is the group multiplication.

   More generally, for any object X in a groupoid $\mathcal{C}$, the set $\mathrm{End}_{\mathcal{C}}(X) = \mathrm{Hom}_{\mathcal{C}}(X, X)$ with the composition $\circ : \mathrm{End}_{\mathcal{C}}(X) \times \mathrm{End}_{\mathcal{C}}(X) \to \mathrm{End}_{\mathcal{C}}(X)$ is a group.
3. For every category $\mathcal{C}$, there is an **opposite category** $\mathcal{C}^{op}$, which has the same objects as $\mathcal{C}$, whose morphisms are given by $\mathrm{Hom}_{\mathcal{C}^{op}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$ and in which the order of the composition is reversed.
4. The **cartesian product** of categories $\mathcal{C}, \mathcal{D}$ is the category $\mathcal{C} \times \mathcal{D}$ with pairs (C, D) of objects in $\mathcal{C}$ and $\mathcal{D}$ as objects, with $\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((C, D), (C', D')) = \mathrm{Hom}_{\mathcal{C}}(C, C') \times \mathrm{Hom}_{\mathcal{D}}(D, D')$ and the composition of morphisms $(h, k) \circ (f, g) = (h \circ f, k \circ g)$.
5. A **subcategory** of a category $\mathcal{C}$ is a category $\mathcal{D}$, such that $\mathrm{Ob}(\mathcal{D}) \subset \mathrm{Ob}(\mathcal{C})$ is a subclass, $\mathrm{Hom}_{\mathcal{D}}(D, D') \subset \mathrm{Hom}_{\mathcal{C}}(D, D')$ for all objects D, D' in $\mathcal{D}$ and the composition of morphisms of $\mathcal{D}$ coincides with their composition in $\mathcal{C}$. A subcategory $\mathcal{D}$ of $\mathcal{C}$ is called a **full subcategory** if $\mathrm{Hom}_{\mathcal{D}}(D, D') = \mathrm{Hom}_{\mathcal{C}}(D, D')$ for all objects D, D' in $\mathcal{D}$.
6. **Quotient categories:** Let $\mathcal{C}$ be a category with an equivalence relation $\sim_{X,Y}$ on each morphism set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ that is compatible with the composition of morphisms:
   $f \sim_{X,Y} g$ and $h \sim_{Y,Z} k$ implies $h \circ f \sim_{X,Z} k \circ g$.

   Then one obtains a category $\mathcal{C}'$, the **quotient category** of $\mathcal{C}$, with the same objects as $\mathcal{C}$ and equivalence classes of morphisms in $\mathcal{C}$ as morphisms.

   The composition of morphisms in $\mathcal{C}'$ is given by $[h] \circ [f] = [h \circ f]$, and the identity morphisms by $[1_X]$. Isomorphisms in $\mathcal{C}'$ are equivalence classes of morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ for which there exists a morphism $g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$ with $f \circ g \sim_{Y,Y} 1_Y$ and $g \circ f \sim_{X,X} 1_X$.

The construction in the last example plays an important role in classification problems, in particular in topology. Classifying the objects of a category $\mathcal{C}$ usually means classifying them up to isomorphism - giving a list of objects in $\mathcal{C}$ such that every object in $\mathcal{C}$ is isomorphic to exactly one object in this list.

This is possible in some contexts - for instance for the category $\mathrm{Vect}_{\mathbb{F}}^{fin}$ of finite dimensional vector spaces over $\mathbb{F}$. In this case the list contains the vector spaces $\mathbb{F}^n$ with $n \in \mathbb{N}_0$. However, it is often too difficult to solve this problem in full generality. In this case, it is sometimes simpler to consider instead a quotient category $\mathcal{C}'$ and to attempt a partial classification.

If two objects are isomorphic in $\mathcal{C}$, they are by definition isomorphic in $\mathcal{C}'$, as any isomorphism $f : X \to Y$ with inverse $g : Y \to X$ yields $[g] \circ [f] = [g \circ f] = [1_X]$ and $[f] \circ [g] = [f \circ g] = [1_Y]$. However, the converse does not hold - the category $\mathcal{C}'$ yields a weaker classification than $\mathcal{C}$.

To relate different categories, one must not only relate their objects but also their morphisms, in a way that is compatible with source and target objects, the composition of morphisms and the identity morphisms. This leads to the concept of a *functor*.

Definition 1.1.5: Let $\mathcal{C}, \mathcal{D}$ be categories. A **functor** $F : \mathcal{C} \to \mathcal{D}$ consists of:

- an assignment of an object $F(C)$ in $\mathcal{D}$ to every object C in $\mathcal{C}$,
- for each pair of objects C, C' in $\mathcal{C}$, a map

$$\mathrm{Hom}_{\mathcal{C}}(C, C') \to \mathrm{Hom}_{\mathcal{D}}(F(C), F(C')), \quad f \mapsto F(f),$$

that is compatible with the composition of morphisms and with the identity morphisms

$$\begin{array}{ll} F(g \circ f) = F(g) \circ F(f) & \forall f \in \mathrm{Hom}_{\mathcal{C}}(C, C'), g \in \mathrm{Hom}_{\mathcal{C}}(C', C'') \\ F(1_C) = 1_{F(C)} & \forall C \in \mathrm{Ob}\, \mathcal{C}. \end{array}$$

- A functor $F : \mathcal{C} \to \mathcal{C}$ is called an **endofunctor**.
- A functor $F : \mathcal{C}^{op} \to \mathcal{D}$ is sometimes called a **contravariant functor** from $\mathcal{C}$ to $\mathcal{D}$.
- The **composite** of two functors $F : \mathcal{B} \to \mathcal{C}$, $G : \mathcal{C} \to \mathcal{D}$ is the functor $GF : \mathcal{B} \to \mathcal{D}$ given by the assignment $B \mapsto GF(B)$ for all objects B in $\mathcal{B}$ and the maps

$$\mathrm{Hom}_{\mathcal{B}}(B, B') \to \mathrm{Hom}_{\mathcal{D}}(GF(B), GF(B')), \quad f \mapsto G(F(f)).$$

Example 1.1.6:

1. For any category $\mathcal{C}$, the identity functor $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ that assigns each object and morphism in $\mathcal{C}$ to itself is an endofunctor of $\mathcal{C}$.

2. The **forgetful functor** $\mathrm{Vect}_{\mathbb{F}} \to \mathrm{Ab}$ assigns to each vector space the underlying abelian group and to each linear map the associated group homomorphism. There are analogous forgetful functors $\mathrm{Vect}_{\mathbb{F}} \to \mathrm{Set}$, $\mathrm{Ring} \to \mathrm{Set}$, $\mathrm{Grp} \to \mathrm{Set}$, $\mathrm{Top} \to \mathrm{Set}$ that assign to each vector space, ring, group, topological space the underlying set and to each morphism the underlying map.

3. Vector space duals define a functor $* : \mathrm{Vect}_{\mathbb{F}} \to \mathrm{Vect}_{\mathbb{F}}^{op}$ that assigns to
   - a vector space V its dual V^*,
   - a linear map $f : V \to W$ its adjoint $f^* : W^* \to V^*$, $\alpha \mapsto \alpha \circ f$.

4. A group G defines a category BG with a single object, the **delooping** of G, with elements of G as morphisms, and with the multiplication of G as the composition.
   - Functors $F : BG \to \mathrm{Set}$ correspond to G-sets $X = F(\bullet)$ with the group action $\rhd : G \times X \to X$, $g \rhd x = F(g)(x)$.
   - Functors $F : BG \to \mathrm{Vect}_{\mathbb{F}}$ correspond to representations of G over $\mathbb{F}$, with the representation space $V = F(\bullet)$ and $\rho = F(g) : G \to \mathrm{Aut}_{\mathbb{F}} V$.

5. Let $\phi : R \to S$ a ring homomorphism. The **restriction functor** $\mathrm{Res} : S\text{-Mod} \to R\text{-Mod}$
   - sends an S-module $(M, \rhd)$ to the R-module $(M, \rhd_\phi)$ with the pulled back module structure $r \rhd_\phi m = \phi(r) \rhd m$.
   - sends every S-linear map $f : M \to M'$ to itself.

6. **Tensor products**: Let R be a ring, M an R-right module and N an R-left module.
   - The functor $M \otimes_R - : R\text{-Mod} \to \mathrm{Ab}$ assigns to
   - an R-left module N the abelian group $M \otimes_R N$,
   - an R-linear map $f : N \to N'$ the group homomorphism $\mathrm{id}_M \otimes f : M \otimes_R N \to M \otimes_R N'$.

- • The functor $-\otimes_R N : R^{op}\text{-Mod} \to \text{Ab}$ assigns to
- an R-right module M the abelian group $M\otimes_R N$,
- an R-linear map $f : M \to M'$ the group homomorphism $f\otimes \mathrm{id}_N : M\otimes_R N \to M\otimes_R N'$.

- • The functor $\otimes_R : R^{op}\text{-Mod} \times R\text{-Mod} \to \text{Ab}$ assigns to
- an R-right module M and an R-left module N the abelian group $M\otimes_R N$,
- an R^{op}-linear map $f : M \to M'$ and R-linear map $g : N \to N'$ the group homomorphism $f\otimes g : M\otimes_R N \to M'\otimes_R N'$.

Note that for *commutative* rings R, any R-left module is an (R, R)-bimodule and these functors can be defined to take values in R-Mod instead of Ab.

7. **Hom-functors**: Let $\mathcal{C}$ be a category and C an object in $\mathcal{C}$.
   - • The functor $\mathrm{Hom}(C, -) : \mathcal{C} \to$ Set assigns to
   - an object C' the set $\mathrm{Hom}_{\mathcal{C}}(C, C')$,
   - a morphism $f : C' \to C''$ the map $\mathrm{Hom}(C, f) : \mathrm{Hom}_{\mathcal{C}}(C, C') \to \mathrm{Hom}_{\mathcal{C}}(C, C'')$, $g \mapsto f \circ g$.
   - • The functor $\mathrm{Hom}(-, C) : \mathcal{C}^{op} \to$ Set assigns to
   - an object C' the set $\mathrm{Hom}_{\mathcal{C}}(C', C)$,
   - a morphism $f : C' \to C''$ the map $\mathrm{Hom}(f, C) : \mathrm{Hom}_{\mathcal{C}}(C'', C) \to \mathrm{Hom}_{\mathcal{C}}(C', C)$, $g \mapsto g\circ f$.
8. The **path component functor** $\pi_0 : \text{Top} \to \text{Set}$ assigns to
   - a topological space X the set $\pi_0(X)$ of its path components,
   - a continuous map $f : X \to Y$ the map $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$, $P(x) \mapsto P(f(x))$.
9. The **fundamental group** defines a functor $\pi_1 : \text{Top}^* \to \text{Grp}$ that assigns to
   - a pointed topological space (x, X) its fundamental group $\pi_1(x, X)$,
   - a morphism $f : (x, X) \to (y, Y)$ of pointed topological spaces the group homomorphism $\pi_1(f) : \pi_1(x, X) \to \pi_1(y, Y)$, $[\gamma] \mapsto [f \circ \gamma]$.
10. **Abelisation**: The abelisation functor $F : \text{Grp} \to \text{Ab}$ assigns to
    - a group G the abelian group $F(G) = G/[G, G]$, where $[G, G]$ is the normal subgroup generated by the set of all elements $ghg^{-1}h^{-1}$ for $g, h \in G$,
    - a group homomorphism $f : G \to H$ the induced group homomorphism $F(f) : G/[G, G] \to H/[H, H]$, $g + [G, G] \mapsto f(g) + [H, H]$.

There is another structure that relates functors. As a functor $F : \mathcal{C} \to \mathcal{D}$ involves maps between the sets $\mathrm{Hom}_{\mathcal{C}}(C, C')$ and $\mathrm{Hom}_{\mathcal{D}}(F(C), F(C'))$, a structure that relates two functors $F, G : \mathcal{C} \to \mathcal{D}$ must in particular relate the sets $\mathrm{Hom}_{\mathcal{D}}(F(C), F(C'))$ and $\mathrm{Hom}_{\mathcal{D}}(G(C), G(C'))$. The simplest way to do this is to assign to each object C in $\mathcal{C}$ a morphism $\eta_C : F(C) \to G(C)$ in $\mathcal{D}$. One then requires compatibility with the images $F(f)$ and $G(f)$ for all morphisms $f : C \to C'$ in $\mathcal{C}$.

Definition 1.1.7: A **natural transformation** $\eta : F \Rightarrow G$ between functors $F, G : \mathcal{C} \to \mathcal{D}$ is an assignment of a morphism $\eta_C : F(C) \to G(C)$ in $\mathcal{D}$ to every object C in $\mathcal{C}$ such that the following diagram commutes for all morphisms $f : C \to C'$ in $\mathcal{C}$

$$\begin{array}{ccc} F(C) & \xrightarrow{\eta_C} & G(C) \\ \big\downarrow{\scriptstyle F(f)} & & \big\downarrow{\scriptstyle G(f)} \\ F(C') & \xrightarrow{\eta_{C'}} & G(C'). \end{array}$$

A **natural isomorphism** is a natural transformation $\eta : F \Rightarrow G$, for which all morphisms $\eta_X : F(X) \to G(X)$ are isomorphisms. Two functors that are related by a natural isomorphism are called **naturally isomorphic**.

Example 1.1.8:

1. For any functor $F : \mathcal{C} \to \mathcal{D}$ the identity natural transformation $\mathrm{id}_F : F \Rightarrow F$ with component morphisms $(\mathrm{id}_F)_X = 1_{F(X)} : F(X) \to F(X)$ is a natural isomorphism.

2. Consider the functors $\mathrm{id} : \mathrm{Vect}_\mathbb{F} \to \mathrm{Vect}_\mathbb{F}$ and $** : \mathrm{Vect}_\mathbb{F} \to \mathrm{Vect}_\mathbb{F}$. Then there is a canonical natural transformation can : $\mathrm{id} \Rightarrow **$, whose component morphisms $\eta_V : V \to V^{**}$ assign to a vector $v \in V$ the unique vector $v^{**} \in V^{**}$ with $v^{**}(\alpha) = \alpha(v)$ for all $\alpha \in V^*$. It is a natural isomorphism if and only if V is finite-dimensional.

3. Consider the category CRing of commutative unital rings and unital ring homomorphisms and the category Grp of groups and group homomorphisms.

   Let $F : \mathrm{CRing} \to \mathrm{Grp}$ the functor that assigns to
   - a commutative unital ring k the group $\mathrm{GL}_n(k)$ of invertible $n \times n$-matrices in k,
   - a unital ring homomorphism $f : k \to l$ the group homomorphism

   $$\mathrm{GL}_n(f) : \mathrm{GL}_n(k) \to \mathrm{GL}_n(l), \quad M = (m_{ij})_{i,j=1,\ldots,n} \mapsto f(M) = (f(m_{ij}))_{i,j=1,..,n}.$$

   Let $G : \mathrm{CRing} \to \mathrm{Grp}$ be the functor that assigns to
   - a commutative unital ring k the group $G(k) = k^\times$ of units in k,
   - a unital ring homomorphism $f : k \to l$ the induced group homomorphism

   $$G(f) = f|_{k^\times} : k^\times \to l^\times.$$

   The determinant defines a natural transformation $\det : F \to G$ with component morphisms $\det_k : \mathrm{GL}_n(k) \Rightarrow k^\times$, because the following diagram commutes for every unital ring homomorphism $f : k \to l$

   $$\begin{array}{ccc} \mathrm{GL}_n(k) & \xrightarrow{\det_k} & k^\times \\ {\scriptstyle \mathrm{GL}_n(f)}\downarrow & & \downarrow{\scriptstyle f|_{k^\times}} \\ \mathrm{GL}_n(l) & \xrightarrow{\det_l} & l^\times. \end{array}$$

4. Let G be a group and BG its delooping. Then functors $F : BG \to \mathrm{Set}$ are G-sets by Example 1.1.6, 4. Natural transformations between them are G-equivariant maps.

   A natural transformation $\eta : F \Rightarrow F'$ has a single component $\eta_\bullet : F(\bullet) \to F'(\bullet)$. The naturality condition states that $\eta_\bullet(g \rhd x) = g \rhd' \eta_\bullet(x)$ for all $g \in G$, $x \in X$.

   Similarly, by Example 1.1.6, 4. functors $F : BG \to \mathrm{Vect}_\mathbb{F}$ are representations of G over $\mathbb{F}$, and natural transformations between them are homomorphisms of representations.

Remark 1.1.9:

1. For any small category $\mathcal{C}$ and category $\mathcal{D}$, the functors $F : \mathcal{C} \to \mathcal{D}$ and natural transformations between them form a category, denoted $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ or $\mathcal{D}^\mathcal{C}$, the **functor category**.

   The composite of natural transformations $\eta : F \Rightarrow G$ and $\kappa : G \Rightarrow H$ is the natural transformation $\kappa \circ \eta : F \Rightarrow H$ with component morphisms $(\kappa \circ \eta)_X = \kappa_X \circ \eta_X : F(X) \to H(X)$ and the identity morphisms are the identity natural transformations $1_F = \mathrm{id}_F : F \Rightarrow F$.

   If $\mathcal{C}$ is a category that is not small, the functor category $\mathcal{D}^\mathcal{C}$ is defined analogously, but no longer locally small.

2. Natural transformations can be composed with functors.

   If $F, F' : \mathcal{C} \to \mathcal{D}$ are functors and $\eta : F \Rightarrow F'$ a natural transformation, then for any functor $G : \mathcal{B} \to \mathcal{C}$ one obtains a natural transformation $\eta G : FG \Rightarrow F'G$ with component morphisms $(\eta G)_B = \eta_{G(B)} : FG(B) \to F'G(B)$. Similarly, any functor $E : \mathcal{D} \to \mathcal{E}$ defines a natural transformation $E\eta : EF \Rightarrow EF'$ with $(E\eta)_C = E(\eta_C) : EF(C) \to EF'(C)$.

The notions of natural transformations and natural isomorphisms are particularly important as they allow one to generalise the notion of an inverse map and of a bijection to functors. An **inverse** of a functor $F : \mathcal{C} \to \mathcal{D}$ is by definition a functor $G : \mathcal{D} \to \mathcal{C}$ with $GF = \mathrm{id}_{\mathcal{C}}$ and $FG = \mathrm{id}_{\mathcal{D}}$, and an **isomorphism** of categories is a functor $F : \mathcal{C} \to \mathcal{D}$ with an inverse.

However, it turns out that there are very few examples of functors with an inverse. A more useful generalisation is obtained by weakening this requirement. Instead of requiring $FG = \mathrm{id}_{\mathcal{D}}$ and $GF = \mathrm{id}_{\mathcal{C}}$, one requires only that these functors are *naturally isomorphic* to the identity functors. This leads to the concept of an equivalence of categories.

Definition 1.1.10: A functor $F : \mathcal{C} \to \mathcal{D}$ is called an **equivalence of categories** if there is a functor $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\kappa : GF \Rightarrow \mathrm{id}_{\mathcal{C}}$ and $\eta : FG \Rightarrow \mathrm{id}_D$. In this case, the categories $\mathcal{C}$ and $\mathcal{D}$ are called **equivalent**.

Sometimes it is easier to use a more direct characterisation of an equivalences of categories in terms of its behaviour on objects and morphisms. This is the categorical equivalent of the statement that a map between sets is an isomorphism if and only if it is injective and surjective. The proof of the following lemma makes use of the axiom of choice and can be found for instance in [K], Chapter XI, Prop XI.1.5.

Lemma 1.1.11: A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if it is:

1. **essentially surjective**:
   for every object D in $\mathcal{D}$ there is an object C of $\mathcal{C}$ such that D is isomorphic to $F(C)$.
2. **fully faithful**:
   all maps $\mathrm{Hom}_{\mathcal{C}}(C, C') \to \mathrm{Hom}_{\mathcal{D}}(F(C), F(C'))$, $f \mapsto F(f)$ are bijections.

Example 1.1.12:

1. The category $\mathrm{Vect}_{\mathbb{F}}^{fin}$ of finite-dimensional vector spaces over $\mathbb{F}$ is equivalent to the category $\mathcal{C}$, whose objects are non-negative integers $n \in \mathbb{N}_0$, whose morphisms $f : n \to m$ are $m \times n$-matrices with entries in $\mathbb{F}$ and with the matrix multiplication as composition.

2. The category Set^{fin} of finite sets is equivalent to the category Ord^{fin}, whose objects are finite **ordinal numbers** $[n] = \{0, 1, ..., n-1\}$ for all $n \in \mathbb{N}_0$ and whose morphisms $f : [m] \to [n]$ are maps $f : \{0, 1, ..., m-1\} \to \{0, 1, ..., n-1\}$ with the composition of maps as the composition of morphisms.

1.2 Universal properties and adjoint functors

Many concepts and constructions from algebra or topology can be generalised straightforwardly to categories. This works, whenever it is possible to characterise them in terms of *universal properties* involving only the *morphisms* in the category.

These universal properties are associated with *adjoint functors*, pairs of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ that define natural bijections between morphism spaces in the two categories $\mathrm{Hom}_{\mathcal{C}}(C, G(D)) \cong \mathrm{Hom}_{\mathcal{D}}(F(C), D)$. These natural bijections encode the universal properties.

In particular, there are concepts of a categorical product and coproduct that generalise cartesian products and disjoint unions of sets and products and sums of topological spaces. We treat products and coproducts as an example that illustrates the use of universal properties. Many more examples of universal properties are given in the following chapters.

Definition 1.2.1: Let $\mathcal{C}$ be a category and $(C_i)_{i\in I}$ a family of objects in $\mathcal{C}$.

1. A **product** of the family $(C_i)_{i\in I}$ is an object $\Pi_{i\in I} C_i$ in $\mathcal{C}$ together with a family of morphisms $\pi_i : \Pi_{j\in I} C_j \to C_i$, such that for all families of morphisms $f_i : W \to C_i$ there is a unique morphism $f : W \to \Pi_{i\in I} C_i$ such that the diagram

$$\begin{array}{ccc} W & \overset{\exists! f}{\dashrightarrow} & \Pi_{j\in I} C_j \\ & \underset{f_i}{\searrow} & \downarrow{\scriptstyle \pi_i} \\ & & C_i \end{array} \qquad (1)$$

   commutes for all $i \in I$. This is called the **universal property** of the product.

2. A **coproduct** of the family $(C_i)_{i\in I}$ is an object $\amalg_{i\in I} C_i$ in $\mathcal{C}$ with a family $(\iota_i)_{i\in I}$ of morphisms $\iota_i : C_i \to \amalg_{j\in I} C_j$, such that for every family $(f_i)_{i\in I}$ of morphisms $f_i : C_i \to Y$ there is a unique morphism $f : \amalg_{i\in I} C_i \to Y$ such that the diagram

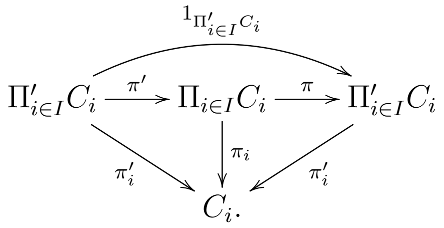
$$\begin{array}{ccc} Y & \overset{\exists! f}{\dashleftarrow} & \amalg_{j\in I} C_j \\ & \underset{f_i}{\nwarrow} & \uparrow{\scriptstyle \iota_i} \\ & & C_i \end{array} \qquad (2)$$

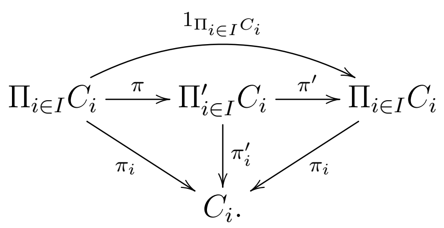
   commutes for all $i \in I$. This is called the **universal property** of the coproduct.

Remark 1.2.2: Products or coproducts do not necessarily exist for a given family of objects $(C_i)_{i\in I}$ in a category $\mathcal{C}$, but if they exist, they are **unique up to unique isomorphism**:

If $(\Pi_{i\in I} C_i, (\pi_i)_{i\in I})$ and $(\Pi'_{i\in I} C_i, (\pi'_i)_{i\in I})$ are two products for a family of objects $(C_i)_{i\in I}$ in $\mathcal{C}$, then there is a unique morphism $\pi' : \Pi'_{i\in I} C_i \to \Pi_{i\in I} C_i$ with $\pi_i \circ \pi' = \pi'_i$ for all $i \in I$, and this morphism is an isomorphism.

By the universal property of the product $\Pi_{i\in I} C_i$ applied to the family $f_i := \pi'_i : \Pi'_{i\in I} C_i \to C_i$, there is a unique morphism $\pi' : \Pi'_{i\in I} C_i \to \Pi_{i\in I} C_i$ such that $\pi_i \circ \pi' = \pi'_i$ for all $i \in I$. Similarly, the universal property of $\Pi'_{i\in I} C_i$ implies that for the family of morphisms $\pi_i : \Pi_{i\in I} C_i \to C_i$ there is a unique morphism $\pi : \Pi_{i\in I} C_i \to \Pi'_{i\in I} C_i$ with $\pi'_i \circ \pi = \pi_i$ for all $i \in I$. It follows that $\pi' \circ \pi : \Pi_{i\in I} C_i \to \Pi_{i\in I} C_i$ is a morphism with $\pi_i \circ \pi \circ \pi' = \pi'_i \circ \pi = \pi_i$ for all $i \in I$. As the identity morphism on $\Pi_{i\in I} C_i$ is another morphism with this property, the uniqueness implies $\pi' \circ \pi = 1_{\Pi_{i\in I} C_i}$. By the same argument one obtains $\pi \circ \pi' = 1_{\Pi'_{i\in I} C_i}$ and hence π' is an isomorphism with inverse π.





Example 1.2.3:

1. The cartesian product of sets is a product in Set, and the disjoint union of sets is a coproduct in Set. The product of topological spaces is a product in Top and the topological sum is a coproduct in Top. In Set and Top, products and coproducts exist for all families of objects.

2. The direct sum of vector spaces is a coproduct and the direct product of vector spaces a product in $\text{Vect}_{\mathbb{F}}$. More generally, direct sums and products of R-left (right) modules over a unital ring R are coproducts and products in R-Mod (Mod-R). Again, products and coproducts exist for all families of objects in R-Mod (Mod-R).

3. The wedge sum is a coproduct in the category Top^* of pointed topological spaces. It exists for all families of pointed topological spaces.

4. The direct product of groups is a product in Grp and the free product of groups is a coproduct in Grp. They exist for all families of groups.

In particular, we can consider categorical products and coproducts over empty index sets I. By definition, a categorical product for an empty family of objects is an object $T = \Pi_\emptyset$ such that for every object C in $\mathcal{C}$ there is a unique morphism $t_C : C \to T$. Similarly, a coproduct over an empty index set I is an object $I := \amalg_\emptyset$ in $\mathcal{C}$ such that for every object C in $\mathcal{C}$, there is a unique morphism $i_C : I \to C$. Such objects are called *terminal* and *initial* objects in $\mathcal{C}$. Initial and terminal objects need not exist in every category $\mathcal{C}$, but if they exist they are unique up to unique isomorphism by the universal property of the products and coproducts.

An object that is both, terminal and initial, is called a *zero object* or *null object.* If it exists, it is unique up to unique isomorphism. It also gives rise to a distinguished morphism, the *zero morphism* $0 = i_{C'} \circ t_C : C \to C'$, between any two objects C, C' in $\mathcal{C}$.

Definition 1.2.4: Let $\mathcal{C}$ be a category. An object X in a category $\mathcal{C}$ is called:

1. A **final** or **terminal object** in $\mathcal{C}$ is an object T in $\mathcal{C}$ such that for every object C in $\mathcal{C}$ there is a unique morphism $t_C : C \to T$.
2. A **cofinal** or **initial object** in $\mathcal{C}$ is an object I in $\mathcal{C}$ such that for every object C in $\mathcal{C}$ there is a unique morphism $i_C : I \to C$,
3. A **null object** or **zero object** in $\mathcal{C}$ is an object 0 in $\mathcal{C}$ that is both final and initial: for every object C in $\mathcal{C}$ there are a unique morphisms $t_C : C \to 0$ and $i_C : 0 \to C$.
4. If $\mathcal{C}$ has a zero object, then the morphism $0 = i_{C'} \circ t_C : C \to 0 \to C'$ is called the **trivial morphism** or **zero morphism** from C to C'.

Example 1.2.5:

1. The empty set is an initial object in Set and the empty topological space an initial object in Top. Any set with one element is a final object in Set and any one point space a final object in Top. The categories Set and Top do not have null objects.

2. The null vector space $\{0\}$ is a null object in the category $\text{Vect}_{\mathbb{F}}$. More generally, for any ring R, the trivial R-module $\{0\}$ is a null object in R-Mod (Mod-R).

3. The trivial group $G = \{e\}$ is a null object in Grp and in Ab.

4. The ring $\mathbb{Z}$ is an initial object in the category Ring, since for every unital ring R, there is exactly one ring homomorphism $f : \mathbb{Z} \to R$, namely the one determined by $f(0) = 0_R$ and $f(1) = 1_R$. The zero ring $R = \{0\}$ with $0 = 1$ is a final object in Ring, but not an initial one. The category Ring has no zero object.

5. The category Field does not have initial or final objects. As any ring homomorphism $f : \mathbb{F} \to \mathbb{K}$ between fields is injective, an initial object in Field would be a subfield of all other fields, and every field would be a subfield of a final field. Either of them would imply that each field has the same characteristic as an initial or final field, a contradiction.

Universal properties of algebraic constructions such as products and coproducts, freely generated modules or abelisation of groups, are encoded in *adjoint functors*. These are pairs of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ that yield natural bijections $\text{Hom}_{\mathcal{C}}(C, G(D)) \cong \text{Hom}_{\mathcal{D}}(F(C), D)$ between the morphism spaces in the two categories. These bijections state the universal properties of the constructions.

Definition 1.2.6: A functor $F : \mathcal{C} \to \mathcal{D}$ is called **left adjoint** to a functor $G : \mathcal{D} \to \mathcal{C}$ and G **right adjoint** to F, $F \dashv G$, if the functors $\text{Hom}(F(-), -) : \mathcal{C}^{op} \times \mathcal{D} \to \text{Set}$ and $\text{Hom}(-, G(-)) : \mathcal{C}^{op} \times \mathcal{D} \to \text{Set}$ are naturally isomorphic.

In other words, there is a family of bijections $\phi_{C,D} : \text{Hom}_{\mathcal{C}}(C, G(D)) \to \text{Hom}_{\mathcal{D}}(F(C), D)$, indexed by objects C in $\mathcal{C}$ and D in $\mathcal{D}$, such that the diagram

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}}(C, G(D)) & \xrightarrow[h \mapsto G(g) \circ h \circ f]{\text{Hom}(f, G(g))} & \text{Hom}_{\mathcal{C}}(C', G(D')) \\ \downarrow \phi_{C,D} & & \downarrow \phi_{C',D'} \\ \text{Hom}_{\mathcal{D}}(F(C), D) & \xrightarrow[h \mapsto g \circ h \circ F(f)]{\text{Hom}(F(f), g)} & \text{Hom}_{\mathcal{D}}(F(C'), D'). \end{array} \tag{3}$$

commutes for all morphisms $f : C' \to C$ in $\mathcal{C}$ and $g : D \to D'$ in $\mathcal{D}$.

Example 1.2.7:

1. **Forgetful functors and freely generated modules:**
For a ring R, the forgetful functor $G : R\text{-Mod} \to \text{Set}$ is right adjoint to the functor $F : \text{Set} \to R\text{-Mod}$ that assigns to
   - a set A the free R-module $F(A) = \langle A \rangle_R$ generated by A,
   - a map $f : A \to B$ the R-linear map $F(f) : \langle A \rangle_R \to \langle B \rangle_R$ with $F(f) \circ \iota_A = \iota_B \circ f$.

Proof:
For every map $f : A \to M$ from a set A into an R-module M, there is a unique R-linear map $\langle f \rangle_R : \langle A \rangle_R \to M$ with $\langle f \rangle_R \circ \iota_A = f$ for the inclusion $\iota_A : A \to \langle A \rangle_R$, $a \mapsto a$. This defines bijections

$$\phi_{A,M} : \mathrm{Hom}_{\mathrm{Set}}(A, G(M)) \to \mathrm{Hom}_{R-\mathrm{Mod}}(F(A), M), \quad f \mapsto \langle f \rangle_R.$$

For all maps $f : A' \to A$, $h : A \to M$ and R-linear maps $g : M \to M'$ we have

$$g \circ \langle h \rangle_R \circ F(f) \circ \iota_{A'} = g \circ \langle h \rangle_R \circ \iota_A \circ f = g \circ h \circ f = \langle g \circ h \circ f \rangle_R \circ \iota_{A'}.$$

This implies $\langle g \circ h \circ f \rangle_R = g \circ \langle h \rangle_R \circ F(f)$. □

2. **Discrete and indiscrete topology**:
The forgetful functor F : Top $\to$ Set is left adjoint to the indiscrete topology functor I : Set $\to$ Top that assigns to
   - a set X the topological space $(X, \mathcal{O}_{ind})$ with the indiscrete topology,
   - a map $f : X \to Y$ the continuous map $f : (X, \mathcal{O}_{ind}) \to (Y, \mathcal{O}_{ind})$.

   The forgetful functor F : Top $\to$ Set is right adjoint to the discrete topology functor D : Set $\to$ Top that assigns
   - to a set X the topological space $(X, \mathcal{O}_{disc})$ with the discrete topology,
   - to a map $f : X \to Y$ the continuous map $f : (X, \mathcal{O}_{disc}) \to (Y, \mathcal{O}_{disc})$.

   The bijections between the Hom-Sets are

$$\Phi_{(W,\mathcal{O}),X} : \mathrm{Hom}_{\mathrm{Top}}((W, \mathcal{O}), (X, \mathcal{O}_{ind})) \to \mathrm{Hom}_{\mathrm{Set}}(W, X), \quad f \mapsto f$$
$$\Phi_{X,(W,\mathcal{O})} : \mathrm{Hom}_{\mathrm{Set}}(X, W) \to \mathrm{Hom}_{\mathrm{Top}}((X, \mathcal{O}_{disc}), (W, \mathcal{O})), \quad f \mapsto f.$$

   The statement that these are bijections expresses the fact that any map $f : W \to X$ from a topological space $(W, \mathcal{O})$ into a set X becomes continuous when X is equipped with the indiscrete topology and any map $f : X \to W$ becomes continuous when X is equipped with the discrete topology. The naturality condition in (3) follows directly.

3. **Forgetful functors without left or right adjoints**:
The forgetful functor V : Field $\to$ Set has no right or left adjoint. If it had a left adjoint F : Set $\to$ Field or a right adjoint G : Set $\to$ Field there would be bijections

$$\Phi_{\emptyset,\mathbb{K}} : \mathrm{Hom}_{\mathrm{Set}}(\emptyset, \mathbb{K}) \to \mathrm{Hom}_{\mathrm{Field}}(F(\emptyset), \mathbb{F}). \quad \Phi_{\mathbb{F},\{x\}} : \mathrm{Hom}_{\mathrm{Field}}(\mathbb{F}, G(\{x\})) \to \mathrm{Hom}_{\mathrm{Set}}(\mathbb{F}, \{x\})$$

   for any field $\mathbb{F}$. This would imply that $F(\emptyset)$ is an initial object in Field, a subfield of any other field $\mathbb{F}$, and that $G(\{x\})$ is a terminal object in Field, a field containing any field $\mathbb{F}$ as a subfield. This would require $\mathrm{char}\,\mathbb{F} = \mathrm{char}\,F(\emptyset) = \mathrm{char}\,G(\{x\})$ for all fields $\mathbb{F}$.

4. **Inclusion functor and abelisation**: The inclusion functor G : Ab $\to$ Grp is right adjoint to the abelisation functor F : Grp $\to$ Ab from Example 1.1.6, 10. (Exercise 3).

5. **Products, coproducts and diagonal functors**:
• Let $\mathcal{C}$ be a category and I a set such that products and coproducts in $\mathcal{C}$ exist for all families of objects indexed by I.

   • Let $\mathcal{C}_I$ be the category with families $(C_i)_{i \in I}$ of objects in $\mathcal{C}$ as objects and families $(f_i : C_i \to C'_i)_{i \in I}$ of morphisms in $\mathcal{C}$ as morphisms, with componentwise composition.

• Let $\Delta : \mathcal{C} \to \mathcal{C}_I$ be the diagonal functor that assigns to an object C and a morphism $f : C \to C'$ in $\mathcal{C}$ the constant families $(C)_{i\in I}$ and $(f)_{i\in I}$.

• Let $\Pi_I : \mathcal{C}_I \to \mathcal{C}$ be the product functor that assigns to a family $(C_i)_{i\in I}$ the product $\Pi_{i\in I} C_i$ and to a family $(f_i)_{i\in I} : (C_i)_{i\in I} \to (C_i')_{i\in I}$ the morphism $\Pi_{i\in I} f_i : \Pi_{i\in I} C_i \to \Pi_{i\in I} C_i'$ with $\pi_i' \circ (\Pi_{i\in I} f_i) = f_i \circ \pi_i$ induced by the universal property of the product.

• Let $\amalg_I : \mathcal{C}_I \to \mathcal{C}$ be the coproduct functor that assigns to a family $(C_i)_{i\in I}$ the coproduct $\amalg_{i\in I} C_i$ and to a family $(f_i)_{i\in I} : (C_i)_{i\in I} \to (C_i')_{i\in I}$ the morphism $\amalg_{i\in I} f_i : \Pi_{i\in I} C_i \to \Pi_{i\in I} C_i'$ with $(\amalg_{i\in I} f_i) \circ \iota_i = \iota_i' \circ f_i$ induced by the universal property of the coproduct.

Then $\Pi_I : \mathcal{C}_I \to \mathcal{C}$ is right adjoint to Δ and $\amalg_I : \mathcal{C}_I \to \mathcal{C}$ is left adjoint to Δ. The bijections between the Hom-sets are given by

$$\Phi_{C,(C_i)_{i\in I}} : \mathrm{Hom}_{\mathcal{C}}(C, \Pi_{i\in I} C_i) \to \mathrm{Hom}_{\mathcal{C}_I}((C)_{i\in I}, \Pi_{i\in I} C_i), \quad f \mapsto (\pi_i \circ f)_{i\in I}$$
$$\Phi^{-1}_{(C_i)_{i\in I},C} : \mathrm{Hom}_{\mathcal{C}}(\amalg_{i\in I} C_i, C) \to \mathrm{Hom}_{\mathcal{C}_I}((C_i)_{i\in I}, (C)_{i\in I}), \quad f \mapsto (f \circ \iota_i)_{i\in I}.$$

The universal property of the (co)product implies that they are bijections, and a short computation shows that they satisfy the naturality condition in (3).

6. **Tensor products and Hom-functors**:
   - For any R-right module M, the functor $M\otimes_R - : R\text{-Mod} \to \mathrm{Ab}$ is left adjoint to the functor $\mathrm{Hom}(M, -) : \mathrm{Ab} \to R\text{-Mod}$.
   - For any R-left module N the functor $- \otimes_R N : R^{op}\text{-Mod} \to \mathrm{Ab}$ is left adjoint to the functor $\mathrm{Hom}(N, -) : \mathrm{Ab} \to R^{op}\text{-Mod}$.

   Proof:
   We prove the claim for R-right modules M. For an abelian group A and R-left module L we equip $\mathrm{Hom}_{\mathrm{Ab}}(M, A)$ with the R-module structure $(r \rhd \phi)(m) = \phi(m \lhd r)$ and define

$$\begin{array}{rlcl} \phi_{L,A} : & \mathrm{Hom}_{R\text{-Mod}}(L, \mathrm{Hom}_{\mathrm{Ab}}(M, A)) & \to & \mathrm{Hom}_{\mathrm{Ab}}(M \otimes_R L, A) \\ & \psi : L \to \mathrm{Hom}_{\mathrm{Ab}}(M, A),\ l \mapsto \psi_l & \mapsto & \chi : M\otimes_R L \to A,\ m\otimes l \mapsto \psi_l(m). \end{array}$$

   The map $\chi : M\otimes_R L \to A$, $m\otimes l \mapsto \psi_l(m)$ is well defined, since the R-linearity of the map $\psi : L \to \mathrm{Hom}_{\mathrm{Ab}}(M, A)$ implies that $\chi' : M \times L \to A$, $(m, l) \to \psi_l(m)$ is R-bilinear: $\chi'(m, r \rhd l) = \psi_{r\rhd l}(m) = (r \rhd \psi_l)(m) = \psi_l(m \lhd r) = \chi'(m \lhd r, l)$ for all $r \in R$, $l \in L$ and $m \in M$. By the universal property of the tensor product, it induces a unique group homomorphism $\chi : M \otimes_R L \to A$ with $\chi(m\otimes l) = \chi'(m, l)$. The inverse of $\phi_{L,A}$ is given by

$$\begin{array}{rlcl} \phi^{-1}_{L,A} : & \mathrm{Hom}_{\mathrm{Ab}}(M \otimes_R L, A) & \to & \mathrm{Hom}_{R\text{-Mod}}(L, \mathrm{Hom}_{\mathrm{Ab}}(M, A)) \\ & \chi : M\otimes_R L \to A, & \mapsto & \psi : L \to \mathrm{Hom}_{\mathrm{Ab}}(M, A),\ l \mapsto \psi_l \text{ with } \psi_l(m) = \chi(m \otimes l). \end{array}$$

   As we have $\psi_{r\rhd l}(m) = \chi(m \otimes (r \rhd l)) = \chi((m \lhd r) \otimes l) = \psi_l(m \lhd r)$, the map ψ_l is indeed R-linear, and a short computation shows that the diagram (3) commutes for all R-linear maps $f : L' \to L$ and all group homomorphisms $g : A \to A'$. □

7. **Restriction, induction and coinduction**:
   Let $\phi : R \to S$ be a ring homomorphism and $\mathrm{Res} : S\text{-Mod} \to R\text{-Mod}$ the **restriction functor** from Example 1.1.6, 5. that sends
   - an S-module $(M, \rhd_S)$ to the R-module $(M, \rhd_R)$ with $r \rhd_R m = \phi(r) \rhd_S m$,
   - every S-linear map $f : M \to M'$ to itself.

The **induction functor** $\mathrm{Ind} = S\otimes_R - : R\text{-Mod} \to S\text{-Mod}$ is left adjoint to Res. It sends

- an R-module M to the S-module $\mathrm{Ind}(M) = S\otimes_R M$ with $s \rhd (s'\otimes m) = (ss')\otimes m$,
- an R-linear map $f : M \to M'$ to the S-linear map $\mathrm{Ind}(f) = \mathrm{id}_S \otimes f$.

The **coinduction functor** $\mathrm{Coind} = \mathrm{Hom}_R(S, -) : R\text{-Mod} \to S\text{-Mod}$ is right adjoint to Res. It sends

- an R-module M to the S-module $\mathrm{Hom}_R(S, M)$ with $(s \rhd f)(s') = f(s' \cdot s)$,
- an R-linear map $f : M \to M'$ to $\mathrm{Hom}_R(S, f) : g \mapsto f \circ g$.

Proof:
To see that Ind is left adjoint to Res, note that the (S, R)-bimodule structure on S given by $s \rhd s' = s\cdot s'$ and $s \lhd r = s\cdot\phi(r)$ defines an S-left-module structure on the abelian group $S\otimes_R M$ given by $s \rhd (s'\otimes m) = (s\cdot s')\otimes m$. For all R-modules M and S-modules N the group homomorphisms

$$\begin{aligned}
&\phi_{M,N} : \mathrm{Hom}_R(M, \mathrm{Res}(N)) \to \mathrm{Hom}_S(\mathrm{Ind}(M), N), && \phi_{M,N}(f)(s\otimes m) = s \rhd f(m)\\
&\psi_{M,N} : \mathrm{Hom}_S(\mathrm{Ind}(M), N) \to \mathrm{Hom}_R(M, \mathrm{Res}(N)), && \psi_{M,N}(g)(m) = g(1\otimes m).
\end{aligned}$$

are mutually inverse and hence bijections. To prove that the diagram (3) commutes, we compute for all R-linear maps $f : M' \to M$, $h : M \to N$ and S-linear maps $g : N \to N'$

$$\begin{aligned}
g\circ\phi_{M,N}(h)\circ(\mathrm{id}_S\otimes f)(s\otimes m') &= g\circ\phi_{M,N}(h)(s\otimes f(m')) = g(s \rhd h\circ f(m'))\\
&= s \rhd (g\circ h\circ f(m')) = \phi_{M',N'}(g\circ h\circ f)(s\otimes m').
\end{aligned}$$

To show that Coind is right adjoint to Res we consider the ring S with the R-left module structure $r \rhd s := \phi(r)\cdot s$ and the abelian group $\mathrm{Hom}_R(S, M)$ with the S-left module structure $(s \rhd f)(s') = f(s'\cdot s)$ and note that the maps

$$\begin{aligned}
&\phi_{M,N} : \mathrm{Hom}_R(\mathrm{Res}(N), M) \to \mathrm{Hom}_S(N, \mathrm{Hom}_R(S, M)), && \phi_{M,N}(f)(s) = f(s \rhd n)\\
&\psi_{M,N} : \mathrm{Hom}_S(N, \mathrm{Hom}_R(S, M)) \to \mathrm{Hom}_R(\mathrm{Res}(N), M), && \psi_{M,N}(g)(n) = g(n)(1).
\end{aligned}$$

are mutually inverse and hence bijections. A short computation shows that $\phi_{M,N}$ makes the diagram (3) commute. □

8. **Induction, coinduction and forgetful functor**:
For every ring S, the induction functor $\mathrm{Ind} = S\otimes_{\mathbb{Z}} - : \mathrm{Ab} \to S\text{-Mod}$ is left adjoint and the coinduction functor $\mathrm{Coind} = \mathrm{Hom}_{\mathbb{Z}}(S, -) : \mathrm{Ab} \to S\text{-Mod}$ is right adjoint to the forgetful functor $\mathrm{Res} : S\text{-Mod} \to \mathrm{Ab}$.

   This is Example 1.2.7, 7. for $R = \mathbb{Z}$, where $\mathrm{Res} : S\text{-Mod} \to \mathrm{Ab}$ is the forgetful functor.

These examples show that adjoint functors arise in many contexts in algebra and topology and are often related to certain canonical constructions such as forgetful functors, freely generated modules or tensoring over a ring. Example 1.2.7, 3. shows that a functor need not have left and right adjoints. However, it seems plausible that if they exist, left or right adjoint functors should be unique, at least up to natural isomorphisms. To address this, we work with an alternative characterisation of left and right adjoints in terms of natural transformations.

Proposition 1.2.8: A functor $F : \mathcal{C} \to \mathcal{D}$ is left adjoint to $G : \mathcal{D} \to \mathcal{C}$ if and only if there are natural transformations $\epsilon : FG \Rightarrow \mathrm{id}_{\mathcal{D}}$ and $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$, the **unit** and **counit** of the adjunction, such that

$$(G\epsilon)\circ(\eta G) = \mathrm{id}_G, \qquad (\epsilon F)\circ(F\eta) = \mathrm{id}_F. \tag{4}$$

Proof:
1. Let $F:\mathcal{C}\to\mathcal{D}$ be left adjoint to $G:\mathcal{D}\to\mathcal{C}$. Then there are bijections

$$\begin{aligned}\phi_{G(D),D}&:\mathrm{Hom}_{\mathcal{C}}(G(D),G(D))\to\mathrm{Hom}_{\mathcal{D}}(FG(D),D)\\ \phi^{-1}_{C,F(C)}&:\mathrm{Hom}_{\mathcal{D}}(F(C),F(C))\to\mathrm{Hom}_{\mathcal{C}}(C,GF(C)).\end{aligned}$$

We define the natural transformations $\epsilon: FG\Rightarrow \mathrm{id}_{\mathcal{D}}$ and $\eta:\mathrm{id}_{\mathcal{C}}\Rightarrow GF$ by specifying their component morphisms:

$$\epsilon_D:=\phi_{G(D),D}(1_{G(D)}):FG(D)\to D\qquad \eta_C:=\phi^{-1}_{C,F(C)}(1_{F(C)}):C\to GF(C).$$

The commuting diagram (3) in Definition 1.2.6 implies

$$\phi_{C'.D'}(G(g)\circ h\circ f)=g\circ\phi_{C,D}(h)\circ F(f)$$

for all morphisms $h:C\to G(D)$, $f:C'\to C$ and $g:D\to D'$. It follows that for every morphism $k:D\to D'$ in $\mathcal{D}$:

$$\begin{aligned}\epsilon_{D'}\circ FG(k)&=\phi_{G(D'),D'}(1_{G(D')})\circ FG(k)\overset{(3)}{=}\phi_{G(D),D}(1_{G(D')}\circ G(k))=\phi_{G(D),D}(G(k))\\ &=\phi_{G(D),D}(G(k)\circ 1_{G(D)})\overset{(3)}{=}k\circ\phi_{G(D),D}(1_{G(D)})=k\circ\epsilon_D.\end{aligned}$$

This shows that the morphisms $\epsilon_D:FG(D)\to D$ define a natural transformation $\epsilon:FG\to\mathrm{id}_{\mathcal{D}}$. Diagram (3) then implies for all objects C in $\mathcal{C}$

$$\begin{aligned}\epsilon_{F(C)}\circ F(\eta_C)&=\phi_{GF(C),F(C)}(1_{GF(C)})\circ F(\phi^{-1}_{C,F(C)}(1_{F(C)}))\\ &\overset{(3)}{=}\phi_{C,F(C)}(1_{GF(C)}\circ\phi^{-1}_{C,F(C)}(1_{F(C)}))=\phi_{C,F(C)}\circ\phi^{-1}_{C,F(C)}(1_{F(C)})=1_{F(C)}.\end{aligned}$$

The proofs for $\eta:\mathrm{id}_{\mathcal{C}}\Rightarrow GF$ and of the identity $G(\epsilon_D)\circ\eta_{G(D)}=1_{G(D)}$ are analogous.

2. Let $\epsilon:FG\Rightarrow\mathrm{id}_{\mathcal{D}}$ and $\eta:\mathrm{id}_{\mathcal{C}}\Rightarrow GF$ be natural transformations that satisfy (4). Consider for all objects C in $\mathcal{C}$ und D in $\mathcal{D}$ the maps

$$\begin{aligned}\phi_{C,D}&=\mathrm{Hom}(1_{F(C)},\epsilon_D)\circ F:\mathrm{Hom}_{\mathcal{C}}(C,G(D))\to\mathrm{Hom}_{\mathcal{D}}(F(C),D),\quad f\mapsto\epsilon_D\circ F(f)\\ \psi_{C,D}&=\mathrm{Hom}(\eta_C,1_{G(D))})\circ G:\mathrm{Hom}_{\mathcal{D}}(F(C),D)\to\mathrm{Hom}_{\mathcal{C}}(C,G(D)),\quad g\mapsto G(g)\circ\eta_C.\end{aligned}$$

Then we have for all morphisms $f:C\to G(D)$ in $\mathcal{C}$ and $g:F(C)\to D$ in $\mathcal{D}$

$$\begin{aligned}\psi_{C,D}\circ\phi_{C,D}(f)&=G(\epsilon_D)\circ GF(f)\circ\eta_C\overset{\text{nat}}{=}G(\epsilon_D)\circ\eta_{G(D)}\circ f\overset{(4)}{=}f\\ \phi_{C,D}\circ\psi_{C,D}(g)&=\epsilon_D\circ FG(g)\circ F(\eta_C)\overset{\text{nat}}{=}g\circ\epsilon_{F(C)}\circ F(\eta_C)\overset{(4)}{=}g.\end{aligned}$$

This shows that $\psi_{C,D}=\phi^{-1}_{C,D}$ and $\phi_{C,D}:\mathrm{Hom}_{\mathcal{C}}(C,G(D))\to\mathrm{Hom}_{\mathcal{D}}(F(C),D)$ is a bijection. To verify that the diagram (3) in Definition 1.2.6 commutes, consider morphisms $f:C'\to C$, $h:C\to G(D)$ in $\mathcal{C}$ and $g:D\to D'$ in $\mathcal{D}$ and compute

$$\phi_{C',D'}(G(g)\circ h\circ f)=\epsilon_{D'}\circ FG(g)\circ F(h)\circ F(f)\overset{\text{nat}}{=}g\circ\epsilon_D\circ F(h)\circ F(f)=g\circ\phi_{C,D}(h)\circ F(f).$$

□

Theorem 1.2.9: Left and right adjoint functors are unique up to natural isomorphisms.

Proof:
Let $F, F' : \mathcal{C} \to \mathcal{D}$ be left adjoint to $G : \mathcal{D} \to \mathcal{C}$. Then by Proposition 1.2.8 there are natural transformations $\epsilon : FG \Rightarrow \mathrm{id}_\mathcal{D}$, $\eta : \mathrm{id}_\mathcal{C} \Rightarrow GF$ and $\epsilon' : F'G \Rightarrow \mathrm{id}_\mathcal{D}$, $\eta' : \mathrm{id}_\mathcal{C} \Rightarrow GF'$ satisfying (4). Consider the natural transformations $\kappa = (\epsilon F') \circ (F\eta') : F \Rightarrow F'$, $\kappa' = (\epsilon' F) \circ (F'\eta) : F' \Rightarrow F$ with component morphisms $\kappa_C = \epsilon_{F'(C)} \circ F(\eta'_C)$ and $\kappa'_C = \epsilon'_{F(C)} \circ F'(\eta_C)$. Then κ_C und κ'_C are inverse to each other since

$$\begin{aligned}
\kappa_C \circ \kappa'_C &\overset{\text{def}\,\kappa}{=} \epsilon_{F'(C)} \circ F(\eta'_C) \circ \kappa'_C \overset{\text{nat}\,\kappa'}{=} \epsilon_{F'(C)} \circ \kappa'_{GF'(C)} \circ F'(\eta'_C) \\
&\overset{\text{def}\,\kappa'}{=} \epsilon_{F'(C)} \circ \epsilon'_{FGF'(C)} \circ F'(\eta_{GF'(C)}) \circ F'(\eta'_C) \overset{\text{nat}\,\epsilon'}{=} \epsilon'_{F'(C)} \circ F'G(\epsilon_{F'(C)}) \circ F'(\eta_{GF'(C)}) \circ F'(\eta'_C) \\
&= \epsilon'_{F'(C)} \circ F'(G(\epsilon_{F'(C)}) \circ \eta_{GF'(C)}) \circ F'(\eta'_C) \overset{(4)}{=} \epsilon'_{F'(C)} \circ F'(\eta'_C) \overset{(4)}{=} 1_{F'(C)},
\end{aligned}$$

and an analogous computation yields $\kappa'_C \circ \kappa_C = 1_{F(C)}$. This shows that κ and κ' are natural isomorphisms and that F is naturally isomorphic to F'. The proof for right adjoints is analogous.
□

1.3 Hom functors and the Yoneda lemma

Besides identity functors and constant functors, the *Hom functors* are the only functors that arise directly from the definition of a category and are defined for all categories. They play a special role, as they allow one to relate and translate claims for a category $\mathcal{C}$ to claims in Set. They are also important in relation to adjoint functors.

Definition 1.3.1: Let $\mathcal{C}$ be a category. The **Hom functors** for an object $C \in \mathrm{Ob}\mathcal{C}$ there are functors $\mathrm{Hom}(C, -) : \mathcal{C} \to \mathrm{Set}$ and $\mathrm{Hom}(-, C) : \mathcal{C}^{op} \to \mathrm{Set}$ that assign

- to an object the sets of morphisms $\mathrm{Hom}_\mathcal{C}(C, D)$ and $\mathrm{Hom}_\mathcal{C}(D, C)$,
- to a morphism $f : D \to D'$ the maps

$$\begin{aligned}
\mathrm{Hom}(C, f) : \mathrm{Hom}_\mathcal{C}(C, D) &\to \mathrm{Hom}_\mathcal{C}(C, D'), & \mathrm{Hom}(f, C) : \mathrm{Hom}_\mathcal{C}(D', C) &\to \mathrm{Hom}_\mathcal{C}(D, C), \\
g &\mapsto f \circ g & g &\mapsto g \circ f.
\end{aligned}$$

Given a functor $F : \mathcal{C} \to \mathrm{Set}$ or $F : \mathcal{C}^{op} \to \mathrm{Set}$ one might ask if this functor is naturally isomorphic to a Hom functor for an object $C \in \mathrm{Ob}\mathcal{C}$. If this is the case, then it singles out an object in $\mathcal{C}$ with special properties. Obvious functors to consider are the forgetful functors $V : \mathcal{C} \to \mathrm{Set}$ in concrete categories.

Definition 1.3.2: Let $\mathcal{C}$ be a category.

1. A functor $F : \mathcal{C} \to \mathrm{Set}$ is called **representable**, , if there is an object $C \in \mathrm{Ob}\,\mathcal{C}$, a **representing object** of F, and a natural isomorphism $\eta : \mathrm{Hom}(C, -) \Rightarrow F$.

2. A functor $F : \mathcal{C}^{op} \to \mathrm{Set}$ is called **representable**, if there is an object $C \in \mathrm{Ob}\,\mathcal{C}$, a **representing object** of F, and a natural isomorphism $\eta : \mathrm{Hom}(-, C) \Rightarrow F$.

Example 1.3.3:

1. The identity functor $\mathrm{id}_{\mathrm{Set}}$: Set $\to$ Set is representable with the singleton set $\{\bullet\}$ as representing object.

   Proof:
   The maps $\eta_X : \mathrm{Hom}_{\mathrm{Set}}(\{\bullet\}, X) \to X$, $f \mapsto f(\bullet)$ for every set X define a natural isomorphism $\eta : \mathrm{Hom}(\{\bullet\}, -) \to \mathrm{id}_{\mathrm{Set}}$. As all maps $g : X \to X'$ the satisfy the identities

   $$g \circ \eta_{X'}(f) = g(f(\bullet)) = \eta_{X'}(g \circ f) = \eta_{X'}(\mathrm{Hom}(\{\bullet\}, g)(f)) = \eta_{X'} \circ \mathrm{Hom}(\{\bullet\}, g)(f),$$

   one has the commuting diagram for a natural transformation

   $$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Set}}(\{\bullet\}, X) & \xrightarrow{\eta_X : f \mapsto f(\bullet)} & X \\ {\scriptstyle \mathrm{Hom}(\{\bullet\}, f) : g \mapsto f \circ g} \downarrow & & \downarrow {\scriptstyle g} \\ \mathrm{Hom}_{\mathrm{Set}}(\{\bullet\}, X') & \xrightarrow[\eta_{X'} : f \mapsto f(\bullet)]{} & X'. \end{array}$$

   □

2. The forgetful functor V : Top $\to$ Set is representable with the singleton space $\{\bullet\}$ as representing object.

   Proof:
   This follows, because continuous maps $g : \{\bullet\} \to X$ are in bijection with points $x \in X$. The maps $\eta_X : \mathrm{Hom}_{\mathrm{Top}}(\{\bullet\}, X) \to X$, $f \mapsto f(\bullet)$ then define a natural isomorphism $\eta : \mathrm{Hom}(\{\bullet\}, -) \to V$. □

3. The forgetful functor $V : R$-Mod $\to$ Set is representable with the ring R as a left module over itself as representing object.

   Proof:
   Any R-linear map $f : R \to M$ satisfies $f(r) = f(r \rhd 1) = r \rhd f(1)$ for all $r \in R$ and hence is determined uniquely by $f(1) \in M$. The maps $\eta_M : \mathrm{Hom}_R(R, M) \to M$, $f \mapsto f(1)$ define a natural isomorphism $\eta : \mathrm{Hom}_{R\text{-Mod}}(R, -) \to V$, because one has for all R-linear maps $g : M \to M'$

   $$\eta_{M'} \circ \mathrm{Hom}(R, f)(g) = \eta_M(f \circ g) = f(g(1)) = f(\eta_M(g)) = f \circ \eta_M(g),$$

   and this yields the commuting diagram for a natural transformation

   $$\begin{array}{ccc} \mathrm{Hom}_{R\text{-Mod}}(R, M) & \xrightarrow{\eta_M : f \mapsto f(1)} & M \\ {\scriptstyle \mathrm{Hom}(R, f) : g \mapsto f \circ g} \downarrow & & \downarrow {\scriptstyle g} \\ \mathrm{Hom}_{R\text{-Mod}}(R, M') & \xrightarrow[\eta_{M'} : f \mapsto f(1)]{} & M'. \end{array}$$

   □

4. The forgetful functor V : Grp $\to$ Set is representable with representing object $\mathbb{Z}$.

   Proof:
   This follows, because any group homomorphism $f : \mathbb{Z} \to G$ is determined uniquely by $f(1)$. This yields a bijection between group homomorphisms $f : \mathbb{Z} \to G$ and elements of G. The maps $\eta_G : \mathrm{Hom}_{\mathrm{Grp}}(\mathbb{Z}, G) \to G$, $f \mapsto f(1)$ define a natural isomorphism $\eta : \mathrm{Hom}(\mathbb{Z}, -) \to V$. □

5. The forgetful functor $V : \mathrm{URing} \to \mathrm{Set}$ is representable with the polynomial ring $\mathbb{Z}[x]$ as representing object. (Exercise)

The definition of a representable functor $F : \mathcal{C} \to \mathrm{Set}$ and the examples raise the question about the uniqueness of the representing object C and the associated natural isomorphism $\eta : \mathrm{Hom}(C, -) \Rightarrow F$. We will see that the former follows from the latter and investigate the uniqueness of natural transformations $\eta : \mathrm{Hom}(C, -) \Rightarrow F$. It is clear that any such natural transformation has a special component morphism, namely the map $\eta_C : \mathrm{Hom}_\mathcal{C}(C, C) \to F(C)$, and a distinguished morphism in its domain, the identity morphism $1_C \in \mathrm{Hom}_\mathcal{C}(C, C)$. It turns out that its image determines η completely, due to naturality.

Theorem 1.3.4: (**Yoneda-Lemma**) Let $\mathcal{C}$ be locally small.
Let $F : \mathcal{C} \to \mathrm{Set}$ be a functor and $C \in \mathrm{Ob}\mathcal{C}$. Then natural transformations $\eta : \mathrm{Hom}(C, -) \Rightarrow F$ form a set $M_{C,F} = \mathrm{Hom}_{\mathrm{Set}^\mathcal{C}}(\mathrm{Hom}(C, -), F)$ and the **Yoneda map** is a bijection

$$Y : M_{C,F} \to F(C), \quad \eta \mapsto \eta_C(1_C)$$

Proof:
1. Injectivity of the Yoneda map:
A natural transformation $\eta : \mathrm{Hom}(C, -) \Rightarrow F$ defines maps $\eta_{C'} : \mathrm{Hom}_\mathcal{C}(C, C') \to F(C')$ for $C' \in \mathrm{Ob}\mathcal{C}$, such that the following diagram commutes for every morphism $f : C' \to C''$

$$\begin{array}{ccc} \mathrm{Hom}_\mathcal{C}(C,C') & \xrightarrow{\eta_{C'}} & F(C') \\ {\scriptstyle g\mapsto f\circ g}\Big\downarrow & & \Big\downarrow{\scriptstyle F(f)} \\ \mathrm{Hom}_\mathcal{C}(C,C'') & \xrightarrow{\eta_{C''}} & F(C''). \end{array}$$

Setting $C = C'$ and $g = 1_C \in \mathrm{Hom}_\mathcal{C}(C, C)$ yields

$$\eta_{C''}(f) = F(f) \circ \eta_C(1_C) \qquad \forall f \in \mathrm{Hom}_\mathcal{C}(C, C'').$$

This shows that $\eta_{C'} : \mathrm{Hom}_\mathcal{C}(C, C') \to F(C')$ is uniquely determined by F and $\eta_C(1_C)$. Hence, the Yoneda map $Y : M_{C,F} \to F(C)$ is injective.

2. Surjectivity of the Yoneda map:
To show that the Yoneda map is surjective, we construct for every element $c \in F(C)$ a natural transformation $\tau : \mathrm{Hom}(C, -) \Rightarrow F$ with $\tau_C(1_C) = c$. We define the component morphisms

$$\tau_C : \mathrm{Hom}_\mathcal{C}(C, C') \to F(C'), \quad h \mapsto F(h)(c),$$

which satisfy $\tau_C(1_C) = F(1_C)(c) = 1_{F(C)}(c) = \mathrm{id}_{F(C)}(c) = c$. The morphisms $\tau_{C'}$ define a natural transformation if and only if the following diagram commutes for all morphisms $f : C' \to C''$

$$\begin{array}{ccc} \mathrm{Hom}_\mathcal{C}(C,C') & \xrightarrow{\tau_{C'}:h\mapsto F(h)(c)} & F(C') \\ {\scriptstyle h\mapsto f\circ h}\Big\downarrow & & \Big\downarrow{\scriptstyle F(f)} \\ \mathrm{Hom}_\mathcal{C}(C,C'') & \xrightarrow[\tau_{C''}:k\mapsto F(k)(c)]{} & F(C''). \end{array}$$

This follows by a direct computation: $F(f)(F(h)(c)) = (F(f) \circ F(h))(c) = F(f \circ h)(c)$ and shows that the Yoneda map is surjective. □

Corollary 1.3.5: Let $\mathcal{C}$ be a category.

1. The functors $\mathrm{Hom}(C,-), \mathrm{Hom}(C',-) : \mathcal{C} \to \mathrm{Set}$ for $C, C' \in \mathrm{Ob}\mathcal{C}$ are naturally isomorphic if and only if the objects C and C' are isomorphic.
2. If $F : \mathcal{C} \to \mathrm{Set}$ is representable, its representing object is unique up to isomorphism.
3. Analogous claims hold for the functors $\mathrm{Hom}(-,C) : \mathcal{C}^{op} \to \mathrm{Set}$ and $F : \mathcal{C}^{op} \to \mathrm{Set}$.

Proof:
1. Every isomorphism $\epsilon : C \to C'$ defines bijections

$$\eta_D : \mathrm{Hom}_{\mathcal{C}}(C,D) \to \mathrm{Hom}_{\mathcal{C}}(C',D),\ g \mapsto g \circ \epsilon^{-1} \qquad \eta_D^{-1} : \mathrm{Hom}_{\mathcal{C}}(C',D) \to \mathrm{Hom}_{\mathcal{C}}(C,D),\ g \mapsto g \circ \epsilon,$$

for all objects D in $\mathcal{C}$. They form a natural isomorphism $\eta : \mathrm{Hom}(C,-) \Rightarrow \mathrm{Hom}(C',-)$, because the following diagram commutes for all morphisms $f : D \to D'$

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}(C,D) & \xrightarrow{\eta_D : g \mapsto g \circ \epsilon^{-1}} & \mathrm{Hom}_{\mathcal{C}}(C',D) \\
{\scriptstyle \mathrm{Hom}(C,f):}\downarrow{\scriptstyle g \mapsto f \circ g} & & {\scriptstyle \mathrm{Hom}(C',f):}\downarrow{\scriptstyle g \mapsto f \circ g} \\
\mathrm{Hom}_{\mathcal{C}}(C,D') & \xrightarrow{\eta_{D'} : g \mapsto g \circ \epsilon^{-1}} & \mathrm{Hom}_{\mathcal{C}}(C',D').
\end{array}$$

Conversely, any natural isomorphism $\eta : \mathrm{Hom}(C,-) \Rightarrow \mathrm{Hom}(C',-)$ defines an isomorphism $f = \eta^{-1}_{C'}(1_{C'}) : C \to C'$ with inverse $f^{-1} = \eta_C(1_C) : C' \to C$, because the diagrams

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}(C,C) & \xrightarrow{\eta_C} & \mathrm{Hom}_{\mathcal{C}}(C',C) \\
{\scriptstyle \mathrm{Hom}(C,f):}\downarrow{\scriptstyle g \mapsto f \circ g} & & {\scriptstyle \mathrm{Hom}(C',f):}\downarrow{\scriptstyle g \mapsto f \circ g} \\
\mathrm{Hom}_{\mathcal{C}}(C,C') & \xrightarrow[\eta_{C'}]{} & \mathrm{Hom}_{\mathcal{C}}(C',C').
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}(C',C') & \xrightarrow{\eta^{-1}_{C'}} & \mathrm{Hom}_{\mathcal{C}}(C,C') \\
{\scriptstyle g \mapsto \eta_C(1_C) \circ g}\downarrow & & \downarrow{\scriptstyle g \mapsto \eta_C(1_C) \circ g} \\
\mathrm{Hom}_{\mathcal{C}}(C',C) & \xrightarrow[\eta^{-1}_{C}]{} & \mathrm{Hom}_{\mathcal{C}}(C,C).
\end{array}$$

commute by naturality of η. Evaluating the diagram on the left on $1_C \in \mathrm{Hom}_{\mathcal{C}}(C,C)$ and the diagram on the right on $1_{C'} \in \mathrm{Hom}_{\mathcal{C}}(C',C')$ yields

$$f \circ \eta_C(1_C) = \eta_{C'}(f \circ 1_C) = \eta_{C'}(f) = 1_{C'}$$
$$\eta_C(1_C) \circ f = \eta_C(1_C) \circ \eta^{-1}_{C'}(1_{C'}) = \eta^{-1}_{C}(\eta_C(1_C) \circ 1_C) = \eta^{-1}_C(\eta_C(1_C)) = 1_C$$

This shows that $f : C \to C'$ is an isomorphism with inverse $f^{-1} = \eta_C(1_C) : C' \to C$.

2. If C, C' are both representing objects of $F : \mathcal{C} \to \mathrm{Set}$, there are natural isomorphisms $\eta : \mathrm{Hom}(C,-) \Rightarrow F$ und $\eta' : \mathrm{Hom}(C',-) \Rightarrow F'$. Because they define a natural isomorphism $\eta'^{-1} \circ \eta : \mathrm{Hom}(C,-) \Rightarrow \mathrm{Hom}(C',-)$, claim 1. implies $C \cong C'$. □

This Corollary and the Yoneda lemma are useful, because they translate claims about functors and natural transformations into claims about morphism sets. They can be applied whenever functors are characterised by claims about sets of morphisms. To illustrate this, we give an alternative proof of the uniqueness of adjoints with the Yoneda lemma.

Corollary 1.3.6:
If a functor $F : \mathcal{C} \to \mathcal{D}$ has a left or right adjoint, it is unique up to natural isomorphisms.

Proof:
Let $F, F' : \mathcal{C} \to \mathcal{D}$ left adjoint to $G : \mathcal{D} \to \mathcal{C}$ with bijections

$$\Phi_{C,D} : \mathrm{Hom}_{\mathcal{C}}(C, G(D)) \to \mathrm{Hom}_{\mathcal{D}}(F(C), D) \qquad \Phi'_{C,D} : \mathrm{Hom}_{\mathcal{C}}(C, G(D)) \to \mathrm{Hom}_{\mathcal{D}}(F'(C), D)$$

as in Definition 1.2.6 for all $C \in \mathrm{Ob}\,\mathcal{C}$ and $D \in \mathrm{Ob}\,\mathcal{D}$. This yields bijections

$$\eta^{(C)}_D = \phi'_{C,D} \circ \phi^{-1}_{C,D} : \mathrm{Hom}_{\mathcal{D}}(F(C), D) \to \mathrm{Hom}_{\mathcal{D}}(F'(C), D).$$

Because the isomorphisms $\phi_{C,D}$ and $\phi'_{C,D}$ satisfy the naturality conditions in Definition 1.2.6, the following diagram commutes for all morphisms $f : D \to D'$ in $\mathcal{D}$

$$\begin{array}{ccccc}
 & & \xrightarrow{\quad\eta^{(C)}_D\quad} & & \\
\mathrm{Hom}_{\mathcal{D}}(F(C), D) & \xrightarrow{\Phi^{-1}_{C,D}} & \mathrm{Hom}_{\mathcal{C}}(C, G(D)) & \xrightarrow{\Phi'_{C,D}} & \mathrm{Hom}_{\mathcal{D}}(F'(C), D) \\
{\scriptstyle \mathrm{Hom}(F(C),f): g\mapsto f\circ g}\downarrow & & {\scriptstyle g\mapsto G(f)\circ g}\downarrow & & {\scriptstyle \mathrm{Hom}(F'(C),f): g\mapsto f\circ g}\downarrow \\
\mathrm{Hom}_{\mathcal{D}}(F(C), D') & \xrightarrow{\Phi^{-1}_{C,D'}} & \mathrm{Hom}_{\mathcal{C}}(C, G(D')) & \xrightarrow{\Phi'_{C,D'}} & \mathrm{Hom}_{\mathcal{D}}(F'(C), D'). \\
 & & \xrightarrow{\quad\eta^{(C)}_{D'}\quad} & &
\end{array}$$

This shows that for every $C \in \mathrm{Ob}\mathcal{C}$ the morphisms $\eta^{(C)}_D$ define a natural isomorphism

$$\eta^{(C)} : \mathrm{Hom}(F(C), -) \Rightarrow \mathrm{Hom}(F'(C), -), \quad \eta^{(C)}_D = \phi'_{C,D} \circ \phi^{-1}_{C,D}.$$

By the proof of Corollary 1.3.5 the morphisms $\tau_C = \eta^{(C)}_{F(C)}(1_{F(C)}) : F'(C) \to F(C)$ are isomorphisms, and the naturality condition from Definition 1.2.6 implies that the following diagram commutes for all morphisms $f : C' \to C$

$$\begin{array}{ccccc}
 & & \xrightarrow{\quad\eta^{(C)}_D\quad} & & \\
\mathrm{Hom}_{\mathcal{D}}(F(C), D) & \xrightarrow{\Phi^{-1}_{C,D}} & \mathrm{Hom}_{\mathcal{C}}(C, G(D)) & \xrightarrow{\Phi'_{C,D}} & \mathrm{Hom}_{\mathcal{D}}(F'(C), D) \\
{\scriptstyle g\mapsto g\circ F(f)}\downarrow & & {\scriptstyle g\mapsto g\circ f}\downarrow & & {\scriptstyle g\mapsto g\circ F'(f)}\downarrow \\
\mathrm{Hom}_{\mathcal{D}}(F(C'), D) & \xrightarrow{\Phi^{-1}_{C',D}} & \mathrm{Hom}_{\mathcal{C}}(C', G(D)) & \xrightarrow{\Phi'_{C',D}} & \mathrm{Hom}_{\mathcal{D}}(F'(C'), D). \\
 & & \xrightarrow{\quad\eta^{(C')}_D\quad} & &
\end{array}$$

Setting $D = F(C)$ and $g = 1_{F(C)}$ yields with the naturality condition from Definition 1.2.6

$$\tau_C \circ F'(f) = \eta^{(C')}_{F(C)}(1_{F(C)} \circ F(f)) = \eta^{(C')}_{F(C)}(F(f) \circ 1_{F(C')}) = F(f) \circ \eta^{(C')}_{F(C')}(1_{F(C')}) = F(f) \circ \tau_{C'},$$

This shows that the isomorphisms $\tau_C : F'(C) \to F(C)$ form a natural isomorphism $\tau : F' \Rightarrow F$.
□

References:

- Chapters I, III and IV in Mac Lane, S. (2013) Categories for the working mathematician,
- Chapters 1 and 2 in Richter, B. (2020) From categories to homotopy theory,
- Chapters 1,2 and 4 in Riehl, E. (2017) Category theory in context.

2 Limits and Colimits

2.1 (Co)limits: definitions and examples

Limits and colimits are ubiquitous in category theory. Limits are generalisations of categorical products and pullbacks. The dual concept, colimits, generalises categorical coproducts and pushouts. Limits and colimits associate to given categorical data - certain families of objects in a category $\mathcal{C}$ and morphisms between them - a distinguished object, the limit or colimit, and a family of distinguished morphisms with a universal property.

The given categorical data can be viewed as a functor $F : \mathcal{J} \to \mathcal{C}$ from a category $\mathcal{J}$ that encodes the combinatorics of the given objects and morphisms. Often, the category $\mathcal{J}$ is small and rather trivial. It determines the shape of the *diagram* for the universal property. The functor $F : \mathcal{J} \to \mathcal{C}$ is therefore called a *diagram*. Limits and colimits can then be viewed as *optimal approximations* of the diagram $F : \mathcal{J} \to \mathcal{C}$ by *constant* functors.

We start by considering constant functors $F : \mathcal{J} \to \mathcal{C}$ and natural transformations between them. Such functors and natural transformations are defined, respectively, by a choice of an object and a morphism in $\mathcal{C}$ and define a functor from $\mathcal{C}$ to the functor category $\mathcal{C}^{\mathcal{J}}$.

Definition 2.1.1: Let $\mathcal{J}, \mathcal{C}$ be categories. The **embedding functor** $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{J}}$ sends

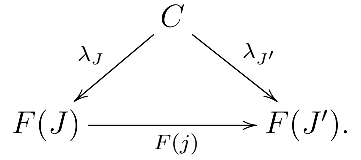
- $C \in \mathrm{Ob}\mathcal{C}$ to the **constant functor** $\Delta(C) : \mathcal{J} \to \mathcal{C}$ with $\Delta(C)(J) = C$ and $\Delta(C)(j) = 1_C$ for all objects J and morphisms j in $\mathcal{J}$,
- $f \in \mathrm{Hom}_{\mathcal{C}}(C, C')$ to the **constant natural transformation** $\Delta(f) : \Delta(C) \Rightarrow \Delta(C')$ with component morphisms $\Delta(f)_J = f : C \to C'$ for all $J \in \mathrm{Ob}\mathcal{J}$.

The only sensible interpretation of an *approximation* of a functor $F : \mathcal{J} \to \mathcal{C}$ by a constant functor $\Delta(C) : \mathcal{J} \to \mathcal{C}$ is a natural transformation between them. The constant functor $\Delta(C)$ is either the source or the target of this natural transformation. In the first case, one calls the natural transformation a *cone over* F, in the second a *cocone* or a *cone under* F. The component morphisms of the natural transformation are called *legs*.

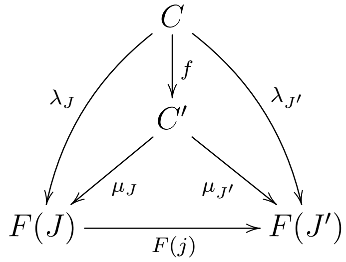
The name *cone* originated in topology, where constructions involving topological cones give rise to categorical cones. One can also justify it by viewing the images of objects and morphisms in $\mathcal{J}$ under F as a diagram in a plane. If the object C is placed *above* the plane, the component morphisms of the natural transformation correspond to downward arrows from C to the vertices in the plane and form a cone. Cocones are the dual concept and hence correspond to an object C *below* this plane with arrows pointing downwards towards C.

Definition 2.1.2: Let $\mathcal{C}, \mathcal{J}$ be categories. A **diagram** of **shape** $\mathcal{J}$ in $\mathcal{C}$ is a functor $F : \mathcal{J} \to \mathcal{C}$.

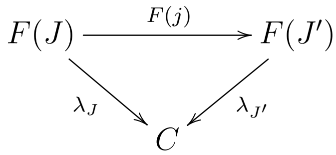
1. A **cone** over a diagram $F : \mathcal{J} \to \mathcal{C}$ with **apex** $C \in \mathrm{Ob}\mathcal{C}$ is a natural transformation $\lambda : \Delta(C) \Rightarrow F$: a family of morphisms $\lambda_J : C \to F(J)$ indexed by $\mathrm{Ob}\mathcal{J}$ such that $F(j) \circ \lambda_J = \lambda_{J'}$ for all morphisms $j : J \to J'$ in $\mathcal{J}$



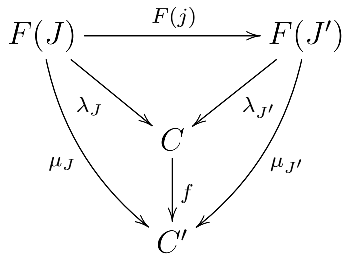
2. A **cone morphism** from $\lambda : \Delta(C) \Rightarrow F$ to $\mu : \Delta(C') \Rightarrow F$ is a morphism $f : C \to C'$ with $\mu \circ \Delta(f) = \lambda$:



3. A **cocone** under a diagram $F : \mathcal{J} \to \mathcal{C}$ with **nadir** $C \in \mathrm{Ob}\mathcal{C}$ is a natural transformation $\lambda : F \Rightarrow \Delta(C)$: a family of morphisms $\lambda_J : F(J) \to C$ indexed by $\mathrm{Ob}\mathcal{J}$ such that $\lambda_{J'} \circ F(j) = \lambda_J$ for all morphisms $j : J \to J'$ in $\mathcal{J}$



4. A **cocone morphism** from $\lambda : F \Rightarrow \Delta(C)$ to $\mu : F \Rightarrow \Delta(C')$ is a morphism $f : C \to C'$ with $\Delta(f) \circ \lambda = \mu$.

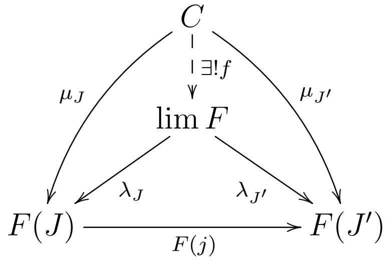


We denote by (co)cone(F) the category of (co)cones for a diagram $F : \mathcal{J} \to \mathcal{C}$ and (co)cone morphisms between them.

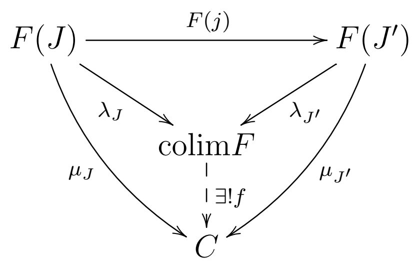
We now define a limit of a functor $F : \mathcal{J} \to \mathcal{C}$ as a cone over F that approximates F as closely as possible from above. Approximation by any other cone must give rise to a cone morphism from that cone to the limit. Thus, we define a limit as a terminal object in the category cone(F). Dually, a colimit of F is a cocone that approximates F as closely as possible from below and is defined as an initial object in the category cocone(F).

Definition 2.1.3: Let $F : \mathcal{J} \to \mathcal{C}$ be a diagram.

1. A **limit** of F is a terminal object in the category cone(F): a cone $\lambda : \Delta(\lim F) \Rightarrow F$ such that for every cone $\mu : \Delta(C) \Rightarrow F$ there is a unique cone morphism $f : C \to \lim F$.



2. A **colimit** of F is an initial object in the category cocone(F): a cocone $\lambda : F \Rightarrow \Delta(\text{colim}F)$ that yields for any cocone $\mu : F \Rightarrow \Delta(C)$ a unique cocone morphism $f : \text{colim}F \to C$.



Remark 2.1.4:

1. To emphasise the shape of the diagram we sometimes write $\lim_{\mathcal{J}} F$ and $\text{colim}_{\mathcal{J}}F$ instead of $\lim F$ and colim$F$. This is the standard convention in the literature.

2. To distinguish the objects $\lim F$, colim$F$ from the (co)cones $\lambda : \Delta(\lim F) \Rightarrow F$ and $\mu : F \Rightarrow \Delta(\text{colim}F)$, we often call the latter **limit cone** and **colimit cone**. In most references the word *(co)limit* refers to both, the object and the natural transformation.

Remark 2.1.5:

1. As terminal (initial) objects, (co)limits are unique up to unique isomorphisms.

   If $\lambda : \Delta(C) \Rightarrow F$ and $\mu : \Delta(C') \Rightarrow F$ are limits of $F : \mathcal{J} \to \mathcal{C}$, then there is a unique cone morphism $f : C \to C'$ and a unique cone morphism $f' : C' \to C$. The universal property of the limit then implies $f \circ f' = 1_{C'}$ and $f' \circ f = 1_C$. An analogous claim holds for colimits.

2. The existence of (co)limits is not guaranteed. It depends on the functor $F : \mathcal{J} \to \mathcal{C}$.

3. Colimits of diagrams $F : \mathcal{J} \to \mathcal{C}$ can be viewed as limits of the opposite diagram $F' : \mathcal{J}^{op} \to \mathcal{C}^{op}$ and vice versa.

   Every natural transformation $\mu : F \Rightarrow G$ a defines a natural transformation $\mu' : G' \Rightarrow F'$. Thus, (initial) cocones for $F : \mathcal{J} \to \mathcal{C}$ are (terminal) cones for $F' : \mathcal{J}^{op} \to \mathcal{C}^{op}$.

We will often use Remark 2.1.5, 3. when proving claims for limits and the dual claims for colimits. Although we state both claims for completeness, it is sufficient to prove one of them.

We now consider examples of (co)limits. In most examples the categories $\mathcal{J}$ have a very simple form and consist of few objects or morphisms. The resulting (co)limit cones then correspond to familiar diagrams characterising the universal properties of certain constructions.

Example 2.1.6:

1. **constant functors:**
   A category $\mathcal{J}$ is called **connected**, if for any two objects $J, J' \in \text{Ob}\mathcal{J}$ there is a finite sequence of objects $J = J_0, J_1, \ldots, J_n = J'$ with $\text{Hom}_{\mathcal{J}}(J_i, J_{i+1}) \cup \text{Hom}_{\mathcal{J}}(J_{i+1}, J_i) \neq \emptyset$ for each $i = 0, ..., n-1$. In a connected category, every constant functor $\Delta(C) : \mathcal{J} \to \mathcal{C}$ has limit and colimit cone $\text{id}_{\Delta(C)} : \Delta(C) \Rightarrow \Delta(C)$ (Exercise 6).

2. **(co)products:**
   Let $\mathcal{J}$ be a small **discrete category**, a category with $\text{Ob}\mathcal{J} = J$ for a set J and only identity morphisms. Then limits of functors $F : \mathcal{J} \to \mathcal{C}$ are products $\Pi_{j \in J} F(j)$ and colimits coproducts $\amalg_{j \in J} F(j)$.

• A functor $F : \mathcal{J} \to \mathcal{C}$ is simply a family of objects $(C_j \in \text{Ob}\mathcal{C})_{j\in J}$ indexed by J.

• Cones $\pi : \Delta(C) \Rightarrow F$ are families of morphisms $(\pi_j : C \to C_j)_{j\in J}$. A cone $\pi : \Delta(C) \Rightarrow F$ is terminal, if for any family of morphisms $(f_j : C' \to C_j)_{j\in J}$, there is a unique morphism $f : C' \to C$ with $\pi_j \circ f = f_j$ for all $j \in J$. Thus, $\lim F = \Pi_{j\in J} C_j$ is a product.

• Cocones $\iota : F \Rightarrow \Delta(C)$ are families of morphisms $(\iota_j : C_j \to C)_{j\in J}$, and a cocone is initial, if for every family of morphisms $(f_j : C_j \to C')_{j\in J}$ there is a unique morphism $f : C \to C'$ with $f \circ \iota_j = f_j$ for all $J \in J$. Thus, $\text{colim} F = \amalg_{j\in J} C_j$ is a coproduct.

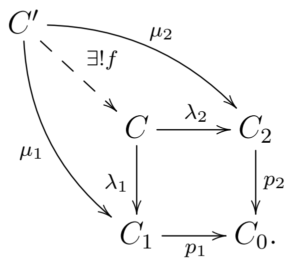
3. **pullbacks and pushouts:**
Let $\mathcal{J}$ be a category with three objects J_0, J_1, J_2 and two non-identity morphisms

$$J_1 \xrightarrow{j_1} J_0 \xleftarrow{j_2} J_2.$$

Then a limit of a functor $F : \mathcal{J} \to \mathcal{C}$ is a **pullback** or **fibre product** and a colimit of a functor $F : \mathcal{J}^{op} \to \mathcal{C}$ a **pushout** in $\mathcal{C}$.

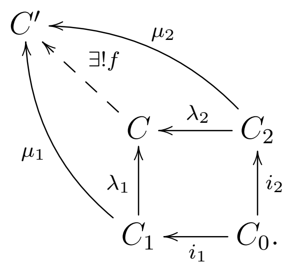
• Functors $F : \mathcal{J} \to \mathcal{C}$ are pairs of morphisms $p_1 : C_1 \to C_0$ and $p_2 : C_2 \to C_0$ in $\mathcal{C}$.

A cone $\lambda : \Delta(C) \Rightarrow F$ is a triple of morphisms $\lambda_j : C \to C_j$ for $j = 0, 1, 2$ with $\lambda_0 = p_1 \circ \lambda_1 = p_2 \circ \lambda_2$ or, equivalently, a pair $\lambda_1 : C \to C_1, \lambda_2 : C \to C_2$ with $p_1 \circ \lambda_1 = p_2 \circ \lambda_2$. It is terminal, if for every pair of morphisms $\mu_1 : C' \to C_1$, $\mu_2 : C' \to C_2$ satisfying $p_1 \circ \mu_1 = p_2 \circ \mu_2$ there is a unique morphism $f : C' \to C$ with $\lambda_1 \circ f = \mu_1$ and $\lambda_2 \circ f = \mu_2$.



• Functors $F : \mathcal{J}^{op} \to \mathcal{C}$ are pairs of morphisms $i_1 : C_0 \to C_1$ and $i_2 : C_0 \to C_2$ in $\mathcal{C}$.

A cocone $\lambda : F \Rightarrow \Delta(C)$ under F is a pair of morphisms $\lambda_1 : C_1 \to C$, $\lambda_2 : C_2 \to C$ with $\lambda_1 \circ i_1 = \lambda_2 \circ i_2$. It is initial, if for every pair of morphisms $\mu_1 : C_1 \to C'$, $\mu_2 : C_2 \to C'$ there is a unique morphism $f : C \to C'$ with $f \circ \lambda_i = \mu_i$.



Example 2.1.7: Examples of (co)products are the following:

1. **terminal and initial objects:**
If $J = \emptyset$ in Example 2.1.6, 2. then the associated product in $\mathcal{C}$ is a **terminal object**, an object $T \in \text{Ob}\mathcal{C}$ such that for every object $C \in \text{Ob}\mathcal{C}$ there is a unique morphism $t_C : C \to T$. The associated coproduct in $\mathcal{C}$ is an **initial object**, an object $I \in \text{Ob}\mathcal{C}$ such that for every $C \in \text{Ob}\mathcal{C}$ there is a unique morphism $i_C : I \to C$.

