Nets of real subspaces on homogeneous spaces and Algebraic Quantum Field Theory *

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Contents

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Introduction

In these notes, we describe an interesting connection between unitary representations of Lie groups and nets of local algebras, as they appear in Algebraic Quantum Field Theory (AQFT). It is based on first translating the axioms for nets of operator algebras parameterized by regions in a spacetime manifold into those for nets of real subspaces, and then study this structure from a perspective based on geometry and representation theory of Lie groups.

This topic owes much of its fascination to the close relations between operator algebraic concepts, such as Kubo–Martin–Schwinger (KMS) conditions and spectral conditions, and the complex geometry related to unitary Lie group representations. To make this a little more concrete, suppose that $U_t = e^{itH}$ is a unitary one-parameter group on the complex Hilbert space $\mathcal{H}$, $H = H^*$ is its selfadjoint generator, and $\xi \in \mathcal{H}$. We are interested in analytic continuations of the orbit map $U^\xi \colon \mathbb{R} \to \mathcal{H}, t \mapsto U_t\xi$. If a bounded analytic extension exists on the upper half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} \colon \operatorname{Im} z > 0\}$, then its range lies in an invariant subspace on which the operator H is non-negative (spectral condition). This is rather restrictive, and it is much more common that U^ξ only extends to the closure of a strip $\mathcal{S}_\beta = \{z \in \mathbb{C} \colon 0 < \operatorname{Im} z < \beta\}$. Here the most interesting context arises if the upper boundary values are coupled to the original map by a conjugation J via

$$JU^\xi(i\beta + t) = U^\xi(t) \quad \text{for} \quad t \in \mathbb{R}.$$

This is precisely the situation one finds in the modular theory of operator algebras if ξ represents a KMS state (thermal state), and the case of positive spectrum corresponds to so-called ground states. Below we shall see that such conditions also specify so-called standard subspaces $\mathtt{V} \subseteq \mathcal{H}$ (for $\beta = \pi$) if $(U_t)_{t\in\mathbb{R}}$ is the corresponding modular group.

On the geometric side, an action σ of a Lie group G on a manifold M often has a "complexification" in the sense that M sits in the boundary of a complex manifold Ξ that locally looks like a tube domain $\mathbb{R}^n + i\Omega \subseteq \mathbb{C}^n$, i.e., $\Omega \subseteq \mathbb{R}^n$ is a pointed open convex cone. In this context, one may also ask for extensions of orbit maps $\sigma^m \colon \mathbb{R} \to M, t \mapsto \exp(tx).m$ ($m \in M, x \in \mathfrak{g} = \mathbf{L}(G)$), to the upper half-plane $\sigma^m \colon \mathbb{C}_+ \to \Xi$, or to a strip $\sigma^m \colon \mathcal{S}_\beta \to \Xi$. In the latter case, we typically have an antiholomorphic involution τ_Ξ satisfying $\tau_\Xi(\sigma^m(i\beta + t)) = \sigma^m(t)$ for $t \in \mathbb{R}$. In the context of semisimple Lie groups, such situations are well-known for non-compactly causal symmetric spaces $M = G/H$, sitting in the boundary of the so-called complex crown of the Riemannian symmetric space G/K ([GK02]). Then the existence of such analytic extensions specifies so-called wedge regions $W \subseteq M$ that can be characterized in many different ways ([NÓ23b]). Here the "imaginary tangent cone", specifying how M sits in the boundary of Ξ, determines the causal structure on M. So M carries similar geometric structures as the spacetimes in Mathematical Physics. Our goal is to connect the analytic extension phenomena in unitary group representations and the underlying geometry with structures in AQFT.

These notes consist of four main sections whose contents are as follows. In **Section 1** we discuss axioms for nets of local observables, as they appear in Algebraic Quantum Field Theory (AQFT). This involves a symmetry group G (a connected Lie group) acting on a manifold M (spacetime in the physics context) and, for each open subset $\mathcal{O} \subseteq M$ a von Neumann algebra $\mathcal{M}(\mathcal{O})$ on some complex Hilbert space $\mathcal{H}$, on which we also have a unitary representation $(U, \mathcal{H})$ of G, i.e., a continuous homomorphism $U \colon G \to \mathrm{U}(\mathcal{H})$.

Open subsets $\mathcal{O} \subseteq M$ may be considered as laboratories, in which experiments are performed that correspond to the evaluation of quantum observables. The corresponding set of observables then depends on $\mathcal{O}$, which leads to families, also called *nets*, of von Neumann algebras $(\mathcal{M}(\mathcal{O}))_{\mathcal{O}\subseteq M}$. Here $\mathcal{M}(\mathcal{O})$ corresponds to observables measurable in the "laboratory" $\mathcal{O} \subseteq M$.

The axioms that we discuss are:

(Iso) **Isotony:** $\mathcal{O}_1 \subseteq \mathcal{O}_2$ implies $\mathcal{M}(\mathcal{O}_1) \subseteq \mathcal{M}(\mathcal{O}_2)$.

(RS) **Reeh–Schlieder property:** There exists a unit vector $\Omega \in \mathcal{H}$ that is cyclic for $\mathcal{M}(\mathcal{O})$ if $\mathcal{O} \neq \emptyset$. This means that the orbit map $\mathcal{M}(\mathcal{O}) \to \mathcal{H}, A \mapsto A\Omega$ is injective with dense range.

(Cov) **Covariance:** $U_g \mathcal{M}(\mathcal{O}) U_g^{-1} = \mathcal{M}(g\mathcal{O})$ for $g \in G$.

(Vi) **Invariance of the vacuum:** $U(g)\Omega = \Omega$ for $g \in G$.

(BW) **Bisognano–Wichmann property:** There exists a Lie algebra element $h \in \mathfrak{g}$ and an open subset $W \subseteq M$ (called a wedge region), such that the orbit map $\mathcal{M}(W) \to \mathcal{H}, A \mapsto A\Omega$ is injective with dense range (Ω is cyclic and separating) and the corresponding modular operator $\Delta = \Delta_{\mathtt{V}_{\mathcal{M}(W),\Omega}}$, associated to the pair $(\mathcal{M}, \Omega)$ by the Tomita–Takesaki Theorem 1.11, satisfies $\Delta^{-it/2\pi} = U(\exp th)$ for $t \in \mathbb{R}$. In this sense, the modular group is geometrically implemented by a one-parameter subgroup of G.

A first step in our analysis is to simplify this situation by replacing the algebra $\mathcal{M}(\mathcal{O})$ by the real subspace

$$\mathsf{H}(\mathcal{O}) := \mathtt{V}_{\mathcal{M}(\mathcal{O}),\Omega} = \overline{\{A\Omega \colon A = A^* \in \mathcal{M}(\mathcal{O})\}}.$$

To formulate our axioms for real subspaces, recall that a closed real subspace $\mathtt{V} \subseteq \mathcal{H}$ is called *standard* if $\mathtt{V} + i\mathtt{V}$ is dense and $\mathtt{V} \cap i\mathtt{V} = \{0\}$. For any standard subspace, there exists a unique positive selfadjoint operator $\Delta_{\mathtt{V}}$ and a conjugation (an antilinear involutive isometry) $J_{\mathtt{V}}$, such that $\mathtt{V} = \mathrm{Fix}(J_{\mathtt{V}}\Delta_{\mathtt{V}}^{1/2})$ (see Definition 1.3 for details).

We are now ready to formulate the axioms for the family $\mathsf{H}(\mathcal{O})$:

(Iso) **Isotony**: $\mathcal{O}_1 \subseteq \mathcal{O}_2$ implies $\mathsf{H}(\mathcal{O}_1) \subseteq \mathsf{H}(\mathcal{O}_2)$

(RS) **Reeh–Schlieder property:** $\mathsf{H}(\mathcal{O})$ is cyclic if $\mathcal{O} \neq \emptyset$.

(Cov) **Covariance:** $U_g \mathsf{H}(\mathcal{O}) = \mathsf{H}(g\mathcal{O})$ for $g \in G$.

(BW) **Bisognano–Wichmann property:** There exists a Lie algebra element $h \in \mathfrak{g}$ and an open connected subset $W \subseteq M$, such that $\mathsf{H}(W)$ is standard and the corresponding modular operator satisfies $\Delta^{-it/2\pi} = U(\exp th)$ for $t \in \mathbb{R}$.

Our goal is to understand such nets and the requirements on the G-space M, its geometry, the structure of G and the representation $(U, \mathcal{H})$ for which such nets exist. Eventually, one would like to "classify" all these nets in a suitable sense, but first one has to specify which structures we are dealing with. Key questions are:

(Q1) Which elements $h \in \mathfrak{g}$ can arise in the Bisognano–Wichmann (BW) condition?

(Q2) What G-invariant structure do we need on M as a fertile ground for nets of real subspaces?

(Q3) How to find the domains $W \subseteq M$, arising in the (BW) condition?

The key result in **Section 2** answers (Q1), namely that h has to be an *Euler element*, i.e., $\mathrm{ad}\, h$ is non-zero and diagonalizable with $\mathrm{Spec}(\mathrm{ad}\, h) \subseteq \{-1, 0, 1\}$. In physics context of the Lorentz and Poincaré group, these are suitably normalized generators of Lorentz boosts. In Section 2.3 we argue that it is natural to require M to carry a *causal structure*, i.e., a field of pointed generating convex cones $C_m \subseteq T_m(M)$, invariant under the G-action. Given an Euler element h and a causal structure on M, the natural candidates for W are the connected components of the *positivity region*

$$W_M^+(h) = \Big\{ m \in M \colon \frac{d}{dt}\Big|_{t=0} \exp(th).m \in C_m^\circ \Big\}.$$

In **Section 2** we discuss these structures for various examples. Since it will play an important role later on in the construction of nets of real subspaces, we also describe the compression semigroups

$$S_W := \{ g \in G \colon g.W \subseteq W \}$$

for some types of wedge regions W. The most important examples of causal homogeneous spaces M are causal symmetric spaces and causal flag manifolds.

In **Section 3** we turn to constructions of nets for a given antiunitary representation $(U, \mathcal{H})$ and an Euler element $h \in \mathfrak{g}$. This is motivated by the consequence of the Euler Element Theorem 2.3, according to which we may assume that the Lie algebra involution $\tau_h^{\mathfrak{g}} = e^{\pi i\, \mathrm{ad}\, h}$ integrates to a group involution τ_h (f.i. if G is simply connected), so that we can form the group

$$G_{\tau_h} = G \rtimes \{\mathrm{id}_G, \tau_h\}$$

and assume that U extends to an antiunitary representation of G_{τ_h}. This specifies in particular a standard subspace $\mathtt{V} = \mathtt{V}(h, U)$ by

$$\Delta_{\mathtt{V}} = e^{2\pi i \partial U(h)} \quad \text{and} \quad J_{\mathtt{V}} = U(\tau_h) \tag{0.1}$$

(Definitions 1.3 and 2.54).

To find a net H satisfying (BW) with $\mathsf{H}(W) = \mathtt{V}$, it is instructive to observe that the elements of $\mathtt{V}$ are characterized by the (abstract) Kubo–Martin–Schwinger (KMS) condition: The orbit map $U^v(t) := U(\exp th)v$ extends analytically to the closure of the strip $\mathcal{S}_\pi = \{ z \in \mathbb{C} \colon 0 < \mathrm{Im}\, z < \pi \}$, such that

$$U^v(\pi i) = J_{\mathtt{V}} v$$

(cf. Proposition 3.26).

This suggests to look for domains $W \subseteq M$ and a complex manifold Ξ with $M \subseteq \partial\Xi$ on which G acts by holomorphic maps, such that W consists of elements $m \in M$ whose orbit map $\alpha^m(t) = \exp(th).m$ extends analytically to a map $\mathcal{S}_\pi \to \Xi$, such that $\alpha^m(\pi) = \overline{\tau}_h(m)$, where $\overline{\tau}_h$ is an antiholomorphic involution on Ξ satisfying $\overline{\tau}_h(g.z) = \tau_h(g).\overline{\tau}_h(z)$ for $z \in \Xi$.

For the case where G is contained in its universal complexification $G_{\mathbb{C}}$, we describe in Section 3 conditions on a domain $\Xi \subseteq G_{\mathbb{C}}$ (crown domains for G), so that the following construction leads to nets. We start with a real subspace $\mathtt{F}$ of $J_{\mathtt{V}}$-fixed vectors v, whose orbit map $U^v \colon G \to \mathcal{H}$ extends analytically to a map $U^v \colon \Xi \to \mathcal{H}$ in such a way that

$$\beta^+(v) = \lim_{t \to \frac{\pi}{2}} U^v(\exp(-ith))$$

exists in the space $\mathcal{H}^{-\infty}(U_h)$ of distribution vectors for the one-parameter group $U_h(t) = U(\exp th)$. Note that we have natural inclusions

$$\mathcal{H}^\infty \subseteq \mathcal{H}^\infty(U_h) \subseteq \mathcal{H} \subseteq \mathcal{H}^{-\infty}(U_h) \subseteq \mathcal{H}^{-\infty}$$

(see Appendix C.2 for details). Then

$$\mathtt{E} := \beta^+(\mathtt{F}) \subseteq \mathcal{H}^{-\infty}(U_h) \subseteq \mathcal{H}^{-\infty}$$

is a real subspace. For $\varphi \in C_c^\infty(G,\mathbb{R})$, we have operators $U^{-\infty}(\varphi) = \int_G \varphi(g)U^{-\infty}(g)\,dg$, mapping $\mathcal{H}^{-\infty}$ to $\mathcal{H}$. By

$$\mathsf{H}^G_{\mathtt{E}}(\mathcal{O}) := \overline{\mathrm{span}_{\mathbb{R}}\{U^{-\infty}(\varphi)\mathtt{E}\colon \varphi \in C_c^\infty(\mathcal{O},\mathbb{R})\}},$$

we thus obtain a net of real subspaces on G satisfying (Iso) and (Cov) for trivial reasons, but also (RS) and (BW). Here the main point is to show that $\mathsf{H}^G_{\mathtt{E}}(W^G) = \mathtt{V}$ for a suitable open subset $W^G \subseteq G$.

Example 0.1. Elementary particles in the sense of E. Wigner [Wi39] are classified by irreducible unitary representations of the Poincaré group $G = \mathbb{R}^{1,d-1} \rtimes \mathrm{SO}_{1,d-1}(\mathbb{R})_e$. We write $V := \mathbb{R}^{1,d-1}$ for the corresponding translation group. For scalar particles, the Hilbert space is of the form $\mathcal{H} = L^2(\mathbb{R}^{1,d-1},\mu)$, where μ is a Lorentz invariant measure on the dual space V^* (often identified with V via the Lorentzian form). Here the space $\mathtt{E} = \mathbb{R}1$ of real-valued constant functions represents distribution vectors, and for test functions $\varphi \in C_c^\infty(V,\mathbb{R})$, we have $U^{-\infty}(\varphi)\mathbf{1} = \widehat{\varphi}$ (Fourier transform). So $\mathsf{H}^V_{\mathtt{E}}(\mathcal{O})$ is generated by Fourier transforms of test functions supported in $\mathcal{O}$.

This leaves us with the question of how to find such domains Ξ and subspaces $\mathtt{F} \subseteq \mathcal{H}^J$. For semisimple groups, this can be done with the theory of crown domains for Riemannian symmetric spaces G/K. They provide natural domains $\Xi \subseteq G_{\mathbb{C}}$ to which orbit maps of K-finite vectors of irreducible representations extend, and a recent result by T. Simon ([Si24]) ensures that they have a sufficiently well-behaved boundary behavior at $\partial\Xi$ to ensure $\mathtt{E} = \beta^+(\mathtt{F}) \subseteq \mathcal{H}^{-\infty}(U_h)$. Here an important point is that **no restriction on** U is required to obtain these nets, and they all descend in a natural way to the *non-compactly causal symmetric spaces* $M = G/H$, associated to the Euler element h (cf. Section 2.7, [MNO23, Thm. 4.21]).

Section 4 develops a global perspective on these results. Here we are dealing with representations that are not necessarily irreducible. Starting with a homogeneous space $M = G/H$, a domain $W \subseteq M$ and an antiunitary representation $(U,\mathcal{H})$, we associate two nets $\mathsf{H}^{\max}_M$ and $\mathsf{H}^{\min}_M$ on M, such that any net H on M satisfying (Iso), (Cov) and $\mathsf{H}(W) = \mathtt{V} = \mathtt{V}(h,U)$, satisfies

$$\mathsf{H}^{\min}_M(\mathcal{O}) \subseteq \mathsf{H}(\mathcal{O}) \subseteq \mathsf{H}^{\max}_M(\mathcal{O})$$

for every open subset $\mathcal{O} \subseteq M$ (Lemma 4.7). From this perspective, the question is whether a net H satisfying (Iso), (Cov) and (BW) exists at all. This is equivalent to $\mathsf{H}^{\max}(W) = \mathtt{V}$, which in turn is equivalent to the inclusion of semigroups

$$S_W = \{g \in G\colon g.W \subseteq W\} \subseteq S_{\mathtt{V}} = \{g \in G\colon U(g)\mathtt{V} \subseteq \mathtt{V}\}. \tag{0.2}$$

The semigroups S_W has already been described in Section 2 for some examples of wedge regions. If $\ker U$ is discrete, then

$$S_{\mathtt{V}} = \exp(C_+)G_{\mathtt{V}}\exp(C_-) \quad \text{for} \quad C_\pm = \pm C_U \cap \mathfrak{g}_{\pm 1}(h), \tag{0.3}$$

where

$$C_U := \{x \in \mathfrak{g}\colon -i\cdot\partial U(x) \geq 0\} \quad \text{with} \quad \partial U(x) = \frac{d}{dt}\Big|_{t=0} U(\exp tx),$$

is the *positive cone of* U and $\mathfrak{g}_\lambda(h) = \ker(\lambda\mathbf{1} - \mathrm{ad}\,h)$ are the eigenspaces of $\mathrm{ad}\,h$ (Section 4.2).

If G is semisimple and M the non-compactly causal symmetric space associated with the Euler element h, then $S_W = G_W$ is a group, so that (0.2) reduces to the inclusion $G_W \subseteq G_{\mathsf{V}}$, which boils down to $g.W = W \Rightarrow U(g)J = JU(g)$, which is equivalent to $\tau_h(g)^{-1}g \in \ker U$.

If S_W is not a group, it is typically of the form

$$S_W = \exp(C_+)G_W \exp(C_-),$$

where $C_\pm = \pm C_{\mathfrak{g}} \cap \mathfrak{g}_{\pm 1}(h)$ for an $\mathrm{Ad}(G)$-invariant cone $C_{\mathfrak{g}} \subseteq \mathfrak{g}$. Comparing with (0.3), we thus obtain $G_W \subseteq G_{\mathsf{V}}$, and the *spectral condition*

$$C_\pm \subseteq C_U$$

on the representation U, i.e., the operators $-i\partial U(x)$ are positive for $x \in \pm C_\pm$. For the Poincaré group, acting on Minkowski space (Remark 1.16), this corresponds to the positivity of the energy.

We conclude these notes with a discussion of perspectives and open problems in Section 5.

Some history: The starting point for the development that led to fruitful applications of modular theory in QFT was the Bisognano–Wichmann Theorem, asserting that the modular automorphisms $\alpha_t(M) = \Delta^{-it/2\pi} M \Delta^{it/2\pi}$ associated to the algebra $\mathcal{M}(W_R)$ of observables corresponding to the right wedge

$$W_R = \{(x_0, x_1, \ldots, x_{d-1}) \colon x_1 > |x_0|\}$$

in d-dimensional Minkowski space $\mathbb{R}^{1,d-1}$ are implemented by the unitary action of a one-parameter group of Lorentz boosts preserving W_R. This geometric implementation of modular automorphisms in terms of Poincaré transformations was an important first step in a rich development based on the work of Borchers and Wiesbrock in the 1990s [Bo92, Bo95, Bo97, Wi92, Wi93, Wi93b]. They managed to distill the abstract essence from the Bisognano–Wichmann Theorem which led to a better understanding of the basic configurations of von Neumann algebras in terms of half-sided modular inclusions and modular intersections. In his survey [Bo00], Borchers described how these concepts have revolutionized quantum field theory. Subsequent developments can be found in [Ar99, BGL02, Lo08, LW11, LL15, JM18, Mo18].

How to read these notes? Each of the four sections has a main part and appendices. The appendices contain more details and discussion of related issues. So they can be skipped on first reading.

Notation

- Strips in the complex plane: $\mathcal{S}_\beta = \{z \in \mathbb{C} \colon 0 < \mathrm{Im}\, z < \beta\}$ and $\mathcal{S}_{\pm\beta} = \{z \in \mathbb{C} \colon |\,\mathrm{Im}\, z| < \beta\}$.
- The neutral element of a Lie group G is denoted e, and G_e is the identity component.
- The Lie algebra of a Lie group G is denoted $\mathbf{L}(G)$ or $\mathfrak{g}$.
- For an involutive automorphism σ of G, we write $G^\sigma = \{g \in G \colon \sigma(g) = g\}$ for the subgroup of fixed points and $G_\sigma := G \rtimes \{\mathrm{id}_G, \sigma\}$ for the corresponding group extension.
- $\mathrm{AU}(\mathcal{H})$ is the group of unitary or antiunitary operators on a complex Hilbert space.
- An antiunitary representation of G_σ is a homomorphism $U \colon G_\sigma \to \mathrm{AU}(\mathcal{H})$ with $U(G) \subseteq \mathrm{U}(\mathcal{H})$ for which $J := U(\sigma)$ is antiunitary, i.e., a *conjugation.* We denote representations as pairs $(U, \mathcal{H})$.

- If G is a group acting on a set M and $W \subseteq M$ a subset, then the stabilizer subgroup of W in G is denoted $G_W := \{g \in G : g.W = W\}$, and the compression semigroup by $S_W := \{g \in G : g.W \subseteq W\}$.
- If $\mathfrak{g}$ is a Lie algebra and $h \in \mathfrak{g}$, then $\mathfrak{g}_\lambda(h) = \ker(\operatorname{ad} h - \lambda\mathbf{1})$ is the λ-eigenspace of $\operatorname{ad} h$ and $\mathfrak{g}^\lambda(h) = \bigcup_k \ker(\operatorname{ad} h - \lambda\mathbf{1})^k$ is the generalized λ-eigenspace. g
- An element x of a Lie algebra $\mathfrak{g}$ is called
  - *hyperbolic* if $\operatorname{ad} x$ is diagonalizable over $\mathbb{R}$
  - *elliptic* or *compact* if $\operatorname{ad} x$ is semisimple with purely imaginary spectrum, i.e., $\overline{e^{\mathbb{R} \operatorname{ad} x}}$ is a compact subgroup of $\operatorname{Aut}(\mathfrak{g})$.
- We write $\mathcal{E}(\mathfrak{g})$ for the set of *Euler elements* $h \in \mathfrak{g}$, i.e., $\operatorname{ad} h$ is non-zero and diagonalizable with $\operatorname{Spec}(\operatorname{ad} h) \subseteq \{-1, 0, 1\}$. We call h *symmetric* if $-h \in \operatorname{Inn}(\mathfrak{g})h$. We write $\tau_h := e^{\pi i \operatorname{ad} h} \in \operatorname{Aut}(\mathfrak{g})$ for the involution of $\mathfrak{g}$ specified by h.
- A *causal G-space* is a smooth G-space M, endowed with a G-invariant *causal structure*, i.e., a field $(C_m)_{m \in M}$ of closed convex cones $C_m \subseteq T_m(M)$.
- For a unitary representation $(U, \mathcal{H})$ of G we write:
  - ◇ $\partial U(x) = \frac{d}{dt}\big|_{t=0} U(\exp tx)$ for the infinitesimal generator of the unitary one-parameter group $(U(\exp tx))_{t \in \mathbb{R}}$ in the sense of Stone's Theorem.
  - ◇ $\mathtt{d}U \colon \mathfrak{g} \to \operatorname{End}(\mathcal{H}^\infty)$ for the representation of the Lie algebra $\mathfrak{g}$ on the space $\mathcal{H}^\infty$ of smooth vectors. Then $\partial U(x) = \overline{\mathtt{d}U(x)}$ (operator closure) for $x \in \mathfrak{g}$.
- For a $*$-algebra $\mathcal{M}$, we write $\mathcal{M}_h := \{A \in \mathcal{M} : A^* = A\}$ for the real subspace of hermitian elements and for $\Omega \in \mathcal{H}$, we put $\mathtt{V}_{M,\Omega} := \overline{\mathcal{M}_h \Omega}$.

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1 Nets of operator algebras and AQFT

Throughout, $\mathcal{H}$ denotes a complex Hilbert space.

1.1 Standard subspaces of Hilbert spaces

In this subsection, we introduce the key concept of a standard subspace $\mathtt{V}$ of a complex Hilbert space $\mathcal{H}$. Standard subspaces are "slanted" real forms in the sense that $\mathtt{V} + i\mathtt{V}$ is dense in $\mathcal{H}$ and $\mathtt{V} \cap i\mathtt{V} = \{0\}$. As we shall see below, they are parametrized by pairs (Δ, J), where $\Delta > 0$ is a selfadjoint operator and J is a *conjugation* (an antilinear isometric involution) satisfying the modular relation

$$J\Delta J = \Delta^{-1}.$$

Standard subspaces appear naturally in the modular theory of operator algebras (Tomita–Takesaki Theorem 1.11) and also in antiunitary representations of Lie groups, where they correspond to antiunitary representations of the multiplicative group $\mathbb{R}^\times \cong \mathbb{R} \times \{\pm 1\}$. This establishes an important link between operator algebras and antiunitary representations.

Definition 1.1. (a) A closed real subspace $\mathtt{V} \subseteq \mathcal{H}$ is called

- *separating* if $\mathtt{V} \cap i\mathtt{V} = \{0\}$,
- *cyclic* if $\mathtt{V} + i\mathtt{V}$ is dense in $\mathcal{H}$,
- *standard* if it is cyclic and separating.

We write $\mathrm{Stand}(\mathcal{H})$ for the set of standard subspaces of $\mathcal{H}$.
(b) For a separating subspace $\mathtt{V}$, we define the antilinear *Tomita involution*

$$T_{\mathtt{V}} \colon \mathtt{V} + i\mathtt{V} \to \mathcal{H}, \quad T_{\mathtt{V}}(v + iw) = v - iw \quad \text{for} \quad v, w \in \mathtt{V}.$$

(c) We write $\gamma(v, w) := \mathrm{Im}\langle v, w\rangle$ for the canonical symplectic form on $\mathcal{H}$. For a real subspace $\mathtt{V} \subseteq \mathcal{H}$, we define its *symplectic orthogonal space* by

$$\mathtt{V}' := \mathtt{V}^{\perp_\gamma} = \{w \in \mathcal{H} \colon \mathrm{Im}\langle v, w\rangle = 0\} = i\mathtt{V}^{\perp_{\mathbb{R}}},$$

where $\mathtt{V}^{\perp_{\mathbb{R}}}$ is the real orthogonal space of $\mathtt{V}$ with respect to the real-valued scalar product $\mathrm{Re}\langle v, w\rangle$. Note that $\langle \mathtt{V}, \mathtt{V}'\rangle \subseteq \mathbb{R}$.

Lemma 1.2. *If $\mathtt{V}$ is standard, then $T_{\mathtt{V}}$ is closed and densely defined.*

Proof. As $\mathtt{V}$ is cyclic, the operator $T_{\mathtt{V}}$ is densely defined. To see that the graph of $T_{\mathtt{V}}$ is closed, suppose that $\xi_n = a_n + ib_n$ is a sequence in $\mathcal{D}(T_{\mathtt{V}}) = \mathtt{V} + i\mathtt{V}$ with $a_n, b_n \in \mathtt{V}$, such that $(\xi_n, T_{\mathtt{V}}\xi_n) = (a_n + ib_n, a_n - ib_n) \to (\xi, \eta)$ in $\mathcal{H} \times \mathcal{H}$. As $\mathtt{V}$ is closed,

$$a_n = \frac{1}{2}(a_n + ib_n + (a_n - ib_n)) = \frac{1}{2}(\xi_n + T_{\mathtt{V}}\xi_n) \to \frac{1}{2}(\xi + \eta) =: a \in \mathtt{V},$$

and

$$b_n = \frac{1}{2i}(a_n + ib_n - (a_n - ib_n)) = \frac{1}{2i}(\xi_n - T_{\mathtt{V}}\xi_n) \to \frac{1}{2i}(\xi - \eta) =: b \in \mathtt{V}.$$

Therefore $\xi = a + ib \in \mathcal{D}(T_{\mathtt{V}})$ satisfies $T_{\mathtt{V}}\xi = a - ib = \eta$. This means that $T_{\mathtt{V}}$ is closed. □

Definition 1.3. We have seen in Lemma 1.2 that, for every standard subspace $\mathtt{V} \subseteq \mathcal{H}$, the Tomita operator

$$T_{\mathtt{V}} \colon \mathcal{D}(T_{\mathtt{V}}) := \mathtt{V} + i\mathtt{V} \to \mathcal{H}, \qquad T_{\mathtt{V}}(v + iw) := v - iw$$

is closed, hence has a polar decomposition ([Sch12, Thm. 7.2], [SZ79, Thm. 9.29] [1]), i.e.,

$$\Delta_{\mathtt{V}} := T_{\mathtt{V}}^* T_{\mathtt{V}}$$

is a positive selfadjoint operator, and there exists an antilinear isometry $J_{\mathtt{V}}$ such that

$$T_{\mathtt{V}} = J_{\mathtt{V}} \Delta_{\mathtt{V}}^{1/2}.$$

The isometry $J_{\mathtt{V}}$ is defined on all of $\mathcal{H}$ because $\Delta_{\mathtt{V}}$ has dense range, which in turn follows from $\mathcal{R}(\Delta_{\mathtt{V}})^\perp = \ker(\Delta_{\mathtt{V}}) = \ker(T_{\mathtt{V}}) = \{0\}$. The relation

$$J_{\mathtt{V}} \Delta_{\mathtt{V}}^{1/2} = T_{\mathtt{V}} = T_{\mathtt{V}}^{-1} = \Delta_{\mathtt{V}}^{-1/2} J_{\mathtt{V}}^{-1} = J_{\mathtt{V}}^{-1}(J_{\mathtt{V}} \Delta_{\mathtt{V}}^{-1/2} J_{\mathtt{V}}^{-1})$$

[1] To obtain the polar decomposition of a closed operator T, the main step is to show that the operator T^*T is selfadjoint. Then the unique positive square root $|T| := \sqrt{T^*T}$ satisfies $\| \, |T|\xi\| = \|T\xi\|$ for all $\xi \in \mathcal{D}(T)$, which easily leads to a partial isometry U from $\overline{\mathcal{R}(|T|)} = \mathcal{N}(|T|)^\perp = \mathcal{N}(T)^\perp$ to $\overline{\mathcal{R}(T)}$ with $T = U|T|$.

and the uniqueness of the polar decomposition now implies that $J_{\mathtt{V}}^2 = \mathbf{1}$ and the *modular relation*

$$J_{\mathtt{V}}\Delta_{\mathtt{V}}J_{\mathtt{V}} = \Delta_{\mathtt{V}}^{-1}. \tag{1.1}$$

The unitary one-parameter group $(\Delta_{\mathtt{V}}^{it})_{t\in\mathbb{R}}$ is called the *modular group of* $\mathtt{V}$. It has the important property that it preserves $\mathtt{V}$ (Remark 1.4(b)) and its true importance is revealed in the Tomita–Takesaki Theorem 1.11.

Remark 1.4. (a) The modular group $\Delta_{\mathtt{V}}^{it}$ commutes with the antiunitary conjugation $J_{\mathtt{V}}$. In fact, the antilinearity of $J_{\mathtt{V}}$ implies that

$$J_{\mathtt{V}}\Delta_{\mathtt{V}}^{z}J_{\mathtt{V}} = \Delta_{\mathtt{V}}^{-\overline{z}} \quad \text{for} \quad z \in \mathbb{C}.$$

In view of [NÓ15, Prop. 3.1], a unitary one-parameter group $(U_t = e^{itH})_{t\in\mathbb{R}}$ commutes with a conjugation J if and only if H is *symmetric* in the sense that there exists a unitary involution S satisfying $SHS^{-1} = -H$.
(b) The fact that the operators $\Delta_{\mathtt{V}}^{it}$ commute with $J_{\mathtt{V}}$ implies that they also commute with $T_{\mathtt{V}}$, hence leave $\mathtt{V}$ invariant.

Proposition 1.5. *The map $\mathtt{V} \mapsto (\Delta_{\mathtt{V}}, J_{\mathtt{V}})$ is a bijection between the set of standard subspaces of $\mathcal{H}$ and the set of pairs (Δ, J), where J is a conjugation and $\Delta > 0$ selfadjoint with $J\Delta J = \Delta^{-1}$. Its inverse is given by $(\Delta, J) \mapsto \mathrm{Fix}(J\Delta^{1/2})$.*

Proof. ([Lo08, Prop. 3.2]) To see that we obtain a bijection, suppose that (Δ, J) is a pair of modular objects, i.e., a positive operator and a conjugation, satisfying the modular relation (1.1). Then $T := J\Delta^{1/2}$ is a closed, densely defined antilinear involution and

$$\mathtt{V} := \mathrm{Fix}(T) := \{\xi \in \mathcal{D}(T) \colon T\xi = \xi\}$$

is a standard subspace with $J_{\mathtt{V}} = J$ and $\Delta_{\mathtt{V}} = \Delta$. Here closedness of T follows from the closedness of the selfadjoint operator $\Delta^{1/2}$, and this implies the closedness of $\mathrm{Fix}(T)$. □

The correspondence between modular objects and standard subspaces is the core of the modular theory of operator algebras. It is the key structure in the Tomita–Takesaki Theorem discussed below.

1.2 Modular theory and the Tomita–Takesaki Theorem

Definition 1.6. For a subset $S \subseteq B(\mathcal{H})$, we write

$$S' := \{A \in B(\mathcal{H}) \colon (\forall M \in S)\, AM = MA\}$$

for its *commutant.* It is a closed subalgebra and $*$-invariant if S has this property.

A *von Neumann algebra* is a $*$-invariant complex subalgebra $\mathcal{M} \subseteq B(\mathcal{H})$ satisfying $\mathcal{M} = \mathcal{M}''$. For a von Neumann algebra $\mathcal{M}$, a unit vector $\Omega \in \mathcal{H}$ is called

- *cyclic,* if $\mathcal{M}\Omega$ is dense in $\mathcal{H}$.
- *separating,* if the orbit map $\mathcal{M} \to \mathcal{H}, M \mapsto M\Omega$ is injective,
- *standard*, if it is cyclic and separating.

Lemma 1.7. *$\Omega \in \mathcal{H}$ is cyclic for $\mathcal{M}$ if and only if it is separating for $\mathcal{M}'$.*

Proof. Suppose first that Ω is cyclic for $\mathcal{M}$. For $A \in \mathcal{M}'$ with $A\Omega = 0$, we then obtain $A\mathcal{M}\Omega = \mathcal{M}A\Omega = \{0\}$, and since $\mathcal{M}\Omega$ is dense in $\mathcal{H}$, it follows that $A = 0$. So Ω is separating for $\mathcal{M}'$.

Suppose, conversely, that Ω is separating for $\mathcal{M}'$. Let $P\colon \mathcal{H} \to \mathcal{H}$ be the orthogonal projection onto $\overline{\mathcal{M}\Omega}$. Then $P \in \mathcal{M}'$ and $(\mathbf{1} - P)\Omega = 0$ imply $\mathbf{1} = P$, so that Ω is cyclic for $\mathcal{M}$. □

Definition 1.8. For $\Omega \in \mathcal{H}$ and $\mathcal{M} \subseteq B(\mathcal{H})$, we consider the closed real subspace

$$\mathtt{V} := \mathtt{V}_{\mathcal{M},\Omega} := \overline{\mathcal{M}_h\Omega} \subseteq \mathcal{H}, \tag{1.2}$$

where $\mathcal{M}_h := \{M \in \mathcal{M} \colon M^* = M\}$ is the real subspace of hermitian elements in $\mathcal{M}$.

Lemma 1.9. *The following assertions hold for a von Neumann algebra $\mathcal{M} \subseteq B(\mathcal{H})$ and a unit vector $\Omega \in \mathcal{H}$.*

(a) $\mathtt{V}_{\mathcal{M},\Omega}$ *is cyclic if and only if Ω is cyclic for $\mathcal{M}$.*

(b) $\mathtt{V}_{\mathcal{M},\Omega}$ *is separating in the sense of* Definition 1.1 *if and only if Ω is separating for the restriction of $\mathcal{M}$ to the cyclic subspace $\overline{\mathcal{M}\Omega}$, i.e., $A\Omega = 0$ implies $A\mathcal{M}\Omega = \{0\}$.*

(c) $\mathtt{V}_{\mathcal{M},\Omega}$ *is standard if and only if Ω is a standard vector for $\mathcal{M}$.*

Note that $\mathtt{V}_{\mathcal{M},\Omega}$ being separating only contains information on the representation of $\mathcal{M}$ on the cyclic subspace $\mathcal{K} := \overline{\mathcal{M}\Omega} \subseteq \mathcal{H}$, but not on the representation of $\mathcal{M}$ on $\mathcal{K}^\perp$. If $\mathcal{H} = \mathbb{C}^2$, $\mathcal{M} \cong \mathbb{C}^2$ is the subalgebra of diagonal operators, and $\Omega = \mathbf{e}_1$, then $\mathtt{V}_{\mathcal{M},\Omega} = \mathbb{R}\mathbf{e}_1$ is separating, but Ω is **not** separating for $\mathcal{M}$. This subtlety does not play a role for (c) because we also assume cyclicity.

Proof. (a) follows immediately from the definitions.
(b) Suppose first that Ω is separating, hence cyclic for $\mathcal{M}'$ (Lemma 1.7). We have for $A \in \mathcal{M}_h$ and $B \in \mathcal{M}'_h$ the relation

$$\langle A\Omega, B\Omega\rangle = \langle \Omega, AB\Omega\rangle = \langle \Omega, BA\Omega\rangle = \langle B\Omega, A\Omega\rangle,$$

so that

$$\langle \mathtt{V}_{\mathcal{M},\Omega}, \mathtt{V}_{\mathcal{M}',\Omega}\rangle \subseteq \mathbb{R}.$$

We conclude that

$$\mathtt{V}_{\mathcal{M},\Omega} \cap i\mathtt{V}_{\mathcal{M},\Omega} \subseteq \mathtt{V}_{\mathcal{M}',\Omega}^{\perp} = (\mathcal{M}'\Omega)^\perp = \{0\},$$

i.e., $\mathtt{V}_{\mathcal{M},\Omega}$ is separating.

Now we assume that $\mathtt{V}_{\mathcal{M},\Omega}$ is separating and derive that Ω is separating for the image of $\mathcal{M}$ on the cyclic subspace $\mathcal{K} := \overline{\mathcal{M}\Omega}$. So let $A \in \mathcal{M}$ with $A\Omega = 0$. For $B \in \mathcal{M}$, we then have

$$A^*B\Omega = (A^*B + B^*A)\Omega \in \mathtt{V}_{\mathcal{M},\Omega},$$

so that $A^*\mathcal{M}\Omega \subseteq \mathtt{V}_{\mathcal{M},\Omega}$ is a complex linear subspace, hence trivial because $\mathtt{V}_{\mathcal{M},\Omega}$ is separating. Thus $A^*\mathcal{K} = \{0\}$, and this implies that $A|_{\mathcal{K}} = 0$. This proves (b).
(c) follows from (a) and (b). □

Remark 1.10. (a) Cyclic vectors play an important role in representation theory because every $*$-representation on a Hilbert space is a direct sum of cyclic representations. Moreover, representations with cyclic unit vector Ω can be reconstructed completely from the corresponding state

$$\omega\colon \mathcal{M} \to \mathbb{C}, \quad \omega(M) := \langle \Omega, M\Omega\rangle.$$

The map $\iota\colon \mathcal{H} \to \mathcal{M}^*, \iota(v)(M) := \langle \Omega, Mv\rangle$ is injective and intertwines the representation on $\mathcal{H}$ with the right translation representation on $\mathcal{M}^*$. The Hilbert space structure on $\iota(\mathcal{H})$, for which ι is isometric, is given by

$$\langle \iota(M\Omega), \iota(N\Omega)\rangle = \omega(M^*N),$$

exhibiting $\iota(\mathcal{H})$ as a reproducing kernel Hilbert space of linear functionals f, satisfying

$$f(M) = \langle M^*\Omega, f\rangle \quad \text{for} \quad M \in \mathcal{M}, f \in \iota(\mathcal{H})$$

(cf. [Ne99, Ch. I]).

(b) If Ω is standard, then the orbit map $\pi^\Omega\colon \mathcal{M} \to \mathcal{H}, M \mapsto M\Omega$ is a dense linear embedding of $\mathcal{M}$ into $\mathcal{H}$, so that we may consider $\mathcal{H}$ as the completion of $\mathcal{M}$ with respect to the scalar product $\langle M, N\rangle := \omega(M^*N)$.

That Ω is separating corresponds to the property of the state ω that $\omega(M^*M) = 0$ implies $M = 0$ (ω is *faithful*). One can show that all normal (cf. Appendix A) faithful states on a von Neumann algebra lead to equivalent GNS representations, called the *standard form representation* ([Bla06, Thm. III.2.6.7]). For more details on these issues, see also the discussion of *symmetric forms representations* in [NÓ17], [Bla06], [BGN20, §3.1] and Remark A.4.