2. **(co)powers:** If $(C_j)_{j\in J}$ in Example 2.1.6, 2. is a constant family with $C_j = C$ for all $j \in J$, then the product $\Pi_{j\in J} C$ is called a **power** of C and denoted C^J and the coproduct $\amalg_{j\in J} C$ is called a **copower** of C and denoted $J \cdot C$. If $\mathcal{C} = \text{Set}$, they coincide with the set C^J of functions from J to C and with the product $J \times C$, respectively (Exercise 8).

3. **Set and Top:** A product of a family $(X_j)_{j\in J}$ of sets (topological spaces) is their usual product $\Pi_{j\in J} X_j$ (with the product topology). Their coproduct is the disjoint union $\dot\cup_{j\in J} X_j$ (with the sum topology).

4. R**-Mod:** The product of a family $(M_j)_{j\in J}$ of R-modules is their usual product, and their coproduct is the direct sum $\oplus_{j\in J} M_j$. *Finite* products and coproducts are isomorphic, and this holds more generally for any abelian category.

5. **Grp:** The product of a family $(G_j)_{j\in J}$ of groups is their direct product $\times_{j\in J} G_j$. Their coproduct is the **free product** $\star_{j\in J} G_j$. Elements of the free product are finite tuples $(g_1, \ldots, g_k)$ of non-trivial elements of the groups G_j such that neighbouring entries are in different groups. The group multiplication is given by

$$(g_1, \ldots, g_k) \cdot (h_1, \ldots, h_l) = \begin{cases} (g_1, \ldots, g_{k-1}) \cdot (h_2, \ldots, h_l) & g_k = h_1^{-1} \in G_i \\ (g_1, \ldots, g_{k-1}, g_k \cdot h_1, h_2, \ldots, h_l) & g_k \neq h_1^{-1} \in G_i \\ (g_1, \ldots, g_k, h_1, \ldots, h_m) & \text{else}, \end{cases}$$

   and the inclusion morphisms by $\iota_i : G_i \to \amalg_{j\in J} G_j$, $g \mapsto (g)$.

6. **Ring:** The product of a family $(x_j)_{j\in J}$ of rings is their direct product. Their coproduct is the **free product** of rings, constructed analogously to the one for groups. It is a quotient of the tensor algebra of $R \oplus S$ over $\mathbb{Z}$ by an ideal that encodes the relations of R, S.

Pullbacks and pushouts are ubiquitous in topology. Pushouts are quotient spaces of topological sums and allow one to build up topological spaces from simpler components. Many of the spaces investigated in topology and geometry are built in this way. Pullbacks are subspaces of product spaces and also arise in many applications. Pullbacks and pushouts also exist for morphisms in familiar algebraic categories such as Grp, Ring, R-Mod, $\text{Alg}_{\mathbb{F}}$.

The **Seifert-van Kampen theorem** states that the functor $\Pi_1 : \text{Top} \to \text{Grpd}$ that assigns to a topological space X its fundamental groupoid $\Pi_1(X)$ and to a continuous map $f : X \to Y$ the induced functor between them, sends pushouts in Top to pushouts in Grpd.

Algebraic examples of pushouts arise, whenever an algebraic structure is presented in terms of generators and relations.This includes presentations of groups, algebras and modules over rings. Other important examples of pullbacks and pushouts are (co)kernel pairs. A **kernel pair** is the pullback and a **cokernel pair** the pushout of two identical morphisms. Kernel pairs detect monomorphisms and cokernel pairs epimorphisms, which generalise injective and surjective maps to categories (cf. Exercise 13).

Example 2.1.8: (**Pullbacks and pushouts in** Top)

1. A **pullback** of a pair of continuous maps $p_1 : X_1 \to X_0$, $p_2 : X_2 \to X_0$ is the set $X_1 \times_{X_0} X_2 = \{(x_1, x_2) \in X_1 \times X_2 \mid p_1(x_1) = p_2(x_2)\}$ with the subspace topology induced by the product topology and with the continuous maps $\lambda_i : X_1 \times_{X_0} X_2 \to X_i$, $(x_1, x_2) \mapsto x_i$.

2. A **pushout** of a pair of continuous maps $i_1 : X_0 \to X_1$, $i_2 : X_0 \to X_2$ is the set $X_1 +_{X_0} X_2 = X_1 \dot{\cup} X_2/\sim$ with[1] $i_1(x_0) \sim i_2(x_0)$ for all $x_0 \in X_0$ with the quotient topology induced by the sum topology and with the maps $\lambda_i : X_i \to X_1 +_{X_0} X_2$, $x \mapsto [\iota_i(x)]$.

3. **wedge product**: If $X_0 = \{x_0\}$, then the continuous maps $i_j : X_0 \to X_j$ are given by two points $x_1 = i_1(x_0) \in X_1$, $x_2 = i_2(x_0) \in X_2$. The equivalence relation $\sim$ identifies x_1 and x_2. In this case $X_1 +_{X_0} X_2$ is called the **wedge product** of X_1 and X_2 and denoted $X_1 \vee X_2$.

4. **attaching topological spaces:**
   If $X_0 \subset X_1$ is a subspace and $i_1 : X_0 \to X_1$ its inclusion, one says that $X_1 +_{X_0} X_2$ is obtained by **attaching** the topological space X_1 to X_2 with the **attaching map** i_2. In this case, the equivalence relation $\sim$ identifies all points $x_1 \in X_0$ with their images $\iota_2(x_1) \in X_2$.

5. **attaching n-cells:**
   If in 4. $X_1 = \amalg_I D^n$ is a topological sum of closed n-discs $D^n = \{x \in \mathbb{R}^n \mid ||x|| \leq 1\}$ and $X_0 = \amalg_I S^{n-1} \subset X_1$ the disjoint union of their boundaries $\partial D^n = S^{n-1}$, one says that $X_1 +_{X_0} X_2$ is obtained by **attaching n-cells** to X_2.

Example 2.1.9: (pullbacks and pushouts in other categories)

1. **Set:** Pullbacks and pushouts in Set are obtained from Example 2.1.8, 1. and 2. by forgetting the topology.

2. **R-Mod:** The pullback of a pair of R-linear maps $p_1 : X_1 \to X_0$ and $p_2 : X_2 \to X_0$ is the submodule $X_1 \times_{X_0} X_2 = \{(x_1, x_2) \in X_1 \times X_2 \mid p_1(x_1) = p_2(x_2)\} \subset X_1 \times X_2$.

   The pushout of a pair of R-linear maps $i_1 : X_0 \to X_1$ and $i_2 : X_0 \to X_2$ is the quotient module $X_1 +_{X_0} X_2 = X_1 \oplus X_2/\langle\{i_1(x_0) - i_2(x_0) \mid x_0 \in X_0\}\rangle$.

3. **Grp and Ring:**
   • Pullbacks in Grp and Ring are the same subsets as in Example 2.1.8, 1. with the induced group or ring structure.

   • Pushouts in Grp are factor groups $C_1 \star C_2/N$ of the free product $C_1 \star C_2$ by the normal subgroup $N \subset C_1 \star C_2$ generated by the set $\{\iota_1(c)\iota_2(c)^{-1} \mid c \in C_0\}$. Pushouts in Ring are given as quotients of the tensor algebra $T(C_1 \oplus C_2)$ by a certain ideal.

Note that all pullbacks in Examples 2.1.8 and 2.1.9 are subsets of products and all pushouts quotients of coproducts. We will see later in Theorem 2.2.2 that this is not a coincidence. To investigate this systematically, we need categorical concepts that generalise the subset $U \subset X$, where two functions $f_1, f_2 : X \to Y$ are equal, and the quotient set $Y/\sim$ obtained by identifying their values. These are *equalisers* and *coequalisers*.

Example 2.1.10: (equalisers and coequalisers)
Let $\mathcal{J}$ be a category with two objects J_s, J_t and two parallel non-identity morphisms

$$J_s \underset{j_2}{\overset{j_1}{\rightrightarrows}} J_t$$

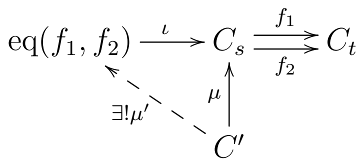
Functors $F : \mathcal{J} \to \mathcal{C}$ correspond to a choice of parallel morphisms $f_1, f_2 : C_s \to C_t$ in $\mathcal{C}$.

[1]Here, we mean by $\iota_1(x_0) \sim \iota_2(x_0)$ for all $x_0 \in X_0$ the smallest equivalence relation $\sim$ on $X := X_1 \dot{\cup} X_2$ generated by the subset $\{(\iota_1(x_0), \iota_2(x_0)) \mid x_0 \in X_0\} \subset X \times X$. We use a similar notation in the following.

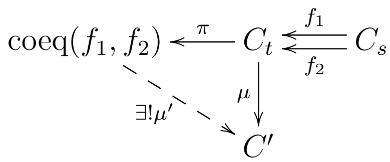
• A cone $\iota : \Delta(C) \Rightarrow F$ corresponds to morphisms $\iota_s : C \to C_s$ and $\iota_t : C \to C_t$ with $f_1 \circ \iota_s = f_2 \circ \iota_s = \iota_t$. This is equivalent to a choice of a single morphism $\iota = \iota_s : C \to C_s$ with $f_1 \circ \iota = f_2 \circ \iota$. A cone given by $\iota : C \to C_s$ is terminal, if for any morphism $\mu : C' \to C_s$ with $f_1 \circ \mu = f_2 \circ \mu$, there is a unique morphism $\mu' : C' \to C$ with $\iota \circ \mu' = \mu$.

• A cocone over F is given by a morphism $\pi : C_t \to C$ with $\pi \circ f_1 = \pi \circ f_2$. It is initial if for any morphism $\mu : C_t \to C'$ with $\mu \circ f_1 = \mu \circ f_2$, there is a unique $\mu' : C \to C'$ with $\mu' \circ \pi = \mu$.

• Thus, a limit of a functor $F : \mathcal{J} \to \mathcal{C}$ is an **equaliser** of the morphisms $f_1, f_2 : C_s \to C_t$: a morphism $\iota : \mathrm{eq}(f_1, f_2) \to C_s$ in $\mathcal{C}$ with $f_1 \circ \iota = f_2 \circ \iota$ and the following universal property: for every $\mu : C' \to C_s$ with $f_1 \circ \mu = f_2 \circ \mu$, there is a unique $\mu' : C' \to \mathrm{eq}(f_1, f_2)$ with $\iota \circ \mu' = \mu$.



• A colimit of $F : \mathcal{J} \to \mathcal{C}$ is a **coequaliser** of the morphisms $f_1, f_2 : C_s \to C_t$: a morphism $\pi : C_t \to \mathrm{coeq}(f_1, f_2)$ in $\mathcal{C}$ with $\pi \circ f_1 = \pi \circ f_2$ and the following universal property: for every $\mu : C_t \to C'$ with $\mu \circ f_1 = \mu \circ f_2$, there is a unique $\mu' : \mathrm{coeq}(f_1, f_2) \to C'$ with $\mu' \circ \pi = \mu$.



Example 2.1.11: (**equalisers and coequalisers**)

1. Let $f_1, f_2 : X \to Y$ be morphisms in Set (Top):
   - Their equaliser is $\mathrm{eq}(f_1, f_2) = \{x \in X \mid f_1(x) = f_2(x)\} \subset X$ (equipped with the subspace topology) with the (continuous) inclusion $\iota : \mathrm{eq}(f_1, f_2) \to X$.
   - Their coequaliser is $\mathrm{coeq}(f_1, f_2) = Y/\sim$ with $f_1(x) \sim f_2(x)$ for all $x \in X$ (equipped with the quotient topology) with the (continuous) surjection $\pi : Y \to \mathrm{coeq}(f_1, f_2)$.
2. Let R be a ring and $f_1, f_2 : M \to N$ morphisms in R-Mod:
   - Their equaliser is the submodule $\mathrm{eq}(f_1, f_2) = \ker(f_1 - f_2) \subset M$ with the inclusion $\iota : \mathrm{eq}(f_1, f_2) \to M$.
   - Their coequaliser is $\mathrm{coeq}(f_1, f_2) = N/\mathrm{im}(f_1 - f_2) = \mathrm{coker}(f_1 - f_2)$ with the canonical surjection $\pi : N \to \mathrm{coeq}(f_1, f_2)$.

   This covers $\mathrm{Ab} = \mathbb{Z}$-Mod, $\mathrm{Vect}_{\mathbb{F}} = \mathbb{F}$-Mod and $\mathrm{Rep}_{\mathbb{F}}(G) = \mathbb{F}[G]$-Mod for a group G.
3. More generally, any two morphisms $f_1, f_2 : A \to A'$ in an abelian category $\mathcal{A}$ have
   - **kernels** as equalisers $\iota : \ker(f_1 - f_2) \to M$,
   - **cokernels** as coequalisers $\pi : N \to \mathrm{coker}(f_1 - f_2)$.
4. For two morphisms $f_1, f_2 : G \to H$ in Grp:
   - the equaliser is the subgroup $\mathrm{eq}(f_1, f_2) = \{g \in G \mid f_1(g) = f_2(g)\} \subset G$ with the inclusion $\iota : \mathrm{eq}(f_1, f_2) \to G$,
   - the coequaliser is the factor group H/N by the normal subgroup N generated by the set $\{f_1(g)f_2(g)^{-1} \mid g \in G\}$ with the canonical surjection $\pi : H \to H/N$.

5. For two morphisms $f_1, f_2 : R \to S$ in Ring
   - the equaliser is $\text{eq}(f_1, f_2) = \{r \in R \mid f_1(r) = f_2(r)\}$ with the inclusion $\iota : \text{eq}(f_1, f_2) \to R$,
   - the coequaliser is the quotient S/I by the ideal I generated by $\{f_1(r) - f_2(r) \mid r \in R\}$ with the canonical surjection $\pi : S \to S/I$.

So far we showed that (co)limits generalise many familiar notions for sets, topological spaces, vector spaces, groups or rings to categories and treat them in a common framework. However, they seem to have little to do with usual notions of limits in calculus and topology. In fact, limits of sequences in a topological space *can* be viewed as special cases of categorical limits. This requires a category of filters on a topological space and hence more background in topology. Exercise 15 shows that categorical (co)limits capture the notions of infimum and supremum.

We can also consider other examples of limits and colimits associated to sequences of objects and morphisms in a category. The appropriate diagram category $\mathcal{J}$ for sequences indexed by non-negative integers is the poset category $(\mathbb{N}_0, \leq)$.

Recall that a **partially ordered set** or **poset** is a set X together with a relation $\preceq$ that is

(i) *reflexive*: $x \preceq x$ for all $x \in X$,
(ii) *antisymmetric:* $x \preceq y$ and $y \preceq x$ implies $x = y$,
(iii) *transitive:* $x \preceq y$ and $y \preceq z$ implies $x \preceq z$.

The **poset category** for a poset $(X, \preceq)$ is a category $\mathcal{C}$ with $\text{Ob}\mathcal{C} = X$ and morphism spaces $\text{Hom}_{\mathcal{C}}(x, y)$ that contain a single morphism if $x \preceq y$ and are empty otherwise.

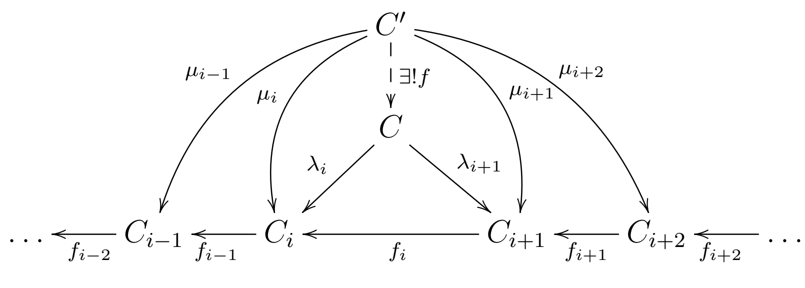
Example 2.1.12: (**sequential (co)limits**)
Let $\mathcal{J}$ be the poset category for $(\mathbb{N}_0, \leq)$.

A functor $F : \mathcal{J} \to \mathcal{C}$ corresponds to a family of objects $(C_i)_{i \in \mathbb{N}_0}$ and a family of morphisms $f_{ij} : C_i \to C_j$ for $i \leq j$ such that $f_{ii} = 1_{C_i}$ and $f_{jk} \circ f_{ij} = f_{ik}$ for all $i \leq j \leq k$. This is equivalent to specifying a family of morphisms $(f_i : C_i \to C_{i+1})_{i \in \mathbb{N}_0}$ with no further relations. Thus:

- A functor $F : \mathcal{J} \to \mathcal{C}$ is a family of morphisms $(f_i : C_i \to C_{i+1})_{i \in \mathbb{N}_0}$.
- A functor $F : \mathcal{J}^{op} \to \mathcal{C}$ is a family of morphisms $(f_i : C_{i+1} \to C_i)_{i \in \mathbb{N}_0}$.

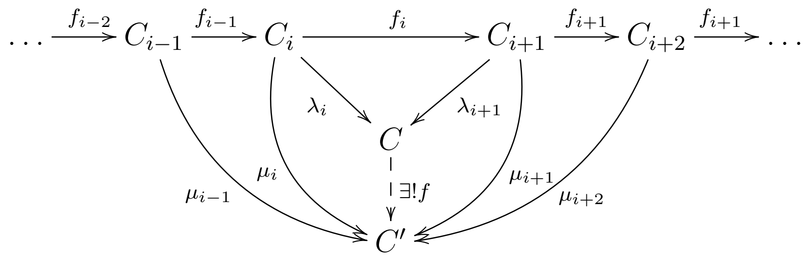
A limit of a functor $F : \mathcal{J}^{op} \to \mathcal{C}$ is called **sequential limit** or **inverse limit** and noted $\lim_{\leftarrow} f_n$.

It is a family of morphisms $(\lambda_i : C \to C_i)_{i \in \mathbb{N}_0}$ with $f_i \circ \lambda_{i+1} = \lambda_i$ for all $i \in \mathbb{N}_0$ and the following universal property: for any family of morphisms $(\mu_i : C' \to C_i)_{i \in \mathbb{N}_0}$ with $f_i \circ \mu_{i+1} = \mu_i$ for all $i \in \mathbb{N}_0$ there is a unique morphism $f : C' \to C$ with $\lambda_i \circ f = \mu_i$ for all $i \in \mathbb{N}_0$.



A colimit of a functor $F : \mathcal{J} \to \mathcal{C}$ is called **sequential colimit** or **direct limit**, $\lim_{\rightarrow} f_n$.

It is a family of morphisms $(\lambda_i : C_i \to C)_{i\in\mathbb{N}_0}$ with $\lambda_{i+1} \circ f_i = \lambda_i$ for all $i \in \mathbb{N}_0$ and the following universal property: for any family of morphisms $(\mu_i : C_i \to C')_{i\in\mathbb{N}_0}$ with $\mu_{i+1} \circ f_i = \mu_i$ for all $i \in \mathbb{N}_0$ there is a unique morphism $f : C \to C'$ with $f \circ \lambda_i = \mu_i$ for all $i \in \mathbb{N}_0$.



Example 2.1.13: Let $\mathcal{J}$ be as in Example 2.1.12.

1. Let $\mathcal{C}$ = Set, Top, Grp, Ring, R-Mod or any other category whose objects are sets and such that all inclusion maps between objects define morphisms in $\mathcal{C}$.

   Then any sequence of objects $X_0 \subset X_1 \subset X_2 \subset \ldots$ in $\mathcal{C}$ together with the inclusions $\iota_i : X_i \to X_{i+1}$ defines a functor $F : \mathcal{J} \to \mathcal{C}$. Its sequential colimit (direct limit) is the object $X = \cup_{n\in\mathbb{N}_0} X_n$ together with the inclusions $j_i : X_i \to X$, $x \mapsto x$ and with the topology, group structure, ring structure or module structure induced by these inclusions.

   - For $\mathcal{C}$ = Top the topology on X is the **weak topology** on X, the **final topology** induced by the inclusions j_i for $i \in \mathbb{N}_0$:
     a subset $U \subset X$ is open if and only if $j_i^{-1}(U) = U \cap X_i$ is open in X_i for all $i \in J$.

   - For $\mathcal{C}$ = Grp, Ring, R-Mod and $x, x' \in X$ there is always a $j \in \mathbb{N}_0$ with $x, x' \in X_j$. The multiplication, addition and R-module structure is then given by the multiplication, addition and R-module structure in X_j. The properties of the inclusions ensure that this is consistent and independent of the choice of j.

2. A **CW-complex** is the sequential colimit (direct limit) of a sequence of topological spaces $X_0 \subset X_1 \subset X_2 \subset \ldots$ such that X_0 is equipped with the discrete topology and X_n is obtained from X_{n-1} by attaching n-cells for all $n \in \mathbb{N}$, cf. Example 2.1.8, 5. The topological space X_n is called the **n-skeleton** of X.

3. The permutation groups S_n for $n \in \mathbb{N}$ with the inclusion maps $\iota_n : S_n \to S_{n+1}$, $\sigma \mapsto \sigma'$ with $\sigma'(n+1) = n+1$ and $\sigma'(i) = \sigma(i)$ for $i \in \{1, \ldots, n\}$ define a functor $S : \mathcal{J} \to$ Grp. Its sequential colimit (direct limit) is the group S_∞.

4. The groups $\mathrm{GL}(n, \mathbb{F})$ with the inclusions $\iota_n : \mathrm{GL}(n, \mathbb{F}) \to \mathrm{GL}(n+1, \mathbb{F})$, $\phi \mapsto \phi'$ with $\phi'(e_{n+1}) = e_{n+1}$ and $\phi'(e_i) = \phi(e_i)$ for $i \in \{1, \ldots, n\}$ define a functor $\mathrm{GL} : \mathcal{J} \to$ Grp. Its sequential colimit (direct limit) is the group $\mathrm{GL}(\infty, \mathbb{F})$.

5. Let $p \in \mathbb{N}$ be a prime and $F : \mathcal{J}^{op} \to$ CRing given by the ring homomorphisms

   $$0 = \mathbb{Z}/\mathbb{Z} \xleftarrow{f_0} \mathbb{Z}/p\mathbb{Z} \xleftarrow{f_1} \mathbb{Z}/p^2\mathbb{Z} \xleftarrow{f_2} \ldots \qquad f_k : \mathbb{Z}/p^{k+1}\mathbb{Z} \to \mathbb{Z}/p^k\mathbb{Z},\ \bar{n} \mapsto \bar{n}$$

   The sequential limit (inverse limit) of F is the ring $\mathbb{Z}_p$ of p**-adic integers**

   $$\mathbb{Z}_p = \left\{(x_k)_{k\in\mathbb{N}_0} \in \Pi_{k\in\mathbb{N}_0} \mathbb{Z}/p^k\mathbb{Z} \mid f_k(x_{k+1}) = x_k \ \forall k \in \mathbb{N}_0\right\}$$

   with componentwise addition and multiplication. It is an integral domain, and its field of fractions is the field of p**-adic numbers**.

We now discuss a less elementary example that is more typical of the way (co)limits are used in modern mathematics, namely *sheaves*. The idea is to describe structures on a topological space X, for instance continuous functions on X or abelian groups associated to X, by defining them *locally*, on open subsets $U \subset X$. Common sense dictates that the data assigned to open subsets must restrict appropriately, whenever one subset is contained in another, and that the data assigned to two subsets must agree on their intersection. The idea is that any local assignment of structure satisfying these conditions should lead to a unique *global* assignment of data to X. This idea can be formulated in terms of limits, namely equalisers and products.

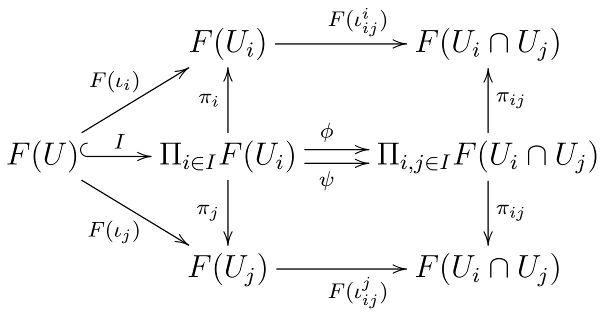
Example 2.1.14: (**sheaves**)
Let X be a topological space and $\mathcal{O}(X)$ the poset category with open subsets $U \subset X$ as objects and the partial ordering $\preceq = \subset$ given by inclusions of subsets.

- A **presheaf** on X with values in a category $\mathcal{C}$ is a functor $F : \mathcal{O}(X)^{op} \to \mathcal{C}$.
- A **morphism of presheaves** from F to F' is a natural transformation $\mu : F \Rightarrow F'$.

Thus a presheaf $F : \mathcal{O}(X)^{op} \to \mathcal{C}$ assigns to each open subset $U \subset X$ an object $F(U)$ in $\mathcal{C}$ and to each inclusion map $\iota_{UV} : U \to V$ a morphism $F(\iota_{UV}) : F(V) \to F(U)$ such that $F(\iota_{UU}) = 1_{F(U)}$ and $F(\iota_{UV}) \circ F(\iota_{UW}) = F(\iota_{UW})$ for all $U \subset V \subset W$.

An example is the presheaf $F : \mathcal{O}(X)^{op} \to \text{Set}$ that assigns to each subset $U \subset X$ the set $C^0(U, \mathbb{R})$ of continuous real functions on U and to an inclusion $\iota_{UV} : U \to V$ the restriction map $F(\iota_{UV}) : C^0(V, \mathbb{R}) \to C^0(U, \mathbb{R})$, $f \mapsto f|_U$.

A presheaf $F : \mathcal{O}(X)^{op} \to \mathcal{C}$ is called a **sheaf** if for all open subsets $U \subset X$ and all open covers $(U_i)_{i \in I}$ of U, the pair $(F(U), I)$ is an equaliser of the morphisms ϕ, ψ in the diagram



where

- the morphisms $\pi_i : \Pi_{i \in I} F(U_i) \to F(U_i)$ and $\pi_{ij} : \Pi_{i,j \in I} F(U_i \cap U_j) \to F(U_i \cap U_j)$ are the projection morphisms of the products,
- $\iota^i_{ij} : U_i \cap U_j \to U_i$ and $\iota_i : U_i \to U$ are the inclusion maps,
- $I : F(U) \to \Pi_{i \in I} F(U_i)$ is the unique morphism with $\pi_i \circ I = F(\iota_i)$ for all $i \in I$,
- ϕ and ψ are the unique morphisms with $\pi_{ij} \circ \phi = F(\iota^i_{ij}) \circ \pi_i$ and $\pi_{ij} \circ \psi = F(\iota^j_{ij}) \circ \pi_j$ for all $i, j \in I$ defined by the universal property of the product.

The condition on a sheaf guarantees that morphisms $f_i : C \to F(U_i)$ that match on all intersections $U_i \cap U_j$ can be glued to a unique morphism $f : C \to F(U)$:

A family $(f_i)_{i \in I}$ of morphisms $f_i : C \to F(U_i)$ matches on all intersections $U_i \cap U_j$, if it satisfies $F(\iota^i_{ij}) \circ f_i = F(\iota^j_{ij}) \circ f_j$ for all $i, j \in I$. By the universal property of the product, each such family defines a unique morphism $f' : C \to \Pi_{i \in I} F(U_i)$ with $\pi_i \circ f' = f_i$ for all $i \in I$.

As the family of morphisms matches on the intersections, one has for all $i, j \in I$

$$\pi_{ij} \circ \phi \circ f' = F(\iota^i_{ij}) \circ \pi_i \circ f' = F(\iota^i_{ij}) \circ f_i = F(\iota^j_{ij}) \circ f_j = F(\iota^j_{ij}) \circ \pi_j \circ f' = \pi_{ij} \circ \psi \circ f',$$

and the universal property of the product implies $\phi \circ f' = \psi \circ f'$. By universal property of the equaliser, there is a unique morphism $f : C \to F(U)$ with $I \circ f = f'$ or, equivalently, $\pi_i \circ I \circ f = f_i$ for all $i \in I$.

It is a good exercise to show that the presheaf $F : \mathcal{O}(X)^{op} \to \text{Set}$ of continuous functions discussed as an example above satisfies the sheaf condition and to determine the morphism $f : C \to F(U)$ induced by a matching family of morphisms $(f_i : C \to F(U_i))_{i \in I}$ in this case. In contrast, the presheaf $G : \mathcal{O}(X)^{op} \to \text{Set}$ that assigns to a subset $U \subset X$ the set $G(U)$ of *bounded* functions $f : U \to \mathbb{R}$ is not necessarily a sheaf (Exercise 16).

We conclude this section with an alternative characterisation of (co)limits that relates their existence to the representability of certain functors from $\mathcal{C}$ to Set. More specifically, for any diagram $F : \mathcal{J} \to \mathcal{C}$ we obtain a functor $\text{cone}(-, F) : \mathcal{C}^{op} \to \text{Set}$ that assigns

- to an object in $\mathcal{C}$ the set of all cones for F with apex C,
- to a morphism $f : C \to C'$ the induced map between these sets of cones.

A limit of F is a representation of $\text{cone}(-, F)$. This encodes the universal property of limits relating cone morphisms from a cone to the limit cone to morphisms from its apex to the limit. An analogous characterisation exists for colimits. This perspective on limits and colimits allows one to apply the Yoneda lemma to prove their uniqueness.

Proposition 2.1.15: Let $\mathcal{J}$ be small, $\mathcal{C}$ locally small and $F : \mathcal{J} \to \mathcal{C}$ a diagram.

1. Cones over F define a functor $\text{cone}(-, F) : \mathcal{C}^{op} \to \text{Set}$ that sends
   - an object C in $\mathcal{C}$ to the set $\text{cone}(C, F)$ of cones over F with apex C,
   - a morphism $f : C \to C'$ to the map $\text{cone}(f, F) : \text{cone}(C', F) \to \text{cone}(C, F)$ that sends a cone $\lambda : \Delta(C') \Rightarrow F$ to the cone $\mu = \lambda \circ \Delta(f) : \Delta(C) \Rightarrow F$.

   A limit of F is a representing object of this functor: $\text{cone}(-, F) \cong \text{Hom}_{\mathcal{C}}(-, \lim F)$.

2. Cocones under F define a functor $\text{cocone}(F, -) : \mathcal{C} \to \text{Set}$ that sends
   - an object C in $\mathcal{C}$ to the set $\text{cocone}(F, C)$ of cocones under F with nadir C,
   - a morphism $f : C \to C'$ to the map $\text{cocone}(F, f) : \text{cocone}(C, F) \to \text{cocone}(C', F)$ that sends a cocone $\lambda : F \Rightarrow \Delta(C)$ to the cocone $\Delta(f) \circ \lambda : F \Rightarrow \Delta(C')$.

   A colimit of F is a representing object of this functor: $\text{cocone}(F, -) \cong \text{Hom}_{\mathcal{C}}(\text{colim} F, -)$.

Proof:
We prove the claim for limits. The claim for colimits follows with Remark 2.1.5, 3.

1. By definition, $\text{cone}(f, F)(\lambda) = \lambda \circ \Delta(f)$ is a natural transformation and hence a cone over F. For $f = 1_C$ we have $\lambda \circ \Delta(f) = \lambda$, which implies $\text{cone}(1_C, F) = \text{id}_{\text{cone}(C,F)}$. For morphisms $f : C \to C'$ and $f' : C' \to C''$ and $\lambda \in \text{cone}(C'', F)$ we have

$$\text{cone}(f' \circ f, F)(\lambda) = \lambda \circ \Delta(f' \circ f) = \lambda \circ \Delta(f') \circ \Delta(f) = \text{cone}(f', F) \circ \Delta(f) = \text{cone}(f, F) \circ \text{cone}(f', F)(\lambda).$$

This shows that $\text{cone}(-, F) : \mathcal{C}^{op} \to \text{Set}$ is a functor.

2. A representation of $\mathrm{cone}(-,F) : \mathcal{C}^{op} \to \mathrm{Set}$ consists of an object C in $\mathcal{C}$ and a natural isomorphism $\eta : \mathrm{Hom}_{\mathcal{C}}(-,C) \Rightarrow \mathrm{cone}(-,F)$. By the Yoneda lemma, such a natural isomorphism is determined uniquely by the cone $\lambda := \eta_C(1_C) : \Delta(C) \Rightarrow F$, and the naturality of η states that the following diagram commutes for all morphisms $f : C' \to C$

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(C',C) & \xrightarrow[\cong]{\eta_{C'}} & \mathrm{cone}(C',F) \\ \uparrow{\scriptstyle h\mapsto h\circ f} & & \uparrow{\scriptstyle \rho_J\mapsto \rho_J\circ f} \\ \mathrm{Hom}_{\mathcal{C}}(C,C) & \xrightarrow[\eta_C]{\cong} & \mathrm{cone}(C,F). \end{array}$$

Inserting the identity morphism $1_C \in \mathrm{Hom}_{\mathcal{C}}(C,C)$ yields for every cone $\mu \in \mathrm{cone}(C',F)$ a unique cone morphism $f = \eta_{C'}^{-1}(\mu) : C' \to C$ with $\lambda_J \circ f = \mu_J$ for all $j \in \mathrm{Ob}\mathcal{J}$. This is precisely the condition that $\lambda = \eta_C(1_C)$ is a terminal cone over F. □

2.2 Existence of (co)limits

In this section, we derive criteria for the existence of limits and colimits of functors $F : \mathcal{J} \to \mathcal{C}$ and relate them to other concepts from category theory. We will show that if $\mathcal{J}$ is small, the existence of products and equalisers in $\mathcal{C}$ is sufficient for the existence of limits. Dually, the existence of coproducts and coequalisers in $\mathcal{C}$ guarantees the existence of colimits for all functors $F : \mathcal{J} \to \mathcal{C}$. In this case, limits and colimits organise into functors $\mathrm{(co)lim} : \mathcal{C}^J \to \mathcal{C}$ that are adjoints of the embedding functor $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{J}}$ from Definition 2.1.1.

Functors $F : \mathcal{J} \to \mathcal{C}$ from *small* categories $\mathcal{J}$ are often called **small diagrams** and the associated (co)limits are called **small colimits**. In some cases, it is sufficient to restrict attention to **finite categories** $\mathcal{J}$, small categories $\mathcal{J}$ with finitely many objects and morphisms. In this case one speaks of **finite diagrams** and **finite (co)limits**.

Definition 2.2.1: A category $\mathcal{C}$ is called

- **(co)complete**, if every small diagram $F : \mathcal{J} \to \mathcal{C}$ has a (co)limit,
- **finitely (co)complete**, if every finite diagram $F : \mathcal{J} \to \mathcal{C}$ has a (co)limit.
- **(finitely) bicomplete**, if $\mathcal{C}$ is (finitely) complete and cocomplete,

It turns out that the existence of (co)products and (co)equalisers is sufficient for the existence of all small (co)limits. The following theorem explains in particular why all pullbacks and pushouts in Examples 2.1.8 and 2.1.9 are given in terms of (co)products and (co)equalisers and generalises the constructions in these examples.

Theorem 2.2.2:
A category $\mathcal{C}$ is (co)complete, if and only if it has all (co)products and (co)equalisers.

Proof:
We prove the claim for cocomplete categories. The claim for complete categories then follows with Remark 2.1.5, 3.

As coproducts and coequalisers are colimits, cocompleteness implies the existence of all coproducts and coequalisers. Suppose now that $\mathcal{C}$ has all coproducts and coequalisers, let $\mathcal{J}$ be small and $F : \mathcal{J} \to \mathcal{C}$ a diagram in $\mathcal{C}$.

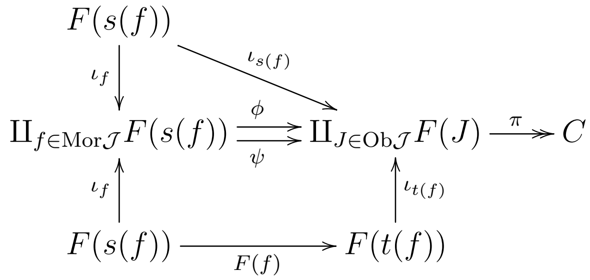
As $\mathcal{J}$ is small, we have sets $\mathrm{Ob}\mathcal{J}$ and $\mathrm{Mor}\mathcal{J} = \bigcup_{J,J' \in \mathrm{Ob}\mathcal{J}} \mathrm{Hom}_{\mathcal{J}}(J, J')$ and can consider the coproducts $\amalg_{J \in \mathrm{Ob}\mathcal{J}} F(J)$ and $\amalg_{f \in \mathrm{Mor}\mathcal{J}} F(s(f))$ with associated inclusion morphisms

$$\iota_I : F(I) \to \amalg_{J \in \mathrm{Ob}\mathcal{J}} F(J) \qquad \iota_g : F(s(g)) \to \amalg_{f \in \mathrm{Mor}\mathcal{J}} F(s(f)).$$

Let $\phi, \psi : \amalg_{f \in \mathrm{Mor}\mathcal{J}} F(s(f)) \to \amalg_{J \in \mathrm{Ob}\mathcal{J}} F(J)$ be the unique morphisms induced by the universal property of the coproducts that satisfy for all morphisms $f : s(f) \to t(f)$ in $\mathcal{J}$

$$\phi \circ \iota_f = \iota_{s(f)} \qquad \psi \circ \iota_f = \iota_{t(f)} \circ F(f)$$

and define $\pi : \amalg_{j \in \mathrm{Ob}\mathcal{J}} F(J) \to C$ as the coequaliser of ϕ and ψ.



(5)

The morphisms $\lambda_J := \pi \circ \iota_J : F(J) \to C$ define a cocone $\lambda : F \Rightarrow \Delta(C)$, as we have for each morphism $j : J \to J'$ in $\mathcal{J}$

$$\lambda_{J'} \circ F(j) = \pi \circ \iota_{J'} \circ F(j) = \pi \circ \psi \circ \iota_j = \pi \circ \phi \circ \iota_j = \pi \circ \iota_J = \lambda_J,$$

where we used the definition of λ_J, of ψ, that π coequalises ϕ, ψ and then the definition of ϕ.

We show that the cocone $\lambda : F \Rightarrow \Delta(C)$ is an initial cocone. Via the universal property of the coproduct, any cocone $\mu : F \Rightarrow \Delta(C')$ defines a morphism $g : \amalg_{J \in \mathrm{Ob}\mathcal{J}} F(J) \to C'$ with $g \circ \iota_J = \mu_J$ for all $J \in \mathrm{Ob}\mathcal{J}$. This morphism satisfies for all morphisms $f : J \to J'$

$$g \circ \phi \circ \iota_f = g \circ \iota_J = \mu_J = \mu_{J'} \circ F(f) = g \circ \iota_{J'} \circ F(f) = g \circ \psi \circ \iota_f, \qquad (6)$$

where we used the definition of ϕ, of g, that μ is a cocone, then the definition of g and ψ. By the universal property of the coproduct, this implies $g \circ \phi = g \circ \psi$. By the universal property of the coequaliser there is a unique morphism $g' : C \to C'$ with $g' \circ \pi = g$. This is a cocone morphism, since the definitions of g' and g imply for all $J \in \mathrm{Ob}\mathcal{J}$

$$g' \circ \lambda_J = g' \circ \pi \circ \iota_J = g \circ \iota_J = \mu_J. \qquad \square$$

The proof of Theorem 2.2.2 also shows that that the existence of (co)equalisers and *finite* (co)products is sufficient for the existence of all *finite* (co)limits. In this case, diagram (5) is still defined for a finite category $\mathcal{J}$, and the rest of the proof proceeds analogously.

Corollary 2.2.3: A category $\mathcal{C}$ is finitely (co)complete if and only if it has all (co)equalisers and all finite (co)products.

The crux in the proof of Theorem 2.2.2 is diagram (5), which looks complicated at first, but this is mainly due to the notation. For *concrete* categories $\mathcal{C}$, categories whose objects are sets, it has a simple interpretation:

It states that the colimit of $F : \mathcal{J} \to \mathcal{C}$ is obtained by taking the coproduct of all objects $F(J)$ for $J \in \mathrm{Ob}\mathcal{J}$ and then identifying for each morphism $j : s(j) \to t(j)$ each element of $F(s(j))$ with its image under the map $F(j)$ in the set $F(t(j))$.

This is precisely the construction used for the pushouts in Examples 2.1.8 and 2.1.9 and in the coequalisers of Example 2.1.11. It is a useful exercise to check that the sequential (co)limits in Example 2.1.13 are also a special case of this construction.

Example 2.2.4:

1. The following categories are bicomplete by Theorem 2.2.2 and Examples 2.1.7 and 2.1.11
   - Set, Top
   - R-Mod for any ring R, in particular $\mathrm{Vect}_{\mathbb{F}}$, Ab, $\mathbb{F}[G]$-Mod.
   - Grp, Ring.
2. Any abelian category is finitely bicomplete by Corollary 2.2.3, because it has (co)equalisers and finite (co)products by definition, see Example 2.1.11, 3.
3. The category Cat of small categories and functors between them and its full subcategory Grpd of groupoids and functors between them are bicomplete:.
   - The product of a family $(\mathcal{C}_j)_{j\in J}$ of small categories is the category $\mathcal{C} = \Pi_{j\in J}\mathcal{C}_j$ with $\mathrm{Ob}\mathcal{C} = \Pi_{j\in J}\mathrm{Ob}\mathcal{C}_j$ and $\mathrm{Hom}_{\mathcal{C}}((C_j)_{j\in J}, (C'_j)_{j\in J}) = \Pi_{j\in J}\mathrm{Hom}_{\mathcal{C}_j}(C_j, C'_j)$, together with the projection functors $\pi_i : \mathcal{C} \to \mathcal{C}_i$.
   - The coproduct of a family $(\mathcal{C}_j)_{j\in J}$ of small categories is the category $\mathcal{C} = \amalg_{j\in J}\mathcal{C}_j$ with $\mathrm{Ob}\mathcal{C} = \dot\cup_{j\in J}\mathrm{Ob}\mathcal{C}_j$ and $\mathrm{Hom}_{\mathcal{C}}(C, C') = \mathrm{Hom}_{\mathcal{C}_j}(C, C')$ for $C, C' \in \mathrm{Ob}\,\mathcal{C}_j$ and $\mathrm{Hom}_{\mathcal{C}}(C, C') = \emptyset$ otherwise, together with the inclusion functors $\iota_i : \mathcal{C}_i \to \mathcal{C}$.
   - The equaliser of two functors $F_1, F_2 : \mathcal{C} \to \mathcal{D}$ between small categories is the subcategory $\mathcal{U} \subset \mathcal{C}$ with $\mathrm{Ob}\mathcal{U} = \{C \in \mathrm{Ob}\mathcal{C} \mid F_1(C) = F_2(C)\}$ and morphism sets $\mathrm{Hom}_{\mathcal{U}}(C, C') = \{f \in \mathrm{Hom}_{\mathcal{C}}(C, C') \mid F_1(f) = F_2(f)\}$, with the inclusions $\iota : \mathcal{U} \to \mathcal{C}$.
   - Coequalisers in Cat and Grpd exist, but are more difficult to describe. We will prove their existence in Corollary 5.4.7 in Section 5.

Even if a category $\mathcal{C}$ is not complete or cocomplete, limits or colimits of functors may still exist for all diagrams $F : \mathcal{J} \to \mathcal{C}$ of a fixed shape $\mathcal{J}$. For instance, a category $\mathcal{C}$ may have all coproducts, but not all coequalisers. In this case, it still makes sense to consider the (co)limits of functors $F : \mathcal{J} \to \mathcal{C}$ and to determine how they interact with natural transformations.

As cones over $F : \mathcal{J} \to \mathcal{C}$ are natural transformations $\lambda : \Delta(C) \Rightarrow F$, post-composing them with natural transformation $\tau : F \Rightarrow F'$ yields cones over F'. This induces a morphisms between the limits of F and F' and allows one to organise limits into a functor. There also is a dual construction for colimits. It turns out that the resulting functors are right adjoint and left adjoint to the embedding functor $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{J}}$ from Definition 2.1.1. Conversely, if this functor has a right or left adjoint, this defines limits or colimits of functors $F : \mathcal{J} \to \mathcal{C}$.

Proposition 2.2.5: Let $\mathcal{J}$ be a small category and $\mathcal{C}$ a category.

1. If all limits of diagrams $F : \mathcal{J} \to \mathcal{C}$ exist, they define a right adjoint $\lim : \mathcal{C}^{\mathcal{J}} \to \mathcal{C}$ to $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{J}}$. If Δ has a right adjoint, all limits of functors $F : \mathcal{J} \to \mathcal{C}$ exist.

2. If all colimits of diagrams $F : \mathcal{J} \to \mathcal{C}$ exist, they define a left adjoint colim : $\mathcal{C}^{\mathcal{J}} \to \mathcal{C}$ to $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{J}}$. If Δ has a left adjoint, all colimits of functors $F : \mathcal{J} \to \mathcal{C}$ exist.

Proof:
We prove the claim for limits. The claim for colimits follows with Remark 2.1.5, 3.

1. Suppose that all functors $F : \mathcal{J} \to \mathcal{C}$ have limit cones $\lambda^F : \Delta(\lim F) \Rightarrow F$.

(a) We define the functor $\lim : \mathcal{C}^{\mathcal{J}} \to \mathcal{C}$:

It assigns to a functor $F : \mathcal{J} \to \mathcal{C}$ the object $\lim F$. A natural transformation $\tau : F \Rightarrow F'$ defines a cone $\tau \circ \lambda^F : \Delta(\lim F) \Rightarrow F'$. Hence, there is a unique morphism $\lim \tau : \lim F \to \lim F'$ with

$$\lambda^{F'} \circ \Delta(\lim \tau) = \tau \circ \lambda^F \tag{7}$$

by the universal property of $\lim F'$. We assign to τ this morphism $\lim \tau : \lim F \to \lim F'$. To show that this defines a functor, note that for $\tau = \mathrm{id}_F$ one has $\lambda^F \circ \Delta(1_{\lim F}) = \lambda^F$ and hence $\lim \mathrm{id}_F = 1_{\lim F}$. For any natural transformation $\tau' : F' \to F''$ we have from (7)

$$\lambda^{F''} \circ \Delta(\lim \tau') \circ \Delta(\lim \tau) = \tau' \circ \lambda^{F'} \circ \Delta(\lim \tau) = \tau' \circ \tau \circ \lambda^F \quad \Rightarrow \quad \lim(\tau' \circ \tau) = \lim \tau' \circ \lim \tau.$$

(b) We show that $\lim : \mathcal{C}^{\mathcal{J}} \to \mathcal{C}$ is right adjoint to $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{J}}$: the maps

$$\begin{aligned} \phi_{C,F} : \mathrm{Hom}_{\mathcal{C}}(C, \lim F) &\to \mathrm{Hom}_{\mathcal{C}^{\mathcal{J}}}(\Delta(C), F) \\ g &\mapsto \lambda^F \circ \Delta(g) : \Delta(C) \Rightarrow F, . \end{aligned} \tag{8}$$

for $C \in \mathrm{Ob}\mathcal{C}$ and functors $F : \mathcal{J} \to \mathcal{C}$ are bijections by the universal property of the limit cone. They are natural, as we have for all $f : C' \to C$ and natural transformations $\mu : F \Rightarrow F'$

$$\begin{aligned} \phi_{C',F'}(\lim(\mu) \circ g \circ f) &= \lambda^{F'} \circ \Delta(\lim(\mu) \circ g \circ f) = \lambda^{F'} \circ \Delta(\lim \mu) \circ \Delta(g) \circ \Delta(f) \\ &\overset{(7)}{=} \mu \circ \lambda^F \circ \Delta(g) \circ \Delta(f) = \mu \circ \phi_{C,F}(g) \circ \Delta(f). \end{aligned}$$

2. Suppose that $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{J}}$ has a right adjoint $G : \mathcal{C}^{\mathcal{J}} \to \mathcal{C}$ with unit $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow G\Delta$ and counit $\epsilon : \Delta G \Rightarrow \mathrm{id}_{\mathcal{C}^{\mathcal{J}}}$ satisfying (i) $\epsilon\Delta \circ \Delta\eta = \mathrm{id}_\Delta$ and (ii) $G\epsilon \circ \eta G = \mathrm{id}_G$.

We prove that for each functor $F : \mathcal{J} \to \mathcal{C}$ the component morphism $\epsilon_F : \Delta(G(F)) \Rightarrow F$ is a limit cone. Let $\mu : \Delta(C) \Rightarrow F$ be a cone over F. Then $f = G(\mu) \circ \eta_C : C \to G(F)$ satisfies

$$\epsilon_F \circ \Delta(f) = \epsilon_F \circ \Delta G(\mu) \circ \Delta(\eta_C) \overset{\text{nat } \epsilon}{=} \mu \circ \epsilon_{\Delta(C)} \circ \Delta(\eta_C) = \mu \circ (\epsilon\Delta \circ \Delta\eta)_C \overset{(i)}{=} \mu \circ (\mathrm{id}_\Delta)_C = \mu.$$

If $f' \in \mathrm{Hom}_{\mathcal{C}}(C, G(F))$ is another morphism with $\epsilon_F \circ \Delta(f') = \mu$ one has

$$f = G(\mu) \circ \eta_C = G(\epsilon_F) \circ G\Delta(f') \circ \eta_C \overset{\text{nat } \eta}{=} G(\epsilon_F) \circ \eta_{G(F)} \circ f' = (G\epsilon \circ \eta G)_F \circ f' \overset{(ii)}{=} 1_{G(F)} \circ f' = f'.$$

□

2.3 (Co)limits in functor category and exchange of (co)limits

In this section, we investigate (co)limits in functor categories. As we will see in the following, this has many applications. An important example are simplicial objects and simplicial morphisms, which we will encounter in Section 5 and which form the foundation of homological algebra as well as many modern developments in category theory. Another example is the category BG for a finite group G, also called the **delooping** of G. Functors $F : BG \to \mathcal{C}$ define objects in $\mathcal{C}$ equipped with G-actions, such as G-sets and representations of G on vector spaces.

Example 2.3.1: Let G be a group and $\mathcal{C}$ a category. The category BG has

- a single object $\bullet$,
- the morphism set $\mathrm{Hom}_{BG}(\bullet, \bullet) = G$ with the group multiplication as composition and the group unit as identity morphism.

The category $\mathcal{C}^{BG}$ is called the category of **G-objects** and **G-equivariant morphisms** in $\mathcal{C}$. Examples are the following:

- $\mathrm{Set}^{BG} = G$-Set has
  - as objects **G-sets**: sets X equipped with a group action $\rhd_X : G \times X \to X$,
  - as morphisms from $(X, \rhd_X)$ to $(Y, \rhd_Y)$ **G-equivariant maps**: maps $f : X \to Y$ with $f(g \rhd_X x) = g \rhd_Y f(x)$ for all $g \in G$, $x \in X$.

- Top^{BG} has
  - as objects topological spaces X equipped with a continuous action $\rhd_X : G \times X \to X$ of the discrete group G,
  - as morphisms from $(X, \rhd_X)$ to $(Y, \rhd_Y)$ continuous G-equivariant maps.

- $\mathrm{Vect}_{\mathbb{F}}^{BG} = \mathrm{Rep}_{\mathbb{F}}(G) = \mathbb{F}[G]$-Mod has
  - as objects representations of G over $\mathbb{F}$,
  - as morphisms morphisms of representations.

To investigate (co)limits in functor categories we identify the functor category $(\mathcal{C}^{\mathcal{D}})^{\mathcal{J}}$ with the functor category $\mathcal{C}^{\mathcal{J} \times \mathcal{D}}$ via the isomorphism of categories that assigns

- to a functor $F : \mathcal{J} \times \mathcal{D} \to \mathcal{C}$ the functor $F' : \mathcal{J} \to \mathcal{C}^{\mathcal{D}}$ that sends
  - $J \in \mathrm{Ob}\mathcal{J}$ to the functor $F(J, -) : \mathcal{D} \to \mathcal{C}$,
  - $j : J \to J'$ to the natural transformation $F(j, -) : F(J, -) \Rightarrow F(J', -)$,
- to a natural transformation $\tau : F \Rightarrow G$ the natural transformation $\tau' : F' \Rightarrow G'$ with component morphisms $\tau'_J = \tau_{J,-} : F(J, -) \Rightarrow G(J, -)$.

This allows us to describe (co)limits of functors $F : \mathcal{J} \to \mathcal{C}^{\mathcal{D}}$ in terms of (co)limits of the associated functors $F : \mathcal{J} \times \mathcal{D} \to \mathcal{C}$. We can evaluate functors $F : \mathcal{J} \times \mathcal{D} \to \mathcal{C}$ on objects $D \in \mathrm{Ob}\,\mathcal{D}$ and consider the (co)limits of the resulting functors $F(-, D) : \mathcal{J} \to \mathcal{C}$. It turns out that if all of the latter have (co)limits, then they combine into a (co)limit of $F : \mathcal{J} \to \mathcal{C}^{\mathcal{D}}$. One says that (co)limits of functors $F : \mathcal{J} \to \mathcal{C}^{\mathcal{D}}$ are computed *pointwise* or *objectwise*.

Proposition 2.3.2: (**Limits and colimits in functor categories are pointwise.**)
Let $\mathcal{J}$ be a small category and $F : \mathcal{J} \to \mathcal{C}^{\mathcal{D}}$ a diagram such that $F_D := F(-, D) : \mathcal{J} \to \mathcal{C}$ has a (co)limit for all objects $D \in \mathrm{Ob}\mathcal{D}$. Then the (co)limit of F is the functor

$$(\mathrm{co})\lim F : \mathcal{D} \to \mathcal{C} \qquad \begin{array}{ccc} D & \longmapsto & (\mathrm{co})\lim F_D \\ \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle (\mathrm{co})\lim F(d)} \\ D' & \longmapsto & (\mathrm{co})\lim F_{D'} \end{array}$$

In particular, if $\mathcal{C}$ is (co)complete or finitely (co)complete, so is $\mathcal{C}^{\mathcal{D}}$.

Proof:
We prove the claim for limits. The claim for colimits follows with Remark 2.1.5, 3.

1. For $D \in \mathrm{Ob}\mathcal{D}$ denote by $\lambda^D : \Delta(\lim F_D) \Rightarrow F_D$ the limit cone of the associated functor $F_D = F(-, D) : \mathcal{J} \to \mathcal{C}$. These limit cones define a functor $\lim F : \mathcal{D} \to \mathcal{C}$ that assigns

- to $D \in \mathrm{Ob}\mathcal{D}$ the object $\lim F(D) = \lim F_D \in \mathrm{Ob}\mathcal{C}$,
- to $d \in \mathrm{Hom}_{\mathcal{D}}(D, D')$ the unique morphism $\lim F(d) : \lim F_D \to \lim F_{D'}$ with

$$\lambda^{D'} \circ \Delta(\lim F(d)) = F_d \circ \lambda^D \tag{9}$$

  from the universal property of the limit and $F_d = F(-, d) : F_D \Rightarrow F_{D'}$.

That $\lim F$ is indeed a functor, follows by applying (9) to an identity morphism $d = 1_D$ and to the composites $d' \circ d$ of morphisms $d : D \to D'$, $d' : D' \to D''$.

2. The limit cones $\lambda^D : \Delta(\lim F_D) \Rightarrow F_D$ define a cone $\lambda : \Delta(\lim F) \Rightarrow F$ over $F : \mathcal{J} \to \mathcal{C}^{\mathcal{D}}$ with legs $(\lambda_J)_D = \lambda^D_J$, because one has for all morphisms $j : J \to J'$ and $d : D \to D'$

$$F(j, d) \circ \lambda^D_J = (F_d)_{J'} \circ F_D(j) \circ \lambda^D_J = (F_d)_{J'} \circ \lambda^D_{J'} = \lambda^{D'}_{J'}$$

where we first used the fact that λ^D is a cone and then (9). We show that the cone λ is terminal.

For this, let $\mu : \Delta(G) \Rightarrow F$ with $G : \mathcal{D} \to \mathcal{C}$ be a cone over F. Then for all $D \in \mathrm{Ob}\mathcal{D}$, this defines a cone $\mu^D : \Delta(G(D)) \Rightarrow F_D$ over F_D. As λ^D is a limit cone, there is a unique morphism $f^D : G(D) \to \lim F_D$ with $\lambda^D \circ \Delta(f^D) = \mu^D$. The morphisms f^D define a natural transformation $f : G \Rightarrow \lim F$, because one has for all morphisms $d : D \to D'$

$$\begin{aligned}\lambda^{D'} \circ \Delta(f^{D'} \circ G(d)) &= \lambda^{D'} \circ \Delta(f^{D'}) \circ \Delta(G(d)) = \mu^{D'} \circ \Delta(G(d)) \overset{\text{nat}\,\mu}{=} F_d \circ \mu^D \\ &= F_d \circ \lambda^D \circ \Delta(f^D) \overset{(9)}{=} \lambda^{D'} \circ \Delta(\lim F(d)) \circ \Delta(f^D) = \lambda^{D'} \circ \Delta(\lim F(d) \circ f^D).\end{aligned}$$

As λ^D is a limit cone, this implies $\lim F(d) \circ f^D = f^{D'} \circ G(d)$ for all morphisms $d : D \to D'$ and hence naturality of f. This shows that $f : G \Rightarrow \lim F$ is a morphism in $\mathcal{C}^{\mathcal{D}}$ with $\lambda \circ \Delta(f) = \mu$, and hence λ is a terminal cone. □

If the category $\mathcal{D}$ has a simple structure, we can think of functors $G : \mathcal{D} \to \mathcal{C}$ as objects in $\mathcal{C}$ that are equipped with an action of the category $\mathcal{D}$. This generalises the group actions on categories for $\mathcal{D} = BG$ in Example 2.3.1. Proposition 2.3.2 then states that the (co)limit of a functor $F : \mathcal{J} \to \mathcal{C}^{\mathcal{D}}$ is obtained by taking (co)limits in $\mathcal{C}$ and equipping them with the induced $\mathcal{D}$-action. This intuition becomes a precise statement for the category $\mathcal{D} = BG$.

Example 2.3.3: Let G be a group.

1. Any (co)limit of a functor $F : \mathcal{J} \to G$-Set is given as the (co)limit of the corresponding functor $F' : \mathcal{J} \to$ Set, equipped with the induced G-action.
2. Any (co)limit of a functor $F : \mathcal{J} \to \mathrm{Rep}_{\mathbb{F}}(G)$ is the (co)limit of the corresponding functor $F' : \mathcal{J} \to \mathrm{Vect}_{\mathbb{F}}$, equipped with the induced representation of G.

As we can view a functor $F : \mathcal{I} \times \mathcal{J} \to \mathcal{C}$ as a functor $F : \mathcal{I} \to \mathcal{C}^{\mathcal{J}}$ or as a functor $F : \mathcal{J} \to \mathcal{C}^{\mathcal{I}}$, we can take (co)limits with respect to both categories $\mathcal{I}$ and $\mathcal{J}$. We can also combine them, for instance, take first a (co)limit of $F : \mathcal{I} \to \mathcal{C}^{\mathcal{J}}$ and then a (co)limit of $(\mathrm{co})\lim_{\mathcal{I}} F : \mathcal{J} \to \mathcal{C}$ or first a (co)limit of $F : \mathcal{J} \to \mathcal{C}^{\mathcal{I}}$ and then a (co)limit of $(\mathrm{co})\lim_{\mathcal{J}} F : \mathcal{I} \to \mathcal{C}$. The question is whether these procedures commute and yield isomorphic (co)limit cones. This depends on the precise combination of limits and colimits. It is true if we either take limits or take colimits with respect to both categories.

Theorem 2.3.4: Let $\mathcal{I}, \mathcal{J}$ be small and $F : \mathcal{I} \times \mathcal{J} \to \mathcal{C}$ be a functor.

1. If $\lim_{\mathcal{I}} \lim_{\mathcal{J}} F$ and $\lim_{\mathcal{J}} \lim_{\mathcal{I}} F$ exist, then there are canonical isomorphisms

$$\lim_{\mathcal{I}} \lim_{\mathcal{J}} F \cong \lim_{\mathcal{J}} \lim_{\mathcal{I}} F \cong \lim_{\mathcal{I}\times\mathcal{J}} F.$$

2. If $\mathrm{colim}_{\mathcal{I}}\mathrm{colim}_{\mathcal{J}} F$ and $\mathrm{colim}_{\mathcal{J}}\mathrm{colim}_{\mathcal{I}} F$ exist, then there are canonical isomorphisms

$$\mathrm{colim}_{\mathcal{I}}\mathrm{colim}_{\mathcal{J}} F \cong \mathrm{colim}_{\mathcal{J}}\mathrm{colim}_{\mathcal{I}} F \cong \mathrm{colim}_{\mathcal{I}\times\mathcal{J}} F.$$

Proof:
We prove the claim for limits. The claim for colimits follows with Remark 2.1.5, 3. As a limit of F is a terminal object in the category cone(F), it is sufficient to show that cone(F), cone$(\lim_{\mathcal{J}} F)$ and cone$(\lim_{\mathcal{I}} F)$ are isomorphic via functors that preserve the apexes of the cones.

We construct a functor $N : \mathrm{cone}(\lim_{\mathcal{J}} F) \to \mathrm{cone}(F)$ with a strict inverse that preserves apexes of cones and is the identity on the morphisms. For this, we denote by $\mu : \Delta(\lim_{\mathcal{J}} F) \Rightarrow F$ the limit cone of the associated functor $F : \mathcal{J} \to \mathcal{C}^{\mathcal{I}}$ whose legs are natural transformations $\mu_J : \lim_{\mathcal{J}} F \Rightarrow F(-, J)$. with component morphisms $(\mu_J)_I : \lim_{\mathcal{J}} F(I) \to F(I, J)$.

The functor $N : \mathrm{cone}(\lim_{\mathcal{J}} F) \to \mathrm{cone}(F)$ sends

- a cone $\sigma : \Delta(C) \Rightarrow \lim_{\mathcal{J}} F$ over $\lim_{\mathcal{J}} F$ to the cone $N(\sigma) = \mu \circ \Delta(\sigma) : \Delta(C) \Rightarrow F$ with

$$N(\sigma)_{I,J} = (\mu_J)_I \circ \sigma_I, \tag{10}$$

- a cone morphism $f : C \to C'$ from σ to σ' to itself, as cone morphism from $N(\sigma)$ to $N(\sigma')$.

By definition $N(\sigma)$ is a natural transformation and hence a cone over F. It is also directly apparent that any cone morphism from σ to σ' defines a cone morphism from $N(\sigma)$ to $N(\sigma')$. The functoriality of N follows directly, as N is the identity on the morphisms.

The inverse $N^{-1} : \mathrm{cone}(F) \to \mathrm{cone}(\lim_{\mathcal{J}} F)$ sends

- a cone $\tau : \Delta(C) \Rightarrow F$ to the unique cone morphism $N^{-1}(\tau) : \Delta(C) \Rightarrow \lim_{\mathcal{J}} F$ in $\mathcal{C}^{\mathcal{I}}$ with $\mu \circ \Delta(N^{-1}(\tau)) = \tau$ or, equivalently,

$$(\mu_J)_I \circ N^{-1}(\tau)_I = \tau_{I,J}, \tag{11}$$

- a cone morphism $f : C \to C'$ from τ to τ' to itself, viewed as a cone morphism from $N^{-1}(\tau)$ to $N^{-1}(\tau')$.

That $N^{-1}(\tau) : \Delta(C) \Rightarrow \lim_{\mathcal{J}} F$ is a cone over $\lim_{\mathcal{J}} F$ follows, as it is a morphism in $\mathcal{C}^{\mathcal{I}}$. That any cone morphism $f : C \to C'$ from τ to τ' is also a cone morphism from $N^{-1}(\tau)$ and $N^{-1}(\tau')$ follows from the definition of $N^{-1}(\tau)$. The functoriality of N^{-1} is again clear.

It remains to show that N^{-1} is the inverse of N. For any cone $\tau : \Delta(C) \Rightarrow F$, one has

$$N(N^{-1}(\tau))_{I,J} \overset{(10)}{=} (\mu_J)_I \circ N^{-1}(\tau)_I \overset{(11)}{=} \tau_{I,J}$$

for all $J \in \mathrm{Ob}\,\mathcal{J}$, $I \in \mathrm{Ob}\,\mathcal{I}$ and hence $N(N^{-1}(\tau)) = \tau$. For a cone $\sigma : \Delta(C) \Rightarrow \lim_{\mathcal{J}} F$, one has

$$(\mu_J)_I \circ N^{-1}(N(\sigma))_I \overset{(11)}{=} N(\sigma)_{I,J} \overset{(10)}{=} (\mu_J)_I \circ \sigma_I,$$

and the universal property of the limit μ implies $N^{-1}(N(\sigma)) = \sigma$. □

If we take a colimit with respect to $\mathcal{I}$ and a limit with respect to $\mathcal{J}$ for a functor $F : \mathcal{I}\times\mathcal{J} \to \mathcal{C}$, then exchanging these two operations is more subtle. It is clear that the result cannot correspond to a limit or colimit with respect to $\mathcal{I} \times \mathcal{J}$, but it is not even guaranteed that the resulting objects are isomorphic. One obtains a canonical morphism from the colimit of the limit to the limit of the colimit, which need not be an isomorphism.

Lemma 2.3.5: For every functor $F : \mathcal{I} \times \mathcal{J} \to \mathcal{C}$ for which these (co)limits exist, there is a canonical morphism

$$g : \operatorname*{colim}_{\mathcal{I}} \lim_{\mathcal{J}} F \to \lim_{\mathcal{J}} \operatorname*{colim}_{\mathcal{I}} F$$

Proof:
We denote by

- $\lambda : \Delta(\lim_{\mathcal{J}} F) \Rightarrow F$ the limit cone of the associated functor $F : \mathcal{J} \to \mathcal{C}^{\mathcal{I}}$,
- $\mu : F \Rightarrow \Delta(\mathrm{colim}_{\mathcal{I}} F)$ the colimit cone of the associated functor $F : \mathcal{I} \to \mathcal{C}^{\mathcal{J}}$,
- $\rho : \Delta(\lim_{\mathcal{J}} \mathrm{colim}_{\mathcal{I}} F) \Rightarrow \mathrm{colim}_{\mathcal{I}} F$ the limit cone of the functor $\mathrm{colim}_{\mathcal{I}} F : \mathcal{J} \to \mathcal{C}$,
- $\sigma : \lim_{\mathcal{J}} F \Rightarrow \Delta(\mathrm{colim}_{\mathcal{I}} \lim_{\mathcal{J}} F)$ the colimit cone of the functor $\lim_{\mathcal{J}} F : \mathcal{I} \to \mathcal{C}$,

The morphisms $\kappa_{I,J} = (\mu_I)_J \circ (\lambda_J)_I : \lim_{\mathcal{J}} F(I) \xrightarrow{(\lambda_J)_I} F(I,J) \xrightarrow{(\mu_I)_J} \mathrm{colim}_{\mathcal{I}} F(J)$ are natural in I and J and hence define cones $\kappa^I : \Delta(\lim_{\mathcal{J}} F(I)) \Rightarrow \mathrm{colim}_{\mathcal{I}} F$ with legs $\kappa^I_J = \kappa_{I,J}$. By the universal property of ρ, there are unique cone morphisms $f_I : \lim_{\mathcal{J}} F(I) \to \lim_{\mathcal{J}} \mathrm{colim}_{\mathcal{I}} F$ with $\rho \circ \Delta(f_I) = \kappa^I$ for $I \in \mathrm{Ob}\mathcal{I}$. Because they are natural in $I \in \mathrm{Ob}\mathcal{I}$, the cone morphisms f_I define a cocone $f : \lim_{\mathcal{J}} F \Rightarrow \Delta(\lim_{\mathcal{J}} \mathrm{colim}_{\mathcal{I}} F)$ over $\lim_{\mathcal{J}} F : \mathcal{I} \to \mathcal{C}$. By the universal property of σ, there is a unique cocone morphism $g : \mathrm{colim}_{\mathcal{I}} \lim_{\mathcal{J}} F \to \lim_{\mathcal{J}} \mathrm{colim}_{\mathcal{I}} F$ with $\Delta(g) \circ \sigma = f$. □

Example 2.3.6:
If $\mathcal{I} = \mathcal{J} = \emptyset$, then $\mathcal{I} \times \mathcal{J} = \emptyset$ and $\mathcal{C}^{\mathcal{I}} = \mathcal{C}^{\mathcal{J}} = \mathcal{C}^{\mathcal{I}\times\mathcal{J}} = \mathcal{C}^{\emptyset}$ contains a single functor that is the empty map on the objects and morphisms. Hence, limits and colimits correspond to terminal and initial objects in $\mathcal{C}$ by Example 2.1.7.

Thus, $\mathrm{colim}_{\mathcal{I}} \lim_{\mathcal{J}} F$ is an initial and $\lim_{\mathcal{J}} \mathrm{colim}_{\mathcal{I}} F$ a terminal object in $\mathcal{C}$. The morphism $g : \mathrm{colim}_{\mathcal{I}} \lim_{\mathcal{J}} F \to \lim_{\mathcal{J}} \mathrm{colim}_{\mathcal{I}} F$ the unique morphism from the initial to the terminal object. This is an isomorphism if and only if $\mathcal{C}$ has a zero object.

This example shows that the morphism in Lemma 2.3.5 need not be an isomorphism, not even for $\mathcal{C} = \mathrm{Set}$ or $\mathcal{C} = \mathrm{Top}$. Sufficient conditions on $\mathcal{I}$ and $\mathcal{J}$ that guarantee that it is an isomorphism for any functor $F : \mathcal{I} \times \mathcal{J} \to \mathrm{Set}$ are that $\mathcal{I}$ is finite and $\mathcal{J}$ is *filtered*, see for instance [Rh, Theorem 3.8.9] and [Rc, Theorem 3.5.6].

2.4 Transformations of (co)limits under functors

In this section, we investigate how (co)limits behave under functors and derive sufficient criteria to guarantee that a functor sends (co)limits to (co)limits. As cones over or under a diagram $F : \mathcal{J} \to \mathcal{C}$ are natural transformations, applying a functor $F : \mathcal{C} \to \mathcal{D}$ to them yields cones over or under the image $GF : \mathcal{J} \to \mathcal{D}$ of this diagram. There are now several ways to logically relate the statements that either the original cone or its image is a (co)limit.

Definition 2.4.1: Let $G : \mathcal{C} \to \mathcal{D}$ be a functor and I a family of diagrams in $\mathcal{C}$.

1. **preserves (co)limits** for I, if for any diagram $F \in I$ and (co)limit cone λ of F, the image $G\lambda$ is a (co)limit cone of GF.
2. **reflects (co)limits** for I, if any (co)cone λ for $F \in I$, whose image $G\lambda$ is a (co)limit cone of GF, is a (co)limit cone of F.
3. **creates (co)limits** for I, if it reflects them and for any $F \in I$ whose image GF has a (co)limit, it has a (co)limit cone of the form $G\lambda$ with a (co)limit cone λ of F.

Remark 2.4.2: (Exercise)

1. The condition that G reflects limits 3. is necessary to ensure that creation of (co)limits implies reflection of (co)limits. Show with an example that the second condition in 3. alone does not guarantee this.
2. If $G : \mathcal{C} \to \mathcal{D}$ creates (co)limits for a family I of diagrams in $\mathcal{C}$ and their images in $\mathcal{D}$ have (co)limits, then the diagrams in I have (co)limits, and G preserves them.

Families of diagrams that are of special interest in Definition 2.4.1 are the families of all small diagrams and of all finite diagrams. Whenever one deals with abelian categories, for instance in homological algebra or in representation theory, the statement that a functor preserves all *finite* limits or colimits has important consequences. The following definition is often stated only for abelian categories, but makes sense in more generality.

Definition 2.4.3: A functor $G : \mathcal{C} \to \mathcal{D}$ is called **left exact**, if it preserves all finite limits, and **right exact**, if it preserves all finite colimits in $\mathcal{C}$.

Example 2.4.4:

1. Any fully faithful functor $G : \mathcal{C} \to \mathcal{D}$ reflects all (co)limits in $\mathcal{D}$.
2. Any equivalence of categories $G : \mathcal{C} \to \mathcal{D}$ preserves and creates all (co)limits in $\mathcal{C}$ or $\mathcal{D}$.
3. A functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories is left (right) exact, if and only if it preserves finite direct sums and (co)kernels. This follows from the proof of Theorem 2.2.2, Corollary 2.2.3 and Example 2.1.11, 3.

It turns out that there are many functors that are right or left exact or even preserve all (co)limits, not just the rather obvious functors in Example 2.4.4. The first important example are Hom functors. Covariant and contravariant Hom functors, respectively, send all limits or colimits that exist in a given category $\mathcal{C}$ to limits in Set.

Proposition 2.4.5: Let $\mathcal{C}$ be a category and $C \in \mathrm{Ob}\mathcal{C}$.

1. The functor $\mathrm{Hom}(C, -) : \mathcal{C} \to \mathrm{Set}$ preserves all limits that exist in $\mathcal{C}$.
2. The functor $\mathrm{Hom}(-, C) : \mathcal{C}^{op} \to \mathrm{Set}$ sends all colimits that exist in $\mathcal{C}$ to limits in Set.

Proof:
We prove the claim for limits. The claim for colimits follows with Remark 2.1.5, 3. if one takes into account that that $\mathrm{Hom}_{\mathcal{C}}(-, C) \cong \mathrm{Hom}_{\mathcal{C}^{op}}(C, -) : \mathcal{C}^{op} \to \mathrm{Set}$.

Let $F : \mathcal{J} \to \mathcal{C}$ be a diagram in $\mathcal{C}$ with a limit cone $\lambda : \Delta(\lim F) \Rightarrow F$ and $C \in \mathrm{Ob}\mathcal{C}$. We show that the natural transformation $\mathrm{Hom}(C, \lambda) : \Delta(\mathrm{Hom}(C, \lim F)) \Rightarrow \mathrm{Hom}(C, F(-))$ with components $\mathrm{Hom}(C, \lambda)_J : \mathrm{Hom}_{\mathcal{C}}(C, \lim F) \to \mathrm{Hom}_{\mathcal{C}}(C, F(J))$, $f \mapsto \lambda_J \circ f$ is a limit cone of $\mathrm{Hom}_{\mathcal{C}}(C, F(-)) : \mathcal{J} \to \mathrm{Set}$.

• A cone ρ over $\mathrm{Hom}_{\mathcal{C}}(C, F(-))$ with apex X is a collection of maps $\rho_J : X \to \mathrm{Hom}_{\mathcal{C}}(C, F(J))$ that satisfy $F(j) \circ \rho_J(x) = \rho_{J'}(x)$ for all $x \in X$ and morphisms $j : J \to J'$. This corresponds to a collection of cones $\rho^x : \Delta(C) \Rightarrow F$, indexed by X, with components $\rho^x_J = \rho_J(x) : C \to F(J)$.

• Cone morphisms from a cone ρ to $\mathrm{Hom}(C, \lambda)$ are maps $f : X \to \mathrm{Hom}_{\mathcal{C}}(C, \lim F)$ satisfying $\rho_J(x) = \lambda_J \circ f(x)$ for all $x \in X$ and $J \in \mathrm{Ob}_{\mathcal{J}}$. This amounts to a collection of morphisms $f^x = f(x) : C \to \lim F$ indexed by X with $\rho^x_J = \lambda_J \circ f^x$ and hence to a collection of cone morphisms f^x from ρ^x to λ_J.

By the universal property of λ, for each $x \in X$ there is a unique cone morphism $f^x : C \to \lim F$ with $\rho^x_J = \lambda_J \circ f^x$. This defines a unique cone morphism $f : X \to \mathrm{Hom}_{\mathcal{C}}(C, \lim F)$, $x \mapsto f^x$ from ρ to $\mathrm{Hom}(C, \lambda)$ and shows that $\mathrm{Hom}(C, \lambda)$ is a limit of $\mathrm{Hom}(C, F(-))$. □

In fact, the proof of Proposition 2.4.5 shows more than the just the claim. Combining it with Proposition 2.3.2 by which (co)limits in functor categories are computed objectwise, we find that any limit cone $\lambda : \Delta(\lim F) \Rightarrow F$ in $\mathcal{C}$ yields a limit cone of the diagram in $\mathrm{Set}^{\mathcal{C}^{op}}$ that sends $J \in \mathrm{Ob}\mathcal{J}$ to the functor $\mathrm{Hom}(-, F(J)) : \mathcal{C}^{op} \to \mathrm{Set}$ and a morphism $j : J \to J'$ to the natural transformation $\mathrm{Hom}(-, F(j)) : \mathrm{Hom}(-, F(J)) \Rightarrow \mathrm{Hom}(-, F(J'))$.

We can state this more succinctly in terms of the co- and contravariant **Yoneda embedding**

$$y : \mathcal{C} \to \mathrm{Set}^{\mathcal{C}^{op}} \qquad \begin{array}{ccc} C & \longmapsto & \mathrm{Hom}(-, C) \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}(-,c)} \\ C' & \longmapsto & \mathrm{Hom}(-, C') \end{array} \qquad y : \mathcal{C}^{op} \to \mathrm{Set}^{\mathcal{C}} \qquad \begin{array}{ccc} C & \longmapsto & \mathrm{Hom}(C, -) \\ {\scriptstyle c}\downarrow & & \uparrow{\scriptstyle \mathrm{Hom}(c,-)} \\ C' & \longmapsto & \mathrm{Hom}(C', -). \end{array} \tag{12}$$

The Yoneda embeddings are fully faithful by the Yoneda Lemma and hence reflect all (co)limits by Example 2.4.4, 1. Combining the proof of Proposition 2.4.5 with Proposition 2.3.2 then shows that the Yoneda embeddings preserve and reflect limits.

Corollary 2.4.6:

1. The covariant Yoneda embedding $y : \mathcal{C} \to \mathrm{Set}^{\mathcal{C}^{op}}$ preserves and reflects limits.
2. The contravariant Yoneda embedding $y : \mathcal{C}^{op} \to \mathrm{Set}^{\mathcal{C}}$ preserves and reflects limits.

Another important class of functors that preserve either limits or colimits are right and left adjoints. Preserving both, limits and colimits, is a much stronger condition, satisfied by all functors that have a left *and* a right adjoint. Roughly speaking, the data of an adjunction allows one to transport cones and cone morphisms between the two categories. This can be used to show that the image of a (co)limit cone remains terminal (initial).

Theorem 2.4.7: Right adjoint functors preserve limits and left adjoint functors colimits.

Proof:
Let $L : \mathcal{C} \to \mathcal{D}$ be left adjoint to $R : \mathcal{D} \to \mathcal{C}$ with unit $\eta : \mathrm{id}_{\mathcal{D}} \Rightarrow RL$ and counit $\epsilon : LR \Rightarrow \mathrm{id}_{\mathcal{C}}$ satisfying (i) $\epsilon L \circ L\eta = \mathrm{id}_L$ and (ii) $R\epsilon \circ \eta R = \mathrm{id}_R$.

We prove the claim for limits. The claim for colimits follows with Remark 2.1.5, 3.

Suppose $\lambda : \Delta(\lim G) \Rightarrow G$ is a limit of $G : \mathcal{J} \to \mathcal{D}$. Then $R\lambda : \Delta(R(\lim G)) \Rightarrow RG$ is a cone over RG. We show that it is terminal. Let $\mu : \Delta(C) \Rightarrow RG$ be another cone over RG. Then $\epsilon G \circ L\mu : \Delta(L(C)) \Rightarrow G$ is a cone over G and hence there is a unique morphism $g : L(C) \to \lim G$ with $\lambda \circ \Delta(g) = \epsilon G \circ L\mu$. The morphism $f = R(g) \circ \eta_C : C \to R(\lim G)$ satisfies

$$R\lambda \circ \Delta(f) = R\lambda \circ \Delta R(g) \circ \Delta(\eta_C) \overset{R\Delta=\Delta R}{=} R\epsilon G \circ RL\mu \circ \Delta(\eta_C) \overset{\text{nat }\eta}{=} R\epsilon G \circ \eta RG \circ \mu \overset{\text{(ii)}}{=} \mu.$$

If $f' : C \to R(\lim G)$ is another morphism with $R\lambda \circ \Delta(f') = \mu$, then one has

$$\lambda \circ \Delta(g) = \epsilon G \circ L\mu = \epsilon G \circ LR\lambda \circ \Delta(L(f')) \overset{\text{nat }\epsilon}{=} \lambda \circ \Delta(\epsilon_{\lim G} \circ L(f')).$$

As g is unique by the universal property of the limit, this implies $g = \epsilon_{\lim G} \circ L(f')$ and

$$f = R(g) \circ \eta_C = R(\epsilon_{\lim G}) \circ RL(f') \circ \eta_C \overset{\text{nat }\eta}{=} R(\epsilon_{\lim G}) \circ \eta_{R(\lim G)} \circ f' \overset{\text{(ii)}}{=} f'. \qquad \square$$

Example 2.4.8:

1. The forgetful functor V : Top $\to$ Set is left adjoint to the indiscrete topology functor R : Set $\to$ Top and right adjoint to the discrete topology functor L : Set $\to$ Top that assign to a set X the indiscrete and discrete topology on X (cf. Example 1.2.7, 2).

   Hence, V preserves all limits and colimits, the indiscrete topology limits and the discrete topology colimits. In particular:

   - the indiscrete topology on a product set $\Pi_{i\in I} X_i$ is the product topology for the indiscrete topologies on the sets X_i,
   - the indiscrete topology on a subset $U \subset X$ coincides with the subspace topology induced by the indiscrete topology on X,
   - the discrete topology on a disjoint union $\dot\cup_{i\in I} X_i$ is the sum topology of the discrete topologies on X_i,
   - the quotient topology on a quotient of a discrete topological space is the discrete topology on the quotient.

2. By Example 1.2.7, 1. the forgetful functor V : R-Mod $\to$ Set is right adjoint to the free generation functor $\langle\ \rangle_R$: Set $\to$ R-Mod, that assigns

   - to a set X the free R-module $\langle X\rangle_R$ generated by X,
   - to a map $f : X \to X'$ the unique R-linear map $\langle f\rangle : \langle X\rangle_R \to \langle X'\rangle_R$, $x \mapsto f(x)$,

   Hence, V preserves limits and $\langle\ \rangle_R$ colimits. In particular, one has $\langle X\dot\cup Y\rangle_R \cong \langle X\rangle_R \oplus \langle Y\rangle_R$.

3. The forgetful functor V : Grp $\to$ Set is right adjoint to the free generation functor $\star$: Set $\to$ Grp that assigns

   - to a set X the free group $F(X) = \star_X \mathbb{Z}$ generated by X,
   - to a map $f : X \to X'$ the unique group homomorphism $F(f) : \star_X\mathbb{Z} \to \star_{X'}\mathbb{Z}$ with $F(f) \circ \iota_x = \iota_{f(x)}$ for all $x \in X$.

   Hence, V preserves limits and $\star$ colimits.

4. The forgetful functor V : H-Set $\to$ Set for a group H is left adjoint to the functor $G = \mathrm{Hom}_{\mathrm{Set}}(H, -)$: Set $\to$ H-Set, that assigns

   - to a set X the set of maps $f : H \to X$ with $(h \rhd f)(k) = f(kh)$,
   - to a map $g : X \to X'$ the H-linear map

   $$\mathrm{Hom}_{\mathrm{Set}}(H, f) : \mathrm{Hom}_{\mathrm{Set}}(H, X) \to \mathrm{Hom}_{\mathrm{Set}}(H, X'), \quad f \mapsto g \circ f.$$

   Hence, V preserves colimits and G limits.

5. By Example 1.2.7, 6. for any R-right module M, the functor $F := M\otimes_R - : R\text{-Mod} \to \text{Ab}$ is left adjoint to $G := \text{Hom}_{\text{Ab}}(M, -) : \text{Ab} \to R\text{-Mod}$. Thus, F is right and G left exact.

   The unit and counit of the adjunction are given by $\eta_N : N \to \text{Hom}_{\text{Ab}}(M, M\otimes_R N)$, $n \mapsto f_n$ with $f_n(m) = m\otimes n$ and $\epsilon_A : M\otimes_R \text{Hom}_{\text{Ab}}(M, A) \to A$, $m\otimes f \mapsto f(m)$.

6. Left (right) adjoint functors are right (left) exact. In particular, this holds for left (right) adjoint functors between abelian categories.

The following example is important in representation theory, where one often considers the inclusion homomorphism $\phi : \mathbb{F}[U] \to \mathbb{F}[H]$ for the group algebra of a subgroup $U \subset H$. This inclusion homomorphism allows one to *restrict* any representation of H to a representation of the subgroup U. Constructing a representation of H out of a representation of U is less obvious. It can be achieved by considering *induced and coinduced representations*. We consider this example in more generality, with rings instead of the group algebras $\mathbb{F}[U]$ and $\mathbb{F}[H]$ and a general ring homomorphism instead of the inclusion.

Example 2.4.9: (**induction** and **coinduction**, cf. Example 1.2.7, 7)
A ring homomorphism $\phi : R \to S$ induces a functor $\text{Res}_\phi : S\text{-Mod} \to R\text{-Mod}$ that assigns

- to an S-module $(M, \rhd)$ the R-module $(M, \rhd_\phi)$ with $r \rhd_\phi m = \phi(r) \rhd m$,
- to an S-linear map $f : (M, \rhd) \to (M', \rhd')$ the R-linear map $f : (M, \rhd_\phi) \to (M', \rhd'_\phi)$.

It is right adjoint to the **induction functor** $\text{Ind}_\phi = S\otimes_R - : R\text{-Mod} \to S\text{-Mod}$ that assigns

- to an R-module M the S-module $S\otimes_R M$ with $s \rhd (s'\otimes m) = ss'\otimes m$ and the R-right module structure on S given by $s \lhd r = s\phi(r)$,
- to an R-linear map $f : M \to M'$ the S-linear map $\text{id}_S\otimes f : S\otimes_R M \to S\otimes_R M'$.

and left adjoint to the **coinduction functor** $\text{Coind}_\phi = \text{Hom}_R(S, -) : R\text{-Mod} \to S\text{-Mod}$ that

- sends an R-module M to the S-module of R-linear maps $f : S \to M$ with $(s\rhd f)(t) = f(ts)$ and the R-module structure on S is given by $r \rhd s = \phi(r)s$,
- an R-linear map $f : M \to M'$ to the S-linear map

$$\text{Hom}_R(S, f) : \text{Hom}_R(S, M) \to \text{Hom}_R(S, M'), \quad g \mapsto f \circ g.$$

Hence Res_ϕ preserves limits and colimits and is left and right exact. The functor Ind_ϕ preserves colimits and is right exact, the functor Coind_ϕ preserves limits and is left exact.

Inspired by Example 2.4.9, one might ask under which conditions it is possible to compute limits and colimits in a subcategory $\mathcal{U} \subset \mathcal{C}$ from limits and colimits in $\mathcal{C}$. It is clear that one should at least require that the subcategory $\mathcal{U} \subset \mathcal{C}$ is a *full* subcategory to have control over its morphisms. However, even in this case it is in general not guaranteed that (co)limit cones in $\mathcal{C}$ define (co)limit cones in $\mathcal{U}$. It holds, if the inclusion functor $\iota : \mathcal{U} \to \mathcal{C}$ has a left adjoint.

Definition 2.4.10: A **reflective subcategory** is a full subcategory $\mathcal{U}$ of $\mathcal{C}$ such that the inclusion functor $\iota : \mathcal{U} \hookrightarrow \mathcal{C}$ has a left adjoint $L : \mathcal{C} \to \mathcal{U}$, the **reflector** or **localisation**.

Example 2.4.11:

1. The category Ab is a reflective subcategory of Grp.

   The reflector is the abelisation functor $L : \mathrm{Grp} \to \mathrm{Ab}$ that sends
   - a group G to its abelisation $L(G) = G/[G, G]$,
   - each group homomorphism $f : G \to G'$ to the induced group homomorphism $L(f) : G/[G, G] \to G'/[G', G']$.

2. The category $\mathrm{CAlg}_{\mathbb{F}}$ of commutative algebras over $\mathbb{F}$ and algebra homomorphisms between them is a reflective subcategory of $\mathrm{Alg}_{\mathbb{F}}$.

   The reflector $L : \mathrm{Alg}_{\mathbb{F}} \to \mathrm{CAlg}_{\mathbb{F}}$ sends
   - an $\mathbb{F}$-algebra A to its symmetrisation $S(A) = A/[A, A]$, its quotient by the two-sided ideal generated the commutators $[a, b] = ab - ba$ for all $a, b \in A$,
   - an algebra homomorphism $f : A \to A'$ to the induced algebra homomorphism $L(f) : A/[A, A] \to A'/[A', A']$.

3. The category Field is a reflective subcategory of the category Int of integral domains and injective ring homomorphisms between them.

   The reflector $L : \mathrm{Int} \to \mathrm{Field}$ assigns
   - to each integral domain I its field of fractions $Q(I)$,
   - to each injective ring homomorphism $f : I \to I'$ the induced field homomorphism $L(f) : Q(I) \to Q(I')$.

4. The category cHaus of compact Hausdorff spaces and continuous maps between them is a reflective subcategory of Top.

   The reflector $L : \mathrm{Top} \to \mathrm{cHaus}$ assigns
   - to a topological space X its **Stone-Čech compactification** $\check{X}$,
   - to each continuous map $f : X \to X'$ the induced map $f : \check{X} \to \check{X}'$.

We now show that reflective subcategories are well-behaved with respect to (co)limits. Whenever a diagram in $\mathcal{U}$ has a (co)limit as a diagram in $\mathcal{C}$, then it has a (co)limit as a diagram in $\mathcal{U}$, and this (co)limit is obtained by applying the reflector and the counit of the adjunction. Hence, the (co)limit of a diagram of abelian groups in Ab coincides with its (co)limit in Grp. Analogous statements hold for commutative algebras and algebras, for fields and integral domains and for compact Hausdorff spaces and topological spaces.