Theorem 1.11. (Tomita–Takesaki Theorem; Tomita, 1967; Takesaki, 1970) *Let $\mathcal{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra and $\Omega \in \mathcal{H}$ be a standard vector for $\mathcal{M}$. Then $\mathtt{V} := \mathtt{V}_{\mathcal{M},\Omega} := \overline{\mathcal{M}_h\Omega}$ is a standard subspace. The corresponding modular objects (Δ, J) satisfy*

(a) *$J\mathcal{M}J = \mathcal{M}'$ and $\Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M}$ for $t \in \mathbb{R}$.*

(b) *$J\Omega = \Omega$, $\Delta\Omega = \Omega$ and $\Delta^{it}\Omega = \Omega$ for $t \in \mathbb{R}$.*

(c) *For $M \in \mathcal{Z}(\mathcal{M}) := \mathcal{M} \cap \mathcal{M}'$, the center of $\mathcal{M}$, we have $JMJ = M^*$ and $\Delta^{it}M\Delta^{-it} = M$ for $t \in \mathbb{R}$.*

It follows in particular, that

$$\alpha_t(A) := \Delta^{it}A\Delta^{-it}$$

defines a one-parameter group of automorphisms of $\mathcal{M}$, called the *modular automorphism group associated to* Ω.

Proof. By Lemma 1.9(c), $\mathtt{V}$ is a standard subspace. We refer to [BR87, Thm. 2.5.14] for the other assertions, whose proof is rather involved. The standard subspace $\mathtt{V}$ already provides Δ and J. The main work consists in the verification of (a). □

An approach to the Tomita–Takesaki Theorem through bounded operators can be found in [RvD77]. For a rather general approach to modular operators for pairs of subspaces of real Hilbert spaces, we refer to [NZ24].

The passage to the commutant of an algebra translates easily into the symplectic orthogonal space $\mathtt{V}'$ (cf. Definition 1.1).

Lemma 1.12. *For a standard vector Ω of $\mathcal{M}$, we have $(\mathtt{V}_{\mathcal{M},\Omega})' = \mathtt{V}_{\mathcal{M}',\Omega}$.*

Proof. Let $J = J_{\mathcal{M},\Omega}$ and $\mathtt{V} := \mathtt{V}_{\mathcal{M},\Omega}$. In view of $J\Omega = \Omega$ and $J\mathcal{M}J = \mathcal{M}'$ (Theorem 1.11), the assertion follows from Lemma 1.18(f) below:

$$\mathtt{V}' \overset{1.18(f)}{=} J\mathtt{V} = J\overline{\mathcal{M}_h\Omega} = \overline{\mathcal{M}_h'J\Omega} = \overline{\mathcal{M}_h'\Omega} = \mathtt{V}_{\mathcal{M}',\Omega}.$$ □

Example 1.13. (a) Let $\mathcal{H} = L^2(X, \mathfrak{S}, \mu)$ for a σ-finite measure space $(X, \mathfrak{S}, \mu)$ and $\mathcal{M} = L^\infty(X, \mathfrak{S}, \mu)$, acting on $\mathcal{H}$ by multiplication operators. Then the normal states of $\mathcal{M}$ (Remark 1.10) are of the form

$$\omega_h(f) = \int_X fh\, d\mu,$$

where $0 \leq h$ satisfies $\int_X h\, d\mu = 1$. Such a state is faithful if and only if $h \neq 0$ holds μ-almost everywhere. Then $\Omega := \sqrt{h} \in \mathcal{H}$ is a corresponding standard unit vector. Let $\mathtt{V} = \mathtt{V}_{\mathcal{M},\Omega}$ be the corresponding standard subspace. As it consists of real-valued functions, we obtain $T_\mathtt{V}(f) = \overline{f}$, which is isometric and therefore $T_\mathtt{V} = J$ and $\Delta_\mathtt{V} = \mathbf{1}$.
(b) Let $\mathcal{H} = B_2(\mathcal{K})$ be the space of Hilbert–Schmidt operators on the complex separable Hilbert space $\mathcal{K}$ and consider the von Neumann algebra $\mathcal{M} = B(\mathcal{K})$, acting on $\mathcal{H}$ by left multiplications. Then $\mathcal{M}' \cong B(\mathcal{K})^{\mathrm{op}}$, the opposite algebra, acting by right multiplications. Normal states of $\mathcal{M}$ are of the form

$$\omega_S(A) = \mathrm{tr}(AS), \quad \text{where} \quad 0 \leq S \quad \text{with} \quad \mathrm{tr}\, S = 1.$$

Such a state is faithful if and only if $\ker S = \{0\}$ (which requires $\mathcal{K}$ to be separable), and then $\Omega := \sqrt{S} \in \mathcal{H}$ is a cyclic separating unit vector. Then $T_\mathtt{V}(M\Omega) = M^*\Omega = (\Omega M)^*$ implies that

$$JA = A^* \quad \text{and} \quad \Delta(A) = \Omega^2 A \Omega^{-2} = SAS^{-1} \quad \text{for} \quad A \in B_2(\mathcal{K}).$$

(c) The prototypical pair (Δ, J) of a modular operator and a modular conjugation arises from the regular representation of a locally compact group G on the Hilbert space $\mathcal{H} = L^2(G, \mu_G)$ with respect to a left Haar measure μ_G. Here the modular operator is given by the multiplication

$$\Delta f = \Delta_G \cdot f,$$

where $\Delta_G \colon G \to \mathbb{R}_+^\times$ is the modular function of G, and the modular conjugation is given by

$$(Jf)(g) = \Delta_G(g)^{-\frac{1}{2}} \overline{f(g^{-1})}.$$

Accordingly, we have for $T = J\Delta^{1/2}$:

$$(Tf)(g) = \Delta_G(g)^{-1} \overline{f(g^{-1})} = f^*(g).$$

The corresponding von Neumann algebra is the algebra $\mathcal{M} \subseteq B(L^2(G, \mu_G))$ generated by the left regular representation. If $M_f h = f * h$ is the left convolution with $f \in C_c(G)$, then the value of the corresponding normal weight ω on $\mathcal{M}$ (Remark A.4) is given by $\omega(M_f) = f(e)$, so that ω corresponds to evaluation in e, which is defined on a weakly dense subalgebra of $\mathcal{M}$.

Remark 1.14. Theorem 1.11(c) asserts that the modular group and J commute with all central projections, and this entails that the whole situation adapts to the canonical central disintegration

$$\mathcal{M} = \int_X^\oplus \mathcal{M}_x \, d\mu(x)$$

of $\mathcal{M}$, for which $\mathcal{Z}(\mathcal{M}) = L^\infty_{\mathrm{loc}}(X, \mathfrak{S}, \mu)$ are the scalar decomposable operators on a locally finite measure space, and almost every von Neumann algebra $\mathcal{M}_x$ is a factor, i.e., $\mathcal{Z}(\mathcal{M}_x) = \mathbb{C}\mathbf{1}$ (cf. Examples A.2(b) and [MN24, §5.4] for more details).

So the modular groups are "direct integrals" of modular groups of factors, and for factors, the modular operators and their spectra are a key tool in Connes' classification of factors and in the characterization of von Neumann algebras by their natural cones by A. Connes [Co73, Co74] (see also [NÓ17, §4.4] and [BR87])

1.3 The axioms for nets of local observables

States of quantum mechanical systems are represented by one-dimensional subspaces $\mathbb{C}\Omega \subseteq \mathcal{H}$ (for unit vectors Ω) and selfadjoint elements of $B(\mathcal{H})$ represent observables. The evaluation of an observable in a state $[\Omega] := \mathbb{C}\Omega$ corresponds to the evaluation of the corresponding state

$$\omega(A) = \langle \Omega, A\Omega \rangle.$$

For some systems, the observables are restricted to selfadjoint elements of a proper von Neumann subalgebra $\mathcal{M} \subseteq B(\mathcal{H})$.

In Algebraic Quantum Field Theory (AQFT) one starts with a "spacetime manifold" M, which, in the simplest case is Minkowski space $M = \mathbb{R}^{1,d-1}$. We write its elements as pairs

$$x = (x_0, \mathbf{x}) = (x_0, x_1, \ldots, x_{d-1})$$

and define the Lorentzian form by

$$\beta(x, y) = x_0 y_0 - \mathbf{x}\mathbf{y} = x_0 y_0 - x_1 y_1 - \cdots - x_{d-1} y_{d-1}.$$

We call $x \in \mathbb{R}^{1,d-1}$ *timelike* if $\beta(x,x) > 0$, *lightlike* if $\beta(x,x) = 0$, and *spacelike* if $\beta(x,x) < 0$. The convex cone

$$\mathrm{V}_+ := \{x \in \mathbb{R}^{1,d-1} \colon x_0 > 0, \beta(x,x) > 0\}$$

is called the *positive lightcone*. Timelike vectors are possible tangent vectors to *worldlines* $\gamma \colon \mathbb{R} \to M$ of massive particles and lightlike vectors are tangent vectors to light-rays (moving with the speed of light). *Causal curves* are specified by $\gamma'(t) \in \overline{\mathrm{V}_+}$ for every t, i.e., they correspond to movements not faster than light.

Examples 1.15. There are also curved homogeneous spacetimes, such as *de Sitter space*

$$\mathrm{dS}^d = \{(x_0, \mathbf{x}) \in \mathbb{R}^{1,d} \colon x_0^2 - \mathbf{x}^2 = -1\}.$$

It provides a model of a spherical (positively curved) expanding universe. This is a hypersurface in the $(d+1)$-dimensional Minkowski space $\mathbb{R}^{1,d}$. The tangent space $T_x(\mathrm{dS}^d)$ can be identified with the hyperplane

$$x^{\perp_\beta} = \{y \in \mathbb{R}^{1,d} \colon \beta(x,y) = 0\}.$$

Since x is spacelike, the restriction of β to this hyperplane is Lorentzian, and this specifies a causal structure on dS^d:

$$C_x = \overline{\mathrm{V}_+} \cap T_x(\mathrm{dS}^d).$$

Anti-de Sitter space is the hypersurface

$$\mathrm{AdS}^d = \{(x_1, x_2, \mathbf{x}) \in \mathbb{R}^{2,d-1} \colon x_1^2 + x_2^2 - \mathbf{x}^2 = 1\}$$

in $\mathbb{R}^{2,d}$, endowed with the symmetric bilinear form

$$\gamma(x,y) = x_1 y_1 + x_2 y_2 - \mathbf{x}\mathbf{y} \quad \text{for} \quad x = (x_1, x_2, \mathbf{x}) \in \mathbb{R}^{2,d-1}.$$

Again, the tangent space $T_x(\mathrm{AdS}^d)$ can be identified with the hyperplane

$$x^{\perp_\gamma} = \{y \in \mathbb{R}^{2,d-1} \colon \gamma(x,y) = 0\}.$$

Since $\gamma(x,x) = 1$, the restriction of γ to this hyperplane is Lorentzian, and it is easy to verify that it is time-orientable (there exists a continuous selection of "positive" light cones) (cf. [NÓ23a, §11]), so that it carries a causal structure on AdS^d. One can also argue by the connectedness of the stabilizer group $\mathrm{SO}_{2,d-1}(\mathbb{R})_e^{\mathbf{e}_1} \cong \mathrm{SO}_{1,d-1}(\mathbb{R})_e$ to see that it leaves both light cones in $T_{\mathbf{e}_1}(\mathrm{AdS}^d)$ invariant.

For a family $\mathcal{M}(\mathcal{O}) \subseteq B(\mathcal{H})$ and a unitary representation $(U, \mathcal{H})$ of a Lie group G on $\mathcal{H}$, acting also on M, we consider the following axioms:

(Iso) **Isotony:** $\mathcal{O}_1 \subseteq \mathcal{O}_2$ implies $\mathcal{M}(\mathcal{O}_1) \subseteq \mathcal{M}(\mathcal{O}_2)$.

(RS) **Reeh–Schlieder property:** There exists a unit vector $\Omega \in \mathcal{H}$ that is cyclic for $\mathcal{M}(\mathcal{O}), \mathcal{O} \neq \emptyset$.

(Cov) **Covariance:** $U_g \mathcal{M}(\mathcal{O}) U_g^{-1} = \mathcal{M}(g\mathcal{O})$ for $g \in G$.

(Vi) **Invariance of the vacuum:** $U(g)\Omega = \Omega$ for $g \in G$.

(BW) **Bisognano–Wichmann property:** There exists a Lie algebra element $h \in \mathfrak{g}$ and a subset $W \subseteq M$ (called a wedge region), such that Ω is cyclic and separating for $\mathcal{M}(W)$ and the corresponding modular operator $\Delta = \Delta_{\mathtt{V}_{\mathcal{M}(W),\Omega}}$ is given by

$$\Delta = e^{2\pi i \cdot \partial U(h)}, \qquad \text{i.e.,} \qquad \Delta^{-it/2\pi} = U(\exp th), t \in \mathbb{R}.$$

(Loc) **Locality:** There exists an open non-empty G-invariant subset $\mathcal{D}_{\text{loc}} \subseteq M \times M$ such that $\mathcal{O}_1 \times \mathcal{O}_2 \subseteq \mathcal{D}_{\text{loc}}$ implies $\mathcal{M}(\mathcal{O}_1) \subseteq \mathcal{M}(\mathcal{O}_2)'$.

(Add) **Additivity:** $\mathcal{M}(\bigcup_j \mathcal{O}_j)$ is generated by the algebras $\mathcal{M}(\mathcal{O}_j), j \in J$.

Remark 1.16. These axioms are an abstract form of the axioms imposed on nets of local algebras on Minkowski space $M = \mathbb{R}^{1,d}$ and the *Poincaré group* $G = \mathbb{R}^{1,d-1} \rtimes \mathrm{SO}_{1,d-1}(\mathbb{R})_e$, acting by affine isometries. We now explain the differences, resp., the specifics of the Minkowski case.
(a) Here h is a generator of a Lorentz boost:

$$h.(x_0, x_1, \ldots, x_{d-1}) = (x_1, x_0, 0, \ldots, 0), \tag{1.3}$$

and the corresponding wedge region is the *Rindler wedge*

$$W_R := \{x \in \mathbb{R}^{1,d-1} \colon x_1 > |x_0|\}, \tag{1.4}$$

the set of all points x, where $h.x$ is positive timelike. The corresponding one-parameter group of G consists of Lorentz boosts

$$e^{th} = \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} \oplus \mathbf{1}_{\mathbb{R}^{d-2}}.$$

(b) The physical interpretation of the Reeh–Schlieder condition is that every state can be measured with arbitrary precision in any laboratory $\mathcal{O}$.
(c) In AQFT, one sometimes assumes, in addition to (Vi), the "irreducibility condition" that the fixed point space $\mathcal{H}^G$ of G is one-dimensional, i.e., $\mathcal{H}^G = \mathbb{C}\Omega$.
(d) For Minkowski space, the subset $\mathcal{D}_{\text{loc}} \subseteq M \times M$ is the set of spacelike pairs

$$\{(x, y) \in \mathbb{R}^{1,d-1} \times \mathbb{R}^{1,d-1} \colon \beta(x - y, x - y) < 0\}$$

for the Lorentzian form $\beta(x, y) = x_0 y_0 - \mathbf{x}\mathbf{y}$. These are the pairs of spacetime events that cannot "exchange" information traveling not faster than light. As a consequence, observables in $\mathcal{O}_1$ and $\mathcal{O}_2$ can be evaluated simultaneously if $\mathcal{O}_1 \times \mathcal{O}_2 \subseteq \mathcal{D}_{\text{loc}}$.

For two selfadjoint operators A_1 and A_2, commuting is equivalent to the non-existence of uncertainties in common measurements (Exercise 1.9). Then there exists a spectral measure P on $\mathbb{R}^2$ with

$$A_1 = \int_{\mathbb{R}^2} x_1 \, dP(x) \quad \text{and} \quad A_2 = \int_{\mathbb{R}^2} x_2 \, dP(x).$$

As a consequence, states can be localized simultaneously with respect to A_1 and A_2 with arbitrary precision.

The monographs of Varadarajan [Va85] and Mackey [Ma78] are excellent references for the connection between observables in Quantum Physics and selfadjoint operators. We also recommend the recent paper [Ba20] by J. Baez on Jordan and Lie structures related to classical and quantum observables.

We would like to understand the configurations specified by the G-action on M, the geometry of M, the unitary representation $U\colon G \to \mathrm{U}(\mathcal{H})$ and the von Neumann algebras $\mathcal{M}(\mathcal{O})$, satisfying these axioms. As the algebra structure of the local algebras $\mathcal{M}(\mathcal{O})$ only enters through the modular groups, it makes sense to strip it off to simplify the situation, with the hope that we arrive at more tractable structures.

So we consider the family

$$\mathsf{H}(\mathcal{O}) = \mathsf{V}_{\mathcal{M}(\mathcal{O}),\Omega} = \overline{\mathcal{M}(\mathcal{O})_h\Omega} \subseteq \mathcal{H} \tag{1.5}$$

of closed real subspaces. If Ω is standard for $\mathcal{M}(\mathcal{O})$, then $\mathsf{H}(\mathcal{O})$ is standard (Lemma 1.9(c)), and the corresponding modular objects can be recovered from $\mathsf{H}(\mathcal{O})$ (Definition 1.3). So we do not lose any information on them.

The axioms for the algebras $\mathcal{M}(\mathcal{O})$ thus turn into the following axioms for the net $\mathsf{H}(\mathcal{O})$ of real subspaces:

(Iso) **Isotony**: $\mathcal{O}_1 \subseteq \mathcal{O}_2$ implies $\mathsf{H}(\mathcal{O}_1) \subseteq \mathsf{H}(\mathcal{O}_2)$

(RS) **Reeh–Schlieder property:** $\mathsf{H}(\mathcal{O})$ is cyclic if $\mathcal{O} \neq \emptyset$.

(Cov) **Covariance:** $U_g\mathsf{H}(\mathcal{O}) = \mathsf{H}(g\mathcal{O})$ for $g \in G$.

(BW) **Bisognano–Wichmann property:** There exists a Lie algebra element $h \in \mathfrak{g}$ and $W \subseteq M$, such that $\mathsf{H}(W)$ is standard and the corresponding modular operator is

$$\Delta_{\mathsf{H}(W)} = e^{2\pi i \cdot \partial U(h)}, \quad \text{i.e.,} \quad \Delta^{-it/2\pi} = U(\exp th), t \in \mathbb{R}.$$

(Loc) **Locality:** There exists an open non-empty G-invariant subset $\mathcal{D}_{\rm loc} \subseteq M \times M$ such that $\mathcal{O}_1 \times \mathcal{O}_2 \subseteq \mathcal{D}_{\rm loc}$ implies $\mathsf{H}(\mathcal{O}_1) \subseteq \mathsf{H}(\mathcal{O}_2)'$.

(Add) **Additivity:** $\mathsf{H}(\bigcup_j \mathcal{O}_j) = \overline{\sum_{j\in J} \mathsf{H}(\mathcal{O}_j)}$.

Remark 1.17. (a) The covariance condition (Cov) for real subspaces follows from the G-invariance of Ω and the covariance condition $U_g\mathcal{M}(\mathcal{O})U_g^{-1} = \mathcal{M}(g\mathcal{O})$.
(b) The subspace $\mathsf{H}(M)$ is G-invariant by (Cov) and cyclic by (RS). If it is also separating, hence standard, then its modular operator $\Delta_{\mathsf{H}(M)}$ and the conjugation $J_M := J_{\mathsf{H}(M)}$ commute with $U(G)$. If (BW) holds, then Proposition 1.20 below implies $\mathsf{H}(W) = \mathsf{H}(M)$, and thus h is central in $\mathfrak{g}$, provided $\ker U$ is discrete. This shows that $\mathsf{H}(M)$ cannot be standard if the net is not very degenerate.

The passage from a net of algebras $\mathcal{M}(\mathcal{O})$ to a net of real subspace $\mathsf{H}(\mathcal{O})$ (which is similar to a forgetful functor) can be "inverted" (in the spirit of an adjoint functor) by procedures of second quantization assigning operator algebras $\Gamma(\mathsf{H})$ to real subspaces $\mathsf{H} \subseteq \mathcal{H}$. We refer to Subsection 1.4.3 below for some more details (see also [Ar63] and [NÓ17]). Therefore any result on nets of real subspaces can also be transformed into a result on nets of local algebras obtained by second quantization (see also [NÓ17, Rem. 4.10]). We note, however, that most second quantization procedures (such as the bosonic and fermionic one) are "free" in the sense that they do not take interaction between particles into account. For a recent systematic construction of twisted second quantization functors, we refer to [CSL23].

As far as the symmetries and the modular groups are concerned, the algebra axioms are faithfully represented by the axioms for their associated real subspaces. Even inclusions are rather well-behaved; we refer to Proposition 1.21 for a precise statement.

1.4 Appendices to Section 1

1.4.1 Background on standard subspaces

Lemma 1.18. *The passage* $\mathtt{V} \mapsto \mathtt{V}'$ *has the following properties:*

(a) $\mathtt{V}'' = \mathtt{V}$.

(b) $\mathtt{V}$ *is cyclic if and only if* $\mathtt{V}'$ *is separating.*

(c) $\mathtt{V}$ *is standard if and only if* $\mathtt{V}'$ *is standard.*

(d) $T_{\mathtt{V}}^* = T_{\mathtt{V}'}$*, i.e.,* $\mathcal{D}(T_{\mathtt{V}}^*) = \mathtt{V}' + i\mathtt{V}'$ *and* $\langle T_{\mathtt{V}}\xi, \eta\rangle = \overline{\langle \xi, T_{\mathtt{V}'}\eta\rangle}$ *for* $\xi \in \mathtt{V} + i\mathtt{V}, \eta \in \mathtt{V}' + i\mathtt{V}'$.

(e) $\Delta_{\mathtt{V}'} = \Delta_{\mathtt{V}}^{-1}$ *and* $J_{\mathtt{V}'} = J_{\mathtt{V}}$.

(f) $J_{\mathtt{V}}\mathtt{V} = \mathtt{V}'$.

Proof. (a) follows immediately from the Hahn–Banach Theorem. Alternatively, we can use that $\mathtt{V}' = i\mathtt{V}^{\perp_{\mathbb{R}}}$ and that multiplication with i is isometric, to obtain $\mathtt{V}'' = i^2(\mathtt{V}^{\perp_{\mathbb{R}}})^{\perp_{\mathbb{R}}} = \mathtt{V}$.

(b) The subspace $(\mathtt{V} + i\mathtt{V})' = \mathtt{V}' \cap i\mathtt{V}'$ vanishes if and only if $\mathtt{V}$ is separating if and only if $\mathtt{V}'$ is cyclic.

(c) If $\mathtt{V}$ is standard, then (b) implies that $\mathtt{V}'$ is separating. That $\mathtt{V}'$ is also cyclic follows from (b) and $(\mathtt{V}')' = \mathtt{V}$. Hence $\mathtt{V}'$ is standard if $\mathtt{V}$ has this property. If $\mathtt{V}'$ is standard, then we now see with (a) that $\mathtt{V} = \mathtt{V}''$ is also standard.

(d) First we show that $T_{\mathtt{V}'} \subseteq T_{\mathtt{V}}^*$. In fact, for $a, b \in \mathtt{V}'$ and $v, w \in \mathtt{V}$, we derive from $\langle \mathtt{V}, \mathtt{V}'\rangle \subseteq \mathbb{R}$ that

$$\langle T_{\mathtt{V}'}(a+ib), v+iw\rangle = \langle a-ib, v+iw\rangle = \langle a, v\rangle - \langle b, w\rangle = \langle a+ib, v-iw\rangle = \langle a+ib, T_{\mathtt{V}}(v+iw)\rangle.$$

Next we observe that, for $\xi \in \mathtt{V}$ and $\eta \in \mathcal{D}(T_{\mathtt{V}}^*)$, we have

$$\langle \xi, T_{\mathtt{V}}^*\eta\rangle = \overline{\langle T_{\mathtt{V}}\xi, \eta\rangle} = \overline{\langle \xi, \eta\rangle}.$$

From the equality of real and imaginary part, we derive that

$$T_{\mathtt{V}}^*\eta - \eta \in \mathtt{V}^{\perp_{\mathbb{R}}} = i\mathtt{V}' \quad \text{and} \quad T_{\mathtt{V}}^*\eta + \eta \in \mathtt{V}'.$$

Therefore $\eta \in \mathtt{V}' + i\mathtt{V}' = \mathcal{D}(T_{\mathtt{V}'})$, and hence that $T_{\mathtt{V}'} = T_{\mathtt{V}}^*$.

(e) From (d) we derive with Exercise 1.10 that

$$T_{\mathtt{V}'} = (T_{\mathtt{V}})^* = (J_{\mathtt{V}}\Delta_{\mathtt{V}}^{1/2})^* \overset{1.10}{=} \Delta_{\mathtt{V}}^{1/2}J_{\mathtt{V}}^* = \Delta_{\mathtt{V}}^{1/2}J_{\mathtt{V}} = J_{\mathtt{V}}\Delta_{\mathtt{V}}^{-1/2}.$$

Thus (e) follows from the uniqueness of the polar decomposition.

(f) If $v \in \mathtt{V}$, then

$$T_{\mathtt{V}'} J_{\mathtt{V}} v = J_{\mathtt{V}} \Delta_{\mathtt{V}}^{-1/2} J_{\mathtt{V}} v = \Delta_{\mathtt{V}}^{1/2} v = J_{\mathtt{V}} v.$$

This shows that $J_{\mathtt{V}}\mathtt{V} \subseteq \mathtt{V}'$. Likewise $J_{\mathtt{V}}\mathtt{V}' = J_{\mathtt{V}'}\mathtt{V}' \subseteq \mathtt{V}'' = \mathtt{V}$, so that $\mathtt{V}' \subseteq J_{\mathtt{V}}\mathtt{V}$, and thus $\mathtt{V}' = J_{\mathtt{V}}\mathtt{V}$. □

Lemma 1.19. ([Lo08, Prop. 3.11]) *Let $U_t = e^{itA}$ be a unitary one-parameter group on $\mathcal{H}$, and $f \colon \mathbb{R} \to \mathbb{C}$ a locally bounded Borel measurable function. If $\mathcal{D} \subseteq \mathcal{D}(f(A))$ is a U-invariant linear subspace dense in $\mathcal{H}$, then it is a core for $f(A)$.*

Proof. We factorize $f = f_0 f_1$ with $f_0(\mathbb{R}) \subseteq \mathbb{T}$ and $f_1 \geq 0$, so that $f(A) = f_0(A) f_1(A)$. Then $f_0(A)$ is bounded and $\mathcal{D} \subseteq \mathcal{D}(f_1(A)) = \mathcal{D}(f(A))$. It therefore suffices to show that $\mathcal{D}$ is a core for $B := f_1(A)$, resp., that the graph $\Gamma(B_0)$ of $B_0 := B|_{\mathcal{D}}$ is dense in the graph of B. This is equivalent to B_0 being essentially selfadjoint.

Replacing B_0 by its closure, whose domain is also U-invariant, we may assume that B_0 is closed and we have to show that $B_0 = B$. As B is selfadjoint, it suffices to verify that $\mathcal{R}(B_0 + i\mathbf{1})$ is dense in $\mathcal{H}$. So let $v \in \mathcal{R}(B_0 + i\mathbf{1})^\perp$. We have to show that $v = 0$.

The closed subspace $\Gamma(B_0) \subseteq \mathcal{H}^2$ is invariant under the diagonal action of the operators $(U_t)_{t \in \mathbb{R}}$, hence also under the operators $U(\varphi) = \int_{\mathbb{R}} \varphi(t) U_t \, dt$ for $\varphi \in L^1(\mathbb{R})$. In view of the relation $U(\varphi) = \widehat{\varphi}(A)$, these include the operators $\psi(A), \psi \in \mathcal{S}(\mathbb{R}, \mathbb{R})$. For all $w \in \mathcal{D}$ and $\psi \in \mathcal{S}(\mathbb{R}, \mathbb{R})$, we thus have

$$v \perp (B_0 + i\mathbf{1})\psi(A)\mathcal{D} = (f(A) + i\mathbf{1})\psi(A)\mathcal{D}.$$

If ψ has compact support, then the operator $f(A)\psi(A)$ is bounded because f is locally bounded. So the density of $\mathcal{D}$ in $\mathcal{H}$ implies that $v \perp (B + i\mathbf{1})\psi(A)\mathcal{H}$. This in turn implies that

$$\psi(A)v \in \mathcal{R}(B + i\mathbf{1})^\perp = \{0\}.$$

Choosing ψ_n in such a way that $0 \leq \psi_n \leq 1$ and $\psi_n|_{[-n,n]} = 1$, then $0 = \psi_n(A)v \to v$ entails that $v = 0$. □

Proposition 1.20. ([Lo08, Prop. 3.10]) *Let $\mathsf{H}_1 \subseteq \mathtt{V} \subseteq \mathsf{H}_2$ be closed subspaces such that $\mathtt{V}$ is standard, H_1 is cyclic and H_2 separating. If $\Delta_{\mathtt{V}}^{it}\mathsf{H}_j = \mathsf{H}_j$ holds for all $t \in \mathbb{R}$, then $\mathsf{H}_1 = \mathtt{V} = \mathsf{H}_2$.*

Proof. Our assumption implies that $\mathsf{H}_1 + i\mathsf{H}_1 = \mathcal{D}(T_{\mathsf{H}_1}) = \mathcal{D}(\Delta_{\mathsf{H}_1}^{1/2})$ is a dense subspace of $\mathcal{H}$, invariant under the modular group $U_t = \Delta_{\mathtt{V}}^{it}$, $t \in \mathbb{R}$. This subspace is contained in $\mathtt{V} + i\mathtt{V} = \mathcal{D}(T_{\mathtt{V}}) = \mathcal{D}(\Delta_{\mathtt{V}}^{1/2})$, hence a core of $\Delta_{\mathtt{V}}^{1/2}$ by Lemma 1.19, and therefore also a core of $T_{\mathtt{V}}$. Since $T_{\mathtt{V}}$ is an extension of T_{H_1}, the closedness of T_{H_1} implies that $T_{\mathsf{H}_1} = T_{\mathtt{V}}$, hence that $\mathsf{H}_1 = \mathtt{V}$.

To deal with H_2, we note that $\mathsf{H}_2' \subseteq \mathtt{V}'$ is cyclic by Lemma 1.18(b). Our assumption now implies that H_2' is invariant under the modular group of $\mathtt{V}'$, and the first part of the proof thus entails $\mathsf{H}_2' = \mathtt{V}'$. Finally, $\mathsf{H}_2 = \mathsf{H}_2'' = \mathtt{V}'' = \mathtt{V}$. □

1.4.2 Cyclic and separating vectors

We collect in this subsection some basic observations on cyclic and separating vectors.

Proposition 1.21. ([Lo08, Prop. 3.24]) *Let $\mathcal{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra with standard vector Ω.*

(a) *If $\mathcal{N}_1, \mathcal{N}_2 \subseteq \mathcal{M}$ are von Neumann algebras with $\mathtt{V}_{\mathcal{N}_1,\Omega} \subseteq \mathtt{V}_{\mathcal{N}_2,\Omega}$, then $\mathcal{N}_1 \subseteq \mathcal{N}_2$.*

(b) *If $\mathcal{N}$ is a von Neumann algebra commuting with $\mathcal{M}$ and $\mathtt{V}_{\mathcal{N},\Omega} = \mathtt{V}'_{\mathcal{M},\Omega}$, then $\mathcal{N} = \mathcal{M}'$.*

Proof. (a) Let $A \in \mathcal{N}_1$ be selfadjoint. As $\mathcal{N}_{1,h}\Omega \subseteq \overline{\mathcal{N}_{2,h}\Omega}$, there exists a sequence of hermitian elements $A_n \in \mathcal{N}_2$ with $A_n\Omega \to A\Omega$. Then $A_nA'\Omega \to AA'\Omega$ for every $A' \in \mathcal{M}'$. Thus $A_n \to A$ strongly on the dense subspace $\mathcal{M}'\Omega$. Since the hermitian operators A_n and A are bounded and Ω is separating, hence cyclic for $\mathcal{M}'$, the dense subspace $\mathcal{M}'\Omega$ is a common core for all of them. With [RS73, Thm. VIII.25] it now follows that $A_n \to A$ holds in the strong resolvent sense, i.e., that $(i\mathbf{1} + A_n)^{-1} \to (i\mathbf{1} + A)^{-1}$ in the strong operator topology. This implies that $(i\mathbf{1} + A)^{-1} \in \mathcal{N}_2$, which entails $A \in \mathcal{N}_2$.
(b) From $\mathcal{N} \subseteq \mathcal{M}'$ and $\mathtt{V}_{\mathcal{N},\Omega} = \mathtt{V}'_{\mathcal{M},\Omega} = \mathtt{V}_{\mathcal{M}',\Omega}$ (Lemma 1.12) we derive with (a) that $\mathsf{N} = \mathcal{M}'$. □

Corollary 1.22. *Let $\mathcal{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra and $\Omega \in \mathcal{H}$ separating for $\mathcal{M}$. To every von Neumann subalgebra $\mathcal{N} \subseteq \mathcal{M}$ we associate the closed real subspace $\mathtt{V}_{\mathcal{N}} := \overline{\mathcal{N}_h\Omega}$. Then $\mathtt{V}_{\mathcal{N}_1} = \mathtt{V}_{\mathcal{N}_2}$ implies $\mathcal{N}_1 = \mathcal{N}_2$ for $\mathcal{N}_1, \mathcal{N}_2 \subseteq \mathcal{M}$.*

Note that the subspace $\mathtt{V}_{\mathcal{N},\Omega}$ is standard if Ω is also cyclic for $\mathcal{N}$.

1.4.3 Weyl operators on the symmetric Fock space

In this subsection, we consider the bosonic Fock space $\mathcal{F}_s(\mathcal{H})$ of the complex Hilbert space $\mathcal{H}$. We want to define natural unitary operators on this space, called the *Weyl operators*. They will form a unitary representation of the *Heisenberg group* $\mathrm{Heis}(\mathcal{H})$.

We start by observing that, for every $v \in \mathcal{H}$, the series

$$\mathrm{Exp}(v) := \sum_{n=0}^{\infty} \frac{1}{n!} v^n\,,$$

defines an element in $\mathcal{F}_s(\mathcal{H})$ and that by

$$\langle v^n, w^n\rangle = n!\langle v, w\rangle^n \quad \text{and} \quad \|v^n\| = \sqrt{n!}\|v\|^n$$

([NÓ17, §6.1]), the scalar product of two such elements is given by

$$\langle \mathrm{Exp}(v), \mathrm{Exp}(w)\rangle = \sum_{n=0}^{\infty} \frac{n!}{(n!)^2}\langle v, w\rangle^n = e^{\langle v,w\rangle}.$$

Lemma 1.23. $\mathrm{Exp}(\mathcal{H})$ *is total in $\mathcal{F}_s(\mathcal{H})$, i.e., it spans a dense subspace.*

Proof. Let $\mathcal{K} \subseteq \mathcal{F}_s(\mathcal{H})$ be the closed subspace generated by $\mathrm{Exp}(\mathcal{H})$. We consider the unitary representation of the circle group $\mathbb{T} \subseteq \mathbb{C}^\times$ on $\mathcal{F}_s(\mathcal{H})$ by

$$U_z(v_1 \vee \cdots \vee v_n) := z^n(v_1 \vee \cdots \vee v_n) \quad \text{for} \quad n \in \mathbb{N}_0, v_j \in \mathcal{H}.$$

The decomposition $\mathcal{F}_s(\mathcal{H}) = \widehat{\bigoplus}_{n=0}^{\infty} S^n(\mathcal{H})$ is the eigenspace decomposition with respect to the operators U_z and it is easy to see that the action of $\mathbb{T}$ on $\mathcal{F}_s(\mathcal{H})$ has continuous orbit maps (Exercise 1.1).

For $\xi \in \mathcal{F}_s(\mathcal{H})$ with $\xi = \sum_{n=0}^{\infty} \xi_n$ and $\xi_n \in S^n(\mathcal{H})$, we have $U_z\xi = \sum_{n=0}^{\infty} z^n\xi_n$, so that

$$\xi_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-2\pi int} U_{e^{it}}\xi\, dt$$

(observe the analogy with Fourier coefficients). It follows that, for $\xi \in \mathcal{K}$, the existence of the above Riemann integral in the closed subspace $\mathcal{K}$ implies $\xi_n \in \mathcal{K}$. We conclude that $v^n \in \mathcal{K}$ for $v \in \mathcal{H}$ and $n \in \mathbb{N}_0$. Therefore it suffices to observe that the subset $\{v^n : v \in \mathcal{H}\}$ is total in $S^n(\mathcal{H})$ (Exercise 1.8). □

For $v, x \in \mathcal{H}$ we have

$$\langle \operatorname{Exp}(v+x), \operatorname{Exp}(w+x)\rangle = e^{\langle v+x, w+x\rangle} = e^{\langle v,w\rangle} e^{\langle x,w\rangle + \frac{\|x\|^2}{2}} e^{\langle v,x\rangle + \frac{\|x\|^2}{2}},$$

so that there exists a well-defined and uniquely determined unitary operator $U(x)$ on $\mathcal{F}_s(\mathcal{H})$ satisfying

$$U(x)\operatorname{Exp}(v) = e^{-\langle x,v\rangle - \frac{\|x\|^2}{2}} \operatorname{Exp}(v+x) \quad \text{for} \quad x, v \in \mathcal{H} \tag{1.6}$$

(Exercise 1.6; the surjectivity of $U(x)$ follows from the totality of $\operatorname{Exp}(\mathcal{H})$). A direct calculation then shows that

$$U(x)U(y) = e^{-i \operatorname{Im}\langle x,y\rangle} U(x+y) \quad \text{for} \quad x, y \in \mathcal{H}. \tag{1.7}$$

In fact, for $v \in \mathcal{H}$, we have

$$\begin{aligned} U(x)U(y)\operatorname{Exp}(v) &= U(x) e^{-\langle y,v\rangle - \frac{\|y\|^2}{2}} \operatorname{Exp}(v+y) \\ &= e^{-\langle y,v\rangle - \frac{\|y\|^2}{2}} e^{-\langle x,v+y\rangle - \frac{\|x\|^2}{2}} \operatorname{Exp}(v+y+x) \\ &= e^{-\langle x+y,v\rangle} e^{-\frac{\|y\|^2}{2} - \frac{\|x\|^2}{2} - \langle x,y\rangle} \operatorname{Exp}(v+y+x) \end{aligned}$$

and

$$\begin{aligned} U(x+y)\operatorname{Exp}(v) &= e^{-\langle x+y,v\rangle - \frac{\|x+y\|^2}{2}} \operatorname{Exp}(v+y+x) \\ &= e^{-\langle x+y,v\rangle - \frac{\|x\|^2}{2} - \frac{\|y\|^2}{2} - \operatorname{Re}\langle x,y\rangle} \operatorname{Exp}(v+y+x) \end{aligned}$$

The relation (1.7) shows that the map $U \colon (\mathcal{H}, +) \to \mathrm{U}(\mathcal{F}_s(\mathcal{H}))$ is not a group homomorphism. Instead, we have to replace the additive group of $\mathcal{H}$ by the *Heisenberg group*

$$\operatorname{Heis}(\mathcal{H}) := \mathbb{T} \times \mathcal{H} \quad \text{with} \quad (z,v)(z',v') := (zz' e^{-i\operatorname{Im}\langle v,v'\rangle}, v+v').$$

For this group, we obtain a unitary representation

$$\widehat{U} \colon \operatorname{Heis}(\mathcal{H}) \to \mathrm{U}(\mathcal{F}_s(\mathcal{H})) \quad \text{by} \quad \widehat{U}(z,v) := zU(v).$$

The operators

$$W(v) := U(iv/\sqrt{2}), \qquad v \in \mathcal{H},$$

are called *Weyl operators*. They satisfy the *Weyl relations*

$$W(v)W(w) = e^{-i\operatorname{Im}\langle v,w\rangle/2} W(v+w) \quad \text{for} \quad v, w \in \mathcal{H}. \tag{1.8}$$

They are an exponentiated form of the "canonical commutation relations" for the corresponding infinitesimal generators.

The *Weyl algebra*

$$W(\mathcal{H}) := C^*(\{W(v) \colon v \in \mathcal{H}\}) \subseteq B(\mathcal{F}_s(\mathcal{H}))$$

is the C^*-subalgebra of $B(\mathcal{F}_s(\mathcal{H}))$ generated by the Weyl operators. It plays an important role in Quantum (Statistical) Mechanics and Quantum Field Theory. This is partly due to the fact that it is a simple C^*-algebra (all ideals are trivial), which implies that all its representations are faithful. Closely related is its universal property: If $\mathcal{A}$ is a unital C^*-algebra and $\varphi \colon \mathcal{H} \to \mathrm{U}(\mathcal{A})$ a map satisfying the Weyl relations in the form

$$\varphi(v)\varphi(w) = e^{-i\operatorname{Im}\langle v,w\rangle/2} \varphi(v+w) \quad \text{for} \quad v, w \in \mathcal{H}, \tag{1.9}$$

then there exists a unique homomorphism $\Phi \colon W(\mathcal{H}) \to \mathcal{A}$ of unital C^*-algebras with $\Phi \circ W = \varphi$. An excellent discussion of the Weyl algebra and its properties can be found in the monograph [BR96] which also describes the physical applications in great detail.

1.4.4 From real subspaces to von Neumann algebras

In this subsection, we describe a mechanism that associates to real subspaces of a Hilbert space $\mathcal{H}$ von Neumann algebras on the symmetric Fock space $\mathcal{F}_s(\mathcal{H})$. This construction plays an important role in recent developments in Algebraic Quantum Field Theory (AQFT) because it provides natural links between the geometric structure of spacetime and operator algebras (see in particular [Ar99, Lo08, Le15]). It has also been of great interest for the classification of factors because it provides very controlled constructions of factors whose type can be determined in some detail ([AW63, AW68]).