Proposition 2.4.12: Let $\mathcal{U}$ be a reflective subcategory of $\mathcal{C}$ with inclusion functor $\iota : \mathcal{U} \hookrightarrow \mathcal{C}$ and left adjoint $L : \mathcal{C} \to \mathcal{U}$. Then:

1. The counit of the adjunction is a natural isomorphism $\epsilon : L\iota \Rightarrow \mathrm{id}_{\mathcal{U}}$.
2. $\mathcal{U}$ has all colimits that exist in $\mathcal{C}$:
   if $F : \mathcal{J} \to \mathcal{U}$ is a diagram and $\iota F : \mathcal{J} \to \mathcal{C}$ has a colimit $\lambda : \iota F \Rightarrow \Delta(C)$, then $L\lambda \circ \epsilon^{-1}F : F \Rightarrow \Delta(L(C))$ is a colimit of F.
3. The inclusion functor $\iota : \mathcal{U} \hookrightarrow \mathcal{C}$ creates all limits that exist in $\mathcal{C}$:
   if $F : \mathcal{J} \to \mathcal{U}$ is a diagram and $\iota F : \mathcal{J} \to \mathcal{C}$ has a limit $\lambda : \Delta(C) \Rightarrow \iota F$, then the cone $\mu := \epsilon F \circ L\lambda : \Delta(L(C)) \Rightarrow F$ is a limit of F and $\iota\mu$ a limit of ιF.

In particular, if $\mathcal{C}$ is (co)complete, so is $\mathcal{U}$.

Proof:
Denote by $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow \iota L$ and $\epsilon : L\iota \Rightarrow \mathrm{id}_{\mathcal{U}}$ the unit and counit of the adjunction with (i) $\iota\epsilon \circ \eta\iota = \mathrm{id}_\iota$ and (ii) $\epsilon L \circ L\eta = \mathrm{id}_L$.

1. As $\mathcal{U}$ is a full subcategory, the embedding functor ι is full and faithful and induces isomorphisms $\mathrm{Hom}_{\mathcal{U}}(U,U') \cong \mathrm{Hom}_{\mathcal{C}}(U,U')$ for all $U,U' \in \mathrm{Ob}\,\mathcal{U}$. Thus, for each $U \in \mathrm{Ob}\,\mathcal{U}$, the morphism $\eta_U : U \to L(U)$ is in $\mathcal{U}$. This implies

$$\epsilon_U \circ \eta_U \overset{\text{(i)}}{=} 1_U \qquad\qquad \eta_U \circ \epsilon_U \overset{\text{nat }\epsilon}{=} \epsilon_{L(U)} \circ L(\eta_U) \overset{\text{(ii)}}{=} 1_{L(U)}.$$

This shows that all component morphisms of ϵ are isomorphisms, and $\epsilon : L\iota \Rightarrow \mathrm{id}_{\mathcal{U}}$ is a natural isomorphism. In particular, this implies that L is full.

2. Let $F : \mathcal{J} \to \mathcal{U}$ be a diagram such that $\iota F : \mathcal{J} \to \mathcal{C}$ has a colimit $\lambda : \iota F \Rightarrow \Delta(C)$. Then $L\lambda : L\iota F \Rightarrow \Delta(L(C))$ is a colimit of $L\iota F$, since left adjoint preserve colimits by Theorem 2.4.7. Pre-composing with the natural isomorphism $\epsilon^{-1}F : F \Rightarrow L\iota F$ then yields a colimit $L\lambda \circ \epsilon^{-1}F : F \Rightarrow \Delta(L(C))$ of F.

3. Suppose $F : \mathcal{J} \to \mathcal{U}$ is a diagram and $\lambda : \Delta(C) \Rightarrow \iota F$ a limit cone of ιF. Then the natural transformation $\mu := \epsilon F \circ L\lambda : \Delta(L(C)) \Rightarrow F$ is a cone over F. We show that it is a limit cone.

(a) We construct a cone isomorphism from $\iota\mu$ to λ:
As $\iota\mu : \Delta(L(C)) \Rightarrow \iota F$ is a cone over ιF, there is a unique morphism $\phi : L(C) \to C$ in $\mathcal{C}$ with $\lambda \circ \Delta(\phi) = \iota\mu$. We show that ϕ is an isomorphism:

• As we have $\iota\mu \circ \Delta(\eta_C) = \iota\epsilon F \circ \iota L\lambda \circ \Delta(\eta_C) = \iota\epsilon F \circ \eta\iota F \circ \lambda = \lambda$ by naturality of η and by (i), it follows that $\lambda \circ \Delta(\phi \circ \eta_C) = \lambda$, and the universal property of λ implies $\phi \circ \eta_C = 1_C$.

• Because ι and L are full, there is a morphism $f : C \to C$ in $\mathcal{C}$ with $\iota L(f) = \eta_C \circ \phi$, and this implies $\eta_C = \eta_C \circ 1_C = \eta_C \circ \phi \circ \eta_C = \iota L(f) \circ \eta_C = \eta_C \circ f$ by naturality of η. This yields $f = \phi \circ \eta_C \circ f = \phi \circ \eta_C = 1_C$. Hence, $\phi : L(C) \to C$ is an isomorphism with inverse $\eta_C : C \to L(C)$ and a cone isomorphism from $\iota\mu$ to λ. It follows that $\iota\mu$ is a limit of ιF.

(b) If $\rho : \Delta(U) \Rightarrow F$ is a cone over F, then $\iota\rho : \Delta(U) \Rightarrow \iota F$ is a cone over ιF, and hence there is a unique morphism $\psi : U \to C$ with $\lambda \circ \Delta(\psi) = \iota\rho$. This implies by naturality of ϵ^{-1}

$$\mu \circ \Delta(L\psi \circ \epsilon_U^{-1}) = \epsilon F \circ L(\lambda \circ \Delta(\psi)) \circ \Delta(\epsilon_U^{-1}) = \epsilon F \circ L\iota\rho \circ \Delta(\epsilon_U^{-1}) = \epsilon F \circ \epsilon^{-1} F \circ \rho = \rho.$$

Suppose now that $\sigma, \tau : U \to L(C)$ are morphisms in $\mathcal{U}$ with $\mu \circ \Delta(\sigma) = \mu \circ \Delta(\tau)$. As $\lambda \circ \Delta(\phi) = \iota\mu$ by (a), this implies $\lambda \circ \Delta(\phi \circ \sigma) = \lambda \circ \Delta(\phi \circ \tau)$ and by the universal property of λ then $\phi \circ \sigma = \phi \circ \tau$. As ϕ is an isomorphism, this yields $\tau = \sigma$. Hence, $\mu : \Delta(L(C)) \Rightarrow F$ is a limit cone of F, such that $\iota\mu : \Delta(L(C)) \Rightarrow \iota F$ is a limit cone of ιF. □

Proposition 2.4.12 allows one to draw conclusions about the (co)completeness of reflective subcategories without explicitly computing their equalisers, coequalisers, products and coproducts. An important example is the category Cat of small categories and functors between them and its full subcategory Grpd of groupoids. Coequalisers in these categories are difficult to describe, as quotients need to be taken in a coherent way for both, objects and morphisms.

Remark 2.4.13: We will show in Section 5, Corollary 5.4.7, that Cat is a reflective subcategory of a functor category $\mathrm{Set}^{\mathcal{C}}$ for a small category $\mathcal{C}$. Propositions 2.3.2 and 2.4.12 then imply that Cat is bicomplete.

References:

- Chapters III and IV in Mac Lane, S. (2013) Categories for the working mathematician,
- Chapter 3 in Richter, B. (2020) From categories to homotopy theory,
- Chapters 3 and 4.5 in Riehl, E. (2017) Category theory in context.

3 Kan extensions

3.1 Kan extensions: definitions and examples

In the last section we *approximated* functors by *constant functors*, which lead to the concepts of *limits* and *colimits* as their closest categorical approximations from above and below. This allowed us to recover familiar concepts from calculus and topology such as infima, suprema and limits of sequences but also included many more advanced constructions from algebra and topology, such as pullbacks and pushouts, CW complexes and direct and inverse limits.

In this section, we *extend* functors, that is, change their domain from one category to another. Extending a function $f : C \to E$ usually means finding another function $f' : D \to E$ with $C \subset D$ and $f'|_C = f$ or, equivalently, $f' \circ \iota = f$ for the inclusion map $\iota : C \to D$. The second condition generalises to arbitrary maps $k : C \to D$. We can define an extension of f along k as a function $f' : D \to E$ with $f' \circ k = f$.

If we replace the maps $f : C \to E$ and $k : C \to D$ by functors $F : \mathcal{C} \to \mathcal{E}$ and $K : \mathcal{C} \to \mathcal{D}$, imposing that there is a functor $F' : \mathcal{D} \to \mathcal{E}$ with $F'K = F$ or even $F'K \cong F$ is too restrictive and destroys interesting examples. Instead, we should require either that there is a natural transformation $\eta : F \Rightarrow F'K$ or that there is a natural transformation $\epsilon : F'K \Rightarrow F$. Just as in the case of cocones and cones, we can view the pairs (F', η) and (F', ϵ) as categorical approximations of the functor F from below and above, respectively.

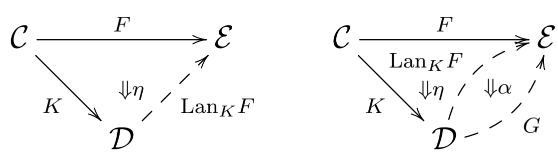
Just as in the definition of (co)limits we then require that these approximations are as close as possible in a categorical sense. In case of an approximation (F, η) from below, *as close as possible* means that for any other approximation (G, γ), consisting of a functor $G : \mathcal{D} \to \mathcal{E}$ and a natural transformation $\gamma : F \Rightarrow GK$, there is a unique natural transformation $\alpha : F' \Rightarrow G$ that relates η and γ. In case of an approximation (F, ϵ) from above, it means that for any other approximation (G, δ), by functor $G : \mathcal{D} \to \mathcal{E}$ and a natural transformation $\delta : GK \Rightarrow F$, there is a unique natural transformation $\beta : G \Rightarrow F'$ that relates ϵ and δ.

This yields the concept of a left and right Kan extension, with left corresponding to approximations from below and right corresponding to approximations from above. As we will see in the following, Kan extensions are a very fundamental concept of category theory that encompasses, among others, all limits and colimits, adjunctions and the Yoneda lemma. As stated by Saunders MacLane: *“The notion of Kan extensions subsumes all the other fundamental concepts of category theory”* or, more briefly: *All concepts are Kan extensions.*

Definition 3.1.1: Let $F : \mathcal{C} \to \mathcal{E}$ and $K : \mathcal{C} \to \mathcal{D}$ be functors.

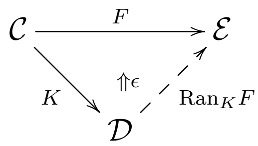
1. A **left Kan extension** of F along K is a functor $\mathrm{Lan}_K F : \mathcal{D} \to \mathcal{E}$ together with a natural transformation $\eta : F \Rightarrow (\mathrm{Lan}_K F)K$ that has the following universal property:

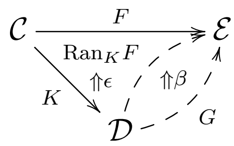
   For every pair (G, γ) of a functor $G : \mathcal{D} \to \mathcal{E}$ and a natural transformation $\gamma : F \Rightarrow GK$, there is a unique natural transformation $\alpha : \mathrm{Lan}_K F \Rightarrow G$ such that $\gamma = (\alpha K) \circ \eta$.



2. A **right Kan extension** of F along K is a functor $\mathrm{Ran}_K F : \mathcal{D} \to \mathcal{E}$ together with a natural transformation $\epsilon : (\mathrm{Ran}_K F)K \Rightarrow F$ that has the following universal property:

   For every pair (G, δ) of a functor $G : \mathcal{D} \to \mathcal{E}$ and a natural transformation $\delta : GK \Rightarrow F$, there is a unique natural transformation $\beta : G \Rightarrow \mathrm{Ran}_K F$ such that $\delta = \epsilon \circ (\beta K)$.





Remark 3.1.2:

1. As they are defined by universal properties, left and right Kan extensions are unique up to unique isomorphisms: if $(\mathrm{Lan}_K F, \eta)$ and $(\mathrm{Lan}'_K F, \eta')$ are left Kan extensions of F along K, there are unique natural transformations $\alpha : \mathrm{Lan}_K F \Rightarrow \mathrm{Lan}'_K F$ with $(\alpha K) \circ \eta = \eta'$ and $\alpha' : \mathrm{Lan}'_K F \Rightarrow \mathrm{Lan}_K F$ with $(\alpha' K) \circ \eta' = \eta$, and α' is the inverse of α.

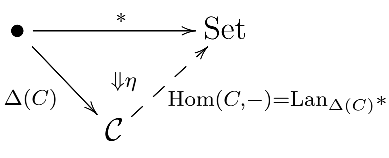
2. Right (left) Kan extensions are left (right) Kan extensions in opposite categories.

   Every functor $F : \mathcal{C} \to \mathcal{D}$ corresponds to a unique functor $F' : \mathcal{C}^{op} \to \mathcal{D}^{op}$ and every natural transformation $\mu : F \Rightarrow G$ to a natural transformation $\mu' : G' \Rightarrow F'$. Thus, a left (right) Kan extension of $F : \mathcal{C} \to \mathcal{E}$ along $K : \mathcal{C} \to \mathcal{D}$ defines a right (left) Kan extension of $F' : \mathcal{C}^{op} \to \mathcal{E}^{op}$ along $K' : \mathcal{C}^{op} \to \mathcal{D}^{op}$.

It should be noted that Kan extensions are categorical extensions and do not generalise extensions of functions in a naive sense. Many claims that one would expect to hold naively are false in general. Even if $K : \mathcal{C} \to \mathcal{D}$ is fully faithful, for instance a categorical inclusion of a full subcategory, the natural transformations of a left or right Kan extension need not be natural isomorphisms. We will derive a sufficient condition for this to hold in Corollary 3.2.11. The next example shows that a Kan extension of a constant functor along a constant functor is not necessarily given by a constant functor.

Example 3.1.3: (Hom-functors are Kan extensions)
Let $\bullet$ be the category with a single object and morphism and $* : \bullet \to \mathrm{Set}$ the functor that sends the object to the singleton set $\{\bullet\}$.

Then a left Kan extension of $*$ along $\Delta(C) : \bullet \to \mathcal{C}$ is given by $\mathrm{Hom}(C, -) : \mathcal{C} \to \mathrm{Set}$ and the map $\eta : \{\bullet\} \to \mathrm{Hom}_{\mathcal{C}}(C, C)$, $\bullet \mapsto 1_C$

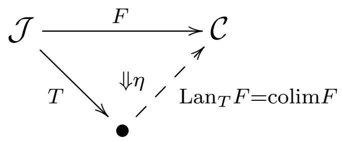


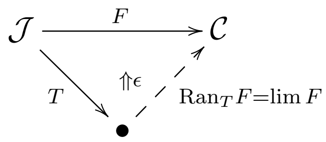
A natural transformation $\gamma : * \Rightarrow G\Delta(C)$ for a functor $G : \mathcal{C} \to \mathrm{Set}$ corresponds to a map $\gamma : \{\bullet\} \to G(C)$ and hence to the choice of an element $x = \gamma(\bullet) \in G(C)$.

By the Yoneda lemma a natural transformation $\alpha : \mathrm{Hom}(C, -) \Rightarrow G$ is given by $\alpha_{C'}(f) = G(f) \circ \alpha_C(1_C)$ for all morphisms $f : C \to C'$ and hence determined by $\alpha_C(1_C)$.

The condition $\gamma = (\alpha\Delta(C)) \circ \eta$ implies $x = \gamma(\bullet) = \alpha_C(\eta(\bullet)) = \alpha_C(1_C)$. This shows that $\mathrm{Hom}(C, -)$ has the universal property of the left Kan extension.

Example 3.1.4: (Limits and Colimits are Kan extensions)
Let $\mathcal{J}$, $\mathcal{C}$ be categories and $T : \mathcal{J} \to \bullet$ the terminal functor. A left (right) Kan extension of a functor $F : \mathcal{J} \to \mathcal{C}$ along T is a colimit (limit) of F:



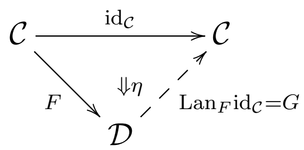


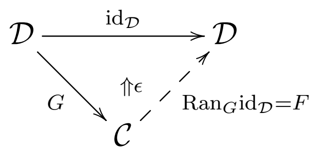
Proof:
1. A pair (G, γ) of a functor $G : \bullet \to \mathcal{C}$ and a natural transformation $\gamma : F \Rightarrow GT$ is a cocone $\gamma : F \Rightarrow \Delta(G(\bullet))$ over F. Thus $(\mathrm{Lan}_T F, \eta)$ is a cocone $\eta : F \Rightarrow \Delta(C)$ with nadir $C = \mathrm{Lan}_T F(\bullet)$. A natural transformation $\alpha : \mathrm{Lan}_T F \Rightarrow G$ with $(\alpha T) \circ \eta = \gamma$ is a cocone morphism. The universal property of the left Kan extension states that $(\mathrm{Lan}_T F, \eta)$ is an initial cocone, a colimit of F.

2. A pair (G, δ) of a functor $G : \bullet \to \mathcal{C}$ and a natural transformation $\delta : GT \Rightarrow F$ is a cone $\gamma : \Delta(G(\bullet)) \Rightarrow F$ over F. Thus $(\mathrm{Ran}_T F, \epsilon)$ is a cone $\epsilon : \Delta(C) \Rightarrow F$ with apex $C = \mathrm{Ran}_T F$. A natural transformation $\beta : G \Rightarrow \mathrm{Ran}_T F$ with $\epsilon \circ (\beta T) = \delta$ is a cone morphism. The universal property of the right Kan extension states that ϵ is a terminal cone, a limit of F. □

Other interesting examples of Kan extensions that could be expected to be trivial at first sight are Kan extensions involving identity functors. The left and right Kan extension of a functor F *along* an identity functor is indeed trivial. It is given by F and the identity natural transformation on F. In contrast, extensions *of* identity functors arise from adjunctions.

Example 3.1.5: (**Adjunctions are Kan extensions**)
Let $F : \mathcal{C} \to \mathcal{D}$ be left adjoint to $G : \mathcal{D} \to \mathcal{C}$ with unit $\eta : \mathrm{id}_\mathcal{C} \Rightarrow GF$ and counit $\epsilon : FG \Rightarrow \mathrm{id}_\mathcal{D}$. Then (G, η) is a left Kan extension of $\mathrm{id}_\mathcal{C}$ along F and (F, ϵ) a right Kan extension of $\mathrm{id}_\mathcal{D}$ along G.





Proof:
1. For a pair (H, γ) of a functor $H : \mathcal{D} \to \mathcal{C}$ and a natural transformation $\gamma : \mathrm{id}_\mathcal{C} \Rightarrow HF$ the unique natural transformation $\alpha : G \Rightarrow H$ with $(\alpha F) \circ \eta = \gamma$ is $\alpha = H\epsilon \circ \gamma G$.

The defining relations for the adjunction (i) $G\epsilon \circ \eta G = \mathrm{id}_G$ and (ii) $\epsilon F \circ F\eta = \mathrm{id}_F$ imply for all natural transformations $\alpha : G \Rightarrow H$ with $(\alpha F) \circ \eta = \gamma$ and $D \in \mathrm{Ob}\mathcal{D}$

$$H(\epsilon_D) \circ \gamma_{G(D)} = H(\epsilon_D) \circ \alpha_{FG(D)} \circ \eta_{G(D)} \overset{\text{nat } \alpha}{=} \alpha_D \circ G(\epsilon_D) \circ \eta_{G(D)} \overset{\text{(i)}}{=} \alpha_D.$$

That $\alpha = H\epsilon \circ \gamma G$ indeed satisfies the condition follows, because for all $C \in \mathrm{Ob}\mathcal{C}$

$$(\alpha F)_C \circ \eta_C = H(\epsilon_{F(C)}) \circ \gamma_{GF(C)} \circ \eta_C \overset{\text{nat } \gamma}{=} H(\epsilon_{F(C)}) \circ HF(\eta_C) \circ \gamma_C = H(\epsilon F \circ F\eta)_C \circ \gamma_C \overset{\text{(ii)}}{=} \gamma_C.$$

This shows that (G, η) has the universal property of the left Kan extension.

2. For a pair (H, δ) of a functor $H : \mathcal{C} \to \mathcal{D}$ and a natural transformation $\delta : HG \Rightarrow \mathrm{id}_\mathcal{C}$ the unique natural transformation $\beta : H \Rightarrow F$ with $\epsilon \circ (\beta G) = \delta$ is $\beta = \delta F \circ H\eta$. This follows analogously to 1. and shows that (F, ϵ) is a right Kan extension. □

The preceding examples show that even Kan extensions along functors or of functors that seem rather trivial give rise to important categorical concepts. However, the main goal is not to rediscover known concepts, but to apply Kan extensions to other mathematical problems.

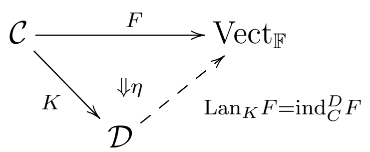
A fairly typical example of the latter are induced and coinduced representations. They answer the question how to canonically extend a representation of a subgroup $C \subset D$ to a representation of the full group D, which is not obvious, if attempted naively. The notion of a Kan extension not only allows one to solve this problem, but also to generalise its solution to sets or other objects with group actions instead of representations (cf. Example 2.3.1).

Example 3.1.6: (**Induced and coinduced representations are Kan extensions**)
Let $\mathcal{E} = \mathrm{Vect}_{\mathbb{F}}$, C, D groups and $\mathcal{C} = BC$, $\mathcal{D} = BD$ the associated categories with a single object $\bullet$ and group elements as morphisms. Then:

- functors $F : \mathcal{C} \to \mathcal{E}$ are representations $F : C \to \mathrm{Aut}_{\mathbb{F}}(V_F)$ on $V_F = F(\bullet)$,
- functors $G : \mathcal{D} \to \mathcal{E}$ are representations $G : D \to \mathrm{Aut}_{\mathbb{F}}(V_G)$ on $V_G = G(\bullet)$,
- functors $K : \mathcal{C} \to \mathcal{D}$ are group homomorphisms $K : C \to D$,
- natural transformations $\mu : F \Rightarrow F' : \mathcal{C} \to \mathrm{Vect}_{\mathbb{F}}$ are morphisms of C-representations,
- natural transformations $\nu : G \Rightarrow G' : \mathcal{D} \to \mathrm{Vect}_{\mathbb{F}}$ are morphisms of D-representations.

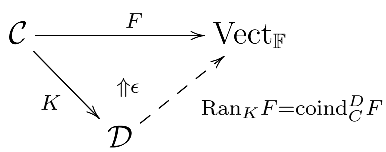
1. The left Kan extension of a functor $F : \mathcal{C} \to \mathrm{Vect}_{\mathbb{F}}$ along $K : \mathcal{C} \to \mathcal{D}$ is given by the induced representation from Example 2.4.9 (Exercise 23):

- the functor $\mathrm{Lan}_K F : \mathcal{D} \to \mathrm{Vect}_{\mathbb{F}}$ with
  - $\mathrm{Lan}_K F(\bullet) = \mathbb{F}[D] \otimes_{\mathbb{F}[C]} V_F$ for the right $\mathbb{F}[C]$-action $d \lhd c = dK(c)$ on $\mathbb{F}[D]$,
  - $\mathrm{Lan}_K F(d)(d' \otimes v) = (dd') \otimes v$,
- the morphism of representations $\eta : V_F \to \mathbb{F}[D] \otimes_{\mathbb{F}[C]} V_F$, $v \mapsto 1 \otimes v$.



2. The right Kan extension of $F : \mathcal{C} \to \mathrm{Vect}_{\mathbb{F}}$ along $K : \mathcal{C} \to \mathcal{D}$ is given by the coinduced representation from Example 2.4.9 (Exercise 23):

- the functor $\mathrm{Ran}_K F : \mathcal{D} \to \mathrm{Vect}_{\mathbb{F}}$ with
  - $\mathrm{Ran}_K F(\bullet) = \mathrm{Hom}_{\mathbb{F}[C]}(\mathbb{F}[D], V_F)$ for the $\mathbb{F}[C]$-action $c \rhd d = K(c)d$ on $\mathbb{F}[D]$,
  - $(\mathrm{Ran}_K F(d) f)(d') = f(d'd)$,
- the morphism of representations $\epsilon : \mathrm{Hom}_{\mathbb{F}[C]}(\mathbb{F}[D], V_F) \to V_F$, $f \mapsto f(1)$.



Induced and coinduced representations were already considered in Example 2.4.9 for general rings and ring homomorphisms. The (co)induced representations from Example 3.1.6 are a special case of Example 2.4.9 with group algebras $R = \mathbb{F}[C]$ and $S = \mathbb{F}[D]$ as rings and a ring homomorphism $\phi : R \to S$ induced by a group homomorphism $K : C \to D$. In this case, the categories R-Mod and S-Mod from Example 2.4.9 are functor categories $\mathrm{Vect}_{\mathbb{F}}^{BC}$ and $\mathrm{Vect}_{\mathbb{F}}^{BD}$.

Example 2.4.9 shows that the induced and coinduced representations define functors $\mathrm{Ind}_K, \mathrm{Coind}_K : \mathrm{Vect}_{\mathbb{F}}^{BC} \to \mathrm{Vect}_{\mathbb{F}}^{BD}$ that are left and right adjoint to the restriction functor $\mathrm{Res}_K : \mathrm{Vect}_{\mathbb{F}}^{BD} \to \mathrm{Vect}_{\mathbb{F}}^{BC}$. The latter is simply pre-composition with the group homomorphism K or, equivalently, with the functor $K : BC \to BD$.

One might ask if this pattern generalises to other functors $K : \mathcal{C} \to \mathcal{D}$ and categories $\mathcal{E}$:

- Do the left and right Kan extensions $\mathrm{Lan}_K F, \mathrm{Ran}_K F : \mathcal{D} \to \mathcal{E}$ of functors $F : \mathcal{C} \to \mathcal{E}$ organise into functors $\mathrm{Lan}_K, \mathrm{Ran}_K : \mathcal{E}^{\mathcal{C}} \to \mathcal{E}^{\mathcal{D}}$?
- If yes, are the functors $\mathrm{Lan}_K, \mathrm{Ran}_K : \mathcal{E}^{\mathcal{C}} \to \mathcal{E}^{\mathcal{D}}$ part of an adjunction?

The natural candidate for an adjoint is the **pre-composition functor** $K^* : \mathcal{E}^{\mathcal{D}} \to \mathcal{E}^{\mathcal{C}}$ that

- sends a functor $F : \mathcal{D} \to \mathcal{E}$ to the functor $FK : \mathcal{C} \to \mathcal{E}$,
- a natural transformation $\mu : F \Rightarrow F'$ to the natural transformation $\mu K : FK \Rightarrow F'K$.

This generalises the group homomorphism $K : C \to D$ from Examples 2.4.9 and 3.1.6. The following proposition shows that the answer to these questions is indeed positive, whenever the left and right Kan extensions along K exist for *all* functors $F : \mathcal{C} \to \mathcal{E}$. Thus, the pattern from Examples 2.4.9 and 3.1.6 holds more generally.

If one ignores largeness issues and does not require that the resulting functor categories are locally small, one can investigate this for arbitrary categories and functors, as we do in the following. If one wants to work with locally small functor categories, one must restrict attention to functors $K : \mathcal{C} \to \mathcal{D}$ between small categories $\mathcal{C}$ and $\mathcal{D}$.

Proposition 3.1.7: Let $\mathcal{E}$ be a category and $K : \mathcal{C} \to \mathcal{D}$ a functor.

1. If the left Kan extension $\mathrm{Lan}_K F : \mathcal{D} \to \mathcal{E}$ exists for all functors $F : \mathcal{C} \to \mathcal{E}$, it defines a left adjoint $\mathrm{Lan}_K : \mathcal{E}^{\mathcal{C}} \to \mathcal{E}^{\mathcal{D}}$ to $K^* : \mathcal{E}^{\mathcal{D}} \to \mathcal{E}^{\mathcal{C}}$ that sends
   - a functor $F : \mathcal{C} \to \mathcal{E}$ to its left Kan extension $\mathrm{Lan}_K F : \mathcal{D} \to \mathcal{E}$ along K,
   - a natural transformation $\mu : F \Rightarrow F'$ between functors $F, F' : \mathcal{C} \to \mathcal{E}$ to the unique natural transformation $\mathrm{Lan}_K \mu : \mathrm{Lan}_K F \Rightarrow \mathrm{Lan}_K F'$ with $(\mathrm{Lan}_K \mu)K \circ \eta = \eta' \circ \mu$.

2. If the right Kan extension $\mathrm{Lan}_K F : \mathcal{D} \to \mathcal{E}$ exists for all functors $F : \mathcal{C} \to \mathcal{E}$, it defines a right adjoint $\mathrm{Ran}_K : \mathcal{E}^{\mathcal{C}} \to \mathcal{E}^{\mathcal{D}}$ to $K^* : \mathcal{E}^{\mathcal{D}} \to \mathcal{E}^{\mathcal{C}}$ that sends
   - a functor $F : \mathcal{C} \to \mathcal{E}$ to its right Kan extension $\mathrm{Ran}_K F : \mathcal{D} \to \mathcal{E}$ along K,
   - a natural transformation $\mu : F \Rightarrow F'$ between functors $F, F' : \mathcal{C} \to \mathcal{E}$ to the unique natural transformation $\mathrm{Ran}_K \mu : \mathrm{Ran}_K F \Rightarrow \mathrm{Ran}_K F'$ with $\epsilon \circ (\mathrm{Ran}_K \mu)K = \mu \circ \epsilon'$.

Proof:
By Remark 3.1.2, 2. it is sufficient to prove the claim for left Kan extensions.

1. The natural transformation $\mathrm{Lan}_K \mu : \mathrm{Lan}_K F \Rightarrow \mathrm{Lan}_K F'$ is defined uniquely by the universal property of the Kan extension $\mathrm{Lan}_K F$, by setting $G = \mathrm{Lan}_K F'$ and $\gamma = \eta' \circ \mu : F \Rightarrow (\mathrm{Lan}_K F')K$

in Definition 3.1.1. To see that Lan_K is a functor, note that for $F = F'$ and $\mu = \mathrm{id}_F$ we have $\eta \circ \mathrm{id}_F = \eta = \mathrm{id}_{\mathrm{Lan}_K F} K \circ \eta$ and hence $\mathrm{Lan}_K \mathrm{id}_F = \mathrm{id}_{\mathrm{Lan}_K F}$. For natural transformations $\mu : F \Rightarrow F'$ and $\mu' : F' \Rightarrow F'$ we obtain

$$(\mathrm{Lan}_K\mu' \circ \mathrm{Lan}_K\mu)K \circ \eta = (\mathrm{Lan}_K\mu')K \circ (\mathrm{Lan}_K\mu)K \circ \eta = (\mathrm{Lan}_K\mu')K \circ \eta' \circ \mu = \eta'' \circ (\mu' \circ \mu),$$
$$\Rightarrow \quad \mathrm{Lan}_K(\mu' \circ \mu) = \mathrm{Lan}_K\mu' \circ \mathrm{Lan}_K\mu.$$

2. We show that Lan_K is left adjoint to K^*.

The unit $\eta : \mathrm{id}_{\mathcal{E}^\mathcal{C}} \Rightarrow K^*\mathrm{Lan}_K$ of the adjunction has as component morphisms the natural transformations $\eta_F : F \Rightarrow (\mathrm{Lan}_K F)K$ in Definition 3.1.1 of the left Kan extension. The counit $\epsilon : \mathrm{Lan}_K K^* \Rightarrow \mathrm{id}_{\mathcal{E}^\mathcal{D}}$ has as component morphism $\epsilon_H : \mathrm{Lan}_K(HK) \Rightarrow H$ the unique natural transformation with $(\epsilon_H K) \circ \eta_{HK} = \mathrm{id}_{HK}$ that is defined by the universal property of the left Kan extension for $G = H : \mathcal{D} \to \mathcal{E}$ and $\gamma = \mathrm{id}_{HK} : HK \Rightarrow HK$. It remains to verify the defining identities of the adjunction

$$K^*\epsilon \circ \eta K^* = \mathrm{id}_{K^*} \qquad \epsilon\mathrm{Lan}_K \circ \mathrm{Lan}_K\eta = \mathrm{id}_{\mathrm{Lan}_K}.$$

The component morphism of the first identity for a functor $H : \mathcal{D} \to \mathcal{E}$ is precisely the defining identity for ϵ_H. To verify the second identity, we compute

$$(\epsilon\mathrm{Lan}_K \circ \mathrm{Lan}_K\eta)_F K \circ \eta_F = (\epsilon_{\mathrm{Lan}_K F}K) \circ K^*\mathrm{Lan}_K(\eta_F) \circ \eta_F \overset{\text{nat}\,\eta}{=} (\epsilon_{\mathrm{Lan}_K F}K) \circ \eta_{(\mathrm{Lan}_K F)K} \circ \eta_F = \eta_F,$$

where we used the defining identity for $\epsilon_{\mathrm{Lan}_K F}$ in the last step. This shows that the natural transformation $\alpha := (\epsilon\mathrm{Lan}_K \circ \mathrm{Lan}_K\eta)_F : \mathrm{Lan}_K F \Rightarrow \mathrm{Lan}_K F$ is a natural transformation with $\alpha K \circ \eta_F = \eta_F$ for the pair $G = \mathrm{Lan}_K F : \mathcal{D} \to \mathcal{E}$ and $\gamma = \eta_F : F \Rightarrow GK$. As $\mathrm{id}_{\mathrm{Lan}_K F}$ is another one, the universal property of $\mathrm{Lan}_K F$ implies $(\epsilon\mathrm{Lan}_K \circ \mathrm{Lan}_K\eta)_F = 1_{\mathrm{Lan}_K F}$. □

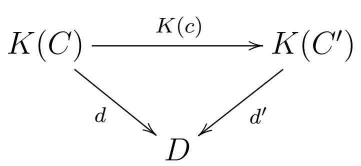
This proposition is useful in two ways. If a left or right adjoint to the pre-composition functor K^* is known, it gives a candidate for the left or right Kan extension along $K : \mathcal{C} \to \mathcal{D}$ for any functor $F : \mathcal{C} \to \mathcal{E}$. Conversely, if the left or right Kan extensions of all functors $F : \mathcal{C} \to \mathcal{E}$ are known, we can infer that the pre-composition functor K^* has a left or right adjoint and compute it explicitly. It also motivates the words *left* and *right* for left and right Kan extensions: left Kan extensions define left and right Kan extensions right adjoints to pre-composition functor.

3.2 Formulas for Kan extensions

In this section, we derive sufficient criteria for the existence of left and right Kan extensions and formulas that describe them in terms of colimits and limits. The central ingredient is the *comma category*, which can be viewed as a generalisation of action groupoids for group actions. It exists in two versions, one for left, one for right Kan extensions. There are also more general versions of comma categories that involve additional functors, but we will not consider them.

Definition 3.2.1: Let $K : \mathcal{C} \to \mathcal{D}$ be a functor.

1. The **comma category** $K \downarrow D$ has
   - as objects pairs (C, d) with $C \in \mathrm{Ob}\mathcal{C}$ and $d \in \mathrm{Hom}_\mathcal{D}(K(C), D)$,
   - as morphisms from (C, d) to (C', d') morphisms $c : C \to C'$ with $d' \circ K(c) = d$.



2. The **comma category** $D \downarrow K$ has
   - as objects pairs (C, d) with $C \in \mathrm{Ob}\mathcal{C}$ and $d \in \mathrm{Hom}_{\mathcal{D}}(D, K(C))$,
   - as morphisms from (C, d) to (C', d') morphisms $c : C \to C'$ with $d' = K(c) \circ d$.

$$\begin{array}{ccc} K(C) & \xrightarrow{K(c)} & K(C') \\ {\scriptstyle d}\nwarrow & & \nearrow{\scriptstyle d'} \\ & D & \end{array}$$

3. The **projection functors** $P^D : K \downarrow D \to \mathcal{C}$ and $P_D : D \downarrow K \to \mathcal{C}$ are given by

$$P^D : \begin{array}{ccc} (C,d) & \longmapsto & C \\ \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle c} \\ (C',d') & \longmapsto & C' \end{array} \qquad\qquad P_D : \begin{array}{ccc} (C,d) & \longmapsto & C \\ \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle c} \\ (C',d') & \longmapsto & C' \end{array}$$

Remark 3.2.2: The comma categories for a functor $K : \mathcal{C} \to \mathcal{D}$ have the following properties:

- Every morphism $d : D \to D'$ in $\mathcal{D}$ defines a functor $d^* : K \downarrow D \to K \downarrow D'$ with $d^*(C, d') = (C, d \circ d')$ and $P^{D'} d^* = P^D$.
- Every morphism $d : D \to D'$ in $\mathcal{D}$ defines a functor $d_* : D' \downarrow K \to D \downarrow K$ with $d_*(C, d') = (C, d' \circ d)$ and $P_D\, d_* = P_{D'}$.
- The morphisms $\tau^D_{(C,d)} = d : K(C) \to D$ form a cone $\tau^D : KP^D \Rightarrow \Delta(D)$ under KP^D with $\tau^{D'} d^* = \Delta(d) \circ \tau^D$ for all morphisms $d : D \to D'$ in $\mathcal{D}$.
- The morphisms $\tau^D_{(C,d)} = d : D \to K(C)$ form a cone $\tau^D : \Delta(D) \Rightarrow KP_D$ over KP_D with $\tau^D d_* = \tau^{D'} \circ \Delta(d)$ for all morphisms $d : D \to D'$ in $\mathcal{D}$.

Although the comma categories seem complicated at first, we can view them as generalisations of action groupoids. Recall that the **action groupoid** or **transformation groupoid** for a left action $\rhd : G \times X \to X$ of a group G on a set X is the category with

- elements $x \in X$ as objects,
- group elements $g \in G$ with $g \rhd x = x'$ as morphisms from x to x'.

Action groupoids arise as special cases of comma categories for functors $K : BG \to \mathcal{C}$. The simplest cases are the ones where $\mathcal{C} = \mathrm{Set}$ or $\mathcal{C} = \mathrm{Set}^{BH}$ for another group H. In the first case, the functor K corresponds to a G-set, in the second to a group homomorphisms $K : G \to H$.

Example 3.2.3: Let G be a group and $K : BG \to \mathrm{Set}$ a functor with $K(\bullet) = X$.

1. The comma category $K \downarrow D$ for a set D has
   - maps $f : X \to D$ as objects,
   - morphism spaces $\mathrm{Hom}_{K\downarrow D}(f, f') = \{g \in G \mid f' \circ K(g) = f\}$.

   This is the action groupoid for the G-action

$$\rhd' : G \times \mathrm{Hom}_{\mathrm{Set}}(X, D) \to \mathrm{Hom}_{\mathrm{Set}}(X, D), \quad g \rhd' f = f \circ K(g^{-1}).$$

2. The comma category $D \downarrow K$ for a set D has
   - maps $f : D \to X$ as objects,
   - morphism spaces $\mathrm{Hom}_{D\downarrow K}(f, f') = \{g \in G \mid f' = K(g) \circ f\}$.

   This is the action groupoid for the G-action

$$\rhd'' : G \times \mathrm{Hom}_{\mathrm{Set}}(D, X) \to \mathrm{Hom}_{\mathrm{Set}}(D, X), \quad g \rhd'' f = K(g) \circ f.$$

The functors P^D and P_D send every object of $K \downarrow D$ and $D \downarrow K$ to $\bullet$ and every morphism to the corresponding element of G.

Example 3.2.4:
Let $K : BC \to BD$ be the functor defined by a group homomorphism $K : C \to D$.

1. The comma categories $K \downarrow \bullet_D$ and $\bullet_D \downarrow K$ have object sets D and morphisms

$$\mathrm{Hom}_{K\downarrow\bullet_D}(d, d') = \{c \in C \mid d' = dK(c)^{-1}\} \qquad \mathrm{Hom}_{\bullet_D\downarrow K}(d, d') = \{c \in C \mid d' = K(c)d\}.$$

   They are the action groupoids for the left actions

$$\rhd_R : C \times D \to D, \quad c \rhd_R d = dK(c)^{-1} \qquad\qquad \rhd_L : C \times D \to D, \quad c \rhd_L d = K(c)d.$$

2. The functors $P^\bullet : K \downarrow \bullet_D \to BG$ and $P_\bullet : \bullet_D \downarrow K \to BG$ send each element $d \in D$ to the object $\bullet_C$ and each morphism $c : d \to d'$ to the corresponding element of C.

In the general case, the comma categories are more complicated, as they keep track of the objects as well as the morphisms in $\mathcal{C}$. Nevertheless, we can think of them as a sort of generalised action categories, where morphisms in $\mathcal{C}$ act on morphisms with a fixed source or target in $\mathcal{D}$.

The comma categories and their projection functors allow one to construct left and right Kan extensions of functors $F : \mathcal{C} \to \mathcal{E}$ along functors $K : \mathcal{C} \to \mathcal{D}$.

Theorem 3.2.5: Let $F : \mathcal{C} \to \mathcal{E}$ and $K : \mathcal{C} \to \mathcal{D}$ be functors.

1. If the functor $FP^D : K \downarrow D \to \mathcal{E}$ has a colimit for every $D \in \mathrm{Ob}\mathcal{D}$, then F has a left Kan extension given by $\mathrm{Lan}_K F(D) = \mathrm{colim}(FP^D)$ on the objects.
2. If the functor $FP_D : D \downarrow K \to \mathcal{E}$ has a limit for every $D \in \mathrm{Ob}\mathcal{D}$, then F has a right Kan extension given by $\mathrm{Ran}_K F(D) = \lim(FP_D)$ on the objects.

Proof:
By Remark 3.1.2, 2. it is sufficient to prove the claim for left Kan extensions. We denote by $\lambda^D : FP^D \Rightarrow \Delta(\mathrm{colim}(FP^D))$ the colimit cone for $D \in \mathrm{Ob}\mathcal{D}$.

1. We define the functor $\mathrm{Lan}_K F : \mathcal{D} \to \mathcal{E}$:

We set $\mathrm{Lan}_K F(D) = \mathrm{colim}(FP^D)$ on the objects of $\mathcal{D}$. As $P^{D'} d^* = P^D$ for all morphisms $d : D \to D'$ by Remark 3.2.2, the natural transformation $\lambda^{D'} d^* : FP^D \Rightarrow \Delta(\mathrm{colim}(FP^{D'}))$ is a cocone under FP^D. The universal property of the colimit yields a unique morphism

$$\mathrm{Lan}_K F(d) : \mathrm{colim}(FP^D) \to \mathrm{colim}(FP^{D'}) \quad \text{with} \quad \Delta(\mathrm{Lan}_K F(d)) \circ \lambda^D = \lambda^{D'} d^*. \tag{13}$$

We show that this defines a functor $\mathrm{Lan}_K F$. Note first that $1_D^* = \mathrm{id}_{K\downarrow D}$ and $(d' \circ d)^* = d'^* \circ d^*$ for all morphisms $d : D \to D'$ and $d' : D' \to D''$. This implies $\mathrm{Lan}_K F(1_D) = 1_{\mathrm{Lan}_K F(D)}$ and

$$\begin{aligned}\lambda^{D''}(d' \circ d)^* = \lambda^{D''} d'^* d^* &\overset{(13)}{=} (\Delta(\mathrm{Lan}_K F(d')) \circ \lambda^D) d^* = \Delta(\mathrm{Lan}_K F(d')) \circ \lambda^D d^* \\ &\overset{(13)}{=} \Delta(\mathrm{Lan}_K F(d')) \circ \Delta(\mathrm{Lan}_K F(d)) \circ \lambda^D = \Delta(\mathrm{Lan}_K F(d') \circ \mathrm{Lan}_K F(d)) \circ \lambda^D,\end{aligned}$$

which yields $\mathrm{Lan}_K F(d' \circ d) = \mathrm{Lan}_K F(d') \circ \mathrm{Lan}_K F(d)$.

2. We define the natural transformation $\eta : F \Rightarrow (\mathrm{Lan}_K F)K$ with component morphisms

$$\eta_C = \lambda^{K(C)}_{(C, 1_{K(C)})} : F(C) \to \mathrm{colim}(FP^{K(C)}). \tag{14}$$

The naturality of η follows, because each morphism $c \in \mathrm{Hom}_{\mathcal{C}}(C, C')$ defines a morphism $c : (C, K(c)) \to (C', 1_{K(C')})$ in $K \downarrow D$ and $\lambda^{K(C')}$ is a cocone:

$$\begin{aligned}\eta_{C'} \circ F(c) &\overset{\text{def } \eta}{=} \lambda^{K(C')}_{(C', 1_{K(C')})} \circ F(c) \overset{\text{cocone}}{=} \lambda^{K(C')}_{(C, K(c))} = \lambda^{K(C')}_{(C, K(c) \circ 1_{K(C)})} = (\lambda^{K(C')} K(c)^*)_{(C, 1_{K(C)})} \\ &\overset{(13)}{=} \Delta(\mathrm{Lan}_K F(K(c))) \circ \lambda^{K(C)}_{(C, 1_{K(C)})} \overset{\text{def } \eta}{=} \Delta(\mathrm{Lan}_K(F) K(c)) \circ \eta_C.\end{aligned}$$

3. We show that $(\mathrm{Lan}_K F, \eta)$ has the universal property of the left Kan extension:
For this, let $\gamma : F \Rightarrow GK$ a natural transformation for a functor $G : \mathcal{D} \to \mathcal{E}$. We construct a natural transformation $\alpha : \mathrm{Lan}_K F \Rightarrow G$ with $(\alpha K) \circ \eta = \gamma$ from the cocones

$$\mu^D = G\tau^D \circ \gamma P^D : FP^D \Rightarrow \Delta(G(D)), \tag{15}$$

where $\tau^D : KP^D \Rightarrow \Delta(D)$ is the cocone from Remark 3.2.2. As μ^D is a cone under FP^D, the universal property of the colimit yields a unique morphism

$$\alpha_D : \mathrm{colim}(FP^D) \to G(D) \quad \text{with} \quad \Delta(\alpha_D) \circ \lambda^D = \mu^D. \tag{16}$$

That the morphisms α_D define a natural transformation $\alpha : \mathrm{Lan}_K F \Rightarrow G$ follows from the universal property of λ^D, as we have for all morphisms $d : D \to D'$ in $\mathcal{D}$

$$\begin{aligned}\Delta(\alpha_{D'} \circ \mathrm{Lan}_K F(d)) \circ \lambda^D &\overset{(13)}{=} \Delta(\alpha_{D'}) \circ \lambda^{D'} d^* \overset{(16)}{=} \mu^{D'} d^* \overset{(15)}{=} (G\tau^{D'} \circ \gamma P^{D'}) d^* \overset{3.2.2}{=} G\tau^{D'} d^* \circ \gamma P^D \\ &\overset{3.2.2}{=} \Delta(G(d)) \circ G\tau^D \circ \gamma P^D \overset{(15)}{=} \Delta(G(d)) \circ \mu^D \overset{(16)}{=} \Delta(G(d)) \circ \Delta(\alpha_D) \circ \lambda^D = \Delta(G(d) \circ \alpha_D) \circ \lambda^D.\end{aligned}$$

It satisfies $(\alpha K) \circ \eta = \gamma$, since we have for all objects $C \in \mathrm{Ob}\mathcal{C}$

$$(\alpha K)_C \circ \eta_C = \alpha_{K(C)} \circ \eta_C \overset{(14)}{=} \alpha_{K(C)} \circ \lambda^{K(C)}_{(C, 1_{K(C)})} \overset{(16)}{=} \mu^{K(C)}_{(C, 1_{K(C)})} \overset{(15)}{=} G(1_{K(C)}) \circ \gamma_C = \gamma_C.$$

To show the uniqueness of α, suppose $\alpha' : \mathrm{Lan}_K F \Rightarrow G$ is another natural transformation with $(\alpha' K) \circ \eta = \gamma$. We then have for all objects (C, d) of $K \downarrow D$

$$\begin{aligned}\mu^D_{(C,d)} &\overset{(15)}{=} (G\tau^D \circ \gamma P^D)_{(C,d)} \overset{3.2.2}{=} G(d) \circ \gamma_C = G(d) \circ \alpha'_{K(C)} \circ \eta_C \overset{(14)}{=} G(d) \circ \alpha'_{K(C)} \circ \lambda^{K(C)}_{(C, 1_{K(C)})} \\ &\overset{\text{nat } \alpha'}{=} \alpha'_D \circ \mathrm{Lan}_K F(d) \circ \lambda^{K(C)}_{(C, 1_{K(C)})} \overset{(13)}{=} \alpha'_D \circ \lambda^D_{(C,d)}\end{aligned}$$

and hence $\Delta(\alpha'_D) \circ \lambda^D = \mu^D = \Delta(\alpha_D) \circ \lambda^D$. The universal property of λ^D implies $\alpha' = \alpha$. □

This theorem reduces Kan extensions to the computation of (co)limits in those cases where the relevant (co)limits exist. In particular, their existence is guaranteed for all functors $F : \mathcal{C} \to \mathcal{E}$ and $K : \mathcal{C} \to \mathcal{D}$, if $\mathcal{C}$ is small, $\mathcal{D}$ locally small and $\mathcal{E}$ (co)complete. In this case, they define adjoints to the pre-composition functor $K^* : \mathcal{E}^{\mathcal{D}} \to \mathcal{E}^{\mathcal{C}}$ by Proposition 3.1.7.

Corollary 3.2.6: Let $K : \mathcal{C} \to \mathcal{D}$ with $\mathcal{C}$ small, $\mathcal{D}$ locally small and $\mathcal{E}$ (co)complete. Then the right (left) Kan extensions along K exist for all functors $F : \mathcal{C} \to \mathcal{E}$, are given by Theorem 3.2.5 and define a right adjoint $\mathrm{Ran}_K : \mathcal{E}^{\mathcal{C}} \to \mathcal{E}^{\mathcal{D}}$ (left adjoint $\mathrm{Lan}_K : \mathcal{E}^{\mathcal{C}} \to \mathcal{E}^{\mathcal{D}}$) to K^*.

Proof:
If $\mathcal{C}$ is small and $\mathcal{D}$ locally small, the comma categories $K \downarrow D$ and $D \downarrow K$ are small for all functors $K : \mathcal{C} \to \mathcal{D}$. As $\mathcal{E}$ is (co)complete, the (co)limit of the functors $FP^D : K \downarrow D \to \mathcal{E}$ and $FP_D : D \downarrow K \to \mathcal{E}$ exist for any functor $F : \mathcal{C} \to \mathcal{D}$, define the left (right) Kan extensions by Theorem 3.2.5 and a left (right) adjoint to K^* by Proposition 3.1.7. □

The benefits of the formula in Theorem 3.2.5 go beyond the concrete expressions for their values on objects. Via Corollary 3.2.6 it gives sufficient criteria for the existence of Kan extensions that involve mainly the existence of (co)limits in the category $\mathcal{E}$. This allows one to define mathematical structures as Kan extensions of certain functors in many situations, also in those cases, where they are difficult to compute explicitly.

The challenges in determining the left or right Kan extensions with the formula in Theorem 3.2.5 depend on the complexity of the relevant comma categories. Generally, it is often sufficient to compute the Kan extensions on the objects. The associated natural transformations are then given by the limit cones, and the action of the functors on morphisms is often obvious from the context. The simplest cases involve categories BG for a group G and yield familiar constructions.

Example 3.2.7:
Let G be a group and $K : BG \to \mathrm{Set}^{BG^{op}}$ be the covariant Yoneda embedding from (12). Then the left Kan extension of a G-set $X : BG \to \mathrm{Set}$ along K sends a G^{op}-set $Y : BG^{op} \to \mathrm{Set}$ to

$$\mathrm{Lan}_K X(Y) = Y \times_G X$$

Proof:
The functor K defines the G^{op}-set $G = K(\bullet) = \mathrm{Hom}(\bullet, \bullet)$ with the right multiplication. The comma category $K \downarrow Y$ for a G^{op}-set Y has

- as objects G^{op}-equivariant maps $f : G \to Y$, or, equivalently, elements $y \in Y$,
- morphism sets $\mathrm{Hom}_{K\downarrow Y}(y, y') = \{g \in G \mid y' \lhd g = y\}$.

The functor $XP^Y : Y \downarrow K \to \mathrm{Set}$ is given by

$$XP^Y: \qquad \begin{array}{ccc} y \in Y & \longmapsto & X \\ \downarrow g & & \downarrow X(g): x \mapsto g \rhd x \\ y \lhd g^{-1} \in Y & \longmapsto & X \end{array}$$

A cocone $\tau : XP^Y \Rightarrow \Delta(Z)$ is a collection of maps $\tau_y : X \to Z$ with $\tau_y(g \rhd x) = \tau_{y \lhd g}(x)$ for all $g \in G$, $y \in Y$ and $x \in X$. This is equivalent to a map $f_\tau : Y \times_G X \to Z$, $(y, x) \mapsto \tau_y(x)$. Thus, the initial cocone is $\lambda : XP^Y \Rightarrow \Delta(Y \times_G X)$ with $\lambda_y : X \mapsto Y \times_G X$, $x \mapsto (y, x)$ and the unique cone morphism from λ to τ is f_τ. Thus, $\mathrm{Lan}_K X(Y) = \mathrm{colim}(XP^Y) = Y \times_G X$. □

Example 3.2.8: Let G be a group and $K : BG \to (\mathrm{Set}^{BG})^{op}$ the contravariant Yoneda embedding from (12). The right Kan extension of a G-set $X : BG \to \mathrm{Set}$ along K sends a G-set $Y : BG \to \mathrm{Set}$ to the set of G-equivariant maps $f : Y \to X$

$$\mathrm{Ran}_K X(Y) = \mathrm{Hom}_{\mathrm{Set}^{BG}}(Y, X).$$

Proof:
The category $(\mathrm{Set}^{BG})^{op}$ has as objects G-sets and as morphisms from a G-set X to a G-set Y G-equivariant maps $f : Y \to X$. The functor K defines the G-set $G = \mathrm{Hom}(\bullet, \bullet)$ with the left multiplication. The comma category $Y \downarrow K$ for a G-set Y has

- as objects G-equivariant maps $f : G \to Y$, or, equivalently, elements $y \in Y$,
- morphism sets $\mathrm{Hom}_{Y\downarrow K}(y, y') = \{g \in G \mid y' = g \rhd y\}$.

The functor $XP_Y : Y \downarrow K \to \mathrm{Set}$ is given by

$$XP^Y : \quad \begin{array}{ccc} y \in Y & \longmapsto & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle X(g): x \mapsto g \rhd x} \\ g \rhd y \in Y & \longmapsto & X \end{array}$$

A cone $\tau : \Delta(Z) \Rightarrow XP_Y$ is a collection of maps $\tau_y : Z \to X$ with $g \rhd \tau_y(z) = \tau_{g \rhd y}(z)$ for all $g \in G$, $y \in Y$ and $z \in Z$. This corresponds to a map $f_\tau : Z \mapsto \mathrm{Hom}_{\mathrm{Set}^{BG}}(Y, X)$, $z \mapsto \tau(z)$ with $\tau(z) : Y \to X$, $y \mapsto \tau_y(z)$. Thus, the terminal cone is $\lambda : \Delta(\mathrm{Hom}_{\mathrm{Set}^{BG}}(Y, X)) \Rightarrow XP_Y$ with legs $\lambda_y : \mathrm{Hom}_{\mathrm{Set}^{BG}}(X, Y) \to X$, $h \mapsto h(y)$, and the unique cone morphism from τ to λ is f_τ. □

Example 3.2.9: Let $K : BC \to BD$ the functor for a group homomorphism $K : C \to D$ and $F : BC \to \mathrm{Vect}_{\mathbb{F}}$ with $F(\bullet) = V$. Then the (co)limit formulas from Theorem 3.2.5 yield the (co)induced representations

$$\mathrm{Lan}_K F(\bullet) = \mathbb{F}[D] \otimes_{\mathbb{F}[C]} V \qquad \mathrm{Ran}_K F(\bullet) = \mathrm{Hom}_{\mathbb{F}[C]}(\mathbb{F}[D], V).$$

Proof:
By Example 3.2.4 the functors $FP^\bullet : K \downarrow \bullet_D \to \mathrm{Vect}_{\mathbb{F}}$ and $FP^\bullet : \bullet_D \downarrow K \to \mathrm{Vect}_{\mathbb{F}}$ send

- each object, given by an element $d \in D$, to $FP^\bullet(d) = V$,
- a morphism $c : d \to d'$ to $F(c) \in \mathrm{Aut}_{\mathbb{F}}(V)$.

• A cocone $\mu : FP^\bullet \Rightarrow \Delta(W)$ is a collection of linear maps $\mu_d : V \to W$ with $\mu_d \circ F(c) = \mu_{dK(c)}$ for all $c \in C$, $d \in D$. Every such cocone induces a unique linear map

$$\phi : \mathbb{F}[D] \otimes_{\mathbb{F}[C]} V \to W, \quad d \otimes v \mapsto \mu_d(v),$$

where $\mathbb{F}[D]$ has the $\mathbb{F}[C]$-right module structure $d \lhd c = dK(c)$. This is a cocone morphism from the cocone λ with legs $\lambda_d : V \to \mathbb{F}[D] \otimes_{\mathbb{F}[C]} V$, $v \mapsto 1_D \otimes v$ to μ.

• A cone $\mu : \Delta(W) \Rightarrow FP_\bullet$ is a collection of linear maps $\mu_d : W \to V$ with $F(c) \circ \mu_d = \mu_{K(c)d}$ for all $c \in C$, $d \in D$. Every such cone induces a linear map

$$\phi : W \to \mathrm{Hom}_{\mathbb{F}[C]}(\mathbb{F}[D], V), w \mapsto f_w \quad \text{with} \quad f_w(d) = \mu_d(w),$$

where the $\mathbb{F}[C]$-module structure on $\mathbb{F}[D]$ is given by $c \rhd d = K(c)d$ and f_w is e $\mathbb{F}[C]$-linear. This is a cone morphism from μ to the cone λ with legs $\lambda_d : f \mapsto f(d)$. □

Another insight is that Theorem 3.2.5 and Corollary 3.2.6 can be used to prove the Yoneda lemma. The basic idea is to consider the right Kan extension of a functor $F : \mathcal{C} \to \mathrm{Set}$ along the identity functor $\mathrm{id}_{\mathcal{C}}$. This is of course given by $\mathrm{Ran}_K F = F$ and $\epsilon = \mathrm{id}_F : F \Rightarrow F$. On the other hand, we can use the comma category $C \downarrow \mathrm{id}_{\mathcal{C}}$ and the limit formula to compute it.

Corollary 3.2.10: (**Yoneda Lemma**)
Let $\mathcal{C}$ be small. Then for any functor $F : \mathcal{C} \to \mathrm{Set}$ and object $C \in \mathrm{Ob}\mathcal{C}$, natural transformations $\tau : \mathrm{Hom}(C, -) \Rightarrow F$ are in bijection with elements of $F(C)$:

$$\mathrm{Nat}(\mathrm{Hom}(C, -), F) \cong F(C).$$

Proof:
As $\mathcal{C}$ is small and Set locally small and complete, the right Kan extension of $F : \mathcal{C} \to \mathrm{Set}$ along the identity functor $\mathrm{id}_{\mathcal{C}}$ is given by Corollary 3.2.6 and the limit formula from Theorem 3.2.5.

The category $C \downarrow \mathrm{id}_{\mathcal{C}}$ has as objects morphisms $d \in \mathrm{Hom}_{\mathcal{C}}(C, C')$ for objects $C' \in \mathrm{Ob}\mathcal{C}$ and as morphisms from d to $d' \in \mathrm{Hom}_{\mathcal{C}}(C, C'')$ morphisms $c \in \mathrm{Hom}_{\mathcal{C}}(C', C'')$ with $d' = c \circ d$.

A cone $\mu : \Delta(X) \Rightarrow FP_C$ has legs $\mu_d : X \to F(C')$ for $d \in \mathrm{Hom}_{\mathcal{C}}(C, C')$ with $F(c) \circ \mu_d = \mu_{c \circ d}$ for all $c \in \mathrm{Hom}_{\mathcal{C}}(C', C'')$. It defines a map $\mu' : X \to \mathrm{Nat}(\mathrm{Hom}(C, -), F)$ that assigns to $x \in X$ the natural transformation $\mu^x : \mathrm{Hom}(C, -) \Rightarrow F$ with components $\mu^x_{C'} : \mathrm{Hom}_{\mathcal{C}}(C, C') \to F(C')$, $d \mapsto \mu_d(x)$. The naturality of μ^x is precisely the cone property of μ

$$\mu^x_{C''} \circ \mathrm{Hom}(C, c)(d) = \mu_{c \circ d}(x) = F(c) \circ \mu_d(x) = F(c) \circ \mu^x_{C'}(d).$$

The map μ' is the unique cone morphism from μ to the terminal cone

$$\lambda : \Delta(\mathrm{Nat}(\mathrm{Hom}(C, -), F)) \Rightarrow FP_C$$
$$\lambda_d : \mathrm{Nat}(\mathrm{Hom}(C, -), F) \to F(C'), \quad \rho \mapsto \rho_{C'}(d) \quad \text{for} \quad d \in \mathrm{Hom}_{\mathcal{C}}(C, C').$$

The limit formula and the uniqueness of the Kan extension then yield the Yoneda lemma:

$$\mathrm{Ran}_{\mathrm{id}_{\mathcal{C}}} F(C) = \mathrm{Nat}(\mathrm{Hom}(C, -), F) \cong F(C).$$ □

These examples show that Kan extensions given by the (co)limit formulas from Theorem 3.2.5 have particularly nice properties. Almost all Kan extensions arising from familiar constructions are of this type. Generally, Kan extensions that are not given by (co)limit formulas are ill-behaved and mostly useless in practice. In particular, we will see in the next section that they interact badly with Hom functors. As the Kan extensions given by (co)limit formulas are computed object by object by taking for each object an appropriate (co)limit, these Kan extensions are called *objectwise* or, more frequently, *pointwise*.

Definition 3.2.11: Let $\mathcal{C}$ be small and $\mathcal{D}$ and $\mathcal{E}$ locally small categories. A right (left) Kan extension of $F : \mathcal{C} \to \mathcal{D}$ along $K : \mathcal{C} \to \mathcal{D}$ is called **pointwise**, if it is given by the (co)limit formula from Theorem 3.2.5.

A first indication that pointwise Kan extensions are the good Kan extensions is that they are indeed extensions in a more naive sense under certain conditions. If a functor $F : \mathcal{C} \to \mathcal{E}$ is extended along a fully faithful functor $K : \mathcal{C} \to \mathcal{D}$, the natural transformation that characterises a *pointwise* Kan extension is a natural isomorphism.

Corollary 3.2.12:

1. If a left Kan extension $(\mathrm{Lan}_K F, \eta)$ is pointwise and $K : \mathcal{C} \to \mathcal{D}$ fully faithful, then $\eta : F \Rightarrow (\mathrm{Lan}_K F)K$ is a natural isomorphism.

2. If a right Kan extension $(\mathrm{Ran}_K F, \epsilon)$ is pointwise and $K : \mathcal{C} \to \mathcal{D}$ fully faithful, then $\epsilon : (\mathrm{Ran}_K F)K \Rightarrow F$ is a natural isomorphism.

Proof:
We prove the claim for left Kan extensions. The claim for right Kan extensions then follows from Remark 3.1.2, 2.

If K is fully faithful, then the canonical functor $H : \mathrm{id}_{\mathcal{C}} \downarrow C \to K \downarrow K(C)$ with $P^C = P^{K(C)}H$

$$H:\quad \begin{array}{ccc} (C', d') & \longmapsto & (C', K(d')) \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c} \\ (C'', d'') & \longmapsto & (C'', K(d'')) \end{array}$$

is an isomorphism of categories, since for all morphisms $d' : C' \to C$, $d'' : C'' \to C$ and $c : C' \to C''$ one has $d' = d'' \circ c$ if and only if $K(d') = K(d'') \circ K(c)$.

It follows that any cocone $\mu : FP^{K(C)} \Rightarrow \Delta(D)$ defines a unique cocone $\mu' : FP^C \Rightarrow \Delta(D)$ with legs $\mu'_{(C',d')} = \mu_{(C',K(d'))}$ and vice versa. Thus, by Theorem 3.2.5 the left Kan extension is determined by the colimit cone $\lambda' : FP^C \Rightarrow \Delta(\mathrm{colim}(FP^C))$.

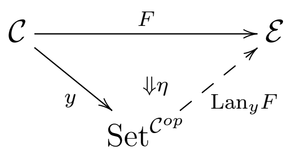
The category $\mathrm{id}_{\mathcal{C}} \downarrow C$ has the terminal object $(C, 1_C)$, since for every pair (C', d'), there is exactly one morphism $t : (C', d') \to (C, 1_c)$, namely $t = d' : C' \to C$. Hence, the colimit cone λ' has the legs $\lambda'_{(C',d')} = F(d') : F(C') \to F(C)$, see Exercise 10, and the left Kan extension is

$$\begin{aligned} &\mathrm{Lan}_K F(K(C)) = \mathrm{colim}(FP^{K(C)}) = \mathrm{colim}(FP^C) = F(C) \\ &\eta_C = \lambda'_{(C,1_C)} = F(1_C) = 1_{F(C)} : F(C) \to F(C). \end{aligned}$$

□

Another important motivation for pointwise Kan extensions are the special properties of pointwise Kan extensions along the Yoneda embeddings (12). Any pointwise left Kan extension of a functor $F : \mathcal{C} \to \mathcal{E}$ along the covariant Yoneda embedding $y : \mathcal{C} \to \mathrm{Set}^{\mathcal{C}^{op}}$ yields a left adjoint $\mathrm{Lan}_y F : \mathrm{Set}^{\mathcal{C}^{op}} \to \mathcal{E}$ to the functor $R = \mathrm{Hom}_{\mathcal{E}}(F(-), -) : \mathcal{E} \to \mathrm{Set}^{\mathcal{C}^{op}}$. One one hand, this can be used to determine this left Kan extension, if a left adjoint of R is known, by using the uniqueness of adjoints. On the other hand, this adjunction is conceptually important. We will see examples of both in Section 5. There is also an analogous statement for right Kan extensions along the contravariant Yoneda embedding, which is less used in practice (Exercise).

Proposition 3.2.13: Let $\mathcal{C}$ be small. Then a pointwise left Kan extension $F : \mathcal{C} \to \mathcal{E}$ along the covariant Yoneda embedding $y : \mathcal{C} \to \mathrm{Set}^{\mathcal{C}^{op}}$



is left adjoint to the functor $R = \mathrm{Hom}_{\mathcal{E}}(F(-), -) : \mathcal{E} \to \mathrm{Set}^{\mathcal{C}^{op}}$ that assigns to

- an object $E \in \mathrm{Ob}\mathcal{E}$ the functor $R(E) = \mathrm{Hom}_{\mathcal{E}}(F(-), E) : \mathcal{C}^{op} \to \mathrm{Set}$,
- a morphism $e \in \mathrm{Hom}_{\mathcal{E}}(E, E')$ the natural transformation $R(e) : R(E) \Rightarrow R(E')$ with components $R(e)_C = \mathrm{Hom}_{\mathcal{E}}(F(C), e) : \mathrm{Hom}_{\mathcal{E}}(F(C), E) \to \mathrm{Hom}_{\mathcal{E}}(F(C), E')$, $f \mapsto e \circ f$.

The natural transformation $\eta : F \Rightarrow \mathrm{Lan}_y F y$ is a natural isomorphism.

Proof:
That $\eta : F \Rightarrow \mathrm{Lan}_y F$ is a natural isomorphism follows from Corollary 3.2.12, because the Yoneda embedding is fully faithful.

We show that $\mathrm{Lan}_y F : \mathrm{Set}^{\mathcal{C}^{op}} \to \mathcal{E}$ is left adjoint to $R : \mathcal{E} \to \mathrm{Set}^{\mathcal{C}^{op}}$ by constructing bijections

$$\mathrm{Hom}_{\mathrm{Set}^{\mathcal{C}^{op}}}(S, R(E)) \cong \mathrm{cocone}(FP^S, E) \cong \mathrm{Hom}_{\mathcal{E}}(\mathrm{Lan}_y F(S), E) \tag{17}$$

that are natural in S and E for all functors $S : \mathcal{C}^{op} \to \mathrm{Set}$ and objects $E \in \mathrm{Ob}\mathcal{E}$. Here, $P^S : y \downarrow S \to \mathcal{C}$ is the projector for the comma category $y \downarrow S$.

1. The second bijection in (17) follows from the colimit formula in Theorem 3.2.5. The universal property of the colimit states that there is a bijection

$$\mathrm{cocone}(FP^S, E) \cong \mathrm{Hom}_{\mathcal{E}}(\mathrm{colim}(FP^S), E) \cong \mathrm{Hom}_{\mathcal{E}}(\mathrm{Lan}_y F(S), E)$$

that is natural in E. Its naturality in S follows from the fact that $\mathrm{Lan}_y F : \mathrm{Set}^{\mathcal{C}^{op}} \to \mathcal{E}$ is a functor and the functoriality of the colimit, see Proposition 2.2.5.

2. To construct the first bijection in in (17), we consider the comma category $y \downarrow S$.

- Its objects are pairs (C, ν) of $C \in \mathrm{Ob}\mathcal{C}$ and natural transformations $\nu : \mathrm{Hom}_{\mathcal{C}}(-, C) \Rightarrow S$. By the Yoneda lemma the latter are in bijection with elements of $S(C)$. For each $s \in S(C)$ the unique natural transformation $\nu^s : \mathrm{Hom}_{\mathcal{C}}(-, C) \Rightarrow S$ with $\nu^s_C(1_C) = s$ has component morphisms are $\nu^s_{C'} : \mathrm{Hom}_{\mathcal{C}}(C', C) \to S(C')$, $f \mapsto S(f)(s)$.
- A morphism $c : (C, \nu) \to (C', \nu')$ in $y \downarrow S$ is a morphism $c : C \to C'$ in $\mathcal{C}$ that satisfies $\nu' \circ \mathrm{Hom}_{\mathcal{C}}(-, c) = \nu$, or, equivalently,

$$\nu'_C(c) = S(c)(\nu'_{C'}(1_{C'})) = \nu_C(1_C). \tag{18}$$

The projector $P^S : y \downarrow S \to \mathcal{C}$ sends each object (C, ν) to C and each morphism to itself.

3. We show that there are bijections $\mathrm{Hom}_{\mathrm{Set}^{\mathcal{C}^{op}}}(S, R(E)) \cong \mathrm{cocone}(FP^S, E)$ that are natural in $E \in \mathrm{Ob}\mathcal{E}$ and $S : \mathcal{C}^{op} \to \mathrm{Set}$. For this, we consider the maps

$$\begin{array}{llcl}
\phi_{S,E}: & \mathrm{Hom}_{\mathrm{Set}^{\mathcal{C}^{op}}}(S, R(E)) & \to & \mathrm{cocone}(FP^S, E) \\
& \alpha : S \Rightarrow R(E) & \mapsto & \lambda^\alpha : FP^S \Rightarrow \Delta(E) \\
& & & \lambda^\alpha_{(C,\nu)} = \alpha_C(\nu_C(1_C)) : F(C) \to E \\
\psi_{S,E}: & \mathrm{cocone}(FP^S, E) & \to & \mathrm{Hom}_{\mathrm{Set}^{\mathcal{C}^{op}}}(S, R(E)) \\
& \lambda : FP^S \Rightarrow \Delta(E) & \mapsto & \beta^\lambda : S \Rightarrow R(E) \\
& & & \beta^\lambda_C : S(C) \to \mathrm{Hom}_{\mathcal{E}}(F(C), E),\ s \mapsto \lambda_{(C,\nu^s)}
\end{array}$$

To show that λ^α is indeed a cocone for each natural transformation $\alpha : S \Rightarrow R(E)$, we compute for a morphism $c : (C, \nu) \to (C', \nu')$ in $y \downarrow S$

$$\begin{aligned}
\lambda^\alpha_{(C',\nu')} \circ F(c) &\overset{\mathrm{def}\,\lambda^\alpha}{=} \alpha_{C'}(\nu'_{C'}(1_{C'})) \circ F(c) \overset{\mathrm{def}\,R(E)}{=} (R(E)(c) \circ \alpha_{C'})(\nu'_{C'}(1_{C'})) \overset{\mathrm{nat}\,\alpha}{=} (\alpha_C \circ S(c))(\nu'_{C'}(1_{C'})) \\
&\overset{(18)}{=} \alpha_C(\nu_C(1_C)) = \lambda^\alpha_{(C,\nu)}.
\end{aligned}$$

To show that $\beta^\lambda : S \Rightarrow R(E)$ is a natural transformation for each cocone $\lambda : FP^S \Rightarrow \Delta(E)$, note that by (18) each morphism $c : C \to C'$ in $\mathcal{C}$ defines a morphism $c : (C, \nu^{S(c)(s')}) \to (C', \nu^{s'})$

in $y \downarrow S$ for all $s' \in S(C')$, because one has $\nu_C^{s'}(c) = S(c)\nu^{s'}(1_C) = S(c)(s') = \nu_C^{S(c)s'}(1_C)$. This implies for all $s' \in S(C')$

$$\beta_C^\lambda \circ S(c)(s') = \beta_C^\lambda(S(c)(s')) = \lambda_{(C,\nu^{S(c)(s')})} \overset{\lambda \text{ cocone}}{=} \lambda_{(C',\nu^{s'})} \circ F(c) = \beta_{C'}^\lambda(s') \circ F(c) = (R(E)(c) \circ \beta_{C'}^\lambda)(s').$$

A direct computation then shows that $\psi_{S,E}$ is the inverse of $\psi_{S,E}$:

$$(\psi_{S,E} \circ \phi_{S,E})(\alpha)_C(s) = \psi_{S,E}(\lambda^\alpha)_C(s) = \lambda^\alpha_{(C,\nu^s)} = \alpha_C(\nu_C^s(1_C)) = \alpha_C(s) \quad \forall\, C \in \mathrm{Ob}\mathcal{C},\ s \in S(C)$$
$$(\phi_{S,E} \circ \psi_{S,E})(\lambda)_{(C,\nu)} = \psi_{S,E}(\lambda)_C(\nu_C(1_C)) = \lambda_{(C,\nu)} \quad \forall\, (C,\nu) \in \mathrm{Ob}(y \downarrow S).$$

The naturality of $\phi_{S,E}$ follows, because we have for each natural transformation $\beta : S' \Rightarrow S$, morphism $e : E \to E'$ and $(C,\nu) \in \mathrm{Ob}(y \downarrow S)$

$$\phi_{S',E'}(\mathrm{Hom}(-,R(e)) \circ \alpha \circ \beta)_{(C,\nu)} = (\mathrm{Hom}(F(C),e) \circ \alpha_C \circ \beta_C)(\nu_C(1_C)) = e \circ \alpha_C(\beta_C(\nu_C(1_C)))$$
$$= \Delta(e) \circ \alpha_C((\beta \circ \nu)_C(1_C)) = (\Delta(e) \circ \phi_{S,E}(\alpha))_{(C,\beta\circ\nu)}.$$

□

The simplest example of the construction in Proposition 3.2.13 arises if we take for $\mathcal{C} = BG$ for a group G. In this case, a functor $F : \mathcal{C} \to \mathrm{Set}$ is a G-set $X = F(\bullet)$ and the left Kan extension is $\mathrm{Lan}_y F = - \times_G X : \mathrm{Set}^{BG^{op}} \to \mathrm{Set}$ by Example 3.2.7. By working with opposite categories, we can also consider the right Kan extension along the contravariant Yoneda embedding and obtain an analogous statement for right Kan extensions. By Example 3.2.8 we then have $\mathrm{Ran}_y F = \mathrm{Hom}_{\mathrm{Set}^{BG}}(-,X) : (\mathrm{Set}^{BG})^{op} \to \mathrm{Set}$ Proposition 3.2.13 then takes the following form.

Example 3.2.14: Let G be a group and X a G-set.

1. The functor $\mathrm{Lan}_y X = - \times_G X : \mathrm{Set}^{BG^{op}} \to \mathrm{Set}$ from Example 3.2.7 that sends
   - a G^{op}-set Y to the set $Y \times_G X$,
   - a G^{op}-equivariant map $f : Y \to Y'$ to the induced map $f \times_G \mathrm{id} : Y \times_G X \to Y' \times_G X$,

   is left adjoint to the functor $\mathrm{Hom}_{\mathrm{Set}}(X,-) : \mathrm{Set} \to \mathrm{Set}^{BG^{op}}$ that sends
   - a set Z to the G^{op}-set of maps $f : X \to Z$ with $(f \lhd g)(x) = f(gx)$ for $x \in X$, $g \in G$,
   - a map $f : Z \to Z'$ to $f_* : \mathrm{Hom}_{\mathrm{Set}}(X,Y) \to \mathrm{Hom}_{\mathrm{Set}}(X,Z')$, $g \mapsto f \circ g$.

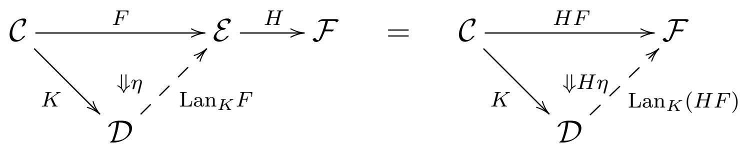
2. The functor $\mathrm{Ran}_y X = \mathrm{Hom}_{\mathrm{Set}^{BG}}(-,X) : (\mathrm{Set}^{BG})^{op} \to \mathrm{Set}$ from Example 3.2.8 that sends
   - a G-set Y to the set $\mathrm{Hom}_{BG}(Y,X)$ of G-equivariant maps $f : Y \to X$,
   - a G-equivariant map $f : Y' \to Y$ to $f^* : \mathrm{Hom}_{BG}(Y,X) \to \mathrm{Hom}_{BG}(Y',X)$, $g \mapsto g \circ f$,

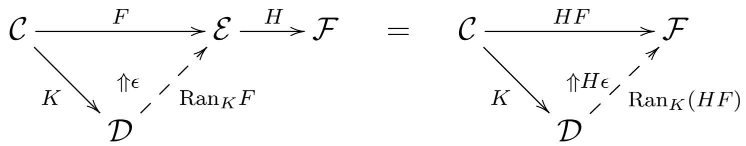
   is right adjoint to the functor $\mathrm{Hom}_{\mathrm{Set}}(-,X) : \mathrm{Set} \to (\mathrm{Set}^{BG})^{op}$ that sends
   - a set Z to the G-set of maps $f : Z \to X$ with $(g \rhd f)(z) = g \rhd f(z)$ for $g \in G$, $z \in Z$,
   - a map $f : Z' \to Z$ to $f^* : \mathrm{Hom}_{\mathrm{Set}}(Z,X) \to \mathrm{Hom}_{\mathrm{Set}}(Z',X)$, $g \mapsto g \circ f$.

3.3 Preservation of Kan extensions

It remains to consider the interaction of Kan extensions with functors. Given a left or right Kan extension $(\mathrm{Lan}_K F, \eta)$ or $(\mathrm{Ran}_K F, \epsilon)$ of a functor $F : \mathcal{C} \to \mathcal{E}$ along $K : \mathcal{C} \to \mathcal{D}$ and a functor $H : \mathcal{E} \to \mathcal{F}$, we can consider the pairs $(H\mathrm{Lan}_K F, H\eta)$ or $(H\mathrm{Ran}_K F, H\epsilon)$. If they are left or right Kan extensions of $HF : \mathcal{C} \to \mathcal{F}$ along $K : \mathcal{C} \to \mathcal{D}$, we say that the Kan extensions are *preserved* by H.

Definition 3.3.1: Let $F : \mathcal{C} \to \mathcal{E}$ and $K : \mathcal{C} \to \mathcal{D}$ be functors. A functor $H : \mathcal{E} \to \mathcal{F}$ **preserves** a left Kan extension $(\mathrm{Lan}_K F, \eta)$ or right Kan extension $(\mathrm{Ran}_K F, \epsilon)$, respectively, if $(H\mathrm{Lan}_K F, H\eta)$ or $(H\mathrm{Ran}_K F, H\epsilon)$ is a left or right Kan extension of HF along K:





As right (left) adjoint functors preserve (co)limits by Theorem 2.4.7, it is plausible that they preserve *pointwise* right (left) Kan extensions, which are given by (co)limit formulas. One could also expect them to preserve Kan extensions in more generality, as the functors in the adjunction and its unit and counit can be used to transport natural transformations between the categories $\mathcal{E}$ and $\mathcal{F}$, as in the Proof of Theorem 2.4.7. This intuition is correct.

Proposition 3.3.2: Left (right) adjoints preserve left (right) Kan extensions.

Proof:
We prove the claim for left Kan extensions. The one for right Kan extensions follows with Remark 3.1.2, 2. because $L : \mathcal{E} \to \mathcal{F}$ left adjoint to $R : \mathcal{F} \to \mathcal{E}$ with unit $\eta : \mathrm{id}_{\mathcal{E}} \Rightarrow RL$ and counit $\epsilon : LR \Rightarrow \mathrm{id}_{\mathcal{F}}$ implies that $L' : \mathcal{E}^{op} \to \mathcal{F}^{op}$ is right adjoint to $R' : \mathcal{F}^{op} \to \mathcal{E}^{op}$ with unit $\epsilon' : \mathrm{id}_{\mathcal{F}^{op}} \Rightarrow L'R'$ and counit $\eta' : R'L' \Rightarrow \mathrm{id}_{\mathcal{E}^{op}}$.

Let $(\mathrm{Lan}_K F, \eta)$ be a left Kan extension of $F : \mathcal{C} \to \mathcal{E}$ along $K : \mathcal{C} \to \mathcal{D}$ and $L : \mathcal{E} \to \mathcal{F}$ left adjoint to $R : \mathcal{F} \to \mathcal{E}$ with unit $\iota : \mathrm{id}_{\mathcal{E}} \Rightarrow RL$ and counit $\nu : LR \Rightarrow \mathrm{id}_{\mathcal{F}}$ of the adjunction satisfying (i) $R\nu \circ \iota R = \mathrm{id}_R$ and (ii) $\nu L \circ L\iota = \mathrm{id}_L$.

We show that $(L\mathrm{Lan}_K F, L\eta)$ is a left Kan extension of $LF : \mathcal{C} \to \mathcal{F}$ along $K : \mathcal{C} \to \mathcal{D}$:
For this, let $G : \mathcal{D} \to \mathcal{F}$ be a functor and $\gamma : LF \Rightarrow GK$ a natural transformation. Then $R\gamma \circ \iota F : F \Rightarrow RLF \Rightarrow RGK$ is a natural transformation with $RG : \mathcal{D} \to \mathcal{E}$, and by the universal property of $(\mathrm{Lan}_K F, \eta)$ there is a unique natural transformation $\beta : \mathrm{Lan}_K F \Rightarrow RG$ with $\beta K \circ \eta = R\gamma \circ \iota F$. Then $\alpha = \nu G \circ L\beta : L\mathrm{Lan}_K F \Rightarrow LRG \Rightarrow G$ satisfies

$$\alpha K \circ L\eta = \nu GK \circ L\beta K \circ L\eta = \nu GK \circ LR\gamma \circ L\iota F \overset{\text{nat}\,\nu}{=} \gamma \circ \nu LF \circ L\iota F \overset{\text{(ii)}}{=} \gamma.$$

Suppose that $\alpha' : L\mathrm{Lan}_K F \Rightarrow G$ is another natural transformation with $\alpha' K \circ L\eta = \gamma$. Then the natural transformation $R\alpha' \circ \iota\mathrm{Lan}_K F : \mathrm{Lan}_K F \Rightarrow RL\mathrm{Lan}_K F \Rightarrow RG$ satisfies

$$(R\alpha' \circ \iota\mathrm{Lan}_K F)K \circ \eta = R\alpha' K \circ \iota\mathrm{Lan}_K FK \circ \eta \overset{\text{nat}\,\iota}{=} R\alpha' K \circ RL\eta \circ \iota F = R\gamma \circ \iota F,$$

and the universal property of $(\mathrm{Lan}_K F, \eta)$ implies $R\alpha' \circ \iota\mathrm{Lan}_K F = \beta$. This yields

$$\alpha = \nu G \circ L\beta = \nu G \circ LR\alpha' \circ L\iota\mathrm{Lan}_K F \overset{\text{nat}\,\nu}{=} \alpha' \circ \nu L\mathrm{Lan}_K F \circ L\iota\mathrm{Lan}_K F \overset{\text{(ii)}}{=} \alpha'$$

and shows that $(L\mathrm{Lan}_K F, L\eta)$ has the universal property of the left Kan extension. □

Example 3.3.3:

1. The forgetful functor $V : \mathrm{Top} \to \mathrm{Set}$ has a left and a right adjoint by Example 2.4.8, 1. and hence preserves left and right Kan extensions.

2. The forgetful functors $V : R\text{-Mod} \to \mathrm{Set}$ and $V : \mathrm{Grp} \to \mathrm{Set}$ preserve right Kan-extensions, as they have left adjoints by Example 2.4.8, 2. and 3.

3. The forgetful functor $V : H\text{-Set} \to \mathrm{Set}$ for a group H preserves left Kan extensions, as it has a right adjoint by Example 2.4.8, 4.

4. The functor $M \otimes_R - : R\text{-Mod} \to \mathrm{Ab}$ for an R-right module M preserves left Kan extensions and the functor $\mathrm{Hom}_{\mathrm{Ab}}(M, -) : \mathrm{Ab} \to R\text{-Mod}$ for an R-right module M preserves right Kan extensions by Example 2.4.8, 6.

5. By Example 2.4.9, for a ring homomorphism $\phi : R \to S$ the associated induction functor $\mathrm{Ind}_\phi : R\text{-Mod} \to S\text{-Mod}$ preserves left Kan extensions, the coinduction functor $\mathrm{Coind}_\phi : R\text{-Mod} \to S\text{-Mod}$ preserves right Kan extensions and the restriction functor $\mathrm{Res}_\phi : S\text{-Mod} \to R\text{-Mod}$ preserves left and right Kan extensions.

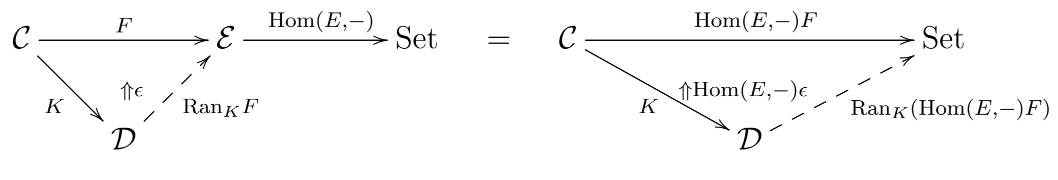
6. The inclusion $\iota : \mathcal{U} \to \mathcal{C}$ for a reflective subcategory $U \subset \mathcal{C}$ preserves right Kan extensions and its reflector $L : \mathcal{C} \to \mathcal{U}$ left Kan extensions, see Example 2.4.11 for concrete examples.

The examples involving forgetful functors $V : \mathcal{E} \to \mathrm{Set}$ seem somewhat trivial, but they often allow one to guess a Kan extension of a functor $F : \mathcal{C} \to \mathcal{E}$ from the corresponding Kan extension of $VF : \mathcal{C} \to \mathrm{Set}$. The latter can often be promoted to a Kan extension with values in $\mathcal{E}$ by equipping it with additional structure. It is tempting to employ a similar strategy for Kan extension in other categories by applying a Hom functor instead of a forgetful functor.

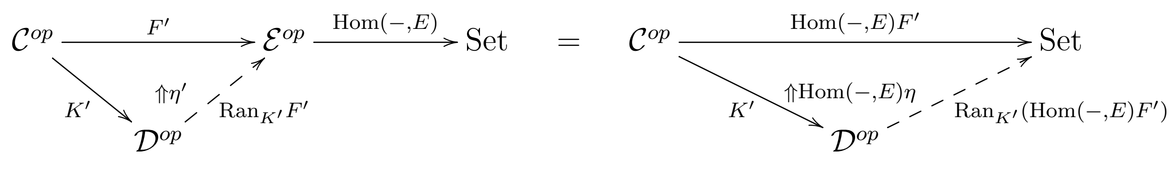
This raises the question which Kan extensions are preserved by Hom-functors. As every left Kan extension of $F : \mathcal{C} \to \mathcal{E}$ along $K : \mathcal{C} \to \mathrm{Set}$ corresponds to a right Kan extension of $F' : \mathcal{C}^{op} \to \mathcal{E}^{op}$ along $K' : \mathcal{C}^{op} \to \mathcal{E}^{op}$ by Remark 3.1.2, 2, it is sufficient to consider this question for right Kan extensions. The relevant Hom-functors are then *covariant* Hom-functors $\mathrm{Hom}(E, -) : \mathcal{E} \to \mathrm{Set}$. They preserve precisely the *pointwise* right Kan extensions.

Theorem 3.3.4: Let $F : \mathcal{C} \to \mathcal{E}$ and $K : \mathcal{C} \to \mathcal{D}$ be functors.

1. A right Kan extension $(\mathrm{Ran}_K F, \epsilon)$ is pointwise if and only if it is preserved by all functors $\mathrm{Hom}(E, -) : \mathcal{E} \to \mathrm{Set}$



2. A left Kan extension $(\mathrm{Lan}_K F, \eta)$ is pointwise if and only if the associated right Kan extension from Remark 3.1.2 is preserved by all functors $\mathrm{Hom}(-, E) : \mathcal{E}^{op} \to \mathrm{Set}$



Proof:
We prove the claim for right Kan extensions. The proof for left Kan extensions follows with Remark 3.1.2, 2. , as $\mathrm{Hom}_{\mathcal{E}^{op}}(E,-) \cong \mathrm{Hom}_{\mathcal{E}}(-,E) : \mathcal{E}^{op} \to \mathrm{Set}$.

1. If the right Kan extension $(\mathrm{Ran}_K F, \epsilon)$ is pointwise, then by Theorem 3.2.5 it is given by the limit cone $\lambda^D : \Delta(\lim(FP_D)) \Rightarrow FP_D$. By Proposition 2.4.5 its image $\mathrm{Hom}(E,-)\lambda^D$ is a limit cone of $\mathrm{Hom}(E,-)FP_D$ and by Theorem 3.2.5 it defines the right Kan extension of $\mathrm{Hom}(E,-)F$ along K.

2. Suppose now that $(\mathrm{Ran}_K F, \epsilon)$ is a right Kan extension of F along K such that for all $E \in \mathrm{Ob}\mathcal{E}$ the pair $(\mathrm{Hom}(E,-)\mathrm{Ran}_K F, \mathrm{Hom}(E,-)\epsilon)$ is a right Kan extension of $\mathrm{Hom}(E,-)F$ along K.

To prove that $(\mathrm{Ran}_K F, \epsilon)$ is pointwise, we have to show that $\mathrm{Ran}_K F(D)$ is a limit of the functor $FP_D : D \downarrow K \to \mathcal{E}$ or, equivalently, that the set $\mathrm{cone}(E, FP_D)$ of cones over FP_D with apex E is in bijection with the set $\mathrm{Hom}_{\mathcal{E}}(E, \mathrm{Ran}_K F(D))$. For this, note first that

$$\begin{aligned}\mathrm{Hom}_{\mathcal{E}}(E, \mathrm{Ran}_K F(D)) &\cong \mathrm{Hom}_{\mathrm{Set}^{\mathcal{D}}}(\mathrm{Hom}(D,-), \mathrm{Hom}(E, \mathrm{Ran}_K F(-))) \\ &\cong \mathrm{Hom}_{\mathrm{Set}^{\mathcal{C}}}(\mathrm{Hom}(D, K(-)), \mathrm{Hom}(E, F(-))),\end{aligned}$$

where we used in the first step the Yoneda Lemma, stating that natural transformations $\mathrm{Hom}(D,-) \Rightarrow \mathrm{Hom}(E, \mathrm{Ran}_K F(-))$ are in bijection with elements of $\mathrm{Hom}_{\mathcal{E}}(E, \mathrm{Ran}_K F(D))$, and in the second step the universal property of the right Kan extension $\mathrm{Hom}(E,-)\mathrm{Ran}_K F$, stating that natural transformations $\delta : \mathrm{Hom}(D,-)K \Rightarrow \mathrm{Hom}(E,-)F$ are in bijection with natural transformations $\beta : \mathrm{Hom}(D,-) \Rightarrow \mathrm{Hom}(E,-)\mathrm{Ran}_K F$.

It remains to show that natural transformations $\phi : \mathrm{Hom}(D, K(-)) \Rightarrow \mathrm{Hom}(E, F(-))$ are in bijection with cones over FP_D with apex E.

The former are collections of maps $\phi_C : \mathrm{Hom}_{\mathcal{D}}(D, K(C)) \to \mathrm{Hom}_{\mathcal{E}}(E, F(C))$, $d \mapsto \phi_C(d)$ for $C \in \mathrm{Ob}\mathcal{C}$ with $\phi_{C'}(K(c) \circ d) = F(c) \circ \phi_C(d)$ for all morphisms $c : C \to C'$ and $d : D \to K(C)$. The latter are collections of morphisms $\lambda_{(C,d)} : E \to F(C)$ indexed by pair of objects $C \in \mathrm{Ob}\mathcal{C}$ and morphisms $d : D \to K(C)$ with $\lambda_{(C', K(c)\circ d)} = F(c) \circ \lambda_{(C,d)}$ for for all morphisms $c : C \to C'$ and $d : D \to K(C)$. The bijection is thus given by

$$\begin{array}{lcl}\mathrm{cone}(E, FP_D) & \to & \mathrm{Hom}_{\mathrm{Set}^{\mathcal{C}}}(\mathrm{Hom}(D, K(-)), \mathrm{Hom}(E, F(-))) \\ \lambda_{(C,d)} : E \to F(C) & \mapsto & \phi_C : d \mapsto \lambda_{(C,d)}.\end{array}$$

□

Thus, we can characterise *pointwise* Kan extensions not only as the Kan extensions given by (co)limit formulas, but also as the Kan extensions that can be related to Kan extensions in Set by applying Hom-functors. As a good interaction with functors is essential for categorical concepts, these are the only Kan extensions that are relevant in practice. By Corollary 3.2.6 this includes in particular all Kan extensions of functors $F : \mathcal{C} \to \mathcal{E}$ from small categories $\mathcal{C}$ into bicomplete categories $\mathcal{E}$ along functors $K : \mathcal{C} \to \mathcal{D}$ into a locally small category $\mathcal{D}$.

References:

- Chapter X in Mac Lane, S. (2013) Categories for the working mathematician,
- Chapter 4 in Richter, B. (2020) From categories to homotopy theory,
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4 Ends and Coends

4.1 Ends and coends: definition and properties

In this section, we introduce ends and their dual concept, coends. They are mathematical tools that allow rather simple and intuitive computations with limits and colimits and gives rise to a more efficient and intuitive formula for Kan extensions. Ends and coends are in principle nothing new and arise as special cases of limits and colimits. Nevertheless, they are useful and important, in roughly the same way as tensor products over rings.

Recall that the tensor product of an R-module N and an R-right module M is, by definition, a quotient of the free abelian group generated by the set $M \times N$ by the subgroup that enforces the R-bilinearity: $M \otimes_R N = \langle M \times N \rangle_{\mathbb{Z}} / U$ where U is the subgroup generated by the elements

- $(m + m', n) - (m, n) - (m', n)$,
- $(m, n + n') - (m, n) - (m, n')$,
- $(m \lhd r, n) - (m, r \rhd n)$

for all $m, m' \in M$, $n, n' \in N$ and $r \in R$. Thus, the tensor product over R is nothing but a coequaliser (quotient) of a coproduct (direct sum, free module), and in principle quotients and direct sums are sufficient to describe it. Nevertheless, it would be very cumbersome to describe tensor products in this way. Rather, one characterises them by a universal property (arising from the ones of quotients and direct sums) and works directly with tensor products.

(Co)ends describe categorical generalisations of this construction and are defined in a similar way as (co)limits. Limits are taken of functors $F : \mathcal{C} \to \mathcal{E}$ and defined as terminal *cones*, natural transformations from a constant functor to F and colimits as initial *cocones*, natural transformations from F to a constant functor. Ends and coends are taken only for functors $F : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{E}$ and associated to wedges and cowedges, dinatural transformations from a constant functor to F or from F to a constant functor.

A dinatural transformation between functors $F, G : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{E}$ can be viewed as a weakening of the concept of a natural transformation. While a natural transformation $\nu : F \Rightarrow G$ is defined by component morphisms $\nu_{C,C'} : F(C, C') \to G(C, C')$ for all $C, C' \in \mathrm{Ob}\mathcal{C}$ and required to be natural in both arguments, the components of a dinatural transformation are defined only on the diagonal, for $C = C'$ and the naturality condition is weakened accordingly. The prefix *di* thus stands for *defined only on the diagonal.*

Definition 4.1.1: Let $F, G : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{E}$ be functors. A **dinatural transformation** $\tau : F \overset{\bullet}{\Rightarrow} G$ is a collection of morphisms $\tau_C : F(C, C) \to G(C, C)$ such that the following diagram commutes for all morphisms $f : C \to C'$

$$\begin{array}{ccccc} F(C',C) & \xrightarrow{F(f,1_C)} & F(C,C) & & \\ \downarrow{\scriptstyle F(1_{C'},f)} & & & \searrow^{\tau_C} & \\ F(C',C') & & & & G(C,C) \\ & \searrow^{\tau_{C'}} & & & \downarrow{\scriptstyle G(1_C,f)} \\ & & G(C',C') & \xrightarrow[G(f,1_{C'})]{} & G(C,C') \end{array}$$

Example 4.1.2:

1. Every natural transformation $\nu : F \Rightarrow G$ for $F, G : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{E}$ defines a dinatural transformation $\nu' : F \overset{\bullet}{\Rightarrow} G$ with components $\nu'_C = \nu_{C,C} : F(C,C) \to G(C,C)$ (Exercise 35).

2. A functor $F : BG^{op} \times BG \to \text{Set}$ for a group G corresponds to a set $X = F(\bullet)$ with a left G-action $\rhd$ defined by $g \rhd x = F(1_\bullet, g)x$ and right G-action $\lhd$ defined by $x \lhd g = F(g, 1_\bullet)x$ with $g \rhd (x \lhd h) = (g \rhd x) \lhd h$ for all $x \in X$ and $g, h \in G$.

   A natural transformation $\nu : F \Rightarrow F'$ is a map $f : X \to X'$ that is equivariant with respect to both actions: $f(g \rhd x) = g \rhd f(x)$ and $f(x \lhd g) = f(x) \lhd g$ for all $g \in G$ and $x \in X$.

   A dinatural transformation $F \overset{\bullet}{\Rightarrow} F'$ is a map $f : X \to X'$ with $g \rhd f(x \lhd g) = f(g \rhd x) \lhd g$ for all $g \in G$ and $x \in X$, or, equivalently, equivariant with respect to the conjugation action $\rhd' : G \times X \to X$, $(g, x) \mapsto g \rhd x \lhd g^{-1}$.

Just as (co)cones for a functor F are defined as a natural transformations between F and a constant functor, (co)wedges for a functor $F : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{E}$ are defined as dinatural transformations between F and a constant functor. There are also notions of (co)wedge morphisms and categories of (co)wedges for the functor F. Just as limits and colimits are terminal cones and initial cocones, ends and coends are defined as terminal wedges and and initial cowedges.

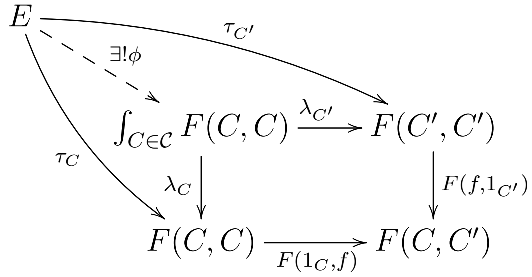
Definition 4.1.3: Let $F : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{E}$ be a functor.

1. A **wedge** over F is a dinatural transformation $\tau : \Delta(E) \overset{\bullet}{\Rightarrow} F$ from a constant functor.
2. A **wedge morphism** from $\tau : \Delta(E) \overset{\bullet}{\Rightarrow} F$ to $\tau' : \Delta(E') \overset{\bullet}{\Rightarrow} F$ is a morphism $e : E \to E'$ in $\mathcal{E}$ with $\tau'_C \circ e = \tau_C$ for all $C \in \text{Ob}\mathcal{C}$.
3. A **cowedge** under F is a dinatural transformation $\tau : F \overset{\bullet}{\Rightarrow} \Delta(E)$ to a constant functor.
4. A **cowedge morphism** from $\tau : F \overset{\bullet}{\Rightarrow} \Delta(E)$ to $\tau' : F \overset{\bullet}{\Rightarrow} \Delta(E')$ is a morphism $e : E \to E'$ in $\mathcal{E}$ with $\tau'_C = f \circ \tau_C$ for all $C \in \text{Ob}\mathcal{C}$.

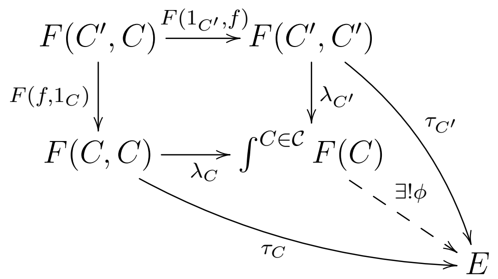
(Co)wedges and (co)wedge morphisms over F form a category (co)wedge(F).

Definition 4.1.4: Let $F : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{E}$ be a functor.

1. An **end** of F is a terminal object in wedge(F): a wedge $\lambda : \Delta(\int_{C \in \mathcal{C}} F(C,C)) \overset{\bullet}{\Rightarrow} F$ such that for every wedge $\tau : \Delta(E) \overset{\bullet}{\Rightarrow} F$ there is a unique wedge morphism $\phi : E \to \int_{C \in \mathcal{C}} F(C,C)$.



2. A **coend** of F is an initial object in cowedge(F): a cowedge $\lambda: F \overset{\bullet}{\Rightarrow} \Delta(\int^{C\in\mathcal{C}} F(C,C))$ such that for each cowedge $\tau: F \overset{\bullet}{\Rightarrow} \Delta(E)$ there is a unique cowedge morphism $\phi: \int^{C\in\mathcal{C}} F(C,C) \to E$.



Remark 4.1.5:

1. As terminal and initial objects, ends and coends are unique up to unique isomorphism.
2. A cowedge under a functor $F : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{E}$ is a wedge over the associated functor $F' : \mathcal{C} \times \mathcal{C}^{op} \to \mathcal{E}^{op}$. This implies that an end or coend of $F : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{E}$ is a coend or end of $F' : \mathcal{C} \times \mathcal{C}^{op} \to \mathcal{E}^{op}$.

The integral notation for ends and coends will be justified in the next subsection. At the moment, we just take it as a notation for the objects that characterise the end or coend.

The diagrams for the universal properties of (co)ends in Definition 4.1.4 are more illuminating. By comparing with the diagrams for pullbacks and pushouts in Example 2.1.6, 3. we see that for each morphism $f : C \to C'$ the end is a *pullback* of $F(1_{C'}, f) : F(C,C) \to F(C,C')$ and $F(f, 1_C) : F(C', C') \to F(C, C')$ and the coend a *pushout* of $F(1_{C'}, f) : F(C', C) \to F(C', C')$ and $F(f, 1_C) : F(C', C) \to F(C, C)$. As pullbacks are equalisers of products and pushouts coequalisers of coproduct, this also fits well with the intuition about tensor products at the beginning of this section. What distinguishes (co)ends from usual pullbacks and pushouts is that they are *simultaneous* pullbacks and pushouts for all morphisms in $\mathcal{C}$.

Example 4.1.6:

1. **group actions:** Let G be a group and $F : BG^{op} \times BG \to \mathrm{Set}$ a set $X = F(\bullet)$ with commuting left G-action $\rhd$ and right G-action $\lhd$, as in Example 4.1.2, 2.

   A wedge over F is a map $\tau : E \to X$ with $g \rhd \tau(e) = \tau(e) \lhd g$ for all $e \in E$, $g \in G$ or, equivalently, a map $\tau' : E \to X^G$ into the fixed point set X^G of the conjugation action $\rhd' : G \times X \to X$, $g \rhd' x = g \rhd x \lhd g^{-1}$. Hence, the end of F is the inclusion $\iota : X^G \to X$ and the unique morphism $\phi : E \to X$ with $\iota \circ \phi = \tau$ is the corestriction τ' of τ.

   A cowedge under F is a map $\tau : X \to E$ with $\tau(g \rhd x) = \tau(x \lhd g)$ or, equivalently, $\tau(g \rhd' x) = x$ for all $x \in X, g \in G$. Hence, the end of F is the orbit set $\mathcal{O} = \{G \rhd' x \mid x \in G\}$ with the canonical surjection $\pi : X \to \mathcal{O}$. The map $\phi : \mathcal{O} \to E$, $G \rhd' x \mapsto \tau(x)$ is the unique map with $\phi \circ \pi = \tau$.

2. **tensor product of G-sets:** As a special case of 1. consider $F : BG^{op} \times BG \to \mathrm{Set}$ with $F(\bullet) = X \times Y$ for a right G-set X and left G-set Y.

   A wedge over F is a map $\tau : E \to X \times Y$ with values in the fixed point set of the action $\rhd' : G \times X \times Y \to X \times Y$, $g \rhd' (x, y) = (x \lhd g^{-1}, g \rhd y)$, and the end its inclusion into $X \times Y$.

   A cowedge over F is a map $\tau : X \times Y \to E$ with $\tau(x \lhd g, y) = \tau(x, g \rhd y)$ for all $x \in X$, $y \in Y$ and $g \in G$. Any such map induces a unique map $\phi' : X \times_G Y \to E$ with $\phi' \circ \pi = \tau$, and hence the coend is $\pi : X \times Y \to X \times_G Y$.

3. **evaluation:** Let k be a commutative ring, N a k-module and consider the functor $F = \mathrm{Hom}_k(-, N)\otimes_k - : k\text{-Mod}^{op} \times k\text{-Mod} \to k\text{-Mod}$ that assigns to
   - a pair of k-modules (M_1, M_2) the k-module $\mathrm{Hom}_k(M_1, N)\otimes_k M_2$,
   - to a pair (f_1, f_2) of k-linear maps $f_1 : M_1' \to M_1$ and $f_2 : M_2 \to M_2'$ the k-linear map
   $$\mathrm{Hom}_k(f_1, N)\otimes f_2 : f\otimes m \mapsto (f \circ f_1)\otimes f_2(m).$$

   A cowedge of F is a collection of k-linear maps $\tau_M : \mathrm{Hom}_k(M, N)\otimes_k M \to E$ satisfying $\tau_{M'}(f \circ g\otimes m') = \tau_M(f\otimes g(m'))$ for all $m' \in M'$ and k-linear $f : M \to N$ and $g : M' \to M$. The coend of F is given by the maps $\lambda_M : \mathrm{Hom}_k(M, N)\otimes M \to N$, $f\otimes m \mapsto f(m)$, and $\phi : N \to E$, $n \mapsto \tau_N(\mathrm{id}_N\otimes n)$ is the unique map with $\phi\circ\lambda_M = \tau_M$ for all k-modules M.

4. **natural transformations:** Let $\mathcal{C}$ be small, $F, G : \mathcal{C} \to \mathcal{E}$ functors and consider the functor $H = \mathrm{Hom}_{\mathcal{E}}(F(-), G(-)) : \mathcal{C}^{op} \times \mathcal{C} \to \mathrm{Set}$.

   A wedge over H is a collection of maps $\tau_C : E \to \mathrm{Hom}_{\mathcal{E}}(F(C), G(C))$, $e \mapsto \tau_C^e$ indexed by objects of C, with $\tau_{C'}^e \circ F(f) = G(f) \circ \tau_C^e$ for all morphisms $f : C \to C'$. This is a map $\tau' : E \to \mathrm{Nat}(F, G)$, $e \mapsto \tau^e$ that assigns to $e \in E$ a natural transformation $\tau^e : F \Rightarrow G$.

   The end of H is given by the maps $\lambda_C : \mathrm{Nat}(F, G) \to \mathrm{Hom}_{\mathcal{E}}(F(C), G(C))$, $\tau \mapsto \tau_C$, and the unique morphism $\phi : E \to \mathrm{Nat}(F, G)$ with $\lambda_C \circ \phi = \tau_C$ is $\phi = \tau'$.

We will now show that (co)wedges are special cases of (co)cones and consequently (co)ends are special cases of (co)limits. For this, we introduce the twisted arrow category, which exists in the literature with various notations.

Definition 4.1.7: Let $\mathcal{C}$ be a category.

1. The **twisted arrow category** $\mathcal{C}^\tau$ has
   - as objects triples (C_1, c, C_2) of objects $C_1, C_2 \in \mathrm{Ob}\mathcal{C}$ and a morphism $c : C_1 \to C_2$,
   - as morphisms from (C_1, c, C_2) to (C_1', c', C_2) pairs (f_1, f_2) of morphisms $f_1 : C_1' \to C_1$ and $f_2 : C_2 \to C_2'$ with $f_2 \circ c \circ f_1 = c'$
   $$\begin{array}{ccc} C_1 & \xrightarrow{\;c\;} & C_2 \\ {\scriptstyle f_1}\big\uparrow & & \big\downarrow{\scriptstyle f_2} \\ C_1' & \xrightarrow[\;c'\;]{} & C_2' \end{array}$$

   Composition of morphisms and identity morphisms are given by the ones in $\mathcal{C}$.

2. The **twisted arrow functor** $\chi : \mathcal{C}^\tau \to \mathcal{C}^{op} \times \mathcal{C}$ assigns to
   - an object (C_1, c, C_2) the pair (C_1, C_2),
   - a morphism $(f_1, f_2) : (C_1, c, C_2) \to (C_1', c', C_2')$ the corresponding morphism $(f_1, f_2) : (C_1, C_2) \to (C_1', C_2')$ in $\mathcal{C}^{op} \times \mathcal{C}$.

To show that (co)ends are special cases of (co)limits, it is sufficient to show that (co)wedges are special cases of (co)cones. By Remark 2.1.5, 3 and Remark 4.1.5, 2. it is sufficient to consider cones, wedges and ends. For a functor $H : \mathcal{C} \to \mathcal{D}$ we denote by $H' : \mathcal{C}^{op} \to \mathcal{D}^{op}$ the associated functor between the opposite categories and identify $(\mathcal{C}^{op} \times \mathcal{C})^{op} = \mathcal{C} \times \mathcal{C}^{op}$. In particular, the twisted arrow functor defines functors $\chi' : (\mathcal{C}^\tau)^{op} \to \mathcal{C} \times \mathcal{C}^{op}$ and $\chi'' : (\mathcal{C}^{\mathrm{op}\,\tau})^{op} \to \mathcal{C}^{op} \times \mathcal{C}$.

Proposition 4.1.8: Let $F : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{E}$ be a functor. Then:

1. The category wedge(F) is equivalent to cone($F\chi$).
2. The category cowedge(F) is equivalent to cocone($F\chi''$).

In particular, an end of F is a limit of $F\chi$ and a coend of F a colimit of $F\chi''$.

Proof:
We prove the first claim. The second claim follows from Remarks 2.1.5, 3 and 4.1.5, 2.
We consider the functor $\phi : \text{cone}(F\chi) \to \text{wedge}(F)$ that assigns to

- a cone $\mu : \Delta(E) \Rightarrow F\chi$ the wedge $\phi(\mu) : \Delta(E) \overset{\bullet}{\Rightarrow} F$ with components $\phi(\mu)_C = \mu_{(C,1_C,C)}$,
- a cone morphism $e : E \to E'$ from μ to μ' to the wedge morphism e from $\phi(\mu)$ to $\phi(\mu')$.

and the functor $\psi : \text{wedge}(F) \to \text{cone}(F\chi)$ that assigns to

- a wedge $\nu : \Delta(E) \overset{\bullet}{\Rightarrow} F$ the cone $\psi(\nu) : \Delta(E) \Rightarrow F\chi$ with $\psi(\nu)_{(C_1,c,C_2)} = F(1_{C_1}, c) \circ \nu_{C_1}$,
- a wedge morphism $e : E \to E'$ from ν to ν' the cone morphism e from $\psi(\nu)$ to $\psi(\nu')$.

To show that ϕ and ψ are defined, note that $c \in \text{Hom}(C_1, C_2)$ defines two morphisms in $\mathcal{C}^\tau$

$$(1_{C_1}, c) : (C_1, 1_{C_1}, C_1) \to (C_1, c, C_2) \qquad (c, 1_{C_2}) : (C_2, 1_{C_2}, C_2) \to (C_1, c, C_2).$$

That $\phi(\mu)$ is a wedge for each cone μ over $F\chi$ then follows from the cone property of μ:

$$\begin{aligned} F(1_{C_1}, c) \circ \phi(\mu)_{C_1} &= F(1_{C_1}, c) \circ \mu_{(C_1,1_{C_1},C_1)} \overset{\text{cone}}{=} \mu_{(C_1,c,C_2)} \overset{\text{cone}}{=} F(c, 1_{C_2}) \circ \mu_{(C_2,1_{C_2},C_2)} \\ &= F(c, 1_{C_2}) \circ \phi(\mu)_{C_2} \end{aligned}$$

To show that $\psi(\nu)$ is a cone over $F\chi$ for each wedge ν over F, we compute for $c \in \text{Hom}_{\mathcal{C}}(C_1, C_2)$, $c' \in \text{Hom}_{\mathcal{C}}(C_1', C_2')$, $f_1 \in \text{Hom}_{\mathcal{C}}(C_1', C_1)$ and $f_2 \in \text{Hom}_{\mathcal{C}}(C_2, C_2')$ with $c' = f_2 \circ c \circ f_1$

$$\begin{aligned} &F(f_1, f_2) \circ \psi(\nu)_{(C_1,c,C_2)} = F(f_1, f_2) \circ F(1_{C_1}, c) \circ \nu_{C_1} \overset{\text{func}}{=} F(f_1, 1_{C_2'}) \circ F(1_{C_1}, f_2 \circ c) \circ \nu_{C_1} \\ &\overset{\text{func}}{=} F(1_{C_1'}, f_2 \circ c) \circ F(f_1, 1_{C_1}) \circ \nu_{C_1} \overset{\text{wedge}}{=} F(1_{C_1'}, f_2 \circ c) \circ F(1_{C_1'}, f_1) \circ \nu_{C_1'} \\ &\overset{\text{func}}{=} F(1_{C_1'}, f_2 \circ c \circ f_1) \circ \nu_{C_1'} = \psi(\nu)_{(C_1', f_2 \circ c \circ f_1, C_2')} = \psi(\nu)_{(C_1', c', C_2')} \end{aligned}$$

That every cone morphism $e : E \to E'$ from μ to μ' defines a wedge morphism from $\phi(\mu)$ to $\phi(\mu')$ and every wedge morphism from ν to ν' a cone morphism from $\psi(\nu)$ to $\psi(\nu')$ follows directly from the definitions. For all $C \in \text{Ob}\mathcal{C}$ and $c \in \text{Hom}_{\mathcal{C}}(C_1, C_2)$ one has

$$\begin{aligned} &\phi(\mu')_C \circ e = \mu'_{(C,1_C,C)} \circ e = \mu_{(C,1_C,C)} = \phi(\mu)_C \\ &\psi(\nu')_{(C_1,c,C_2)} \circ e = F(1_{C_1}, c) \circ \nu'_{C_1} \circ e = F(1_{C_1}, c) \circ \nu_{C_1} = \psi(\nu)_{(C_1,c,C_2)}. \end{aligned}$$

As ϕ and ψ are the identity on the morphisms, it follows that they are functors. To show that ϕ and ψ form an equivalence of categories, we compute for all $C \in \text{Ob}\mathcal{C}$ and $c \in \text{Hom}_{\mathcal{C}}(C_1, C_2)$ cones μ over $F\chi$ and wedges ν over F

$$\begin{aligned} &\psi(\phi(\mu))_{(C_1,c,C_2)} = F(1_{C_1}, c) \circ \phi(\mu)_{C_1} = F(1_{C_1}, c) \circ \mu_{(C_1,1_{C_1},C_1)} \overset{\text{cone}}{=} \mu_{(C_1,c,C_2)} \\ &\phi(\psi(\nu))_C = \psi(\nu)_{(C,1_C,C)} = F(1_C, 1_C) \circ \nu_C = \nu_C. \end{aligned}$$

This shows that $\psi = \phi^{-1}$ and ϕ, ψ are equivalences of categories. As such, they send terminal to terminal objects and initial ones to initial ones and identify limits of $F\chi$ with ends of F. □

In particular, this characterisation of (co)ends as limits yields sufficient conditions for the existence of (co)ends for all functors $F : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{E}$. By applying Proposition 2.2.5 we also obtain a characterisation of (co)ends as functors.

Corollary 4.1.9: If $\mathcal{C}$ is small and $\mathcal{E}$ (co)complete, all (co)ends of functors $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{E}$ exist and define functors $\int_{C\in\mathcal{C}} : \mathcal{E}^{\mathcal{C}^{op}\times\mathcal{C}} \to \mathcal{E}$ and $\int^{C\in\mathcal{C}} : \mathcal{E}^{\mathcal{C}^{op}\times\mathcal{C}} \to \mathcal{E}$.

4.2 (Co)end calculus

After identifying (co)ends as special cases of (co)limits, we now show that they render certain formulas with (co)limits more simple and intuitive. In fact, with the integral notation, many identities for limits and colimits appear as categorical counterparts of well-known theorems in calculus, such as Fubini's or Lebesgue's theorem. All of the following are direct consequences of certain identities for (co)limits and the fact that (co)ends are (co)limits.

Corollary 4.2.1: Let $\mathcal{C}$ be small, $\mathcal{E}$ (co)complete, $F : \mathcal{C} \times \mathcal{C}^{op} \to \mathcal{E}$ a functor and $G : \mathcal{E} \to \mathcal{F}$ a functor that preserves small (co)limits, also called a **(co)continuous** functor. Then:

$$G\left(\int_{C\in\mathcal{C}} F(C,C)\right) \cong \int_{C\in\mathcal{C}} GF(C,C) \qquad G\left(\int^{C\in\mathcal{C}} F(C,C)\right) \cong \int^{C\in\mathcal{C}} GF(C,C).$$

Proof:
This follows directly from the fact that (co)ends of F and GF are (co)limits of $F\chi$ and $GF\chi$, respectively, by Proposition 4.1.8, that the twisted arrow categories $\mathcal{C}^\tau$ are small for all small categories $\mathcal{C}$ and that G preserves small (co)limits. □

Corollary 4.2.2: (**Fubini's Theorem**)
Let $F : (\mathcal{C} \times \mathcal{D})^{op} \times (\mathcal{C} \times \mathcal{D}) \to \mathcal{E}$ be a functor. Whenever all (co)ends are defined one has

$$\int_{C\in\mathcal{C}} \int_{D\in\mathcal{D}} F(C,D,C,D) = \int_{D\in\mathcal{D}} \int_{C\in\mathcal{C}} F(C,D,C,D) = \int_{(C,D)\in\mathcal{C}\times\mathcal{D}} F(C,D,C,D)$$
$$\int^{C\in\mathcal{C}} \int^{D\in\mathcal{D}} F(C,D,C,D) = \int^{D\in\mathcal{D}} \int^{C\in\mathcal{C}} F(C,D,C,D) = \int^{(C,D)\in\mathcal{C}\times\mathcal{D}} F(C,D,C,D).$$

Proof:
This follows, because (co)ends are (co)limits by Proposition 4.1.8, with Theorem 2.3.4 and with the canonical equivalence of categories $(\mathcal{C} \times \mathcal{D})^\tau \cong \mathcal{C}^\tau \times \mathcal{D}^\tau$. □

Corollary 4.2.3: (**Lebesgue's theorem**)
Let $F : \mathcal{B} \times \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{E}$ be a functor. Whenever the (co)ends and (co)limits exist

$$\lim_{\mathcal{B}} \int_{C\in\mathcal{C}} F(-,C,C) \cong \int_{C\in\mathcal{C}} \lim_{\mathcal{B}} F(-,C,C) \qquad \operatorname*{colim}_{\mathcal{B}} \int^{C\in\mathcal{C}} F(-,C,C) \cong \int^{C\in\mathcal{C}} \operatorname*{colim}_{\mathcal{B}} F(-,C,C).$$

Proof:
This follows from Theorem 2.3.4 and because (co)ends are (co)limits by Proposition 4.1.8. □

Further examples that translate claims about (co)limits into integration formulas for (co)ends are given as exercises for this chapter, see Section 8.4. We now derive the last and most important formula of this type, namely a formula for pointwise Kan extensions.

To translate the (co)limit formulas for Kan extensions from Theorem 3.2.5 into integration formulas for (co)ends we first require concrete descriptions of (co)ends in terms of (co)equalisers and (co)products. This is the concrete counterpart of the abstract characterisation of (co)ends as (co)limits in Proposition 4.1.8. It also generalises the concrete formulas for (co)ends in Example 4.1.6 and the observations about pullbacks and pushouts after Remark 4.1.5.

Proposition 4.2.4: If $\mathcal{C}$ is small and $\mathcal{E}$ (co)complete, every functor $F : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{E}$ has a (co)end, and it is given by

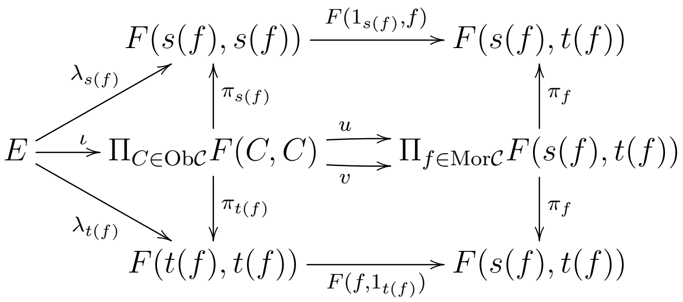
$$\int_{C\in\mathcal{C}} F(C,C) \cong \mathrm{eq}\left(\Pi_{C\in\mathrm{Ob}\mathcal{C}} F(C,C) \underset{v}{\overset{u}{\rightrightarrows}} \Pi_{f\in\mathrm{Mor}\mathcal{C}} F(s(f),t(f))\right), \tag{19}$$
$$\int^{C\in\mathcal{C}} F(C,C) \cong \mathrm{coeq}\left(\Pi_{f\in\mathrm{Mor}\mathcal{C}} F(t(f),s(f)) \underset{v'}{\overset{u'}{\rightrightarrows}} \Pi_{C\in\mathrm{Ob}\mathcal{C}} F(C,C)\right),$$

where u, v, u', v' are the morphisms induced by the universal property of the (co)product with

- $\pi_f \circ u = F(1_{s(f)}, f) \circ \pi_{s(f)}$ and $\pi_f \circ v = F(f, 1_{t(f)}) \circ \pi_{t(f)}$,
- $u' \circ \iota_f = \iota_{s(f)} \circ F(f, 1_{s(f)})$ and $v' \circ \iota_f = \iota_{t(f)} \circ F(1_{t(f)}, f)$

and π_f, π_C, ι_f, ι_C are the projection and inclusion morphisms for the products and coproducts.

Proof:
It is sufficient to prove the claim for ends; the one for coends follows with Remarks 2.1.5, 3 and 4.1.5, 2. We write $\iota : E \to \Pi_{C\in\mathrm{Ob}\mathcal{C}} F(C,C)$ for the equaliser in (19)



and show that the morphisms $\lambda_C = \pi_C \circ \iota : E \to F(C,C)$ for $C \in \mathrm{Ob}\mathcal{C}$ define the end of F.

To show that they define a wedge over F we compute for all morphisms $f : C \to C'$

$$F(f, 1_{C'}) \circ \lambda_{C'} = F(f, 1_{C'}) \circ \pi_{C'} \circ \iota = \pi_f \circ v \circ \iota \overset{\mathrm{eq}}{=} \pi_f \circ u \circ \iota = F(1_C, f) \circ \pi_C \circ \iota = F(1_C, f) \circ \lambda_C.$$

To show that the wedge λ is terminal, note that any wedge $\tau : \Delta(E') \overset{\bullet}{\Rightarrow} F$ induces a unique morphism $\tau' : E' \to \Pi_{C\in\mathrm{Ob}\mathcal{C}} F(C,C)$ with $\pi_C \circ \tau' = \tau_C$ for all $C \in \mathrm{Ob}\mathcal{C}$ by the universal property of the product. The morphism τ' satisfies for all morphisms $f : C \to C'$

$$\pi_f \circ u \circ \tau' = F(1_C, f) \circ \pi_C \circ \tau' = F(1_C, f) \circ \tau_C = F(f, 1_{C'}) \circ \tau_{C'} = F(f, 1_{C'}) \circ \pi_{C'} \circ \tau' = \pi_f \circ v \circ \tau',$$

and the universal property of the product implies $u \circ \tau' = v \circ \tau'$. By the universal property of the equaliser, there is a unique morphism $\tau'' : E' \to E$ with $\iota \circ \tau'' = \tau'$. This implies $\tau_C = \pi_C \circ \tau' = \pi_C \circ \iota \circ \tau'' = \lambda_C \circ \tau''$ for all $C \in \mathrm{Ob}\mathcal{C}$. Conversely, any morphism $\rho : E' \to E$ with $\tau_C = \lambda_C \circ \rho$ for all $C \in \mathrm{Ob}\mathcal{C}$ satisfies $\pi_C \circ \tau' = \pi_C \circ \tau_C = \pi_C \circ \iota \circ \rho$ for all $C \in \mathrm{Ob}\mathcal{C}$ and by the universal property of the product $\iota \circ \rho = \tau'$. By the universal property of the equaliser this implies $\tau'' = \rho$ and hence uniqueness of τ''. □

The formulas for ends and coends in Proposition 4.2.4 allow one to organise the (co)limit formulas for Kan extensions from Theorem 3.2.5 in a more efficient way. In principle, the relevant (co)limits in Theorem 3.2.5 can be computed in terms of (co)equalisers of (co)product via formula (5) and its dual, see the proof of Theorem 2.2.2. However, this is rather complicated in practice. With formulas (19) for the (co)ends, one can compute these (co)equalisers of (co)products in terms of *(co)ends* and *(co)powers*.

Definition 4.2.5: Let $\mathcal{C}$ be a category in which all (co)products exist.

1. The **power functor** $\sqcap : \mathrm{Set}^{op} \times \mathcal{C} \to \mathcal{C}$ assigns to
   - an object (X, C) the product $C^X = \Pi_{x \in X} C$,
   - a morphism $(f, c) : (X, C) \to (Y, D)$ the unique morphism $c^f : C^X \to D^Y$ with $\pi_y \circ c^f = c \circ \pi_{f(y)}$ for all $y \in Y$, where $\pi_x : C^X \to C$ and $\pi_y : D^Y \to D$ are the projection morphisms of the product.

2. The **copower functor** $\sqcup : \mathrm{Set} \times \mathcal{C} \to \mathcal{C}$ assigns to
   - an object (X, C) the coproduct $X \cdot C = \amalg_{x \in X} C$,
   - a morphism $(f, c) : (X, C) \to (Y, D)$ the unique morphism $fc : X \cdot C \to Y \cdot C$ with $fc \circ \iota_x = \iota_{f(x)} \circ c$ for all $x \in X$, where $\iota_x : C \to X \cdot C$ and $\iota_y : D \to Y \cdot D$ are the inclusion morphisms for the coproducts.

The benefit of the (co)end formula (19) and the (co)power functors is that they allow one to split (co)products in the limit formula (5) into two separate (co)products. One of them defines the (co)power functor and the other is absorbed by the (co)end. This yields the following result.

Proposition 4.2.6: Let $F : \mathcal{C} \to \mathcal{E}$ and $K : \mathcal{C} \to \mathcal{D}$ functors.

1. A left Kan extension of F along K is pointwise if and only if it is given by
$$\mathrm{Lan}_K F(D) = \int^{C \in \mathcal{C}} \mathrm{Hom}_{\mathcal{D}}(K(C), D) \sqcup F(C)$$

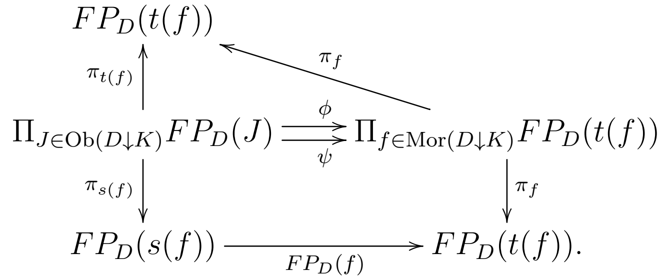
2. A right Kan extension of F along K is pointwise if and only if it is given by
$$\mathrm{Ran}_K F(D) = \int_{C \in \mathcal{C}} \mathrm{Hom}_{\mathcal{D}}(D, K(C)) \sqcap F(C)$$

Proof:
We prove the claim for right Kan extensions. The claim for left Kan extensions follows with Remark 2.1.5, 3, Remark 3.1.2, 2 and Remark 4.1.5, 2.

By Definition 3.2.11 a right Kan extension is pointwise if and only if it is given by the limit formula $\mathrm{Ran}_K F(D) = \lim(FP_D)$, where $FP_D : D \downarrow K \to \mathcal{E}$ is the functor that assigns to

- an object (C, d) with $d \in \mathrm{Hom}_{\mathcal{D}}(D, K(C))$ the object $F(C)$,
- a morphism $f : (C, d) \to (C', d')$, given by $f \in \mathrm{Hom}_{\mathcal{C}}(C, C')$ with $d' = K(f) \circ d$, the morphism $F(f) : F(C) \to F(C')$.

By the proof of Theorem 2.2.2 the limit is given by the dual of formula (5), as the equaliser of the morphisms ϕ, ψ defined by the diagram



By Definition 3.2.1 of the comma category $D\downarrow K$ the products in this formula become

$$\begin{aligned}
&\Pi_{J\in \mathrm{Ob}(D\downarrow K)} FP_D(J) = \Pi_{C\in\mathrm{Ob}\mathcal{C}}\Pi_{d\in \mathrm{Hom}_{\mathcal{D}}(D,K(C))}F(C) = \Pi_{C\in \mathrm{Ob}\mathcal{C}}\mathrm{Hom}_{\mathcal{D}}(D,K(C))\sqcap F(C)\\
&\Pi_{f\in \mathrm{Mor}(D\downarrow K)} FP_D(t(f)) = \Pi_{f\in\mathrm{Mor}\mathcal{C}}\Pi_{d\in \mathrm{Hom}_{\mathcal{D}}(D,K(s(f)))}F(t(f))\\
&\qquad\qquad\qquad\qquad\qquad\quad = \Pi_{f\in \mathrm{Mor}\mathcal{C}}\mathrm{Hom}_{\mathcal{D}}(D,K(s(f)))\sqcap F(t(f)).
\end{aligned}$$

Applying Formula (19) to $G = \mathrm{Hom}_{\mathcal{D}}(D, K(-))\sqcap F(-) : \mathcal{C}^{op}\times\mathcal{C}\to\mathcal{E}$ proves the claim. □

The formulas for Kan extensions in Proposition 4.2.6 are somewhat more intuitive than the (co)limit formulas in Theorem 3.2.5 and exhibit analogies with integration theory. The formula for the right Kan extension resembles the formula $\int_X f(x)d\mu(x)$ for the integral of a μ-measurable function $f : X\to\mathbb{R}$ with respect to a measure $d\mu$ on X. The functor $F : \mathcal{C}\to\mathcal{E}$ replaces the function $f : X\to\mathbb{R}$ and the set of morphisms $\mathrm{Hom}_{\mathcal{D}}(D, K(C))$ the measure $d\mu(x)$. The two are combined via the power functor.

The functor $K : \mathcal{C}\to\mathcal{D}$ along which F is extended thus defines a categorical weight assigned to each object $C\in\mathrm{Ob}\mathcal{C}$. This weight is given by set of morphisms $f : D\to K(C)$ and suppresses the contributions of objects $C\in\mathrm{Ob}\mathcal{C}$ for which the corresponding object $K(C)$ cannot be reached by morphisms from D. The integral is dominated by the images $F(C)$ of those objects $C\in\mathrm{Ob}\mathcal{C}$ for which there are many morphisms $f : D\to K(C)$.

We illustrate the formalism with a simple example that was already treated in Example 3.2.9 with the (co)limit formula from Theorem 3.2.5, namely induced and coinduced representations.

Example 4.2.7: A functor $F : BG^{op}\times BG\to\mathrm{Vect}_{\mathbb{F}}$ corresponds to a vector space $V = F(\bullet)$ with a left action $\rhd$ and right action $\lhd$ of the group G on V by linear automorphisms such that $g\rhd v = F(1_\bullet, g)v$, $v\lhd g = F(g, 1_\bullet)v$ and $(g\rhd v)\lhd h = g\rhd(v\lhd h)$ for all $g, h\in G$, $v\in V$.

1. The (co)power functors $\sqcap : \mathrm{Set}^{op}\times\mathrm{Vect}_{\mathbb{F}}\to\mathrm{Vect}_{\mathbb{F}}$ and $\sqcup : \mathrm{Set}\times\mathrm{Vect}_{\mathbb{F}}\to\mathrm{Vect}_{\mathbb{F}}$ are given by

$$X\sqcap V = \Pi_X V\cong \mathrm{Hom}_{\mathbb{F}}(\langle X\rangle_{\mathbb{F}}, V)\qquad X\sqcup V = \amalg_X V\cong\langle X\rangle_{\mathbb{F}}\otimes_{\mathbb{F}}V,$$

with projection morphisms $\pi_x : \mathrm{Hom}_{\mathbb{F}}(\langle X\rangle_{\mathbb{F}}, V)\to V$, $f\mapsto f(x)$ and inclusion morphisms $\iota_x : V\to\langle X\rangle_{\mathbb{F}}\otimes_{\mathbb{F}}V$, $x\mapsto x\otimes v$ for $x\in X$. The (co)end formulas (19) yield

$$\begin{aligned}
V^G &= \int_{\bullet\in BG} F(\bullet,\bullet) = \mathrm{eq}\left(V \underset{v\mapsto(g\mapsto v\lhd g)}{\overset{v\mapsto(g\mapsto g\rhd v)}{\rightrightarrows}} \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}[G], V)\right)\\
V_G &= \int^{\bullet\in BG} F(\bullet,\bullet) = \mathrm{coeq}\left(\mathbb{F}[G]\otimes_{\mathbb{F}}V \underset{g\otimes v\mapsto v\lhd g}{\overset{g\otimes v\mapsto g\rhd v}{\rightrightarrows}} V\right)
\end{aligned}$$

Thus, the end and coend are given by the linear maps

$$\begin{aligned}
&\iota : V^G = \{v\in V\mid g\rhd v = v\lhd g\ \forall g\in G\}\to V, &&v\mapsto v\\
&\pi : V\to V_G = V/\langle\{g\rhd v - v\lhd g\mid v\in V, g\in G\}\rangle, &&v\mapsto[v].
\end{aligned}$$

These are precisely the inclusions for the invariants V^G and the projection on the coinvariants V_G of the representation $\rhd' : G\times V\to V$, $g\rhd' v = g\rhd v\lhd g^{-1}$.

2. Consider now the Kan extensions of a C-representation $X : BC\to\mathrm{Vect}_{\mathbb{F}}$ on $X(\bullet) = W$ along the functor $K : BC\to BD$ induced by a group homomorphism $K : C\to D$.

In this case, we have $\mathrm{Hom}_{BD}(K(\bullet), \bullet) = \mathrm{Hom}_{BD}(\bullet, K(\bullet)) = D$.

• The power functor $\mathrm{Hom}_{BD}(\bullet, K(-)) \sqcap X(-) : BC^{op} \times BC \to \mathrm{Vect}_F$ in Proposition 4.2.6 corresponds to the vector space $V = \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}[D], W)$ with C-left and C-right actions given by

$$(c \rhd f)(d) = c \rhd f(d) \qquad (f \lhd c)(d) = f(K(c)d)$$

for all $c \in C$, $d \in D$ and $\mathbb{F}$-linear maps $f : \mathbb{F}[D] \to W$. Thus, by step 1. we have

$$\mathrm{Ran}_K X(\bullet) = \{f \in \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}[D], W) \mid f(K(c)d) = c \rhd f(d)\ \forall c \in C, d \in D\} = \mathrm{Hom}_{\mathbb{F}[C]}(\mathbb{F}[D], W).$$

• The copower functor $\mathrm{Hom}_{BD}(K(-), \bullet) \sqcup X(-) : BC^{op} \times BC \to \mathrm{Vect}_{\mathbb{F}}$ in Proposition 4.2.6 corresponds to the vector space $V = \mathbb{F}[D] \otimes_{\mathbb{F}} W$ with the C-left and right C-actions given by

$$c \rhd (d \otimes w) = d \otimes (c \rhd w) \qquad (d \otimes w) \lhd d = dK(c) \otimes w$$

for all $c \in C$, $d \in D$ and $w \in W$. Thus, by step 1. we have

$$\mathrm{Lan}_K X(\bullet) = \mathbb{F}[D] \otimes_{\mathbb{F}} W / \mathrm{span}_{\mathbb{F}}\{dK(c) \otimes w - d \otimes (c \rhd w) \mid c \in C, d \in D, w \in W\} = \mathbb{F}[D] \otimes_{\mathbb{F}[C]} W.$$

The formulas in Proposition 4.2.6 together with Formula (19) for the (co)ends thus yield a concrete way of computing the left and right Kan extension via (co)products and (co)equalisers.

References:

- Chapters 1 and 2 in Loregian, F. (2015) Coend calculus. arXiv preprint arXiv:1501.02503,
- Chapters IX.4,5,6, X.4 in Mac Lane, S.(2013) Categories for the working mathematician,
- Chapters 4.4 and 4.5 in Richter, B. (2020) From categories to homotopy theory.

5 Simplicial objects

5.1 Simplexes and simplicial complexes

Simplexes and simplicial complexes are widely used in topology and geometry and come in various flavours - affine, singular or combinatorial. They were originally used in topology and geometry to build and investigate topological spaces. Their usefulness comes from the fact that they are simple building blocks that exist in all dimensions and can be glued together in a controlled way. The simplest variant are affine simplexes in Euclidean space.

Although they are geometric objects, they are mostly characterised by the combinatorics of their vertices, as any affine linear map $f : \mathbb{R}^n \to \mathbb{R}^k$ is determined uniquely by its value on any $n+1$ points that are not contained in an affine plane of dimension $< n$.

Definition 5.1.1:

1. An **affine m-simplex** $\Delta \subset \mathbb{R}^n$ is the convex hull of $m+1$ points $v_0, ..., v_m \in \mathbb{R}^n$
$$\Delta = \mathrm{conv}(\{v_0, ..., v_m\}) = \{\Sigma_{i=0}^m \lambda_i v_i \mid 0 \leq \lambda_i \leq 1, \Sigma_{i=0}^m \lambda_i = 1\}.$$
It is called **degenerate**, if there is an affine plane P of dimension $< m$ with $\Delta \subset P$.
2. The k-simplexes $\mathrm{conv}(\{v_{i_0}, ..., v_{i_k}\})$ for subsets $\{i_0, ..., i_k\} \subset \{0, ..., m\}$ with $k+1$ elements are called the **k-faces** of Δ, the 0-faces are called **vertices** and the 1-faces **edges**.
3. An **ordered m-simplex** is an affine m-simplex with an ordering of its vertices. We write $[v_0, ..., v_m]$ for $\Delta = \mathrm{conv}(\{v_0, ..., v_m\})$ with ordering $v_0 < v_1 < ... < v_m$.

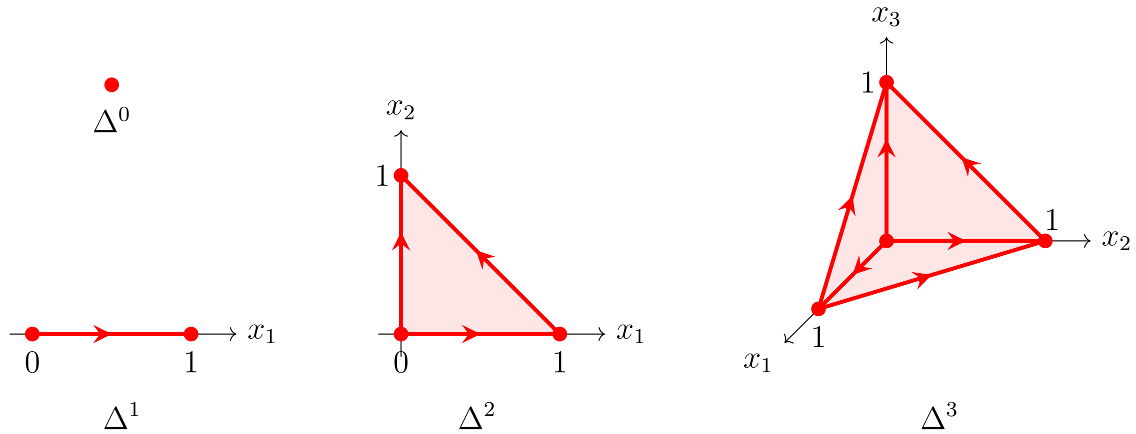
Affine n-simplexes can be glued together via affine linear maps that identify some of their $(n-1)$-faces. If one is only interested in the resulting polyhedra, it is sufficient to consider affine simplexes of a particular simple form, the *standard n-simplexes* with the origin and the elements $e_1, \ldots, e_n$ of the standard basis as vertices. By applying affine transformations, one can relate any affine n-simplex to a standard simplex. In particular, we require affine linear maps that embed standard simplexes into faces of higher-dimensional standard simplexes and collapse them to lower-dimensional ones.

Definition 5.1.2: Let $(e_1, ..., e_n)$ be the standard basis of $\mathbb{R}^n$ and $e_0 := 0 \in \mathbb{R}^n$.

1. The **standard n-simplex** $\Delta^n \subset \mathbb{R}^n$ is the ordered n-simplex $[e_0, ..., e_n]$.
2. For $n \in \mathbb{N}$ and $i \in \{0, ..., n\}$ the ith **face map** $f_i^n : \Delta^{n-1} \to \Delta^n$ and **degeneracy map** $s_i^n : \Delta^{n+1} \to \Delta^n$ are the affine linear maps given by
$$f_i^n(e_j) = \begin{cases} e_j & j < i \\ e_{j+1} & j \geq i. \end{cases} \qquad s_i^n(e_j) = \begin{cases} e_j & j \leq i \\ e_{j-1} & j > i \end{cases}.$$

The name *face map* is due to the fact that the ith face map f_i^n sends the simplex Δ^{n-1} to the $(n-1)$-face $[e_0, ...e_{i-1}, e_{i+1}, ..., e_n]$ in Δ^n opposite the vertex e_i. In contrast, the ith *degeneracy map* s_i^n sends the simplex Δ^{n+1} to Δ^n by collapsing the edge $[e_i, e_{i+1}] \subset \Delta^{n+1}$. We can also interpret its image as a *degenerate* $(n+1)$-simplex $s_i^n(\Delta^{n+1}) = [e_0, \ldots, e_{i-1}, e_i, e_i, e_{i+1}, \ldots, e_n]$.

The ordering of an affine m-simplex is pictured by drawing an arrow on each edge that points from its vertex of lower order to its vertex of higher order. Note that the face maps and degeneracies respect the ordering of vertices in the standard n-simplexes. They omit or repeat vertices but do not change their ordering. Hence, the ordering of the vertices in the $(n-1)$-face $f_i^n(\Delta^{n-1}) \subset \Delta^n$ induced by the ordering of Δ^{n-1} coincides with the one induced by the ordering of Δ^n and analogously for lower-dimensional faces.



The standard n-simplexes for $n = 0, 1, 2, 3$.

Instead of just glueing affine simplexes via affine linear maps, one can also distort them and use continuous maps that are injective in the interior of the simplexes. This allows for a more flexible construction of topological spaces with simplexes and leads to the concepts of *(semi)simplicial complexes*. Many familiar spaces from topology have the structure of a semisimplicial complex.

Definition 5.1.3:

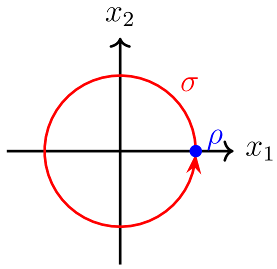
1. A (finite) **Δ-complex** or **semisimplicial complex** is a topological space X, together with a (finite) family $\{\tau_\alpha\}_{\alpha\in I}$ of continuous maps $\tau_\alpha : \Delta^{n_\alpha} \to X$ such that:
   (S1) The maps $\tau_\alpha|_{\mathring{\Delta}^{n_\alpha}} : \mathring{\Delta}^{n_\alpha} \to X$ are injective for all $\alpha \in I$.
   (S2) For every point $x \in X$ there is a unique $\alpha \in I$ with $x \in \tau_\alpha(\mathring{\Delta}^{n_\alpha})$.
   (S3) For every $\alpha \in I$ and $i \in \{0, ..., n_\alpha\}$ there is a $\beta \in I$ with $\tau_\alpha \circ f_i^{n_\alpha} = \tau_\beta : \Delta^{n_\alpha - 1} \to X$.
   (S4) The topology on X is the final topology induced by the family $\{\tau_\alpha\}_{\alpha\in I}$:
   A subset $A \subset X$ is open if and only if $\tau_\alpha^{-1}(A) \subset \Delta^{n_\alpha}$ is open for all $\alpha \in I$.

2. A semisimplicial complex is called a **simplicial complex** if
   (S5) For each $\alpha \in I$ the images of the vertices of Δ^{n_α} under τ_α are all distinct:
   $\tau_\alpha(e_i) \neq \tau_\alpha(e_j)$ for all $i \neq j \in \{0, ..., n_\alpha\}$.
   (S6) $\{\tau_\alpha(e_0), ..., \tau_\alpha(e_{n_\alpha})\} = \{\tau_\beta(e_0), ..., \tau_\beta(e_{n_\beta})\}$ implies $\alpha = \beta$.

3. The **n-skeleton** of a (semi)simplicial complex $(X, \{\tau_\alpha\}_{\alpha\in I})$ is the (semi)simplicial complex $\text{sk}_n X = X \setminus (\cup_{n_\alpha > n} \tau_\alpha(\mathring{\Delta}^{n_\alpha}))$ with the family $\{\tau_\alpha\}_{\alpha\in I, n_\alpha \leq n}$.

Example 5.1.4:

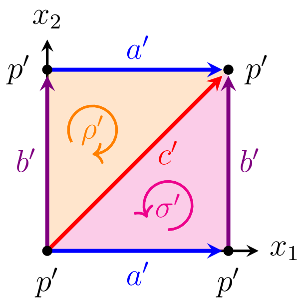
1. A semisimplicial structure on the circle S^1 is given by any continuous map $\sigma : [0, 1] \to S^1$ with $\sigma(0) = \sigma(1) = 1$ and $\sigma|_{(0,1)} : (0, 1) \to S^1$ injective and $\rho : \{0\} \to S^1, 0 \mapsto 1$. Its 1-skeleton is S^1 and its 0-skeleton the point $(1, 0)$.



2. The torus is the quotient $T = [0,1]^{\times 2}/\sim$ with the equivalence relation $(x,0)\sim(x,1)$ and $(0,x)\sim(1,x)$ for all $x\in[0,1]$. It has the structure of a semisimplicial complex with two 2-simplexes, three 1-simplexes and one 0-simplex. They are the composites of the canonical surjection $\pi:[0,1]\times[0,1]\to T$ with the affine linear maps

$$\begin{array}{lll}\rho:[e_0,e_1,e_2]\to[e_0,e_2,e_1+e_2], & \sigma:[e_0,e_1,e_2]\to[e_0,e_1,e_1+e_2], & p:[e_0]\to[e_0]\\ a:[e_0,e_1]\to[e_0,e_1], & b:[e_0,e_1]\to[e_0,e_2], & c:[e_0,e_1]\to[e_0,e_1+e_2].\end{array}$$

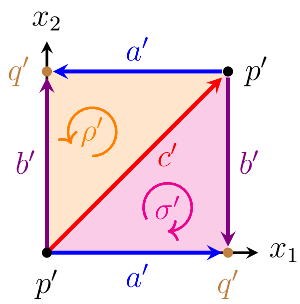
Its 2-skeleton is T, its 1-skeleton is the union of the images of a',b',c', a bouquet with three circles, and its 0-skeleton is the image of p', a point.



3. Real projective space $\mathbb{RP}^2$ is the quotient $\mathbb{RP}^2=[0,1]\times[0,1]/\sim$ with the equivalence relation $(x,1)\sim(1-x,0)$ and $(0,x)\sim(1,1-x)$ for all $x\in[0,1]$. It has a semisimplicial structure with two 2-simplexes, three 1-simplexes and two 0 -simplexes which are obtained by composing the canonical surjection $\pi:[0,1]\times[0,1]\to\mathbb{RP}^2$ with the affine simplexes

$$\begin{array}{l}\rho:[e_0,e_1,e_2]\to[e_0,e_1+e_2,e_2], \qquad \sigma:[e_0,e_1,e_2]\to[e_0,e_1+e_2,e_1],\\ a:[e_0,e_1]\to[e_0,e_1], \quad b:[e_0,e_1]\to[e_0,e_2], \quad c:[e_0,e_1]\to[e_0,e_1+e_2],\\ p:[e_0]\to[e_0], \quad q:[e_0]\to[e_1].\end{array}$$

Its 2-skeleton is $\mathbb{RP}^2$, its 1-skeleton the union of the images of a',b',c', two points connected by three different edges, and its 0-skeleton contains two points, the images of p',q'.



A given topological space may have many (semi)simplicial complex structures. The notion of a *simplicial complex* is more restrictive than the one of a *semisimplicial complex*. However, every semisimplicial complex can be transformed into a simplicial one by a subdivision procedure.

Axiom (S5) forbids that the images of distinct vertices of an n-simplex coincide, and condition (S6) forbids that the vertex sets of different simplexes coincide.

This allows one to describe a simplicial complex in a purely combinatorial way. Every k-face in a simplicial complex is determined uniquely by its vertices. If one is interested only in the resulting topological space, one can discard all the information about the continuous maps $\sigma_\alpha : \Delta^{n_\alpha} \to X$ and the standard simplexes and retain only the information how the simplexes are contained in each other or glued together. This leads to *combinatorial simplicial complexes.*

Definition 5.1.5:

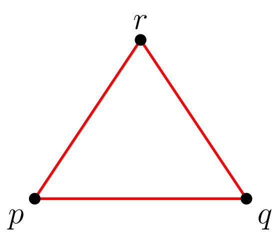
1. A **combinatorial simplicial complex** is a subset $\mathcal{K} \subset \mathcal{P}(V)$ for some set V such that
   (i) $M \in \mathcal{K}$ implies $M \neq \emptyset$ and M finite,
   (ii) $\emptyset \neq M \subset N$ with $N \in \mathcal{K}$ implies $M \in \mathcal{K}$.

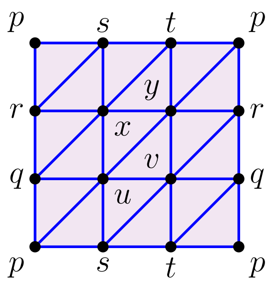
2. A combinatorial simplicial complex is called **ordered**, if the set V is ordered.

3. The n**-skeleton** of $\mathcal{K}$ is the combinatorial simplicial complex

$$\mathrm{sk}_n(\mathcal{K}) = \mathcal{K} \setminus \{M \in \mathcal{K} \mid |M| > n+1\}.$$

Example 5.1.6:





1. The first picture describes a simplicial complex structure and an associated combinatorial simplicial complex $\mathcal{K} = \{\{p\}, \{q\}, \{r\}, \{p,q\}, \{q,r\}, \{p,r\}\}$ with $V = \{p,q,r\}$.

2. The second picture describes a simplicial and combinatorial simplicial complex $\mathcal{K}$ for the torus with vertex set $V = \{p,q,r,s,t,u,v,w,x\}$. The elements of $\mathcal{K}$ are
   - the 9 singleton sets containing elements of V,
   - the 27 sets of pairs of distinct neighbouring vertices connected by a blue line,
   - the 18 sets of neighbouring triples of vertices that form triangles.

5.2 The simplex category and (co)simplicial objects

The idea is now to organise all combinatorial information about the standard n-simplexes and the face maps and degeneracies between them in a common framework. As we need to treat all n-simplexes Δ^n for $n \in \mathbb{N}_0$, this suitable framework is a category, whose objects correspond to the n-simplexes. To account for the $n+1$ ordered vertices of the simplexes Δ^n, we choose as objects the finite ordinals $[n+1] = \{0, 1, ..., n\}$ for $n \in \mathbb{N}_0$.

The category must also encode as morphisms all maps between standard simplexes obtained by composing face maps and degeneracies. As the face maps and degeneracies are monotonic, such composites must be monotonic as well. Hence, we describe them as weakly monotonic maps $f : [m] \to [n]$ between ordinals. The counterparts of the face maps are the face morphisms, strictly monotonic maps $\delta^i_n : [n] \to [n+1]$ that skip the number $i \in [n+1]$. The counterparts of the degeneracies are the degeneracy morphisms, weakly monotonic maps $\sigma^i_n : [n+1] \to [n]$ that send $i, i+1 \in [n+1]$ to $i \in [n]$ and are injective on the other elements.

Definition 5.2.1:

1. The **simplex category** or **simplicial category** Δ has
   - as objects the finite non-empty **ordinal numbers** $[n] = \{0, 1, ..., n-1\}$ for $n \in \mathbb{N}$,
   - as morphisms $f : [m] \to [n]$ weakly monotonic maps $f : \{0, ..., m-1\} \to \{0, ..., n-1\}$,

   with the composition of maps as composition and the identity as identity morphisms.

2. The ith **face morphism** $\delta^i_n : [n] \to [n+1]$ for $i \in [n+1]$ and jth **degeneracy morphism** $\sigma^j_n : [n+1] \to [n]$ for $j \in [n]$ are the morphisms

$$\delta^i_n(k) = \begin{cases} k & 0 \leq k < i \\ k+1 & i \leq k < n \end{cases} \qquad \sigma^j_n(k) = \begin{cases} k & 0 \leq k \leq j \\ k-1 & j < k \leq n. \end{cases}$$

We now show that any monotonic map between finite ordinals is a composite of face and degeneracy morphisms. If one imposes a fixed ordering of the latter, this composite even becomes unique. We also describe relations between different face maps and degeneracies. This leads to a characterisation of the simplex category in terms of generating morphisms and relations.

Proposition 5.2.2: (**factorisation in the simplex category**)

1. Every morphism $f : [m] \to [n]$ in Δ can be expressed uniquely as a composite

$$f = \delta^{i_1}_{n-1} \circ ... \circ \delta^{i_k}_{m-l} \circ \sigma^{j_1}_{m-l} \circ ... \circ \sigma^{j_l}_{m-1} \tag{20}$$
$$n = m - l + k, \quad 0 \leq i_k < ... < i_1 < n, \quad 0 \leq j_1 < ... < j_l < m-1$$

2. The morphisms $\delta^i_n : [n] \to [n+1]$ and $\sigma^j_n : [n+1] \to [n]$ satisfy the relations

$$\begin{aligned} &\delta^i_{n+1} \circ \delta^j_n = \delta^{j+1}_{n+1} \circ \delta^i_n && \text{for} \quad i \leq j \\ &\sigma^j_n \circ \sigma^i_{n+1} = \sigma^i_n \circ \sigma^{j+1}_{n+1} && \text{for} \quad i \leq j \\ &\sigma^j_n \circ \delta^i_n = \begin{cases} \delta^i_{n-1} \circ \sigma^{j-1}_{n-1} & i < j \\ 1_{[n]} & i \in \{j, j+1\} \\ \delta^{i-1}_{n-1} \circ \sigma^j_{n-1} & i > j+1. \end{cases} \end{aligned} \tag{21}$$

Proof:
Every monotonic map $f : [m] \to [n]$ is determined uniquely by the sets

$$M_\delta = \{i_1, ..., i_k\} = [n] \setminus \text{im}(f) \qquad M_\sigma = \{j_1, ..., j_l\} = \{x \in [m-1] \mid f(x) = f(x+1)\}$$

with $n - k = m - l$. If $0 \leq i_k < ... < i_1 < n$, $0 \leq j_1 < ... < j_l < m-1$ and $\text{im}(f) = \{l_1, ..., l_{n-k}\}$ with $0 \leq l_1 < ... < l_{n-k}$, then f factorises uniquely as $f = g \circ h$ with an injective monotonic map $g : [m-l] \to [n]$ and a surjective monotonic map $h : [m] \to [m-l]$ given by $g(r) = l_{r+1}$ for $r \in [m-l]$ and $h(r) = r - s$ for $j_s < r \leq j_{s+1}$, $h(r) = r$ for $r \leq j_1$, $h(r) = r - l$ for $r > j_l$. This implies $g = \delta^{i_1}_{n-1} \circ ... \circ \delta^{i_k}_{m-l}$ and $h = \sigma^{j_1}_{m-l} \circ ... \circ \sigma^{j_l}_{m-1}$. The relations follow by a direct computation from the definitions. □

Remark 5.2.3: As the relations (21) allow one to transform any composite of the morphisms δ_n^i and σ_n^j into the form (20) and the factorisation in (20) is unique, all relations between the morphisms δ_n^i and σ_n^j are composites of the relations (21). One says that Δ is **generated as a category** or **presented as a category** by the morphisms δ_n^i and σ_n^j with the **relations** (21).

The simplex category encodes the combinatorics of the standard n-simplexes and face maps and degeneracies between them. More precisely, we can describe the latter as a functor from the simplex category into the category Top that sends the object $[n+1]$ to the standard n-simplex Δ^n and the face and degeneracy morphisms to the associated face and degeneracy maps from Definition 5.1.2. This idea is not restricted to the category Top. We can consider analogous functors into any category $\mathcal{C}$.

Definition 5.2.4: Let $\mathcal{C}$ be a category.

1. A **simplicial object** in $\mathcal{C}$ is a functor $S : \Delta^{op} \to \mathcal{C}$ and a **simplicial morphism** from a simplicial object S to simplicial object S' a natural transformation $\lambda : S \Rightarrow S'$.
2. A **cosimplicial object** in $\mathcal{C}$ is a functor $S : \Delta \to \mathcal{C}$ and a **cosimplicial morphism** from a cosimplicial object S to cosimplicial object S' a natural transformation $\lambda : S \Rightarrow S'$.

A (co)simplicial object in Set is also called a **(co)simplicial set** and a (co)simplicial morphism in Set a **(co)simplicial map**. Simplicial sets and maps form the category $\mathrm{SSet} = \mathrm{Set}^{\Delta^{op}}$.

More generally, simplicial objects and morphisms in $\mathcal{C}$ form the functor category $\mathcal{C}^{\Delta^{op}}$ and cosimplicial objects and morphisms in $\mathcal{C}$ the functor category $\mathcal{C}^{\Delta}$. The nomenclature for Set is used analogously for common categories in algebra and topology. For instance, a simplicial object in Top is called a simplicial space, a simplicial object in Grp a simplicial group and a simplicial object in R-Mod a simplicial module.

As the morphisms $\delta_n^i : [n] \to [n+1]$ and $\sigma_n^j : [n+1] \to [n]$ from Proposition 5.2.2 generate the simplex category Δ subject to the relations (21), a (co)simplicial object is determined uniquely by the images of the objects $[n]$ for $n \in \mathbb{N}$ and the images of the morphisms δ_n^i and σ_n^j, which must satisfy relations analogous or dual to (21).

Remark 5.2.5: Let $\mathcal{C}$ be a category.

1. A simplicial object $C : \Delta^{op} \to \mathcal{C}$ is given by
   - a family $(C_n)_{n\in\mathbb{N}_0}$ of objects $C_n \in \mathrm{Ob}\mathcal{C}$,
   - families of morphisms $d_n^i : C_n \to C_{n-1}$ for $n \in \mathbb{N}_0$, $0 \leq i \leq n$, the **face maps**,
   - families of morphisms $s_n^i : C_n \to C_{n+1}$ for $n \in \mathbb{N}_0$ and $0 \leq i \leq n$, the **degeneracies**

   with $C_n = C([n+1])$, $d_n^i = C(\delta_n^i)$ and $s_n^i = C(\sigma_{n+1}^i)$, satisfying the **simplicial relations**

$$\begin{aligned} d_n^j \circ d_{n+1}^i &= d_n^i \circ d_{n+1}^{j+1} && \text{for} \quad i \leq j \\ s_{n+1}^i \circ s_n^j &= s_{n+1}^{j+1} \circ s_n^i && \text{for} \quad i \leq j \\ d_{n+1}^i \circ s_n^j &= \begin{cases} s_{n-1}^{j-1} \circ d_n^i & i < j \\ 1_{C_n} & i \in \{j, j+1\} \\ s_{n-1}^j \circ d_n^{i-1} & j+1 < i \leq n+1. \end{cases} \end{aligned} \tag{22}$$

2. A simplicial morphism $\mu : C \Rightarrow C'$ is a collection of maps $\mu_n : C_n \to C'_n$ with

$$\mu_{n-1} \circ d_n^i = d'^i_n \circ \mu_n \qquad \mu_{n+1} \circ s_n^i = s'^i_n \circ \mu_n, \qquad n \in \mathbb{N}_0,\ 0 \leq i \leq n.$$

3. A cosimplicial object $C : \Delta \to \mathcal{C}$ is given by
    - a family $(C^n)_{n\in\mathbb{N}_0}$ of objects $C^n \in \mathrm{Ob}\mathcal{C}$,
    - families of morphisms $d_i^n : C^{n-1} \to C^n$ for $n \in \mathbb{N}_0$, $0 \leq i \leq n$, the **face maps**,
    - families of morphisms $s_i^n : C^{n+1} \to C^n$ for $n \in \mathbb{N}_0$ and $0 \leq i \leq n$, the **degeneracies**,

    with $C^n = C([n+1])$, $d_i^n = C(\delta_n^i)$ and $s_i^n = C(\sigma_{n+1}^i)$, satisfying relations dual to (22).

4. A cosimplicial morphism $\mu : C \Rightarrow C'$ is a collection of maps $\mu^n : C^n \to C'^n$ with

$$\mu^n \circ d_i^n = d_i'^n \circ \mu^{n-1} \qquad \mu^n \circ s_i^{n+1} = s_i'^n \circ \mu^{n+1}, \qquad n \in \mathbb{N}_0,\ 0 \leq i \leq n.$$

The shift in indices between the ordinal $[n+1]$ and the sets S_n and C_n are unfortunate. It arises, because we use the algebraist's convention, in which the ordinal $[n]$ has n elements. In the topologist's convention, the ordinal $[n]$ has $n+1$ elements, just as the standard n-simplex has $n+1$ vertices. Thus, $[n+1]$ in our convention corresponds to $[n]$ in the topologist's convention.

Example 5.2.6:

1. The standard n-simplexes and the face maps and degeneracies between them from Definition 5.1.2 define a cosimplicial object $T : \Delta \to \mathrm{Top}$ with

$$T_n = \Delta^n, \qquad T(\delta_n^i) = f_i^n : \Delta^{n-1} \to \Delta^n, \qquad T(\sigma_{n+1}^i) = s_i^n : \Delta^{n+1} \to \Delta^n.$$

    It assigns to a monotonic map $f : [n+1] \to [m+1]$ the unique affine linear map $T(f) : \Delta^m \to \Delta^n$ with $T(f)(e_i) = e_{f(i)}$ for all $i \in [m+1]$.

2. Denote by $[n]'$ the poset category for the set $[n] = \{0, 1, ..., n-1\}$ with the ordering $\preceq = \leq$. Then every monotonic map $f : [m] \to [n]$ defines a functor $f : [m]' \to [n]'$. This yields a cosimplicial object $C : \Delta \to \mathrm{Cat}$ with

$$C_n = [n+1]' \qquad d_i^n = \delta_n^i : C_{n-1} \to C_n \qquad s_i^n = \sigma_{n+1}^i : C_{n+1} \to C_n.$$

3. The identity functor $\mathrm{id}_\Delta : \Delta \to \Delta$ defines a cosimplicial object in Δ.

Although these cosimplicial objects seem rather trivial, they give rise to important constructions in topology and algebra. They allow one to associate a simplicial set to every object in the categories Top, Cat and Δ via the nerve construction, by combining them with a Hom functor.

Example 5.2.7: (nerves)
Any *cosimplicial* object $F : \Delta \to \mathcal{C}$ in a category $\mathcal{C}$ yields a functor $\mathrm{Hom}_\mathcal{C}(F(-), -) : \mathcal{C} \to \mathrm{SSet}$ that assigns to

- an object C the simplicial set $\mathrm{Hom}_\mathcal{C}(F(-), C) : \Delta^{op} \to \mathrm{Set}$,
- a morphism $c : C \to C'$ the simplicial map

$$\mathrm{Hom}_\mathcal{C}(F(-), c) : \mathrm{Hom}_\mathcal{C}(F(-), C) \Rightarrow \mathrm{Hom}_\mathcal{C}(F(-), C').$$

Such a functor is called a **nerve** of $\mathcal{C}$. For the cosimplicial objects in Example 5.2.6 this yields

1. For $\mathcal{C} = \mathrm{Top}$, $F = T : \Delta \to \mathrm{Top}$ the **singular nerve** $\mathrm{Sing} = \mathrm{Hom}(T(-), -) : \mathrm{Top} \to \mathrm{SSet}$ that assigns to

- a topological space X the simplicial set $S^X : \Delta^{op} \to \text{Set}$ with $S^X_n = \text{Hom}_{\text{Top}}(\Delta^n, X)$,

$$d^i_n : \text{Hom}(\Delta^n, X) \to \text{Hom}(\Delta^{n-1}, X), \quad \sigma \mapsto \sigma \circ f^n_i,$$
$$s^i_n : \text{Hom}(\Delta^n, X) \to \text{Hom}(\Delta^{n+1}, X), \quad \sigma \mapsto \sigma \circ s^n_i,$$

- a continuous map $f : X \to Y$ the simplicial map $S^f : S^X \Rightarrow S^Y$ given by the maps $S^f_n : \text{Hom}_{\text{Top}}(\Delta^n, X) \to \text{Hom}_{\text{Top}}(\Delta^n, Y)$, $\sigma \mapsto f \circ \sigma$.

2. For $\mathcal{C} = \text{Cat}$ and $F = C : \Delta \to \text{Cat}$ the **simplicial nerve** or **categorical nerve** $N = \text{Hom}_{\text{Cat}}(C(-), -) : \text{Cat} \to \text{SSet}$ that sends
   - a category $\mathcal{C}$ to the simplicial set $S^{\mathcal{C}} : \Delta^{op} \to \text{Set}$ with $S^{\mathcal{C}}_n = \text{Hom}_{\text{Cat}}([n+1]', \mathcal{C})$,
   - a functor $G : \mathcal{C} \to \mathcal{D}$ to the simplicial map $S^G : S^{\mathcal{C}} \Rightarrow S^{\mathcal{D}}$ with components $S^G_n : \text{Hom}_{\text{Cat}}([n+1]', \mathcal{C}) \to \text{Hom}_{\text{Cat}}([n+1]', \mathcal{D})$, $H \mapsto GH$.

3. For $\mathcal{C} = \Delta$ and $F = \text{id}_\Delta : \Delta \to \Delta$, the functor $\Delta^\bullet : \Delta \to \text{SSet}$ that sends
   - the object $[n+1]$ in Δ to the simplicial set $\Delta^n = \text{Hom}_\Delta(-, [n+1]) : \Delta^{op} \to \text{Set}$, the **standard n-simplex** in SSet,
   - a morphism $\alpha : [m+1] \to [n+1]$ to the simplicial map $\text{Hom}(-, \alpha) : \Delta^m \Rightarrow \Delta^n$ with components $\text{Hom}([k+1], \alpha) : \text{Hom}([k+1], [m+1]) \to \text{Hom}([k+1], [n+1])$ $\tau \mapsto \alpha \circ \tau$.

   The simplicial set $\Delta^n : \Delta^{op} \to \text{Set}$ play a special role in the category SSet. By the Yoneda lemma simplicial maps $\alpha : \Delta^n \Rightarrow S$ into a simplicial set $S : \Delta^{op} \to \text{Set}$ are in bijection with elements of S_n.

Other important examples are the simplicial objects that underly the definition of Hochschild homology and group homology, cf. Section 5.5. In this case, the input data are an algebra and a bimodule. If the algebra is a group algebra, one usually assumes that the left module structure is trivial and works with a right module over the group algebra. One may also take the algebra as a bimodule over itself with the left or right multiplication or the underlying field $\mathbb{F}$ as a trivial module over the group algebra.

Example 5.2.8: Let A be an algebra over $\mathbb{F}$ and M an (A, A)-bimodule.
This defines a simplicial vector space $S : \Delta^{op} \to \text{Vect}_{\mathbb{F}}$ with $S_n = M \otimes_{\mathbb{F}} A^{\otimes n}$ for $n \in \mathbb{N}_0$ and face maps and degeneracies

$$d^i_n : S_n \to S_{n-1}, \quad m \otimes a_1 \otimes \ldots \otimes a_n \mapsto \begin{cases} (m \lhd a_1) \otimes a_2 \otimes \ldots \otimes a_n & i = 0 \\ m \otimes a_1 \otimes \ldots \otimes (a_i a_{i+1}) \otimes \ldots \otimes a_n & 1 \leq i \leq n-1 \\ (a_n \rhd m) \otimes a_1 \otimes \ldots \otimes a_{n-1} & i = n \end{cases}$$
$$s^i_n : S_n \to S_{n+1}, \quad m \otimes a_1 \otimes \ldots \otimes a_n \mapsto m \otimes a_1 \otimes \ldots \otimes a_i \otimes 1_A \otimes a_{i+1} \otimes \ldots \otimes a_n.$$

If $A = \mathbb{F}[G]$ for a group G and M is equipped with the trivial $\mathbb{F}[G]$-left module structure, this reduces to $S_n = M \otimes_{\mathbb{F}} \mathbb{F}[G^{\times n}]$ with face maps and degeneracies

$$d^i_n : S_n \to S_{n-1}, \quad m \otimes (g_1, \ldots, g_n) \mapsto \begin{cases} (m \lhd g_1) \otimes (g_2, \ldots, g_n) & i = 0 \\ m \otimes (g_1, \ldots, (g_i g_{i+1}), \ldots \otimes g_n) & 1 \leq i \leq n-1 \\ m \otimes (g_1, \ldots, g_{n-1}) & i = n \end{cases}$$
$$s^i_n : S_n \to S_{n+1}, \quad m \otimes (g_1, \ldots, g_n) \mapsto m \otimes (g_1, \ldots, g_i, 1_G, g_{i+1}, \ldots, g_n).$$

Besides these structural examples that play a fundamental role in topology and homological algebra, there are also concrete examples of simplicial objects, arising for instance from ordered combinatorial simplicial complexes.

Example 5.2.9: Any ordered combinatorial simplicial complex $\mathcal{K} \subset \mathcal{P}(V)$ as in Definition 5.1.5 defines a simplicial set $S^{\mathcal{K}} : \Delta^{op} \to \text{Set}$ given by

$$\begin{aligned}
&S^{\mathcal{K}}_n = \{(v_0, \ldots, v_n) \,|\, \{v_0, \ldots, v_n\} \in \mathcal{K},\ v_0 \leq \ldots \leq v_n\} \\
&d^i_n : S^{\mathcal{K}}_n \to S^{\mathcal{K}}_{n-1}, \quad (v_0, \ldots, v_n) \mapsto (v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n) \\
&s^i_n : S^{\mathcal{K}}_n \to S^{\mathcal{K}}_{n+1}, \quad (v_0, \ldots, v_n) \mapsto (v_0, \ldots, v_{i-1}, v_i, v_i, v_{i+1}, \ldots, v_n).
\end{aligned}$$

This example illustrates the role of the degeneracies in a simplicial set. Here, a point $(v_0, \ldots, v_n) \in S_n$ is in the image of a degeneracy map if and only if it contains repeated vertices. If the vertices are taken as points in $\mathbb{R}^n$, this corresponds to a degenerate n-simplex contained in an affine plane of dimension $< n$. If V is *finite* with k vertices, every element of S_n is degenerate for all $n \geq k$. A similar phenomenon occurs when one realises a k-dimensional manifold as a combinatorial simplicial complex. In this case, all n-simplexes for $k > n$ are degenerate. The elements in the images of degeneracy maps carry only redundant information.

In the following we will use simplicial sets and other simplicial objects in two ways, to *construct* mathematical objects in a category $\mathcal{E}$ and to *investigate* objects in a category $\mathcal{E}$. The simplicial objects used in the former and in the latter are adjoints.

The left adjoint is a left Kan extension $\text{Lan}_y F : \text{SSet} \to \mathcal{E}$ of a cosimplicial object $F : \Delta \to \mathcal{E}$ along the Yoneda embedding $y : \Delta \to \text{SSet}$ and serves to *construct* objects in the category $\mathcal{E}$ from simplicial sets. The right adjoint is the nerve functor $\text{Hom}(F(-), -) : \mathcal{E} \to \text{SSet}$ from Example 5.2.7. It assigns to each object of $\mathcal{E}$ a simplicial set that can be used to *analyse* the category $\mathcal{E}$. Essentially everything in the next sections is based on the following corollary, which is a direct consequence of Corollary 3.2.6 and Proposition 3.2.13.

Corollary 5.2.10: Let $\mathcal{E}$ be cocomplete.

1. Every cosimplicial object $F : \Delta \to \mathcal{E}$ has a pointwise left Kan extension $\text{Lan}_y F : \text{SSet} \to \mathcal{E}$ along the Yoneda embedding $y : \Delta \to \text{SSet}$.

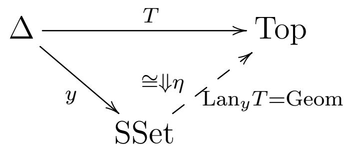
2. The left Kan extension is left adjoint to the nerve $\text{Hom}(F(-), -) : \mathcal{E} \to \text{SSet}$.

5.3 Geometric realisation

In this section, we consider the left adjoint in Corollary 5.2.10 for the category $\mathcal{E} = \text{Top}$. In this case, $F : \Delta \to \text{Top}$ is the cosimplicial object from Example 5.2.6, 1. defined by the standard n-simplexes. Its left Kan extension $\text{Lan}_y F : \text{SSet} \to \text{Top}$ is called the *geometric realisation*. It associates to every simplicial set a topological space and is left adjoint to the *singular nerve* $\text{Sing} : \text{Top} \to \text{SSet}$ from Example 5.2.7, 1.

The geometric realisation is important, because it allows one to investigate simplicial sets with methods from topology. We will see in the following sections how this construction can be applied to characterise objects in different categories and relate them to topological spaces.

Definition 5.3.1: The **geometric realisation** $\mathrm{Geom} = \mathrm{Lan}_y T : \mathrm{SSet} \to \mathrm{Top}$ is the left Kan extension of the cosimplicial object $T : \Delta \to \mathrm{Top}$ from Example 5.2.6, 1. along the covariant Yoneda embedding $y : \Delta \to \mathrm{SSet}$



Its right adjoint is called the **singular nerve** $\mathrm{Sing} = \mathrm{Hom}_{\mathrm{Top}}(T(-), -) : \mathrm{Top} \to \mathrm{SSet}$.

While the existence of the geometric realisation is guaranteed by Corollary 5.2.10, it is desirable to have a more concrete picture of the topological spaces and continuous maps that this functor associates to simplicial sets and simplicial maps, respectively. The first step is to derive a formula for this Kan extension with the coend formula from Proposition 4.2.6.

Proposition 5.3.2: The geometric realisation $\mathrm{Geom} : \mathrm{SSet} \to \mathrm{Top}$ assigns to

- a simplicial set $S : \Delta^{op} \to \mathrm{Set}$ the topological space

$$\mathrm{Geom}(S) = |S| = (\amalg_{n \in \mathbb{N}_0} S_n \times \Delta^n) \,/ \sim \qquad\qquad (S(f)(s), x) \sim (s, T(f)(x))$$

  for $f : [m+1] \to [n+1]$, $x \in \Delta^m$ and $s \in S_n$, with the topology induced by the metric topology on Δ^n and the discrete topology on S_n,
- a simplicial map $\alpha : S \Rightarrow S'$ the continuous map $\mathrm{Geom}(\alpha) = |\alpha| : |S| \to |S'|$ with $\alpha([s,x]) = [\alpha_n(s), x]$ for $s \in S_n$, $x \in \Delta^n$.

Proof:
By Proposition 4.2.6 the left Kan extension $\mathrm{Lan}_y T(S)$ is the coend of the functor

$$F = \mathrm{Hom}_{\mathrm{SSet}}(y(-), S) \sqcup T(-) : \Delta^{op} \times \Delta \to \mathrm{Top}.$$

By the Yoneda lemma we have $\mathrm{Hom}_{\mathrm{SSet}}(y([n+1]), S) = \mathrm{Hom}_{\mathrm{SSet}}(\mathrm{Hom}_\Delta(-, [n+1]), S) \cong S_n$ for all $n \in \mathbb{N}_0$ and hence $F([n+1], [m+1]) = S_n \sqcup \Delta^m = \amalg_{S_n} \Delta^m \cong S_n \times \Delta^m$ for all $m, n \in \mathbb{N}_0$. Formula (19) for the coend then gives

$$\mathrm{Geom}(S) = \int^{[n] \in \Delta} F([n],[n]) = \mathrm{coequ}\left(\amalg_{f:[m+1]\to[n+1]} S_n \times \Delta^m \underset{(s,x)\mapsto(S(f)s,x)}{\overset{(s,x)\mapsto(s,T(f)x)}{\rightrightarrows}} \amalg_{m \in \mathbb{N}_0} S_m \times \Delta^m \right),$$

where the first coproduct runs over all morphisms f in Δ and $[m+1] := s(f)$ and $[n+1] := t(f)$. A simplicial map $\alpha : S \Rightarrow S'$ induces a natural transformation $\mu : F \Rightarrow F'$ with components

$$\mu_{n,m} : S_n \times \Delta^m \to S'_n \times \Delta^m, \quad (s,x) \mapsto (\alpha_n(s), x) \quad \text{for } s \in S_n,\ x \in \Delta^m,$$

and by Example 4.1.2, 1. a dinatural transformation $\mu' : F \overset{\bullet}{\Rightarrow} F'$ with $\mu'_n = \mu_{n,n}$. The universal property of the coend then defines the morphism $\mathrm{Geom}(\alpha) : \mathrm{Geom}(S) \to \mathrm{Geom}(S')$. □

Proposition 5.3.2 provides a formula for the geometric realisation of a simplicial set. However, it is still difficult to have a concrete picture. In particular, the formula necessarily involves standard n-simplexes for all $n \in \mathbb{N}_0$. It is therefore not obvious how it describes simple topological spaces such as tori or spheres. Also, while it is intuitive that the face maps glue different standard n-simplexes along their faces, it remains unclear what is the role of the degeneracies.

It turns out that their role is similar to Example 5.2.9, where they define *degenerate* higher simplexes obtained by repeating the vertices of lower dimensional ones. With the equivalence relation from Proposition 5.3.2 we see that degenerate affine simplexes in the geometric realisation correspond to elements of the simplicial sets in the images of the degeneracy maps. We call an element $s \in S_n$ of a simplicial set $S : \Delta^{op} \to \mathrm{Set}$ **degenerate**, if it is in the image of one of the degeneracy maps. It turns out that these elements are redundant in the geometric realisation, as every point in $\mathrm{Geom}(S)$ is represented by a non-degenerate element of S.

Theorem 5.3.3: Let S be a simplicial set with geometric realisation $\mathrm{Geom}(S) = |S|$.

1. For every $p \in |S|$ there are unique $x \in \mathring{\Delta}^n$ and non-degenerate $s \in S_n$ with $[s, x] = p$.
2. $|S|$ inherits a semisimplicial complex structure and a CW complex structure, such that n-simplexes and an n-cells are in bijection with non-degenerate elements of S_n.

Proof:
We prove only the first part. The second is a rather direct consequence of the first, but requires more topology. For each point $p \in |S|$, there is an $m \in \mathbb{N}_0$ and $r \in S_m$, $z \in \Delta^m$ with $p = [r, z]$.