We write

$$\gamma(v,w) := \operatorname{Im}\langle v,w\rangle \quad \text{for} \quad v,w \in \mathcal{H}$$

and observe that γ is skew-symmetric and non-degenerate, so that the underlying real Hilbert space $\mathcal{H}^{\mathbb{R}}$ carries the structure of a *symplectic vector space* $(\mathcal{H}^{\mathbb{R}},\gamma)$.

Using the Weyl operators, we associate to every real linear subspace $V \subseteq \mathcal{H}$ a von Neumann subalgebra

$$\mathcal{R}(V) := W(V)'' = \{W(v) \colon v \in V\}'' \subseteq B(\mathcal{F}_s(\mathcal{H})).$$

Lemma 1.24. *We have*

(i) $\mathcal{R}(V) \subseteq \mathcal{R}(W)'$ *if and only if* $V \subseteq W'$.

(ii) $\mathcal{R}(V)$ *is commutative if and only if* $V \subseteq V'$.

(iii) $\mathcal{R}(\mathcal{H}) = B(\mathcal{F}_s(\mathcal{H}))$*, i.e., the representation of* $\operatorname{Heis}(\mathcal{H})$ *on* $\mathcal{F}_s(\mathcal{H})$ *is irreducible.*

(iv) $\mathcal{R}(V) = \mathcal{R}(\overline{V})$.

(v) $\Omega = \operatorname{Exp}(0) \in \mathcal{F}_s(\mathcal{H})$ *is cyclic for* $\mathcal{R}(V)$ *if and only if* $V + iV$ *is dense in* $\mathcal{H}$.

(vi) $\Omega = \operatorname{Exp}(0) \in \mathcal{F}_s(\mathcal{H})$ *is separating for* $\mathcal{R}(V)$ *if and only if* $\overline{V} \cap i\overline{V} = \{0\}$.

Proof. (i) follows directly from the Weyl relations (1.8).
(ii) follows from (i).
(iii) follows from [BR96, Prop. 5.2.4(3)].
(iv) follows from the fact that $\mathcal{H} \to B(\mathcal{F}_s(\mathcal{H})), v \mapsto W_v$ is strongly continuous and $\mathcal{R}(V)$ is closed in the weak operator topology.
(v) Let $\mathcal{K} := \overline{V + iV}$. Then $\mathcal{R}(V)\Omega \subseteq \mathcal{F}_s(\mathcal{K})$, so that Ω cannot be cyclic if $\mathcal{K} \neq \mathcal{H}$.

Suppose, conversely, that $\mathcal{K} = \mathcal{H}$ and that $f \in (\mathcal{R}(V)\Omega)^{\perp}$. Then the holomorphic function $\widehat{f}(v) := \langle f, \operatorname{Exp}(v)\rangle$ on $\mathcal{H}$ vanishes on iV, hence also on $V + iV$, and since this subspace is dense in $\mathcal{H}$, we obtain $f = 0$ because $\operatorname{Exp}(\mathcal{H})$ is total in $\mathcal{F}_s(\mathcal{H})$.
(vi) In view of (iv), we may assume that V is closed. Let $0 \neq w \in \mathcal{K} := V \cap iV$. To see that Ω is not separating for $\mathcal{R}(V)$, it suffices to show that, for the one-dimensional Hilbert space $\mathcal{H}_0 := \mathbb{C}w$, the vector Ω is not separating for $\mathcal{R}(\mathbb{C}w) = B(\mathcal{F}_s(\mathbb{C}w))$ (see (iii)). This is obviously the case because $\dim \mathcal{F}_s(\mathbb{C}w) > 1$.

Suppose that $\mathcal{K} = \{0\}$. As $\mathcal{K} = V'' \cap (iV'') = (V' + iV')'$, it follows that $V' + iV'$ is dense in $\mathcal{H}$. By (v), Ω is cyclic for $\mathcal{R}(V')$ which commutes with $\mathcal{R}(V)$. Therefore Ω is separating for $\mathcal{R}(V)$. $\square$

Theorem 1.25. ([Ar63]) (Araki's Duality Theorem) *For closed real subspaces* V, W, V_j *of* $\mathcal{H}$*, the following assertions hold:*

(i) $\mathcal{R}(V) \subseteq \mathcal{R}(W)$ *if and only if* $V \subseteq W$.

(ii) $\mathcal{R}\big(\bigcap_{j\in J} V_j\big) = \bigcap_{j\in J} \mathcal{R}(V_j)$.

(iii) $\mathcal{R}(V)' = \mathcal{R}(V')$ *(Duality).*

(iv) $Z(\mathcal{R}(V)) = \mathcal{R}(V \cap V')$. *In particular, $\mathcal{R}(V)$ is a factor if and only if $V \cap V' = \{0\}$.*

Proof. We only comment on some of these statements:
(i) That $V \subseteq W$ implies $\mathcal{R}(V) \subseteq \mathcal{R}(W)$ is clear, but the converse is non-trivial. It can be derived from the duality property (iii), which is a deep result, basically the main result of the paper [Ar63].
(ii) here "$\subseteq$" is easy.
(iii) is a deep theorem.
(iv) follows from (ii) and (iii). $\square$

The preceding theorem asserts in particular that

- $\mathcal{R}(V)$ is a factor if and only if $V \cap V' = \{0\}$. This means that the form $\gamma|_{V \times V}$ is non-degenerate, i.e., that (V, γ) is a symplectic vector space.

Subspaces with this property are very easy to construct. In [Ar64b] many results on the types of the so-obtained factors have been derived. In particular, it is shown that factors of type II do not arise from this construction and [Ar64] provides an explicit criterion for $\mathcal{R}(V)$ to be of type I. "Generically", the so-obtained factors are of type III. We refer to [Sa71] for details on the type of a von Neumann algebra.

1.4.5 Standard subspaces and graphs

Let $\mathtt{V} \subseteq \mathcal{H}$ be a standard subspace and recall that $\mathtt{V} + i\mathtt{V} = \mathcal{D}(\Delta^{1/2})$. The natural Hilbert space structure on this dense subspace of $\mathcal{H}$ is obtained from the isomorphism with the graph

$$\Gamma(\Delta^{1/2}) = \{(v, \Delta^{1/2}v) \colon v \in \mathcal{D}(\Delta^{1/2})\} \subseteq \mathcal{H} \oplus \mathcal{H}$$

which is a closed subspace.

Proposition 1.26. *Let $\mathcal{H}$ be a complex Hilbert space. Consider the complex structure on $\mathcal{H}^{\oplus 2}$ defined by $I(v, w) := (iv, -iw)$. For any positive selfadjoint operator $A > 0$ on $\mathcal{H}$, the graph $\Gamma(A) \subseteq \mathcal{H}^{\oplus 2}$ is a standard subspace whose Tomita operator is given by*

$$T(v, w) = (A^{-1}w, Av),$$

its modular operator by

$$\Delta(v, w) = A^2 \oplus A^{-2},$$

and its modular conjugation by

$$J(v, w) = (w, v).$$

Proof. Let $\mathsf{H} := \Gamma(A)$. We first observe that

$$I\mathsf{H} = \{(iv, -iAv) \colon v \in \mathcal{D}(A)\} = \{(v, -Av) \colon v \in \mathcal{D}(A)\} = \Gamma(-A).$$

Therefore

$$\mathsf{H} \cap I\mathsf{H} = \Gamma(A) \cap \Gamma(-A) = \ker(A) \oplus \{0\} = \{(0, 0)\}. \tag{1.10}$$

Next we observe that

$$\Gamma(A)^{\perp_{\mathbb{R}}} = \{(-Av, v) \colon v \in \mathcal{D}(A)\} =: \Gamma^{\mathrm{flip}}(-A).$$

So

$$(\mathsf{H} + I\mathsf{H})^{\perp_{\mathbb{R}}} = \mathsf{H}^{\perp_{\mathbb{R}}} \cap I\mathsf{H}^{\perp_{\mathbb{R}}} = \Gamma^{\mathrm{flip}}(-A) \cap I\Gamma^{\mathrm{flip}}(-A) = \Gamma^{\mathrm{flip}}(-A) \cap \Gamma^{\mathrm{flip}}(A) = \{(0,0)\}.$$

In view of (1.10), this proves that H is standard.

To identify the corresponding modular objects, we claim that

$$\mathsf{H} + I\mathsf{H} = \mathcal{D}(A) \oplus \mathcal{D}(A^{-1}).$$

Clearly, $\Gamma(\pm A) \subseteq \mathcal{D}(A) \oplus \mathcal{R}(A) = \mathcal{D}(A) \oplus \mathcal{D}(A^{-1})$, so that "$\subseteq$" holds. For the converse, let $v \in \mathcal{D}(A)$, $w \in \mathcal{D}(A^{-1})$ and put $u := A^{-1}w$. Then

$$(v, w) = (v, Au) = \Big(\frac{v+u}{2}, A\frac{v+u}{2}\Big) + \Big(\frac{v-u}{2}, -A\frac{v-u}{2}\Big) \in \mathsf{H} + I\mathsf{H}.$$

The domain of the modular operator T_{H} is $\mathsf{H} + I\mathsf{H} = \mathcal{D}(A) \oplus \mathcal{D}(A^{-1})$. On this domain the prescription

$$T(v, w) := (A^{-1}w, Av)$$

defines an I-antilinear involution with

$$\mathrm{Fix}(T) = \Gamma(A) = \mathsf{H}.$$

This implies that $T = T_{\mathsf{H}}$ is the Tomita operator of the standard subspace H.

It is easy to see that the adjoint operator is given by

$$T^*(v, w) = (Aw, A^{-1}v) \quad \text{with domain} \quad \mathcal{D}(A^{-1}) \oplus \mathcal{D}(A).$$

We thus obtain

$$\Delta_{\mathsf{H}}(v, w) = (T^*T)(v, w) = T^*(A^{-1}w, Av) = (A^2 v, A^{-2} w),$$

and therefore $\Delta_{\mathsf{H}} = A^2 \oplus A^{-2}$. Finally, we obtain

$$J_{\mathsf{H}}(v, w) = T_{\mathsf{H}}\Delta_{\mathsf{H}}^{-1/2}(v, w) = T_{\mathsf{H}}(A^{-1}v, Aw) = (w, v). \qquad \square$$

Example 1.27. Let $\mathtt{V} \subseteq \mathcal{H}$ be a standard subspace. Then $\mathtt{V} + i\mathtt{V} = \mathcal{D}(\Delta^{1/2})$ and the embedding

$$\mathtt{V}_{\mathbb{C}} \to \Gamma(\Delta^{1/2}), \quad (v, w) \mapsto (v + iw, \Delta^{1/2}(v + iw))$$

identifies $\mathtt{V}_{\mathbb{C}}$ with a standard subspace of $\mathcal{H}^{\oplus 2}$, endowed with the complex structure $I(v, w) = (iv, -iw)$. Its modular operator takes the form

$$\Delta_{\mathtt{V}_{\mathbb{C}}} = \Delta_{\mathtt{V}} \oplus \Delta_{\mathtt{V}}^{-1}.$$

Example 1.28. (Standard subspaces for the translation representation) We consider $\mathcal{H} = L^2(\mathbb{R})$, $\beta > 0$, and the standard subspace $\mathtt{V} \subseteq L^2(\mathbb{R})$, specified by

$$Jf = \overline{f} \quad \text{and} \quad (\Delta^{-it/2\beta} f)(x) = f(x + t), \quad x, t \in \mathbb{R}.$$

Then $\mathcal{D}(\Delta^{1/2})$ consists of the space of boundary values of elements of the Hardy space

$$H^2(\mathcal{S}_\beta) := \Big\{F \in \mathcal{O}(\mathcal{S}_\beta) \colon \sup_{0<y<\beta} \|F(\cdot + iy)\|_2 < \infty\Big\}$$

(cf. [Go69, Prop 5.1]). For $f \in \mathcal{D}(\Delta^{1/2})$ we then have (almost everywhere in the sense of L^2-functions)

$$(\Delta^{1/2} f)(x) = f(x + i\beta)$$

(the upper boundary values on $\mathbb{R} + i\beta$), so that f is fixed by $J\Delta^{1/2}$ if and only if $f^\sharp = f$, where

$$f^\sharp(x) := \overline{f(x + i\beta)} \quad \text{for} \quad x \in \mathbb{R}.$$

This shows that

$$\mathtt{V} = \{f \in \mathcal{D}(\Delta^{1/2}) \colon f^\sharp = f\}. \tag{1.11}$$

Endowed with the graph topology, we have $\mathcal{D}(\Delta^{1/2}) \cong \Gamma(\Delta^{1/2})$, and this further leads to

$$\Gamma(\Delta^{1/2}) \cong H^2(\mathcal{S}_\beta) \subseteq L^2(\mathbb{R})^{\oplus 2},$$

where we identify $H^2(\mathcal{S}_\beta)$ via the boundary value map $F \mapsto (F|_{\mathbb{R}}, F|_{\mathbb{R}+i\beta})$ with a closed subspace of $L^2(\mathbb{R})^{\oplus 2}$.

In this picture, the Tomita involution $T_{\mathtt{V}}$ corresponds to the involution on $H^2(\mathcal{S}_\beta)$, given by

$$f^\sharp(z) = \overline{f(\beta i + \overline{z})} \quad \text{for} \quad z \in \mathcal{S}_\beta, \tag{1.12}$$

and the lower boundary value map thus induces an isometry

$$H^2(\mathcal{S}_\beta)^\sharp := \{f \in H^2(\mathcal{S}_\beta) \colon f^\sharp = f\} \to \mathtt{V}, \quad f \mapsto f|_{\mathbb{R}} \tag{1.13}$$

(cf. [NÓ17, Ex. 3.16]). On the pairs $(f_1, f_2) = (f, \Delta^{1/2} f) \in \Gamma(\Delta^{1/2}) \subseteq L^2(\mathbb{R})^{\oplus 2}$ of boundary values of elements of $H^2(\mathcal{S}_\beta)$, the involution $\sharp$ then takes the form

$$(f_1, f_2)^\sharp = (\overline{f_2}, \overline{f_1}).$$

1.4.6 Endomorphisms of standard subspaces and von Neumann algebras

Let $\Omega \in \mathcal{H}$ be a standard vector for the von Neumann algebra $\mathcal{M}$, let $\mathtt{V} = \mathtt{V}_{\mathcal{M},\Omega}$ be the corresponding standard subspace, and let $G \subseteq \mathrm{U}(\mathcal{H})$ be a subgroup.

We note that the inclusion

$$S_{\mathcal{M},\Omega} = \{g \in G \colon g\mathcal{M}g^{-1} \subseteq \mathcal{M}, g\Omega = \Omega\} \subseteq S_{\mathtt{V},\Omega} = \{g \in G \colon g\mathtt{V} \subseteq \mathtt{V}, g\Omega = \Omega\}$$

may be proper.

Example 1.29. [2] (a) We consider the Hilbert space $\mathcal{H} := M_n(\mathbb{C})$ of matrices, endowed with the Hilbert–Schmidt scalar product $\langle A, B\rangle := \mathrm{tr}(A^*B)$. By matrix multiplications from the left, we obtain a von Neumann subalgebra $\mathcal{M} \subseteq B(\mathcal{H})$, isomorphic to $M_n(\mathbb{C})$, and its commutant $\mathcal{M}'$ consists of right multiplications. The unit vector $\Omega := \frac{1}{\sqrt{n}}\mathbf{1}_n$ is cyclic and separating, and the corresponding standard subspaces for $\mathcal{M}$ and $\mathcal{M}'$ coincide with

$$\mathtt{V}_{\mathcal{M}} = \mathtt{V}_{\mathcal{M}'} = \mathrm{Herm}_n(\mathbb{C})$$

of hermitian matrices. Now $\theta(A) := A^\top$ defines a unitary operator on $\mathcal{H}$ preserving Ω and the standard subspace $\mathtt{V}_{\mathcal{M}} = \mathtt{V}_{\mathcal{M}'}$, and satisfying $\theta\mathcal{M}\theta^{-1} = \mathcal{M}'$. For $G = \mathrm{U}(\mathcal{H})$, we therefore obtain $S_{\mathtt{V}} \neq S_{\mathcal{M}}$.

[2] We thank Yoh Tanimoto for the discussion that led to this example.

(b) In the situation above, when $\mathcal{M}$ is given, the G-orbit of $\mathcal{M}$ in the space of von Neumann subalgebras of $B(\mathcal{H})$ can be identified with the homogeneous space $G/G_{\mathcal{M}}$, and similarly, $G/G_{\mathtt{V}} \hookrightarrow \mathrm{Stand}(\mathcal{H}), gG_{\mathtt{V}} \mapsto g\mathtt{V}$ is an embedding. The discrepancy between both spaces comes from the fact that the von Neumann algebra $\mathcal{M}$ need not be invariant under the stabilizer group $G_{\mathtt{V}}$ of $\mathtt{V}$.

Related questions have been analyzed by Y. Tanimoto in [Ta10]. He refines the picture by considering the closed convex cone

$$\mathtt{V}_{\mathcal{M}}^{+} = \overline{\{M\Omega \colon 0 \leq M = M^* \in \mathcal{M}\}} \subseteq \mathtt{V}_{\mathcal{M}},$$

which leads to the inclusions

$$S_{\mathcal{M},\Omega} \hookrightarrow S_{\mathtt{V}_{\mathcal{M}}^{+},\Omega} = \{g \in G \colon g\mathtt{V}_{\mathcal{M}}^{+} \subseteq \mathtt{V}_{\mathcal{M}}^{+}, g\Omega = \Omega\} \subseteq S_{\mathtt{V}_{\mathcal{M}}}.$$

The semigroup $S_{\mathtt{V}_{\mathcal{M}}^{+},\Omega}$ appears to be much closer to $S_{\mathcal{M},\Omega}$ than $S_{\mathtt{V},\Omega}$. From [Ta10, Thm. 2.10] it follows in particular that, if $\mathcal{M}$ is purely infinite, then $S_{\mathtt{V}_{\mathcal{M}}^{+},\Omega} = S_{\mathcal{M},\Omega}$. Let $\mathcal{M}_*$ denote the predual of the von Neumann algebra $\mathcal{M}$ (the space of normal linear functionals) and $\mathcal{M}_*^{+}$ the convex cone of positive normal functionals. In this context, it is also interesting to note that the map

$$\mathtt{V}_{\mathcal{M}}^{+} \to \mathcal{M}_*^{+}, \quad \xi \mapsto \omega_\xi, \qquad \omega_\xi(M) = \langle \xi, M\xi \rangle$$

is bijective by [Ko80, Thm. 1.2]. Accordingly, every element $g \in S_{\mathtt{V}_{\mathcal{M}}^{+}}$ induces a continuous map on the convex cone $\mathcal{M}_*^{+}$.

We refer to [Ta10] and [Co74] for more detailed information.

1.4.7 Positive definite functions on $\mathbb{R}$ satisfying a KMS condition

This subsection has only illustrative character. It explains how the KMS condition that classically appears in the context of KMS states for C^*-algebraic dynamical systems, can be formulated independently of C^*-algebras as a condition for functions on $\mathbb{R}$ with values in spaces of bilinear forms.

Definition 1.30. Let V be a real vector space and $\mathrm{Bil}(V)$ be the space of real bilinear maps $V \times V \to \mathbb{C}$. A function $\psi \colon \mathbb{R} \to \mathrm{Bil}(V)$ is said to be *positive definite* if the kernel $\psi(t-s)(v,w)$ on $\mathbb{R} \times V$ is positive definite. We say that a positive definite function $\psi \colon \mathbb{R} \to \mathrm{Bil}(V)$ satisfies the *KMS condition* for $\beta > 0$ if ψ extends to a function $\overline{\mathcal{S}_\beta} \to \mathrm{Bil}(V)$ which is pointwise continuous and pointwise holomorphic on the interior $\mathcal{S}_\beta$, and satisfies

$$\psi(i\beta + t) = \overline{\psi(t)} \quad \text{for} \quad t \in \mathbb{R}. \tag{1.14}$$

The central idea in the classification of positive definite functions satisfying a KMS condition is to relate them to standard subspaces. A key result in [NÓ19] is the following characterization of the KMS condition in terms of standard real subspaces. Here we write $\mathrm{Bil}^{+}(V) \subseteq \mathrm{Bil}(V)$ for the convex cone of all those bilinear forms f for which the sesquilinear extension to $V_{\mathbb{C}} \times V_{\mathbb{C}}$ is positive semidefinite.