1. There is a unique injection $\delta : [k+1] \to [m+1]$ and a unique $y \in \mathring{\Delta}^k$ with $z = T(\delta)y$:

If $z \in \mathring{\Delta}^m$, then $\delta = 1_{[m+1]}$ and $y = z$. Otherwise, z is contained in a $(m-1)$-face, and there is a face map $f^m_{i_1} = T(\delta^{i_1}_m) : \Delta^{m-1} \to \Delta^m$ and an $y_1 \in \Delta^{m-1}$ with $z = f^m_{i_1}(y_1)$. If $y_1 \in \mathring{\Delta}^{m-1}$ we set $\delta = \delta^{i_1}_m$ and $y = y_1$. Otherwise, there is a face map $f^{m-1}_{i_2} = T(\delta^{i_2}_{m-1}) : \Delta^{m-2} \to \Delta^{m-1}$ and $y_2 \in \Delta^{m-2}$ with $y_1 = f^{m-1}_{i_2}(y_2)$. If $y_2 \in \mathring{\Delta}^{m-2}$, we set $y = y_2$ and $\delta = \delta^{i_1}_m \circ \delta^{i_2}_{m-1}$. The procedure terminates after at most m steps and yield a unique $y \in \mathring{\Delta}^k$ and monotonic injection $\delta : [k+1] \to [m+1]$ with $z = T(\delta)y$.

2. There is a unique surjection $\sigma : [n+1] \to [k+1]$ in Δ and a unique non-degenerate $s \in S_n$ with $S(\delta)r = S(\sigma)(s)$:

If $S(\delta)r$ is non-degenerate, then $s = S(\delta)r$ and $\sigma = 1_{[k+1]}$. Otherwise, there is a surjection $\sigma^{j_1}_k : [k+1] \to [k]$ and an $s_1 \in S_{k-1}$ with $S(\delta)r = S(\sigma^{j_1}_k)(s_1)$. If s_1 is non-degenerate, set $s = s_1$ and $\sigma = \sigma^{j_1}_k$. Otherwise there is a surjection $\sigma^{j_2}_{k-1} : [k] \to [k-1]$ with $s_1 = S(\sigma^{j_2}_{k-1})(s_2)$. If s_2 is non-degenerate, set $\sigma = \sigma^{j_2}_{k-1} \circ \sigma^{j_1}_k$ and $s = s_2$. This terminates after at most k steps and yields a unique non-degenerate $s \in S_n$ and surjection $\sigma : [k+1] \to [n+1]$ with $S(\delta)r = S(\sigma)s$.

3. Combining 1. and 2. yields $p = [r, z] = [r, T(\delta)y] = [S(\delta)r, y] = [S(\sigma)s, y] = [s, T(\sigma)y]$. As

$$y \in \mathring{\Delta}^k = \{\Sigma^k_{j=0} t_j e_j \mid \Sigma^k_{j=0} t_j = 1, t_j > 0 \ \forall j \in \{0, ..., k\}\}$$

one has $T(\sigma)y \in \mathring{\Delta}^n$ by definition of the degeneracy maps. Hence, $p = [s, x]$ with $s \in S_n$ and $x = S(\sigma)r \in \mathring{\Delta}^n$. The uniqueness is clear from the construction. □

Example 5.3.4:

1. The **geometric realisation** of an ordered combinatorial simplicial complex $\mathcal{K} \subset \mathcal{P}(V)$ is defined as the geometric realisation of its simplicial set $S^{\mathcal{K}} : \Delta^{op} \to \mathrm{Set}$ from Example 5.2.9. The n-simplexes $\tau_\alpha : \Delta^n \to |S^{\mathcal{K}}|$ in the semisimplicial complex $|S^{\mathcal{K}}|$ are in bijection with elements $\alpha = (v_0, \ldots, v_n) \in S^{\mathcal{K}}_n$ with $v_0 < \ldots < v_n$.

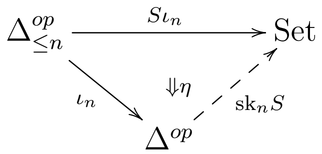
2. The geometric realisation of the simplicial set $\Delta^n = \mathrm{Hom}(-, [n+1]) : \Delta^{op} \to \mathrm{Set}$ from Example 5.2.7, 3. is the topological standard n-simplex Δ^n. (Exercise 43)

Theorem 5.3.3 gives a more concrete and intuitive picture of the geometric realisation of a simplicial set, and Example 5.3.4 allow one to understand it in terms of combinatorial simplicial complexes. Nevertheless, some questions remain. For instance, for a semisimplicial complex or a combinatorial simplicial complex one can construct the n-skeleta by discarding all simplexes of dimension greater than n, see Definitions 5.1.3 and 5.1.5.

The question is if a similar procedure can be applied to simplicial sets and how it interacts with forming the n-skeleton of its geometric realisation. The definition of a simplicial set enforces non-empty sets S_n for all $n \in \mathbb{N}_0$ as soon as $S_0 \neq \emptyset$. However, one can fill the sets S_k for $k > n$ with only *degenerate* simplexes. This is achieved by first restricting a simplicial set to the full subcategory of Δ that contains only the ordinals $[k]$ for $k \leq n+1$ and then performing a left Kan extension along the inclusion functor.

Definition 5.3.5: Let $\Delta_{\leq n} \subset \Delta$ the full subcategory with objects $[1], [2], \ldots, [n+1]$ and $\iota_n : \Delta_{\leq n} \to \Delta$ the inclusion functor.

1. The n**-truncation** of a simplicial set $S : \Delta^{op} \to \text{Set}$ is the functor $S\iota_n : \Delta^{op}_{\leq n} \to \text{Set}$.
2. The n**-skeleton** $\text{sk}_n S : \Delta^{op} \to \text{SSet}$ of a simplicial set $S : \Delta^{op} \to \text{Set}$ is the left Kan extension of $S\iota_n : \Delta^{op}_{\leq n} \to \text{Set}$ along the inclusion $\iota_n : \Delta^{op}_{\leq n} \to \Delta^{op}$



Proposition 5.3.6: In the n-skeleton $\text{sk}_n S$ of a simplicial set S the sets $(\text{sk}_n S)_k$ for $k > n$ contain only degenerate elements. One has $|\text{sk}_n S| \cong \text{sk}_n |S|$

Proof:
1. With the coend formula (19) and the formula from Proposition 4.2.6 one computes

$$(\text{sk}_n S)_p = (\amalg_{m \leq n} \text{Hom}_\Delta([p+1], [m+1]) \times S_m)/\sim$$

with the equivalence relation $(\beta \circ \alpha, s) \sim (\alpha, S(\beta)s)$ for $\beta : [m+1] \to [l+1]$ with $l, m \leq n$, $s \in S_l$ and $\alpha : [p+1] \to [m+1]$. On the morphisms the left Kan extension is given by

$$(\text{sk}_n S)(\gamma) : (\text{sk}_n S)_p \to (\text{sk}_n S)_q, \quad [\alpha, s] \mapsto [\alpha \circ \gamma, s]$$

for all $\gamma : [q+1] \to [p+1]$, $\alpha : [p+1] \to [m+1]$, $s \in S_m$.

If $p > n$ and $m \leq n$, the canonical factorisation of the simplex category (20) allows one to express any morphism $\alpha : [p+1] \to [m+1]$ as a composite $\alpha = \delta \circ \sigma$ with an injective monotonic map $\delta : [k+1] \to [m+1]$ and a surjective monotonic map $\sigma : [p+1] \to [k+1]$ that is a *non-trivial* composite of degeneracies. This implies $[\alpha, s] = [\delta \circ \sigma, s] = (\text{sk}_n S)(\sigma)[\delta, \alpha]$ for all $s \in S_m$, $\alpha : [p+1] \to [m+1]$ and hence any element in $(\text{sk}_n S)_p$ for $p > n$ is degenerate.

In contrast, if $p \leq n$, then any morphism $\alpha : [p+1] \to [m+1]$ is a morphism in $\Delta_{\leq n}$, and one has $[\alpha, s] = [1_{[p+1]}, S(\alpha)s]$. This induces bijections $\phi_p : (\text{sk}_n S)_p \to S_p$, $[\alpha, s] \mapsto S(\alpha)s$ with inverses $\phi_p^{-1} : S_p \to (\text{sk}_n S)_p$, $s \mapsto [1_{[p+1]}, s]$ that satisfy

$$\phi_q \circ (\text{sk}_n S)(\gamma)([\alpha, s]) = \phi_q([\alpha \circ \gamma, s]) = S(\gamma)S(\alpha)s = S(\gamma) \circ \phi_p([\alpha, s])$$

for all $\gamma : [q+1] \to [p+1]$, $\alpha : [p+1] \to [m+1]$ and $s \in S_m$ with $m, p, q \le n$. Hence, an element in $[\alpha, s] \in (\mathrm{sk}_n S)_p$ is degenerate if and only if $\phi_p([\alpha, s])$ is degenerate. Theorem 5.3.3 then implies $|\mathrm{sk}_n S| = \mathrm{sk}_n |S|$ for all $n \in \mathbb{N}_0$. □

Another option is to consider **semisimplicial sets**, functors $s : \Delta_{\mathrm{inj}}^{op} \to \mathrm{Set}$ from the subcategory $\Delta_{\mathrm{inj}} \subset \Delta$ with the same objects but only *injective* monotonic maps as morphisms. For semisimplicial sets, one can define a counterpart of the geometric realisation, the fat realisation, which is more intuitive. In this approach, the sets $s_k = s([k+1])$ may be empty for $k \ge n$ and some fixed n, and elements of s_k correspond bijectively to k-cells in the fat realisation, cf. Exercises 44 and 47. This was the first approach, but also has drawbacks compared with simplicial sets. In particular, it becomes difficult to collapse simplexes and to include degenerate data.

5.4 Simplicial nerve and homotopy category

In this section, we consider the left Kan extension $\mathrm{Lan}_y F : \mathrm{SSet} \to \mathrm{Cat}$ from Corollary 5.2.10 for the cosimplicial object $F : \mathrm{Cat} \to \mathrm{Set}$ from Example 5.2.6, 2. This Kan extension assigns small categories to simplicial sets and functors to simplicial maps. Its right adjoint, the *simplicial nerve* $N : \mathrm{Cat} \to \mathrm{SSet}$ from Example 5.2.7, 2. assigns simplicial sets to small categories and simplicial maps to functors between them .

As we have not yet shown that the category Cat is cocomplete, we cannot argue that this left Kan extension exists by cocompleteness of Cat. Instead, we consider the simplicial nerve, construct its left adjoint explicitly and show that this adjunction identifies Cat with a reflective subcategory of SSet. As SSet is bicomplete, this implies that Cat is bicomplete as well. It follows that the left Kan extension exists, is pointwise and the left adjoint of the simplicial nerve. To work with the simplicial nerve we require a more concrete description. We characterise functors from poset categories into a small category $\mathcal{C}$ as composable sequences of objects in $\mathcal{C}$.

Lemma 5.4.1: Let $C : \Delta \to \mathrm{Cat}$ be the cosimplicial object from Example 5.2.6 that sends

- an ordinal $[n]$ to the poset category $[n]'$,
- a monotonic map $f : [m] \to [n]$ to the associated functor $f' : [m]' \to [n]'$.

The **simplicial nerve** $N = \mathrm{Hom}_{\mathrm{Cat}}(C(-), -) : \mathrm{Cat} \to \mathrm{SSet}$

- sends a category $\mathcal{C}$ to the simplicial set $N(\mathcal{C}) : \Delta^{op} \to \mathrm{Set}$, where
  - $N(\mathcal{C})_0 = \mathrm{Ob}\mathcal{C}$,
  - $N(\mathcal{C})_n = \{(f_1, \ldots, f_n) \mid f_j : C_{j-1} \to C_j\}$ for $n \ge 1$ is the set of sequences of n composable morphisms in $\mathcal{C}$,
  - face maps and degeneracies are given by

$$d_1^0 : f_1 \mapsto C_1, \qquad d_1^1 : f_1 \mapsto C_0, \tag{23}$$

$$d_n^i : (f_1, \ldots, f_n) \mapsto \begin{cases} (f_2, \ldots, f_n) & i = 0 \\ (f_1, \ldots, f_{i+1} \circ f_i, \ldots, f_n) & 1 \le i \le n-1 \\ (f_1, \ldots, f_{n-1}) & i = n \end{cases} \qquad n > 1,$$

$$s_0^0 : C \mapsto 1_C \qquad s_n^i : (f_1, \ldots, f_n) \mapsto (f_1, \ldots, f_i, 1_{C_i}, f_{i+1}, \ldots, f_n) \qquad n \ge 1,$$

- sends a functor $F : \mathcal{C} \to \mathcal{C}'$ to the simplicial map $N(F) : N(\mathcal{C}) \Rightarrow N(\mathcal{C}')$ with components

$$N(F)_n : N(\mathcal{C})_n \to N(\mathcal{C}')_n,\ (f_1, \ldots, f_n) \mapsto (F(f_1), \ldots, F(f_n)).$$

This lemma gives a more explicit expression for the simplicial nerve that could be used in principle to compute it. The simplest examples are nerves of poset categories and simplicial nerves of groups, viewed as categories with a single object.

Example 5.4.2: The simplicial nerve of the poset category $\mathcal{C} = [n+1]'$ is given by

$$N(\mathcal{C})_k = \mathrm{Hom}_{\mathrm{Cat}}([k+1]', [n+1]') \cong \mathrm{Hom}_{\Delta}([k+1], [n+1])$$
$$N(\mathcal{C})(\tau) : N(\mathcal{C})_l \to N(\mathcal{C})_k, \; \sigma \mapsto \sigma \circ \tau \quad \text{for} \quad \tau : [k+1] \to [l+1].$$

Thus, $N([n+1]') = \mathrm{Hom}(-, [n+1]) : \Delta^{op} \to \mathrm{Set}$ is the simplicial set from Example 5.2.7, 3.

Example 5.4.3: The simplicial nerve of the category BG for a group G is given by

$$N(BG)_n = G^{\times n}$$
$$s^i_n : G^{\times n} \to G^{\times (n+1)}, \quad (g_1, \ldots, g_n) \mapsto (g_1, \ldots, g_{i-1}, 1, g_i, \ldots, g_n)$$
$$d^i_n : G^{\times n} \to G^{\times (n-1)}, \quad (g_1, \ldots, g_n) \mapsto \begin{cases} (g_2, \ldots, g_n) & i = 0 \\ (g_1, \ldots, g_{i+1} \cdot g_i, \ldots, g_n) & 1 \leq i \leq 0 \\ (g_1, \ldots, g_{n-1}) & i = n \end{cases}$$

Clearly, the simplicial nerve $N : \mathrm{Cat} \to \mathrm{SSet}$ is injective on the objects and fully faithful and hence identifies the category Cat with a full subcategory of SSet. We will now show that it has a left adjoint $h : \mathrm{SSet} \to \mathrm{Cat}$, the *homotopy functor*. Thus, the simplicial nerve identifies Cat with a *reflective* subcategory of SSet, see Definition 2.4.10.

Definition 5.4.4: The **homotopy functor** $h : \mathrm{SSet} \to \mathrm{Cat}$ sends

- a simplicial set $S : \Delta^{op} \to \mathrm{Set}$ to the category $\mathcal{C} = h(S)$ with $\mathrm{Ob}\mathcal{C} = S_0$, elements $y \in S_1$ as generating morphisms $y : d^1_1(y) \to d^0_1(y)$ and relations

$$1_x = s^0_0(x) \text{ for all } x \in S_0 \qquad\qquad d^0_2(z) \circ d^2_2(z) = d^1_2(z) \text{ for all } z \in S_2, \tag{24}$$

- a simplicial map $\alpha : S \Rightarrow S'$ to the functor $h(\alpha) : h(S) \to h(S')$ given by $\alpha_0 : S_0 \to S'_0$ on the objects and $\alpha_1 : S_1 \to S'_1$ on the morphisms.

The category $h(S)$ for a simplicial set S is called the **homotopy category** of S.

Remark 5.4.5:

1. Only non-degenerate elements of S_2 give rise to non-trivial relations in $h(S)$:
   If $z = s^0_1(y)$ for some $y \in S_1$, then relations (22) and the first relation in (24) imply $d^0_2(z) = d^1_2(z) = y$ and $d^2_2(z) = 1_{d^1_1(y)}$. If $z = s^1_1(y)$, then one obtains $d^1_2(z) = d^2_2(z) = y$ and $d^0_2(z) = 1_{d^1_0(y)}$. In both cases, the second relation in (24) is satisfied trivially.
2. In general, the morphisms of $h(S)$ are hard to describe concretely. They are equivalence classes of sequences $(y_1, \ldots, y_n)$ with $y_i \in S_1$ and $d^1_1(y_i) = d^0_1(y_{i-1})$ for $1 \leq i \leq n$, with the concatenation as composition and the equivalence relation defined by (24).

3. The naturality of α ensures that the functor $h(\alpha) : h(S) \to h(S')$ respects the source and targets of morphisms and the relations (24). Compatibility with composition of morphisms is then satisfied trivially, as composition is given by concatenation.

We now show that the homotopy functor is left adjoint to the simplicial nerve. The key observation is that for any simplicial map $\alpha : S \Rightarrow N(\mathcal{C})$ the component morphisms $\alpha_o : S_0 \to \mathrm{Ob}\mathcal{C}$ and $\alpha_1 : S_1 \to \mathrm{Mor}\mathcal{C}$ satisfy the relations in (24). Hence, the adjunction must identify a simplicial map with the functor $h(S) \to \mathcal{C}$ induced by the component morphisms α_0 and α_1.

Proposition 5.4.6: The homotopy functor $h : \mathrm{SSet} \to \mathrm{Cat}$ is left adjoint to the simplicial nerve $N : \mathrm{Cat} \to \mathrm{SSet}$ with $hN \cong \mathrm{id}_{\mathrm{Cat}}$.

Proof:
1. The adjunction is given by the maps

$$\phi_{S,\mathcal{C}} : \mathrm{Hom}_{\mathrm{SSet}}(S, N(\mathcal{C})) \to \mathrm{Hom}_{\mathrm{Cat}}(h(S), \mathcal{C})$$

that send a simplicial map $\alpha : S \Rightarrow N(\mathcal{C})$ to the functor $\phi_{S,\mathcal{C}}(\alpha) : h(S) \to \mathcal{C}$ induced by the maps $\alpha_0 : S_0 \to \mathrm{Ob}\mathcal{C}$ and $\alpha_1 : S_1 \to \mathrm{Mor}\mathcal{C}$, where $\mathrm{Mor}\mathcal{C}$ is the set of all morphisms of $\mathcal{C}$. That $\phi_{S,C}(\alpha)$ is indeed a functor follows, because the naturality of α and formulas (23) for the simplicial nerve imply for all $x \in S_0$, $y \in S_1$ and $z \in S_2$

$$\begin{aligned}
&t(\phi_{S,\mathcal{C}}(\alpha)(y)) \overset{(23)}{=} d_1^0 \circ \alpha_1(y) \overset{\text{nat }\alpha}{=} \alpha_0 \circ d_1^0(y) = \phi_{S,\mathcal{C}}(\alpha)(d_1^0(y)) = \phi_{S,\mathcal{C}}(\alpha)(t(y)) \\
&s(\phi_{S,\mathcal{C}}(\alpha)(y)) \overset{(23)}{=} d_1^1 \circ \alpha_1(y) \overset{\text{nat }\alpha}{=} \alpha_0 \circ d_1^1(y) = \phi_{S,\mathcal{C}}(\alpha)(d_1^1(y)) = \phi_{S,\mathcal{C}}(\alpha)(s(y)) \\
&\phi_{S,\mathcal{C}}(\alpha)(s_0^0(x)) = \alpha_1 \circ s_0^0(x) \overset{\text{nat }\alpha}{=} s_0^0 \circ \alpha_0(x) \overset{(23)}{=} 1_{\alpha_0(x)} = \phi_{S,\mathcal{C}}(\alpha)(1_x) \\
&\phi_{S,\mathcal{C}}(\alpha)(d_2^1(z)) = \alpha_1 \circ d_2^1(z) \overset{\text{nat }\alpha}{=} d_2^1 \circ \alpha_2(z) \overset{(23)}{=} d_2^0(\alpha_2(z)) \circ d_2^2(\alpha_2(z)) \overset{\text{nat }\alpha}{=} \alpha_1(d_2^0(z)) \circ \alpha_1(d_2^2(z)) \\
&\qquad = \phi_{S,\mathcal{C}}(\alpha)(d_2^0(z) \circ d_2^2(z)),
\end{aligned} \tag{25}$$

where $s(f)$ and $t(f)$ denote the source and the target of a morphism f in $\mathcal{C}$ or $h(S)$. Naturality in S and $\mathcal{C}$ follows directly from the definitions.

2. The inverse of $\phi_{S,C}$ sends a functor $F : h(S) \to \mathcal{C}$ given by $F_0 : S_0 \to \mathrm{Ob}\mathcal{C}$ and $F_1 : S_1 \to \mathrm{Mor}\mathcal{C}$ to the simplicial map $\alpha = \phi_{S,C}^{-1}(F)$ given by $\alpha_0 = F_0$, $\alpha_1 = F_1$ and

$$\alpha_n = (F_1(f_n^1), \ldots, F_1(f_n^n)) : S_n \to N(\mathcal{C})_n,\ s \mapsto (F_1(f_n^1(s)), \ldots, F_1(f_n^n(s))) \quad \text{for } n > 2, \tag{26}$$

where $f_n^j = S(\tau_n^j)$ is the image of the map

$$\tau_n^j : [2] \to [n+1], \quad 0 \mapsto j-1, 1 \mapsto j \qquad 1 \leq j \leq n. \tag{27}$$

3. To show that this defines a simplicial map, it remains to prove the naturality of α. Note that the fact that F is a functor implies for all $x \in S_0$ and $y \in S_1$

$$\begin{aligned}
&d_1^0 \circ \alpha_1(y) = t(F_1(y)) = F_0(t(y)) = \alpha_0 \circ d_1^0(y) \\
&d_1^1 \circ \alpha_1(y) = s(F_1(y)) = F_0(s(y)) = \alpha_0 \circ d_1^1(y) \\
&s_0^0 \circ \alpha_0(x) = 1_{\alpha_0(x)} = 1_{F_0(x)} = F_1(1_x) = F_1(s_0^0(x)) = \alpha_1 \circ s_0^0(x).
\end{aligned}$$

It remains to show that for all $n \geq 2$ and $i \in [n+1]$, $k \in [n]$

$$\alpha_n \circ d_{n+1}^i = d_{n+1}^i \circ \alpha_{n+1} \qquad \alpha_n \circ s_{n-1}^k = s_n^k \circ \alpha_{n-1}.$$

With some computations we obtain

$$\delta^i_{n+1}\circ\tau^j_n = \begin{cases} \tau^j_{n+1} & i<j \\ \rho^j_n\circ\delta^1_2 & i=j \\ \tau^{j+1}_{n+1} & i>j \end{cases} \qquad \sigma^k_n\circ\tau^j_n = \begin{cases} \tau^{j-1}_{n-1} & k<j-1 \\ \tau^{j-1}_{n-1}\circ\sigma^0_1 & k=j-1 \\ \tau^j_{n-1} & k\geq j \end{cases}$$

$$\rho^j_n : [3]\to[n+2], 0\mapsto j-1, 1\mapsto j, 2\mapsto j+1,$$

and this yields with (26), (27) and $d^i_{n+1}=S(\delta^i_{n+1})$

$$\alpha_n\circ d^0_{n+1} = (F_1(f^2_{n+1}),\ldots,F_1(f^{n+1}_{n+1}))$$
$$\alpha_n\circ d^{n+1}_{n+1} = (F_1(f^1_{n+1}),\ldots,F_1(f^n_{n+1}))$$
$$\alpha_n\circ s^k_{n-1} = (F_1(f^1_{n-1}),\ldots,F_1(f^{k-1}_{n-1}),1_{d^1_0\circ f^{k-1}_{n-1}},F_1(f^k_{n-1}),\ldots F_1(f^{n-1}_{n-1})).$$

As $\rho^j_n\circ\delta^0_2=\tau^{j+1}_{n+1}$ and the relations (21) imply $\rho^j_n\circ\delta^2_2=\tau^j_{n+1}$, we also obtain with relation (24) and the identity $F_1(d^1_2(z))=F_1(d^0_2(z))\circ F_1(d^2_2(z))$ for all $z\in S_2$

$$\begin{aligned}\alpha_n\circ d^i_{n+1} &= \alpha_n\circ S(\delta^i_{n+1}) = (F_1(f^1_{n+1}),\ldots,F_1(f^{i-1}_{n+1}),F_1S(\rho^i_n\circ\delta^1_2)),F_1(f^{i+1}_{n+1})\ldots,F_1(f^{n+1}_{n+1}))\\ &= (F_1(f^1_{n+1}),\ldots,F_1(d^1_2\circ S(\rho^i_n)),\ldots,F_1(f^{n+1}_{n+1}))\\ &= (F_1(f^1_{n+1}),\ldots,F_1(d^0_2\circ S(\rho^i_n))\circ F_1(d^2_2\circ S(\rho^i_n)),\ldots,F_1(f^{n+1}_{n+1}))\\ &= (F_1(f^1_{n+1}),\ldots,F_1S(\rho^i_n\circ\delta^0_2)\circ F_1S(\rho^i_n\circ\delta^2_2),\ldots,F_1(f^{n+1}_{n+1}))\\ &= (F_1(f^1_{n+1}),\ldots,F_1S(\tau^{j+1}_{n+1})\circ F_1S(\tau^j_{n+1}),\ldots,F_1(f^{n+1}_{n+1}))\\ &= (F_1(f^1_{n+1}),\ldots,F_1(f^{i+1}_{n+1})\circ F(f^i_{n+1}),\ldots,F_1(f^{n+1}_{n+1})) \qquad 1\leq 1\leq n-1.\end{aligned}$$

Comparing with the formulas in Lemma 5.4.1 then shows that α is a simplicial map.

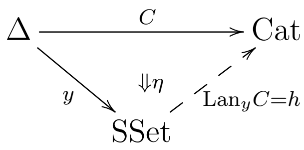
4. To show that $hN\cong \mathrm{id}_{\mathrm{Cat}}$ we consider the functors $\beta_{\mathcal{C}}:\mathcal{C}\to hN(\mathcal{C})$ for each small category $\mathcal{C}$ that send each object to itself and each morphism f in $\mathcal{C}$ to the equivalence class of $(f)\in N(\mathcal{C})_1$ in $hN(\mathcal{C})$. Their inverses are the functors $\beta^{-1}_{\mathcal{C}}:hN(\mathcal{C})\to\mathcal{C}$ that send each object to itself and the equivalence class of a composable n-tuple $(f_1,\ldots,f_n)$ of morphisms in $\mathcal{C}$ to the morphism $f_n\circ\ldots\circ f_1$. That $\beta^{-1}_{\mathcal{C}}$ is indeed inverse to $\beta_{\mathcal{C}}$ follows from the relations of $hN(\mathcal{C})$, which relate any composable n-tuple $(f_1,\ldots,f_n)$ of morphisms in $\mathcal{C}$ to their composite $f_n\circ\ldots\circ f_1$. □

As the simplicial nerve $N:\mathrm{Cat}\to\mathrm{SSet}$ is injective on the objects, fully faithful and right adjoint to the homotopy functor $h:\mathrm{SSet}\to\mathrm{Cat}$, it identifies Cat with a reflective subcategory of SSet. As SSet is bicomplete by Proposition 2.3.2, Proposition 2.4.12 implies that Cat is cocomplete, as anticipated in Remark 2.4.13 and at the beginning of this section.

Corollary 5.4.7: The category Cat is a reflective subcategory of SSet with the embedding functor $N:\mathrm{Cat}\to\mathrm{SSet}$. In particular, Cat is bicomplete.

Proposition 5.4.6 also allows us to identify the homotopy functor $h:\mathrm{SSet}\to\mathrm{Cat}$ as the left Kan extension $\mathrm{Lan}_yC:\mathrm{SSet}\to\mathrm{Cat}$ of the functor $C:\Delta\to\mathrm{Top}$ from Example 5.2.6, 2. along the Yoneda embedding $y:\Delta\to\mathrm{SSet}$. As Δ is small and Cat cocomplete, this Kan extension exists, is pointwise and left adjoint to the simplicial nerve $N:\mathrm{Cat}\to\mathrm{SSet}$ by Corollary 5.2.10. As the left adjoint is unique, it follows that the homotopy functor is naturally isomorphic to this left Kan extension, $h\cong\mathrm{Lan}_yC$, and hence a left Kan extension of C along y.

Corollary 5.4.8: The homotopy functor $h : \mathrm{SSet} \to \mathrm{Cat}$ is the left Kan extension of the functor $C : \Delta \to \mathrm{Cat}$ along the Yoneda embedding $y : \Delta \to \mathrm{SSet}$.



Thus, the nerve $N : \mathrm{Cat} \to \mathrm{SSet}$ and the homotopy functor $h : \mathrm{SSet} \to \mathrm{Cat}$ are the categorical counterparts of the singular nerve $\mathrm{Sing} : \mathrm{Top} \to \mathrm{SSet}$ and the geometric realisation functor $\mathrm{Geom} : \mathrm{SSet} \to \mathrm{Top}$ from Definition 5.3.1. The functor $C : \Delta \to \mathrm{Cat}$ from Example 5.2.6, 2. replaces the functor $T : \Delta \to \mathrm{Top}$ from Example 5.2.6, 1 and the poset categories for ordinals take the role of the standard n-simplexes. Both Kan extensions and adjunctions are just different manifestations of Corollary 5.2.10.

Beyond the proof of the bicompleteness of Cat, the nerve and the homotopy functor have other applications. As the nerve assigns to each category a simplicial set and to each functor a simplicial map, one can combine it with the geometric realisation functor to assign to each category a topological space and to each functor a continuous map. This allows one to apply techniques from topology to investigate categories and functors between them.

Definition 5.4.9: The **classifying space functor** for a small category $\mathcal{C}$ is the functor $B = \mathrm{Geom}N : \mathrm{Cat} \to \mathrm{Top}$ that assigns

- to a small category $\mathcal{C}$ its **classifying space** $B\mathcal{C} = |N(\mathcal{C})|$,
- to a functor $F : \mathcal{C} \to \mathcal{D}$ the associated continuous map $B(F) : |B\mathcal{C}| \to |B\mathcal{D}|$.

Example 5.4.10:

1. A set J as a discrete category $\mathcal{J}$ with only identity morphisms has the classifying space $B\mathcal{J} = J$ with the discrete topology.

2. The **classifying space** of a group G is the classifying space of the associated category BG.

3. The poset category $[n+1]'$ for $n \in \mathbb{N}_0$ has as classifying space the standard n-simplex $B[n+1] = \Delta^n$. This follows from Example 5.3.4, 2. and Example 5.4.2.

5.5 Chain complexes and homologies

In this section, we discuss an important application of simplicial objects, namely chain complexes and their homologies. Chain complexes play a fundamental role in algebra, topology and geometry - they are often the best and most practical way to characterise mathematical objects. They are complicated enough to store considerable amounts of mathematical information, but simple to handle due to their linear nature.

We will show that simplicial objects in the category R-Mod of modules over a ring R define chain complexes in R-Mod and simplicial morphisms define chain maps between them. In fact, this holds more generally for simplicial objects and morphisms in abelian categories. We will even see that up to isomorphism, all positive chain complexes in R-Mod arise from simplicial objects in R-Mod. This is the famous Dold-Kan correspondence. Its proof can be viewed as an exercise in Kan extensions. It consists mainly of computing a Kan extension.

In particular, this correspondence between chain complexes and simplicial objects in R-Mod allows one to construct chain complex from simplicial sets. Any simplicial set $S : \Delta^{op} \to \text{Set}$ can be promoted to a simplicial R-module by considering the R-modules generated by the sets S_n for $n \in \mathbb{N}_0$ and the R-linear maps between them induced by its face maps and degeneracies.

We start by introducing chain complexes, chain maps and their homologies and then investigate their relation with simplicial objects. They play a role in almost every part of mathematics.

Definition 5.5.1: Let R be a ring.

1. A **chain complex** $C_\bullet$ in R-Mod is a family $(C_n)_{n\in\mathbb{Z}}$ of R-modules and a family $(d_n)_{n\in\mathbb{Z}}$ of R-linear maps $d_n : C_n \to C_{n-1}$, the **boundary operators**, with $d_{n-1} \circ d_n = 0$ for $n \in \mathbb{Z}$.
$$C_\bullet = \ldots \xrightarrow{d_{n+2}} C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \ldots$$
2. A **chain map** $f_\bullet : C_\bullet \to C'_\bullet$ is a family $(f_n)_{n\in\mathbb{Z}}$ of R-linear maps $f_n : C_n \to C'_n$ with $d'_n \circ f_n = f_{n-1} \circ d_n$ for all $n \in \mathbb{Z}$
$$\begin{array}{ccccccccc} \ldots & \xrightarrow{d_{n+2}} & C_{n+1} & \xrightarrow{d_{n+1}} & C_n & \xrightarrow{d_n} & C_{n-1} & \xrightarrow{d_{n-1}} & \ldots \\ & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\ \ldots & \xrightarrow[d'_{n+2}]{} & C'_{n+1} & \xrightarrow[d'_{n+1}]{} & C'_n & \xrightarrow[d'_n]{} & C'_{n-1} & \xrightarrow[d'_{n-1}]{} & \ldots \end{array}$$

The category of chain complexes and chain maps, with the composition $(g_\bullet \circ f_\bullet)_n = g_n \circ f_n$ and identity morphisms $(1_{C_\bullet})_n = \text{id}_{C_n}$, is denoted $\text{Ch}_{R\text{-Mod}}$.

A chain complex is called **positive** if $C_n = 0$ for all $n < 0$. The full subcategory of positive chain complexes is denoted $\text{Ch}_{R\text{-Mod}\geq 0} \subset \text{Ch}_{R\text{-Mod}}$.

Remark 5.5.2:

1. Chain complexes and chain maps between them can be defined in more generality, namely in *abelian categories*.
2. Sequences of trivial modules are omitted from a chain complex. A chain complex that ends with 0 on the left, the right or both is to be supplemented by trivial modules.
3. Elements of the R-modules C_n are called n-**chains**, elements of the kernel $\ker(d_n) \subset C_n$ are called n-**cycles** and elements of the image $\text{im}(d_{n+1}) \subset C_n$ are called n-**boundaries**.
4. A chain complex $C_\bullet$ is called **exact in** n, if $\ker(d_n) = \text{im}(d_{n+1})$, and **exact**, if it is exact in all $n \in \mathbb{Z}$.

The essential quantities associated to a chain complex are its homologies. The condition $d_n \circ d_{n+1} = 0$ for all boundary operators implies that $\text{im}(d_{n+1}) \subset \ker(d_n)$ is a submodule. Thus, one can form the quotient module $H_n(C_\bullet) = \ker(d_n)/\text{im}(d_{n+1})$, the *nth homology* of $C_\bullet$. It vanishes if and only if $C_\bullet$ is exact in n and therefore measures the non-exactness of $C_\bullet$ in n.

The compatibility conditions between boundary operators and chain maps ensures that $f_n(\ker(d_n)) \subset \ker(d'_n)$ and $f_n(\text{im}(d_{n+1})) \subset \text{im}(d'_{n+1})$ for all chain maps $f_\bullet : C_\bullet \to C'_\bullet$. Thus, any chain map induces an R-linear map $H_n(f_\bullet) : H_n(C_\bullet) \to H_n(C'_\bullet)$. This allows one to view the homologies as functors $H_n : \text{Ch}_{R\text{-Mod}} \to R\text{-Mod}$.

Definition 5.5.3: Let R be a ring.
For $n \in \mathbb{Z}$ the nth **homology** is the functor $H_n : \mathrm{Ch}_{R\text{-Mod}} \to R\text{-Mod}$ that assigns to

- a chain complex $C_\bullet$ in R-Mod the R-module $H_n(C_\bullet) = \ker(d_n)/\mathrm{im}(d_{n+1})$,
- a chain map $f_\bullet : C_\bullet \to C'_\bullet$ the induced map $H_n(f_\bullet) : H_n(C_\bullet) \to H_n(C'_\bullet)$, $[c] \mapsto [f_n(c)]$.

Example 5.5.4:

1. The chain complex $C_\bullet = \ldots \xrightarrow{\bar{x}\mapsto\overline{2x}} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\bar{x}\mapsto\overline{2x}} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\bar{x}\mapsto\overline{2x}} \ldots$ in Ab has the homologies $H_n(C_\bullet) = \ker(\bar{x} \mapsto \overline{2x})/\mathrm{im}(\bar{x} \mapsto \overline{2x}) = \{\bar{0}, \bar{2}\}/\{\bar{0}, \bar{2}\} = 0$ for all $n \in \mathbb{N}_0$ and is exact.
2. A chain complex of the form $0 \to L \xrightarrow{\iota} M \xrightarrow{\pi} N \to 0$ is exact if and only if ι is injective and π surjective with $\ker(\pi) = \mathrm{im}(\iota)$. This is equivalent to the statement that L is isomorphic to a submodule of M and N to the associated quotient module. An exact chain complex of this form is called a **short exact sequence**.

Chain complexes and chain maps in a category R-Mod are closely related to simplicial objects and simplicial morphisms in R-Mod. Any simplicial object $S : \Delta^{op} \to R$-Mod defines a chain complex $S_\bullet$, whose modules are the objects S_n for $n \in \mathbb{N}_0$ and zero if $n < 0$. Its boundary morphisms are given as alternating sums over the face maps. The simplicial relations (21) ensure that they satisfy the relation $d_{n-1} \circ d_n = 0$ for all $n \in \mathbb{Z}$. Due to their naturality, the component morphisms $\alpha_n : S_n \to S'_n$ of a simplicial morphism $\alpha : S \Rightarrow S'$ define a chain map $\alpha_\bullet : S_\bullet \to S'_\bullet$. This gives rise to a functor from the category of simplicial objects and morphisms in R-Mod to the category of chain complexes and chain maps in R-Mod.

Proposition 5.5.5: Let R be a ring.
The **standard chain complex** functor $\bullet : R\text{-Mod}^{\Delta^{op}} \to \mathrm{Ch}_{R\text{-Mod}}$ assigns to

- a simplicial object $S : \Delta^{op} \to R$-Mod the chain complex $S_\bullet$ with boundary operators
$$d_n = \Sigma_{i=0}^n (-1)^i S(\delta_n^i) : S_n \to S_{n-1},$$
- a simplicial morphism $\alpha : S \Rightarrow S'$ in R-Mod the chain map $\alpha_\bullet : S_\bullet \to S'_\bullet$ with components
$$\alpha_n : S_n \to S'_n.$$

Proof:
1. We show that each simplicial object $S : \Delta^{op} \to R$-Mod yields a chain complex $S_\bullet$:

$$\begin{aligned} d_{n-1} \circ d_n &= \Sigma_{j=0}^{n-1}\Sigma_{i=0}^{n} (-1)^{i+j} S(\delta_n^i \circ \delta_{n-1}^j) \\ &= \Sigma_{0\le j<i\le n} (-1)^{i+j} S(\delta_n^i \circ \delta_{n-1}^j) + \Sigma_{0\le i\le j\le n} (-1)^{i+j} S(\delta_n^i \circ \delta_{n-1}^j) \\ &\overset{(21)}{=} \Sigma_{0\le j<i\le n} (-1)^{i+j} S(\delta_n^i \circ \delta_{n-1}^j) + \Sigma_{0\le i\le j\le n} (-1)^{i+j} S(\delta_n^{j+1} \circ \delta_{n-1}^i) \\ &= \Sigma_{0\le j<i\le n} (-1)^{i+j} S(\delta_n^i \circ \delta_{n-1}^j) - \Sigma_{0\le j<i\le n} (-1)^{i+j} S(\delta_n^j \circ \delta_{n-1}^i) = 0. \end{aligned}$$

2. That any simplicial morphism $\alpha : S \Rightarrow S'$ defines a chain map $\alpha_\bullet : S_\bullet \to S'_\bullet$ follows, because

$$\alpha_{n-1} \circ d_n = \Sigma_{i=0}^n (-1)^i \alpha_{n-1} \circ S(\delta_n^i) \overset{\text{nat}\,\alpha}{=} \Sigma_{i=0}^n (-1)^i S'(\delta_n^i) \circ \alpha_n = d_n \circ \alpha_n.$$

This defines a functor, because the composition and identity morphisms of simplicial maps and chain maps are both componentwise. □

Simplicial objects and morphisms in a category R-Mod can be constructed easily from simplicial sets and maps. One simply composes them with the functor $\langle\ \rangle_R : \text{Set} \to R\text{-Mod}$ that assigns

- to a set X the free R-module $\langle X\rangle_R$,
- to a map $f : X \to Y$ the R-linear map $\langle f\rangle_R : \langle X\rangle_R \to \langle Y\rangle_R$ with $\langle f\rangle_R|_X = f$.

This yields a post-composition functor $F_R = \langle\ \rangle_{R\,*} : \text{SSet} \to R\text{-Mod}^{\Delta^{op}}$ that assigns to

- a simplicial set $S : \Delta^{op} \to \text{Set}$ the simplicial object $\langle\ \rangle_R S : \Delta^{op} \to R\text{-Mod}$,
- a simplicial map $\alpha : S \Rightarrow S'$ the simplicial morphism $\langle\ \rangle_R\,\alpha : \langle\ \rangle_R\, S \Rightarrow \langle\ \rangle_R\, S'$.

Many chain complexes that are important in algebraic topology and homological algebra are constructed in this way. For practical reasons and ease of computation one usually restricts attention to commutative rings $R = k$ or even the ring $R = \mathbb{Z}$.

Example 5.5.6: Let k be a commutative ring.

1. Post-composing the singular nerve $\text{Sing} : \text{Top} \to \text{SSet}$ from Example 5.2.7, 1. with the functor $F_k : \text{SSet} \to k\text{-Mod}^{\Delta^{op}}$ and the standard chain complex functor from Proposition 5.5.5 yields the **singular chain complex** functor $C_\bullet(-, k) : \text{Top} \to \text{Ch}_{k\text{-Mod}}$. It sends
   - a topological space X to the chain complex $C_\bullet(X, k)$ with
   $$C_n(X, k) = \langle \text{Hom}_{\text{Top}}(\Delta^n, X)\rangle_k$$
   $$d_n : C_n(X, k) \to C_{n-1}(X, k), \quad \sigma \mapsto \Sigma_{i=0}^n (-1)^i \sigma \circ f_i^n,$$
   - a continuous map $f : X \to Y$ to the chain map $C_\bullet(f, k) : C_\bullet(X, k) \to C_\bullet(Y, k)$ with
   $$f_n : C_n(X, k) \to C_n(Y, k), \quad \sigma \mapsto f \circ \sigma,$$

   The associated homologies $H_n(X, k) = H_n C_\bullet(X, k)$ are called **singular homologies** of X. They play a fundamental role in algebraic topology - most algebraic topology lectures are essentially an investigation of this functor.

2. Composing the simplicial nerve $N : \text{Cat} \to \text{SSet}$ from Example 5.2.7, 2. with the functor $F_k : \text{SSet} \to k\text{-Mod}^{\Delta^{op}}$ and then the standard chain complex functor from Proposition 5.5.5 yields a functor $C_\bullet(-, k) : \text{Cat} \to \text{Ch}_{k\text{-Mod}}$ that assigns
   - to a category $\mathcal{C}$ the chain complex $C_\bullet(\mathcal{C}, k)$ with
   $$C_n(\mathcal{C}, k) = \langle N(\mathcal{C})_n\rangle_k \qquad d = \Sigma_{i=0}^n (-1)^i \langle N(\delta_n^i)\rangle_k : C_n(\mathcal{C}, k) \to C_{n-1}(\mathcal{C}, k)$$
   - to a functor $F : \mathcal{C} \to \mathcal{D}$ the chain map $C_\bullet(F, k) : C_\bullet(\mathcal{C}, k) \to C_\bullet(\mathcal{D}, k)$ with components $C_n(F, k) = \langle N(F)_n\rangle_k : \langle N(\mathcal{C})_n\rangle_k \to \langle N(\mathcal{D})_n\rangle_k$.

3. Let $\iota : \text{Grp} \to \text{Cat}$ be the inclusion functor that assigns to a group G its delooping BG. Composing it with the functor $C_\bullet(-, \mathbb{Z}) : \text{Cat} \to \text{Ch}_{\text{Ab}}$ from 2. yields a functor $C_\bullet(-, \mathbb{Z}) : \text{Grp} \to \text{Ch}_{\text{Ab}}$ that assigns
   - to a group G the chain complex $C_\bullet(G, \mathbb{Z})$ with $C_n(G, \mathbb{Z}) = \mathbb{Z}[G^{\times n}]$ and boundary operator $d_n = \Sigma_{i=0}^n (-1)^i d_n^i$ given by Example 5.4.3
   $$d_n^i(g_1, \ldots, g_n) = \begin{cases} (g_2, \ldots, g_n) & i = 0 \\ (g_1, \ldots, g_{i+1} \cdot g_i, \ldots, g_n) & 1 \leq i < n \\ (g_1, \ldots, g_{n-1}) & i = n, \end{cases}$$

- to a group homomorphism $f : G \to H$ the chain map $C_\bullet(f,\mathbb{Z}) : C_\bullet(G,\mathbb{Z}) \to C_\bullet(H,\mathbb{Z})$ given by $C_n(f,\mathbb{Z}) : C_n(G,\mathbb{Z}) \to C_n(H,\mathbb{Z})$, $(g_1,\ldots,g_n) \to (f(g_1),\ldots,f(g_n))$.

The homologies of $C_\bullet(G,\mathbb{Z})$ are called the **group homologies** of G and denoted $H_n(G,\mathbb{Z})$.

4. Let $S : \mathrm{SSet} \to \mathrm{Vect}_\mathbb{F}$ be the simplicial object from Example 5.2.8 for an algebra A as a bimodule over itself. Composing it with the standard chain complex functor from Proposition 5.5.5 yields a functor $C_\bullet(-) : \mathrm{Alg}_\mathbb{F} \to \mathrm{Ch}_{\mathrm{Vect}_\mathbb{F}}$ that assigns to
   - an algebra A over $\mathbb{F}$ the chain complex $C_\bullet(A)$ with $C_n(A) = A^{\otimes(n+1)}$ and boundary operator $d_n = \Sigma_{i=0}^n(-1)^i d_n^i$

$$d_n^i(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = \begin{cases} a_0 \otimes a_1 \otimes \ldots \otimes (a_i a_{i+1}) \otimes \ldots \otimes a_n & 0 \le i \le n-1 \\ (a_n a_0) \otimes a_1 \otimes \ldots \otimes a_{n-1} & i = n, \end{cases}$$

   - an algebra homomorphism $f : A \to B$ the chain map $C_\bullet(f) : C_\bullet(A) \to C_\bullet(B)$ with $C_n(f) : A^{\otimes(n+1)} \to B^{\otimes(n+1)}$, $a_0 \otimes \ldots \otimes a_n \mapsto f(a_0) \otimes \ldots \otimes f(a_n)$.

The homologies of $C_\bullet(A)$ are called **Hochschild homology** of A and denoted $H_n(A)$.

Example 5.5.6 covers the most important chain complexes that arise in topology and algebra. The point of these constructions is that they assign a chain complex to each topological space, category, group or algebra in a systematic way that is compatible with the morphisms. This allows one to use these chain complexes and their homologies in classification problems. For instance, if two groups, algebras or topological spaces are isomorphic, this must also hold for their chain complexes and the associated homologies. Homologies can thus be used to compare or distinguish algebraic objects or topological spaces.

The following examples show that the homologies of the chain complexes in Example 5.5.6 contain relevant information about the underlying algebraic object or topological space. In general, the information encoded in the nth homology becomes more complicated and difficult to access for increasing n. The lowest homologies have a simple interpretation.

Example 5.5.7:

1. The zeroth group homology $H_0(G,\mathbb{Z}) = \ker(d_0)/\mathrm{im}(d_1) \cong \mathbb{Z}/0 = \mathbb{Z}$, and the first group homology is the abelisation of G

$$H_1(G,\mathbb{Z}) = \ker(d_1)/\mathrm{im}(d_2) = \mathbb{Z}[G]/\langle gh - g - h \mid g,h \in G\rangle = G/[G,G] = \mathrm{Ab}(G).$$

2. The 0th and 1st Hochschild homologies are

$$H_0(A) = \ker(d_0)/\mathrm{im}(d_1) = A/\langle ab - ba \mid a,b \in A\rangle = A/[A,A]$$
$$H_1(A) = \frac{\{a \otimes b \in A \otimes A \mid ab = ba\}}{\langle ab \otimes c - a \otimes bc + ca \otimes b \mid a,b,c \in A\rangle}$$

3. The zeroth singular homology of a topological space X is the free k-module generated by its path components: $H_0(X,k) = \langle \pi_0(X)\rangle_k$.

   Hurewicz's theorem states that the first homology of a path-connected topological space X for $k = \mathbb{Z}$ is the abelisation of its fundamental group: $H_1(X,\mathbb{Z}) = \mathrm{Ab}\,\pi_1(X)$.

The boundary operators of the chain complex for a simplicial object from Proposition 5.5.5 depend only on its face maps and do not involve the degeneracies. This is similar to the geometric realisation, where only the face maps of a simplicial set are used to glue different simplexes along their faces. The simplexes of the resulting semisimplicial complex are then in bijection with *non-degenerate elements* of the simplicial set.

The standard chain complex $S_\bullet$ associated to a simplicial object follows a similar pattern. It can be decomposed as a direct sum or coproduct of the *degenerate chain complex* $DS_\bullet$ that contains the images of the degeneracies and a *normalised chain complex* $NS_\bullet$. One can show that only the latter contributes to the homologies and contains non-trivial information.

Proposition 5.5.8: Let R be a ring and $S:\Delta^{op}\to R\text{-Mod}$ a simplicial object.

1. For all $n\in\mathbb{N}_0$ the associated standard chain complex $S_\bullet$ satisfies

$$S_n = NS_n\oplus DS_n \qquad \text{with} \qquad NS_n=\cap_{i=0}^{n-1}\ker(d_n^i),\;\; DS_n=+_{i=0}^{n-1}\mathrm{im}(s_{n-1}^i)\subset S_n$$

   This defines chain complexes $NS_\bullet\subset S_\bullet$ and $DS_\bullet\subset S_\bullet$, the **normalised chain complex** and **degenerate chain complex**, with $H_n(S_\bullet)=H_n(NS_\bullet)$ and $H_n(DS_\bullet)=0$ for $n\in\mathbb{N}_0$.

2. This induces functors $N,D: R\text{-Mod}^{\Delta^{op}}\to \mathrm{Ch}_{R\text{-Mod}}$ that assign
   - to a simplicial object $S:\Delta^{op}\to R\text{-Mod}$ the chain complexes $NS_\bullet$ and $DS_\bullet$,
   - to a simplicial morphism $\alpha: S\Rightarrow S'$ the induced chain maps $N\alpha_\bullet: NS_\bullet\to NS'_\bullet$ and $D\alpha_\bullet: DS_\bullet\to DS'_\bullet$ induced by the components $\alpha_n: S_n\to S'_n$.

Proof:
1. We show that the boundary operators of $S_\bullet$ restrict to $NS_\bullet$ and $DS_\bullet$:

This follows by a direct computation with the simplicial relations (22). We have for all $x\in NS_n$, $y=s_{n-1}^k(z)\in DS_n$ and $i=0,...,n-2$

$$\begin{aligned}
&d_{n-1}^i\circ d_n(x)=(-1)^n d_{n-1}^i\circ d_n^n(x)\overset{(22)}{=}(-1)^n d_{n-1}^{n-1}\circ d_n^i(x)=0 &&\Rightarrow\quad d_n(x)\in NS_{n-1}\\
&d_n(y)=\Sigma_{j=0}^n(-1)^j d_n^j\circ s_{n-1}^k(z)\\
&\quad\overset{(22)}{=}\Sigma_{j=0}^{k-1}(-1)^j s_{n-2}^{k-1}\circ d_{n-1}^j(z)+\Sigma_{j=k+2}^n(-1)^j s_{n-2}^k\circ d_{n-1}^{j-1}(z) &&\Rightarrow\quad d_n(y)\in +_{i=0}^{n-2}\mathrm{im}(s_{n-2}^i).
\end{aligned}$$

2. We show that for each $n\in\mathbb{N}_0$ one has $S_n=NS_n\oplus DS_n$:

2.(a) To see that $DS_n\cap NS_n=\{0\}$, let $0\neq x\in DS_n\cap NS_n$ and set

$$j=\max\{k\in\{0,...,n-1\}\mid x\in +_{i=k}^{n-1}\mathrm{im}(s_{n-1}^i)\}.$$

Then we have $x=\Sigma_{i=j}^{n-1}s_{n-1}^i(x_i)$ with $s_{n-1}^j(x_j)\neq 0$. As $x\in NS_n$, we obtain

$$0=d_n^j(x)=\Sigma_{i=j}^{n-1}d_n^j\circ s_{n-1}^i(x_i)=x_j+\Sigma_{i=j+1}^{n-1}s_{n-2}^{i-1}\circ d_n^j(x_i).$$

If $j=n-1$, it follows that $x_{n-1}=0$ and $x=s_{n-1}^{n-1}(x_{n-1})=0$. If $j<n-1$ it follows that

$$s_{n-1}^j(x_j)=-\Sigma_{i=j+1}^{n-1}s_{n-1}^j\circ s_{n-2}^{i-1}\circ d_n^j(x_i)=-\Sigma_{i=j+1}^{n-1}s_{n-1}^i\circ s_{n-2}^j\circ d_n^j(x_i)\in +_{i=j+1}^{n-1}\mathrm{im}(s_{n-1}^i)$$

and $x\in +_{k=j+1}^{n-1}\mathrm{im}(s_{n-1}^i)$, in contradiction to the maximality of j. Hence, $NS_n\cap DS_n=\{0\}$.

2.(b) To show that $S_n = NS_n + DS_n$, let $x \in S_n$ and $j_x = \min\{k \in \{0, ..., n\} \mid d_n^k(x) \neq 0\}$. If $j_x = n$, then $x \in NS_n$. If $j_x = j < n$, we have $x = x_1 + y_1 = (x - s_{n-1}^j \circ d_n^j(x)) + s_{n-1}^j \circ d_n^j(x)$

$$d_n^j(x_1) = d_n^j(x - s_{n-1}^j \circ d_n^j(x)) = d_n^j(x) - d_n^j(x) = 0$$
$$d_n^k(x_1) = d_n^k(x - s_{n-1}^j \circ d_n^j(x)) = d_n^k(x) - s_{n-2}^{j-1} \circ d_{n-1}^k \circ d_n^j(x) = -s_{n-2}^{j-1} \circ d_{n-1}^{j-1} \circ d_n^k(x) = 0$$

for $k \in \{0, ..., j-1\}$. We decomposed $x = x_1 + y_1$ with $y_1 \in DS_n$ and an element $x_1 \in S_n$ with $j_{x_1} \geq j_x + 1$. By iterating this procedure, we obtain elements $y_1, ..., y_k \in DS_n$ and $x_1, ..., x_k \in S_n$ with $x_i = x_{i+1} + y_{i+1}$ and $x_k \in NS_n$. This shows that $x = x_k + \Sigma_{i=1}^k y_i \in NS_n + DS_n$.

3. Due to naturality, the components of a simplicial morphism $\alpha : S \Rightarrow S'$ satisfy

$$d_n'^i \circ \alpha_n = \alpha_{n-1} \circ d_n^i \quad \Rightarrow \quad \alpha_n(\ker(d_n^i)) \subset \ker d_n'^i \quad \Rightarrow \quad \alpha_n(NS_n) \subset NS_n'$$
$$\alpha_n \circ s_{n-1}^i = s_{n-1}'^i \circ \alpha_{n-1} \quad \Rightarrow \quad \alpha_n(\mathrm{im}(s_{n-1}^i)) \subset \mathrm{im}(s_{n-1}'^i) \quad \Rightarrow \quad \alpha_n(DS_n) \subset DS_n'.$$

This shows that every simplicial morphism $\alpha : S \Rightarrow S'$ induces chain maps $N\alpha_\bullet : NS_\bullet \to NS'_\bullet$ and $D\alpha_\bullet : DS_\bullet \to DS'_\bullet$. The functoriality follows directly from Proposition 5.5.5.

4. The proof that $H_n(DS_\bullet) = 0$ and, consequently, $H_n(S_\bullet) = H_n(NS_\bullet)$ for $n \in \mathbb{N}_0$ requires more techniques from homological algebra. We refer to [GJ, Chapter III.2, Theorem 2.4] for an elegant and to [Me, Proposition 5.2.1] for a more pedestrian proof. □

Proposition 5.5.5 tells us that every simplicial object in R-Mod defines a positive chain complex, and Proposition 5.5.8 allows us to decompose this chain complex into a degenerate chain complex with trivial homologies and a normalised chain complex.

This raises the question, which positive chain complexes in a category R-Mod can be obtained as normalised chain complexes of a simplicial object, up to isomorphism. A fundamental and surprising result by Dold and Kan states that these are *all* positive chain complexes in R-Mod.

Theorem: (**Dold-Kan correspondence**)
The functor $N : R\text{-Mod}^{\Delta^{op}} \to \mathrm{Ch}_{R\text{-Mod}\geq 0}$ is an equivalence of categories.

In fact, this theorem is valid in more generality, namely in all categories, where one can define chain complexes - abelian categories. It explains the emphasis we placed on simplicial objects. Instead of the category of positive chain complexes and chain maps in an abelian category $\mathcal{A}$ we can consider the category of simplicial objects and morphisms in $\mathcal{A}$.

We prove it for the category R-Mod by constructing a functor $F : \mathrm{Ch}_{R\text{-Mod}\geq 0} \to R\text{-Mod}^{\Delta^{op}}$ with $FN \cong \mathrm{id}$ and $NF \cong \mathrm{id}$. It needs to assign a simplicial object $S : \Delta^{op} \to R$-Mod to any positive chain complex $C_\bullet$ in R-Mod. It is reasonable to take as its modules the component modules of the chain complex: $S_n = S([n+1]) = C_n$ for all $n \in \mathbb{N}_0$.

The difficulty is to define the face maps and degeneracies. The boundary operators of $C_\bullet$ $d_n : C_n \to C_{n-1}$ provide just one morphism in each degree, which lowers the degree and hence must be related to face maps. However, there are no apparent morphisms that raise the degree and could be related to degeneracies.

To solve this problem, we employ a similar strategy as for the truncation and skeleta of simplicial sets in Definition 5.3.5. We consider the subcategory $\Delta_{\mathrm{inj}} \subset \Delta$ of the simplex category that contains only *injective* monotonic maps, composites of the face morphisms and then extend functors $S : \Delta_{\mathrm{inj}}^{op} \to \mathcal{E}$ along the inclusion functor $\iota : \Delta_{\mathrm{inj}}^{op} \to \Delta^{op}$ to functors $\mathrm{Lan}_\iota S : \Delta^{op} \to \mathcal{E}$.

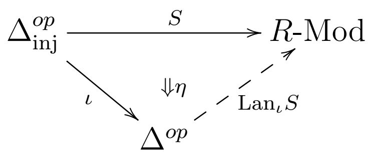
Definition 5.5.9: Let $\Delta_{\text{inj}} \subset \Delta$ the subcategory of Δ with the same objects but only injective monotonic maps as morphisms and $\iota : \Delta_{\text{inj}}^{op} \to \Delta$ the inclusion functor.

A functor $S : \Delta_{\text{inj}}^{op} \to \mathcal{E}$ is called a **semisimplicial object** and a natural transformation $\alpha : S \Rightarrow S'$ between semisimplicial objects a **semisimplicial morphism** in $\mathcal{E}$.

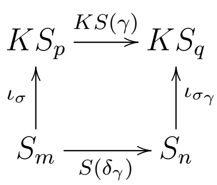
Given a semisimplicial object $S : \Delta_{\text{inj}}^{op} \to R\text{-Mod}$ we can apply Corollary 3.2.6 to extend it along the inclusion $\iota : \Delta_{\text{inj}}^{op} \to \Delta^{op}$ to a simplicial object $\text{Lan}_\iota S : \Delta^{op} \to R\text{-Mod}$. To work with these Kan extensions, we need a more explicit description, which is obtained from the formula in Proposition 4.2.6. To keep notation simple, we write $\sigma : [p+1] \twoheadrightarrow [m+1]$ for a monotonic surjection and $\delta : [n+1] \hookrightarrow [m+1]$ for a monotonic injection in the category Δ.

Lemma 5.5.10:

1. The left Kan extensions of semisimplicial objects $S : \Delta_{\text{inj}}^{op} \to R\text{-Mod}$ along $\iota : \Delta_{\text{inj}}^{op} \to \Delta^{op}$ define a left adjoint $K = \text{Lan}_\iota : R\text{-Mod}^{\Delta_{\text{inj}}^{op}} \to R\text{-Mod}^{\Delta^{op}}$ to $\iota^* : R\text{-Mod}^{\Delta^{op}} \to R\text{-Mod}^{\Delta_{\text{inj}}^{op}}$.



2. The functor $K = \text{Lan}_\iota : R\text{-Mod}^{\Delta_{\text{inj}}^{op}} \to R\text{-Mod}^{\Delta^{op}}$ assigns to
   - $S : \Delta_{\text{inj}}^{op} \to R\text{-Mod}$ the simplicial object $KS : \Delta^{op} \to R\text{-Mod}$ with
     - $KS_p = \amalg_{\sigma:[p+1]\twoheadrightarrow[m+1]} S_m$, where the coproduct runs over all monotonic surjections σ with source $[p+1]$ and any target $[m+1] := t(\sigma)$,
     - $KS(\gamma) : KS_p \to KS_q$ for $\gamma : [q+1] \to [p+1]$ is the unique morphism for which the following diagram commutes,



$$\delta_\gamma : [n+1] \hookrightarrow [m+1], \quad \sigma_\gamma : [q+1] \twoheadrightarrow [n+1], \qquad \text{with} \quad \sigma \circ \gamma = \delta_\gamma \circ \sigma_\gamma.$$

   - a semisimplicial morphism $\alpha : S \Rightarrow S'$ the simplicial morphism $K\alpha : KS \Rightarrow KS'$ with components satisfying $K\alpha_p \circ \iota_\sigma = \iota_\sigma \circ \alpha_m$ for all $\sigma : [p+1] \twoheadrightarrow [m+1]$.

3. Every element in the image of an inclusion $\iota_\sigma : S_m \to KS_p$ with $m < p$ is degenerate.

Proof:

1. As R-Mod is cocomplete and Δ small, the first claim follows directly from Corollary 3.2.6.

2. To derive the concrete formulas for K we use the coend formulas (19) and Proposition 4.2.6. To determine KS on an ordinal $[p+1]$ we need to compute the coend of the functor

$$\text{Hom}_\Delta([p+1], -) \sqcup S(-) : \Delta \times \Delta^{op} \to R\text{-Mod}.$$

The coend formula (19) yields

$$KS_p = \left(\amalg_{\alpha:[p+1]\to[m+1]} S_m\right) / \sim \tag{28}$$
$$\text{with } \iota_{\delta\circ\alpha}(s) \sim \iota_\alpha \circ S(\delta)(s) \;\; \text{for } \delta : [m+1] \hookrightarrow [n+1],\; \alpha : [p+1] \to [m+1],\; s \in S_n,$$

where the coproduct runs over all morphisms α in Δ with fixed source $s(\alpha) = [p+1]$ and any target $[m+1] := t(\alpha)$. The morphism $KS(\gamma) : KS_p \to KS_q$ for a morphism $\gamma : [q+1] \to [p+1]$ in Δ is given by $KS(\gamma) \circ \iota_\alpha = \iota_{\alpha\circ\gamma}$.

By Proposition 5.2.2 each $\alpha : [p+1] \to [m+1]$ can be factorised uniquely as $\alpha = \delta_\alpha \circ \sigma_\alpha$ with an injection $\delta_\alpha : [n+1] \hookrightarrow [m+1]$ and a surjection $\sigma_\alpha : [p+1] \twoheadrightarrow [n+1]$. With (28) this yields

$$KS_p = \amalg_{\sigma:[p+1]\twoheadrightarrow[n+1]} S_n,$$

where the coproduct runs over all monotonic surjections with source $[p+1]$ and any target $[n+1] := t(\sigma)$. For any $\gamma : [q+1] \to [p+1]$ and surjection $\sigma : [p+1] \twoheadrightarrow [m+1]$ there is a unique injection $\delta_\gamma : [n+1] \hookrightarrow [m+1]$ and surjection $\sigma_\gamma : [q+1] \twoheadrightarrow [n+1]$ with $\delta_\gamma \circ \sigma_\gamma = \sigma \circ \gamma$. This yields $KS(\gamma) \circ \iota_\sigma = \iota_{\sigma_\gamma} \circ S(\delta_\gamma)$.

By the proof of Theorem 3.2.5 each semisimplicial morphism $\alpha : S \Rightarrow S'$ induces a simplicial morphism $K\alpha : KS \Rightarrow KS'$ with components $K\alpha_p : KS_p \to KS'_p$ satisfying $K\alpha_p \circ \iota_\sigma = \iota_\sigma \circ \alpha_m$ for all $\sigma : [p+1] \twoheadrightarrow [m+1]$.

3. For any monotonic surjection $\sigma : [p+1] \twoheadrightarrow [m+1]$ the commuting diagram for $KS(\sigma)$ yields

$$\begin{array}{ccc} KS_m & \xrightarrow{KS(\sigma)} & KS_p \\ \uparrow{\scriptstyle \iota_{1_{[m+1]}}} & & \uparrow{\scriptstyle \iota_\sigma} \\ S_m & \xrightarrow[\mathrm{id}_{S_m}]{} & S_m \end{array}$$

This shows that the image of ι_σ is contained in the image of $KS(\sigma)$ and hence degenerate if $\sigma \neq 1_{[p+1]}$ or, equivalently, $m < p$. □

We now apply this extension procedure to construct an equivalence of categories with the functor $N : R\text{-Mod}^{\Delta^{op}} \to \mathrm{Ch}_{R\text{-Mod}\geq 0}$ from Proposition 5.5.8. We proceed as follows:

1. As the definition of N involves only face morphisms, we can define it analogously for semisimplicial objects. This yields a functor $N' : R\text{-Mod}^{\Delta^{op}_{\mathrm{inj}}} \to \mathrm{Ch}_{R\text{-Mod}\geq 0}$ with $N'\iota^* \cong N$.
2. We construct a left adjoint $G : \mathrm{Ch}_{R\text{-Mod}\geq 0} \to R\text{-Mod}^{\Delta^{op}_{\mathrm{inj}}}$ to N' with $GN' \cong \mathrm{id}$.
3. We compose G with the functor K from Lemma 5.5.10 and show that the resulting functor $KG : \mathrm{Ch}_{R\text{-Mod}\geq 0} \to R\text{-Mod}^{\Delta^{op}}$ is left adjoint to $N : R\text{-Mod}^{\Delta^{op}} \to \mathrm{Ch}_{R\text{-Mod}\geq 0}$.
4. To prove Dold-Kan correspondence, we show that KG and N form an adjoint equivalence.

For step 1, we simplify notation by working with a different sign convention. We set $d_n = S(\delta^n_n)$ instead of $d_n = (-1)^n S(\delta^n_n)$ as in Proposition 5.5.8. This has no consequences, as we will see in the following that the resulting functors are naturally isomorphic.

Definition 5.5.11: Let $N' : R\text{-Mod}^{\Delta^{op}_{\mathrm{inj}}} \to \mathrm{Ch}_{R\text{-Mod}\geq 0}$ be the functor that assigns to

- a semisimplicial object $S : \Delta^{op}_{\mathrm{inj}} \to R\text{-Mod}$ the chain complex $N'S_\bullet$ with modules $N'S_n = \cap_{i=0}^{n-1} \ker S(\delta^i_n)$ and boundary operator $d_n = S(\delta^n_n) : N'S_n \to N'S_{n-1}$,
- a semisimplicial morphism $\alpha : S \Rightarrow T$ the chain map $\alpha_\bullet : N'S_\bullet \to N'T_\bullet$ with components $\alpha_n = \alpha_{[n+1]} : N'S_n \to N'T_n$.

We now implement step 2 and construct a functor $G : \mathrm{Ch}_{R\text{-Mod}\geq 0} \to R\text{-Mod}^{\Delta^{op}_{\mathrm{inj}}}$ that is left adjoint to N with $N'G \cong \mathrm{id}$. It must associate to each positive chain complex $C_\bullet$ a semisimplicial object $GC : \Delta^{op}_{\text{inj}} \to \text{Set}$. The simplest choice is to set $GC_n = C_n$. On the morphisms, the condition $N'G \cong \text{id}$ suggests $GC(\delta^n_n) = d_n : C_n \to C_{n-1}$ and $GC(\delta^i_n) = 0$ for all $i \neq n$.

Lemma 5.5.12:
The functor $N' : R\text{-Mod}^{\Delta^{op}_{\text{inj}}} \to \text{Ch}_{R\text{-Mod}\geq 0}$ has a left adjoint $G : \text{Ch}_{R\text{-Mod}\geq 0} \to R\text{-Mod}^{\Delta^{op}_{\text{inj}}}$ that

- assigns to a chain complex $C_\bullet$ the semisimplicial object $GC : \Delta^{op}_{\text{inj}} \to R\text{-Mod}$ with $GC_n = C_n$, $GC(\delta^i_n) = 0 : C_n \to C_{n-1}$ for $i < n$ and $GC(\delta^n_n) = d_n : C_n \to C_{n-1}$,
- to a chain map $f_\bullet : C_\bullet \to C'_\bullet$ the semisimplicial morphism $Gf : GC \Rightarrow GC'$ with components $Gf_n = f_n : C_n \to C'_n$.

The adjunction satisfies $N'G \cong \text{id}$ and $N'\iota^* \cong N$, where $\iota^* : R\text{-Mod}^{\Delta^{op}} \to R\text{-Mod}^{\Delta^{op}_{\text{inj}}}$ is the restriction functor for the inclusion $\iota : \Delta^{op}_{\text{inj}} \to \Delta^{op}$.

Proof:
The adjunction is given by the bijections

$$\begin{aligned} \Phi_{C_\bullet,S} : \text{Hom}_{R\text{-Mod}^{\Delta^{op}_{\text{inj}}}}(G(C_\bullet), S) \quad &\to \quad \text{Hom}_{\text{Ch}_{R\text{-Mod}\geq 0}}(C_\bullet, N'S_\bullet) \\ \alpha : G(C_\bullet) \Rightarrow S \quad &\mapsto \quad \alpha_\bullet : C_\bullet \to N'S_\bullet,\ \alpha_n = \alpha_{[n+1]} : C_n \to N'S_n. \end{aligned}$$

That this yields a chain map $\alpha_\bullet : C_\bullet \to N'S_\bullet$ for every semisimplicial morphism $\alpha : G(C_\bullet) \Rightarrow S$ follows from the naturality of α and the definition of G, which imply for $i \neq n$

$$S(\delta^i_n) \circ \alpha_n = \alpha_{n-1} \circ GC(\delta^i_n) = \alpha_{n-1} \circ 0 = 0 \qquad S(\delta^n_n) \circ \alpha_n = \alpha_{n-1} \circ GC(\delta^n_n) = \alpha_{n-1} \circ d_n.$$

The same identities ensure that $\Phi_{C_\bullet,S}$ is invertible and that its inverse associates to each chain map $\alpha_\bullet : C_\bullet \to N'S_\bullet$ a semisimplicial morphism $\alpha : G(C_\bullet) \Rightarrow S$. The naturality of $\phi_{C_\bullet,S}$ in $C_\bullet$ and S is also evident from the definitions, and the same holds for the identity $N'G \cong \text{id}$. A natural isomorphism $\tau : N \Rightarrow N'\iota^*$ has as components for a simplicial object $S : \Delta^{op} \to R\text{-Mod}$

$$(\tau_S)_{2k+1} = (-1)^{k+1}\text{id}_{NS_{2k+1}}, \quad (\tau_S)_{2k} = (-1)^k \text{id}_{NS_{2k}} \qquad k \in \mathbb{N}_0. \qquad \square$$

We can now implement steps 3 and 4. We apply Lemma 5.5.10 to extend the semisimplicial objects from Lemma 5.5.12 to simplicial objects. This amounts to composing the functor G from Lemma 5.5.12 with the functor K from Lemma 5.5.10.

Theorem 5.5.13: (**Dold-Kan correspondence**)
The functor $N : R\text{-Mod}^{\Delta^{op}} \to \text{Ch}_{R\text{-Mod}\geq 0}$ is an equivalence of categories.

Proof:
1. We show that $KG : \text{Ch}_{R\text{-Mod}\geq 0} \to R\text{-Mod}^{\Delta^{op}}$ is left adjoint to N with $NKG \cong \text{id}$:

From Lemmas 5.5.12 and 5.5.10 we have a chain of natural isomorphisms

$$\begin{aligned} \text{Hom}_{R\text{-Mod}^{\Delta^{op}}}(KG(C_\bullet), S) &\cong \text{Hom}_{R\text{-Mod}^{\Delta^{op}_{\text{inj}}}}(G(C_\bullet), \iota^*(S)) \\ &\cong \text{Hom}_{\text{Ch}_{R\text{-Mod}\geq 0}}(C_\bullet, N'\iota^*(S)) \\ &\cong \text{Hom}_{\text{Ch}_{R\text{-Mod}\geq 0}}(C_\bullet, N(S)) = \text{Hom}_{\text{Ch}_{R\text{-Mod}\geq 0}}(C_\bullet, NS_\bullet) \end{aligned}$$

where we used first that K is left adjoint to ι^* by Lemma 5.5.10, then that G is left adjoint to N' by Lemma 5.5.12 and in the last step that $N \cong N'\iota^*$ by Lemma 5.5.12. This shows that KG is left adjoint to N.

2. We show that $NKG = \mathrm{id}$:

By Lemma 5.5.10, 3. any element in the image of an inclusion $\iota_\sigma : G(C_\bullet)_m \to KG(C_\bullet)_p$ for a monotonic surjection $\sigma : [p+1] \twoheadrightarrow [m+1]$ with $m < p$ and a positive chain complex $C_\bullet$ is degenerate. This implies $\amalg_{\sigma:[p+1]\twoheadrightarrow[m+1],m<p} C_m \subset DKG(C_\bullet)$. With the identity $G(\delta_p^i) = 0$ for $i < p$ one obtains that $NKG(C_\bullet)_p = NG(C_\bullet)_p = C_p$. The commuting diagram in Lemma 5.5.10 for $\sigma = 1_{[p+1]}$ and $\gamma = \delta_p^p$ then implies $NKG(C_\bullet) = C_\bullet$. With the expression for $KG(f_\bullet)$ for chain maps $f_\bullet : C_\bullet \to C'_\bullet$ in Lemma 5.5.10 it follows that $NKG = \mathrm{id}$.

3. It remains to show that $KGN \cong \mathrm{id}$ or, equivalently, $KGN'\iota^* \cong \mathrm{id}$:

3.(a) Note that the adjunctions from Lemmas 5.5.10 and 5.5.12 yield natural transformations $\epsilon : GN' \Rightarrow \mathrm{id}$ and $\epsilon' : K\iota^* \Rightarrow \mathrm{id}$ and hence a natural transformation $\eta = \epsilon' \circ K\epsilon\iota^* : KGN'\iota^* \Rightarrow \mathrm{id}$. By Lemmas 5.5.10 and 5.5.12 its component morphisms $\eta^S : KGN'\iota^*(S) \Rightarrow S$ are given by

$$\eta^S_{[p+1]} \circ \iota_\sigma = S(\sigma) \circ i_m \qquad (29)$$

for all $\sigma : [p+1] \twoheadrightarrow [m+1]$ and simplicial objects $S : \Delta^{op} \to R$-Mod, where $i_m : NS_m \to S_m$ is the inclusion and $\iota_\sigma : NS_m \to \amalg_{\sigma:[p+1]\twoheadrightarrow[m+1]} NS_m$ the inclusion for the coproduct.

3.(b) We show that all component morphisms $\eta^S_{[p+1]}$ are injective:
For each $\sigma : [p+1] \twoheadrightarrow [m+1]$ we can use the relations (21) in Δ to construct an injective monotonic map $\delta : [m+1] \hookrightarrow [p+1]$ with $\sigma \circ \delta = 1_{[m+1]}$. With (29) this yields

$$S(\delta) \circ \eta^S_{[p+1]} \circ \iota_\sigma \overset{(29)}{=} S(\delta) \circ S(\sigma) \circ i_m = S(\sigma \circ \delta) \circ i_m = S(1_{[m+1]}) \circ i_m = i_m.$$

As i_m is injective, $\eta^S_{[p+1]} \circ \iota_\sigma$ is injective and $\eta^S_{[p+1]}$ is injective as well.

3.(c) We show by induction over p that all morphisms $\eta^S_{[p+1]}$ are surjective:

$p = 0$: In this case $m = 0$, $i_0 = \mathrm{id} : NS_0 \to S_0$ and $\sigma = 1_{[1]} : [1] \to [1]$ is the only surjective morphism indexing the coproduct $\amalg_{\sigma:[p+1]\twoheadrightarrow[m+1]} NS_m$. This yields $\eta^S_{[1]} = \mathrm{id}_{S_0} : S_0 \to S_0$.

$p - 1 \Rightarrow p$: Suppose we showed that $\eta^S_{[k+1]}$ is an isomorphism for all $k \leq p-1$. With the naturality of η^S and the induction hypothesis we then obtain

$$s^j_{p-1} = S(\sigma^j_p) = \eta^S_{[p+1]} \circ KGN\iota^*(\sigma^j_p) \circ (\eta^S_{[p]})^{-1} \quad \Rightarrow \quad \mathrm{im}(s^j_{p-1}) \subset \mathrm{im}(\eta^S_{[p+1]})\ \forall j \in \{0, ..., p-1\},$$

and choosing $\sigma = 1_{[p+1]}$ in (29) yields

$$\eta^S_{[p+1]} \circ \iota_{1_{[p+1]}} = i_p \quad \Rightarrow \quad NS_p \subset \mathrm{im}(\eta^S_{[p+1]}).$$

As $S_p = NS_p \oplus DS_p$ with $DS_p = +_{j=0}^{m-1} \mathrm{im}(s^j_{m-1})$ by Proposition 5.5.8, $\eta^S_{[p+1]}$ is surjective. □

The proof of Theorem 5.5.13 not only shows that simplicial objects capture the essential information of chain complexes, but also that Kan extensions are a useful and essential concept. Without the concept of a Kan extension, it would be almost impossible to find and motivate the left adjoint functor K from Theorem 5.5.13. If one is familiar with Kan extensions and previous examples, then the idea is rather simple. The concrete formula for this functor is derived by routine computations, namely the ones with the Kan extension formula in Lemma 5.5.10.

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6 Homotopies

6.1 Homotopies in Top

In this section we recall some background on homotopies and homotopy groups of topological spaces. A homotopy between continuous maps $f, f' : X \to Y$ with the same source and target can be viewed as a continuous deformation from f into f'. Thus, one considers a parameter $t \in [0,1]$ and a family of continuous maps $h_t : X \to Y$ that depend continuously on t, such that $h_0 = f$ and $h_1 = f'$. This is equivalent to the following definition.

Definition 6.1.1: Let X and Y be topological spaces and $f, f' : X \to Y$ continuous maps.

1. A **homotopy** from f to f' is a continuous map $h : [0,1] \times X \to Y$ with $h(0,x) = f(x)$ and $h(1,x) = f'(x)$ for all $x \in X$.

$$\begin{array}{ccccc} X & \xrightarrow{\iota^0} & [0,1]\times X & \xleftarrow{\iota^1} & X \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle h} & \swarrow{\scriptstyle f'} & \\ & & Y & & \end{array} \qquad \iota^i : X \to [0,1] \times X,\ x \mapsto (i,x).$$

   If there is a homotopy from f to f' one calls f and f' **homotopic** and writes $f \sim f'$.

2. The spaces X and Y are called **homotopy equivalent** or of the same **homotopy type**, if there are continuous maps $f : X \to Y$ and $g : Y \to X$ with $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$. Then f and g are called a **homotopy equivalence**, and one writes $X \simeq Y$.

3. If X is homotopy equivalent to a one-point space, it is called **contractible**.

Remark 6.1.2:

1. For all topological spaces X, Y being homotopic is an equivalence relation on $\mathrm{Hom}_{\mathrm{Top}}(X,Y)$:
   - For any continuous map $f : X \to Y$ a homotopy from f to f is given by
   $$h_f : [0,1] \times X \to Y, \quad (t,x) \mapsto f(x).$$
   - If $h : [0,1] \times X \to Y$ is a homotopy from f to f', then a homotopy from f' to f is
   $$\bar{h} : [0,1] \times X \to Y, \quad (t,x) \mapsto h(1-t,x).$$
   - If $h : [0,1] \times X \to Y$ is a homotopy from f to f' and $h' : [0,1] \times X \to Y$ a homotopy from f' to f'', then a homotopy $h'' : [0,1] \times X \to Y$ from f to f'' is given by
   $$h''(t,x) = \begin{cases} h(2t,x) & t \in [0, \frac{1}{2}] \\ h'(2t-1,x) & t \in [\frac{1}{2}, 1]. \end{cases}$$

2. Being homotopic is compatible with the composition of morphisms:
   If $h : [0,1] \times X \to Y$ is a homotopy from f to f' and $k : [0,1] \times Y \to Z$ a homotopy from g to g' then a homotopy from $g \circ f$ to $g' \circ f'$ is given by
   $$l : [0,1] \times X \to Z, \quad (t,x) \mapsto k(t, h(t,x)).$$

3. This defines a category $K(\text{Top})$, the **homotopy category** of topological spaces with
   - topological spaces as objects,
   - homotopy classes of continuous maps as morphisms.

   Isomorphisms in $K(\text{Top})$ are homotopy classes of homotopy equivalences.

In many settings the category $K(\text{Top})$ is preferable to Top. If one works in the category Top, one considers and classifies topological spaces up to homeomorphisms, the isomorphisms in Top. This is very difficult, even with simplifying assumptions. It turns out that classifying topological spaces in $K(\text{Top})$, up to homotopy equivalences rather than homeomorphisms, is more intuitive, easier in practice and often more suitable, as many quantities that characterise topological spaces depend only on their homotopy type.

Many homotopy equivalences can be easily visualised and correspond more directly to intuitive ideas of shapes. For instance, the following spaces are all homotopy equivalent, but none of them is homeomorphic to any of the others

- the n-sphere $S^n = \{x \in \mathbb{R}^{n+1} \mid ||x|| = 1\}$,
- an open n-sphere of finite thickness $OS^n = \{x \in \mathbb{R}^{n+1} \mid \frac{1}{2} < ||x|| < \frac{3}{2}\}$,
- a closed n-sphere of finite thickness $CS^n = \{x \in \mathbb{R}^n \mid \frac{1}{2} \leq ||x|| \leq \frac{3}{2}\}$,
- the open n-sphere of infinite thickness $\mathbb{R}^{n+1} \setminus \{0\}$.

It is intuitive that these spaces roughly have the same shape, although some of their topological properties such as openness, closedness, compactness are different. Homotopy equivalences and homotopies can often be visualised as stretching, melting or denting spaces or blowing them up to infinite thickness. They are not sensitive to openness, closedness and compactness and give rise to a coarser classification than homeomorphisms. Examples are the following.

Example 6.1.3:

1. Homeomorphic topological spaces are homotopy equivalent.

2. Two maps $f, f' : \{\bullet\} \to X$ from the one-point space into a topological space X are homotopic if and only if there is a continuous map $h : [0,1] \to X$ with $h(0) = f(\bullet)$ and $h(1) = f'(\bullet)$, a **path** in X from $f(\bullet)$ to $f'(\bullet)$.

   Thus, the homotopy class of a map $f : \{\bullet\} \to X$ can be identified with the set of points $x' \in X$ for which there is a path from $f(\bullet)$ to x', the **path component** $\pi(f(\bullet))$. The set of homotopy classes of maps from $\{\bullet\}$ to X is the set of path components of X

$$\pi_0(X) = \{\pi(x) \mid x \in X\}.$$

3. The spaces $\mathbb{R}^n \setminus \{0\}$ and S^{n-1} are homotopy equivalent, but not homeomorphic.

   The maps $f : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$, $x \mapsto x/||x||$ and $g : S^{n-1} \to \mathbb{R}^n \setminus \{0\}$, $x \mapsto x$ form a homotopy equivalence, as $f \circ g = \mathrm{id}_{S^{n-1}}$ and a homotopy from $\mathrm{id}_{\mathbb{R}^n \setminus \{0\}}$ to $g \circ f$ is given by

$$h : [0,1] \times \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}, \qquad (t, x) \mapsto tx/||x|| + (1-t)x.$$

4. Any topological space X is homotopy equivalent to the cylinder $C = X \times [0,1]$.

   The maps $f : C \to X$, $(x, z) \mapsto x$ and $g : X \to C$, $x \mapsto (x, 0)$ form a homotopy equivalence, since $f \circ g = \mathrm{id}_X$ and a homotopy from $g \circ f$ to id_C is given by

$$h : [0,1] \times C \to C, \quad (t, x, z) \mapsto (x, tz).$$

5. Any convex subspace $X \subset \mathbb{R}^n$ is contractible.

   For any point $p \in X$ the maps $f: X \to \{p\}$, $x \mapsto p$ and $g: \{p\} \to X$, $p \mapsto p$ form a homotopy equivalence, since $f \circ g = \mathrm{id}_{\{p\}}$ and a homotopy from id_X to $g \circ f$ is given by

$$h: [0,1] \times X \to X, \quad x \mapsto (1-t)x + tp.$$

In homotopy theory and, more generally, algebraic topology, one characterises topological spaces by algebraic quantities such as groups, abelian groups or modules. An important example of such quantities are the homotopy groups of a topological space. They probe a topological space X with continuous maps $f: S^n \to X$ that send a reference point $p \in S^n$ to a fixed point $x \in X$. As it is simpler in practice, one works instead with continuous maps from a unit cube $C = [0,1]^{\times n} \subset \mathbb{R}^n$ into X that send the boundary ∂C to $x \in X$. Such maps are considered up to homotopies that preserve the point x.

Definition 6.1.4:
Let $f, g: X \to Y$ be continuous maps and $A \subset X$ a subspace with $f(a) = g(a)$ for all $a \in A$.

A **homotopy** from f to g **relative to** A is a homotopy $h: [0,1] \times X \to Y$ from f to g with $h(t,a) = f(a) = g(a)$ for all $t \in [0,1]$ and $a \in A$. If there is such a homotopy, one writes $f \sim_A g$ and calls f and g **homotopic relative to** A.

Clearly, a homotopy in the sense of Definition 6.1.1 is just a homotopy relative to $\emptyset \subset X$. The idea is now to consider continuous maps $f: C \to X$ with $f(\partial C) = \{x\}$, up to homotopies relative to ∂C. One then defines a concatenation of such maps that induces a group structure on their relative homotopy classes.

The simplest case is the one one where $n = 1$ and the unit cube is the unit interval $C = [0,1]$. In this case continuous maps $f: C \to X$ with $f(\partial C) = \{x\}$ are paths that start and end at x. Given two paths $\gamma: [0,1] \to X$ and $\delta: [0,1] \to X$ with $\gamma(\{0,1\}) = \delta(\{0,1\}) = \{x\}$, one has an obvious way to compose them, namely to first traverse the path γ and then δ, each at twice the usual speed. There is also a candidate for a neutral element, namely the constant path $x: [0,1] \to X$, $t \mapsto x$ and for an inverse, namely the path $\bar{\gamma}: [0,1] \to X$, $t \mapsto \gamma(1-t)$ which goes backwards. For higher n, there is no such obvious composition or inverse. However, it turns out we can use the same definitions and just apply them to one of the coordinates.

Theorem 6.1.5: Let X be a topological space, $x \in X$ and $n \in \mathbb{N}$.

1. Homotopy classes of continuous maps $f: [0,1]^{\times n} \to X$ with $f(\partial[0,1]^{\times n}) = \{x\}$ relative to $\partial[0,1]^{\times n}$ form a group $\pi_n(x, X)$ with the multiplication

$$[g] \circ [f] = [g \star f] \qquad (g \star f)(t_1, ..., t_n) = \begin{cases} f(2t_1, t_2, ..., t_n) & t_1 \in [0, \frac{1}{2}] \\ g(2t_1 - 1, t_2, \ldots, t_n) & t_1 \in [\frac{1}{2}, 1], \end{cases} \tag{30}$$

   the nth **homotopy group** at x or, for $n = 1$, the **fundamental group** of X at x.

2. For $n > 1$ the homotopy group $\pi_n(x, X)$ is abelian.

Proof:
To simplify notation we write $C := [0,1]^{\times n}$ for the unit cube.

1. We show that the group multiplication is well-defined: If $h_1 : [0,1] \times C \to X$ is a homotopy from f to f' relative to ∂C and $h_2 : [0,1] \times C \to X$ a homotopy from g to g' relative to ∂C, then the continuous map

$$k : [0,1] \times C \to X, \quad (s, t_1, ..., t_n) \mapsto \begin{cases} h_1(s, 2t_1, ..., t_n) & t_1 \in [0, \frac{1}{2}] \\ h_2(s, 2t_1 - 1, ..., t_n) & t_1 \in [\frac{1}{2}, 1] \end{cases}$$

satisfies $k(0,t) = (g \star f)(t)$, $k(1,t) = (g' \star f')(t)$ for all $t \in C$ and $k(s,t) = x$ for all $t \in \partial C$ and hence is a homotopy relative to ∂C from $g \star f$ to $g' \star f'$.

2. We show that the group multiplication is associative: Let $e, f, g : C \to X$ be continuous with $e(\partial C) = f(\partial C) = g(\partial C) = \{x\}$. Then we have

$$(g \star (f \star e))(t_1, ..., t_n) = \begin{cases} e(4t_1, t_2, ..., t_n) & t_1 \in [0, \frac{1}{4}] \\ f(4t_1 - 1, t_2, ..., t_n) & t_1 \in [\frac{1}{4}, \frac{1}{2}] \\ g(2t_1 - 1, t_2, ..., t_n) & t_1 \in [\frac{1}{2}, 1] \end{cases}$$

$$((g \star f) \star e)(t_1, ..., t_n) = \begin{cases} e(2t_1, t_2, ..., t_n) & t_1 \in [0, \frac{1}{2}] \\ f(4t_1 - 2, t_2, ..., t_n) & t_1 \in [\frac{1}{2}, \frac{3}{4}] \\ g(4t_1 - 3, t_2, ..., t_n) & t_1 \in [\frac{3}{4}, 1] \end{cases}$$

and the following map is a homotopy from $g \star (f \star e)$ to $(g \star f) \star e$ relative to ∂C

$$h : [0,1] \times C \to X, \quad (s, t_1, ..., t_n) \mapsto \begin{cases} e(\frac{4t_1}{1+s}, t_2, ..., t_n) & t_1 \in [0, \frac{1}{4}(1+s)] \\ f(4t_1 - 1 - s, t_2, ..., t_n) & t_1 \in [\frac{1}{4}(1+s), \frac{1}{4}(2+s)] \\ g(\frac{4t_1 - 2 - s}{2-s}, t_2, ..., t_n) & t_1 \in [\frac{1}{4}(2+s), 1]. \end{cases}$$

3. Denote by $x : C \to X$, $t \mapsto x$ the constant map to x. We show that the homotopy class of x is the neutral element in $\pi_n(x, X)$. This follows, because for any continuous map $f : C \to X$ with $f(\partial C) = \{x\}$ the following are homotopies from $f \star x$ and from $x \star f$ to f relative to ∂C

$$h_1 : [0,1] \times C \to X, \quad (s, t_1, ..., t_n) \mapsto \begin{cases} x & t_1 \in [0, \frac{1}{2}(1-s)] \\ f(\frac{2t_1 - 1 + s}{1+s}, t_2, ..., t_n) & t_1 \in [\frac{1}{2}(1-s), 1] \end{cases}$$

$$h_2 : [0,1] \times C \to X, \quad (s, t_1, ..., t_n) \mapsto \begin{cases} f(\frac{2t_1}{1+s}, t_2, ..., t_n) & t_1 \in [0, \frac{1}{2}(1+s)] \\ x & t_1 \in [\frac{1}{2}(1+s), 1]. \end{cases}$$

4. We show that for any continuous map $f : C \to X$ with $f(\partial C) = \{x\}$ the homotopy class of $\bar{f} : C \to X$, $(t_1, ..., t_n) \mapsto f(1 - t_1, ..., t_n)$ is the inverse of the homotopy class of f. For this it, is sufficient to note that the following map is a homotopy from $\bar{f} \star f$ to x relative to ∂C

$$h : [0,1] \times C \to X, \quad (s, t_1, ..., t_n) \mapsto \begin{cases} f(\frac{2t_1}{1-s}, t_2, ..., t_n) & t_1 \in [0, \frac{1}{2}(1-s)] \\ \bar{f}(\frac{2t_1 - 1 + s}{1-s}, t_2, ..., t_n) & t_1 \in [\frac{1}{2}(1-s), 1-s] \\ x & t_1 \in [1-s, 1]. \end{cases}$$

5. We show that $\pi_n(x, X)$ is abelian for $n \geq 2$ by using an **Eckmann-Hilton argument**:

If $\circ, \bullet : A \times A \to A$ are two binary operations on a set A with units $1_\circ$ and $1_\bullet$ that satisfy $(a \circ b) \bullet (c \circ d) = (a \bullet c) \circ (b \bullet d)$ for all $a, b, c, d \in A$, then $\circ = \bullet$ is commutative and associative.

One has $1_\circ = 1_\circ \circ 1_\circ = (1_\circ \bullet 1_\bullet) \circ (1_\bullet \bullet 1_\circ) = (1_\circ \circ 1_\bullet) \bullet (1_\bullet \circ 1_\circ) = 1_\bullet \bullet 1_\bullet = 1_\bullet =: 1$, and hence $a \circ b = (a \bullet 1) \circ (1 \bullet b) = (a \circ 1) \bullet (1 \circ b) = a \bullet b$ for all $a, b \in A$. This implies commutativity $a \circ b = (1 \circ a) \circ (b \circ 1) = (1 \circ b) \circ (a \circ 1) = b \circ a$ for all $a, b \in A$ and associativity $(a \circ b) \circ c = (a \circ b) \circ (1 \circ c) = (a \circ 1) \circ (b \circ c) = a \circ (b \circ c)$ for all $a, b, c \in A$.

We apply the Eckmann-Hilton argument to the group multiplication $\circ$ of $\pi_n(x, X)$ and

$$[g] \bullet [f] = [g \star_2 f] \qquad (g \star_2 f)(t_1, ..., t_n) = \begin{cases} f(t_1, 2t_2, t_3, ..., t_n) & t_2 \in [0, \frac{1}{2}] \\ g(t_1, 2t_2 - 1, t_3, \ldots, t_n) & t_2 \in [\frac{1}{2}, 1], \end{cases}$$

whose unitality follows as in 3. Then the condition in the Eckmann-Hilton argument is satisfied for all maps $e, f, g, h : C \to X$ that send ∂C to x, as we have

$$(e \star f) \star_2 (g \star h) = (e \star_2 g) \star (f \star_2 h) = \begin{cases} h(2t_1, 2t_2, t_3, ..., t_n) & t_1, t_2 \in [0, \frac{1}{2}] \\ g(2t_1 - 1, 2t_2, t_3, ..., t_n) & t_1 \in [\frac{1}{2}, 1], t_2 \in [0, \frac{1}{2}] \\ f(2t_1, 2t_2 - 1, t_3, ..., t_n) & t_1 \in [0, \frac{1}{2}], t_2 \in [\frac{1}{2}, 1] \\ e(2t_1 - 1, 2t_2 - 1, t_3, ..., t_n) & t_1, t_2 \in [\frac{1}{2}, 1]. \end{cases}$$

□

Remark 6.1.6:

1. For $n = 0$ one defines $\pi_0(X)$ as the set of path-components of X, cf. Example 6.1.3, 1. In this case, there is no group structure.
2. The quotient space $[0, 1]^{\times n}/\partial[0, 1]^{\times n}$ is homeomorphic to S^n. Thus, one can identify continuous maps $f : [0, 1]^{\times n} \to X$ that satisfy $f(\partial[0, 1]^{\times n}) = \{x\}$ with continuous maps $f : S^n \to X$ that send a point $p \in S^n$ to $f(p) = x$.

Note that group structure of the homotopy groups in Theorem 6.1.5 requires that one works with homotopy classes of maps. The concatenation $\star$ of maps defined in (30) is neither associative nor unital, only associative and unital up to reparametrisations. The homotopy classes are also required to ensure that the group multiplication does not depend on the specific choice of the coordinate that is concatenated in (30).

We now consider how the homotopy groups of a topological space X depend on the choice of the basepoint. The idea is to transport continuous maps $f : [0, 1]^{\times n} \to X$ with $f(\partial[0, 1]^{\times n}) = \{x\}$ along paths in X to relate them to such maps for other basepoints $x' \in X$. If $n = 1$, this is simple and intuitive. One chooses a path γ from x to x', concatenates it with a path from x to x and then again with the path γ, traversed in the other direction.

For $n > 1$ the idea is similar, but the concatenation of paths and maps $f : [0, 1]^{\times n} \to X$ with $f(\partial[0, 1]^{\times n}) = \{x\}$ and the associated homotopies become more complicated. In all cases, one finds that this transport along paths induces group homomorphisms between the homotopy groups that depend only on their homotopy classes.

Proposition 6.1.7: Let X be a topological space.

1. Every path γ in X from x to x' induces group isomorphisms
$$\gamma^* : \pi_n(x', X) \to \pi_n(x, X), \quad [f] \mapsto \gamma^*[f].$$
2. The group isomorphisms γ^* depend only on the homotopy class of γ relative to $\{0, 1\}$ and define a right action $\lhd : \pi_n(x, X) \times \pi_1(x, X) \to \pi_n(x, X)$ with $[f] \lhd [\gamma] = \gamma^*[f]$.

Proof:
We denote by $C_r(p) = \{x \in \mathbb{R}^n \mid ||x-p||_\infty \leq r\}$ the cube of side length $2r$ around $p \in \mathbb{R}^n$ and set $C := [0,1]^n = C_{1/2}(p)$ with $p := (\frac{1}{2}, ..., \frac{1}{2}) \in \mathbb{R}^n$. We parametrise points in C as $p + sy$ with $s \in [0,1]$ and $p + y \in \partial C$.

Let $f : C \to X$ be a continuous map with $f(\partial C) = \{x'\}$ and $\gamma : [0,1] \to X$ a path with $\gamma(0) = x$ and $\gamma(1) = x'$. We define a continuous map $\gamma f : C \to X$ with $\gamma f(\partial C) = \{x\}$ by

$$\gamma f(p + sy) = \begin{cases} \gamma(2s) & s \in [0, \frac{1}{2}] \\ f(q + (2-2s)y) & s \in [\frac{1}{2}, 1]. \end{cases}$$

By defining suitable homotopies analogous to the ones in the proof of Theorem 6.1.5 (Exercise) one can show that for all continuous maps $e, f, g : C \to X$ with $e(\partial C) = f(\partial C) = \{x'\}$, $g(\partial C) = \{x''\}$, paths γ from x to x' and δ from x' to x'' one has

$$\gamma(e \star f) \sim \gamma e \star \gamma f \qquad\qquad x' \star f \sim f \qquad\qquad \gamma(\delta g) \sim (\delta \star \gamma) g.$$

If $h : [0,1] \times C \to X$ is a homotopy from e to f relative to ∂C and $h_s = h(s, -) : C \to X$, then the continuous map $h' : [0,1] \times C \to C$, $(s,t) \mapsto (\gamma h_s)(t)$ is a homotopy from γe to γf relative to ∂C. This shows that every path γ from x to x' defines a group homomorphism

$$\gamma^* : \pi_n(x', X) \to \pi_n(x, X), \quad [f] \mapsto [\gamma f]$$

with $x^* = \mathrm{id}_{\pi_n(x,X)}$ and $(\delta \star \gamma)^* = \gamma^* \circ \delta^*$. To prove the second claim, it remains to show that these group homomorphism depend only on the homotopy class of the path and are invertible.

If β, γ are paths from x to x' and $h : [0,1] \times [0,1] \to X$ a homotopy from β to γ relative to $\{0,1\}$, then for any continuous $f : C \to X$ with $f(\partial C) = \{x'\}$ the map $h' : [0,1] \times C \to X$, $(s,t) \mapsto h_s f$ is a homotopy from βf to γf relative to ∂C. This implies $\beta^*[f] = [\beta f] = [\gamma f] = \gamma^*[f]$ for all continuous maps $f : C \to X$ with $f(\partial C) = \{x'\}$ and hence $\beta^* = \gamma^*$.

As one has $\bar{\gamma} \star \gamma \sim x$ and $\gamma \star \bar{\gamma} \sim x'$ for all paths γ from x to x', the group homomorphisms $\gamma^* : \pi_n(x', X) \to \pi_n(x, X)$ and $\bar{\gamma}^* : \pi_n(x, X) \to \pi_n(x', X)$ satisfy the identities $\gamma^* \bar{\circ} \gamma^* = \mathrm{id}_{\pi_n(x,X)}$ and $\bar{\gamma}^* \circ \gamma^* = \mathrm{id}_{\pi_n(x',X)}$ and hence are mutually inverse isomorphisms. □

For a path-connected topological space X, Proposition 6.1.7 implies $\pi_n(x, X) \cong \pi_n(x', X)$ for all $n \in \mathbb{N}$ and $x, x' \in X$. In this case, one often writes $\pi_n(X)$ instead of $\pi_n(x, X)$ and suppresses the basepoint. However, if one wants to view homotopy groups as functors, one needs to be careful about the basepoints. In this case, one works with the category Top* of pointed topological spaces, whose objects are pairs (x, X) of a topological space X and a point $x \in X$ and whose morphisms from (x, X) to (y, Y) are continuous maps $f : X \to Y$ with $f(x) = y$.

Being homotopic relative to $\{x\}$ then defines an equivalence relation on the set of morphisms from (x, X) to (y, Y) that is compatible with composition: If $h : [0,1] \times X \to Y$ is a homotopy from f to f' relative to $\{x\}$ and $k : [0,1] \times Y \to Z$ a homotopy from g to g' relative to $\{f(x)\}$, then $l : [0,1] \times X \to Y$, $(s, x) \mapsto k(s, h(s, x))$ is a homotopy from $g \circ f$ to $g' \circ f'$ relative to $\{x\}$.

Thus, one obtains a homotopy category $K(\mathrm{Top}^*)$ of pointed topological spaces, whose objects are pointed topological spaces and whose morphisms from (x, X) to (y, Y) homotopy classes relative to $\{x\}$ of continuous maps $f : X \to Y$ with $f(x) = y$.

With these considerations, we can view the nth homotopy groups as a functor from the category Top^* to Grp, or, if we restrict attention to $n > 1$, to the category Ab of abelian groups. The fact that post-composition of a homotopy relative to $\{x\}$ with a continuous map yields again a homotopy relative to $\{x\}$ implies that π_n descends to the homotopy category $K(\text{Top}^*)$.

Theorem 6.1.8:

1. The nth homotopy groups define functors $\pi_n : \text{Top}^* \to \text{Grp}$ that assign
   - to a pointed topological space (x, X) the nth homotopy group $\pi_n(x, X)$,
   - to a continuous map $f : (x, X) \to (f(x), Y)$ the group homomorphism
   $$\pi_n(f) : \pi_n(x, X) \to \pi_n(f(x), Y), \quad [g] \mapsto [f \circ g]$$
2. If $f, f' : X \to Y$ are homotopic relative to $\{x\}$, then
   $$\pi_n(f) = \pi_n(f') : \pi_n(x, X) \to \pi_n(f(x), X).$$
3. The nth homotopy groups induce functors $\pi_n : K(\text{Top}^*) \to \text{Grp}$. In particular, path-connected spaces of the same homotopy type have isomorphic homotopy groups.

Proof:
1. Let $C = [0, 1]^{\times n}$. If $f : X \to Y$ is continuous and $g : C \to X$ continuous with $g(\partial C) = \{x\}$, then $f \circ g : C \to Y$ is continuous with $f \circ g(\partial C) = \{f(x)\}$. For any homotopy $h : [0, 1] \times C \to X$ from g to g' relative to ∂C, the map $f \circ h : [0, 1] \times C \to Y$ is a homotopy from $f \circ g$ to $f \circ g'$ relative to ∂C. Thus, f induces a map $\pi_n(f) : \pi_n(x, X) \to \pi_n(f(x), Y)$, $[g] \mapsto [f \circ g]$

As $f \circ (g \star g') = (f \circ g) \star (f \star g')$ for all continuous $g, g' : C \to X$ with $g(\partial C) = g'(\partial C) = \{x\}$,

$$\pi_n(f)([g]\circ[g']) = \pi_n(f)([g\star g']) = [f\circ(g\star g')] = [(f\circ g)\star(f\circ g')]] = [f\circ g]\circ[f\circ g'] = \pi_n(f)([g])\circ\pi_n(f)([g'])$$

and the map $\pi_n(f)$ is a group homomorphism. It follows directly from the definition that $\pi_n(\text{id}_X) = \text{id}_{\pi_n(x,X)}$ and $\pi_n(f' \circ f) = \pi_n(f') \circ \pi_n(f)$ for all continuous $f : X \to Y$, $f' : X' \to Y'$. This shows that π_n is a functor.

2. If $h : [0, 1] \times X \to Y$ is a homotopy from $f : X \to Y$ to $f' : X \to Y$ relative to $\{x\}$, then for all continuous $g : C \to X$ with $g(\partial C) = \{x\}$, the map $h' : [0, 1] \times C \to Y$, $(s, t) \mapsto h(s, g(t))$ is a homotopy relative to ∂C from $f \circ g$ to $f' \circ g$. Hence, $\pi_n(f)([g]) = [f \circ g] = [f' \circ g] = \pi_n(f')]([g])$ for all continuous $g : C \to X$ with $g(\partial C) = \{x\}$, which implies $\pi_n(f) = \pi_n(f')$.

3. The first claim follows from 2. The second claim follows, because the homotopy groups of a path-connected topological space in all basepoints are isomorphic by Proposition 6.1.7. □

Example 6.1.9:

1. If X is contractible, one has $\pi_n(X) = \{1\}$ for all $n \in \mathbb{N}$, as X is homotopy equivalent to a point. This holds in particular for convex subsets $X \subset \mathbb{R}^n$ such as linear subspaces or discs.

2. Fundamental groups of path-connected topological spaces be computed by choosing a CW-complex structure on X and applying the **Seifert-van Kampen theorem**. Every 1-cell contributes a generator of the fundamental group $\pi_1(X)$ and every 2-cell a relation. This yields, for instance, $\pi_1(S^1) = \mathbb{Z}$ and $\pi_1(S^n) = \{1\}$ for all $n \in \mathbb{N}_0$. The fundamental group of a torus is $\pi_1(T) = \mathbb{Z} \times \mathbb{Z}$ and for a surface of genus $g \geq 2$ one has the presentation $\pi_1(\Sigma) = \langle a_1, b_1, ..., a_g, b_g \mid [b_g, a_g] \cdots [b_1, a_1] = 1\rangle$.

3. Higher homotopy groups are very difficult to compute. One has $\pi_n(S^n) = \mathbb{Z}$ for all $n \in \mathbb{N}$, but many homotopy groups $\pi_k(S^n)$ with $k > n$ are currently unknown.

6.2 Simplicial homotopies

The aim is now to generalise the concept of a homotopy from the category Top to the categories $\mathcal{C}^{\Delta^{op}}$ for suitable categories $\mathcal{C}$, in particular $\mathcal{C} = \text{Set}$, and subsequently to Cat and $\text{Ch}_{R\text{-Mod}}$. For this, recall from Definition 6.1.1 that a homotopy from a continuous map $f : X \to Y$ to a continuous map $g : X \to Y$ is a continuous map $h : [0,1] \times X \to Y$ with $h \circ \iota^0 = f$ and $h \circ \iota^1 = g$. This makes it natural to define a homotopy from a simplicial morphism $\alpha : S \Rightarrow T$ to a simplicial morphism $\beta : S \Rightarrow T$ as a simplicial morphism $h : [0,1] \times S \Rightarrow T$, where $[0,1] \times S$ should be a simplicial object constructed from S that replaces the product space $[0,1] \times X$. Moreover, one should replace the inclusions $\iota^0, \iota^1 : X \to [0,1] \times X$ from Definition 6.1.1 with simplicial morphisms $\iota^0, \iota^1 : S \Rightarrow [0,1] \times S$ and impose the conditions $h \circ \iota^0 = \alpha$ and $h \circ \iota^1 = \beta$.

The first step is to replace the unit interval $[0,1]$ by a corresponding simplicial object and to define its product with S. For this, recall that the counterpart of the standard n-simplex $\Delta^n \subset \mathbb{R}^n$ in SSet is the simplicial set $\Delta^n = \text{Hom}(-, [n+1]) : \Delta^{op} \to \text{Set}$ from Example 5.2.7, 3. In particular, $[0,1] = \Delta^1 \subset \mathbb{R}$ corresponds to the simplicial set $\Delta^1 = \text{Hom}(-, [2]) : \Delta^{op} \to \text{Set}$. The sets $\Delta^1_n = \text{Hom}_\Delta([n+1], [2])$ contain exactly $n+2$ elements, namely the maps

$$\rho^i_n : [n+1] \to [2], \quad k \mapsto \begin{cases} 0 & k < i \\ 1 & k \geq i \end{cases} \qquad i = 0, \ldots, n+1. \tag{31}$$

In a general category $\mathcal{C}$ there are usually no counterparts of the simplicial sets $\Delta^n : \Delta^{op} \to \text{Set}$, so one cannot form a naive categorical product $[0,1] \times S$ of a simplicial object $[0,1]$ in $\mathcal{C}$ with another simplicial object $S : \Delta^{op} \to \mathcal{C}$. However, if $\mathcal{C}$ has coproducts, we can use the copower functor $\sqcup : \text{Set} \times \mathcal{C} \to \mathcal{C}$ from Definition 4.2.5 to form products of sets with objects in $\mathcal{C}$.

Definition 6.2.1: Let $\mathcal{C}$ be a category with all coproducts.

The **copower functor** $\sqcup : \text{SSet} \times \mathcal{C}^{\Delta^{op}} \to \mathcal{C}^{\Delta^{op}}$ assigns

- to a simplicial set $X : \Delta^{op} \to \text{Set}$ and a simplicial object $S : \Delta^{op} \to \mathcal{C}$ the simplicial object $X \sqcup S : \Delta^{op} \to \mathcal{C}$ that sends
  - an ordinal $[n+1]$ to the set $X_n \sqcup S_n = \amalg_{X_n} S_n$,
  - a morphism $\tau : [m+1] \to [n+1]$ to the morphism $X(\tau) \sqcup S(\tau) : X_n \sqcup S_n \to X_m \sqcup S_m$ given by the following commuting diagram for $n \in \mathbb{N}_0$ and $x \in X_n$

$$\begin{array}{ccc} \amalg_{X_n} S_m & \xrightarrow{X(\tau)\sqcup S(\tau)} & \amalg_{X_m} S_m \ . \\ \uparrow{\scriptstyle \iota_x} & & \uparrow{\scriptstyle \iota_{\tau(x)}} \\ S_n & \xrightarrow[S(\tau)]{} & S_m, \end{array} \tag{32}$$

- to a simplicial map $\alpha : X \Rightarrow X'$ and a simplicial morphism $\beta : S \Rightarrow S'$ the simplicial morphism $\alpha \sqcup \beta : X \sqcup S \Rightarrow X' \sqcup S'$ defined by the following commuting diagrams

$$\begin{array}{ccc} \amalg_{X_n} S_n & \xrightarrow{(\alpha\sqcup\beta)_n} & \amalg_{X'_n} S'_n \\ \uparrow{\scriptstyle \iota_x} & & \uparrow{\scriptstyle \iota_{\alpha_n(x)}} \\ S_n & \xrightarrow[\beta_n]{} & S'_n. \end{array}$$

It is a good exercise to verify that this is indeed a functor. Note also that it is sufficient that $\mathcal{C}$ is *finitely cocomplete*, if one considers only simplicial sets X with *finite* components X_n, such as the simplicial sets $\Delta^n = \mathrm{Hom}(-, [n+1]) : \Delta^{op} \to \mathrm{Set}$ from Example 5.2.7, 3. This ensures that the following constructions work for all abelian categories, which have *finite* coproducts by definition. Note also that for $\mathcal{C} = \mathrm{Set}$, this construction reduces to the usual product of simplicial sets defined by Proposition 2.3.2.

Example 6.2.2: If $\mathcal{C} = \mathrm{Set}$, then the product $X \sqcup S$ is just the product $X \times S : \Delta^{op} \to \mathrm{Set}$ given by the sets $(X \times S)_n = X_n \times S_n$ and the maps

$$(X \times S)(\alpha) = X(\alpha) \times S(\alpha) : X_n \times S_n \to X_m \times S_m, \quad (x, s) \mapsto (X(\alpha)x, S(\alpha)s)$$

for every $\alpha : [m+1] \to [n+1]$. The inclusions are $\iota_x : S_n \to X_n \times S_n$, $s \mapsto (x, s)$ for $x \in X_n$.

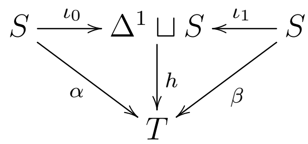
With Definition 6.2.1 we can define the categorical counterpart of the product space $[0,1] \times X$ for a topological space X in any category $\mathcal{C}$ with finite coproducts. We replace

- the topological space X by a simplicial object $S : \Delta^{op} \to \mathcal{C}$,
- the interval $[0,1]$ by the simplicial set $\Delta^1 = \mathrm{Hom}_\Delta(-, [2]) : \Delta^{op} \to \mathrm{Set}$,
- the product space $[0,1] \times X$ by the simplicial object $\Delta^1 \sqcup S : \Delta^{op} \to \mathcal{C}$,
- $0, 1 \in [0,1]$ by $\rho_n^{n+1}, \rho_n^0 \in \Delta_n^1$ from (31) with $\rho_n^{n+1}(k) = 0$, $\rho_n^0(k) = 1$ for all $k \in [n+1]$.

This suggests replacing the inclusions $\iota^i : X \to [0,1] \times X$, $x \mapsto (i, x)$ for $i = 0, 1$ by the simplicial morphisms whose components are the inclusions for $\rho_n^{n+1}, \rho_n^0 \in \Delta_n^1$

$$\begin{aligned} &\iota^0 : S \Rightarrow \Delta^1 \sqcup S && \iota^1 : S \Rightarrow \Delta^1 \sqcup S \\ &\iota_n^0 = \iota_{\rho_n^{n+1}} : S_n \to \amalg_{\Delta_n^1} S_n, && \iota_n^1 = \iota_{\rho_n^0} : S_n \to \amalg_{\Delta_n^1} S_n. \end{aligned} \tag{33}$$

Definition 6.2.3: Let $\mathcal{C}$ be a category with finite coproducts, $S, T : \Delta^{op} \to \mathcal{C}$ simplicial objects in $\mathcal{C}$ and $\alpha, \beta : S \Rightarrow T$ simplicial morphisms. A **simplicial homotopy** $h : \alpha \Rrightarrow \beta$ is a simplicial morphism $h : \Delta^1 \sqcup S \Rightarrow T$ with $h \circ \iota^0 = \alpha : S \Rightarrow T$ and $h \circ \iota^1 = \beta : S \Rightarrow T$.



Although Definition 6.2.3 is conceptual and allows one to see simplicial homotopies as analogues of topological homotopies, it is sometimes difficult to use in practice. It is often convenient to use the following definition, which is more technical, but also more explicit. An additional advantage is that it can be formulated for any category $\mathcal{C}$.

Definition 6.2.4: Let $S, T : \Delta^{op} \to \mathcal{C}$ be simplicial objects in $\mathcal{C}$ and $\alpha, \beta : S \Rightarrow T$ simplicial morphisms. A **simplicial homotopy** $h : \alpha \Rrightarrow \beta$ from α to β is a collection of morphisms

$$h_n^i : S_n \to T_{n+1} \qquad i = 0, \ldots, n$$

that satisfy for all $n \in \mathbb{N}_0$

$$d_{n+1}^0 \circ h_n^0 = \beta_n, \qquad d_{n+1}^{n+1} \circ h_n^n = \alpha_n \tag{34}$$

$$d_{n+1}^i \circ h_n^j = \begin{cases} h_{n-1}^{j-1} \circ d_n^i & i < j \\ d_{n+1}^i \circ h_n^{i-1} & i = j \neq 0 \\ h_{n-1}^j \circ d_n^{i-1} & i > j+1 \end{cases} \qquad s_{n+1}^i \circ h_n^j = \begin{cases} h_{n+1}^{j+1} \circ s_n^i & i \leq j \\ h_{n+1}^j \circ s_n^{i-1} & i > j. \end{cases}$$

That Definitions 6.2.3 and 6.2.4 yield the same notion of simplicial homotopy is far from obvious. Some careful computations show that this is indeed the case. Beware of mistakes and different conventions for simplicial homotopies in the literature. Sometimes the source and target of the homotopy are switched, and the proofs do not always match the conventions.

Proposition 6.2.5: Let $\mathcal{C}$ be a category with finite coproducts and $\alpha, \beta : S \Rightarrow T$ simplicial morphisms in $\mathcal{C}$. Then A simplicial homotopy $k : \alpha \Rrightarrow \beta$ in the sense of Definition 6.2.3 defines a simplicial homotopy $h : \alpha \Rrightarrow \beta$ in the sense of Definition 6.2.4 and vice versa.

Proof:
A direct computation shows that the maps $\rho^i_n : [n+1] \to [2]$ with $\rho^i_n(k) = 0$ for $k < i$ and $\rho^i_n(k) = 1$ for $k \geq i$ from (31) satisfy

$$\rho^0_n \circ \tau = \rho^0_m, \quad \rho^{n+1}_n \circ \tau = \rho^{m+1}_m \qquad \text{for all} \quad \tau : [m+1] \to [n+1] \tag{35}$$
$$\rho^j_n \circ \delta^i_n = \begin{cases} \rho^{j-1}_{n-1} & 0 \leq i < j < n+1 \\ \rho^j_{n-1} & 0 < j \leq i < n+1 \end{cases} \qquad \rho^j_n \circ \sigma^i_{n+1} = \begin{cases} \rho^{j+1}_{n+1} & 0 \leq i < j < n+1 \\ \rho^j_{n+1} & 0 < j \leq i < n+1. \end{cases}$$

1. Given a family of morphisms $h^i_n : S_n \to T_{n+1}$ satisfying the conditions in Definition 6.2.4, we define the morphisms $k_n : \Delta^1_n \sqcup S_n \to T_n$ via the inclusions $\iota_{\rho^i_n} : S_n \to \Delta^1_n \sqcup S_n$ for the coproduct

$$k_n \circ \iota_{\rho^i_n} = d^i_{n+1} \circ h^i_n = d^i_{n+1} \circ h^{i-1}_n \qquad i = 0, \ldots, n-1. \tag{36}$$

Then we have by definition

$$k_n \circ \iota^0_n = k \circ \iota_{\rho^{n+1}_n} = d^{n+1}_{n+1} \circ h^n_n = \alpha_n \qquad k_n \circ \iota^1_n = k \circ \iota_{\rho^0_n} = d^0_{n+1} \circ h^0_n = \beta_n.$$

To show that this defines a simplicial morphism $k : \Delta^1 \sqcup S \Rightarrow S$ one computes for any morphism $\tau : [m+1] \to [n+1]$

$$k_m \circ (\Delta^1(\tau) \sqcup S(\tau)) \circ \iota_{\rho^0_n} \overset{(32)}{=} k_m \circ \iota_{\rho^0_n \circ \tau} \circ S(\tau) \overset{(35)}{=} k_m \circ \iota_{\rho^0_m} \circ S(\tau) \overset{(36)}{=} d^0_{m+1} \circ h^0_m \circ S(\tau)$$
$$= \beta_m \circ S(\tau) = T(\tau) \circ \beta_n = T(\tau) \circ d^0_{n+1} \circ h^0_n \overset{(36)}{=} T(\tau) \circ k_n \circ \iota_{\rho^0_n}$$
$$k_m \circ (\Delta^1(\tau) \sqcup S(\tau)) \circ \iota_{\rho^{n+1}_n} \overset{(32)}{=} k_m \circ \iota_{\rho^{n+1}_n \circ \tau} \circ S(\tau) \overset{(35)}{=} k_m \circ \iota_{\rho^{m+1}_m} \circ S(\tau) \overset{(36)}{=} d^{m+1}_{m+1} \circ h^m_m \circ S(\tau)$$
$$= \alpha_m \circ S(\tau) = T(\tau) \circ \alpha_n = T(\tau) \circ d^{n+1}_{n+1} \circ h^n_n \overset{(36)}{=} T(\tau) \circ k_n \circ \iota_{\rho^{n+1}_n}$$

For the remaining cases, one verifies the relations $d^i_n \circ k_n = k_{n-1} \circ d^i_n$ and $s^i_n \circ k_n = k_{n+1} \circ s^i_n$ by direct computations using formula (36), the relations (34) and (35) and the simplicial relations (22). For instance, one has for $0 < j \leq i < n+1$

$$k_{n-1} \circ d^i_n \circ \iota_{\rho^j_n} \overset{(32)}{=} k_{n-1} \circ \iota_{\rho^j_n \circ \delta^i_n} \circ d^i_n \overset{(35)}{=} k_{n-1} \circ \iota_{\rho^j_{n-1}} \circ d^i_n \overset{(36)}{=} d^j_n \circ h^{j-1}_{n-1} \circ d^i_n \overset{(34)}{=} d^j_n \circ d^{i+1}_{n+1} \circ h^{j-1}_n$$
$$\overset{(22)}{=} d^i_n \circ d^j_{n+1} \circ h^{j-1}_n \overset{(36)}{=} d^i_n \circ k_n \circ \iota_{\rho^j_n}.$$

and for $0 \leq i < j < n+1$

$$k_{n-1} \circ d^i_n \circ \iota_{\rho^j_n} \overset{(32)}{=} k_{n-1} \circ \iota_{\rho^j_n \circ \delta^i_n} \circ d^i_n \overset{(35)}{=} k_{n-1} \circ \iota_{\rho^{j-1}_{n-1}} \circ d^i_n \overset{(36)}{=} d^{j-1}_n \circ h^{j-2}_{n-1} \circ d^i_n \overset{(34)}{=} d^{j-1}_n \circ h^{j-1}_{n-1} \circ d^i_n$$
$$\overset{(34)}{=} d^{j-1}_n \circ d^i_{n+1} \circ h^j_n \overset{(22)}{=} d^i_n \circ d^j_{n+1} \circ h^j_n \overset{(36)}{=} d^i_n \circ k_n \circ \iota_{\rho^j_n}$$

The computations for the degeneracies are analogous, and this shows that the maps k_n define a simplicial homotopy $k : \alpha \Rrightarrow \beta$ in the sense of Definition 6.2.3.

2. Let $k : \Delta^1 \sqcup S \Rightarrow T$ be a simplicial homotopy from α to β in the sense of Definition 6.2.3. We consider the morphisms

$$h^i_n = k_{n+1} \circ \iota_{\rho^{i+1}_{n+1}} \circ s^i_n : S_n \to T_{n+1} \tag{37}$$

which satisfy

$$d^0_{n+1} \circ h^0_n \overset{(37)}{=} d^0_{n+1} \circ k_{n+1} \circ \iota_{\rho^1_{n+1}} \circ s^0_n \overset{(32)}{=} k_n \circ \iota_{\rho^1_{n+1} \circ \delta^0_{n+1}} \circ d^0_{n+1} \circ s^0_n \overset{(22),(35)}{=} k_n \circ \iota_{\rho^0_n} = k_n \circ \iota^1 = \beta$$
$$d^{n+1}_{n+1} \circ h^n_n \overset{(37)}{=} d^{n+1}_{n+1} \circ k_{n+1} \circ \iota_{\rho^{n+1}_{n+1}} \circ s^n_n \overset{(32)}{=} k_n \circ \iota_{\rho^{n+1}_{n+1} \circ \delta^{n+1}_{n+1}} \circ d^{n+1}_{n+1} \circ s^n_n \overset{(22),(35)}{=} k_n \circ \iota_{\rho^{n+1}_n} = k_n \circ \iota^0 = \alpha$$

The relations in (34) then follow by direct computations using (37), the fact that k is a simplicial morphism, the relations (35) and the simplicial relations (22). For instance, we have for $i < j$

$$\begin{aligned}
i < j: \quad & d^i_{n+1} \circ h^j_n \overset{(37)}{=} d^i_{n+1} \circ k_{n+1} \circ \iota_{\rho^{j+1}_{n+1}} \circ s^j_n \overset{(32)}{=} k_n \circ \iota_{\rho^{j+1}_{n+1} \circ \delta^i_{n+1}} \circ d^i_{n+1} \circ s^j_n \\
& \overset{(22),(35)}{=} k_n \circ \iota_{\rho^j_n} \circ s^{j-1}_{n-1} \circ d^i_n \overset{(37)}{=} h^{j-1}_{n-1} \circ d^i_n, \\
i = j \neq 0: \quad & d^i_{n+1} \circ h^i_n \overset{(37)}{=} d^i_{n+1} \circ k_{n+1} \circ \iota_{\rho^{i+1}_{n+1}} \circ s^i_n = k_n \circ \iota_{\rho^{i+1}_{n+1} \circ \delta^i_{n+1}} \circ d^i_{n+1} \circ s^i_n \overset{(22),(35)}{=} k_n \circ \iota_{\rho^i_n} \\
& \overset{(22),(35)}{=} k_n \circ \iota_{\rho^i_{n+1} \circ \delta^i_{n+1}} \circ d^i_{n+1} \circ s^{i-1}_n = d^i_{n+1} \circ k_n \circ \iota_{\rho^i_{n+1}} \circ s^{i-1}_n \overset{(37)}{=} d^i_{n+1} \circ h^{i-1}_n, \\
j < i-1: \quad & d^i_{n+1} \circ h^j_n \overset{(37)}{=} d^i_{n+1} \circ k_{n+1} \circ \iota_{\rho^{j+1}_{n+1}} \circ s^j_n \overset{(32)}{=} k_n \circ \iota_{\rho^{j+1}_{n+1} \circ \delta^i_{n+1}} \circ d^i_{n+1} \circ s^j_n \\
& \overset{(22),(35)}{=} k_n \circ \iota_{\rho^{j+1}_n} \circ s^j_{n-1} \circ d^{i-1}_n \overset{(37)}{=} h^j_{n-1} \circ d^{i-1}_n.
\end{aligned}$$

The relations for the degeneracies in (34) are verified analogously, and this shows that the maps h_n form a simplicial homotopy $h : \alpha \Rrightarrow \beta$ in the sense of Definition 6.2.4.

A direct computation shows that applying first (37) and then (36) or the other way around yields again the original simplicial homotopy . □

Remark 6.2.6:

1. Being simplicially homotopic in the sense of Definition 6.2.3 or Definition 6.2.4 is a reflexive relation (Exercise 55). It also follows from Definition 6.2.4 that it is compatible with the composition of simplicial morphisms:

   If $\alpha, \beta : S \Rightarrow T$ are simplicial maps and $h : \alpha \Rrightarrow \beta$ a simplicial homotopy, then

   - the maps $h^i_n \circ \gamma_n : R_n \to T_{n+1}$ define a simplicial homotopy $h\gamma : \alpha \circ \gamma \Rrightarrow \beta \circ \gamma$ for every simplicial morphism $\gamma : R \Rightarrow S$,
   - the maps $\delta_{n+1} \circ h^i_n : S_n \to U_{n+1}$ define a simplicial homotopy $\delta h : \delta \circ \alpha \Rrightarrow \delta \circ \beta$ for for every simplicial morphism $\delta : T \Rightarrow U$

2. However, being simplicially homotopic is *not necessarily an equivalence relation.* In particular, it is in general not an equivalence relation for simplicial sets. We will derive a sufficient condition for this to hold in Section 7.2.

3. If $\mathcal{C} = R$-Mod for a ring R or, more generally, an abelian category, then being simplicial homotopic in the sense of Definition 6.2.3 or Definition 6.2.4 is an equivalence relation on $\mathrm{Hom}_{\mathcal{C}^{\Delta^{op}}}(S,T)$ for all simplicial objects $S, T : \Delta^{op} \to \mathcal{C}$ (Exercise 55).

We defined simplicial homotopies in analogy to topological homotopies by replacing continuous maps by simplicial morphisms and product spaces by suitable products of simplicial sets and simplicial objects. However, there is a more direct relation between continuous and simplicial maps. The singular nerve Sing : Top → SSet from Example 5.2.7, 1. and Definition 5.3.1 assigns

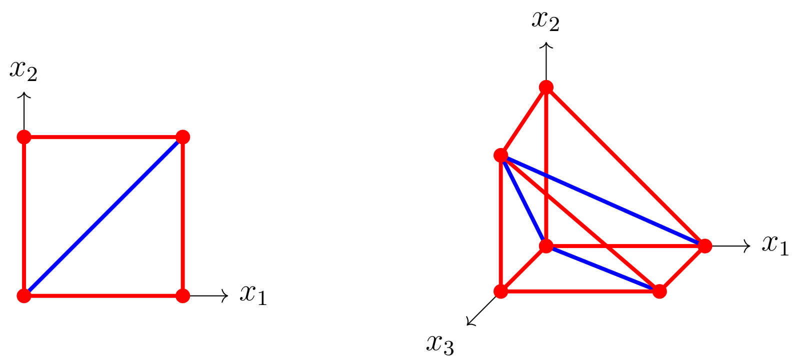


Figure 1: The prism maps $P_i^n : \Delta^{n+1} \to [0,1] \times \Delta^n$ for $n = 1, 2$.

- to a topological space X the simplicial set $\mathrm{Sing}(X) = \mathrm{Hom}_{\mathrm{Top}}(T(-), X) : \Delta^{op} \to \mathrm{Set}$ with components $\mathrm{Sing}(X)_n = \mathrm{Hom}_{\mathrm{Top}}(\Delta^n, X)$ and face maps and degeneracies

$$d_n^i : \mathrm{Sing}(X)_n \to \mathrm{Sing}(X)_{n-1},\ \sigma \mapsto \sigma \circ f_i^n \qquad s_n^i : \mathrm{Sing}(X)_n \to \mathrm{Sing}(X)_{n+1},\ \sigma \mapsto \sigma \circ s_i^n,$$

- to a continuous map $f : X \to Y$ the simplicial map $\mathrm{Sing}(f) : \mathrm{Sing}(X) \Rightarrow \mathrm{Sing}(Y)$ with components $\mathrm{Sing}(f)_n : \mathrm{Sing}(X)_n \to \mathrm{Sing}(Y)_n$, $\sigma \mapsto f \circ \sigma$,

where Δ^n is the standard n-simplex and $f_i^n : \Delta^{n-1} \to \Delta^n$ and $s_i^n : \Delta^{n+1} \to \Delta^n$ be the affine linear face maps and degeneracies from Definition 5.1.2.

It is then natural to ask if a continuous homotopy from f to g give rise to a simplicial homotopy between the singular nerves $\mathrm{Sing}(f)$ and $\mathrm{Sing}(g)$. This is an important consistency check. To relate continuous and simplicial homotopies, one uses the *prism maps*, the affine linear counterparts the maps $\rho_n^j : [n+1] \to [2]$ from (31). They describe the systematic subdivision of a prism into $(n+1)$-simplexes and give a geometrical interpretation to the maps $\rho_n^j : [n+1] \to [2]$.

Proposition 6.2.7: Every continuous homotopy $h : [0,1] \times X \to Y$ from $f : X \to Y$ to $g : X \to Y$ induces a simplicial homotopy $\mathrm{Sing}(h) : \mathrm{Sing}(f) \Rrightarrow \mathrm{Sing}(g)$.

Proof:
Let $f, g : X \to Y$ be continuous maps and $h : [0,1] \times X \to Y$ is a homotopy from f to g. The **prism maps** are the affine linear maps

$$P_i^n : \Delta^{n+1} \to [0,1] \times \Delta^n, \quad e_k \mapsto (\rho_{n+1}^{i+1}(k), s_i^n(e_k)) = \begin{cases} (0, e_k) & k \leq i \\ (1, e_{k-1}) & k > i \end{cases} \qquad i = 0, \ldots, n, \quad (38)$$

where $\rho_n^j : [n+1] \to [2]$ are the maps from (31). They describe the subdivision of the prism $[0,1] \times \Delta^n$ into $(n+1)$ different $(n+1)$-simplexes and are shown in Figure 1. A direct computation (Exercise) shows that they satisfy the relations

$$P_0^n \circ f_0^{n+1} = i_1 : e_k \mapsto (1, e_k) \qquad P_n^n \circ f_{n+1}^{n+1} = i_0 : e_k \mapsto (0, e_k) \quad (39)$$

$$P_j^n \circ f_i^{n+1} = \begin{cases} (\mathrm{id} \times f_i^n) \circ P_{j-1}^n & i < j \\ P_{i-1}^n \circ f_i^{n+1} & i = j \neq 0 \\ (\mathrm{id} \times f_{i-1}^n) \circ P_j^n & i > j+1 \end{cases} \qquad P_j^n \circ s_i^n = \begin{cases} (\mathrm{id} \times s_i^n) \circ P_{j+1}^n & i \leq j \\ (\mathrm{id} \times s_{i-1}^n) \circ P_j^n & i > j. \end{cases}$$

We define the maps

$$h_n^j : \mathrm{Sing}(X)_n \to \mathrm{Sing}(Y)_{n+1}, \quad \sigma \mapsto h \circ (\mathrm{id} \times \sigma) \circ P_j^n$$

The relations (39) for the prism maps then imply that they satisfy the relations (34) and define a simplicial homotopy $\mathrm{Sing}(h) : \mathrm{Sing}(g) \Rrightarrow \mathrm{Sing}(f)$ (Exercise). □

This proposition shows that the singular nerve Sing : Top → SSet respects homotopies. It sends homotopic maps in Top to homotopic simplicial maps. One might ask if an analogous statement also holds for its left adjoint, the geometric realisation functor Geom : SSet → Top from Definition 5.3.1 and Proposition 5.3.2. This is indeed the case and allows one to transform simplicial homotopies into continuous homotopies.

Proposition 6.2.8: Let $S, T : \Delta^{op} \to \mathrm{Set}$ simplicial sets and $\alpha, \beta : S \Rightarrow T$ simplicial maps. A simplicial homotopy $h : \alpha \Rrightarrow \beta$ induces a homotopy $\mathrm{Geom}(h) : [0,1] \times \mathrm{Geom}(S) \to \mathrm{Geom}(T)$ from $\mathrm{Geom}(\alpha)$ to $\mathrm{Geom}(\beta)$ as in Definition 6.1.1.

Proof:
We define the continuous homotopy $\mathrm{Geom}(h)$ with the prism maps from (38). For this, note that for every point $(t,x) \in [0,1] \times \Delta^n$, there is a prism map $P_i^n : \Delta^{n+1} \to [0,1] \times \Delta^n$ with $i \in \{0, \ldots, n\}$ and a point $y \in \Delta^{n+1}$ with $(t,x) = P_i^n(y)$.

If (t,x) is contained in a single affine $(n+1)$-simplex $P_i^n(\Delta^{n+1})$, then i and y are unique. Otherwise, (t,p) is contained in an n-face shared by two affine $(n+1)$-simplexes $P_i^n(\Delta^{n+1})$ and $P_{i+1}^n(\Delta^{n+1})$. In this case (38) yields a $z \in \Delta^n$ with $(t,x) = P_i^n(y) = P_{i+1}^n(y)$ and $y = f_{i+1}^{n+1}(z)$.

For a simplicial homotopy $h : \alpha \Rrightarrow \beta$ given by a collection of maps $h_n : S_n \to T_{n+1}$ as in Definition 6.2.4, we define the continuous homotopy $\mathrm{Geom}(h)$ by

$$\mathrm{Geom}(h) : [0,1] \times \mathrm{Geom}(S) \to \mathrm{Geom}(T), \quad (t, [s,x]) \mapsto [h_n^i(s), y],$$

where $s \in S_n$ and $x \in \mathring{\Delta}^n$ form the unique non-degenerate representative of $[s,x] \in \mathrm{Geom}(S)$ from Theorem 5.3.3 and $(t,p) = P_i^n(y)$. The map $\mathrm{Geom}(h)$ is well-defined and continuous, because one has for $(t,p) = P_i^n(y) = P_{i+1}^n(y)$ with $y = f_{i+1}^{n+1}(z)$

$$[h_n^i(s), y] = [h_n^i, f_{i+1}^{n+1}(z)] = [d_{n+1}^{i+1} \circ h_n^i(s), z] \overset{(34)}{=} [d_{n+1}^{i+1} \circ h_n^{i+1}(s), z] = [h_n^{i+1}(s), f_{i+1}^{n+1}(z)] = [h_n^{i+1}(s), y].$$

It is a homotopy from $\mathrm{Geom}(\beta)$ to $\mathrm{Geom}(\alpha)$, because (38) implies $(0,p) = P_n^n(f_{n+1}^{n+1}(p))$ and $(1,p) = P_0^n(f_0^{n+1}(p))$ and consequently

$$h(0,[s,p]) = [h_n^n(s), f_{n+1}^{n+1}(p)] = [d_{n+1}^{n+1} \circ h_n^n(s), p] \overset{(34)}{=} [\alpha_n(s), p] = \mathrm{Geom}(\alpha)[s,p]$$
$$h(1,[s,p]) = [h_n^0(s), f_0^{n+1}(p)] = [d_{n+1}^0 \circ h_n^0(s), p] \overset{(34)}{=} [\beta_n(s), p] = \mathrm{Geom}(\beta)[s,p].$$

□

We have thus shown that both, the singular nerve Sing : Top → SSet and its left adjoint, the geometric realisation functor Geom : SSet → Top, send homotopies to homotopies.

Besides these two functors, we encountered another pair of adjoints with values in SSet, namely the simplicial nerve $N : \mathrm{Cat} \to \mathrm{SSet}$ from Example 5.2.7, 2. and Lemma 5.4.1 and its left adjoint, the homotopy functor $h : \mathrm{SSet} \to \mathrm{Cat}$ from Definition 5.4.4. It is then natural to ask if there is a notion of homotopy in Cat that corresponds to simplicial homotopies under this adjunction.

As continuous and simplicial homotopies relate morphisms in Top and SSet with the same source and target, a homotopy in Cat should relate functors with the same source and target. The only obvious candidate for this are natural transformations. We investigate how they behave under the nerve $N : \mathrm{Cat} \to \mathrm{SSet}$ and the classifying space functor $B = \mathrm{Geom}N : \mathrm{Cat} \to \mathrm{Top}$.

Theorem 6.2.9: Let $\mathcal{C}, \mathcal{D}$ be small categories and $F, G : \mathcal{C} \to \mathcal{D}$ functors.

1. A natural transformation $\nu : F \Rightarrow G$ induces a simplicial homotopy $h : N(F) \Rrightarrow N(G)$ and a homotopy $h : [0,1] \times B\mathcal{C} \to B\mathcal{D}$ from $BF : B\mathcal{C} \to B\mathcal{D}$ to $BG : B\mathcal{C} \to B\mathcal{D}$.
2. If F has a left or right adjoint, then the classifying spaces $B\mathcal{C}$ and $B\mathcal{D}$ are homotopy equivalent. In particular, this holds if F is an equivalence of categories.

Proof:
1. Let $[1]'$ be the poset category for the ordinal $[1]$ and $[2]'$ the poset category for $[2]$ with objects $0, 1$ and a single non-identity morphism $d : 0 \to 1$.

Denote by $I^i : \mathcal{C} \to [2]' \times \mathcal{C}$ for $i = 0, 1$ the functors given by $I^i(C) = (i, C)$ and $I^i(f) = (1_i, f)$ for all objects C and morphisms $f : C \to C'$.

Then natural transformations $\tau : F \Rightarrow G$ correspond bijectively to functors $T : [2]' \times \mathcal{C} \to \mathcal{D}$ with $TI^0 = F$ and $TI^1 = G$ or, equivalently,

- $T(0, C) = F(C)$ and $T(1, C) = G(C)$ on the objects,
- $T(1_0, f) = F(f)$, $T(1_1, f) = G(f)$, $T(d, f) = \tau_{C'} \circ F(f) = G(f) \circ \tau_C$ for $f \in \mathrm{Hom}_{\mathcal{C}}(C, C')$.

Applying the nerve $N : \mathrm{Cat} \to \mathrm{SSet}$ yields a simplicial homotopy $N(T) : N(F) \Rrightarrow N(G)$:

For this, recall from Example 5.4.2 that $N([n+1]') = \Delta^n = \mathrm{Hom}(-, [n+1]) : \Delta^{op} \to \mathrm{Set}$ for all $n \in \mathbb{N}_0$. As a right adjoint, the simplicial nerve preserves products by Theorem 2.4.7, and this yields simplicial maps $N(T) : \Delta^1 \times N(\mathcal{C}) \Rightarrow N(\mathcal{D})$ and $N(I^i) : N(\mathcal{C}) \to \Delta^1 \times N(\mathcal{C})$ with $N(T)N(I^0) = N(F)$ and $N(T)N(I^1) = N(G)$. With Example 6.2.2, Definition 6.2.3 and equation (33) it follows that $N(T)$ is a simplicial homotopy from $N(F)$ to $N(G)$.

2. We prove the claim for a right adjoint. The proof for a left adjoint is analogous.
If $F : \mathcal{C} \to \mathcal{D}$ has a right adjoint $R : \mathcal{D} \to \mathcal{C}$ then the unit and counit of the adjunction are natural transformations $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow RF$ and $\epsilon : FR \Rightarrow \mathrm{id}_{\mathcal{D}}$. By 1. they induce simplicial homotopies $N(\eta) : \mathrm{id}_{N(\mathcal{C})} \Rrightarrow N(R)N(F)$ and $N(\epsilon) : N(F)N(R) \Rrightarrow \mathrm{id}_{N(\mathcal{D})}$. By Proposition 6.2.8, these simplicial homotopies induce continuous homotopies between the geometrical realisations.

If $B = \mathrm{Geom}N : \mathcal{C} \to \mathrm{Top}$ denotes the classifying space functor from Definition 5.4.9, these are continuous maps $BF : B\mathcal{C} \to B\mathcal{D}$ and $BR = B\mathcal{D} \to B\mathcal{C}$ such that $(BR)(BF) \sim \mathrm{id}_{B\mathcal{C}}$ and $(BF)(BR) \sim \mathrm{id}_{B\mathcal{D}}$. Hence, the classifying spaces $B\mathcal{C}$ and $B\mathcal{D}$ are homotopy equivalent. □

This theorem identifies natural transformations as categorical counterparts of homotopies and adjunctions as the categorical counterparts of homotopy equivalences. This allows one to apply homotopy theory to categories. If one is only interested in quantities that are homotopy invariant such as homotopy groups and homologies of their classifying spaces, then one can identify categories related by adjunctions. On the other hand, one can deduce statements about their classifying spaces from the properties of the categories.

Corollary 6.2.10: (Exercise 57) Let $\mathcal{C}$ be a small category.

1. If $\mathcal{C}$ has an initial or terminal object, then $B\mathcal{C}$ is contractible.
2. If $\mathcal{C}$ has binary products or coproducts, then $B\mathcal{C}$ is contractible.

The question if the left adjoint of the simplicial nerve, the homotopy functor $h : \mathrm{SSet} \to \mathrm{Cat}$ from Definition 5.4.4, also preserves homotopies, is less important in practice. One wants to use simplicial sets and topological spaces to investigate categories, not the other way around. Nevertheless, it is not difficult to show that the homotopy functor sends simplicial homotopies to natural transformations. This follows from the fact that the homotopy functor preserves products and sends the simplicial sets Δ^0 and Δ^1 to the poset categories $[1]'$ and $[2]'$.

Corollary 6.2.11: Let $\alpha, \beta : S \Rightarrow T$ be simplicial maps. The functor $h : \text{SSet} \to \text{Cat}$ sends a simplicial homotopy $k : \Delta^1 \times S \Rightarrow T$ from α to β to a natural transformation $h(k) : h(\alpha) \Rightarrow h(\beta)$.

Proof:
By Definition 5.4.4 the functor h sends the simplicial set $\Delta^n = \text{Hom}(-, [n+1]) : \Delta^{op} \to \text{Set}$ to the associated poset category $[n+1]'$. From Definition 5.4.4 it is also apparent that it sends products in SSet to products in Cat. A direct computation shows that it sends the inclusions $\iota^0, \iota^1 : S \to \Delta^1 \times S$ from (33) to the functors $I^0, I^1 : h(S) \to [2]' \times h(S)$ from the proof of Theorem 6.2.9. Hence, the image of a simplicial homotopy $k : \Delta^1 \times S \Rightarrow T$ is a functor $h(k) : [2]' \times h(S) \to h(T)$ with $h(k)I^0 = h(\alpha)$ and $h(k)I^1 = h(\beta)$. By the proof of Theorem 6.2.9 this defines a natural transformation $h(k) : h(\alpha) \Rightarrow h(\beta)$. □

6.3 Simplicial homotopies and chain homotopies

As shown in Section 5.5, simplicial objects and simplicial morphisms in a category R-Mod define positive chain complexes and chain maps between them via the standard chain complex functor from Proposition 5.5.5 and the normalised chain complex functor from Proposition 5.5.8. The Dold-Kan correspondence in Theorem 5.5.13 states that the normalised chain complex functor is an equivalence of categories. This implies that there must be a counterpart of the concept of a simplicial homotopy in the category $\text{Ch}_{R\text{-Mod}}$ of chain complexes in R-Mod and chain maps between them. As we will see, this is the concept of a *chain homotopy*.

Definition 6.3.1: Let $X_\bullet, X'_\bullet$ be chain complexes in R-Mod and $f_\bullet, f'_\bullet : X_\bullet \to X'_\bullet$ chain maps.

1. A **chain homotopy** $h_\bullet : f_\bullet \Rightarrow f'_\bullet$ is a family $(h_n)_{n\in\mathbb{Z}}$ of morphisms $h_n : X_n \to X'_{n+1}$ with

$$f'_n - f_n = h_{n-1} \circ d_n + d'_{n+1} \circ h_n \qquad \forall n \in \mathbb{Z}.$$

   If there is a chain homotopy $h_\bullet : f_\bullet \Rightarrow f'_\bullet$, then $f_\bullet$ und $f'_\bullet$ are called **chain homotopic**, and one writes $f_\bullet \sim f'_\bullet$.

2. A chain map $f_\bullet : X_\bullet \to X'_\bullet$ is called a **chain homotopy equivalence** if there is a chain map $g : X'_\bullet \to X_\bullet$ with $g_\bullet \circ f_\bullet \sim 1_{X_\bullet}$ and $f_\bullet \circ g_\bullet \sim 1_{X'_\bullet}$.

   In this case the chain complexes $X_\bullet$ und $X'_\bullet$ are called **chain homotopy equivalent** and one writes $X_\bullet \simeq X'_\bullet$.

Remark 6.3.2:

1. For given chain complexes $X_\bullet$, $X'_\bullet$, the chain maps $f_\bullet : X_\bullet \to X'_\bullet$ and chain homotopies between them form an abelian groupoid.

   The composite of two chain homotopies $h : f_\bullet \Rightarrow f'_\bullet$ and $h'_\bullet : f'_\bullet \Rightarrow f''_\bullet$ is the chain homotopy $h'_\bullet \circ h_\bullet = (h_n + h'_n)_{n\in\mathbb{Z}} : f_\bullet \Rightarrow f''_\bullet$. The identity morphisms are trivial chain homotopies $1_{f_\bullet} = (0)_{n\in\mathbb{Z}}$ and the inverse of $h_\bullet : f_\bullet \Rightarrow f'_\bullet$ is $h^{-1}_\bullet = (-h_n)_{n\in\mathbb{Z}} : f'_\bullet \Rightarrow f_\bullet$.

2. By 1. being chain homotopic defines an equivalence relation on the set of chain maps from $X_\bullet$ to $X'_\bullet$. It is compatible with the composition of morphisms:

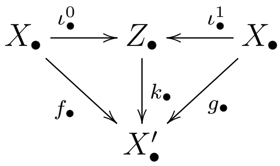
   For all chain maps $f_\bullet, f'_\bullet : X_\bullet \to X'_\bullet$ and $g_\bullet, g'_\bullet : X'_\bullet \to X''_\bullet$ and chain homotopies $h_\bullet : f_\bullet \Rightarrow f'_\bullet$ and $h'_\bullet : g_\bullet \Rightarrow g'_\bullet$, the family of morphisms $k_\bullet = (g_{n+1} \circ h_n + h'_n \circ f'_n)_{n\in\mathbb{Z}}$ is a chain homotopy $k_\bullet : g_\bullet \circ f_\bullet \Rightarrow g'_\bullet \circ f'_\bullet$ (Exercise).

3. This yields the **homotopy category of chain complexes** $K(R\text{-Mod})$ with
   - chain complexes in R-Mod as objects,
   - chain homotopy classes of chain maps in R-Mod as morphisms.

   Isomorphisms in $K(R\text{-Mod})$ are chain homotopy classes of chain homotopy equivalences.

Just as Definition 6.2.4 of a simplicial homotopy, Definition 6.3.1 is technical and not very illuminating. One wonders if chain homotopies $h_\bullet : f_\bullet \Rightarrow g_\bullet$ can also be formulated as *chain maps* between certain chain complexes, as it was done for simplicial homotopies in Definition 6.2.3 and continuous homotopies in Definition 6.1.1. This is indeed possible.

Lemma 6.3.3: Let $(X_\bullet, d_\bullet)$, $(X'_\bullet, d'_\bullet)$ be chain complexes in R-Mod and $f_\bullet, g_\bullet : X_\bullet \to X'_\bullet$ chain maps between them. Then chain homotopies $h_\bullet : f_\bullet \Rightarrow g_\bullet$ are in bijection with chain maps $k_\bullet : Z_\bullet \to X'_\bullet$ such that $k_\bullet \circ \iota^0_\bullet = f_\bullet$ and $k_\bullet \circ \iota^1_\bullet = g_\bullet$



where

- $Z_\bullet$ is the chain complex with

$$\begin{aligned} &Z_n = X_n \oplus X_n \oplus X_{n-1} \\ &d_n : Z_n \to Z_{n-1}, \quad (x, x', x'') \mapsto (d_n(x) - x'', d_n(x') + x'', -d_{n-1}(x'')), \end{aligned}$$

- the inclusions $\iota^i_\bullet : X_\bullet \to Z_\bullet$ for $i = 0, 1$ are given by

$$\iota^0_n : X_n \to Z_n, \quad x \mapsto (x, 0, 0) \qquad \iota^1_n : X_n \to Z_n, \quad x \mapsto (0, x, 0)$$

Proof:
A direct computation shows that $Z_\bullet$ is indeed a chain complex and the inclusions ι^0_n, ι^1_n define chain maps $\iota^0_\bullet, \iota^1_\bullet : X_\bullet \to Z_\bullet$.

By the universal property of the direct sum, an R-linear map $k_n : Z_n \to X'_n$ with $k_n \circ \iota^0_n = f_n$ and $k_n \circ \iota^1_n = g_n$ is given by $f_n, g_n : X_n \to X'_n$ and an R-linear map $h_{n-1} : X_{n-1} \to X'_n$ as

$$k_n(x, x', x'') = f_n(x) + g_n(x') + h_{n-1}(x'') \qquad x, x' \in X_n,\ x'' \in X_{n-1}.$$

The R-linear maps k_n define a chain map if and only if

$$\begin{aligned} &d'_n \circ k_n(x, x', x'') = d'_n \circ f_n(x) + d'_n \circ g_n(x') + d'_n \circ h_{n-1}(x'') \\ =&k_{n-1} \circ d_n(x, x', x'') = f_{n-1} \circ d_n(x) - f_{n-1}(x'') + g_{n-1} \circ d_n(x') + g_{n-1}(x'') - h_{n-2} \circ d_{n-1}(x''). \end{aligned}$$

By setting $x' = x'' = 0$, $x = x'' = 0$ and $x = x' = 0$, one finds that is the case if and only if $f_\bullet$ and $g_\bullet$ are chain maps and $h_\bullet : f_\bullet \Rightarrow g_\bullet$ is a chain homotopy. □

It is also possible to understand the chain complex $Z_\bullet$ from Lemma 6.3.3 as a counterpart of the topological space $[0,1]\times X$ in Definition 6.1.1 and the simplicial object $\Delta^1\sqcup S$ in Definition 6.2.3. This requires the tensor product of chain complexes and yields $Z_\bullet=\Delta^1_\bullet\otimes X_\bullet$ with a chain complex $\Delta^1_\bullet$ that describes the unit interval. This makes the relation more direct, but requires additional background on tensor products and monoidal categories.

The central property of chain homotopies that makes them fundamental in homological algebra is that they do not affect the induced maps between the homologies: chain homotopic chain maps induce the same maps between the homologies. Thus, the homologies descend to functors from the homotopy category of chain complexes from Remark 6.3.2, 3. to the category R-Mod. In particular, chain homotopy equivalent chain complexes have isomorphic homologies.

Proposition 6.3.4: Let R be a ring.

1. Chain homotopic chain maps in R-Mod induce the same morphisms on the homologies: if $f_\bullet\sim g_\bullet$ then $H_n(f_\bullet)=H_n(g_\bullet)$ for all $n\in\mathbb{Z}$.
2. The nth homology induces a functor $H_n: K(R\text{-Mod})\to R\text{-Mod}$ for all $n\in\mathbb{Z}$.
3. Chain homotopy equivalences induce isomorphisms on the homologies: if $X_\bullet\simeq X'_\bullet$, then $H_n(X_\bullet)\cong H_n(X'_\bullet)$ for all $n\in\mathbb{Z}$.

Proof:
Let $f_\bullet, g_\bullet: X_\bullet\to X'_\bullet$ be chain maps and $h_\bullet: f_\bullet\Rightarrow g_\bullet$ a chain homotopy. Then one has

$$\begin{aligned}H_n(g_\bullet)[x]-H_n(f_\bullet)[x]&=[g_n(x)]-[f_n(x)]=[g_n(x)-f_n(x)]=[h_{n-1}\circ d_n(x)+d'_{n+1}\circ h_n(x)]\\&=[h_{n-1}(d_n(x))]=[h_{n-1}(0)]=0 \qquad \forall x\in\ker(d_n)\end{aligned}$$
$$\Rightarrow\quad H_n(f_\bullet)=H_n(g_\bullet).$$

The second and third claim follow directly from the first. □

With the standard chain complex functor from Proposition 5.5.5 and the normalised chain complex functor from Proposition 5.5.8 we can now relate simplicial homotopies to chain homotopies. This extends the Dold-Kan correspondence from Theorem 5.5.13 to simplicial homotopies and chain homotopies.

Theorem 6.3.5: Let $S,T:\Delta^{op}\to R$-Mod simplicial objects and $\alpha,\beta: S\Rightarrow T$ simplicial morphisms in R-Mod.

1. For every simplicial homotopy $h:\alpha\Rrightarrow\beta$, the morphisms $h_n=\Sigma_{i=0}^n(-1)^i h_n^i: S_n\to T_{n+1}$ define chain homotopies $h_\bullet:\alpha_\bullet\Rightarrow\beta_\bullet$ and $Nh_\bullet: N\alpha_\bullet\Rightarrow N\beta_\bullet$.
2. Every chain homotopy $h_\bullet: N\alpha_\bullet\Rightarrow N\beta_\bullet$ arises from a simplicial homotopy $h:\alpha\Rrightarrow\beta$.

Proof:
1. That the morphisms h_n define a chain homotopy $h_\bullet:\alpha_\bullet\Rightarrow\beta_\bullet$ follows by a direct computation from the defining relations (34) of the simplicial homotopy. For this, one splits the sums occurring in the boundary operators according to the three cases in (34)

$$d_{n+1}\circ h_n+h_{n-1}\circ d_n=\Sigma_{j=0}^n\Sigma_{i=0}^{n+1}(-1)^{i+j}d^i_{n+1}\circ h^j_n+\Sigma_{j=0}^{n-1}\Sigma_{i=0}^{n}(-1)^{i+j}h^j_{n-1}\circ d^i_n$$
$$=\Sigma_{j=0}^n\Sigma_{i=0}^{j-1}(-1)^{i+j}d^i_{n+1}\circ h^j_n+\Sigma_{j=0}^n d^j_{n+1}\circ h^j_n-d^{j+1}_{n+1}\circ h^j_n+\Sigma_{j=0}^n\Sigma_{i=j+2}^{n+1}(-1)^{i+j}d^i_{n+1}\circ h^j_n$$
$$+\Sigma_{j=0}^{n-1}\Sigma_{i=0}^{j}(-1)^{i+j}h^j_{n-1}\circ d^i_n+\Sigma_{j=0}^{n-1}\Sigma_{i=j+1}^{n}(-1)^{i+j}h^j_{n-1}\circ d^i_n$$

$$\overset{(34)}{=} \Sigma_{j=0}^{n}\Sigma_{i=0}^{j-1}(-1)^{i+j}h_{n-1}^{j-1}\circ d_n^i + d_{n+1}^0\circ h_n^0 - d_{n+1}^{n+1}\circ h_{n+1} + \Sigma_{j=0}^{n}\Sigma_{i=j+2}^{n+1}h_{n-1}^{j}\circ d_n^{i-1}$$
$$+ \Sigma_{j=0}^{n-1}\Sigma_{i=0}^{j}(-1)^{i+j}h_{n-1}^{j}\circ d_n^i + \Sigma_{j=0}^{n-1}\Sigma_{i=j+1}^{n}(-1)^{i+j}h_{n-1}^{j}\circ d_n^i$$
$$= d_{n+1}^0\circ h_n^0 - d_{n+1}^{n+1}\circ h_n^n = \beta_n - \alpha_n.$$

It is then sufficient to show that $h_n(NS_n)\subset NT_{n+1}$ for all $n\in\mathbb{N}_0$. With the defining relations (34) of the simplicial homotopy, we obtain

$$d_{n+1}^0\circ h_n = \Sigma_{j=0}^{n}(-1)^j d_{n+1}^0\circ h_n^j \overset{(34)}{=} d_{n+1}^0\circ h_n^0 + \Sigma_{j=1}^{n}(-1)^j h_{n-1}^{j-1}\circ d_n^0 = f_n + \Sigma_{j=1}^{n}(-1)^j h_{n-1}^{j-1}\circ d_n^0$$
$$\Rightarrow\quad h_n(\ker(d_n^0))\subset\ker(d_{n+1}^0)$$
$$d_{n+1}^i\circ h_n = \Sigma_{j=0}^{n}(-1)^j d_{n+1}^i\circ h_n^j \qquad 0<i\leq n$$
$$= \Sigma_{j=0}^{i-2}(-1)^j d_{n+1}^i\circ h_n^j + (-1)^i d_{n+1}^i\circ h_n^i + (-1)^{i-1}d_{n+1}^i\circ h_n^{i-1} + \Sigma_{j=i+1}^{n}d_{n+1}^i\circ h_n^j$$
$$\overset{(34)}{=} \Sigma_{j=0}^{i-2}(-1)^j h_n^j\circ d_n^{i-1} + \Sigma_{j=i+1}^{n}(-1)^j h_{n-1}^{j-1}\circ d_n^i$$
$$\Rightarrow\quad h_n(\ker(d_n^i))\subset\ker(d_{n+1}^i)$$

This shows that $h_n(NS_n)\subset NT_{n+1}$ and proves the claim for $N\alpha_\bullet, N\beta_\bullet : NS_\bullet\to NT_\bullet$.

2. We do not give this proof here, because it is lengthy and technical. For a sketch of proof, see [W, Section 8.4]. □

Theorem 6.3.5 is useful whenever one has a functor that assigns simplicial objects in R-Mod to objects in a given category $\mathcal{C}$ and a notion of homotopy in $\mathcal{C}$ that induces simplicial homotopies in R-Mod. An example is the singular chain complex functor $C_\bullet(-,k):\mathrm{Top}\to\mathrm{Ch}_{k\text{-Mod}}$ from Example 5.5.6, 1. that defines the singular homologies of a topological space with coefficients in a commutative ring k. Theorem 6.3.5 implies that this functor sends continuous homotopies in Top to chain homotopies. It follows that the singular homologies of a topological space depend only on its homotopy type.

Corollary 6.3.6: Let k be a commutative ring and X, Y topological spaces.

1. Homotopic maps $f, g: X\to Y$ induce the same morphisms on the singular homologies:

$$f\sim g \quad\Rightarrow\quad H_n(f,k)=H_n(g,k): H_n(X,k)\to H_n(Y,k)\quad \forall n\in\mathbb{N}_0.$$

2. If X and Y are homotopy equivalent, then their homologies are isomorphic

$$X\simeq Y\quad\Rightarrow\quad H_n(X,k)\cong H_n(Y,k)\quad\forall n\in\mathbb{N}_0.$$

Proof:
By Example 5.5.6, 1. the singular chain complex functor $C_\bullet(-,k):\mathrm{Top}\to\mathrm{Ch}_{k\text{-Mod}}$ is the composite of the functor Sing : Top → SSet from Example 5.2.7, 1, the free generation functor $F_k:\mathrm{SSet}\to k\text{-Mod}^{\Delta^{op}}$ and the standard chain complex functor from Proposition 5.5.5.

1. By Proposition 6.2.7 a continuous homotopy $h:[0,1]\times X\to Y$ from f to g induces a simplicial homotopy $\mathrm{Sing}(h):\mathrm{Sing}(f)\Rrightarrow\mathrm{Sing}(g)$. Post-composition with $F_k:\mathrm{SSet}\to k\text{-Mod}^{\Delta^{op}}$ yields a simplicial homotopy between simplicial maps in k-Mod and post-composition with the standard chain complex functor then a chain homotopy $C_\bullet(h,k):C_\bullet(f,k)\Rightarrow C_\bullet(g,k)$ by Theorem 6.3.5. Proposition 6.3.4, 1. then implies 1.

2. If $f : X \to Y$ and $f' : Y \to X$ are continuous maps with $f' \circ f \sim \mathrm{id}_X$ and $f \circ f' \sim \mathrm{id}_Y$ then by 1. the induced chain maps $C_\bullet(f,k) : C_\bullet(X,k) \to C_\bullet(Y,k)$ and $C_\bullet(f',k) : C_\bullet(Y,k) \to C_\bullet(X,k)$ form a chain homotopy equivalence, and Proposition 6.3.4, 3. implies claim 2. □

Another example where Theorem 6.3.5 can be applied is the functor $C_\bullet(-,k) : \mathrm{Cat} \to \mathrm{Ch}_{k\text{-Mod}}$ from Example 5.5.6, 2. that assigns to

- a small category $\mathcal{C}$ the chain complex $C_\bullet(\mathcal{C},k)$ with
$$C_n(\mathcal{C},k) = \langle N(\mathcal{C})_n \rangle_k \qquad d = \Sigma_{i=0}^n (-1)^i \langle N(\delta_n^i)\rangle_k : C_n(\mathcal{C},k) \to C_{n-1}(\mathcal{C},k)$$
- a functor $F : \mathcal{C} \to \mathcal{D}$ the chain map $C_\bullet(F,k) : C_\bullet(\mathcal{C},k) \to C_\bullet(\mathcal{D},k)$ with components $C_n(F,k) = \langle N(F)_n \rangle_k : \langle N(\mathcal{C})_n \rangle_k \to \langle N(\mathcal{D})_n \rangle_k$.

The homologies of these chain complexes define functors $H_n(-,k) : \mathrm{Cat} \to k\text{-Mod}$. By applying Theorem 6.3.5 one can show that these functors do not distinguish categories that are related by adjunctions. This can be used to show that a given functor cannot have a left or right adjoint or, more generally, that there cannot be an adjunction between two categories.

Corollary 6.3.7:
Let k be a commutative ring, $\mathcal{C}, \mathcal{D}$ be small categories and $F, G : \mathcal{C} \to \mathcal{D}$ functors.

- If there is a natural transformation $\tau : F \Rightarrow G$, then F and G induce the same maps on the homologies: $H_n(F,k) = H_n(G,k) : H_n(\mathcal{C},k) \to H_n(\mathcal{D},k)$ for all $n \in \mathbb{N}_0$,
- If $\mathcal{C}$ and $\mathcal{D}$ are related by an adjunction, then $H_n(\mathcal{C},k) \cong H_n(\mathcal{D},k)$ for all $n \in \mathbb{N}_0$.

Proof:
By Example 5.5.6, 2. the functor $C_\bullet(-,k) : \mathrm{Cat} \to \mathrm{Ch}_{k\text{-Mod}}$ is obtained by composing the simplicial nerve $N : \mathrm{Cat} \to \mathrm{SSet}$ from Example 5.2.7, 2. and Lemma 5.4.1 with the functor $F_k : \mathrm{SSet} \to k\text{-Mod}^{\Delta^{op}}$ and then the standard chain complex functor from Proposition 5.5.5.

By Theorem 6.2.9 a natural transformation $\tau : F \Rightarrow G$ induces a simplicial homotopy $h : N(F) \Rrightarrow N(G)$. Post-composition with $F_k : \mathrm{SSet} \to k\text{-Mod}^{\Delta^{op}}$ yields a simplicial homotopy between the induced simplicial morphisms in k-Mod and post-composition with the standard chain complex functor a chain homotopy $C_\bullet(\tau,k) : C_\bullet(F,k) \Rightarrow C_\bullet(G,k)$ by Theorem 6.3.5. Proposition 6.3.4, 1. implies the first claim. The second follows as in Corollary 6.3.6. □

For topological spaces that are classifying spaces $B\mathcal{C}$ or BG of a category $\mathcal{C}$ or a group G, the homologies of the singular chain complexes from Corollary 6.3.6 and the homologies from Corollary 6.3.7 are in fact related. This result is well-known for classifying spaces of groups.

Remark 6.3.8: For any group G the group homologies coincide with the singular homologies of its classifying space
$$H_n(G,\mathbb{Z}) \cong H_n(BG,\mathbb{Z}) \qquad \forall n \in \mathbb{N}_0.$$

This follows, because the classifying spaces are geometric realisations of the nerve from Example 5.2.7, 2. and Lemma 5.4.1. Theorem 5.3.3 states that they are CW-complexes whose cells are in bijection with non-degenerate elements of the simplicial nerve. By applying techniques from algebraic topology one can efficiently compute the homologies of CW-complexes by counting the cells in each dimension. This relates the singular homologies $H_n(BG,\mathbb{Z})$ of the classifying space BG to the homologies of the associated chain complex $C_\bullet(G,\mathbb{Z})$ from Corollary 6.3.6. A sketch of proof is given in [W, Example 8.2.3]. For more background see a textbook on group (co)homology, such as [B] or [AM] or the book [Mo] on classifying spaces of categories.

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7 Kan complexes and quasicategories

7.1 Kan complexes

In this section we generalise the concept of homotopy groups from topological spaces to certain simplicial sets. The simplicial sets for which this is possible are called *Kan complexes*. Their homotopy groups generalise the homotopy groups of topological spaces from Theorem 6.1.5 and are related to them via the singular nerve and the geometric realisation functor.

Studying Kan complexes will also lead to a deeper understanding of the simplicial sets arising as nerves of topological spaces, groups and categories and finally to quasicategories or ∞-categories, a generalisation of categories that is a current research topic in mathematics. Our main references for this chapter are the books [M] and [GJ] and the preprint [F].

The homotopy groups of topological spaces in Section 6.1 were obtained by considering continuous maps $f : [0,1]^{\times n} \to X$ with $f(\partial[0,1]^{\times n}) = \{x\}$, up to homotopies relative to $\partial[0,1]^{\times n}$. To generalise this to simplicial sets, note that the choice of the unit cube $[0,1]^{\times n}$ as the domain of these maps is not essential. It is convenient for working with coordinates, but continuous maps $f : \Delta^n \to X$ with $f(\partial\Delta^n) = \{x\}$ would yield the same homotopy groups.

Hence, it makes sense to replace continuous maps $f : [0,1]^{\times n} \to X$ with simplicial maps $\alpha : \Delta^n \Rightarrow X$. To generalise the condition $f(\partial[0,1]^{\times n}) = \{x\}$, we need to define a boundary $\partial\Delta^n$. This is achieved by considering simplicial subsets. Just as a subspace $U \subset X$ of a topological space X is given by a monomorphism $\iota : U \to X$ in Top, one can define a simplicial subset of $S : \Delta^{op} \to$ Set as a simplicial set $T : \Delta^{op} \to$ Set with a monomorphism $\iota : T \Rightarrow S$ in SSet. As monomorphisms can be realised as limits (Exercise 13) and limits in the functor category SSet are pointwise by Proposition 2.3.2 a monomorphism $\iota : T \Rightarrow S$ in SSet is a simplicial map $\iota : S \Rightarrow T$ such that $\iota_n : T_n \to S_n$ is injective for all $n \in \mathbb{N}_0$. This allows one to identify T_n with a subset of S_n for all $n \in \mathbb{N}_0$ and yields the following definition.

Definition 7.1.1: Let $S : \Delta^{op} \to$ Set be a simplicial set.

1. A **simplicial subset** $T \subset S$ is a simplicial set $T : \Delta^{op} \to S$ with $T_n \subset S_n$ for all $n \in \mathbb{N}_0$ and $S(\alpha)(t) = T(\alpha)(t)$ for all monotonic maps $\alpha : [m+1] \to [n+1]$ and $t \in T_n$.
2. The simplicial subset $\langle X \rangle \subset S$ **generated by** a subset $X \subset S_n$ for some $n \in \mathbb{N}_0$ is the smallest simplicial subset of S containing X. For all $m \in \mathbb{N}_0$ one has

$$\langle X \rangle_m = \{S(\tau)x \mid x \in X, \tau : [m+1] \to [n+1] \text{ monotonic}\}.$$

We now apply Definition 7.1.1 to define the faces and boundaries of the simplicial sets $\Delta^n = \mathrm{Hom}(-, [n+1]) : \Delta^{op} \to$ Set from Example 5.2.7, 3. The boundary $\partial\Delta^n$ of a topological standard n-simplex Δ^n is the union of its faces. The faces are the images of Δ^{n-1} under the face maps $f_i^n : \Delta^{n-1} \to \Delta^n$ from Definition 5.1.2.

The counterparts of the face maps in SSet are the simplicial maps $\mathrm{Hom}(-, \delta_n^i) : \Delta^{n-1} \Rightarrow \Delta^n$. We thus define the ith face of the standard n-simplex Δ^n in SSet as the simplicial subset generated by $\{\delta_n^i\} \subset \Delta_{n-1}^n$ and its boundary $\partial\Delta^n$ as the simplicial subset generated by $\{\delta_n^0, ..., \delta_n^n\} \subset \Delta_{n-1}^n$. As we will see in the following, it is also important to consider subsets of the boundary with exactly one face removed, the *horns* of Δ^n.

Definition 7.1.2: Let $\Delta^n = \mathrm{Hom}_\Delta(-, [n+1]) : \Delta^{op} \to \mathrm{Set}$ and $k \in \{0, \ldots, n\}$.

1. The kth **face** $\partial_k \Delta^n$ is the simplicial subset of Δ^n generated by $\delta^k_n \in \Delta^n_{n-1}$.
2. The **boundary** $\partial\Delta^n$ is the simplicial subset of Δ^n generated by $\{\delta^0_n, ..., \delta^n_n\} \subset \Delta^n_{n-1}$. It is also called the **simplicial** $(n-1)$**-sphere**.
3. The kth **horn** Λ^k_n is the simplicial subset of Δ^n generated by $\{\delta^0_n, ..., \delta^{k-1}_n, \delta^{k+1}_n, ..., \delta^n_n\}$. The horns Λ^1_n, ... Λ^{n-1}_n are called **inner horns**, the horns Λ^0_n and Λ^n_n **outer horns**.

Remark 7.1.3:

1. For $n = 0$ one has $\partial\Delta^0 = \emptyset$, the empty simplicial set with $(\partial\Delta^0)_k = \emptyset$ for all $k \in \mathbb{N}_0$.
2. The boundary $\partial\Delta^n$ is the $(n-1)$-skeleton of Δ^n:
   One has $(\partial\Delta^n)_k = \Delta^n_k$ for all $k < n$ and all elements of $(\partial\Delta^n)_k$ for $k \geq n$ are degenerate. The set $(\partial\Delta^n)_k$ contains precisely the non-surjective monotonic maps $\alpha : [k+1] \to [n+1]$ (Exercise 59).
3. The simplicial set $\partial\Delta^n$ can also be realised as the coequaliser
   $$\partial\Delta^n = \mathrm{coequ}\left(\amalg_{0\leq i<j\leq n}\Delta^{n-2} \underset{v}{\overset{u}{\rightrightarrows}} \amalg_{0\leq i\leq n}\Delta^{n-1}\right),$$
   where $u \circ \iota_{i,j} = \iota_j \circ \mathrm{Hom}(-, \delta^i_{n-1})$ and $v \circ \iota_{ij} = \iota_i \circ \mathrm{Hom}(-, \delta^{j-1}_{n-1})$ (Exercise 60). This corresponds to taking an $(n-1)$-simplex for each of the n faces of Δ^n and identifying them along their shared $(n-2)$-faces in Δ^n.
4. The sets $(\Lambda^k_n)_j$ for the kth horn Λ^k_n contain exactly the monotonic maps $\alpha : [j+1] \to [n+1]$ with $\{0, 1, \ldots, k-1, k+1, \ldots, n\} \not\subset \mathrm{im}(\alpha)$.
5. The kth horn can also be defined as the coequaliser
   $$\Lambda^k_n = \mathrm{coequ}\left(\amalg_{0\leq i<j\leq n}\Delta^{n-2} \underset{v}{\overset{u}{\rightrightarrows}} \amalg_{0\leq i\neq k\leq n}\Delta^{n-1}\right)$$
   with u, v as in 3. This corresponds to taking an $(n-1)$-simplex for each face of Δ^n except the face opposite k and identifying them along shared $(n-2)$-faces in Δ^n.
6. The geometric realisation of the n-simplex, faces, boundary and horns are given by the corresponding quantities for the topological n-simplex from Definition 5.1.2
   $$|\Delta^n| = \Delta^n, \qquad |\partial_k\Delta^n| = f^n_k(\Delta^{n-1}), \qquad |\partial\Delta^n| = \partial\Delta^n, \quad |\Lambda^n_k| = \cup_{0\leq j\neq k\leq n} f^n_j(\Delta^{n-1})$$
   This follows with Theorem 5.3.3 by determining the non-degenerate elements in the simplicial sets Δ^n, $\partial_k\Delta^n$, $\partial\Delta^n$, Λ^k_n.

In Section 6.1 we defined the homotopy groups of a topological space X as homotopy classes of continuous maps $f : [0,1]^{\times n} \to X$ with $f(\partial[0,1]^{\times n}) = \{x\}$ or, equivalently, homotopy classes of continuous maps $f : S^n \to X$ relative to a basepoint. It is therefore plausible to consider simplicial maps $\alpha : \partial\Delta^{n+1} \Rightarrow S$ in SSet and to interpret the simplicial set $\partial\Delta^{n+1}$ as a simplicial counterpart of the n-sphere $S^n = \partial D^{n+1}$ in Top.

For the homotopy groups in Top it is essential, if a continuous map $f : S^{n-1} \to X$ can be extended to a map $g : D^n \to X$ with $g|_{\partial D^n} = f$. If this is the case, then f is homotopic to a constant map $f' : S^{n-1} \to X$ and $[f] = [f'] \in \pi_{n-1}(x, X)$ is trivial (Exercise 53). Only continuous maps $f : S^{n-1} \to X$ that cannot be extended to continuous maps $g : D^n \to X$ give rise to non-trivial elements of homotopy groups.

Analogously, we can ask if a simplicial map $\alpha : \partial\Delta^n \Rightarrow S$ can be *filled*, extended to a simplicial map $\beta : \Delta^n \Rightarrow S$. One can also consider simplicial maps $\alpha : \Lambda^k_n \Rightarrow S$ for the n-horns of the standard n-simplex and ask if they can be *completed* to simplicial maps $\alpha : \partial\Delta^n \Rightarrow S$ or *filled* to simplicial maps $\alpha : \Delta^n \Rightarrow S$.

Definition 7.1.4: Let $S : \Delta^{op} \to \text{Set}$ be a simplicial set.

1. A simplicial $(n-1)$**-cycle** or $(n-1)$**-sphere** on S is a simplicial map $\alpha : \partial\Delta^n \Rightarrow S$.
2. A simplicial $(n-1)$-cycle $\alpha : \partial\Delta^n \Rightarrow S$ is called a simplicial $(n-1)$**-boundary**, if there is a simplicial map $\beta : \Delta^n \Rightarrow S$ with $\beta \circ \iota = \alpha$ for the inclusion $\iota : \partial\Delta^n \Rightarrow \Delta^n$

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{\alpha} & S \\ {\scriptstyle\iota}\big\downarrow & \nearrow_{\beta} & \\ \Delta^n & & \end{array}$$

3. A (k,n)**-horn** on S is a simplicial map $\alpha : \Lambda^n_k \Rightarrow S$. It is called an **outer horn** for $k \in \{0, n\}$ and else an **inner horn**.
4. A **completion** of a (k,n)-horn $\alpha : \Lambda^k_n \Rightarrow S$ is a simplicial map $\beta : \partial\Delta^n \Rightarrow S$ with $\beta \circ \iota = \alpha$ for the inclusion $\iota : \Lambda^k_n \Rightarrow \partial\Delta^n$

$$\begin{array}{ccc} \Lambda^k_n & \xrightarrow{\alpha} & S \\ {\scriptstyle\iota}\big\downarrow & \nearrow_{\beta} & \\ \partial\Delta^n & & \end{array}$$

5. A **filler** of a (k,n)-horn $\alpha : \Lambda^k_n \Rightarrow S$ is a simplicial map $\beta : \Delta^n \Rightarrow S$ with $\beta \circ \iota = \alpha$ for the inclusion $\iota : \Lambda^k_n \Rightarrow \Delta^n$

$$\begin{array}{ccc} \Lambda^k_n & \xrightarrow{\alpha} & S \\ {\scriptstyle\iota}\big\downarrow & \nearrow_{\beta} & \\ \Delta^n & & \end{array}$$

Remark 7.1.5:

1. A simplicial $(n-1)$-cycle $\alpha : \partial\Delta^n \Rightarrow S$ corresponds to an $(n+1)$-tuple $(x_0, ..., x_n)$ of elements $x_i = \alpha_{n-1}(\delta^i_n) \in S_{n-1}$ with $d^i_{n-1}(x_j) = d^{j-1}_{n-1}(x_i)$ for all $0 \le i < j \le n$.

   This is shown in Exercise 60 and also follows from Remark 7.1.3, 3.

2. A simplicial $(n-1)$-cycle as in 1. is a simplicial $(n-1)$-boundary if and only if there is an element $y \in S_n$ with $d^i_n(y) = x_i$ for all $0 \le i \le n$.

   Simplicial maps $\beta : \Delta^n \Rightarrow S$ are in bijection with elements $y = \beta_n(1_{[n+1]}) \in S_n$ by the Yoneda lemma. The condition $\beta \circ \iota = \alpha$ then states that

$$x_i = \alpha_{n-1}(\delta^i_n) = \beta_{n-1}(\delta^i_n) = \beta_{n-1}(1_{[n+1]} \circ \delta^i_n) = S(\delta^i_n)\beta_n(1_{[n+1]}) = d^i_n(y) \qquad i = 0, ..., n.$$

3. A (k,n)-horn $\alpha : \Lambda^k_n \Rightarrow S$ on S corresponds to an n-tuple $(x_0, ..., x_{k-1}, x_{k+1}, ..., x_n)$ of elements $x_j = \alpha_{n-1}(\delta^j_n) \in S_{n-1}$ such that $d^i_{n-1}(x_j) = d^{j-1}_{n-1}(x_i)$ for $0 \leq i < j \leq n$ and $i, j \neq k$. This follows by expressing Λ^k_n as a coequaliser as in Remark 7.1.3, 5.