Theorem 1.31. (Characterization of the KMS condition; [NÓ19, Thm. 2.6]) *Let V be a real vector space and $\psi \colon \mathbb{R} \to \mathrm{Bil}(V)$ be a pointwise continuous positive definite function. Then the following are equivalent:*

(i) *ψ satisfies the KMS condition for $\beta > 0$.*

(ii) *There exists a standard real subspace* $\mathtt{V}$ *in a Hilbert space* $\mathcal{H}$ *and a linear map* $j\colon V \to \mathtt{V}$ *such that*

$$\psi(t)(v,w) = \langle j(v), \Delta^{-it/\beta} j(w)\rangle \quad \textit{for} \quad t \in \mathbb{R}, v, w \in V. \tag{1.15}$$

(iii) *There exists a* $\mathrm{Bil}^+(V)$*-valued regular Borel measure* μ *on* $\mathbb{R}$ *satisfying*

$$\psi(t) = \int_{\mathbb{R}} e^{it\lambda}\, d\mu(\lambda), \quad \textit{where} \quad d\mu(-\lambda) = e^{-\beta\lambda} d\overline{\mu}(\lambda).$$

If these conditions are satisfied, then the function $\psi\colon \overline{\mathcal{S}_\beta} \to \mathrm{Bil}(V)$ *is pointwise bounded.*

The equivalence of (i) and (ii) in the preceding theorem describes the tight connection between the KMS condition and the modular objects associated to a standard real subspace. Part (iii) provides an integral representation that can be viewed as a classification result.

Corollary 1.32. *For a standard subspace* $\mathtt{V} \subseteq \mathcal{H}$ *and the modular operator* $\Delta_{\mathtt{V}}$*, the function*

$$\psi\colon \mathbb{R} \to \mathrm{Bil}(\mathtt{V}), \quad \psi(t)(v,w) := \langle v, \Delta_{\mathtt{V}}^{-it/2\pi} w\rangle$$

satisfies the KMS condition for $\beta = 2\pi$.

Remark 1.33. Important special cases arise from C^*-dynamical systems $(\mathcal{A}, \mathbb{R}, \alpha)$, where $\mathcal{A}$ is a C^*-algebra and $\alpha\colon \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$ defines a strongly continuous $\mathbb{R}$-action on $\mathcal{A}$. Let

$$V := \mathcal{A}_h := \{A \in \mathcal{A}\colon A^* = A\}$$

and consider an α-invariant state ω on $\mathcal{A}$. Such a state is a β-KMS state if and only if

$$\psi\colon \mathbb{R} \to \mathrm{Bil}(\mathcal{A}_h), \quad \psi(t)(A,B) := \omega(A\alpha_t(B))$$

satisfies the KMS condition for $\beta > 0$ (cf. [NÓ15, Prop. 5.2], [RvD77, Thm. 4.10], [BR96]). If $(\pi_\omega, U^\omega, \mathcal{H}_\omega, \Omega)$ is the corresponding covariant GNS representation of $(\mathcal{A}, \mathbb{R})$, then

$$\omega(A) = \langle \Omega, \pi_\omega(A)\Omega\rangle \quad \text{for} \quad A \in \mathcal{A} \quad \text{and} \quad U^\omega_t\Omega = \Omega \quad \text{for} \quad t \in \mathbb{R}.$$

For $A, B \in \mathcal{A}_h$, we thus obtain

$$\begin{aligned}\psi(t)(A,B) &= \omega(A\alpha_t(B)) = \langle \Omega, \pi_\omega(A\alpha_t(B))\Omega\rangle \\ &= \langle \Omega, \pi_\omega(A) U^\omega_t \pi_\omega(B) U^\omega_{-t}\Omega\rangle = \langle \pi_\omega(A)\Omega, U^\omega_t \pi_\omega(B)\Omega\rangle\end{aligned}$$

The corresponding standard real subspace of $\mathcal{H}_\omega$ is $\mathtt{V}_{\mathcal{A},\Omega} := \overline{\pi_\omega(\mathcal{A}_h)\Omega}$. Here we use that the KMS condition implies that Ω is a separating vector for the von Neumann algebra $\pi_\omega(\mathcal{A})''$ (cf. [Si23] and [BR87]).

1.5 Exercises for Section 1

Exercise 1.1. Let X be a topological space, $\mathcal{H}$ be a Hilbert space and $\gamma\colon X \to \mathcal{H}$ be a map. Show that γ is continuous if and only if the corresponding kernel function

$$K\colon X \times X \to \mathbb{C}, \quad K(x,y) := \langle \gamma(x), \gamma(y)\rangle$$

is continuous.

Exercise 1.2. Let $(U_t = e^{itA})_{t\in\mathbb{R}}$ be a unitary one-parameter group on the complex Hilbert space $\mathcal{H}$ and consider on the complex Hilbert space $\mathcal{H}^\sharp := \mathcal{H} \oplus \overline{\mathcal{H}}$ the unitary one-parameter group

$$U_t^\sharp := U_t \oplus U_t.$$

Show that the flip involution $J^\sharp(v,w) := (w,v)$ and the positive operator

$$\Delta^\sharp := e^A \oplus e^{-A}$$

form a modular pair of a standard subspace $\mathtt{V} \subseteq \mathcal{H}^\sharp$ (cf. Proposition 1.26).

Exercise 1.3. If $\mathtt{V} \subseteq \mathcal{H}$ is a standard subspace, we consider the antiunitary representation of $\mathbb{R}^\times$, defined by

$$\gamma_{\mathtt{V}}(e^t) := \Delta_{\mathtt{V}}^{it}, \quad \gamma_{\mathtt{V}}(-1) := J_{\mathtt{V}}.$$

Show that we thus obtain a bijection between the set $\mathrm{Stand}(\mathcal{H})$ of standard subspaces of $\mathcal{H}$ and the set of antiunitary (strongly continuous) representations $\gamma\colon \mathbb{R}^\times \to \mathrm{AU}(\mathcal{H})$.

Exercise 1.4. Let $\mathcal{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra. For two unit vector $\Omega_1, \Omega_2 \in \mathcal{H}$, the states ω_{Ω_1} and ω_{Ω_2} coincide if and only if there exists an $\mathcal{M}$-equivariant isometry

$$U\colon \overline{\mathcal{M}\Omega_1} \to \overline{\mathcal{M}\Omega_2} \quad \text{with} \quad U\Omega_1 = \Omega_2.$$

Conclude further that, if $\mathcal{M} \neq B(\mathcal{H})$, then there exist linearly independent unit vectors Ω_1 and Ω_2, defining the same state on $\mathcal{M}$. Hint: $\mathcal{M} \neq B(\mathcal{H})$ is equivalent to $\mathcal{M}'$ being non-trivial.

Exercise 1.5. (The Brunetti–Guido–Longo (BGL) construction) Let G be a Lie group, $\sigma \in \mathrm{Aut}(G)$ be an involution and $G_\sigma := G \rtimes \{\mathbf{1}, \sigma\}$ the corresponding semidirect product. We consider an antiunitary representation $U\colon G_\sigma \to \mathrm{AU}(\mathcal{H})$, i.e., $U(G) \subseteq \mathrm{U}(\mathcal{H})$ and $U(\sigma)$ antilinear.

We consider the set

$$\mathcal{G}(G_\sigma) := \{(x,\tau) \in \mathfrak{g} \times G\sigma\colon\ \mathrm{Ad}(\tau)x = x, \tau^2 = e\}.$$

Show that:

(a) Each (x,τ) defines a morphism

$$\gamma\colon \mathbb{R}^\times \to G_\sigma, \quad \gamma(e^t) := \exp(tx), \quad \gamma(-1) := \tau.$$

(b) For each pair (x,τ) there exists a unique standard subspace $\mathtt{V} \subseteq \mathcal{H}$ with

$$J_{\mathtt{V}} = U(\tau) \quad \text{and} \quad \Delta_{\mathtt{V}} = e^{2\pi i\cdot \partial U(x)}.$$

Exercise 1.6. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert space, X be a set and $\gamma_j\colon X \to \mathcal{H}_j$, $j = 1,2$, be maps with total range. Then the following are equivalent:

(a) There exists a unitary operator $U\colon \mathcal{H}_1 \to \mathcal{H}_2$ with $U \circ \gamma_1 = \gamma_2$.

(b) $\langle \gamma_2(x), \gamma_2(y)\rangle = \langle \gamma_1(x), \gamma_1(y)\rangle$ for all $x, y \in X$.

Exercise 1.7. Let V and W be $\mathbb{K}$-vector spaces, $\beta\colon V^n \to W$ be a symmetric n-linear map and $\gamma(v) := \beta(v, \cdots, v)$. Show that β is completely determined by the values on the diagonal $\beta(v, \dots, v)$, $v \in V$.
Hint: Consider

$$\gamma(t_1 v_1 + \ldots + t_n v_n) = \sum_{m_1+\ldots+m_n=n} \frac{n!}{m_1! \cdots m_n!} t_1^{m_1} \cdots t_n^{m_n} \beta(v_1^{m_1}, \dots, v_n^{m_n})$$

and recover $\beta(v_1, \dots, v_n)$ as a suitable partial derivative. Alternatively, one can verify the following explicit formula:

$$\beta(v_1, \dots, v_n) = \frac{1}{n!\, 2^n} \sum_{\varepsilon_1,\dots,\varepsilon_n \in \{1,-1\}} \varepsilon_1 \cdots \varepsilon_n \, \gamma(\varepsilon_1 v_1 + \cdots + \varepsilon_n v_n). \tag{1.16}$$

Exercise 1.8. Let V be $\mathbb{K}$-vector space and $S^n(V) := (V^{\otimes n})^{S_n}$ be the nth symmetric power of V. Show that

$$S^n(V) = \mathrm{span}\{v^{\otimes n} \colon v \in V\}.$$

Hint: Use the same technique as in Exercise 1.7.

Exercise 1.9. (Abstract uncertainty principle) Let A and B be bounded selfadjoint operator on $\mathcal{H}$ and $\Omega \in \mathcal{H}$. Then Ω defines a state whose expectation values for the observable A is given by

$$c_A := \omega_\Omega(A) = \langle \Omega, A\Omega \rangle.$$

The variance of the observable A in the state ω_Ω is given by the expectation value

$$\sigma_A := \omega_\Omega((A - c_A \mathbf{1})^2)^{1/2} = \|(A - c_A\mathbf{1})\Omega\|.$$

It vanishes if and only if $A\Omega = c_A\Omega$, i.e., if Ω is an eigenvector of A.

Verify the abstract uncertainty principle:

$$\sigma_A \sigma_B \geq \frac{1}{2} |\langle \Omega, [A, B]\Omega \rangle|. \tag{1.17}$$

Exercise 1.10. Let $A\colon \mathcal{D}(A) \to \mathcal{H}$ and $B\colon \mathcal{D}(B) \to \mathcal{H}$ be densely defined unbounded operators on the real Hilbert space $\mathcal{H}$, so that their adjoints

$$A^*\colon \mathcal{D}(A^*) \to \mathcal{H}, \quad B^*\colon \mathcal{D}(B^*) \to \mathcal{H}$$

are also defined by

$$\langle A^* v, w \rangle = \langle v, Aw \rangle \quad \text{for} \quad w \in \mathcal{D}(A), v \in \mathcal{D}(A^*).$$

The product is defined on $\mathcal{D}(AB) = B^{-1}\mathcal{D}(A)$ by composition. Show that:

(a) If $\mathcal{D}(AB)$ is dense, then $(AB)^*$ is an extension of B^*A^*.

(b) If A is invertible, then $(AB)^* = B^*A^*$.

Exercise 1.11. Let $\mathtt{V} \subset \mathcal{H}$ be a standard subspace and $U \in \mathrm{AU}(\mathcal{H})$ be a unitary or an antiunitary operator. Show that $U\mathtt{V}$ is also standard and $U\Delta_{\mathtt{V}}U^{-1} = \Delta_{U\mathtt{V}}$ and $UJ_{\mathtt{V}}U^{-1} = J_{U\mathtt{V}}$.

2 Euler elements and causal homogeneous spaces

We have seen in Section 1 how nets of real subspace arise from nets of algebras of local observables. Eventually, one would like to “classify” all these nets in a suitable sense, but first one has to specify which structures we are dealing with. Key points are

(Q1) Which elements $h \in \mathfrak{g}$ can arise in the Bisognano–Wichmann (BW) condition?

(Q2) Which G-invariant structure do we need on M as a fertile ground for nets of real subspaces?

(Q3) How to find the domains $W \subseteq M$, arising in the (BW) condition?

As we shall see below, these questions are highly intertwined, in particular when we also discuss which unitary representations $(U, \mathcal{H})$ of G admit nets satisfying the axioms (Iso), (Cov), (RS) and (BW).

2.1 The Euler Element Theorem

Definition 2.1. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. We call $h \in \mathfrak{g}$ an *Euler element* if $\operatorname{ad} h$ is **non-zero** and diagonalizable with $\operatorname{Spec}(\operatorname{ad} h) \subseteq \{-1, 0, 1\}$, i.e., if

$$\mathfrak{g} = \mathfrak{g}_1(h) \oplus \mathfrak{g}_0(h) \oplus \mathfrak{g}_{-1}(h).$$

Then $\tau_h := e^{\pi i \operatorname{ad} h} \in \operatorname{Aut}(\mathfrak{g})$ is an involution of $\mathfrak{g}$. We write $\mathcal{E}(\mathfrak{g})$ for the set of Euler elements in $\mathfrak{g}$. An Euler element h is called *symmetric* if $-h \in \mathcal{O}_h := \operatorname{Inn}(\mathfrak{g})h$.

Remark 2.2. We observe that $\mathcal{E}(\mathfrak{g}) + \mathfrak{z}(\mathfrak{g}) = \mathcal{E}(\mathfrak{g})$.

The following theorem provides a very satisfying answer to question (Q1).

Theorem 2.3. (Euler Element Theorem; [MN24]) *Let G be a connected finite-dimensional Lie group with Lie algebra $\mathfrak{g}$ and $h \in \mathfrak{g}$. Let $(U, \mathcal{H})$ be a unitary representation of G with discrete kernel. Suppose that $\mathtt{V} \subseteq \mathcal{H}$ is a standard subspace and $N \subseteq G$ an identity neighborhood such that*

(a) $U(\exp(th)) = \Delta_{\mathtt{V}}^{-it/2\pi}$ *for $t \in \mathbb{R}$, i.e., $\Delta_{\mathtt{V}} = e^{2\pi i\, \partial U(h)}$, and*

(b) $\mathtt{V}_N := \bigcap_{g \in N} U(g)\mathtt{V}$ *is cyclic.*

Then h is an Euler element or central, and the conjugation $J_{\mathtt{V}}$ satisfies

$$J_{\mathtt{V}} U(\exp x) J_{\mathtt{V}} = U(\exp \tau_h(x)) \quad \text{for} \quad \tau_h = e^{\pi i \operatorname{ad} h}, x \in \mathfrak{g}. \tag{2.1}$$

Corollary 2.4. *If $\mathsf{H}(\mathcal{O})_{\mathcal{O} \subseteq M}$ is a net of real subspaces on open subsets of M satisfying* (Iso), (Cov), (RS) *and* (BW), *and U has discrete kernel, then $h \in \mathfrak{g}$ is an Euler element or central.*

Proof. Let $\mathcal{O} \subseteq W$ be a non-empty open, relatively compact subset. Then $\overline{\mathcal{O}}$ is a compact subset of the open set W, so that

$$N := \{g \in G \colon g^{-1}.\overline{\mathcal{O}} \subseteq W\}$$

is an open e-neighborhood in G. For every $g \in N$ we have by (Cov) and (Iso),

$$g^{-1}.\mathsf{H}(\mathcal{O}) = \mathsf{H}(g^{-1}.\mathcal{O}) \subseteq \mathsf{H}(W) \overset{\text{(BW)}}{=} \mathtt{V}.$$

This implies that $\mathsf{H}(\mathcal{O}) \subseteq \mathtt{V}_N$. Now (RS) implies that $\mathsf{H}(\mathcal{O})$ is cyclic, hence standard because it is contained in $\mathtt{V}$ and thus also separating. Now the assertion follows from Theorem 2.3. $\square$

Remark 2.5. (a) The relation (2.1) implies that for the representations we are dealing with, we may replace G by its simply connected covering group $\widetilde{G}$ or by the quotient group $G/\ker(U)$ to ensure that the involution $\tau_h^{\mathfrak g} = e^{\pi i \operatorname{ad} h}$ on $\mathfrak g$ integrates to an involution τ_h on G, so that we can form the semidirect product

$$G_{\tau_h} = G \rtimes \{\mathrm{id}_G, \tau_h\}.$$

Then (2.1) ensures that U extends to an antiunitary representation of G_{τ_h} by $U(\tau_h) := J$.
(b) If $\mathtt{V}_N = \mathtt{V}$ holds in the Euler Element Theorem, then $U(g)\mathtt{V} \supseteq \mathtt{V}$ for all $g \in N$, hence $U(g)\mathtt{V} = \mathtt{V}$ for all $g \in N \cap N^{-1}$. If G is connected, this implies that $U(G)$ fixes $\mathtt{V}$ and hence that h is central in $\mathfrak g$.

Problem 2.6. In view of the preceding discussion, the following question is fundamental: Suppose that $h \in \mathfrak g$ is an Euler element, G is a corresponding connected Lie group, for which G_{τ_h} exists, and $M = G/H$ a homogeneous space. When does there exist an antiunitary representation $(U, \mathcal H)$ of G_{τ_h}, a connected open subset $W \subseteq M$ and a net $\mathsf{H}(\mathcal O)_{\mathcal O \subseteq M}$ on open subsets of M, satisfying (Iso), (Cov), (RS) and (BW)?

Below we shall see that this is always the case if G is reductive and M is the non-compactly causal symmetric space associated to G and h (cf. Theorem 3.17). If G is solvable, the corresponding question is open (cf. [BN25]).

2.2 First examples of Euler elements

Before we descend deeper into structures related to Euler elements, let us discuss some key examples.

Example 2.7. If E is a finite-dimensional vector space and $D \in \mathrm{End}(E)$ a diagonal endomorphism with eigenvalues contained in $\{1, 0, -1\}$, then we form the solvable Lie algebra $\mathfrak g := E \rtimes_D \mathbb R$. Here $h := (0, 1)$ is an Euler element of $\mathfrak g$.

Example 2.8. (a) In $\mathfrak g = \mathfrak{sl}_2(\mathbb R)$ the diagonal matrix

$$h = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.2}$$

is an Euler element. Conversely, every Euler element $h' \in \mathfrak{sl}_2(\mathbb R)$ must be diagonalizable on $\mathbb R^2$ (Exercise 2.1) and the difference between its eigenvalues must be 1. In view of $\mathrm{tr}(h') = 0$, it is conjugate to h. The set of Euler elements in $\mathfrak{sl}_2(\mathbb R)$ is

$$\mathcal E(\mathfrak{sl}_2(\mathbb R)) = \big\{x \in \mathfrak{sl}_2(\mathbb R) \colon \det(x) = -\tfrac14\big\} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a^2 + bc = \frac14 \right\}$$

and $\mathrm{Inn}(\mathfrak g) \cong \mathrm{SO}_{1,2}(\mathbb R)_e$ acts transitively on this set. In the following, we shall also use the Euler element

$$k = \frac12 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.3}$$

The element

$$z_{\mathfrak k} := \frac12 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = [h, k] \quad \text{satisfies} \quad [z_{\mathfrak k}, h] = -k,$$

so that we have

$$e^{-\frac{\pi}{2} \operatorname{ad} z_{\mathfrak k}} h = -[z_{\mathfrak k}, h] = k \quad \text{and} \quad e^{\pm\pi \operatorname{ad} z_{\mathfrak k}} h = -h. \tag{2.4}$$

(b) If $\mathcal{A}$ is a real unital associative algebra, then $h = \frac{1}{2}\operatorname{diag}(1,-1)$ is also Euler in the Lie algebra $\mathfrak{gl}_2(\mathcal{A})$. If $\sigma \in \operatorname{Aut}(\mathcal{A})$ is an involutive automorphism, then σ extends to a Lie algebra automorphism of $\mathfrak{gl}_2(\mathcal{A})$ and $\mathfrak{g} = \mathfrak{gl}_2(\mathcal{A})^\sigma$ contains the Euler element h with $\mathfrak{g}_1(h) \cong \mathcal{A}^\sigma$. For the involution $\tau := \sigma\tau_h$, we also find a Lie algebra with $\mathfrak{g}_1(h) \cong \mathcal{A}^{-\sigma}$.

This provides a rich supply of Lie algebras with Euler elements. This construction even works for Jordan algebras $\mathcal{A}$, hence in particular also for alternative algebras. We refer to [KSTT19], [dG17] and [Be24] for recent classification results in small dimensions.