4. A completion of a (k,n)-horn that is given by an n-tuple $(x_0, ..., x_{k-1}, x_{k+1}, ..., x_n)$ as in 3. is an $(n-1)$-cycle $(x_0, ..., x_{k-1}, x_k, x_{k+1}, ..., x_n)$ with $x_i \in S_{n-1}$.

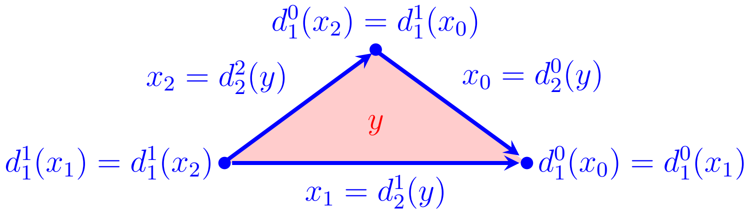
5. A filler of a (k,n)-horn given by an n-tuple $(x_0, ..., x_{k-1}, x_{k+1}, ...x_n)$ as in 3. is an element $y \in S_n$, whose boundary is a completion of the horn:

$$d_n(y) := (d^0_n(y), ..., d^n_n(y)) = (x_0, ..., x_{k-1}, d^k_n(y), x_{k+1}..., x_n).$$

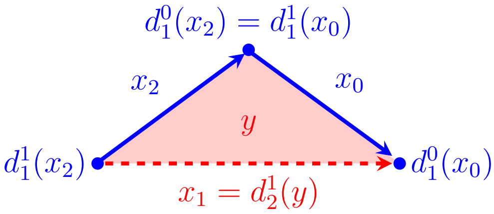
   This follows as in 2, because simplicial maps $\beta : \Delta^n \Rightarrow S$ are in bijection with elements $x = \beta_n(1_{[n+1]}) \in S_n$ by the Yoneda lemma.

For $n = 2$, these structures can be visualised as follows, and in the case $n = 3$ there are analogous pictures involving tetrahedra.

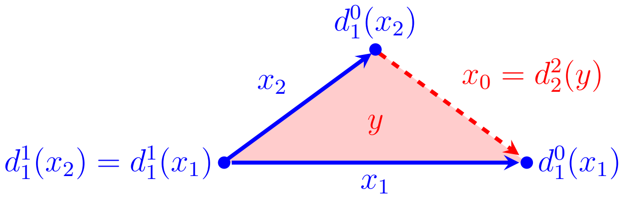
simplicial 1-boundary $d(y) = (x_0, x_1, x_2)$



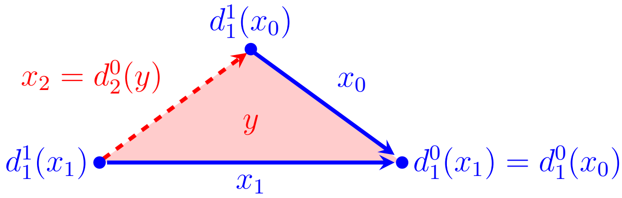
filler y of (1,2)-horn (x_0, x_2)



filler y of (0,2)-horn (x_1, x_2)



filler y of (2,2)-horn (x_0, x_1)



The names *boundaries* and *cycles* are reminiscent of simplicial homologies, and this is not a coincidence. One can show that the boundary $d_n(y) = (d^0_n(y), ..., d^n_n(y))$ for an element $y \in S_n$ is always a simplicial $(n-1)$-cycle (Exercise 60).

Remarks 7.1.5, 2. and 5. also show what is special about simplicial sets, simplicial maps and simplicial standard simplexes Δ^n. The category SSet and the standard n-simplex in SSet allow one to describe simplicial maps $\alpha : \Delta^n \Rightarrow S$ via the Yoneda lemma and hence to identify them with elements of the set S_n. This is a useful feature that will be important in the following.

Using Definition 7.1.4 and Remark 7.1.5 we can now state the essential property that distinguishes Kan complexes from more general simplicial sets.

Definition 7.1.6: A simplicial set $X : \Delta^{op} \to \mathrm{Set}$ is called **fibrant** or a **Kan complex** if it satisfies the **Kan condition**: every horn $\alpha : \Lambda^k_n \Rightarrow X$ with $k \in \{0, \ldots, n\}$ has a filler.

We denote by Kan $\subset$ SSet the full subcategory of Kan complexes in SSet.

Remark 7.1.7: By Remark 7.1.5, a simplicial set X is a Kan complex if and only if for each $n \in \mathbb{N}$ and each n-tuple $(x_0, ..., x_{k-1}, x_{k+1}, ..., x_n)$ with $x_j \in X_{n-1}$ and $d^i_{n-1}(x_j) = d^{j-1}_{n-1}(x_i)$ for $0 \leq i < j \leq n$, $i, j \neq k$, there is an $y \in X_n$, the **horn filler**, with $x_i = d^i_n(y)$ for $0 \leq i \neq k \leq n$.

Example 7.1.8:

1. The simplicial set $\Delta^n = \mathrm{Hom}(-, [n+1]) : \Delta^{op} \to \mathrm{Set}$ is not a Kan complex for $n > 0$:

   The horn $\alpha : \Lambda^0_2 \Rightarrow \Delta^n$ given by the pair (x_1, x_2) with $x_1 : [2] \to [n+1]$, $0, 1 \mapsto 0$ and $x_2 : [2] \to [n+1]$, $0 \mapsto 0$, $1 \mapsto 1$ satisfying $d^1_1(x_1) = x_1 \circ \delta^1_1 = x_2 \circ \delta^1_1 = d^1_1(x_2) : 0 \mapsto 0$ does not have a filler. A map $x : [3] \to [n+1]$ with $d^1_2(x) = x \circ \delta^1_2 = x_1$ and $d^2_2(x) = x \circ \delta^2_2 = x_2$ cannot be monotonic, because

   $$x(0) = x \circ \delta^2_2(0) = x_2(0) = 0,\; x(1) = x \circ \delta^2_2(1) = x_2(1) = 1,\; x(2) = x \circ \delta^1_2(1) = x_1(1) = 0.$$

2. For the same reasons, the simplicial sets $\partial\Delta^n$, Λ^k_n do not satisfy the Kan condition.

3. With the same argument one can show that the simplicial set $S^{\mathcal{K}} : \Delta^{op} \to \mathrm{Set}$ defined by an ordered combinatorial simplicial complex $\mathcal{K}$ is not a Kan complex, unless it contains only degenerate simplexes in all degrees $n > 0$.

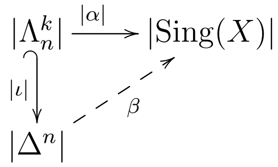
Despite these counterexamples, many important examples of simplicial sets are Kan complexes. In particular, this includes all singular nerves of topological spaces, see Example 5.2.7, 1, and all simplicial sets obtained from simplicial groups by forgetting their group structure. A special case of the latter are simplicial R-modules for any ring R.

Proposition 7.1.9:
For every topological space X, the simplicial set $\mathrm{Sing}(X) : \Delta^{op} \to X$ is a Kan complex.

Proof:
As the singular functor Sing : Top $\to$ SSet is right adjoint to the geometric realisation functor Geom $= |\,|$: SSet $\to$ Top, we can use the adjunctions to transport fillers from Top to SSet.

For this, let $\eta : \mathrm{id} \Rightarrow \mathrm{SingGeom}$ and $\epsilon : \mathrm{GeomSing} \Rightarrow \mathrm{id}$ be the unit and counit of the adjunction with (i) $\mathrm{Sing}(\epsilon_X) \circ \eta_{\mathrm{Sing}(X)} = \mathrm{id}_{\mathrm{Sing}(X)}$ and (ii) $\epsilon_{|S|} \circ |\eta_S| = \mathrm{id}_{|S|}$ for all topological spaces X and simplicial sets S.

The image of the horn extension diagram under the functor Geom $= |\ \ | : \mathrm{SSet} \to \mathrm{Top}$ is



By Remark 7.1.3, 6. the topological horn $|\Lambda^k_n| \subset |\Delta^n|$ is a retract of $|\Delta^n|$: there is a continuous map $r : |\Delta^n| \to |\Lambda^k_n|$ with $r \circ |\iota| = \mathrm{id}$. Thus, $\beta = |\alpha| \circ r : |\Delta^n| \to |\mathrm{Sing}(X)|$ is a horn filler. The simplicial map $\beta' = \mathrm{Sing}(\epsilon_X) \circ \mathrm{Sing}(\beta) \circ \eta_{\Delta^n} : \Delta^n \Rightarrow \mathrm{Sing}(X)$ is a horn filler for α, as we have

$$\begin{aligned}\beta' \circ \iota &= \mathrm{Sing}(\epsilon_X) \circ \mathrm{Sing}(\beta) \circ \eta_{\Delta^n} \circ \iota \overset{\text{nat } \eta}{=} \mathrm{Sing}(\epsilon_X) \circ \mathrm{Sing}(\beta) \circ \mathrm{Sing}(|\iota|) \circ \eta_{\Lambda^k_n} \\ &= \mathrm{Sing}(\epsilon_X) \circ \mathrm{Sing}(\beta \circ |\iota|) \circ \eta_{\Lambda^k_n} = \mathrm{Sing}(\epsilon_X) \circ \mathrm{Sing}(|\alpha|) \circ \eta_{\Lambda^k_n} \overset{\text{nat } \eta}{=} \mathrm{Sing}(\epsilon_X) \circ \eta_{\mathrm{Sing}(X)} \circ \alpha \overset{\text{(i)}}{=} \alpha.\end{aligned}$$

□

Proposition 7.1.10: For every simplicial group $S : \Delta^{op} \to \mathrm{Grp}$, the underlying simplicial set $S : \Delta^{op} \to \mathrm{Set}$ is a Kan complex.

Proof:
Let $(x_0, ..., x_{k-1}, x_{k+1}, ..., x_n)$ be an n-tuple of elements $x_i \in S_{n-1}$ with $d^i_{n-1}(x_j) = d^{j-1}_{n-1}(x_i)$ for all $0 \leq i < j \leq n$ and $i, j \neq k$. We construct a sequence of elements $y_{-1}, y_1, \ldots, y_n = x \in S_n$ with $d^i_n(y_r) = x_i$ for all $0 \leq i \leq r$, $i \neq k$. We start with $y_{-1} = e \in S_n$. Suppose we already constructed $y_{-1}, ..., y_{r-1}$ satisfying this condition.

If $r = k$, we choose $y_k = y_{k-1}$. If $r \neq k$, we consider the element $z := x_r^{-1} \cdot d^r_n(y_{r-1}) \in S_{n-1}$ and compute for $k \neq i < r$

$$d^i_{n-1}(z) = d^i_{n-1}(x_r)^{-1} \cdot d^i_{n-1} \circ d^r_n(y_{r-1}) = d^i_{n-1}(x_r)^{-1} \cdot d^{r-1}_{n-1} \circ d^i_n(y_{r-1}) = d^i_{n-1}(x_r)^{-1} \cdot d^{r-1}_{n-1}(x_i) = e,$$

where we used first the compatibility between the face maps and the group multiplication in a simplicial group and in the last step the identity $d^i_{n-1}(x_r) = d^{r-1}_{n-1}(x_i)$ for $k \neq i < r$. This shows that $d^i_{n-1}(z) = e$ for $k \neq i < r$ and consequently $d^i_n \circ s^r_{n-1}(z) = s^{r-1}_{n-2} \circ d^i_{n-1}(z) = e$ by (22).

We define $y_r := y_{r-1} \cdot s^r_{n-1}(z)^{-1}$ and obtain with (22) for all $k \neq i \leq r$

$$\begin{aligned} i < r \qquad & d^i_n(y_r) = d^i_n(y_{r-1}) \cdot d^i_n \circ s^r_{n-1}(z) = d^i_n(y_{r-1}) = x_i \\ i = r \qquad & d^r_n(y_r) = d^r_n(y_{r-1}) \cdot d^r_n \circ s^r_{n-1}(z) = d^r_n(y_{r-1}) \cdot z^{-1} = d^r_n(y_{r-1}) \cdot d^r_n(y_{r-1})^{-1} x_r = x_r. \end{aligned}$$ □

Propositions 7.1.9 and 7.1.10 show that Kan complexes are abundant. Together with the counterexamples in Example 7.1.8, they also provide an intuition what properties of an object are required in order to obtain a Kan complex via a nerve construction. The Kan property requires that the elements of the sets S_n are invertible in a certain sense.

Roughly speaking, as soon as one has a composition one can fill the *inner* horns. However, in order to fill the *outer* horns, one needs to *invert* some $(n-1)$-simplexes. This is possible for nerves of topological spaces, due to the invertibility of paths, and for simplicial groups, as their elements are invertible by definition. It cannot be done for an ordered combinatorial complex or the standard n-simplex in SSet, because it creates problems with the ordering.

The following theorem confirms this intuition and provides additional motivation for Kan complexes, as well as a reason to weaken this concept eventually.

Theorem 7.1.11:

1. A simplicial set S is isomorphic to the nerve $N(\mathcal{C})$ of a small category $\mathcal{C}$ if and only if every inner horn on S has a *unique* filler.
2. A simplicial set S is isomorphic to the nerve $N(\mathcal{G})$ of a groupoid $\mathcal{G}$ if and only if every n-horn on S with $n \geq 2$ has a *unique* filler.
3. The nerve of a small category $\mathcal{C}$ is a Kan complex, if and only if $\mathcal{C}$ is a groupoid.

Proof:
1.1 Let $S = N(\mathcal{C})$ be the nerve of a small category $\mathcal{C}$ and $\alpha : \Lambda^k_n \Rightarrow S$ with $k \in \{1, ..., n-1\}$ an inner horn on S.

By Remark 7.1.5, 3. the inner horn α is given by a collection $(x_0, ..., x_{k-1}, x_{k+1}, ..., x_n)$ of points $x_i = \alpha_{n-1}(\delta^i_n) \in N(\mathcal{C})_{n-1}$ with $d^i_{n-1}(x_j) = d^{j-1}_{n-1}(x_i)$ for all $0 \leq i < j \leq n$, $i, j \neq k$.

By Lemma 5.4.1 the elements in $N(\mathcal{C})_{n-1}$ are sequences of $n-1$ composable morphisms in $\mathcal{C}$, and the boundary operators d^i_n are given by composing two consecutive morphisms for $1 \leq i \leq n-2$ or removing the first and last morphism, respectively, for $i = 0$ and $i = n-1$. The condition $d^i_{n-1}(x_j) = d^{j-1}_{n-1}(x_i)$ for all $0 \leq i < j \leq n-1$, $i, j \neq k$ then implies that there are objects $C_0, ..., C_n \in \text{Ob}\mathcal{C}$ and morphisms $f_i : C_{i-1} \to C_i$ such that

$$x_0 = \alpha_{n-1}(\delta^0_n) = C_1 \xrightarrow{f_2} \ldots \xrightarrow{f_n} C_n$$
$$x_i = \alpha_{n-1}(\delta^i_n) = C_0 \xrightarrow{f_1} \ldots \xrightarrow{f_{i-1}} C_{i-1} \xrightarrow{f_{i+1} \circ f_i} C_{i+1} \xrightarrow{f_{i+2}} \ldots \xrightarrow{f_n} C_n \qquad 0 \leq i \leq n, i \neq k$$

The Yoneda lemma states that a simplicial map $\beta : \Delta^n \Rightarrow N(\mathcal{C})$ is determined uniquely by $y = \beta_n(1_{[n+1]}) \in N(\mathcal{C})_n$, and the unique horn filler y with $d^i_n(y) = x_i$ for all $0 \leq i \leq n$, $i \neq k$ is

$$y = C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \ldots \xrightarrow{f_{n-1}} C_{n-1} \xrightarrow{f_n} C_n.$$

1.2. Suppose $S : \Delta^{op} \to \text{Set}$ is a simplicial set in which every inner horn has a unique filler.

1.2.(a) We construct a small category $\mathcal{C}$ with $N(\mathcal{C}) \cong S$. This is analogous to the construction of the homotopy category $h(S)$ from Definition 5.4.4, but with the composition of morphisms given by horn fillers:

- $\text{Ob}\mathcal{C} = S_0$,
- $\text{Mor}\mathcal{C} = S_1$ with elements $y \in S_1$ as morphisms $y : d^1_1(y) \to d^0_1(y)$,
- identity morphisms $1_x = s^0_0(x)$ for $x \in S_0 = \text{Ob}\mathcal{C}$,
- the composition of morphisms $x_0, x_2 \in S_1$ with $d^0_1(x_2) = d^1_1(x_0)$ given by $x_0 \circ x_2 = d^1_2(z)$, where $z \in S_2$ is the unique horn filler for (x_0, x_2).

The composition of morphisms defined this way has the right source and target. To show that $\mathcal{C}$ is a category, it remains to check (i) that $1_x = s^0_0(x)$ for $x \in S_0$ is indeed an identity morphism and (ii) the associativity of the composition.

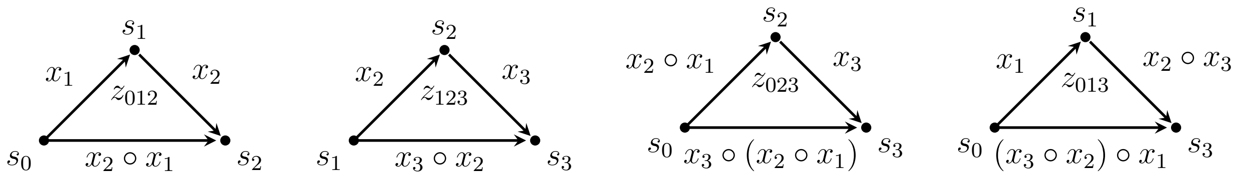
(i) For a morphism $x : d^1_1(x) \to d^0_1(x)$, the composites $1_{d^0_1(x)} \circ x$ and $x \circ 1_{d^1_1(x)}$ are given by the fillers of $(s^0_0 \circ d^0_1(x), x)$ and $(x, s^0_0 \circ d^1_1(x))$, respectively. They are $z = s^1_1(x)$ and $z' = s^0_1(x)$ with

$$d^2_2(z) = d^1_2(z) \overset{(22)}{=} x \qquad\qquad d^0_2(z) = d^0_2 \circ s^1_1(x) \overset{(22)}{=} s^0_0 \circ d^0_1(x)$$
$$d^0_2(z') = d^1_2(z') \overset{(22)}{=} x \qquad\qquad d^2_2(z') = d^2_2 \circ s^0_1(x) \overset{(22)}{=} s^0_0 \circ d^1_1(x).$$

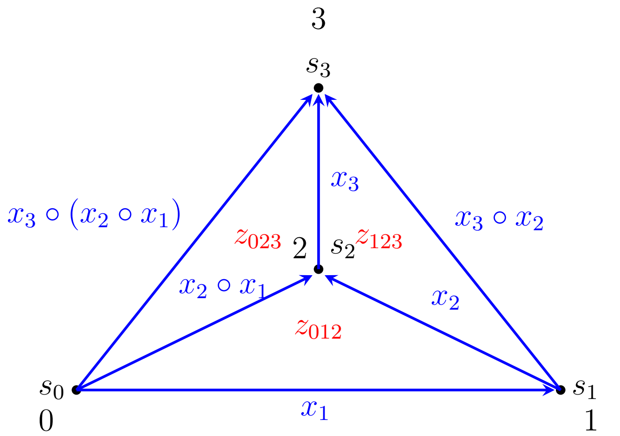
This shows that $1_{d_1^0(x)} \circ x = d_2^1(z) = x$ and $x \circ 1_{d_1^1(x)} = d_2^1(z') = x$.

(ii) To show associativity of the composition, consider composable morphisms $x_1 : s_0 \to s_1$, $x_2 : s_1 \to s_2$ and $x_3 : s_2 \to s_3$ given by elements $x_1, x_2, x_3 \in S_1$ and $s_0, s_1, s_2, s_3 \in S_0$. Let

- $z_{012} \in S_2$ be the filler of the inner horn (x_2, x_1) with $d_2^0(z_{012}) = x_2$, $d_2^2(z_{012}) = x_1$,
- $z_{123} \in S_2$ be the filler of the inner horn (x_3, x_2) with $d_2^0(z_{123}) = x_3$, $d_2^2(z_{123}) = x_2$,
- $z_{023} \in S_2$ be the filler of $(x_3, d_2^1(z_{012}))$ with $d_2^0(z_{023}) = x_3$ and $d_2^2(z_{023}) = d_2^1(z_{012})$,
- $z_{013} \in S_2$ be the filler of $(d_2^1(z_{123}), x_1)$ with $d_2^0(z_{013}) = d_2^1(z_{123})$ and $d_2^2(z_{013}) = x_1$.



Then $(z_{123}, z_{023}, z_{012})$ is a $(2,3)$-horn as we have $d_2^0(z_{023}) = d_2^0(z_{123})$ and $d_2^0(z_{012}) = d_2^2(z_{123})$. This is visualised by the following tetrahedron, whose top three faces are z_{012}, z_{023} and z_{123}.



(40)

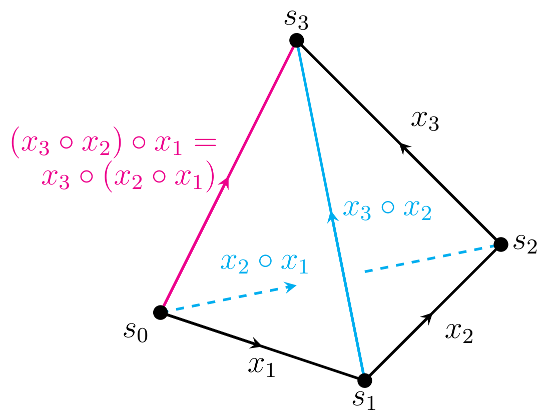
This horn has a unique filler $z \in S_3$ with $d_3^0(z) = z_{123}$, $d_3^1(z) = z_{023}$ and $d_3^3(z) = z_{012}$, and

$$d_2^0 \circ d_3^2(z) \overset{(22)}{=} d_2^1 \circ d_3^0(z) = d_2^1(z_{123}) \qquad d_2^2 \circ d_3^2(z) \overset{(22)}{=} d_2^2 \circ d_3^3(z) = d_2^2(z_{012}) = x_1.$$

Thus, by uniqueness of the filler we have $d_3^2(z) = z_{013}$, and this yields

$$(x_3 \circ x_2) \circ x_1 = d_2^1(z_{013}) = d_2^1 \circ d_3^2(z) \overset{(22)}{=} d_2^1 \circ d_3^1(z) = d_2^1(z_{023}) = x_3 \circ (x_2 \circ x_1).$$

The filler corresponds to the tetrahedron in (40), and the element z_{013} to its bottom face, which is invisible in (40). The associated three-dimensional picture is



1.2.(b) We construct a simplicial isomorphism $\phi : S \Rightarrow N(\mathcal{C})$:
The simplicial map $\phi_n : S_n \to N(\mathcal{C})_n$ assigns to $s \in S_n$ the element $C_0 \xrightarrow{f_1} \ldots \xrightarrow{f_n} C_n \in N(\mathcal{C})_n$ given by $C_i = S(\pi_i)(s) \in S_0$ and $f_j = S(\tau_j)(s) \in S_1$ for the morphisms

$$\pi_i : [1] \to [n+1],\ 0 \mapsto i \qquad \tau_j : [2] \to [n+1],\ 0 \mapsto j-1, 1 \mapsto j \quad 0 \leq i \leq n, 1 \leq j \leq n. \quad (41)$$

That it is a simplicial map can be verified either by direct computations or by noting that it defines a functor $[n]' \to \mathcal{C}$ for the associated poset category $[n]'$.

We show by induction that ϕ is a simplicial isomorphism. We have $\phi_0 = \mathrm{id}_{S_0} : S_0 \to N(\mathcal{C})_0$ and $\phi_1 = \mathrm{id}_{S_1} : S_1 \to N(\mathcal{C})_1$ by definition of $\mathcal{C}$. Suppose we already showed that $\phi_k : S_k \to N(\mathcal{C})_k$ is bijective for $0 \leq k \leq n-1$. Then we have for $1 \leq k \leq n-1$ a commuting diagram

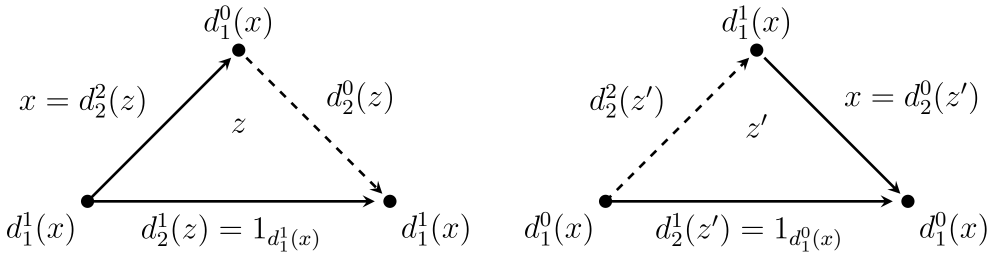
$$\begin{array}{ccc} S_n \cong \mathrm{Hom}_{\mathrm{SSet}}(\Delta^n, S) & \xrightarrow{\alpha \mapsto \phi_n \circ \alpha} & \mathrm{Hom}_{\mathrm{SSet}}(\Delta^n, N(\mathcal{C})) \cong N(\mathcal{C})_n \\ {\scriptstyle \beta \mapsto \beta \circ \iota} \downarrow \cong & & \cong \downarrow {\scriptstyle \beta \mapsto \beta \circ \iota} \\ \mathrm{Hom}_{\mathrm{SSet}}(\Lambda^k_n, S) & \xrightarrow[\gamma \mapsto \phi_{n-1} \circ \gamma]{} & \mathrm{Hom}_{\mathrm{SSet}}(\Lambda^k_n, N(\mathcal{C})) \end{array}$$

in which the lower horizontal arrow is an isomorphism by the induction hypothesis. The vertical arrows are isomorphisms, because every horn has a *unique* filler, which holds for S by assumption and for the nerve $N(\mathcal{C})$ by 1.1. This implies that the upper horizontal arrow is a bijection as well.

2.(a) Let now S be a simplicial set such that *every* horn $\alpha : \Lambda^n_k \Rightarrow S$ with $n \geq 2$ has a unique filler. We show that every morphism in the category $\mathcal{C}$ from 1.2 has an inverse. For this, consider a morphism $x : d^1_1(x) \to d^0_1(x)$ given by $x \in S_1$. Then the pair $(s^0_0 \circ d^1_1(x), x)$ is a $(0, 2)$-horn and the pair $(x, s^0_0 \circ d^0_1(x))$ a $(2, 2)$-horn due to the identities

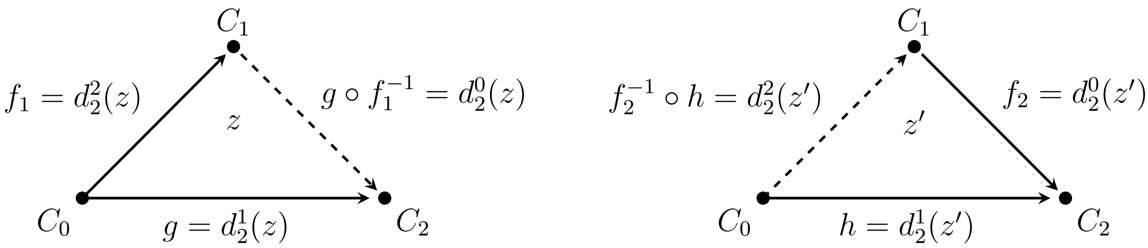
$$d^1_1 \circ s^0_0 \circ d^1_1(x) = d^1_1(x) \qquad d^1_1 \circ s^0_0 \circ d^0_1(x) = d^0_1(x).$$

Their unique fillers z and z' define morphisms $d^0_2(z) : d^0_1(x) \to d^1_1(x)$ and $d^2_2(z') : d^1_1(x) \to d^0_1(x)$ with $d^2_2(z) \circ x = 1_{d^1_1(x)}$ and $x \circ d^1_2(z') = 1_{d^0_1(x)}$.



Thus every morphism $f : C \to C'$ in $\mathcal{C}$ has a left and right inverse. A standard argument then shows that the left and right inverses are equal: for $g, h : C' \to C$ with $g \circ f = 1_C$ and $f \circ h = 1_{C'}$ one has $g = g \circ 1_{C'} = g \circ f \circ h = 1_C \circ h = h$.

2.(b) Suppose that $S = N(\mathcal{G})$ is the nerve of a groupoid $\mathcal{G}$. Then an outer horn $\alpha : \Lambda_2^0 \Rightarrow N(\mathcal{G})$ is given by a pair (g, f_1) and an outer horn $\beta : \Lambda_2^2 \Rightarrow N(\mathcal{G})$ by a pair (f_2, h) of morphisms with the unique horn fillers z and z' as shown below



$$z = C_0 \xrightarrow{f_1} C_1 \xrightarrow{g \circ f_1^{-1}} C_2 \qquad\qquad z' = C_0 \xrightarrow{f_2^{-1} h} C_1 \xrightarrow{f_2} C_2$$

The horn fillers for horns $\alpha : \Lambda_n^0 \Rightarrow N(\mathcal{G})$ and $\beta : \Lambda_n^n \Rightarrow N(\mathcal{G})$ for $n > 2$ are defined analogously.

Horn fillers for the outer horns $\alpha : \Lambda_1^0 \Rightarrow N(\mathcal{C})$ or $\beta : \Lambda_1^1 \Rightarrow \mathcal{C}$ are simply morphisms $f : x \to y$ in $\mathcal{C}$ from or to the given object in the horn. They exist for any category $\mathcal{C}$, as one can take the identity morphisms, but they are in general non-unique. □

7.2 Simplicial homotopy groups

In this section we generalise homotopy groups from topological spaces to Kan complexes. We will see that the Kan condition is needed to ensure that being simplicially homotopic is an equivalence relation. This does not hold for general simplicial maps by Remark 6.2.6, 2.

The general idea is to replace continuous maps $f : [0,1]^{\times n} \to X$ with $f(\partial[0,1]^{\times n}) = \{x\}$ in a pointed topological space (x, X) by simplicial maps $\alpha : \Delta^n \Rightarrow X$ from the standard n-simplex $\Delta^n = \mathrm{Hom}(-, [n+1])$ to a Kan complex X and homotopies by simplicial homotopies. This requires the notion of a simplicial basepoint and a simplicial homotopy relative to $\partial\Delta^n$.

We start by generalising the notion of a basepoint. A choice of a point $x \in X$ in a topological space X is equivalent to the choice of a continuous map $f : \{\bullet\} \to X$, where $\{\bullet\}$ is the terminal object in Top. To generalise this to simplicial sets, we consider the terminal object $T : \Delta^{op} \to \mathrm{Set}$ in SSet and simplicial maps $\alpha : T \Rightarrow X$.

As limits in functor categories are pointwise by Proposition 2.3.2, the terminal simplicial set $T : \Delta^{op} \to \mathrm{Set}$ is given by singleton sets $T_n = \{t_n\}$ for all $n \in \mathbb{N}_0$. A simplicial map $\star : T \Rightarrow X$ is given by a collection of points $\star_n = \star_n(t_n) \in X_n$ satisfying $d_n^i(\star_n) = \star_{n-1}$ and $s_n^i(\star_n) = \star_{n+1}$ for all $n \in \mathbb{N}_0$ and $0 \leq i \leq n$. Thus, specifying a simplicial map $\star : T \Rightarrow X$ amounts to specifying an element $\star = \star_0 \in S_0$ and setting $\star_n = s_{n-1}^0 \circ \ldots \circ s_0^0(\star_0)$ for all $n \in \mathbb{N}_0$.

Definition 7.2.1: A **pointed Kan complex** $(\star, X)$ is a Kan complex $X : \Delta^{op} \to \mathrm{Set}$ together with a simplicial map $\star : T \Rightarrow X$, the **basepoint**, given by a collection of points $\star_n \in X_n$ with $X(\tau)(\star_m) = \star_n$ for all monotonic maps $\tau : [n+1] \to [m+1]$.

Given this definition of a pointed Kan complex, it is plausible to replace continuous maps $f : [0,1]^{\times n} \to X$ with $f(\partial[0,1]^{\times n}) = \{x\}$ for a pointed topological space (x, X) by simplicial maps $\alpha : \Delta^n \Rightarrow X$ with $\alpha(\tau) = \star_m$ for all elements $\tau \in \partial\Delta_m^n \subset \Delta_m^n$.

By the Yoneda lemma, simplicial maps $\alpha : \Delta^n \Rightarrow X$ are in bijection with elements $x \in X_n$ and are given by $\alpha_m(\rho) = X(\rho)(x)$ for all monotonic maps $\rho : [m+1] \to [n+1]$. The condition $\alpha(\tau) = \star_m$ for $\tau \in \partial\Delta_m^n$ translates into the condition $d_n^i(x) = X(\delta_n^i)(x) = \star_{n-1}$ for all $0 \leq i \leq n$. For a point $x \in X_n$ with $d_n^i(x) = \star_{n-1}$ for $0 \leq i \leq n$ we denote by $\bar{x} : \Delta^n \Rightarrow X$ the associated simplicial map with $\bar{x}_m(\rho) = X(\rho)x$ for all monotonic maps $\rho : [m+1] \to [n+1]$.

In analogy with the continuous homotopies relative to $\partial[0,1]^{\times n}$ from Theorem 6.1.5, we then define a simplicial homotopy *relative to* $\partial\Delta^n$ from $\bar{x} : \Delta^n \Rightarrow X$ to $\bar{x}' : \Delta^n \Rightarrow X$ as a simplicial homotopy $h : \bar{x} \Rrightarrow \bar{x}'$ in the sense of Definitions 6.2.3 and 6.2.4 such that the associated maps $h_m : \Delta_m^n \to X_{m+1}$ satisfy an appropriate boundary condition on $\partial\Delta_m^n$.

Definition 7.2.2: Let $(\star, X)$ be a pointed Kan complex and $\bar{x}, \bar{x}' : \Delta^n \Rightarrow X$ the simplicial maps defined by elements $x, x' \in X_n$ with $d_n^i(x) = d_n^i(x') = \star_{n-1}$ for all $0 \leq i \leq n$.

A **homotopy relative to** $\partial\Delta^n$ from $\bar{x}$ to $\bar{x}'$ is a simplicial homotopy $h : \bar{x} \Rrightarrow \bar{x}'$ that satisfies $h_m^i(\tau) = \star_{m+1}$ for all $m \in \mathbb{N}_0$, $0 \leq i \leq m$ and $\tau \in \partial\Delta_m^n$.

If there is a homotopy relative to $\partial\Delta^n$ from $\bar{x}$ to $\bar{x}'$, then x, x' are called **homotopic**, $x \sim x'$.

This notion of relative simplicial homotopy is conceptual and a direct generalisation of the corresponding concept for continuous maps in Theorem 6.1.5. However, it is rather abstract and difficult to work with. We need a definition that gives the same notion of simplicial homotopy relative to $\partial\Delta^n$ but is formulated entirely in terms of *elements* of the sets X_n. This is given by the following definition, which also makes it easier to show that being simplicially homotopic is an equivalence relation.

Definition 7.2.3: Let $(\star, X)$ be a pointed Kan complex and $x, x' \in X_n$ elements satisfying $d_n^i(x) = d_n^i(x') = \star_{n-1}$ for $0 \leq i \leq n$. Then x and x' are called **homotopic**, $x \sim x'$, if there is a $y \in X_{n+1}$, a **homotopy** from x to x', with

$$d_{n+1}(y) := (d_{n+1}^0(y), \ldots, d_{n+1}^{n+1}(y)) = (\star_n, \ldots, \star_n, x, x').$$

Lemma 7.2.4: Being homotopic in the sense of Definition 7.2.3 is equivalence relation.

Proof:
1. $x \sim x$: For every $x \in X_n$ with $d_n^i(x) = \star_{n-1}$ for $0 \leq i \leq n$, the element $y = s_n^n(x)$ is a homotopy from x to x, as (22) implies

$$d_{n+1}^i \circ s_n^n(x) = s_{n-1}^{n-1} \circ d_n^i(x) = s_{n-1}^{n-1}(\star_{n-1}) = \star_n \; 0 \leq i \leq n-1, \quad d_{n+1}^n \circ s_n^n(x) = d_{n+1}^{n+1} \circ s_n^n(x) = x.$$

2. $x \sim x' \wedge x \sim x'' \Rightarrow x' \sim x''$: Suppose $y \in X_{n+1}$ is a homotopy from x to x' and $y' \in X_{n+1}$ a homotopy from x to x''. Then the $(n+2)$-tuple

$$w = (w_0, \ldots, w_{n+1}) = (\star_{n+1}, ..., \star_{n+1}, y, y'),$$

is an $(n+2, n+2)$-horn on X, as we have

$$\begin{aligned}
&d_{n+1}^i(w_j) = \star_n = d_{n+1}^{j-1}(w_i) && 0 \leq i < j \leq n-1\\
&d_{n+1}^{n-1}(w_i) = \star_n = d_{n+1}^i(y) = d_{n+1}^i(w_n) && 0 \leq i \leq n-1\\
&d_{n+1}^n(w_i) = \star_n = d_{n+1}^i(y') = d_{n+1}^i(w_{n+1}) && 0 \leq i \leq n-1\\
&d_{n+1}^n(w_{n+1}) = d_{n+1}^n(y') = x = d_{n+1}^n(y) = d_{n+1}^n(w_n).
\end{aligned}$$

By the Kan condition, there is a filler $z \in X_{n+2}$ with

$$d_{n+2}(z) = (\star_n, ..., \star_n, y, y', d_{n+2}^{n+2}(z)),$$

and $d_{n+2}^{n+2}(z) \in X_{n+1}$ is a homotopy from x' to x'', as we have with (22) for $0 \leq i \leq n-1$

$$\begin{aligned}
&d_{n+1}^i \circ d_{n+2}^{n+2}(z) \overset{(22)}{=} d_{n+1}^{n+1} \circ d_{n+2}^i(z) = d_{n+1}^{n+1}(\star_{n+1}) = \star_n && 0 \leq i \leq n-1\\
&d_{n+1}^n \circ d_{n+2}^{n+2}(z) \overset{(22)}{=} d_{n+1}^{n+1} \circ d_{n+2}^n(z) = d_{n+1}^{n+1}(y) = x'\\
&d_{n+1}^{n+1} \circ d_{n+2}^{n+2}(z) \overset{(22)}{=} d_{n+1}^{n+1} \circ d_{n+2}^{n+1}(z) = d_{n+1}^{n+1}(y') = x''.
\end{aligned}$$

3. $x \sim x' \Rightarrow x' \sim x$: Suppose $x \sim x'$. By 1. we also have $x \sim x$, and 2. implies $x' \sim x$. □

The proof of Lemma 7.2.4 shows why homotopy is defined for pointed Kan complexes rather than pointed simplicial sets. The Kan condition guarantees that simplicial homotopy in the sense of Definition 7.2.3 is an equivalence relation. While reflexivity holds for any pointed simplicial set, the composition of homotopies is achieved with horn fillers.

We are now ready to show that being homotopic in the sense of Definition 7.2.3 agrees with being homotopic in the sense of Definition 7.2.2.

Lemma 7.2.5: Let $(\star, X)$ be a pointed Kan complex and $x, x' \in X_n$ with $d_n^i(x) = d_n^i(x') = \star_{n-1}$ for $0 \leq i \leq n$. Then x and x' are homotopic in the sense of Definition 7.2.2 if and only if they are homotopic in the sense of Definition 7.2.3.

Proof:
1. Let $x, x' \in X_n$ be homotopic in the sense of Definition 7.2.3 and let $y \in X_{n+1}$ be an element with $d_{n+1}(y) = (\star_n, ..., \star_n, x, x')$. We define a homotopy $h : \bar{x} \Rrightarrow \bar{x}'$ by specifying the maps $h_m^i : \Delta_m^n \to X_{m+1}$ as in Definition 6.2.4 with $h_m^i(\tau) = \star_{m+1}$ for all $\tau \in \Delta_m^n$.

This condition already defines h^i_m on all non-surjective monotonic maps $\tau : [m+1] \to [n+1]$ by Remark 7.1.3, 2. We can use the relations involving the degeneracies in (34) to define it on the degeneracies, and it remains to define it on the identity morphism $1_{[n+1]}$, where we set

$$h^i_n(1_{[n+1]}) = s^i_n(x) \quad 0 \leq i < n, \qquad h^n_n(1_{[n+1]}) = y.$$

Then we have

$$\begin{aligned}
&d^0_{n+1} \circ h^0_n(1_{[n+1]}) = d^0_{n+1} \circ s^0_n(x) = x = \bar{x}(1_{[n+1]}) \qquad d^{n+1}_{n+1} \circ h^n_n(1_{[n+1]}) = d^{n+1}_{n+1}(y) = x' = \bar{x}'(1_{[n+1]})\\
&d^j_{n+1} \circ h^j_n(1_{[n+1]}) = d^j_{n+1} \circ s^j_n(x) \overset{(22)}{=} d^{j+1}_{n+1} \circ s^j_n(x) = d^{j+1}_{n+1} \circ h^j_n(1_{[n+1]})\\
&d^i_{n+1} \circ h^j_n(1_{[n+1]}) = d^i_{n+1} \circ s^j_n(x) \overset{(22)}{=} \star_n \quad i \neq j, j+1 \qquad h^j_{n-1} \circ d^i_n = \star_n,
\end{aligned}$$

which shows that the relations on the degeneracies in (34) are satisfied. Thus, the maps h^j_n define a homotopy from $\bar{x}$ to $\bar{x}'$ in the sense of Definition 6.2.4 relative to $\partial\Delta^n$ and hence a homotopy in the sense of Definition 7.2.2.

2. To prove that being homotopic in the sense of Definition 7.2.2 implies being homotopic in the sense of Definition 7.2.3, we use the following auxiliary statement:

If $u, v \in X_n$ with $d^i_n(u) = d^i_n(v) = \star_{n-1}$ for all $i = 0, ..., n$ and there is an element $z \in X_{n+1}$ with $u = d^k_{n+1}(z)$, $v = d^{k+1}_{n+1}(z)$ and $d^i_{n+1}(z) = \star_n$ for all $i \notin \{k, k+1\}$ and some $k \in \{0, ..., n\}$, then u, v are homotopic in the sense of Definition 7.2.3.

To prove this, note that the element $(\star_{n+1}, ..., \star_n, s^{k+1}_{n+1}(v), z, s^k_n(v), \star_n, ..., \star_n)$ is a $(k, n+1)$-horn by the assumptions on u, v, z and by (22). As X is a Kan complex, it has a filler $w \in X_{n+2}$ with

$$d_{n+2}(w) = (\star_{n+1}, ..., \star_{n+1}, d^k_{n+2}(w), s^{k+1}_{n+1}(v), z, s^k_{n+1}(v), \star_{n+1}, ..., \star_{n+1}).$$

By applying the simplicial relations (22), one finds that the element $z' = d^k_{n+2}(w)$ satisfies $d^i_{n+1}(z') = \star_n$ for $i \neq k+1, k+2$ as well as $d^{k+1}_n(z') = u$ and $d^{k+2}_n(z') = v$. Iterating this procedure finally yields an element $t \in X_{n+1}$ with $d^n_{n+1}(t) = u$, $d^{n+1}_{n+1}(t) = v$ and $d^i_{n+1}(t) = \star_n$ for all $0 \leq i \leq n-1$, a homotopy from u to v in the sense of Definition 7.2.3.

3. Let now $h : \bar{x} \Rrightarrow \bar{x}'$ be a simplicial homotopy relative to $\partial\Delta^n$ in the sense of Definition 7.2.2. Then the associated maps $h^i_m : \Delta^n_m \to X_{m+1}$ satisfy (34) and $h^i_m(\tau) = \star_{m+1}$ for all non-surjective monotonic maps $\tau : [m+1] \to [n+1]$. We set

$$y_i = h^i_n(1_{[n+1]}) \in X_{n+1} \qquad 0 \leq i \leq n.$$

The relations (34) then imply $d^0_{n+1}(y_0) = x$, $d^{n+1}_{n+1}(y_n) = x'$. Together with (22) and some additional computations they yield

$$\begin{aligned}
&d^{i+1}_{n+1}(y_{i+1}) = d^{i+1}_{n+1} \circ h^{i+1}_n(1_{[n+1]}) \overset{(34)}{=} d^{i+1}_{n+1} \circ h^i_n(1_{[n+1]}) = d^{i+1}_{n+1}(y_i) && \\
&d^j_n \circ d^{i+1}_{n+1}(y_i) = d^j_n \circ d^i_{n+1}(y_i) = 0 && 0 \leq i, j, \leq n\\
&d^j_{n+1}(y_i) = \star_n && j \notin \{i, i+1\}.
\end{aligned} \tag{42}$$

To construct a homotopy from x to x' in the sense of Definition 7.2.3, we apply 2. to the elements $z = y_i \in X_{n+1}$. This yields $d^i_{n+1}(y_i) \sim d^{i+1}_{n+1}(y_i)$ for all $i = 0, \ldots, n-1$, and we obtain a chain of homotopies in the sense of Definition 7.2.3

$$x = d^0_{n+1}(y_0) \sim d^1_{n+1}(y_0) \overset{(42)}{=} d^1_{n+1}(y_1) \sim \ldots \sim d^n_{n+1}(y_{n-1}) \overset{(42)}{=} d^n_{n+1}(y_n) \sim d^{n+1}_{n+1}(y_n) = x'. \qquad \square$$

This establishes that the two notions of simplicial homotopy relative to $\partial\Delta^n$ in Definitions 7.2.2 and 7.2.3 agree. In particular, it shows that for each pointed Kan complex $(\star, X)$, simplicial homotopy relative to $\partial\Delta^n$ in the sense of Definition 7.2.2 is an equivalence relation on the set of simplicial maps $\bar{x} : \Delta^n \Rightarrow X$ with $\bar{x}_m(\tau) = \star_m$ for all $\tau \in \Delta^n_m$.

This is a more general pattern that also holds for simplicial homotopies in the sense of Definition 6.2.4 that are not defined relative to a boundary or restricted to simplicial maps $\bar{x} : \Delta^n \Rightarrow X$.

Remark 7.2.6: If X is a Kan complex, then for all simplicial sets S, homotopy of simplicial maps $\alpha : S \Rightarrow X$ is an equivalence relation on the set $\mathrm{Hom}_{\mathrm{SSet}}(S, X)$.

Proof:
As the proof is lengthy and requires more preparation, we will not prove this result. A proof can be found in [M, Chapter 1.6], Corollary 6.11, and a more modern one in [GJ, Chapter 1.6], see in particular Corollary 6.2. □

With the notion of simplicial homotopy from Definitions 7.2.2 and 7.2.3 we can define path components of a Kan complex X and homotopy groups of a pointed Kan complex $(\star, X)$.

For the former, we simply define a path in X from $x \in X_0$ to $x' \in X_0$ as a homotopy from x to x' in the sense of Definition 7.2.3 or, equivalently, as a homotopy from the constant simplicial map $\bar{x} : \Delta^0 \Rightarrow X$ to $\bar{x}' : \Delta^0 \Rightarrow X$ in the sense of Definition 7.2.2. In this case, the basepoints in Definitions 7.2.2 and 7.2.3 play no role and can be omitted. The proof of Lemma 7.2.4 generalises to $n = 0$. Hence, being connected by a path is an equivalence relation on X_0.

Definition 7.2.7: Let X be a Kan complex.

1. A **path** in X from $x \in X_0$ to $x' \in X_0$ is an element $z \in X_1$ with
$$d_1(z) = (d_1^0(z), d_1^1(z)) = (x, x').$$
2. The **path component** of $x \in X_0$ is the set
$$\pi_0(x) = \{x' \in X_0 \mid \exists z \in X_1 \text{ with } d_1(z) = (x, x')\}.$$
3. The set of path components of X is $\pi_0(X) = \{\pi_0(x) \mid x \in X\}$.

For the counterparts of the homotopy groups in Theorem 6.1.5 we need to replace the group multiplication in Theorem 6.1.5 by a suitable multiplication of homotopy classes of simplicial maps relative to the boundary. As this must be compatible with the composition of simplicial homotopies in the proofs of Lemmas 7.2.4 and 7.2.5, we must define it via horn fillers:

For points $x, y \in X_n$ with $d_n^i(x) = d_n^i(y) = \star_{n-1}$ for all $0 \leq i \leq n$, we consider the $(n, n+1)$-horn $(\star_n, ..., \star_n, x, y)$. Its filler replaces the concatenation of maps $f : [0,1]^{\times n} \to X$ in Theorem 6.1.5.

Definition 7.2.8: Let $(\star, X)$ be a pointed Kan complex and $n \in \mathbb{N}$.

The nth **homotopy group** of X with basepoint $\star \in K_0$ is the set
$$\pi_n(\star, X) = \{x \in X_n \mid d_n^i(x) = \star_{n-1} \ \forall i = 0, ..., n\}/\sim$$
with the multiplication $[x] \cdot [y] = [d_{n+1}^n(z)]$, where $z \in X_{n+1}$ is a filler of $(\star_n, \ldots, \star_n, x, y)$:
$$d_{n+1}(z) := (d_{n+1}^0(z), \ldots, d_{n+1}^{n+1}(z)) = (\star_n, \ldots, \star_n, x, d_{n+1}^n(z), y)$$

Similarly to Theorem 6.1.5 one can show that the homotopy groups of a pointed Kan complex are indeed groups and abelian for $n \geq 2$. This requires some computations with horn fillers that replace the corresponding computations for the concatenation of maps in the proof of Theorem 6.1.5. Although the concrete computations are different, the arguments are directly analogues of the ones in the proof of Theorem 6.1.5.

Theorem 7.2.9: Let $(\star, X)$ be a pointed Kan complex.

1. For all $n \in \mathbb{N}$ the nth homotopy group $\pi_n(\star, X)$ is a group.
2. It is abelian for $n \geq 2$.

Proof:
1.(a) We show that the group multiplication is well-defined:

• As the element $d^n_{n+1}(z)$ satisfies $d^i_n \circ d^n_{n+1}(z) = d^{n-1}_n \circ d^i_{n+1}(z) = \star_{n-1}$ for $0 \leq i \leq n-1$ and $d^n_n \circ d^n_{n+1}(z) = d^n_n \circ d^{n+1}_{n+1}(z) = d^n_n(y) = \star_{n-1}$, we have $[d^n_{n+1}(z)] \in \pi_n(\star, X)$.

• The equivalence class $[d^n_{n+1}(z)]$ does not depend on the choice of z:

Suppose that $x, y \in X_n$ with $d^i_n(x) = d^i_n(y) = \star_{n-1}$ for $i = 0, \ldots, n$ and z, z' are fillers of the $(n, n+1)$-horn $(\star_n, ..., \star_n, x, y)$. Then a direct computation with (22) shows that they form an $(n, n+2)$-horn $(\star_{n+1}, ..., \star_{n+1}, s^n_n \circ d^{n-1}_{n+1}(z), z, z')$, which has a filler $w \in X_{n+2}$:

$$d_{n+2}(w) = (\star_n, ..., \star_n, s^n_n \circ d^{n-1}_{n+1}(z), d^n_{n+2}(w), z, z').$$

Then $d^n_{n+2}(w)$ is a homotopy from $d^n_{n+1}(z)$ to $d^n_{n+1}(z')$, as one has for $0 \leq i \leq n-1$ with (22)

$$d^i_{n+1} \circ d^n_{n+2}(w) = d^{n-1}_{n+1} \circ d^i_{n+2}(w) = \star_n = s^{n-1}_{n-1} \circ d^{n-1}_n \circ d^i_{n+1}(z) = s^{n-1}_{n-1} \circ d^i_n \circ d^n_{n+1}(z)$$
$$d^n_{n+1} \circ d^n_{n+2}(w) = d^n_{n+1} \circ d^{n+1}_{n+2}(w) = d^n_{n+1}(z) \qquad d^{n+1}_{n+1} \circ d^n_{n+2}(w) = d^n_{n+1} \circ d^{n+2}_{n+2}(w) = d^n_{n+1}(z'),$$

and this implies $[d^n_{n+1}(z)] = [d^n_{n+1}(z')]$.

• $[d^n_{n+1}(z)]$ does not depend on the choice of the representatives $x, y \in X_n$:

Let $x, y, y' \in X_n$ with $d^i_n(x) = d^i_n(y) = d^i_n(y') = \star_{n-1}$ for $i = 0, \ldots, n$ and suppose that $y \sim y'$. Then there is a $w \in X_{n+1}$ with $d_{n+1}(w) = (\star_n, ..., \star_n, y', y)$. Let $z' \in X_{n+1}$ be a filler of the $(n, n+1)$-horn $(\star_n, ..., \star_n, x, y')$. Then a direct computation with the definitions and the relations (22) shows that we have an $(n+1, n+2)$-horn $(\star_{n+1}, ..., \star_{n+1}, s^{n-1}_n(x), z', w)$, which has a filler $u \in X_{n+2}$ by the Kan condition

$$d_{n+2}(u) = (\star_{n+1}, ..., \star_{n+1}, s^{n-1}_n(x), z', d^{n+1}_{n+2}(u), w).$$

Then $z := d^{n+1}_{n+2}(u) \in X_{n+1}$ is a filler of the $(n, n+1)$-horn $(\star_n, ..., \star_n, x, y)$ that satisfies the condition $d^n_{n+1}(z) = d^n_{n+1}(z')$, as we get with (22)

$$d^i_{n+1}(z) = d^i_{n+1} \circ d^{n+1}_{n+2}(u) = d^n_{n+1} \circ d^i_{n+2}(u) = \star_n \qquad 0 \leq i \leq n-2$$
$$d^{n-1}_{n+1}(z) = d^{n-1}_{n+1} \circ d^{n+1}_{n+2}(u) = d^n_{n+1} \circ d^{n-1}_{n+2}(u) = d^n_{n+1} \circ s^{n-1}_n(x) = x$$
$$d^n_{n+1}(z) = d^n_{n+1} \circ d^{n+1}_{n+2}(u) = d^n_{n+1} \circ d^n_{n+2}(u) = d^n_{n+1}(z')$$
$$d^{n+1}_{n+1}(z) = d^{n+1}_{n+1} \circ d^{n+1}_{n+2}(u) = d^{n+1}_{n+1} \circ d^{n+2}_{n+2}(u) = d^{n+1}_{n+1}(w) = y.$$

This shows that $[x] \cdot [y]$ does not depend on the choice of the representative y. An analogous argument shows that it does not depend on the choice of the representative x.

1.(b) We show that $\pi_n(\star, X)$ is a group:

• The neutral element is $[\star_n]$:

For every $x \in X_n$ with $d_n^i(x) = \star_{n-1}$ for $i = 0, \ldots, n$, the element $z = s_n^n(x) \in X_{n+1}$ is filler of the $(n, n+1)$-horn $(\star_n, ..., \star_n, \star_n, x)$ with $d_{n+1}^n \circ s_n^n(x) = x$, and this implies $[\star] \cdot [x] = [d_{n+1}^n(z)] = [x]$. Likewise, the element $z' = s_n^{n-1}(x) \in X_{n+1}$ is a filler of the $(n, n+1)$-horn $(\star_n, ..., \star_n, x, \star_n)$, and this implies $[x] \cdot [\star_n] = [d_{n+1}^n(z)] = [x]$.

• Existence of inverses:

For every element $x \in X_n$ with $d_n^i(x) = \star_{n-1}$ for $i = 0, ..., n$ the $(n-1, n+1)$-horn $(\star_n, ..., \star_n, \star_n, x)$ and the $(n+1, n+1)$-horn $(\star_n, ..., \star_n, x, \star_n)$ have fillers $z, z' \in X_{n+1}$ with

$$d_{n+1}(z) = (\star_n, ..., \star_n, d_{n+1}^{n-1}(z), \star_n, x) \qquad d_{n+1}(z') = (\star_n, ..., \star_n, x, \star_n, d_{n+1}^{n+1}(z'))$$

These are also fillers of the $(n, n+1)$-horns $(\star_n, ..., \star_n, d_{n+1}^{n-1}(z), x)$ and $(\star_n, ..., \star_n, x, d_{n+1}^{n+1}(z'))$, respectively, and hence $[d_{n+1}^{n-1}(z)] \cdot [x] = [d_{n+1}^n(z)] = [\star_n]$ and $[x] \cdot [d_{n+1}^{n+1}(z')] = [d_{n+1}^n(z')] = [\star_n]$.

• Associativity:

Let $x, y, z \in X_n$ with $d_n^i(x) = d_n^i(y) = d_n^i(z) = \star_{n-1}$ for $i = 0, \ldots, n$ and choose fillers

- $p \in X_{n+1}$ of the $(n, n+1)$-horn $(\star_n, ..., \star_n, x, y)$,
- $q \in X_{n+1}$ of the $(n, n+1)$-horn $(\star_n, ..., \star_n, y, z)$,
- $u \in X_{n+1}$ of the $(n, n+1)$-horn $(\star_{n+1}, ..., \star_{n+1}, d_{n+1}^n(p), z)$,
- $v \in X_{n+2}$ of the $(n, n+2)$-horn $(\star_{n+1}, ..., \star_{n+2}, p, u, q)$.

Then we have $d_{n+2}(v) = (\star_{n+1}, ..., \star_{n+1}, p, d_{n+2}^n(v), u, q)$, and $d_{n+2}^n(v) \in X_{n+1}$ is a filler of the $(n, n+1)$-horn $(\star_n, ..., \star_n, x, d_{n+1}^n(q))$, as (22) implies

$$\begin{aligned} d_{n+1}^i \circ d_{n+2}^n(v) &= d_{n+1}^{n-1} \circ d_{n+2}^i(v) = \star_n \quad 0 \leq i \leq n-2, \\ d_{n+1}^{n-1} \circ d_{n+2}^n(v) &= d_{n+1}^{n-1} \circ d_{n+2}^{n-1}(v) = d_{n+1}^{n-1}(p) = x, \\ d_{n+1}^{n+1} \circ d_{n+2}^n(v) &= d_{n+1}^n \circ d_{n+2}^{n+2}(v) = d_{n+1}^n(q). \end{aligned}$$

This proves that the multiplication is associative:

$$\begin{aligned} [x] \cdot ([y] \cdot [z]) &= [x] \cdot [d_{n+1}^n(q)] = [d_{n+1}^n \circ d_{n+2}^n(v)] \overset{(22)}{=} [d_{n+1}^n \circ d_{n+2}^{n+1}(v)] = [d_{n+1}^n(u)] = [d_{n+1}^n(p)] \cdot [z] \\ &= ([x] \cdot [y]) \cdot [z]. \end{aligned}$$

2. We show that $\pi_n(\star, X)$ is abelian for $n \geq 2$.

Let $n \geq 2$ and $w, y \in X_n$ with $d_n^i(w) = d_n^i(y) = \star_{n-1}$ for $i = 0, ..., n$. The claim follows from the auxiliary identities

$$\begin{aligned} &t_{n+1} \in X_{n+1} \text{ with } d_{n+1}(t) = (\star_n, ..., \star_n, w, x, y, \star_n) && \Rightarrow && [y] \cdot [w] = [x] && (43) \\ &u_n \in X_{n+1} \quad \text{with } d_{n+1}(u_n) = (\star_n, ..., \star_n, w, \star_n, y, z) && \Rightarrow && [w] \cdot [y] = [z] && (44) \\ &v \in X_{n+1} \quad \text{with } d_{n+1}(v) = (\star_n, ..., \star_n, w, x, y, z) && \Rightarrow && [x] \cdot [z] = [w] \cdot [y] && (45) \end{aligned}$$

for all $w, x, y, z \in X_n$ with $d_n^i(w) = d_n^i(x) = d_n^i(y) = d_n^i(z) = \star_{n-1}$ for $i = 0, ..., n$. By extending the $(n-1, n+1)$-horn $(\star_n, ..., \star_n, w, y, \star_n)$, one obtains an element t_{n+1} with

$$d_{n+1}(t_{n+1}) = (\star_n, ..., \star_n, w, x, y, \star_n) \qquad x = d_{n+1}^{n-1}(t_{n+1}).$$

This implies $[x] = [y] \cdot [w]$ by (43) and $[x] = [w] \cdot [y]$ by (45) and proves that $\pi_n(\star, X)$ is abelian.

We prove the auxiliary identities (43) to (45):

• Proof of (43):
Let t_{n+1} as in (43) and $t_{n-1} \in X_{n+1}$ the filler of the $(n-1, n+1)$-horn $(\star_n, ..., \star_n, x, w)$

$$d_{n+1}(t_{n-1}) = (\star_n, ..., \star_n, \star_n, q, x, w) \qquad q = d_{n+1}^{n-1}(u_{n+1}). \tag{46}$$

Then direct computations with the definitions of q, t_{n+1}, t_{n-1} and the relations (22) show that $(\star_{n+1}, ..., \star_{n+1}, s_n^n(w), t_{n-1}, t_{n+1}, s_n^{n-2}(w))$ is an $(n, n+2)$-horn and has a filler $t \in X_{n+2}$

$$d_{n+2}(t) = (\star_{n+1}, ..., \star_{n+1}, s_n^n(w), t_{n-1}, t_n, t_{n+1}, s_n^{n-2}(w)) \qquad t_n = d_{n+2}^n(t).$$

Applying again the relations (22) shows that $d_{n+1}(t_n) = (\star_n, ..., \star_n, q, y, \star_n)$, which implies $[q] \cdot [\star_n] = [q] = [y]$ by definition of the multiplication in $\pi_n(\star, X)$. Likewise, equation (46) implies $[q] \cdot [w] = [y] \cdot [w] = [x]$.

• Proof of (44):
Let u_n be as in (44) and and $u_{n-1} \in X_{n+1}$ the filler of the $(n, n+1)$-horn $(\star_n, ..., \star_n, w, \star_n, \star_n)$

$$d_{n+1}(u_{n-1}) = (\star_n, ..., \star_n, w, \star_n, r, \star_n) \qquad r = d_{n+1}^n(u_{n-1}). \tag{47}$$

Then direct computations with the definition of u_{n-1}, u_n, r and relations (22) show that $(\star_{n+1}, ..., \star_{n+1}, s_n^{n-2}(w), u_{n-1}, u_n, s_n^n(z))$ is an $(n+1, n+2)$-horn and has a filler $u \in X_{n+2}$

$$d_{n+2}(u) = (\star_{n+1}, ..., \star_{n+1}, s_n^{n-2}(w), u_{n-1}, u_n, u_{n+1}, s_n^n(z)) \qquad u_{n+1} = d_{n+2}^{n+1}(u).$$

Applying again the relations (22) one finds that $d_{n+1}(u_{n+1}) = (\star_n, ..., \star_n, r, y, z)$, and this implies $[r] \cdot [z] = [y]$. Applying (43) to (47) yields $[r] \cdot [w] = [\star_n]$ and hence $[r] = [w]^{-1}$ and $[z] = [y] \cdot [w]$.

• Proof of (45):
Let $v_{n+2} \in X_{n+2}$ be as in (45) and $v_{n-2} \in X_{n+1}$ a filler of the $(n-2, n+1)$-horn $(\star_n, ..., \star_n, w)$

$$d_{n+1}(v_{n-2}) = (\star_n, ..., \star_n, p, \star_n, \star_n, w) \qquad p = d_{n+1}^{n-2}(v_{n+2}). \tag{48}$$

Choose a filler $v_{n-1} \in X_{n+1}$ of the $(n, n+1)$-horn $(\star_n, ..., \star_n, p, \star_n, x)$

$$d_{n+1}(v_{n-1}) = (\star_n, ..., \star_n, p, \star_n, s, x) \qquad s = d_{n+1}^n(v_{n-1}). \tag{49}$$

By applying the definition of $v_{n-2}, v_{n-1}, v_{n+2}$ and some computations with relations (22) one finds that $(\star_n, ..., \star_n, v_{n-2}, v_{n-1}, s_n^n(y), v_{n+2})$ is an $(n+1, n+2)$-horn and has a filler $v \in X_{n+2}$

$$d_{n+2}(v) = (\star_{n+1}, ..., \star_{n+1}, v_{n-2}, v_{n-1}, s_n^n(y), v_{n+1}, v_{n+2}) \qquad v_{n+1} = d_{n+2}^{n+1}(v).$$

With relations (22) one finds $d_{n+1}(v_{n+1}) = (\star_n, ..., \star_n, \star_n, s, y, z)$, which implies $[s] \cdot [z] = [y]$ by definition of the multiplication in $\pi_n(\star, X)$. Applying (44) to (48) yields $[p] \cdot [\star_n] = [p] = [w]$, and applying (44) to (49) yields $[p] \cdot [s] = [x]$. Hence, we have $[w] \cdot [s] = [x]$ and $[s] \cdot [z] = [y]$, which implies $[x] \cdot [z] = [w] \cdot [y]$. □

We thus generalised the homotopy groups of pointed topological spaces to simplicial homotopy groups of pointed Kan complexes. It remains to investigate their behaviour under simplicial maps that preserve the basepoints.

In Theorem 6.1.8 we showed that the homotopy groups of pointed topological spaces (x, X) define a functor $\pi_n : \text{Top}^* \to \text{Grp}$ that depends only on the homotopy classes of basepoint preserving maps: $\pi_n(f) = \pi_n(g)$ for all continuous maps $f, g : X \to X'$ with $f \sim g$.

One thus expects an analogous statement for the simplicial homotopy groups of pointed Kan complexes. For this, we consider the category Kan^* with

- pointed Kan complexes $(\star, X)$ as objects,
- simplicial maps $\alpha : X \Rightarrow X'$ with $\alpha_n(\star_n) = \star'_n$ for all $n \in \mathbb{N}_0$ as morphisms.

In fact, it is sufficient to impose that the simplicial maps satisfy $\alpha_0(\star_0) = \star'_0$, as we have $X(\tau)(\star_n) = \star_m$ and $X'(\tau)(\star'_n) = \star'_m$ for all monotonic maps $\tau : [m+1] \to [n+1]$. We call such simplicial maps basepoint preserving.

We also need a notion of basepoint preserving homotopy $h : \alpha \Rrightarrow \beta$ between basepoint preserving simplicial maps $\alpha, \beta : (\star, X) \Rightarrow (\star', X')$. We define this as a homotopy $h : \alpha \Rrightarrow \beta$ in the sense of Definition 6.2.4 such that the maps $h^i_n : X_n \to X'_{n+1}$ satisfy $h^i_n(\star_n) = \star'_{n+1}$ for all $n \in \mathbb{N}_0$ and $0 \leq i \leq n$. These definitions yield a simplicial counterpart of Theorem 6.1.8.

Theorem 7.2.10:

1. For all $n \in \mathbb{N}$ the nth homotopy group defines a functor $\pi_n : \text{Kan}^* \to \text{Grp}$.
2. If two basepoint preserving simplicial maps $\alpha, \beta : (\star, X) \Rightarrow (\star', X')$ are homotopic with a basepoint preserving homotopy, then $\pi_n(\alpha) = \pi_n(\beta)$ for all $n \in \mathbb{N}_0$.

Proof:
1. We define the functor $\pi_n : \text{Kan}^* \to \text{Grp}$ on morphisms and to show that it is indeed functorial.

Let $(\star, X)$ and $(\star', X')$ be pointed Kan complexes and $\alpha : X \Rightarrow X'$ a simplicial map with $\alpha(\star_n) = \star'_n$ for all $n \in \mathbb{N}_0$. We define the functor $\pi_n : \text{Kan}^* \to \text{Grp}$ on the morphisms by

$$\pi_n(\alpha) : \pi_n(\star, X) \to \pi_n(\star', X'), \quad [x] \mapsto [\alpha_n(x)].$$

This assignment is well-defined, as any homotopy $y \in X_{n+1}$ from $x \in X_n$ to $x' \in X_n$ with

$$d_{n+1}(y) = (s^{n-1}_{n-1} \circ d^0_n(x), ..., s^{n-1}_{n-1} \circ d^{n-1}_n(x), x, x')$$

defines a homotopy $\alpha_{n+1}(y) \in X'_{n+1}$ from $\alpha_n(x)$ to $\alpha_n(x')$

$$\begin{aligned} d_{n+1} \circ \alpha_{n+1}(y) &= (d^0_{n+1} \circ \alpha_{n+1}(y), ..., d^{n+1}_{n+1} \circ \alpha_{n+1}(y)) = (\alpha_n \circ d^0_{n+1}(y), ..., \alpha_n \circ d^{n+1}_{n+1}(y)) \\ &= (\alpha_n(\star_n), ..., \alpha_n(\star_n), \alpha_n(x), \alpha_n(x')) = (\star'_n, ..., \star'_n, \alpha_n(x), \alpha_n(x')) \end{aligned}$$

To see that this yields a group homomorphism, let $x, y \in X_n$ with $d^i_n(x) = d^i_n(y) = \star_{n-1}$ for all $i = 0, ..., n$ and let $z \in X_{n+1}$ be a filler of the $(n, n+1)$-horn $(\star_n, ..., \star_n, x, y)$

$$d_{n+1}(z) = (\star_n, ..., \star_n, x, d^n_{n+1}(z), y).$$

Then we have $[x] \cdot [y] = [d^n_{n+1}(z)]$. The image $z' = \alpha_{n+1}(z)$ is a filler of the $(n, n+1)$-horn $(\star_n, ..., \star_n, \alpha_n(x), \alpha_n(y))$ with $d^n_{n+1}(z') = \alpha_n \circ d^n_{n+1}(z)$, as we have

$$\begin{aligned} d^n_{n+1} \circ \alpha_{n+1}(z) &= (d^0_{n+1} \circ \alpha_{n+1}(z), ..., d^{n+1}_{n+1} \circ \alpha_{n+1}(z)) = (\alpha_n \circ d^0_{n+1}(z), ..., \alpha_n \circ d^{n+1}_{n+1}(z)) \\ &= (\alpha_n(\star_n), ..., \alpha_n(\star_n), \alpha_n(x), \alpha_n(d^n_{n+1}(z)), \alpha_n(y)) \\ &= (\star'_n, ..., \star'_n, \alpha_n(x), \alpha_n(d^n_{n+1}(z)), \alpha_n(y)). \end{aligned}$$

This shows that $\pi_n(\alpha) : \pi_n(\star, X) \to \pi_n(\star', X')$ is a group homomorphism:

$$\pi_n(\alpha)([x]) \cdot \pi_n(\alpha)([y]) = [\alpha_n(x)] \cdot [\alpha_n(y)] = [\alpha_n(d^n_{n+1}(z))] = \pi_n(\alpha)([d^n_{n+1}(z)]) = \pi_n(\alpha)([x] \cdot [y]).$$

Then this defines a functor then follows directly from the identities

$$\pi_n(\alpha') \circ \pi_n(\alpha)[x] = \pi_n(\alpha')[\alpha_n(x)] = [\alpha'_n(\alpha_n(x))] = [(\alpha' \circ \alpha)_n(x)] = \pi_n(\alpha' \circ \alpha)[x]$$
$$\pi_n(\mathrm{id}_X)[x] = [\mathrm{id}_{X_n}(x)] = [x]$$

for all basepoint preserving simplicial maps $\alpha : X \Rightarrow X'$, $\alpha' : X' \Rightarrow X''$ and $x \in X_n$.

2. Let now $\alpha, \beta : (\star, X) \Rightarrow (\star', X')$ basepoint preserving simplicial maps and $h : \alpha \Rrightarrow \beta$ a basepoint preserving homotopy, given by maps $h^i_n : X_n \to X'_{n+1}$ with $h_n(\star_n) = \star'_{n+1}$ for all $n \in \mathbb{N}_0$ and $0 \leq i \leq n$ that satisfy the identities in (34).

Let $x \in X_n$ and $\bar{x} : \Delta^n \Rightarrow X$ the associated simplicial map. Then by Remark 6.2.6, 1. the maps $h^i_k \circ \bar{x}_k : \Delta^n_k \to X'_{k+1}$ define a simplicial homotopy from $\alpha \circ \bar{x} : \Delta^n \Rightarrow X'$ to $\beta \circ \bar{x} : \Delta^n \Rightarrow X'$. As $h^i_k(\star_k) = \star'_{k+1}$, we have $h^i_m \circ \bar{x}_m(\tau) = h^i_m(\star_m) = \star'_{m+1}$ for all $\tau \in (\partial\Delta^n)_m$, and this is a homotopy relative to $\partial\Delta^n$. With Lemma 7.2.5 this implies that $\alpha_n(x)$ is homotopic to $\beta_n(x)$ for all $n \in \mathbb{N}_0$ and hence $\pi_n(\alpha)[x] = [\alpha_n(x)] = [\beta_n(x)] = \pi_n(\beta)$ for all $n \in \mathbb{N}$. □

To conclude our investigation of simplicial homotopy groups, we relate them to the simplicial homologies from Section 5.5. By Proposition 5.5.5 that a simplicial object $S : \Delta^{op} \to R$-Mod defines a chain complex $S_\bullet$, its standard chain complex, with boundary operators

$$d_n : S_n \to S_{n-1}, \quad s \mapsto \Sigma^n_{i=0}(-1)^i d^i_n(s).$$

The homologies of this chain complex are called the simplicial homologies of S.

Given a pointed Kan complex $(\star, X)$, we can construct a simplicial object in R-Mod by composing the simplicial set $X : \Delta^{op} \to$ Set with the functor $\langle\ \rangle_R :$ Set $\to R$-Mod that assigns to a set M the free R-module $\langle M \rangle_R$ generated by M and to a map $f : M \to M'$ the R-linear map $\langle f \rangle_R : \langle M \rangle_R \to \langle M' \rangle_R$ given by f on the basis.

To relate the simplicial homotopy groups of $(\star, X)$ to its simplicial homologies, we need to modify this construction slightly to take into account the basepoint. In each degree n we take the quotient by the submodule $\langle \star_n \rangle$ generated by $\star_n \in X_n$ and write $x + \star_n$ for the elements of this quotient module. As $d^i_n(\star_n) = \star_{n-1}$ for all $n \in \mathbb{N}$, this defines a chain complex $X^\star_\bullet$ with

$$X^\star_n = \langle X_n \rangle_R / \langle \star_n \rangle \qquad n \in \mathbb{N}_0$$
$$d_n : X^\star_n \to X^\star_{n-1}, \quad x + \star_n \mapsto \Sigma^n_{i=0}(-1)^i d^i_n(x) + \star_n.$$

We denote by $H_n(X^\star, R)$ the homologies of $X^\star_\bullet$. The relation between the homotopy groups $\pi_n(\star, X)$ and the homologies of $X^\star_\bullet$ is then given by Hurewicz's Theorem. It can be viewed as a generalisation of Hurewicz's Theorem for topological spaces, which relates the homotopy group of a pointed topological space to its singular homologies from Example 5.5.6, 1.

Theorem 7.2.11: (**Hurewicz theorem**)
Let $(\star, X)$ be a a pointed Kan complex and R a ring.

1. For all $n \in \mathbb{N}$ there is a group homomorphism, the **Hurewicz map**,

$$\phi_n : \pi_n(\star, X) \to H_n(X^\star, R), \quad [x]_\pi \mapsto [x]_{H_n}.$$

2. If $\pi_k(\star, X) = \{1\}$ for all $0 \leq k < n$ then this induces an isomorphism

$$\pi_n(\star, X) : \mathrm{Ab}\,\pi_n(\star, X) \to H_n(X^\star, \mathbb{Z}).$$

Proof:
1.(a) We show that the maps $\phi_n : \pi_n(\star, X) \to H_n(x)$ are well-defined:

Suppose $x, x' \in X_n$ with $d_n^i(x) = d_n^i(x') = \star_{n-1}$ are homotopic in the sense of Definition 7.2.3. Then there is a $z \in X_{n+1}$ with

$$d_{n+1}^i(z) = \star_n \quad 0 \leq i \leq n-1, \qquad d_{n+1}^n(z) = x, \qquad d_{n+1}^{n+1}(z) = x'.$$

This implies $d_{n+1}(z + \star_{n+1}) = \Sigma_{i=0}^{n+1}(-1)^i d_n^i(z) + \star_n = (-1)^{n+1}(x' - x) + \star_n$ and hence

$$[x]_H - [x']_H = [x - x']_H = (-1)^{n+1}[d_{n+1}(z)]_H = 0.$$

1.(b) We show that the maps $\phi_n : \pi_n(\star, X) \to H_n(x)$ are group homomorphisms.

Let $x, y \in X_n$ with $d_n^i(x) = d_n^i(y) = \star_{n-1}$ and $z \in X_{n+1}$ a filler of the $(n, n+1)$-horn $(\star_n, ..., \star_n, x, y)$, which exists by the Kan condition

$$d_{n+1}^i(z) = \star_n \quad 0 \leq i < n-1, \qquad d_{n+1}^{n-1}(z) = x, \qquad d_{n+1}^{n+1}(z) = y.$$

Then $[x]_\pi \cdot [y]_\pi = [d_{n+1}^n(z)]_\pi$ by the definition of the group multiplication in $\pi_n(\star, X)$ and

$$d_{n+1}(z + \star_{n+1}) = \Sigma_{i=0}^{n+1}(-1)^i d_{n+1}^i(z) + \star_n = (-1)^{n+1}(x + x' - d_{n+1}^n(z)) + \star_n$$

This implies $[d_{n+1}^n(z)]_H = [x]_H + [y]_H$ and

$$\phi_n([x]_\pi \cdot [y]_\pi) = \phi_n([d_{n+1}^n(z)]_\pi) = [d_{n+1}^n(z)]_H = [x + x']_H = [x]_H + [x']_H = \phi_n([x]_\pi) + \phi_n([y]_\pi).$$

2. We will not prove this result, as it requires extensive preparations and more background. A proof can be found in [M, Chapter II], Theorem 13.6, and in [GJ, Chapter III], Theorem 3.7. □

7.3 Quasicategories

In the two preceding subsection, we encountered many examples of Kan complexes: singular nerves of topological spaces (Proposition 7.1.9), simplicial sets arising from simplicial groups, in particularly from simplicial modules (Proposition 7.1.10) and simplicial nerves of groupoids, in particular of groups (Theorem 7.1.11). However, by Theorem 7.1.11, simplicial nerves of categories are not covered in this setting. In this case, every inner horn has a filler, but not necessarily the outer horns. This is a motivation to relax the Kan condition and require fillers only for inner horns. This leads to the concept of a quasicategory or weak Kan complex.

However, there is a another important motivation to consider quasicategories, namely higher categories. Higher categories have become very important research topic in the last twenty years in algebra, topology and mathematical physics. A higher category has not only objects and morphisms, often called 1-morphisms, but also 2-morphisms between 1-morphisms, 3-morphisms between 2-morphisms and so on, up to a top degree n. Each new layer of morphisms comes with an additional composition and new set of identity morphisms, subject to coherence axioms.

Examples of higher categories are abundant. Many of the structures encountered so far define higher categories. We formulate this precisely for bicategories, which involve objects, 1-morphisms and 2-morphisms and two compositions.

Definition 7.3.1: A **bicategory** $\mathcal{B}$ consists of

- a class of objects $\mathrm{Ob}\mathcal{B}$,
- for each pair of objects $X, Y \in \mathrm{Ob}\mathcal{B}$ a category $\mathcal{B}(X,Y)$, whose objects are called **morphisms** or **1-morphisms** and whose morphisms are called **2-morphisms** of $\mathcal{B}$,
- for each $X \in \mathrm{Ob}\mathcal{B}$ an object $1_X \in \mathcal{B}(X,X)$, the **identity 1-morphism** on X,
- for each triple of objects $X, Y, Z \in \mathrm{Ob}\mathcal{B}$ a functor $\otimes_{Z,Y,X} : \mathcal{B}(Y,Z) \times \mathcal{B}(X,Y) \to \mathcal{B}(X,Z)$,
- for each pair of objects X, Y natural isomorphisms, the left and right **unitors**,

$$\lambda^L_{(X,Y)} : (1_Y \otimes_{Y,Y,X} \mathrm{id}_{\mathcal{B}(X,Y)}) \Rightarrow \mathrm{id}_{\mathcal{B}(X,Y)} \qquad \lambda^R_{X,Y} : (\mathrm{id}_{\mathcal{B}(X,Y)} \otimes_{Y,X,X} 1_X) \Rightarrow \mathrm{id}_{\mathcal{B}(X,Y)},$$

- for each quadruple of objects $W, X, Y, Z \in \mathrm{Ob}\mathcal{B}$ a natural isomorphism, the **associator**,

$$a^{Z,Y,X,W} : \otimes_{Z,X,W}(\otimes_{Z,Y,X} \times \mathrm{id}_{\mathcal{B}(X,W)}) \Rightarrow \otimes_{Z,Y,W}(\mathrm{id}_{\mathcal{B}(Z,Y)} \times \otimes_{Y,X,W})$$

such that the following conditions are satisfied

(B1) **pentagon axiom**: for all objects $V, W, X, Y, Z \in \mathrm{Ob}\mathcal{B}$ and 1-morphisms $f \in \mathcal{B}(Y,Z)$, $g \in \mathcal{B}(X,Y)$, $h \in \mathcal{B}(W,X)$, $k \in \mathcal{B}(V,W)$ the following diagram commutes

$$\begin{array}{ccccc} ((f\otimes g)\otimes h)\otimes k & \xrightarrow{a_{f\otimes g,h,k}} & (f\otimes g)\otimes(h\otimes k) & \xrightarrow{a_{f,g,h\otimes k}} & f\otimes(g\otimes(h\otimes k)) \\ \downarrow{\scriptstyle a_{f,g,h}\otimes 1_k} & & & \nearrow{\scriptstyle 1_f\otimes a_{g,h,k}} & \\ (f\otimes(g\otimes h))\otimes k & \xrightarrow[a_{f,g\otimes h,k}]{} & f\otimes((g\otimes h)\otimes k), & & \end{array} \tag{50}$$

(B2) **triangle axiom**: for all objects X, Y, Z and 1-morphisms $f \in \mathcal{B}(Y,Z)$ and $g \in \mathcal{B}(X,Y)$ the following diagram commutes

$$\begin{array}{ccc} (f\otimes 1_Y)\otimes g & \xrightarrow{a_{f,1_Y,g}} & f\otimes(1_Y\otimes g) \\ {\scriptstyle \lambda^R_f\otimes 1_g}\searrow & & \swarrow{\scriptstyle 1_f\otimes\lambda^L_g} \\ & f\otimes g. & \end{array} \tag{51}$$

The composition $\otimes$ is called **horizontal composition** and the composition in $\mathcal{B}(X,Y)$ **vertical composition** of $\mathcal{B}$.

If the associators and unitors are trivial, one speaks of a **strict bicategory** or **2-category**.

Example 7.3.2:

1. Small categories, functors and natural transformations form a 2-category CAT.

   The vertical composition is the composition of natural transformations and the horizontal composition given by the composition of functors and composition of functors with natural transformations.

2. For every ring R chain complexes in R-Mod, chain maps between them and chain homotopies form a 2-category.

   The vertical composition is composition of chain homotopies from Remark 6.3.2, 1, and the horizontal composition is given by the composition of chain maps and the composition of chain maps and chain homotopies from Remark 6.3.2, 2.

3. A bicategory with a single object is called a **monoidal category** and a 2-category with a single object a **strict monoidal category**.

   Monoidal categories arise from the representation theory of groups and, more generally, Hopf algebras, such as (q-deformed) universal enveloping algebras of Lie algebras and tensor algebras of vector spaces.

4. A **crossed module** $(A, B, \rhd, \partial)$ consists of groups A, B, a group action $\rhd : B\times A \to A$ by automorphisms and a group homomorphism $\partial : A \to B$ satisfying the **Peiffer identities**

$$\partial(b \rhd a) = b\partial(a)b^{-1} \qquad \partial(a) \rhd a' = aa'a^{-1} \qquad \forall a, a' \in A, b \in B.$$

   Every crossed module defines a strict monoidal category, a 2-category with a single object, with 1-morphisms given by elements of B and with 2-morphisms $(a, b) : b \to \partial(a)b$ for $a \in A$, $b \in B$. The horizontal composition is $(a, b)\otimes(a', b') = (a(b \rhd a'), bb')$ and the vertical composition $(a', \partial(a)b) \circ (a, b) = (aa', b)$.

   Examples arise from normal subgroups $A \subset B$ with the inclusion $\partial = \iota : A \to B$ and the conjugation action $\rhd : B \times A \to A$, $b \rhd a = bab^{-1}$ of B on A.

5. For every topological space there is a 2-category $\Pi_2(X)$, whose
   - objects are points $x \in X$,
   - 1-morphisms $\gamma : x \to x'$ are paths $\gamma : [0, 1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = x'$,
   - 2-morphisms from $\gamma : x \to x'$ to $\gamma' : x \to x'$ are homotopy classes relative to $\partial[0, 1]^{\times 2}$ of homotopies from γ to γ' relative to $\{0, 1\}$.

It is useful to compare the last example with the **fundamental groupoid** $\Pi_1(X)$ of a topological space, which has points $x \in X$ as objects and *homotopy classes of paths* relative to the endpoints as morphisms. There, the composition of paths is strictly associative and unital, whereas in the fundamental 2-groupoid composition of paths is associative only up to (homotopy classes of) homotopies, which form the associator. In contrast, the composition of 2-morphisms needs to be strictly associative and unital, which requires that one takes homotopy classes of homotopies on the top level.

This is the general pattern: while the vertical composition of 2-morphisms is strictly associative and unital, the horizontal composition is only associative up to specified 2-isomorphisms, namely the component 2-morphisms of associators and unitors. The pentagon and triangle relations are essential in this, as they ensure *coherence*.

If one forms composites of the component 2-morphisms of associators, unitors and identity morphisms via the horizontal and vertical composition, then the resulting 2-morphisms are equal whenever they have the same source and target 1-morphisms. This is the **coherence theorem** for bicategories which is a generalisation of Mac Lane's coherence theorem for monoidal categories. This allows one to largely ignore rebracketings and insertions of units, as the results of different choices are equal, whenever they have the same source and target 1-morphisms.

Definition 7.3.1 of a bicategory can be generalised inductively to of n-categories with higher morphisms up to degree n for all $n \in \mathbb{N}$. Roughly speaking, one defines an n-category $\mathcal{C}$ as a collection of objects Ob$\mathcal{C}$ and for each pair of objects X, Y an $(n-1)$-category $\mathcal{C}(X, Y)$ together with a composition $(n - 1)$-functors $\otimes_{Z,Y,X} : \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \to \mathcal{C}(X, Z)$ and coherence n-morphisms that encode the associativity and unitality of the composition $\otimes_{Z,Y,X}$, subject to coherence conditions that generalise the pentagon and triangle axioms.

An n-category in this sense has objects X, 1-morphisms $f : X \to Y$ between objects X, Y, 2-morphisms $\alpha : f \Rightarrow g$ between 1-morphisms $f, g : X \to Y$ with the same source and target, 3-morphisms $h : \alpha \Rrightarrow \beta$ between 2-morphisms $\alpha, \beta : f \Rightarrow g$ with the same source and target and so on up to level n. On each level $k \leq n$ there are unit k-morphisms on $(k-1)$-morphisms as well as coherent associators for the n compositions.

The problem with this approach is that the coherence data and coherence axioms become more and more complicated with growing n and are essentially unmanageable for $n > 3$. This lead to a new approach, namely to admit k-morphisms between $(k-1)$-morphisms for all $k \in \mathbb{N}$, but to require that all of these higher morphisms are *coherent isomorphisms* for $k \geq n$, for some fixed level $n \in \mathbb{N}$. This leads to the concept of an (∞, n)-category. In the following we focus on the simplest case, namely $(\infty, 1)$-categories and their concrete realisations as quasicategories. This is one of the main motivations to consider quasicategories.

Definition 7.3.3: A simplicial set $X : \Delta^{op} \to \text{Set}$ is called a

- an ∞-**category** or a **quasicategory**, if every inner horn $\alpha : \Lambda^k_n \Rightarrow X$ has a filler,
- an ∞-**groupoid** or **quasigroupoid**, if it is a Kan complex.

The idea is to take elements of X_0 as objects, elements of X_1 as morphisms and elements of X_k for $k \geq 2$ as higher isomorphisms in the quasicategory. Composites of morphisms are defined by elements of X_2 and identity morphisms by degeneracies. The construction thus resembles the one of the homotopy category of a simplicial set from Definition 5.4.4, but the composition is given by horn fillers and no longer required to be unique, in contrast to Theorem 7.1.11.

Definition 7.3.4: Let X be a quasicategory.

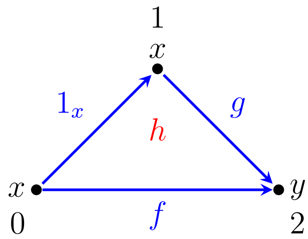
1. Elements $x \in X_0$ are called **objects** of X.
2. Elements $f \in X_1$ are called **morphisms** of X.
3. The **source** of $f \in X_1$ is $d^1_1(f) \in X_0$, its **target** $d^0_1(f) \in X_0$. One writes $f : d^1_1(f) \to d^0_1(f)$.
4. The **identity morphism** $1_x : x \to x$ on an object $x \in S_0$ is the element $s^0_0(x) : x \to x$.
5. If $f, g \in X_1$ form an inner horn (g, f) with filler $z \in X_2$, then one calls $d^1_2(z) = g \circ f$ a **composite** of f and g **witnessed** by z.

Remark 7.3.5:

1. Neither composites of morphisms nor the fillers that witness them are in general unique. By Theorem 7.1.11 uniqueness holds if and only if the quasicategory X is simplicially isomorphic to the nerve of a small category. Thus, quasicategories *generalise* (nerves of) small categories.

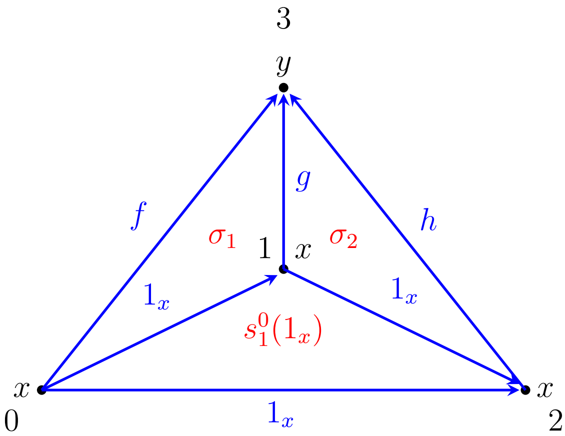
2. We sketch how to realise n-morphisms between $(n-1)$-morphisms for $n \geq 2$ in a quasicategory X and how to form their composites and their inverses. This works by considering degenerate simplexes in which some faces are identities:

- A **2-morphism** from a 1-morphism $f : x \to y$ to a 1-morphism $g : x \to y$ is an element $h \in X_2$ with $d_2(h) = (g, f, 1_x)$. One writes $h : f \Rightarrow g$.



- If $\sigma_1 : f \Rightarrow g$ and $\sigma_2 : g \Rightarrow h$ are 2-morphisms between 1-morphisms $f, g, h : x \to y$, then a **composite** $\sigma_2 \circ \sigma_1 : f \Rightarrow h$ is given by a filler of the (1,3)-horn $(\sigma_2, \sigma_1, s_1^0(1_x))$: $d_3(z) = (\sigma_2, \rho, \sigma_1, s_1^0(1_x))$. As we have $d_2(\rho) = (d_2^0(\sigma_2), d_2^1(\sigma_1), d_2^1 \circ s_1^0(1_x)) = (h, f, 1_x)$, the element $\rho \in X_2$ is a 2-morphism $\rho = \sigma_2 \circ \sigma_1 : f \Rightarrow h$.

  The filler z and the 2-morphism ρ are visualised by a tetrahedron, whose vertices are associated with objects, edges with morphisms and faces with 2-morphisms. The face opposite the ith vertex corresponds to the ith element of the filled horn.



- An **identity 2-morphism** $\sigma : f \Rightarrow f$ for a 1-morphism $f : x \to y$ is the 2-morphism $s_1^0(f) \in X_2$ with $d_2(s_1^0(f)) = (f, f, 1_x)$.

- Every 2-morphism $\sigma : f \Rightarrow g$ between 1-morphisms $f, g : x \to y$ has a **right inverse**:

  There is a 2-morphism $\rho : g \Rightarrow f$ given by a filler $z \in X_3$ of the $(2,3)$-horn $(\sigma, s_1^0(g), s_1^0(1_x))$ with $d_3(z) = (\sigma, s_1^0(g), \rho, s_1^0(1_x))$. This implies $\sigma \circ \rho = 1_g$.

  There is also a 2-morphism $\sigma' : f \Rightarrow g$ given by a filler $z' \in X_3$ of the $(2,3)$-horn $(\rho, s_1^0(f), s_1^0(1_x))$ with $d_3(z') = (\rho, s_1^0(f), \sigma', s_1^0(1_x))$. and $\rho \circ \sigma' = 1_f$.

  Combining these identities yields $\sigma = \sigma \circ 1_f = \sigma \circ (\rho \circ \sigma')$ and $\sigma' = 1_g \circ \sigma' = (\sigma \circ \rho) \circ \sigma'$. Thus, up to associativity, σ and σ' agree.

- The definition of k-**morphisms** for $k > 2$ is analogous: they are defined as k-simplexes $x \in X_k$ such that the first two entries in $d_k(x) = (d_k^0(x), ..., d_k^k(x))$ are the source and target $(k-1)$-morphisms and all other entries are degenerate.

  Their composites and inverses are defined as inner horn fillers. However, the combinatorics become increasingly complicated, so that the simplicial viewpoint in terms of horn fillers is much easier to handle.

This remark illustrates the general features of an $(\infty,1)$-category: composites of morphisms are not unique, only unique up to 2-morphisms. In this sense, composites are coherent choices of structure or data, rather than properties of the category. Associativity and unitality also hold only up to higher morphisms. We will prove that the composition of 1-morphisms is associative and unital up to 2-morphisms in Lemma 7.3.8 below. Morphisms of degree ≥ 2 are invertible, but their inverses are again unique only up to coherent choices of higher morphisms.

Example 7.3.6:

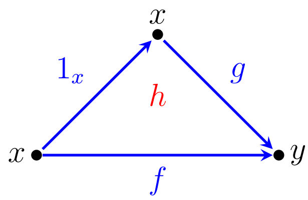
1. The simplicial nerve $N(\mathcal{C})$ of a small category $\mathcal{C}$ is a quasicategory by Theorem 7.1.11. It is an quasigroupoid if and only if $\mathcal{C}$ is a groupoid.
2. Singular nerves of topological spaces and simplicial sets underlying simplicial groups are always quasigroupoids.
3. For every topological space X, there is a quasicategory $S = \Pi_{\leq\infty}(X) : \Delta^{op} \to \text{Set}$, whose
   - objects are points $x \in X$,
   - 1-morphisms $\gamma : x \to x'$ are paths $\gamma : [0,1] \to X$ with $\gamma(0) = x$, $\gamma(1) = x'$,
   - 2-morphisms $h : \gamma \Rightarrow \gamma'$ for paths $\gamma, \gamma' : x \to x'$ are homotopies between paths with fixed endpoints: continuous maps $h : [0,1]^{\times 2} \to X$ satisfying for all $s, t \in [0,1]$
   $$h(0,t) = \gamma(t), \qquad h(1,t) = \gamma'(t), \qquad h(s,0) = x, \qquad h(s,1) = x'.$$
   - 3-morphisms $\sigma : h \Rrightarrow h'$ between homotopies $h, h' : \gamma \Rightarrow \gamma'$ are homotopies between homotopies: continuous maps $\sigma : [0,1]^{\times 3} \to X$ satisfying for all $r, s, t \in [0,1]$
   $$\begin{aligned} &\sigma(0,s,t) = h(s,t), && \sigma(1,s,t) = h'(s,t) \\ &\sigma(r,0,t) = \gamma(t), && \sigma(r,1,t) = \gamma'(t) \\ &\sigma(r,s,0) = \gamma(0) = \gamma'(0), && \sigma(r,s,1) = \gamma(1) = \gamma'(1) \end{aligned}$$
   - n-morphisms are defined analogously.

The fundamental groupoid $\Pi_1(X)$ is obtained from the quasicategory $\Pi_{\leq\infty}(X)$ in Example 7.3.6, 3. by keeping the objects (the points of X) and identifying those 1-morphisms (paths with fixed endpoints) that are related by 2-morphisms (homotopies relative to their endpoints).

This can be viewed as a higher analogue of the construction of a skeleton of a category. The latter is obtained by identifying all objects in a category $\mathcal{C}$ that are related by an isomorphism. The resulting skeleton of $\mathcal{C}$ is a category with isomorphism classes of objects in $\mathcal{C}$ as objects and only identity morphisms. It gives the answer to a classification problem, namely to classify the objects of $\mathcal{C}$ up to isomorphism.

One can attempt a similar identification for general quasicategories. One keeps their objects, elements of X_0, and identifies morphisms, elements of X_1, if and only if they are related by a 2-morphism, an element of X_2. The invertibility of 2-morphisms should guarantee the symmetry of this identification, the existence of unit 2-morphisms its reflexivity and the existence of composites its transitivity. The expected result should be a category in the usual sense. To emphasise the connection with fundamental groupoids and, more generally, topology, we call a 2-morphism $h : f \Rightarrow g$ between 1-morphisms $f, g : x \to y$ a *homotopy* from f to g.

Definition 7.3.7: Let X be a quasicategory. Morphisms $f, g : x \to y$ are called **homotopic**, $f \sim g$, if there is an $h \in X_2$, a **homotopy** from f to g, with $d_2(h) = (g, f, 1_x)$.



To show that identifying homotopic morphisms in a quasicategory X does indeed give rise to a category, we need to prove that being homotopic is an equivalence relation on X_1 and that it is compatible with identity morphisms and the composition of morphisms.

Lemma 7.3.8: Let X be a quasicategory. For all objects $x, y \in X_0$, being homotopic is an equivalence relation on the set of morphisms $f : x \to y$.

It satisfies:

(a) $g \circ f \sim g' \circ f'$ for all composites, if $f \sim f' : x \to y$ and $g \sim g' : y \to z$,

(b) $f \circ 1_x \sim f \sim 1_y \circ f$ for all morphisms $f : x \to y$ and all composites.

(c) $(g \circ f) \circ e \sim g \circ (f \circ e)$ for all composable morphisms e, f, g and all composites.

(d) if $g \sim f \circ e$, then there is a $u \in X_2$ with $d_2(u) = (f, g, e)$.

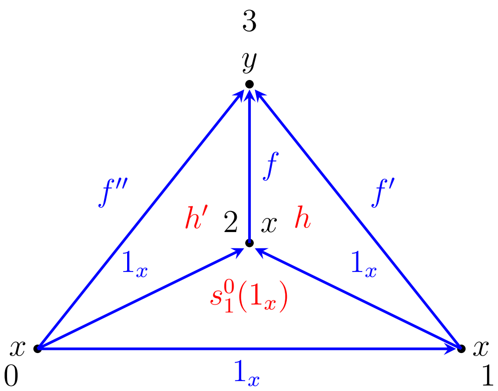
Proof:
• Being homotopic is reflexive:

For all morphisms $f : x \to y$, the element $h = s_1^0(f)$ satisfies

$$d_2(h) = (d_2^0 \circ s_1^0(f), d_2^1 \circ s_1^0(f), d_2^2 \circ s_1^0(f)) \overset{(22)}{=} (f, f, s_0^0 \circ d_1^1(f)) = (f, f, s_0^0(x)) = (f, f, 1_x).$$

• Being homotopic is skew transitive: $f' \sim f \wedge f'' \sim f \Rightarrow f'' \sim f'$:

Let $h, h' \in X_2$ with $d_2(h) = (f, f', 1_x)$ and $d_2(h') = (f, f'', 1_x)$. Then the element $(h, h', s_1^0(1_x))$ is a $(2,3)$-horn, as $d_2^0(h) = d_2^0(h') = f$ and $d_2^2(h) = d_2^2(h') = 1_x = d_2^0 \circ s_1^0(1_x) = d_2^1 \circ s_1^0(1_x)$. This is visualised by the following tetrahedron:



It has a filler $u \in X_3$ with $d_3(u) = (h, h', w, s_1^0(1_x))$, where $w := d_3^2(u)$ corresponds to the bottom face of the tetrahedron. This implies $d_2(w) = (d_2^1(h), d_2^1(h'), d_2^2 \circ s_1^0(1_x)) = (f', f'', 1_x)$.

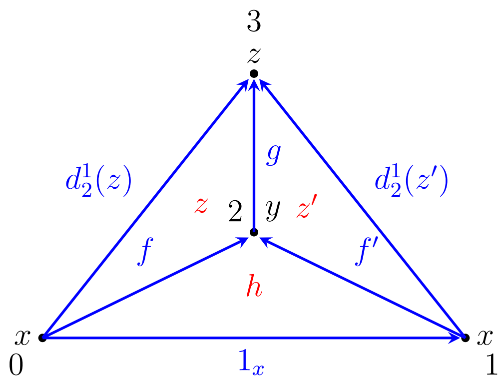
• It is symmetric: if $f \sim f'$, then we have $f' \sim f'$ by reflexivity and $f' \sim f$ by skew transitivity.

(a) We show that $f \sim f' : x \to y$ implies $g \circ f \sim g \circ f'$ for all $g : y \to x$ and all composites. The proof for $g \sim g'$ is analogous.

If $f \sim f'$ there is an $h \in X_2$ with $d_2(h) = (f', f, 1_x)$. Let $z, z' \in X_2$ with $d_2(z) = (g, d_2^1(z), f)$ and $d_2(z') = (g, d_2^1(z'), f')$ filler witnessing the composites. Then (z', z, h) is a $(2,3)$-horn, as $d_2^0(z') = g = d_2^0(z)$, $d_2(z') = f' = d_2^0(h)$ and $d_2^2(z) = f = d_2^1(h)$.

It has a filler $u \in X_3$ with $d_3(u) = (z', z, d_3^2(u), h)$, and the element $w := d_3^2(u)$ corresponding to the bottom face of the tetrahedron satisfies

$$d_2^0(w) = d_2^1(z'), \qquad d_2^1(w) = d_2^1(z), \quad d_2^2(w) = d_2^2(h) = 1_x \quad \Rightarrow \quad d_2^1(z) \sim d_2^1(z').$$

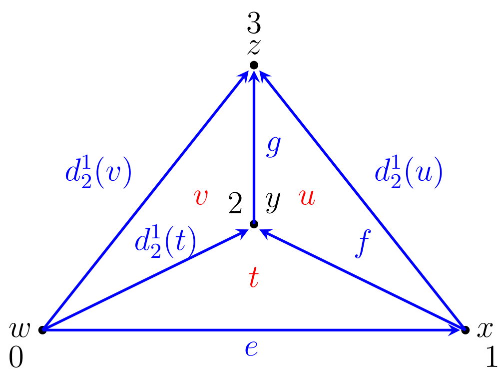


(b) We show that $f \circ 1_x \sim f \sim 1_y \circ f$ for all morphisms $f : x \to y$ and all composites:

By (22), the element $z = s_1^0(f) \in X_2$ satisfies $d_2(z) = (f, f, s_1^0 \circ d_2^1(f))) = (f, f, 1_x)$ and the element $z' = s_1^1(f)$ satisfies $d_2(z') = (s_0^0 \circ d_1^0(f), f, f) = (1_y, f, f)$. This shows that f is a composite of the morphisms f and 1_x and of the morphisms 1_y and f. From (a) we then have $f \circ 1_x \sim f \sim 1_y \circ f$ for all composites.

(c) Let $e : w \to x$, $f : x \to y$ and $g : y \to z$ be morphisms and $t, u, v \in X_2$ fillers witnessing the composites $f \circ e$, $g \circ f$ and $g \circ (f \circ e)$, respectively

$$d_2(t) = (f, d_2^1(t), e), \qquad d_2(u) = (g, d_2^1(u), f), \qquad d_2(v) = (g, d_2^1(v), d_2^1(t)).$$



Then (u, v, t) is a $(2,3)$-horn, as $d_2^0(u) = d_2^0(v) = g$, $d_2^0(t) = d_2^2(u) = f$ and $d_2^1(t) = d_2^2(v)$. It has a filler $w \in X_3$ with $d_3(w) = (u, v, r, t)$, and $r = d_3^2(w)$ satisfies

$$d_2^0(r) = d_2^1(u), \quad d_2^1(r) = d_2^1(v), \quad d_2^2(r) = d_2^2(t) = e.$$

As $d_2^1(u)$ is a composite $g \circ f$, the element r witnesses $(g \circ f) \circ e$, but due to $d_2^1(r) = d_2^1(v)$ it also witnesses $g \circ (f \circ e)$. Thus we have $g \circ (f \circ e) \sim (g \circ f) \circ e$ by 1 and (a).

(d) Exercise 61. □

As being homotopic is an equivalence relation on X_1 that is compatible with the composition of morphisms by Lemma 7.3.8 (a), we obtain a composition on the set of equivalence classes of morphisms in X_1. Lemma 7.3.8 (b) and (c) ensure that the resulting composition on the quotient set is *strictly* associative and unital. Lemma 7.3.8 (d) gives control over the content of the equivalence classes. The result is a category we encountered already in Definition 5.4.4: the homotopy category of the simplicial set X.

Proposition 7.3.9: Let X be a quasicategory.

1. There is a category hX, the **homotopy category** of X with
   - elements of X_0 as objects,
   - homotopy classes of morphisms $f : x \to y$ in X as morphisms from x to y.

   The identity morphisms and the composition in hX are given by the homotopy classes of identity morphisms and composites in X.

2. The homotopy category of hX coincides with the homotopy category of the simplicial set X from Definition 5.4.4.

Proof:
1. That the composition of morphisms is well-defined and associative follows directly from Lemma 7.3.8, and the same holds for the conditions on the identity morphisms.

2. That the homotopy category hX coincides with the homotopy category from Definition 5.4.4 follows by comparing the relations in (24) with the ones in Definition 7.3.7 and Lemma 7.3.8.

Let hX' be the homotopy category from Definition 5.4.4. We obtain a functor $F : hX' \to hX$ that is the identity on objects and given by $F(f) = [f]$ on the generating morphisms $f \in X_1$ of $h'X$. It is well-defined, because the definition of identity morphisms and composition in hX in Definition (7.3.7) and 7.3.4 respects the defining relations (24) in hX'. We also obtain a functor $G : hX \to hX'$ that is the identity on objects and $G([f]) = f$ for $f \in X_1$. It is well defined, because the relations (24) in hX' ensure that it does not depend on the choice of the representative of $[f]$. By definition, the functors F and G are inverses of each other. □

Proposition 7.3.9 justifies the name *homotopy category.* Although the category from Definition 5.4.4 is defined in more generality for all simplicial sets, it arises from an equivalence relation that can be viewed as a generalised notion of homotopy if the simplicial set is a quasicategory.

This has important consequences, namely more control over the equivalence classes. As noted in Remark 5.4.5 it is in general very difficult to describe morphisms in the homotopy category of a simplicial set X explicitly: they are equivalence classes of composable sequence of elements of X_1 with respect to the equivalence relation given in (24). It is in general not clear if a given morphism in hX can be represented as the equivalence class of a single element $f \in X_1$ or what is the minimum length of the composable sequence. In contrast, if X is a quasicategory, Proposition 7.3.9 shows that every morphism in hX is given as the equivalence class of a (not necessarily unique) morphism $f \in X_1$.

Together with Proposition 5.4.6 this also shows that the construction of the homotopy quasicategory is consistent with the simplicial nerve $N : \mathrm{Cat} \to \mathrm{SSet}$. The identity $hN \cong \mathrm{id}_{\mathrm{Cat}}$ states that forming the homotopy category of the nerve of an ordinary category gives back the original category. Thus starting with a quasicategory, we only obtain one category, up to equivalence, by iterating simplicial nerves and homotopy categories.

The homotopy category hX of a quasicategory X defines a distinguished class of morphisms in X, namely those morphisms whose equivalence classes in hX are isomorphisms in hX. Such morphisms are called *equivalences* and play a similar role to equivalences of categories. In fact, we will use them in the following to define equivalences of quasicategories.

Definition 7.3.10: A morphism $f : x \to y$ is a quasicategory X is called an **equivalence** if the associated morphism $[f] : x \to y$ in hX is an isomorphism.

Remark 7.3.11: By Lemma 7.3.8 (d) a morphism $f : x \to y$ in a quasicategory X is an equivalence if and only if there is a morphism $g : y \to x$ and $z, z' \in X_2$ with $d_2(z) = (f, 1_y, g)$ and $d_2(z') = (g, 1_x, f)$.

By definition of an equivalence, quasicategories in which *every* morphism is an equivalence are precisely those quasicategories whose homotopy categories are groupoids. This suggests defining a *quasigroupoid* as a quasicategory in which every morphism is an equivalence, and not as a Kan complex as in Definition 7.3.3. Thankfully, the two definitions agree.

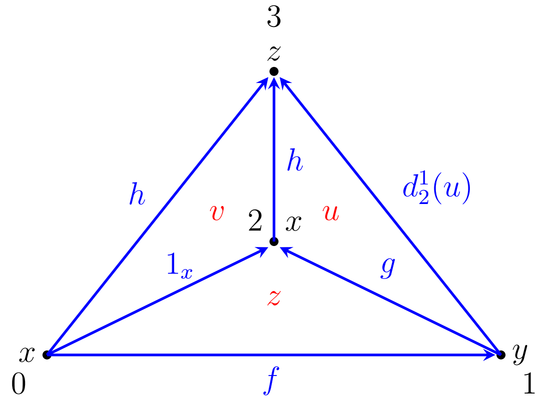
We already noted in the proof of Theorem 7.1.11 that the existence of inner horn fillers corresponds to a notion of composition, whereas the existence of outer horn fillers corresponds to the existence of inverses. It is directly apparent that fillers of $(0, 2)$- and $(2, 2)$-horns are directly related to the existence of inverses of 1-morphisms. For $n > 2$ this becomes more complicated due to combinatorial issues. We refer to the article [J] by Joyal for the proof.

Proposition 7.3.12:
A quasicategory X is a quasigroupoid if and only if its homotopy category hX is a groupoid.

Proof:
1. If X is a quasigroupoid, then all morphisms $f : x \to y$ the $(0, 2)$-horn $(1_x, f)$ and the $(2, 2)$-horn $(f, 1_y)$ have fillers $z, z' \in X_2$ with $d_2(z) = (g, 1_x, f)$ and $d_2(z') = (f, 1_y, g')$, and this implies $[g] \circ [f] = 1_x$ and $[f] \circ [g'] = 1_y$. It follows that $[g] = [g] \circ 1_y = [g] \circ [f] \circ [g'] = 1_x \circ [g'] = [g']$ and hence $[g]$ is an inverse of $[f]$ in hX.

2. If X is a quasicategory, then every inner horn has a filler, and the same holds for any $(0, 1)$-horn or $(1, 1)$-horn. If every morphism $[f] : x \to y$ has an inverse $[g] : y \to x$ in hX, then there are $z, z' \in X_2$ with $d_2(z) = (g, 1_x, f)$ and $d_2(z') = (f, 1_y, g)$ by Lemma 7.3.8 (d).

Suppose that (h, f) is an $(0, 2)$-horn, $d^1_1(h) = d^1_1(f) = x$, and let $v \in X_2$ with $d_2(v) = (h, h, 1_x)$ and $u \in X_2$ with $d_2(u) = (h, d^1_2(u), g)$ a filler of the inner horn (h, g). Then (u, v, z) is a $(2, 3)$-horn and has a filler $w \in X_3$ with $d_3(w) = (u, v, d^2_3(w), z)$. The element $t := d^2_3(w)$ satisfies $d_2(t) = (d^1_2(u), h, f)$ and fills the $(0, 2)$-horn (h, f).The proof that every $(2, 2)$-horn (f, k) can be filled is analogous (Exercise). Showing that every $(0, n)$-horn or (n, n)-horn for $n > 2$ can be filled requires more advanced techniques. A proof is given in [J], see Theorem 1.3, Corollary 1.4 and Theorem 2.2.



□

The next step is to define functors between quasicategories and natural transformations between them, which are needed in particular for a notion of equivalence of quasicategories. It is plausible to define a functor between quasicategories as a simplicial map. The appropriate concept for a natural transformation is less clear. One could attempt to define it as a simplicial homotopy, but this would be too naive and correspond to introducing an artificial cutoff. It is clear that one should not just consider homotopies, but also homotopies between homotopies etc.

The guiding principles can be the following.

- The construction should be compatible with simplicial nerves and with homotopies: functors between categories should induce functors of quasicategories via the nerve and functors of quasicategories should induce functors between their homotopy categories.
- The construction should be coherent and give rise to functor quasicategories between quasicategories that generalise the functor categories between categories.

We take the latter as the starting point. To define functor quasicategories, we consider the corresponding notions for topological spaces. Given two topological spaces X, Y we can consider continuous maps $f : X \to Y$, homotopies between them, homotopies between homotopies and so on. Considering all this data simultaneously amounts to considering continuous maps $h : [0,1]^{\times n} \times X \to Y$ or, equivalently, continuous maps $h : \Delta^n \times X \to Y$ for all $n \in \mathbb{N}_0$.

This can be generalised to simplicial sets by replacing topological spaces by simplicial maps and topological standard n-simplexes Δ^n by the simplicial sets $\Delta^n = \text{Hom}(-, [n+1])$. We thus consider simplicial maps $h : \Delta^n \times X \Rightarrow Y$ for fixed simplicial sets X, Y and all $n \in \mathbb{N}_0$. To organise them into a simplicial set note that the standard n-simplexes Δ^n can be organised into a functor $\Delta^\bullet : \Delta \to \text{SSet}$ that corresponds to the Hom functor with two arguments $\text{Hom}_\Delta(-,-) : \Delta^{op} \times \Delta \to \text{Set}$ and assigns

- to the ordinal $[n+1]$ the simplicial set $\Delta^\bullet([n+1]) = \Delta^n : \Delta^{op} \to \text{Set}$,
- to a monotonic map $\tau : [m+1] \to [n+1]$ the simplicial map $\Delta^\bullet(\tau) : \Delta^n \Rightarrow \Delta^m$ with component morphisms $\Delta^\bullet(\tau)_k : \Delta^n_k \to \Delta^m_k$, $\alpha \mapsto \tau \circ \alpha$.

With these preparations, we can define the quasicategory counterpart of a functor category. This is the *function complex* that is defined in more generality, for any two simplicial sets.

Definition 7.3.13: Let X, Y be simplicial sets.

The **function complex** $\mathrm{Map}(X,Y)$ is the simplicial set with

$$\begin{aligned}
&\mathrm{Map}(X,Y)_n = \mathrm{Hom}_{\mathrm{SSet}}(\Delta^n \times X, Y)\\
&d_n^i : \mathrm{Map}(X,Y)_n \to \mathrm{Map}(X,Y)_{n-1}, \quad \alpha \mapsto \alpha \circ (\Delta^\bullet(\delta_n^i) \times \mathrm{id}_X)\\
&s_n^i : \mathrm{Map}(X,Y)_n \to \mathrm{Map}(X,Y)_{n+1}, \quad \alpha \mapsto \alpha \circ (\Delta^\bullet(\sigma_{n+1}^i) \times \mathrm{id}_X).
\end{aligned}$$

The **evaluation map** $\mathrm{ev} : X \times \mathrm{Map}(X,Y) \Rightarrow Y$ is the simplicial map with components

$$\mathrm{ev}_n : X_n \times \mathrm{Map}(X,Y)_n \to Y_n, \quad (x,\alpha) \mapsto \alpha_n(1_{[n+1]}, x).$$

Remark 7.3.14:
The function complex defines a functor $\mathrm{Map} : \mathrm{SSet}^{op} \times \mathrm{SSet} \to \mathrm{SSet}$ that assigns

- to a pair of simplicial sets $X, Y : \Delta^{op} \to \mathrm{Set}$ the simplicial set $\mathrm{Map}(X,Y)$,
- to a pair of simplicial maps $\alpha : X' \Rightarrow X$ and $\beta : Y \Rightarrow Y'$ the simplicial map

$$\begin{aligned}
&\mathrm{Map}(\alpha,\beta) : \mathrm{Map}(X,Y) \Rightarrow \mathrm{Map}(X',Y')\\
&\mathrm{Map}(\alpha,\beta)_n : \mathrm{Map}(X',Y')_n \to \mathrm{Map}(X,Y)_n, \quad \gamma \mapsto \beta \circ \gamma \circ (\mathrm{id}_{\Delta^n} \times \alpha)
\end{aligned}$$

The function complex of a simplicial set behaves intuitively, similar to maps between sets or topological spaces. For instance, for any set X the functor $\mathrm{Hom}(X,-) : \mathrm{Set} \to \mathrm{Set}$ is right adjoint to the functor $X \times - : \mathrm{Set} \to \mathrm{Set}$. This adjunction identifies $f : Y \to \mathrm{Hom}_{\mathrm{Set}}(X,Z)$ from a set X to the set of maps from X to Z with maps $f' : X \times Y \to Z$. Similarly, for any abelian group M, the functor $\mathrm{Hom}(M,-) : \mathrm{Ab} \to \mathrm{Ab}$ for an is right adjoint of the functor $M \otimes - : \mathrm{Ab} \to \mathrm{Ab}$. This identifies group homomorphisms $f : B \to \mathrm{Hom}(M,A)$ with group homomorphisms $f : M \otimes B \to A$. The function complex defines a similar adjunction, an *exponential law* for SSet.

Proposition 7.3.15: (**exponential law**)

1. For any simplicial set X the functor $\mathrm{Map}(X,-) : \mathrm{SSet} \to \mathrm{SSet}$ is right adjoint to the functor $X \times - : \mathrm{SSet} \to \mathrm{SSet}$:

$$\mathrm{Hom}_{\mathrm{SSet}}(Y, \mathrm{Map}(X,Z)) \cong \mathrm{Hom}_{\mathrm{SSet}}(X \times Y, Z)$$

2. For all simplicial sets X, Y, Z there are simplicial isomorphisms that are natural in X, Y, Z

$$\mathrm{Map}(X \times Y, Z) \cong \mathrm{Map}(X, \mathrm{Map}(Y,Z)).$$

Proof:
1. The natural bijections

$$\begin{aligned}
&\phi_{Y,Z} : \mathrm{Hom}_{\mathrm{SSet}}(Y, \mathrm{Map}(X,Z)) \to \mathrm{Hom}_{\mathrm{SSet}}(X \times Y, Z), \quad \alpha \mapsto \phi_{Y,Z}(\alpha)\\
&\phi_{Y,Z}^{-1} : \mathrm{Hom}_{\mathrm{SSet}}(X \times Y, Z) \to \mathrm{Hom}_{\mathrm{SSet}}(Y, \mathrm{Map}(X,Z)), \quad \beta \mapsto \phi_{Y,Z}^{-1}(\beta)
\end{aligned}$$

that characterise the adjunction are given by

$$\phi_{Y,Z}(\alpha)_n(x,y) = \alpha_n(y)_n(1_{[n+1]}, x) \qquad [\phi_{Y,Z}^{-1}(\beta)_n(y)]_m(\tau, x') = \beta_m(x', Y(\tau)y)$$

for all $n \in \mathbb{N}_0$, $x \in X_n$, $x' \in X_m$, $y \in Y_n$ and monotonic maps $\tau : [m+1] \to [n+1]$. That the morphisms $\phi_{Y,Z}(\alpha)_n : X_n \times Y_n \to Z_n$ define a simplicial map follows by a direct computation:

$$\begin{aligned}\phi_{Y,Z}(\alpha)_m(X(\tau)x, Y(\tau)y) &= \alpha_m(Y(\tau)y)_m(1_{[m+1]}, X(\tau)x) \\ &\overset{\text{nat }\alpha}{=} [\alpha_n(y) \circ (\Delta^\bullet(\tau) \times \mathrm{id})]_m(1_{[m+1]}, X(\tau)x) = \alpha_n(y)_m(\tau, X(\tau)x) \\ &= Z(\tau) \circ \alpha_n(y)_n(1_{[n+1]}, x) = Z(\tau) \circ \phi_{Y,Z}(\alpha)_n(x,y),\end{aligned}$$

where we used that $\alpha_n(y) \in \mathrm{Hom}_{\mathrm{SSet}}(\Delta^n \times X, Z)$ to pass to the third line. Likewise, we have for monotonic maps $\rho : [n+1] \to [p+1]$, $\tau : [m+1] \to [n+1]$, $y' \in Y_p$ and $x' \in X_m$

$$\begin{aligned}[\phi^{-1}_{Y,Z}(\beta)_p(y') \circ (\Delta^\bullet(\rho) \times \mathrm{id})]_m(\tau, x') &= [\phi^{-1}_{Y,Z}(\beta)_p(y')]_m(\rho \circ \tau, x') = \beta_m(x', Y(\rho \circ \tau)y') \\ &= \beta_m(x', Y(\tau)(Y(\rho)y')) = [\phi^{-1}_{Y,Z}(\beta)_n(Y(\rho)y')]_m(\tau, x')\end{aligned}$$

which shows that the morphisms $\phi^{-1}_{Y,Z}(\beta)_n$ define a simplicial map. A similar routine computation shows that $\phi^{-1}_{Y,Z}$ is indeed inverse to $\phi_{Y,Z}$:

$$\begin{aligned}&[(\phi^{-1}_{Y,Z} \circ \phi_{Y,Z}(\alpha))_n(y)]_m(\tau, x') = \phi_{Y,Z}(\alpha)_m(x', Y(\tau)y) = \alpha_m(Y(\tau)y)(1_{[m+1]}, x') = \alpha_n(y)(\tau, x') \\ &[\phi_{Y,Z} \circ \phi^{-1}_{Y,Z}(\beta)]_n(x,y) = \phi^{-1}_{Y,Z}(\beta)_n(y)_n(1_{[n+1]}, x) = \beta_n(x,y).\end{aligned}$$

To show the naturality of $\phi_{Y,Z}$ consider simplicial maps $\gamma : Y' \Rightarrow Y$ and $\delta : Z \Rightarrow Z'$ and denote by $\delta_* : \mathrm{Map}(X,Z) \Rightarrow \mathrm{Map}(X,Z')$ the simplicial map given by post-composition with δ. Then one has for all $x \in X_n$, $y' \in Y'_n$ and $\alpha : Y \Rightarrow \mathrm{Map}(X,Z)$

$$\begin{aligned}[\delta \circ \phi_{Y,Z}(\alpha) \circ (\mathrm{id}_{X_n} \times \gamma)]_n(x,y') &= \delta_n \circ \alpha_n(\gamma_n(y'))_n(1_{[n+1]}, x) = \delta_n \circ (\alpha \circ \gamma)_n(y')(1_{[n+1]}, x) \\ &= (\delta_* \circ \alpha \circ \gamma)_n(y)_n(1_{[n+1]}, x) = \phi_{Y',Z'}(\delta_* \circ \alpha \circ \gamma)_n(x, y').\end{aligned}$$

2. The proof of 2. with 1. is Exercise 62. □

Having defined the function complex of a simplicial set, we now restrict attention to quasicategories and quasigroupoids. It turns out that the complex of functions into a quasicategory (quasigroupoid) is not just a simplicial set, but again a quasicategory (quasigroupoid), and thus satisfies our consistency requirements. The proof of this statement will be omitted, because it requires more background on fibrations and lifting properties. An accessible proof is given in [Rz], see Theorem 20.4 and the required background in Sections 14 to 19.

Proposition 7.3.16: Let X, Y be simplicial sets.

1. If Y is a quasicategory, then $\mathrm{Map}(X,Y)$ is a quasicategory.
2. If Y is quasigroupoid, then $\mathrm{Map}(X,Y)$ is a quasigroupoid.

With the function complex we can define functors and natural transformations between quasicategories in analogy to the objects and morphisms of a quasicategory. Functors from a quasicategory X to a quasicategory Y correspond to elements of $\mathrm{Map}(X,Y)_0$ and thus to simplicial maps $\alpha : \Delta^0 \times X \Rightarrow Y$ or, equivalently, simplicial maps $\alpha : X \Rightarrow Y$. Consequently, we define a natural transformation $h : \alpha \Rrightarrow \beta$ as an element $h \in \mathrm{Map}(X,Y)_1$ with $d^1_1(h) = \alpha$ and $d^0_1(h) = \beta$. Natural isomorphisms must then correspond to equivalences in the quasicategory $\mathrm{Map}(X,Y)$.

Definition 7.3.17: Let X, Y be quasicategories.

1. A **functor of quasicategories** from X to Y is a simplicial map $\alpha : X \Rightarrow Y$ or, equivalently, an element of $\mathrm{Map}_0(X, Y)$.
2. The **identity functor** on a quasicategory X is the simplicial map $\mathrm{id}_X : X \Rightarrow X$.
3. Let $\alpha, \beta : X \Rightarrow Y$ be functors of quasicategories. A **natural transformation** $h : \alpha \Rrightarrow \beta$ is an element $h \in \mathrm{Map}(X, Y)_1$ with $d_1^1(h) = \alpha$ and $d_1^0(h) = \beta$.
4. A natural transformation $h \in \mathrm{Map}(X, Y)_1$ is called a **natural isomorphism** if it is an equivalence in the quasicategory $\mathrm{Map}(X, Y)$:
   there are $g \in \mathrm{Map}(X, Y)_1$, $z, z' \in \mathrm{Map}(X, Y)_2$ with $d_2(z) = (h, 1_\beta, g)$, $d_2(z') = (g, 1_\alpha, h)$.

Identity natural transformations and composition of natural transformations are of course defined exactly as in Definition 7.3.4, namely as identity morphisms and composite morphisms in the quasicategory $\mathrm{Map}(X, Y)$. Equivalences of quasicategories can then be defined in analogy to equivalences of categories, as pairs of functors between the two quasicategories together with natural isomorphisms that relate their composites to the identity.

Definition 7.3.18: An **equivalence of quasicategories** is a functor $\alpha : X \Rightarrow Y$ of quasicategories such that there is a functor $\beta : Y \Rightarrow X$ of quasicategories, with natural isomorphisms $h : \alpha \circ \beta \Rrightarrow \mathrm{id}_Y$ and $h' : \beta \circ \alpha \Rrightarrow \mathrm{id}_X$.

It follows directly from Definition 5.4.4 of the homotopy functor $h : \mathrm{SSet} \to \mathrm{Cat}$ that every functor $\alpha : X \Rightarrow Y$ of quasicategories defines a functor $h(\alpha) : hX \to hY$ and that this is compatible with the composition of functors and the identity functors. Moreover, by definition a natural transformation $k : \alpha \Rightarrow \beta$ between functors of quasicategories is a simplicial homotopy from α to β. By Corollary 6.2.11 its image under the homotopy functor is a natural transformation $h(k) : h(\alpha) \Rightarrow h(\beta)$, and one can show (Exercise) that it is a natural isomorphism, whenever k is a natural isomorphism. This implies, see also [Rz, 22.3],

Corollary 7.3.19: If $\alpha : X \Rightarrow Y$ is an equivalence of quasicategories, then $h(\alpha) : hX \to hY$ is an equivalence of categories.

To conclude our discussion of functors and natural transformations between quasicategories, we show that they are not only compatible with the homotopy functor $h : \mathrm{SSet} \to \mathrm{Cat}$ but also with the simplicial nerve $N : \mathrm{Cat} \to \mathrm{SSet}$. This is an important justification of these concepts and shows that they can indeed be seen as generalisations of functors and natural transformations for categories.

Proposition 7.3.20:
Let $\mathcal{C}, \mathcal{D}$ be categories. Then there is a simplicial isomorphism that is natural in $\mathcal{C}$ and $\mathcal{D}$

$$N(\mathcal{D}^{\mathcal{C}}) \cong \mathrm{Map}(N(\mathcal{C}), N(\mathcal{D})).$$

Proof:
From the definition of N in Example 5.2.7, 2. we obtain a chain of natural isomorphisms

$$\begin{aligned} N(\mathcal{D}^{\mathcal{C}})_n &= \mathrm{Hom}_{\mathrm{Cat}}([n+1]', \mathcal{D}^{\mathcal{C}}) \\ &\cong \mathrm{Hom}_{\mathrm{Cat}}([n+1]' \times \mathcal{C}, \mathcal{D}) \\ &\cong \mathrm{Hom}_{\mathrm{SSet}}(N([n+1]' \times \mathcal{C}), N(\mathcal{D})) \\ &\cong \mathrm{Hom}_{\mathrm{SSet}}(\Delta^n \times N(\mathcal{C}), N(\mathcal{D})) \\ &\cong \mathrm{Map}(N(\mathcal{C}), N(\mathcal{D})), \end{aligned}$$

where we used

- the identification of functor categories from Proposition 2.3.2 to pass to the second line,
- then the fact that the nerve is full faithful to pass to the third line,
- that the nerve preserves products as a right adjoint and the identity $\Delta^n = N([n]')$ from Example 5.2.7, 2. and 3. to pass to the fourth line,
- Definition 7.3.13 of the function complex to pass to the fifth line.

□

References:

- **Kan complexes:**
  - Chapter 7 in Friedman, G. (2008),
    An elementary illustrated introduction to simplicial sets,
  - Chapter I.3 in Goerss, P. G., & Jardine, J. F. (2009) Simplicial homotopy theory,
  - Chapter I in May, J. P. (1992) Simplicial objects in algebraic topology,
  - Chapter 7 in Rezk, C. (2022) Introduction to quasicategories,
  - Chapter 10.12 in Richter, B. (2020) From categories to homotopy theory.

- **Simplicial homotopy groups:**
  - Chapter 9 in Friedman, G. (2008),
    An elementary illustrated introduction to simplicial sets,
  - Chapter I.6-I.8 in Goerss, P. G., & Jardine, J. F. (2009) Simplicial homotopy theory,
  - Chapter I.3.2 in Lurie, J. Kerodon,
  - Chapter I in May, J. P. (1992) Simplicial objects in algebraic topology.

- **Quasicategories:**
  - Chapters I.1.3 and I.1.4 in Lurie, J. Kerodon,
  - Chapter 10.13 in Richter, B. (2020) From categories to homotopy theory,
  - Chapters 8-13 and 20 in Rezk, C. (2022) Introduction to quasicategories,

8 Exercises

8.1 Exercises for Chapter 1

Exercise 1: Let A, B, C be sets.

- A **relation** between A and B is a subset $R \subset A \times B$.
- A relation $R \subset A \times B$ is called a **map** from A to B, if for every $a \in A$ there is a unique $b \in B$ with $(a, b) \in R$.
- The composite of two relations $R \subset A \times B$ and $S \subset B \times C$ is the relation

$$S \circ R = \{(a, c) \in A \times C \mid \exists b \in B : (a, b) \in R, (b, c) \in S\} \subset A \times C.$$

(a) Show that sets and relations form a category **Rel** with $\mathrm{Hom}_{\mathrm{Rel}}(A, B) = \mathcal{P}(A \times B)$.

(b) Determine the isomorphisms in **Rel**.

(c) Show that the disjoint union of sets defines both, a product and a coproduct in Rel

Exercise 2: Let $\mathcal{C}$ be a small category and $\mathcal{D}$ a category in which products (coproducts) exist for all (finite) families $(D_i)_{i \in I}$ of objects in $\mathcal{D}$. Show that then products (coproducts) exist for any (finite) family $(F_i)_{i \in I}$ of objects in the functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Exercise 3: The **abelisation** of a group G is the factor group $G/[G, G]$ with respect to the normal subgroup $[G, G] = \{[g, g'] \mid g, g' \in G\} \subset G$, where $[g, g'] = gg'g^{-1}g'^{-1}$.

(a) Show that the abelisation has the following universal property:
For every group homomorphism $f : G \to A$ into an abelian group A, there is a unique group homomorphism $f' : G/[G, G] \to A$ with $f' \circ \pi = f$, where $\pi : G \to G/[G, G]$, $g \mapsto g[G, G]$ is the canonical surjection.

(b) The abelisation defines a functor $\mathrm{Ab} : \mathrm{Grp} \to \mathrm{Ab}$.

(c) The functor $\mathrm{Ab} : \mathrm{Grp} \to \mathrm{Ab}$ is left adjoint to the inclusion functor $I : \mathrm{Ab} \to \mathrm{Grp}$.

Exercise 4: Let k be a commutative ring. We consider

- the functor $F : \mathrm{Grp} \to \mathrm{Alg}_k$ that assigns to a group G its group algebra $k[G]$ and to a group homomorphism $f : G \to H$ the induced group homomorphism $k[f] : k[G] \to k[H]$, $g \mapsto f(g)$,
- the functor $G : \mathrm{Alg}_k \to \mathrm{Grp}$ that assigns to an algebra A its group $A^\times$ of units and to an algebra homomorphism $f : A \to B$ the induced group homomorphism $f^\times : A^\times \to B^\times$.

Show that F is left adjoint to G.

Exercise 5: Determine, if the forgetful functor $V : \mathrm{Ring} \to \mathrm{Set}$ from the category of unital rings and unital ring homomorphisms is representable. If yes, determine its representing object.

8.2 Exercises for Chapter 2

Exercise 6: A category $\mathcal{J}$ is called **connected**, if for any two objects $J, J' \in \mathrm{Ob}\mathcal{J}$ there is a finite sequence of objects $J = J_0, J_1, \ldots, J_n = J'$ with $\mathrm{Hom}_{\mathcal{J}}(J_i, J_{i+1}) \cup \mathrm{Hom}_{\mathcal{J}}(J_{i+1}, J_i) \neq \emptyset$ for each $i = 0, ..., n-1$.

Show that for a connected category $\mathcal{J}$, every constant functor $\Delta(C) : \mathcal{J} \to \mathcal{C}$ has limit and colimit cone $\mathrm{id}_{\Delta(C)} : \Delta(C) \Rightarrow \Delta(C)$. Show that this is not true for a non-connected category.

Exercise 7: Determine which coproducts exist in the category Grp^{fin} of finite groups and group homomorphisms between them.

Exercise 8: Let J, C be sets. Show that the power $C^J = \Pi_J C$ and the copower $J \cdot C = \amalg_J C$ in the category Set are isomorphic to the set of functions $f : J \to C$ and the product set $J \times C$, respectively. Use the universal property of the (co)products in Set.

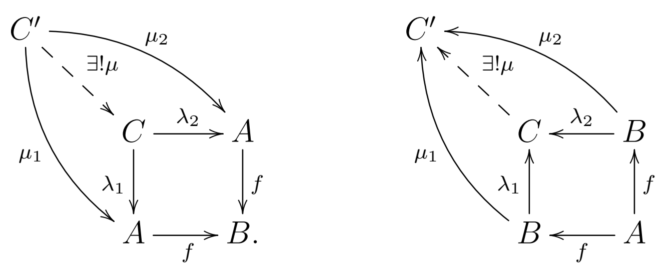
Exercise 9: Let A be a commutative algebra over a field $\mathbb{F}$ and X a finite set. Show that the copower $X \cdot A$ in the category $\mathrm{CommAlg}_{\mathbb{F}}$ of commutative algebras over $\mathbb{F}$ is isomorphic to the $|X|$-fold tensor product $A^{\otimes |X|}$.

Exercise 10: Let $\mathcal{J}$ be a small category with an initial (terminal) object and $\mathcal{C}$ a category. Compute the limit (colimit) of a functor $F : \mathcal{J} \to \mathcal{C}$.

Exercise 11: Let $\mathcal{C} = \mathrm{Ab} = \mathbb{Z}\text{-Mod}$. Determine the pullback and pushout of the group homomorphisms $f_1 : \mathbb{Z} \to \mathbb{Z}$, $z \mapsto mz$ and $f_2 : \mathbb{Z} \to \mathbb{Z}$, $z \mapsto nz$ for $m, n \in \mathbb{N}$.

Exercise 12: A **monomorphism** in a category $\mathcal{C}$ is a morphism $\iota : C \to C'$ in $\mathcal{C}$ such that $\iota \circ f_1 = \iota \circ f_2$ for morphisms $f_1, f_2 : X \to C$ implies $f_1 = f_2$. An **epimorphism** in $\mathcal{C}$ is a morphism $\pi : C \to C'$ such that $g_1 \circ \pi = g_2 \circ \pi$ for two morphisms $g_1, g_2 : C' \to Y$ implies $g_1 = g_2$. Show that equalisers are monomorphisms and coequalisers epimorphisms.

Exercise 13: Let $f : A \to B$ be a morphism in category $\mathcal{C}$. A **(co)kernel pair** of f is a pullback (pushout) of the form



Show that

(a) f is a monomorphism if and only if the kernel pair of f exists and is of the form $\lambda_1 = \lambda_2 : C \to A$.
(b) f is an epimorphism if and only if the cokernel pair of f exists and is of the form $\lambda_1 = \lambda_2 : B \to C$.

Exercise 14: (Co)products of two objects X_1, X_2 in a category $\mathcal{C}$ are also called **binary** coproducts.

(a) Show that a category $\mathcal{C}$ has all binary products and equalisers if and only if it has all binary products and all pullbacks.
(b) Show that a category $\mathcal{C}$ has all binary coproducts and coequalisers if and only if it has all binary coproducts and all pushouts.

Exercise 15: Let $(X, \preceq)$ be a partially ordered set and $S \subset X$ a subset.

- An **infimum** of S is an element $x \in X$ with $x \preceq s$ for all $s \in S$ and $x' \preceq x$ for all $x' \in X$ with $x' \preceq s$ for all $s \in S$.
- A **supremum** of S is an element $x \in X$ with $s \preceq x$ for all $s \in S$ and $x \preceq x'$ for all $x' \in X$ with $s \preceq x'$ for all $s \in S$.

Show that a diagram $F : \mathcal{J} \to \mathcal{C}$ in a poset category $\mathcal{C}$ has a limit (colimit) if and only if the set $S = \{F(J) \mid J \in \mathrm{Ob}\mathcal{J}\}$ has an infimum (supremum). Investigate the existence of limits and colimits for diagrams in the following poset categories

(a) $\mathcal{C} = (\mathbb{R}, \leq)$
(b) $\mathcal{C} = (\mathcal{P}(X), \subseteq)$ where $\mathcal{P}(X)$ denotes the power set of a set X.

Exercise 16: Let X be a topological space.

(a) Show that the presheaf $F : \mathcal{O}(X)^{op} \to \mathrm{Set}$ that assigns to an open subset $U \subset X$ the set of continuous maps $f : U \to \mathbb{R}$ is a sheaf.
(b) Show that the presheaf $G : \mathcal{O}(X)^{op} \to \mathrm{Set}$ that assigns to an open subset $U \subset X$ the set $G(U)$ of *bounded* continuous functions $f : U \to \mathbb{R}$ is not a sheaf.

Exercise 17: Let X be a topological space and $x \in X$ a point. Fix a set S and a one-point set $\{\bullet\}$. Show that there is a presheaf $F : \mathcal{O}(X)^{op} \to \mathrm{Set}$ with $F(U) = S$ if $x \in U$ and $F(U) = \{\bullet\}$ otherwise. Determine if this is a sheaf.

Exercise 18: Let G be a group and BG the associated category with a single object $\bullet$ and $\mathrm{Hom}_{BG}(\bullet, \bullet) = G$ with the group multiplication as composition.

(a) Show that a functor $F : BG \to \mathrm{Set}$ is a G-Set and a natural transformation $\eta : F \Rightarrow F'$ between such functors a G-equivariant map.
(b) Compute the limit and colimit of a functor $F : BG \to \mathrm{Set}$.
(c) Compute the right and left adjoint of the functor $\Delta : \mathrm{Set} \to \mathrm{Set}^{BG}$.

Exercise 19: Let $\mathcal{J}$ be a small category, $\mathcal{C}$ a complete category and $\mathcal{D}$ the category with two objects I, T and a single non-identity morphism $d : I \to T$.

(a) Describe the functor category $\mathcal{C}^{\mathcal{D}}$ using only quantities from $\mathcal{C}$.
(b) Describe the functor category $(\mathcal{C}^{\mathcal{D}})^{\mathcal{J}}$ using only quantities from $\mathcal{C}$ and $\mathcal{J}$.
(c) Describe the limit cone of a functor $F : \mathcal{J} \to \mathcal{C}^{\mathcal{D}}$ using only quantities from $\mathcal{C}$ and $\mathcal{J}$.

Exercise 20: Let $\mathcal{I}$ and $\mathcal{J}$ be discrete categories and $F : \mathcal{I} \times \mathcal{J} \to \mathbb{R}$ a functor into the poset category $(\mathbb{R}, \leq)$. Formulate the statement of Lemma 2.3.5 without using categorical concepts for this case and give an example in which the morphism g in Lemma 2.3.5 is not an isomorphism.

Exercise 21: Let $f : A \to B$ be a map and $f^{-1} : \mathcal{B} \to \mathcal{A}$ be the functor between the the poset categories $\mathcal{A} = (\mathcal{P}(A), \subset)$ and $\mathcal{B} = (\mathcal{P}(B), \subset)$ that assigns to each subset $X \subset B$ its preimage $f^{-1}(X) \subset A$.

(a) Show that f^{-1} has both, a left and a right adjoint.
(b) Determine how they behave with respect to unions and intersections.

Exercise 22: Denote by $A \times B$ the product and by $A + B$ the disjoint union of sets A, B. Show that for all sets A, B, C there are natural isomorphisms

$$A \times (B + C) \cong A \times B + A \times C \qquad (B \times C)^A \cong B^A \times C^A.$$

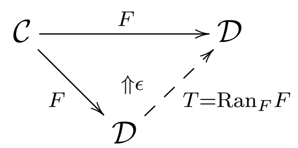
Hint: Consider the functors $(-)^A : \mathrm{Set} \to \mathrm{Set}$ and $A \times - : \mathrm{Set} \to \mathrm{Set}$.

8.3 Exercises for Chapter 3

Exercise 23: Show that the left and right Kan extensions given in Example 3.1.6 have the universal property of the left and right Kan extensions.

Exercise 24: Let [2] and [3] be the poset categories for $[2] = \{0, 1\}$ and $[3] = \{0, 1, 2\}$ and $K : [2] \to [3]$ the functor with $K(0) = 0$ and $K(1) = 2$. Show that for any functor $F : [2] \to \mathcal{C}$ the left and right Kan extensions exist by constructing them explicitly.

Exercise 25: Suppose $F : \mathcal{C} \to \mathcal{D}$ has a right Kan extension (T, ϵ) along itself



Show that $T : \mathcal{D} \to \mathcal{D}$ is a **monad**:
there are natural transformations $\eta : \mathrm{id}_{\mathcal{D}} \Rightarrow T$ and $\mu : T^2 \Rightarrow T$ with

$$\mu \circ \mu T = \mu \circ T\mu \qquad \mu \circ T\eta = \mu \circ \eta T = \mathrm{id}_T.$$

Remark: The monad T is called the **codensity monad** of F.

Exercise 26: We consider the poset categories $(\mathbb{Q}, \leq)$, $(\mathbb{R}_{>0}, \leq)$ and $(\mathbb{R}, \leq)$. Then the map $F : \mathbb{Q} \to \mathbb{R}_{>0}$, $q \mapsto 2^q$ defines a functor $F : \mathbb{Q} \to \mathbb{R}_{>0}$ between the associated poset categories. Determine the left and right Kan extensions $\mathrm{Lan}_\iota F : \mathbb{R} \to \mathbb{R}_{>0}$ and $\mathrm{Ran}_\iota F : \mathbb{R} \to \mathbb{R}_{>0}$ of F along the inclusion functor $\iota : \mathbb{Q} \to \mathbb{R}$.

Exercise 27: Let $\mathcal{C}$, $\mathcal{D}$ be poset categories, $K : \mathcal{C} \to \mathcal{D}$ and $F : \mathcal{C} \to (\mathbb{R}, \leq)$ a functor into the poset category $(\mathbb{R}, \leq)$. Compute the left and right Kan extensions of F along K with the (co)limit formula.

Exercise 28: Let $\mathcal{C}$ and $\mathcal{D}$ be small discrete categories given by sets $\mathrm{Ob}\mathcal{C} = C$ and $\mathrm{Ob}\mathcal{D} = D$, $K : \mathcal{C} \to \mathcal{D}$ the functor given by a map $K : C \to D$ and $F : \mathcal{C} \to \mathcal{E}$ be a functor into a bicomplete category $\mathcal{E}$.

(a) Compute the left and right Kan extension of F with the (co)limit formula.
(b) Determine the natural transformations $\eta : F \Rightarrow \mathrm{Lan}_K FK$ and $\epsilon : \mathrm{Ran}_K FK \Rightarrow F$ and show that the pairs $(\mathrm{Lan}_K F, \eta)$ and $(\mathrm{Ran}_K F, \epsilon)$ have the universal property of the left and right Kan extension.

Exercise 29: Consider the categories

- $\mathcal{A}$ with a single object 0 and 1_0 as the only morphism,
- $\mathcal{B}$ with two objects, 0 and 1, and only identity morphisms 1_0 and 1_1,
- $\mathcal{C}$ with two objects, 0 and 1, identity morphisms 1_0, 1_1 and a morphism $d : 0 \to 1$.

Construct a left Kan extension that is not pointwise.

Exercise 30: The **comma category** $F \downarrow G$ for functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{E} \to \mathcal{D}$ is the category with

- triples (C, E, f) of $C \in \mathrm{Ob}\mathcal{C}$, $E \in \mathrm{Ob}\mathcal{E}$ and a morphism $f : F(C) \to G(E)$ as objects,
- pairs (c, e) of morphisms $c : C \to C'$ and $e : E \to E'$ with $G(e) \circ f = f' \circ F(c)$ as morphisms from (C, E, f) to (C', E', f').

(a) Show that the categories $H \downarrow D$ and $D \downarrow H$ for a functor $H : \mathcal{C} \to \mathcal{D}$ are special cases of comma categories $F \downarrow G$.
(b) Show that the categories $\mathrm{cone}(H)$ and $\mathrm{cocone}(H)$ for a functor $H : \mathcal{C} \to \mathcal{D}$ are special cases of comma categories $F \downarrow G$.

Exercise 31:

- A **directed graph** $\Gamma = (E, V, s, t)$ consists of sets E, V and maps $s, t : E \to V$.
- A **morphism of directed graphs** $\phi = (\phi_E, \phi_V) : \Gamma \to \Gamma'$ is a pair of maps $\phi_E : E \to E'$ and $\phi_V : V \to V'$ with $t' \circ \phi_E = \phi_V \circ t$ and $s' \circ \phi_E = \phi_V \circ s$.
- $\mathcal{G}$ denotes the category of directed graphs and morphisms of directed graphs.
- $\mathcal{C}$ denotes the category with $\mathrm{Ob}\mathcal{C} = \{I, T\}$ and two non-identity morphisms $\sigma, \tau : I \to T$

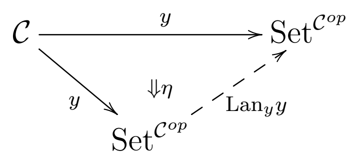
$$I \underset{\sigma}{\overset{\tau}{\rightrightarrows}} T$$

(a) Show that the functor categories $\mathrm{Set}^{\mathcal{C}^{op}} \cong \mathrm{Set}^{\mathcal{C}}$ are isomorphic to $\mathcal{G}$.

(b) Determine the comma category $y \downarrow \Gamma$ for $\Gamma \in \mathrm{Ob}\mathcal{G}$ and the covariant Yoneda embedding $y : \mathcal{C} \to \mathrm{Set}^{\mathcal{C}^{op}}$. Interpreting its objects as vertices and its morphisms as edges yields a directed graph constructed from Γ whose edges and vertices are partitioned into two sets.

(c) Determine the left Kan extension $\mathrm{Lan}_y F$ for a functor $F : \mathcal{C} \to \mathrm{Set}$ that corresponds to
   (i) a graph with a single edge connecting two distinct vertices,
   (ii) a graph with a single vertex and no edges.

Exercise 32: Let $\mathcal{C}$ be a small category. Determine the left Kan extension of the covariant Yoneda embedding along itself:



Exercise 33: Let $F : \mathcal{C} \to \mathrm{Cat}$ be a functor.
The **Grothendieck construction** is the category $\mathcal{C} \int F$, where

- objects are pairs (C, X) of objects $C \in \mathrm{Ob}\mathcal{C}$ and $X \in \mathrm{Ob}F(C)$,
- morphisms from (C_1, X_1) to (C_2, X_2) are pairs (f, g) of a morphism $f : C_1 \to C_2$ in $\mathcal{C}$ and a morphism $g : F(f)(X_1) \to X_2$ in $F(C_2)$,
- composition of $(f_1, g_1) : (C_1, X_1) \to (C_2, X_2)$ and $(f_2, g_2) : (C_2, X_2) \to (C_3, X_3)$ is given by $(f_2, g_2) \circ (f_1, g_1) = (f_2 \circ f_1, g_2 \circ F(f_2)(g_1))$.

(a) Show that any natural transformation $\mu : F \Rightarrow F'$ between functors $F, F' : \mathcal{C} \to \mathrm{Cat}$ induces a functor $\mathcal{C} \int \mu : \mathcal{C} \int F \to \mathcal{C} \int F'$.

(b) Show that a functor $G : \mathcal{C} \int F \to \mathcal{D}$ into a category $\mathcal{D}$ corresponds to the following data:
   - a functor $G_C : F(C) \to \mathcal{D}$ for every $C \in \mathrm{Ob}\mathcal{C}$,
   - a natural transformation $G_f : G_{C_1} \Rightarrow G_{C_2}$ for every morphism $f : C_1 \to C_2$ in $\mathcal{C}$, such that $G_{1_C} = \mathrm{id}_{G_C}$ and $G_{f_2 \circ f_1} = G_{f_2} \circ G_{f_1}$ for all composable morphisms f_1, f_2.

(c) Let N, H be groups and $\phi : G \to \mathrm{Aut}(N)$ a group homomorphism. Show that there is a functor $F : BH \to \mathrm{Cat}$ with $F(\bullet) = BN$ and that the associated Grothendieck construction is the category $B(N \rtimes_\phi H)$.

Exercise 34: Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors. Show the following: If (G, η) is a left Kan extension of $\mathrm{id}_{\mathcal{C}}$ along F and F preserves this left Kan extension, then F is left adjoint to G with the unit of the adjunction given by η.

8.4 Exercises for Chapter 4

Exercise 35: Show that every natural transformation $\nu : F \Rightarrow G$ with $F, G : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{E}$ defines a dinatural transformation $\nu' : F \overset{\bullet}{\Rightarrow} G$ with $\nu'_C = \nu_{C,C} : F(C, C) \to G(C, C)$.

Exercise 36: Show that a (co)limit of a functor $F : \mathcal{C} \to \mathcal{D}$ can be expressed as a (co)end of an appropriate functor $G : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$:

$$\int_{C\in\mathcal{C}} G(C,C) \cong \lim F \qquad \int^{C\in\mathcal{C}} G(C,C) \cong \mathrm{colim} F.$$

Exercise 37: Let $F : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be a functor with an end and a coend. Prove that there are natural isomorphisms

$$\mathrm{Hom}_{\mathcal{D}}(D, \int_{C\in\mathcal{C}} F(C,C)) \cong \int_{C\in\mathcal{C}} \mathrm{Hom}_{\mathcal{D}}(D, F(C,C))$$
$$\mathrm{Hom}_{\mathcal{D}}(\int^{C\in\mathcal{C}} F(C,C), D) \cong \int^{C\in\mathcal{C}} \mathrm{Hom}_{\mathcal{D}}(F(C,C), D)$$

Exercise 38: Let $\mathcal{D}$ be a small category, $\mathcal{C}$ the discrete category with $\mathrm{Ob}\mathcal{C} = \mathrm{Ob}\mathcal{D}$ and only identity morphisms and $K : \mathcal{C} \to \mathcal{D}$ the inclusion functor. Let $F : \mathcal{C} \to \mathcal{E}$ be a functor into a bicomplete category $\mathcal{E}$.

(a) Determine the end and coend of a functor $H : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{E}$.
(b) Compute $\mathrm{Lan}_K F(D)$ and $\mathrm{Ran}_K F(D)$ for $D \in \mathrm{Ob}\mathcal{D}$ with the (co)end formula.
(c) Determine the functors $\mathrm{Lan}_K F, \mathrm{Ran}_K F : \mathcal{D} \to \mathcal{E}$ on the morphisms and the natural transformations $\eta : F \Rightarrow (\mathrm{Lan}_K F)K$ and $\epsilon : (\mathrm{Ran}_K F)K \Rightarrow F$.

Exercise 39: Let R be a ring and $F : R\text{-Mod} \to \mathrm{Set}$ the forgetful functor.

(a) Determine $\mathrm{Lan}_F F : \mathrm{Set} \to \mathrm{Set}$ with the coend formula.
(b) Show that $\mathrm{Lan}_F F \cong \mathrm{Hom}_{\mathrm{Set}}(R, -) : \mathrm{Set} \to \mathrm{Set}$ and verify the universal property of the left Kan extension.

8.5 Exercises for Chapter 5

Exercise 40: Give a combinatorial simplicial complex that describes the cylinder $[0,1] \times S^1$.

Exercise 41: Give the canonical factorisation for the following morphisms in Δ.

(a) $\tau_i : [1] \to [n+1]$, $0 \mapsto i$ for $0 \leq i \leq n$,
(b) $\rho^i_n : [n+1] \to [2]$ with $(\rho^i_n)^{-1}(0) = \{0, 1, \ldots, i-1\}$,
(c) $\tau : [4] \to [5]$, $0 \mapsto 2$, $1, 2 \mapsto 3$, $3 \mapsto 4$,
(d) $\sigma^1_3 \circ \sigma^2_4 \circ \delta^3_4 \circ \delta^2_3 : [3] \to [3]$.

Exercise 42: Consider the simplicial set $\Delta^n = \mathrm{Hom}(-, [n+1]) : \Delta^{op} \to \mathrm{Set}$ for $n \in \mathbb{N}_0$. Show that there are exactly $\binom{n+m+1}{m}$ simplicial maps $\alpha : \Delta^n \Rightarrow \Delta^m$. Give explicitly the 10 simplicial maps $\alpha : \Delta^1 \Rightarrow \Delta^3$ and visualise them with standard n-simplexes in Top.

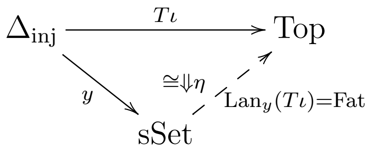
Exercise 43: Consider the simplicial set $\Delta^n = \mathrm{Hom}(-, [n+1]) : \Delta^{op} \to \mathrm{Set}$.

(a) Determine the non-degenerate elements in the sets $\Delta^n_k = \mathrm{Hom}([k+1], [n+1])$ for $k \in \mathbb{N}_0$.
(b) Determine the geometric realisation of Δ^n with Theorem 5.3.3.

Exercise 44: Denote by $\Delta_{\mathrm{inj}} \subset \Delta$ the subcategory of Δ with the same objects and only *injective* monotonic maps as morphisms and by $\iota : \Delta_{\mathrm{inj}} \to \Delta$ the inclusion functor.

- A functor $S : \Delta^{op}_{\mathrm{inj}} \to \mathrm{Set}$ is called a **semisimplicial set** and a natural transformation $\mu : F \Rightarrow F'$ between such functors a **semisimplicial map**.
- We denote by $\mathrm{sSet} = \mathrm{Set}^{\Delta^{op}_{\mathrm{inj}}}$ the category of semisimplicial sets and maps.

The **fat realisation** of semisimplicial set is the left Kan extension $\mathrm{Fat} = \mathrm{Lan}_y(T\iota) : \mathrm{sSet} \to \mathrm{Top}$ of the functor $T\iota : \Delta_{\mathrm{inj}} \to \mathrm{Top}$ along the covariant Yoneda embedding $y : \Delta_{\mathrm{inj}} \to \mathrm{sSet}$



(a) Show that every ordered combinatorial simplicial complex $\mathcal{K} \subset \mathcal{P}(V)$ defines a semisimplicial set $s : \Delta_{\mathrm{inj}} \to \mathrm{Set}$ with

$$s_n = \{(v_0, \ldots, v_n) \mid \{v_0, \ldots, v_n\} \in \mathcal{K}, v_0 < \ldots < v_n\}$$
$$d^i_n : s_n \to s_{n-1},\ (v_0, \ldots, v_n) \mapsto (v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n) \qquad 0 \leq i \leq n.$$

where we take d^i_n as the empty map, if $s_n = \emptyset$.
(b) Give a description of the fat realisation analogous to the one for the geometric realisation in Proposition 5.3.2. Show that every point in $\mathrm{Fat}(s)$ for a semisimplicial set s is represented by a unique pair (s, x) with $s \in s_n$, $x \in \mathring{\Delta}^n$.
(c) Describe the semisimplicial sets $\Delta^n = \mathrm{Hom}_{\Delta_{\mathrm{inj}}}(-, [n+1]) : \Delta^{op}_{\mathrm{inj}} \to \mathrm{Set}$ for $n \in \mathbb{N}$ and visualise them for $n = 0, 1, 2, 3$.
(d) Describe the right adjoint of $\mathrm{Fat} : \mathrm{sSet} \to \mathrm{Top}$.

Exercise 45: Let $\rhd : G \times X \to X$ be an action of a group G on a set X and $X//G$ the associated action groupoid with

- elements of X as objects,
- elements $g \in G$ with $g \rhd x = x'$ as morphisms from x to x'.

Determine the nerve of $X//G$.

Exercise 46: Let $(J, \preceq)$ a poset and $S : \Delta^{op} \to \mathrm{Set}$ the simplicial set given by

$$S_n = \{(j_0, \ldots, j_n) \mid j_0, \ldots, j_n \in J \text{ with } j_0 \preceq j_1 \preceq \ldots \preceq j_n\} \qquad n \in \mathbb{N}_0$$
$$d^i_n(j_0, \ldots, j_n) = (j_0, \ldots, j_{i-1}, j_{i+1}, \ldots, j_n) \qquad n \in \mathbb{N}_0, 0 \leq i \leq n$$
$$s^i_n(j_0, \ldots, j_n) = (j_0, \ldots, j_{i-1}, j_i, j_i, j_{i+1}, \ldots, j_n).$$

Determine its homotopy category $h(S)$.

Exercise 47: Consider the *injective* standard n-simplex $\Delta^n = \mathrm{Hom}_{\Delta_{\mathrm{inj}}}(-, [n{+}1]) : \Delta_{\mathrm{inj}}^{op} \to \mathrm{Set}$ and denote by $\cup\Delta^n = \dot\cup_{k=0}^n \Delta^n_k = \dot\cup_{k=0}^n \mathrm{Hom}_{\Delta_{\mathrm{inj}}}([k+1],[n+1])$ the set of all elements of Δ^n.

(a) Show that $\cup\Delta^n$ has the structure of a poset.

(b) Denote by $N^n : \Delta^{op} \to \mathrm{Set}$ the nerve of the poset category for $\cup\Delta^n$. Describe the sets $ND^n_k \subset N^n_k$ of non-degenerate elements in N^n_k.

(c) Relate the sets ND^n_k from (b) to the **barycentric subdivision** of an affine simplex and draw the associated barycentric subdivision of an affine 2-simplex.

(d) Show that the sets $ND^n_k \subset N^n_k$ define a functor $B : \Delta_{\mathrm{inj}} \to \mathrm{sSet}$ with $B([n+1]) = ND^n$.

(e) Define the barycentric subdivision functor $B' : \mathrm{sSet} \to \mathrm{sSet}$ for general semisimplicial sets.

Remark:

- The barycentre of an affine k-simplex $w = [w_0, \ldots, w_k]$ is $b(w) = \frac{1}{k+1}(w_0 + \ldots + w_k)$.
- The barycentric subdivision of w is given by the k-simplexes $[b(f_0), \ldots, b(f_k)]$, where $b(f_i)$ is the barycentre of the i-face f_i of w with $f_0 \subset f_1 \subset \ldots \subset f_k$.

Exercise 48: Determine the classifying spaces of the following categories:

(a) the category with two objects I, T and two non-identity morphisms $\sigma, \tau : I \to T$,

(b) the category with three objects C_0, C_1, C_2 and two non-identity morphisms $f_1 : C_0 \to C_1$ and $f_2 : C_0 \to C_2$.

Exercise 49: Let X be a topological space and $S := \mathrm{Sing}(X) : \Delta^{op} \to \mathrm{Set}$ the associated simplicial set with

$$\begin{aligned} S_n &= \mathrm{Hom}_{\mathrm{Top}}(\Delta^n, X)\rangle_{\mathbb{Z}} = \{\tau : \Delta^n \to X \text{ continuous}\} \\ d^i_n &= S(\delta^i_n) : S_n \to S_{n-1},\ \tau \mapsto \tau \circ f^n_i \\ s^i_n &= S(\sigma^i_{n+1}) : S_n \to S_{n+1},\ \tau \mapsto \tau \circ s^n_i, \end{aligned}$$

where $f^n_i : \Delta^{n-1} \to \Delta^n$ and $s^n_i : \Delta^{n+1} \to \Delta^n$ are the affine linear face maps and degeneracies. Determine the homotopy category $h(S)$.

Exercise 50: Let k be a commutative ring. Consider the sets and maps

$$\begin{aligned} &S_n = \{\tau : [n+1] \to [4] \mid \tau \text{ monotonic and not surjective}\} && n \in \mathbb{N}_0 \qquad (52) \\ &d^i_n : S_n \to S_{n-1},\ \tau \mapsto \tau \circ \delta^i_n && n \in \mathbb{N} \\ &s^j_n : S_n \to S_{n+1},\ \tau \mapsto \tau \circ \sigma^j_{n+1} && n \in \mathbb{N}_0. \end{aligned}$$

(a) Show that this defines a simplicial set $S : \Delta^{op} \to \mathrm{Set}$ and describe its geometric realisation.

(b) Let $S' : \Delta^{op} \to R\text{-Mod}$ the associated simplicial module with $S'_n = \langle S_n \rangle_R$ and face maps and degeneracies given by (52). Determine its normalised chain complex $NS'_\bullet$.

(c) Compute the homologies of $NS'_\bullet$.

Hint: The proof of Proposition 5.5.8 tells you how to compute $NS'_\bullet$.

Exercise 51: Let k be a commutative ring. Let $\mathcal{C}$ be the category with two objects I, T and two non-identity morphisms $\sigma, \tau : I \to T$ and $C_\bullet(\mathcal{C}, k)$ the associated chain complex from Example 5.5.6, 2. Compute its homologies.

Hint: Use Proposition 5.5.8 and the associated normalised chain complex.

8.6 Exercises for Chapter 6

Exercise 52: Let $X \subset \mathbb{R}^n$ with the standard topology and

$$CX = \{(tx, 1-t) \mid t \in (0,1], x \in X\} \subset \mathbb{R}^{n+1} \qquad CX' = \{(tx, 1-t) \mid t \in [0,1], x \in X\} \subset \mathbb{R}^{n+1}.$$

Show that CX is homotopy equivalent to X and that CX' is contractible.

Exercise 53: Let X be a topological space and $f : S^n \to X$ and $g : D^{n+1} \to X$ continous maps with $g|_{\partial D^{n+1}} = f$. Show that for any $q \in S^n$ the map f is homotopic to a constant map relative to $\{q\}$.

Exercise 54:
In this exercise, we show that $\pi_k(S^n)$ is trivial for $0 \leq k < n$ via the following steps:

1. **C^0 maps are homotopic to C^1 maps:** Let $h : \mathbb{R}^k \to [0, \infty)$ be
   i) of compact *support* $\text{supp}(h) := \overline{\{x \in \mathbb{R}^k \mid h(x) \neq 0\}}$,
   ii) continuously differentiable,
   iii) with $\int_{\mathbb{R}^k} h(x)\, dx = 1$.

   (a) Show that the *convolution* $h*f : \mathbb{R}^k \to \mathbb{R}$, $h*f(x) := \int_{\mathbb{R}^k} h(x-y) f(y)\, dy$ of a continuous function $f : \mathbb{R}^k \to \mathbb{R}$ of compact support with h is continuously differentiable.
   (b) Show that $h_\epsilon(x) := \epsilon^{-k} h(x/\epsilon)$ with $\epsilon \in (0,1]$ has the same properties i)-iii) as h and that $h_\epsilon * f(x)$ depends continuously on ϵ, uniformly in $x \in \mathbb{R}^k$.
   (c) Show that $\lim_{\epsilon \to 0} h_\epsilon * f(x) = f(x)$, uniformly in $x \in \mathbb{R}^k$.

   Remark: Applying (a)-(c) to the components yields analogous results for continuous maps $f : \mathbb{R}^k \to \mathbb{R}^n$. By using partitions of unity one can thus homotope continuous maps between manifolds to continuously differentiable maps.

2. **$\pi_k(S^n)$ is trivial for $0 \leq k < n$:**

   (a) Let $Q := [0,1]^{\times k} \subseteq \mathbb{R}^k$ and $\sim$ the equivalence relation that identifies the points of ∂Q. Show that $Q/\sim$ is homeomorphic to the sphere S^k.
   (b) To show that $\pi_0(S^n)$ is trivial, prove that S^n is path-connected if $n > 0$.
   (c) Let $f : S^k \to S^n$ be continuously differentiable. Show that f is not surjective.

(d) Use (c) and stereographic projection

$$\Phi_n : S^n \backslash \{e_{n+1}\} \to \mathbb{R}^n \quad , \quad \Phi_n(x) := \frac{2}{1 - x_{n+1}} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \tag{53}$$

to show that f is homotopic to a constant map $S^k \to S^n$.

(e) Using Part 1 and the Remark, show that every continuous map $g : S^k \to S^n$ is homotopic to a continuously differentiable map $f : S^k \to S^n$.

Exercise 55: Let $\mathcal{C} = R$-Mod. Show that for all simplicial objects $S, T : \Delta^{op} \to \mathcal{C}$ being simplicially homotopic is an equivalence relation on $\mathrm{Hom}_{\mathcal{C}^{\Delta^{op}}}(S, T)$. Proceed as follows:

(a) Show that being simplicially homotopic is reflexive.
(b) Show that for all simplicial homotopies $h : \alpha \Rrightarrow \beta$ and $h' : \alpha' \Rrightarrow \beta'$ there are simplicial homotopies $k : -\alpha \Rrightarrow -\beta$ and $k' : \alpha + \alpha' \Rrightarrow \beta + \beta'$.
(c) Show that being simplicially homotopic is symmetric and transitive.

Exercise 56: For a simplicial set $S : \Delta^{op} \to \mathrm{Set}$ we consider the simplicial sets

- $CS : \Delta^{op} \to \mathrm{Set}$ with $CS_n = S_0$ for $n \in \mathbb{N}_0$ and $CS(\alpha) = \mathrm{id}_{S_0}$ for all morphisms α in Δ^{op},
- $PS : \Delta^{op} \to \mathrm{Set}$ with $PS_n = S_{n+1}$ and $PS(\delta^i_n) = S(\delta^i_{n+1})$ and $PS(\sigma^j_n) = S(\sigma^j_{n+1})$ for all $n \in \mathbb{N}_0$, $0 \leq i \leq n$, $0 \leq j \leq n-1$.

(a) Show that $\gamma_n = S(\delta^0_{n+1} \circ \ldots \circ \delta^0_1) : S_{n+1} \to S_0$ and $\rho_n = S(\sigma^0_1 \circ \ldots \circ \sigma^0_{n+1}) : S_0 \to S_{n+1}$ define simplicial maps $\gamma : PS \Rightarrow CS$ and $\rho : CS \Rightarrow PS$ with $\gamma\rho = \mathrm{id}_{CS}$,
(b) Construct a simplicial homotopy $h : \rho\gamma \Rrightarrow \mathrm{id}_{PS}$.

Exercise 57: Let $\mathcal{C} \neq \emptyset$ be a small category. Prove the following:

(a) If $\mathcal{C}$ has an initial or terminal object, $B\mathcal{C}$ is contractible.
(b) If $\mathcal{C}$ has binary products or coproducts, $B\mathcal{C}$ is contractible.

Hint: in (b) consider the functor $C \times - : \mathcal{C} \to \mathcal{C}$ for a fixed $C \in \mathrm{Ob}\mathcal{C}$.

Exercise 58: Let $f, f' : G \to H$ be group homomorphisms and $Bf, Bf' : BG \to BH$ the associated functors.

(a) Show that natural transformations $\tau : Bf \Rightarrow Bf'$ are in bijection with elements $h \in H$ such that $f'(g) = hf(g)h^{-1}$ for all $g \in G$,
(b) Determine the simplicial homotopy $N(h) : N(Bf) \Rrightarrow N(Bf')$ induced by a natural transformation as in (a) by specifying its components $N(h)^i_n : G^{\times n} \to H^{\times(n+1)}$,
(c) Determine the induced chain homotopy $h_\bullet : C_\bullet(f, \mathbb{Z}) \Rightarrow C_\bullet(f', \mathbb{Z})$ between the chain maps $C_\bullet(f, \mathbb{Z}), C_\bullet(f', \mathbb{Z}) : C_\bullet(G, \mathbb{Z}) \to C_\bullet(H, \mathbb{Z})$ from Example 5.5.6, 3.

8.7 Exercises for Chapter 7

Exercise 59: Consider the simplicial set $\partial\Delta^n : \Delta^{op} \to \text{Set}$. Show that

$$(\partial\Delta^n)_k = \{\alpha : [k+1] \to [n+1] \mid \alpha \text{ monotonic and non-surjective}\}$$

and that $\partial\Delta^n$ is isomorphic to the $(n-1)$-skeleton of $\Delta^n = \text{Hom}(-, [n+1]) : \Delta^{op} \to \text{Set}$.

Exercise 60: Let $S : \Delta^{op} \to \text{Set}$ be a simplicial set.

(a) Show that simplicial $(n-1)$-cycles $\beta : \partial\Delta^n \Rightarrow S$ are in bijection with $(n+1)$-tuples $(x_0, ..., x_n)$ with $x_i \in S_{n-1}$ and $d^i_{n-1}(x_j) = d^{j-1}_{n-1}(x_i)$ for $i < j$.

(b) Show that for every $y \in S_n$ the element $d_n(y) = (d^0_n(y), ..., d^n_n(y))$ is a simplicial $(n-1)$-cycle.

(c) Show that $\partial\Delta^n$ is the coequaliser

$$\partial\Delta^n = \text{coequ}\left(\amalg_{0\leq i<j\leq n}\Delta^{n-2} \underset{v}{\overset{u}{\rightrightarrows}} \amalg_{0\leq i\leq n}\Delta^{n-1}\right),$$

where $u \circ \iota_{i,j}(\tau) = \iota_j \circ \delta^i_{n-1} \circ \tau$ and $v \circ \iota_{ij}(\tau) = \iota_i \circ \delta^{j-1}_{n-1} \circ \tau$ for all $\tau : [k+1] \to [n-1]$.

Exercise 61: Let $e : x \to y$, $f : y \to z$ and $g : x \to z$ be morphisms in a quasicategory X such that $g \sim f \circ e$. Show that there is a $t \in X_2$ with $d_2(t) = (f, g, e)$, witnessing g as a composite of f and e.

Conclude that two morphisms $f, g : x \to y$ in a quasicategory X are homotopic, if and only if there is an $h \in X_2$ with $d_2(h) = (1_y, f, g)$.

Exercise 62: Let X, Y, Z be simplicial sets.

(a) Show that $\text{ev} : X \times \text{Map}(X, Y) \Rightarrow Y$ is natural in X and Y.

(b) Show that for all simplicial sets X, Y, Z there are simplicial isomorphisms

$$\text{Map}(X \times Y, Z) \cong \text{Map}(X, \text{Map}(Y, Z)).$$

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Index

G-equivariant maps, 40
G-equivariant morphisms, 40
G-objects, 40
G-sets, 40
Δ-complex, 80
∞-category, 149
∞-groupoid, 149
k-morphisms
  quasicategory, 150
n-boundaries, 96
n-truncation, 90
p-adic integers, 33
p-adic numbers, 33
(co)limit
  finite, 36
  sequential, 32
  small, 36
1-morphism, 147
2-category, 147
2-morphism, 147, 150

abelisation, 10
action groupoid, 57
adjoint functor, 15
adjunction, 15
affine simplex, 79
apex, 25
associator, 147
attaching
  n-cells, 30
  map, 30
  topological space, 30

barycentric subdivision, 169
basepoint, 137
bicategory, 147
bicomplete, 36
binary, 163
boundary
  simplicial set, 128, 129
boundary operators, 96

cartesian product of categories, 8
categorical nerve, 86
category, 6
  concrete, 7
  finite, 36
  locally small, 6
  small, 6
chain complex, 96
chain homotopic, 121
chain homotopy, 121
chain homotopy equivalence, 121
chain homotopy equivalent, 121
chain map, 96
chains, 96
classifying space, 95
  category, 95
  group, 95
cocomplete, 36
cocone, 26
cocone morphism, 26
cocontinuous functor, 74
codensity monad, 164
coend, 71
coequaliser, 30, 31
cofinal object, 14
coherence theorem, 148
coinduction, 47
coinduction functor, 18, 47
cokernel, 31
cokernel pair, 29
colimit, 27
colimit cone, 27
combinatorial simplicial complex, 82
comma category, 56, 57, 165
complete, 36
completion
  horn, 129
composite, 150
  2-morphisms, 150
  morphisms is quasicategory, 149
composition
  functors, 9
  morphisms, 6
  natural transformation with functor, 12
  natural transformations, 11
concrete category, 7
cone, 25
cone morphism, 26
connected, 27, 162
continuous functor, 74
contractible, 107

contravariant functor, 9
contravariant Yoneda embedding, 45
copower, 29
  of simplicial set and simplicial object, 114
copower functor, 76
coproduct in category, 13, 27
cosimplicial map, 84
cosimplicial morphism, 84
cosimplicial object, 84
cosimplicial set, 84
counit
  adjunction, 18
covariant Yoneda embedding, 45
cowedge, 70
cowedge morphism, 70
crossed module, 148
CW-complex, 33
cycle
  simplicial set, 129
cycles, 96

degeneracies, 84
degeneracy
  morphism, 83
  standard n-simplex, 79
degenerate, 79, 89
degenerate chain complex, 100
delooping, 9, 39
diagram, 25
  finite, 36
  small, 36
dinatural transformation, 69
direct limit, 32
directed graph, 165
discrete category, 27
Dold-Kan correspondence, 101, 104

Eckmann-Hilton argument, 110
edges, 79
embedding functor, 25
end, 70
endofunctor, 9
endomorphism in category, 6
epimorphism, 162
equaliser, 30, 31
equivalence
  of categories, 12
equivalence of quasicategories, 159
essentially surjective functor, 12
evaluation map
  function complex, 157
exact, 96
exact in n, 96
exponential law, 157

face, 79
  morphism, 83
  simplicial set, 128
face maps, 84
  standard n-simplex, 79
factorisation
  simplex category, 83
fat realisation, 168
fibrant, 131
fibre product, 28
filler, 129
final object, 14
final topology, 33
finite (co)limit, 36
finite category, 36
finite diagram, 36
finitely cocomplete, 36
finitely complete, 36
forgetful functor, 9
free product
  groups, 29
  rings, 29
Fubini's Theorem, 74
full subcategory, 8
fully faithful functor, 12
function complex, 157
functor, 9
  cocontinuous, 74
  continuous, 74
  creates (co)limits, 44
  preserves (co)limits, 44
  reflects (co)limits, 44
  representable, 20
functor category, 11
functor of quasicategories, 159
fundamental group, 10, 109
fundamental groupoid, 148

generators
  category, 84
geometric realisation, 88
  combinatorial simplicial complex, 89
Grothendieck construction, 166
group, 8
group homologies, 99

groupoid, 8

Hochschild homology, 99
Hom functors, 20
Hom-functors, 10
homology, 97
homotopic, 107
    Kan complex, 137
homotopic relative to A, 109
homotopy, 107
    Kan complex, 137
    morphisms in quasicategory, 152
homotopy category, 92, 108
    quasicategory, 154
homotopy category of chain complexes, 122
homotopy equivalence, 107
homotopy equivalent, 107
homotopy functor, 92
homotopy group, 109
homotopy type, 107
homotopy, relative to a subspace, 109
horizontal composition, 147
horn, 129
    simplicial set, 129
    standard n-simplex, 128
horn filler, 131
Hurewicz map, 145
Hurewicz theorem, 145
Hurewicz's theorem, 99

identity 1-morphism, 147
identity 2-morphism, 150
identity morphism, 6
induction, 47
induction functor, 18, 47
initial object, 28
inner horn, 128
inverse
    functor, 12
inverse limit, 32
isomorphic, objects in category, 6
isomorphism
    of categories, 12
    in category, 6

Kan complex, 131
Kan condition, 131
Kan extension, 51
    pointwise, 62
kernel, 31
kernel pair, 29

Lebesgue's theorem, 74
left adjoint functor, 15
left exact, 44
left Kan extension, 51
limit, 26
limit cone, 27
localisation, 47
locally small, 6

monad, 164
monoid, 8
monoidal category, 148
monomorphism, 162
morphism, 6
    equivalence, 155
morphism of directed graphs, 165
morphisms
    quasicategory, 149

n-skeleton, 33
nadir, 26
natural isomorphism, 10
    quasicategories, 159
natural transformation, 10
    functors between quasicategories, 159
naturally isomorphic, 10
normalised chain complex, 100
null object, 14

object, 6
object:quasicategory, 149
opposite category, 8
ordered affine simplex, 79
ordinal numbers, 12, 83
outer horn, 128, 129

pairs of topological spaces, 7
partially ordered set, 32
path, 108
    Kan complex, 140
path component, 108
    Kan complex, 140
Peiffer identities, 148
pentagon axiom, 147
pointed Kan complex, 137
pointed topological spaces, 7
pointwise Kan extension, 62
poset, 32
poset category, 32

positive chain complex, 96
power, 29
power functor, 76
pre-composition functor, 55
presentation
category, 84
preserves
Kan extension, 66
presheaf, 34
prism maps, 118
product in category, 13, 27
projection functors
comma category, 57
pullback, 28
pushout, 28

quasicategory, 149
quasigroupoid, 149
quotient category, 8

reflective subcategory, 47
reflector, 47
relation, 161
relations
category, 84
relative homotopy
simplicial maps, 137
representable functor, 20
restriction functor, 9, 17
right adjoint functor, 15
right exact, 44
right inverse
2-morphism, 150
right Kan extension, 52

Seifert-van Kampen theorem, 29, 113
semisimplicial complex, 80
semisimplicial map, 168
semisimplicial morphism, 102
semisimplicial object, 102
semisimplicial set, 168
semisimplicial sets, 91
sequential colimit, 32
sequential limit, 32
shape, 25
sheaf, 34
short exact sequence, 97
simplex category, 83
simplicial category, 83
simplicial complex, 80
combinatorial, 82
simplicial homotopy, 115
simplicial map, 84
simplicial morphism, 84
simplicial nerve, 86, 91
simplicial object, 84
simplicial relations, 84
simplicial set, 84
fibrant, 131
simplicial sphere, 128, 129
simplicial subset, 127
generated by set, 127
singular chain complex, 98
singular homologies, 98
singular nerve, 85, 88
skeleton
combinatorial simplicial complex, 82
semisimplicial complex, 80
simplicial set, 90
small, 6
small colimit, 36
small diagram, 36
source
morphism in quasicategory, 149
source of morphism, 6
standard n-simplex, 79
simplicial set, 86
standard chain complex, 97
Stone-Čech compactification, 48
strict bicategory, 147
strict monoidal category, 148
subcategory, 8
reflective, 47

target
morphism in quasicategory, 149
target of morphism, 6
terminal object, 14, 28
transformation groupoid, 57
triangle axiom, 147
trivial
morphism, 14
truncation
simplicial set, 90
twisted arrow category, 72
twisted arrow functor, 72

unique up to unique isomorphism, 13
unit
adjunction, 18

unitors, 147
universal property
  coproduct, 13
  product, 13

vertical composition, 147
vertices, 79

weak topology, 33
wedge, 70
wedge morphism, 70
wedge product, 30
witness
  quasicategory, 149

Yoneda embedding, 45
Yoneda map, 22
Yoneda-Lemma, 22

zero morphism, 14
zero object, 14