Examples 2.9. (a) In the simple Lie algebra $\mathfrak{g} := \mathfrak{sl}_n(\mathbb{R})$, we write $n \times n$-matrices as block 2×2-matrices according to the partition $n = k + (n-k)$. Then

$$h_k := \frac{1}{n}\begin{pmatrix} (n-k)\mathbf{1}_k & 0 \\ 0 & -k\mathbf{1}_{n-k} \end{pmatrix} \tag{2.5}$$

is diagonalizable with the two eigenvalues $\frac{n-k}{n} = 1 - \frac{k}{n}$ and $-\frac{k}{n}$. Therefore h_k is an Euler element (Exercise 2.1) whose 3-grading is given by

$$\mathfrak{g}_0(h) = \Big\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a \in \mathfrak{gl}_k(\mathbb{R}), d \in \mathfrak{gl}_{n-k}(\mathbb{R}), \operatorname{tr}(a) + \operatorname{tr}(d) = 0 \Big\},$$
$$\mathfrak{g}_1(h) = \begin{pmatrix} 0 & M_{k,n-k}(\mathbb{R}) \\ 0 & 0 \end{pmatrix}, \quad \mathfrak{g}_{-1}(h) \cong \begin{pmatrix} 0 & 0 \\ M_{n-k,k}(\mathbb{R}) & 0 \end{pmatrix}.$$

It is easy to see that $h_1, \ldots, h_{n-1}$ represent the conjugacy class of Euler elements in $\mathfrak{sl}_n(\mathbb{R})$, whose restricted root system is of type A_{n-1}. This matches the general Classification Theorem 2.45 below.

The Euler element h_k is symmetric, i.e., $-h_k \in \operatorname{Inn}(\mathfrak{g})h_k$, if and only if $n = 2k$. In fact, if h_k is symmetric, then its eigenvalues have to be symmetric, which is equivalent to $n = 2k$. That this condition is sufficient follows by embedding h_k into an $\mathfrak{sl}_2(\mathbb{R})$-subalgebra of block matrices with entries in $M_k(\mathbb{R})$ and using Example 2.8.
(b) In the reductive Lie algebra $\mathfrak{gl}_n(\mathbb{R})$, we infer from (a) that all conjugacy classes of Euler elements are represented by elements of the form

$$h = \lambda\mathbf{1} + h_k, \quad k = 1, \ldots, n-1.$$

They are symmetric if and only if $\lambda = 0$ and $n = 2k$.

These elements are also Euler in the semidirect sum $\mathfrak{g} := \mathbb{R}^n \rtimes \mathfrak{gl}_n(\mathbb{R})$ if and only if $\lambda = \frac{k}{n}$ or $\lambda = \frac{k}{n} - 1$, which leads to

$$h' = \begin{pmatrix} \mathbf{1}_k & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad h'' = \begin{pmatrix} 0 & 0 \\ 0 & -\mathbf{1}_{n-k} \end{pmatrix}.$$

In the first case,

$$\mathfrak{g}_1 \cong \mathbb{R}^k \oplus M_{k,n-k}(\mathbb{R}), \quad \mathfrak{g}_0 = \mathfrak{gl}_k(\mathbb{R}) \oplus (\mathbb{R}^{n-k} \rtimes \mathfrak{gl}_{n-k}(\mathbb{R})) \quad \text{and} \quad \mathfrak{g}_{-1} = M_{n-k,k}(\mathbb{R}),$$

whereas in the second case

$$\mathfrak{g}_1 \cong M_{k,n-k}(\mathbb{R}), \quad \mathfrak{g}_0 = (\mathbb{R}^k \rtimes \mathfrak{gl}_k(\mathbb{R})) \oplus \mathfrak{gl}_{n-k}(\mathbb{R}) \quad \text{and} \quad \mathfrak{g}_{-1} = \mathbb{R}^{n-k} \oplus M_{n-k,k}(\mathbb{R}).$$

Clearly, none of these Euler elements is symmetric.

Examples 2.10. (a) In the Poincaré Lie algebra $\mathfrak{g} = \mathbb{R}^{1,d} \rtimes \mathfrak{so}_{1,d}(\mathbb{R})$, every Euler element h is conjugate to the generator $h \in \mathfrak{so}_{1,d}(\mathbb{R})$ of a Lorentz boost:

$$h.\mathbf{e}_0 = \mathbf{e}_1, \quad h.\mathbf{e}_1 = \mathbf{e}_0 \quad \text{and} \quad h.\mathbf{e}_j = 0 \quad \text{for} \quad j > 1$$

(Lemma 2.49 and $\mathfrak{z}(\mathfrak{g}) = \{0\}$; see also Remark 1.16).
(b) The split oscillator group is

$$G := \mathrm{Heis}(\mathbb{R}^2) \rtimes_\alpha \mathbb{R} \quad \text{with} \quad \alpha_t = e^{th}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

so that

$$\mathfrak{g} = \mathfrak{heis}(\mathbb{R}^2) \rtimes \mathbb{R}h$$

and h is an Euler element in $\mathfrak{g}$. We choose a basis $p, q, z \in \mathfrak{heis}(\mathbb{R}^2)$ with

$$[q,p] = z, \quad [h,q] = 1, \quad [h,p] = -1, \quad [h,z] = 0.$$

The corresponding involution is given by

$$\tau_h(z,q,p,t) = (z,-q,-p,t).$$

The Euler element h is not symmetric, and all Euler elements of $\mathfrak{g}$ are, up to sign, conjugate to elements of the form

$$h_\lambda = \lambda z + h.$$

This Lie algebra can be realized as a subalgebra of $\mathfrak{sl}_3(\mathbb{R})$, where

$$h = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad p = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that h is an Euler element of $\mathfrak{sl}_3(\mathbb{R})$, i.e., $V := \mathbb{R}^3$ is a 2-graded $\mathfrak{g}$-module (cf. Example 2.9(a)):

$$V = V_{1/3} \oplus V_{-2/3}, \quad V_{1/3} = \mathbb{R}\mathbf{e}_1 + \mathbb{R}\mathbf{e}_3, \quad V_{-2/3} = \mathbb{R}\mathbf{e}_2.$$

Remark 2.11. (a) If V is a non-trivial irreducible $\mathfrak{sl}_2(\mathbb{R})$-module and $h \in \mathfrak{g} := V \rtimes \mathfrak{sl}_2(\mathbb{R})$ an Euler element, then the semisimple element h is conjugate to an element of $\mathfrak{sl}_2(\mathbb{R})$ (Lemma 2.49 and $\mathfrak{z}(\mathfrak{g}) = \{0\}$), so that we may assume that $h = \frac{1}{2}\operatorname{diag}(1,-1)$ (Example 2.8(a)). This leaves only the possibility that $\dim V = 3$ is the adjoint module.

We obtain more freedom if we replace $\mathfrak{sl}_2(\mathbb{R})$ by $\mathfrak{gl}_2(\mathbb{R})$. Then Example 2.9(b) also provides Euler elements in $\mathbb{R}^2 \rtimes \mathfrak{gl}_2(\mathbb{R})$. We may also consider non-trivial 1-dimensional representations of $\mathfrak{gl}_2(\mathbb{R})$, for which

$$\mathbb{R} \rtimes \mathfrak{gl}_2(\mathbb{R}) \cong (\mathbb{R} \rtimes \mathbb{R}\mathbf{1}) \oplus \mathfrak{sl}_2(\mathbb{R}).$$

This example shows already how restrictive the existence of a 3-grading is for semidirect sums.
(b) If $h \in \mathfrak{g}$ is an Euler element contained in a subalgebra $\mathfrak{s} \cong \mathfrak{sl}_2(\mathbb{R})$, then all simple $\mathfrak{s}$-submodules of $\mathfrak{g}$ must be 1 or 3-dimensional. If h is contained in a subalgebra $\mathfrak{l} \cong \mathfrak{gl}_2(\mathbb{R})$, then also 2-dimensional irreducible submodules may occur (cf. Lemma 2.52 below).

2.3 Causal structures and wedge regions

The Euler Element Theorem 2.3 provides us with the information that Euler elements are the natural candidates for the elements h in (BW), but it provides no information on how to find appropriate regions $W \subseteq M$?

Motivated by the Bisognano–Wichmann property (BW) in AQFT, the modular flow on $W \subseteq M$, given by $\alpha_t^W(m) = \exp(th).m$ should, in a suitable sense, correspond to the "flow of time" on the spacetime region W. This is mainly based on the interpretation of the modular group in the context of the Tomita–Takesaki Theorem as the dynamics of the corresponding quantum system, the **thermal time hypothesis**, a point of view advocated by A. Connes and C. Rovelli (cf. [CR94]). References for the AQFT perspective on this issue are [BB99, BMS01, Bo09], [CLRR22, §3]. For a perspective from non-commutative geometry, see [KG09], [Kot19] and [He25].

To formulate what it means that a vector field generates on an open domain $W \subseteq M$ a flow that qualifies as a "flow of time" requires a *causal structure on the manifold* M, i.e., in each tangent space $T_m(M)$, we specify a pointed, generating closed convex cone $C_m \subseteq T_m(M)$. [3] We think of elements in the interior C_m° as *timelike*, i.e., tangent vectors to curves describing the dynamics on a region in M (following the "flow of time").

Assumption: For simplicity, we also assume that M is a homogeneous space $M = G/H$, for a closed subgroup $H \subseteq G$ with Lie algebra $\mathfrak{h}$. Then the tangent space $T_{eH}(M)$ in the base point identifies naturally with the quotient space $\mathfrak{q} := \mathfrak{g}/\mathfrak{h}$. Hence the existence of a G-invariant causal structure on M is equivalent to the existence of an $\mathrm{Ad}_{\mathfrak{q}}(H)$-invariant pointed generating cone $C_{\mathfrak{q}} \subseteq \mathfrak{q}$ (cf. [HÓ97]). Then

$$C_{gH} := g.C_{eH} = g.C_{\mathfrak{q}} \quad \text{for} \quad g \in G,$$

is the corresponding causal structure on $M = G/H$. Here we write $G \times TM \to TM, (g, v) \mapsto g.v$ for the induced action of G on the tangent bundle TM.

Coming back to the question of how to find W, let us fix an Euler element $h \in \mathfrak{g}$. Then we call

$$X_h^M(m) := \frac{d}{dt}\Big|_{t=0} \exp(th).m \tag{2.6}$$

the corresponding *modular vector field*. In view of the "flow of time"-philosophy, W should be contained in the *positivity region*

$$W_M^+(h) := \{m \in M \colon X_h^M(m) \in C_m^\circ\}, \tag{2.7}$$

which is the largest open subset on which the flow is "future-directed". For $m = gH \in M = G/H$ and the projection $p_{\mathfrak{q}} \colon \mathfrak{g} \to \mathfrak{q} = \mathfrak{g}/\mathfrak{h} \cong T_{eH}(M)$, we have

$$X_h^M(gH) = \frac{d}{dt}\Big|_{t=0} \exp(th).gH = \frac{d}{dt}\Big|_{t=0} gg^{-1}\exp(th).gH = g.p_{\mathfrak{q}}(\mathrm{Ad}(g)^{-1}h). \tag{2.8}$$

By G-invariance of the causal structure, this calculation shows that $X_h^M(gH) \in C_{gH}^\circ$ is equivalent to $p_{\mathfrak{q}}(\mathrm{Ad}(g)^{-1}h) \in C^\circ$, so that we obtain the Lie algebraic description

$$W_M^+(h) = \{gH \in G/H \colon \mathrm{Ad}(g)^{-1}h \in p_{\mathfrak{q}}^{-1}(C^\circ)\}. \tag{2.9}$$

[3] A closed convex cone C in a finite-dimensional vector space V is called *pointed* if $C \cap -C = \{0\}$, and *generating* if $C - C = V$, i.e., if C has interior points.

Definition 2.12. A *wedge region* for h on the causal homogeneous space M is a connected component W of the positivity region $W_M^+(h)$.

At this point it is not clear why to focus on connected components and not the whole positivity region. As the concrete examples where $W_M^+(h)$ is not connected shows, the inclusions $\mathsf{H}(W) \subseteq \mathsf{H}(W_M^+(h))$ are often proper and $\mathsf{H}(W) = \mathtt{V}$, so that $\mathsf{H}(W_M^+(h))$ can not be separating by Proposition 1.20. Therefore the connected components turn out to be the better choice for wedge regions. In this context, Theorem 4.29 is also of some interest, it shows that small open $\exp(\mathbb{R}h)$-invariant subsets may already satisfy (BW).

Example 2.13. In Minkowski space $M = \mathbb{R}^{1,d-1}$ (Remark 1.16), the causal structure is given by the constant cone field $C_x = C$ for $x \in M$ and

$$C = \overline{\mathrm{V}_+} = \{x \in \mathbb{R}^{1,d-1} \colon x_0 \geq 0, \beta(x,x) = x_0^2 - \mathbf{x}^2 \geq 0\}.$$

Note that M is a homogeneous space of the Poincaré group $G = \mathbb{R}^{1,d-1} \rtimes \mathrm{SO}_{1,d-1}(\mathbb{R})_e$ with base point 0, whose stabilizer is the Lorentz group $\mathrm{SO}_{1,d-1}(\mathbb{R})_e$.

For the Lorentz boost $h(x) = (x_1, x_0, 0, \cdots, 0)$, the corresponding vector field is linear, i.e.,

$$X_h^M(x) = h(x),$$

and these vectors are positive timelike, i.e., contained in $C^\circ = \mathrm{V}_+$ if and only if $x_1 > |x_0|$, which specifies the Rindler wedge W_R.

Lemma 2.14. *Any wedge region $W \subseteq W_M^+(h)$ is invariant under the identity component G_e^h of the centralizer*

$$G^h := \{g \in G \colon \mathrm{Ad}(g)h = h\}$$

of the Euler element h, hence in particular under $\exp(\mathbb{R}h)$.

The following proposition provides a sufficient criterion for the positivity region on M being non-empty. Note that the condition $h \in \mathfrak{h}$ is equivalent to the base point being fixed under the modular flow.

Proposition 2.15. (Sufficient conditions for the existence of wedge regions) *Suppose that $M = G/H$, that $h \in \mathfrak{h}$ is an Euler element and that $\tau_h \in \mathrm{Aut}(G)$ fixes H and induces an anti-causal map, i.e., $\tau_h^M(C_m) = -C_{\tau_h^M(m)}$ for $m \in M$. Then $W_M^+(h) \neq \emptyset$.*

Proof. For the action of the one-parameter group $e^{\mathbb{R}\,\mathrm{ad}\,h}$ on $\mathfrak{q} := \mathfrak{g}/\mathfrak{h}$, we write $\mathfrak{q}_j$, $j = 1, 0, -1$, for the corresponding eigenspace and [4]

$$C_\pm := \pm C \cap \mathfrak{q}_{\pm 1}.$$

In view of (2.9), it suffices to show that, for $x_{\pm 1} \in C_\pm^\circ$, there exists $t > 0$ such that

$$g_t := \exp(tx_{-1})\exp(tx_1)$$

satisfies $\mathrm{Ad}(g_t)^{-1}h \in p_{\mathfrak{q}}^{-1}(C^\circ)$. Note that $-\tau_h.C = C$ implies that

$$C_+^\circ - C_-^\circ = (C_+ - C_-)^\circ \subseteq C^\circ$$

(cf. Lemma 2.17 below).

[4] For the linear vector field defined by h on $\mathfrak{q}$, the positivity region is $W_{\mathfrak{q}}^+(h) = C_+^\circ + \mathfrak{q}_0 + C_-^\circ$ (cf. (2.11)). This is why we consider these two cones.

For $t > 0$ we then have $e^{-t \operatorname{ad} x_{-1}} h = h - t[x_{-1}, h] = h - tx_{-1}$ because $(\operatorname{ad} x_{-1})^2 h \in \mathfrak{g}_{-2}(h) = \{0\}$. We thus obtain

$$\begin{aligned}\operatorname{Ad}(g_t)^{-1} h &= e^{-t \operatorname{ad} x_1} e^{-t \operatorname{ad} x_{-1}} h = e^{-t \operatorname{ad} x_1}(h - tx_{-1}) = h + tx_1 - te^{-t \operatorname{ad} x_1} x_{-1} \\ &= h + t(x_1 - x_{-1}) - t(e^{-t \operatorname{ad} x_1} - \mathbf{1})x_{-1}.\end{aligned}$$

As $p_{\mathfrak{q}}(h) = 0$, this element is contained in $p_{\mathfrak{q}}^{-1}(C^\circ)$ if and only if this is the case for

$$x_1 - x_{-1} - (e^{t \operatorname{ad} x_1} - \mathbf{1})x_{-1}.$$

For $t \to 0$, this expression tends to $x_1 - x_{-1} \in C^\circ$, so that for some $t > 0$, we have $g_t H \in W_M^+(h)$. $\square$

Remark 2.16. For a homogeneous space $M = G/H$, the positivity region $W_M^+(h)$ is non-empty if there exists an open subset $\mathcal{O} \subseteq G$ such that $p_{\mathfrak{q}}(\operatorname{Ad}(H\mathcal{O})h) \subseteq \mathfrak{q}$ is contained in a pointed open convex cone. This depends very much on the geometry of the adjoint orbit $\mathcal{O}_h$, the H-action on this orbit and its position with respect to $\mathfrak{h} = \ker p_{\mathfrak{q}}$.

To understand how wedge regions look like, we first discuss some simple classes of examples.

2.3.1 One-parameter groups on affine causal spaces

To develop the key facts on modular flows on causal homogeneous spaces, we start in this subsection with the case of causal affine spaces, i.e., pairs (E, C), where E is a finite-dimensional vector space and $C \subseteq E$ a pointed generating closed convex cone.

Specifically, we consider the following data (cf. [NÓØ21]):

(A1) E is a finite-dimensional real vector space.

(A2) $h \in \operatorname{End}(E)$ is diagonalizable with eigenvalues $\{-1, 0, 1\}$ and $\tau_h := e^{\pi i h}$.

(A3) $C \subseteq E$ is a pointed, generating closed convex cone invariant under $e^{\mathbb{R}h}$ and $-\tau_h$.

Writing $E_\lambda = E_\lambda(h) := \ker(h - \lambda\mathbf{1})$ for the h-eigenspaces and $E^\pm := \ker(\tau_h \mp \mathbf{1})$ for the τ_h-eigenspaces, (A2) implies

$$E = E_1 \oplus E_0 \oplus E_{-1}, \quad E^- = E_1 \oplus E_{-1}, \quad \text{and} \quad E^+ = E_0. \tag{2.10}$$

We put $C_\pm := C \cap E_{\pm 1}$. For $x \in E$, we write $x = x_1 + x_0 + x_{-1}$ for the decomposition into h-eigenvectors.

Lemma 2.17. *For the projections*

$$p_{\pm 1} : E \to E_{\pm 1}, x \mapsto x_{\pm 1}, \quad \textit{and} \quad p^- : E \to E_1 \oplus E_{-1} = E^-, x \mapsto x_1 + x_{-1} = \frac{1}{2}(x - \tau_h x),$$

the following assertions hold:

(i) $p_{\pm 1}(C) = \pm C_\pm$ *and* $p_{\pm 1}(C^\circ) = \pm C_\pm^\circ \neq \emptyset$.

(ii) $p^-(C) = C \cap E^- = C_+ \oplus -C_-$ *and* $p^-(C^\circ) = C^\circ \cap E^- = C_+^\circ \oplus -C_-^\circ$.

(iii) $C \subseteq C_+ \oplus E_0 \oplus -C_-$.

Proof. (i) As $\pm C_\pm \subset C$, we have $\pm C_\pm \subset p_{\pm 1}(C)$. Using the e^{th}-invariance of C and writing $x = x_1 + x_0 + x_{-1}$ as before, $e^{th}x = e^t x_1 + x_0 + e^{-t}x_{-1}$. Now take the limit $t \to \infty$ to see that

$$C \ni e^{-t}e^{th}x = x_1 + e^{-t}x_0 + e^{-2t}x_{-1} \to x_1 \quad \text{as} \quad t \to \infty.$$

We likewise get $x_{-1} = \lim_{t\to-\infty} e^t e^{th}x \in C$. It follows that $x_\pm \in \pm C_\pm$, so that $p_{\pm 1}(C) = \pm C_\pm$. As $p_{\pm 1}$ are projections and $C^\circ \neq \emptyset$, it follows that $p_{\pm 1}(C^\circ) \subseteq \pm C_\pm^\circ$. To obtain equality, it suffices to observe that $C_+^\circ \oplus -C_-^\circ \subseteq (E^- \cap C)^\circ \subseteq C^\circ$ follows from $-\tau_h(C) = C$.
(ii) The two leftmost equalities follow from $-\tau_h(C) = C$, and the second two rightmost equalities from (i) and $p^- = p_1 + p_{-1}$.
(iii) follows from (ii). □

As the linear vector field on E corresponding to h is given by $X_h^E(x) = x_1 - x_{-1}$, Lemma 2.17(ii) implies that its positivity domain is the wedge region

$$W_E^+(h) = C_+^\circ \oplus E_0 \oplus C_-^\circ \quad \text{for} \quad C_\pm = \pm C \cap E_{\pm 1}. \tag{2.11}$$

In particular, it is not empty. Here (A3) is important to ensure that C° intersects $E^- = \mathrm{im}(h)$. Otherwise we would include cones of the form $C = C_1 + C_0 + C_{-1}$ with $C_j \subseteq E_j$. Any such cone is invariant under $e^{\mathbb{R}h}$, but for such cones $C^\circ \cap (E_{+1} + E_{-1}) = \emptyset$ implies that $W_E^+(h) = \emptyset$.

Example 2.18. (The affine group on $\mathbb{R}$) We endow $M = \mathbb{R}$ with the canonical causal structure given by $C_x = \mathbb{R}_{\geq 0}$ for $x \in \mathbb{R}$. Then the connected *affine group* $G = \mathrm{Aff}(\mathbb{R})_e = \mathbb{R} \rtimes \mathbb{R}_+$ is 2-dimensional. Its elements are denoted (b, a), and they act by the affine, orientation preserving maps $(b, a)x = ax + b$ on the real line.

Here $h = (0, 1) \in \mathfrak{g}$ is an Euler element whose flow is given by $\alpha_t(x) = e^t x$. Therefore its positivity region is

$$W_{\mathbb{R}}^+(h) = \{x \in \mathbb{R} : x > 0\} = \mathbb{R}_+$$

and the corresponding reflection is $\tau_h(x) = -x$.

All other Euler elements are of the form $h' = (x, \pm 1)$, where $\mathcal{O}_h = \mathbb{R} \times \{1\}$ and $\mathcal{O}_{-h} = \mathbb{R} \times \{-1\}$. The corresponding positivity regions are the proper unbounded open intervals in $\mathbb{R}$.

2.3.2 More examples of wedge regions

The first example refers also to an affine causal space, but now the linear part of the automorphism group is larger.

Example 2.19. (Poincaré group and Rindler wedges) The example arising most prominently in physics is the connected *Poincaré group*

$$G := \mathcal{P}_+^\uparrow := \mathbb{R}^{1,d-1} \rtimes \mathrm{SO}_{1,d-1}(\mathbb{R})_e.$$

It acts on d-dimensional Minkowski space $\mathbb{R}^{1,d-1}$ as an isometry group of the Lorentzian metric given by $(x, y) = x_0 y_0 - \mathbf{x}\mathbf{y}$ for $x = (x_0, \mathbf{x}) \in \mathbb{R}^{1,d-1}$. The G-action preserves the constant cone field defined by the closure $C = \overline{\mathrm{V}_+}$ of the open future light cone

$$\mathrm{V}_+ = \{(x_0, \mathbf{x}) \in \mathbb{R}^{1,d-1} : x_0 > 0, x_0^2 > \mathbf{x}^2\}.$$

The generator $h \in \mathfrak{so}_{1,d-1}(\mathbb{R})$ of the Lorentz boost on the (x_0, x_1)-plane

$$h(x_0, x_1, x_2, \ldots, x_{d-1}) = (x_1, x_0, 0, \ldots, 0)$$

is an Euler element and $e^{\pi i h}$ acts by the reflection $\tau_h(x) = (-x_0, -x_1, x_2, \ldots, x_{d-1})$, for which $-\tau_h(C) = C$. In view of the Classification Theorem 2.45, the fact that the restricted root system of $\mathfrak{so}_{1,d-1}(\mathbb{R})$ is of type A_1 implies that there exists only one conjugacy class of Euler elements in $\mathfrak{so}_{1,d-1}(\mathbb{R})$. With Lemma 2.49 it follows that the same holds for the Poincaré algebra because its center is trivial. So Euler elements in this Lie algebra are precisely the Lorentz boosts in different affine coordinate systems.

By (2.11), the positivity region of h is

$$W_M^+(h) = \mathbb{R}_+(\mathbf{e}_0 + \mathbf{e}_1) - \mathbb{R}_+(\mathbf{e}_0 - \mathbf{e}_1) + \operatorname{span}\{\mathbf{e}_2, \ldots, \mathbf{e}_{d-1}\} = \{x \in \mathbb{R}^{1,d-1} : |x_0| < x_1\}.$$

It is called the *standard right wedge* or *Rindler wedge* W_R and plays a key role in AQFT as a localization region for a uniformly accelerated observer, represented by an orbit of the modular flow in W_R ([BGL02, LL15]; see also Remark 1.16).

The following example is the smallest compact one. It is a causal flag manifold. We refer to Subsection 2.6 for more on this class of examples.

Example 2.20. (The action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{S}^1 \cong \mathbb{R}_\infty$) The group $G := \mathrm{SL}_2(\mathbb{R})$ acts on the one-point compactification $M = \mathbb{R}_\infty = \mathbb{R} \cup \{\infty\} \cong \mathbb{S}^1$ by

$$g.x := \frac{ax+b}{cx+d} \quad \text{for} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

The subgroup $\mathrm{SO}_2(\mathbb{R})$ acts transitively by

$$\rho(t).z := \begin{pmatrix} \cos(t/2) & \sin(t/2) \\ -\sin(t/2) & \cos(t/2) \end{pmatrix}.x = \frac{\cos(t/2)\cdot x + \sin(t/2)}{-\sin(t/2)\cdot x + \cos(t/2)},$$

generated by the vector field $X^M(x) = \frac{1}{2}(1+x^2)$. As this flow is 2π-periodic, it induces a diffeomorphism $\mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}_\infty$. This shows that the natural causal structure on $\mathbb{R}$ extends to M in a G-invariant fashion.

In $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ we consider the Euler element $h = \frac{1}{2}\operatorname{diag}(1,-1)$ (cf. Example 2.8). The flow it generates on $\mathbb{R}_\infty$ is given by $\alpha_t(x) = e^t x$, where 0 and ∞ are fixed. Accordingly,

$$W_M^+(h) = \mathbb{R}_+ \subseteq \mathbb{R}_\infty.$$

As G acts transitively of the set $\mathcal{O}_h = \mathcal{E}(\mathfrak{sl}_2(\mathbb{R}))$ of Euler elements in $\mathfrak{sl}_2(\mathbb{R})$ (Example 2.8), their positivity regions in $\mathbb{S}^1$ are precisely the non-dense open intervals.

The Cayley transform

$$C : \mathbb{R}_\infty \to \mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\}, \quad C(x) := \frac{i-x}{i+x}, \qquad C(\infty) := -1,$$

is a homeomorphism, identifying $\mathbb{R}_\infty$ with the circle. Its inverse is

$$C^{-1} : \mathbb{S}^1 \to \mathbb{R}_\infty, \quad C^{-1}(z) = i\frac{1-z}{1+z}$$

(cf. Exercise 2.3). It maps the upper semicircle $\{z \in \mathbb{S}^1 \colon \operatorname{Im} z > 0\}$ to the positive half line $\mathbb{R}_+$. The Cayley transform intertwines the action of $\mathrm{SL}_2(\mathbb{R})$ with the action of $\mathrm{SU}_{1,1}(\mathbb{C})$ on the circle $\mathbb{S}^1 \subseteq \mathbb{C}$ by fractional linear transformations. This action preserves the causal structure on $\mathbb{S}^1$ specified by $C_z = \mathbb{R}_{\geq 0} iz \subseteq T_z(\mathbb{S}^1) = i\mathbb{R}$ for $z \in \mathbb{S}^1$.

Example 2.21. The Lie group $G := \mathrm{SL}_2(\mathbb{R})$ has three classes of causal homogeneous spaces. In Example 2.20 we have already seen its action on the 1-dimensional circle $\mathbb{S}^1$, a flag manifold of $\mathrm{SL}_2(\mathbb{R})$.

Observing that $\mathrm{Ad}(\mathrm{SL}_2(\mathbb{R})) \cong \mathrm{SO}_{1,2}(\mathbb{R})_e$ (Exercise 2.4), we obtain two other examples:

- Two-dimensional de Sitter space

$$\mathrm{dS}^2 = \{(x_0, x_1, x_2) \in \mathbb{R}^{1,2} \colon x_0^2 - x_1^2 - x_2^2 = -1\}$$

carries an $\mathrm{SO}_{1,2}(\mathbb{R})_e$-invariant causal structure with the positive cone in the base point $\mathbf{e}_1$ given by

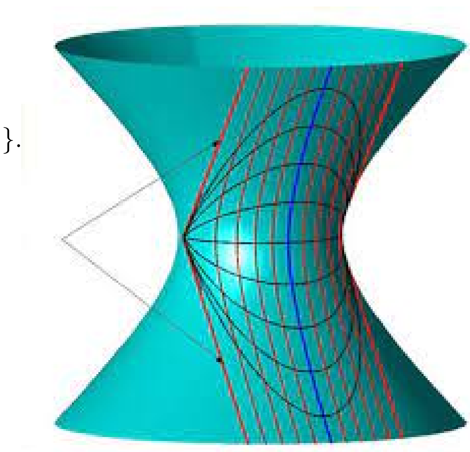
$$C_{\mathbf{e}_1} := \{(x_0, 0, x_2) \colon x_0 \geq |x_2|\} \subseteq T_{\mathbf{e}_1}(\mathrm{dS}^2) = \mathbb{R}\mathbf{e}_0 + \mathbb{R}\mathbf{e}_2.$$

The inversion $-\mathbf{1}$ on dS^2 is an anti-causal map.

For the Euler element defined by $h(x_0, x_1, x_2) = (x_1, x_0, 0)$, we obtain the connected wedge region

$$W = W^+_{\mathrm{dS}^2}(h) = \{(x_0, x_1, x_2) \in \mathrm{dS}^2 \colon x_1 > |x_0|\}.$$

The wedge region W and the orbits of the modular flow in W are marked in the picture on the right.



- Two-dimensional anti-de Sitter space

$$\mathrm{AdS}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^{2,1} \colon x_1^2 + x_2^2 - x_3^2 = 1\}$$

carries an $\mathrm{SO}_{2,1}(\mathbb{R})_e$-invariant causal structure with the positive cone in the base point $\mathbf{e}_2$ given by

$$C_{\mathbf{e}_2} := \{(x_1, 0, x_3) \colon x_3 \geq |x_1|\} \subseteq T_{\mathbf{e}_2}(\mathrm{AdS}^2) = \mathbb{R}\mathbf{e}_1 + \mathbb{R}\mathbf{e}_3.$$

The inversion $-\mathbf{1}$ on AdS^2 is a causal map. For the Euler element defined by $h(x_1, x_2, x_3) = (0, x_3, x_2)$, we obtain the positivity region

$$W^+_{\mathrm{AdS}^2}(h) = \{(x_1, x_2, x_3) \in \mathrm{AdS}^2 \colon x_1 x_3 > 0, |x_2| < |x_3|\}.$$

It has two connected components, specified by the sign of x_1 ([NÓ23a, Lemma 11.3]). Note that $|x_2| < |x_3|$ specifies the region on which h, as a vector field on AdS^2, is timelike. This region has four connected components, and $x_1 x_3 > 0$ selects the two on which it is positive. They are exchanged by the inversion $-\mathbf{1}$.

As homogeneous spaces, both can be identified with the adjoint orbit $\mathcal{O}_h \cong G/G^h \cong \mathcal{E}(\mathfrak{sl}_2(\mathbb{R}))$, where $h := \frac{1}{2}\operatorname{diag}(1, -1)$ is an Euler element in $\mathfrak{sl}_2(\mathbb{R})$ (cf. Example 2.8). However, both carry natural causal structures, and these are non-isomorphic because AdS^2 admits closed causal curves and dS^2 does not.

2.4 The compression semigroup of a wedge region

Let $M = G/H$ be a causal homogeneous space with causal structure given by the cone field $(C_m)_{m\in M}$. The set

$$C_M := \{y \in \mathfrak{g}\colon (\forall m \in M) X^M_y(m) \in C_m\} = \bigcap_{g\in G} \mathrm{Ad}(g)p_{\mathfrak{q}}^{-1}(C) \tag{2.12}$$

of those Lie algebra elements whose vector fields on M are everywhere positive (cf. (2.8)) is a closed convex $\mathrm{Ad}(G)$-invariant cone in $\mathfrak{g}$. It consists of all $y \in \mathfrak{g}$ corresponding to everywhere "positive" vector fields on M. If G acts effectively on M, then it is also pointed because elements in $C_M \cap -C_M$ correspond to vanishing vector fields on M. This cone is a geometric analog of the positive cone C_U of a unitary representation of G (see (4.8)).[5] The following observation shows that it behaves in many respects similarly (cf. [Ne22]).

As any connected component $W \subseteq W^+_M(h) \subseteq M$ is invariant under $G^h_e \supseteq \exp(\mathbb{R}h)$,[6] the same holds for the closed convex cone

$$C_W := \{y \in \mathfrak{g}\colon (\forall m \in W)\ X^M_y(m) \in C_m\} \supseteq C_M. \tag{2.13}$$

Below we show that this cone determines the tangent wedge of the compression semigroup of W.

Proposition 2.22. *For a connected component $W \subseteq W^+_M(h)$, its compression semigroup*

$$S_W := \{g \in G\colon g.W \subseteq W\}$$

is a closed subsemigroup of G with $G_W := S_W \cap S_W^{-1} \supseteq G^h_e$ and

$$\mathbf{L}(S_W) := \{x \in \mathfrak{g}\colon \exp(\mathbb{R}_+x) \subseteq S_W\} = \mathfrak{g}_0(h) + C_{W,+} + C_{W,-}, \quad \text{with} \quad C_{W,\pm} := \pm C_W \cap \mathfrak{g}_{\pm 1}(h).$$

In particular, the convex cone $\mathbf{L}(S_W)$ has interior points if C_M does.

Proof. As $W \subseteq M$ is an open subset, its complement $W^c := M \setminus W$ is closed, and thus

$$S_W = \{g \in G\colon g^{-1}.W^c \subseteq W^c\}$$

is a closed subsemigroup of G, so that its tangent wedge $\mathbf{L}(S_W)$ is a closed convex cone in $\mathfrak{g}$ ([HN93]).

Let $m = gH \in W$, so that $p_{\mathfrak{q}}(\mathrm{Ad}(g)^{-1}h) \in C^\circ$. For $x \in \mathfrak{g}_{\pm 1}(h)$ we then derive from $\mathfrak{g}_{\pm 2}(h) = \{0\}$ that

$$e^{\mathrm{ad}\,x}h = h + [x, h] = h \mp x.$$

This leads to

$$\begin{aligned} p_{\mathfrak{q}}(\mathrm{Ad}(\exp(x)g)^{-1}h) &= p_{\mathfrak{q}}(\mathrm{Ad}(g)^{-1}e^{-\mathrm{ad}\,x}h) = p_{\mathfrak{q}}(\mathrm{Ad}(g)^{-1}(h \pm x)) \\ &= p_{\mathfrak{q}}(\mathrm{Ad}(g)^{-1}h) \pm p_{\mathfrak{q}}(\mathrm{Ad}(g)^{-1}x). \end{aligned}$$

For $x \in C_{W,\pm}$, we have $p_{\mathfrak{q}}(\pm \mathrm{Ad}(g)^{-1}x) \in C$, so that $p_{\mathfrak{q}}(\mathrm{Ad}(\exp(x)g)^{-1}h) \in C^\circ$, which in turn implies that $\exp(x).m \in W$ for $m \in W$. So $\exp(C_{W,\pm}) \subseteq S_W$, and thus $C_{W,\pm} \subseteq \mathbf{L}(S_W)$. The invariance of W under the identity component G^h_e of the centralizer of h further entails $\mathfrak{g}_0(h) \subseteq \mathbf{L}(S_W)$, so that

$$C_{W,+} + \mathfrak{g}_0(h) + C_{W,-} \subseteq \mathbf{L}(S_W). \tag{2.14}$$

[5] Note that the existence of a pointed generating invariant cone in a Lie algebra $\mathfrak{g}$ has strong structural implications (cf. [Ne99]). If, f.i., $\mathfrak{g}$ is simple, then it must be hermitian.

[6] Recall that $G^h = \{g \in G\colon \mathrm{Ad}(g)h = h\}$.

We now prove the converse inclusion. Let $x \in \mathfrak{g}_1(h)$. If $X^M_x(m) \not\in C_m$, i.e., $p_\mathfrak{q}(\mathrm{Ad}(g)^{-1}x) \not\in C$, then there exists a $t_0 > 0$ with

$$p_\mathfrak{q}(\mathrm{Ad}(g)^{-1}h) + t_0 \cdot p_\mathfrak{q}(\mathrm{Ad}(g)^{-1}x) \not\in C$$

([Ne99, Prop. V.1.6]), so that $\exp(t_0 x).m \not\in W$. We conclude that

$$\mathbf{L}(S_W) \cap \mathfrak{g}_1(h) = C_{W,+}.$$

Further, the invariance of the closed convex cone $\mathbf{L}(S_W)$ under $e^{\mathbb{R}\,\mathrm{ad}\,h}$ implies that, for

$$x = x_{-1} + x_0 + x_1 \in \mathbf{L}(S_W) \quad \text{and} \quad x_j \in \mathfrak{g}_j(h),$$

we have

$$x_{\pm 1} = \lim_{t\to\infty} e^{\mp t} e^{\pm t\,\mathrm{ad}\,h} x \in \mathbf{L}(S_W) \cap \mathfrak{g}_{\pm 1}(h) = C_{W,\pm},$$

which implies the other inclusion $\mathbf{L}(S_W) \subseteq C_{W,+} + \mathfrak{g}_0(h) + C_{W,-}$, hence equality by (2.14).

Let $p_\pm \colon \mathfrak{g} \to \mathfrak{g}_{\pm 1}(h)$ denote the projection along the other eigenspaces of $\mathrm{ad}\,h$. Then

$$C_{W,\pm} \supseteq C_{M,\pm} := \pm C_M \cap \mathfrak{g}_{\pm 1}(h) = \pm p_\pm(C_M)$$

also follows from [NÓØ21, Lemma 3.2]. Therefore $C_M^\circ \neq \emptyset$ implies $C_{W,\pm}^\circ \neq \emptyset$, and this is equivalent to $\mathbf{L}(S_W)^\circ \neq \emptyset$. □

The Rindler wedge in Minkowski space

Let $G = P(d)_e$ be the identity component of the *Poincaré group* $P(d) := \mathbb{R}^{1,d-1} \rtimes \mathrm{O}_{1,d-1}(\mathbb{R})$ and $h \in \mathfrak{g}$ the Euler element corresponding to the Lorentz boost in the $(\mathbf{e}_0, \mathbf{e}_1)$-plane with wedge region

$$W_R = \{x \in \mathbb{R}^{1,d-1} \colon x_1 > |x_0|\}$$

(Example 2.19). The corresponding reflection is $\tau_h = \mathrm{diag}(-1, -1, 1, \ldots, 1)$.

Lemma 2.23. *The stabilizer group of* W_R *is*

$$G_{W_R} \cong E(d-2)_+ \times \mathrm{SO}_{1,1}(\mathbb{R})_e \cong (E_R \rtimes \mathrm{SO}_{d-2}(\mathbb{R})) \times \mathrm{SO}_{1,1}(\mathbb{R})_e, \tag{2.15}$$

where $E(d-2)_+$ *denotes the connected group of proper euclidean motions on*

$$E_R := \mathrm{span}\{\mathbf{e}_2, \ldots, \mathbf{e}_{d-1}\} \cong \mathbb{R}^{d-2}$$

and $\mathrm{SO}_{1,1}(\mathbb{R})$ *acts on* $\mathrm{span}\{\mathbf{e}_0, \mathbf{e}_1\}$. *The compression semigroup of* W_R *is*

$$S_{W_R} := \{g \in P(d) \colon gW_R \subseteq W_R\} = \overline{W_R} \rtimes \mathrm{O}_{1,d-1}(\mathbb{R})_{W_R}.$$

Proof. The stabilizer group $P(d)_{W_R}$ contains the translation group corresponding to the edge E_R, and $gW_R = W_R$ implies $g(0) \in E_R$, so that

$$P(d)_{W_R} \cong E_R \rtimes \mathrm{O}_{1,d-1}(\mathbb{R})_{W_R}.$$

Further, each $g \in \mathrm{O}_{1,d-1}(\mathbb{R})$ preserving E_R also preserves its orthogonal complement, so that

$$\mathrm{O}_{1,d-1}(\mathbb{R})_{W_R} = \mathrm{O}_{d-2}(\mathbb{R}) \times \mathrm{O}_{1,1}(\mathbb{R})_{W_R} = \mathrm{O}_{d-2}(\mathbb{R}) \times (\mathrm{SO}_{1,1}(\mathbb{R})_e\{\mathbf{1}, r_1\}),$$

where $r_1 = \mathrm{diag}(1, -1, 1, \ldots, 1)$.

Next we use Lemma 2.24 below to see that

$$S_{W_R} = \overline{W_R} \rtimes \{g \in \mathrm{SO}_{1,d-1}(\mathbb{R})_e \colon gW_R \subseteq W_R\}.$$

Any $g \in \mathrm{SO}_{1,d-1}(\mathbb{R})_e$ with $gW_R \subseteq W_R$ satisfies $gE_R = E_R$ because g is injective and $\dim E_R < \infty$. This in turn implies that g commutes with $\tau_h = \mathrm{diag}(-1, -1, 1, \ldots, 1)$, so that $g = g_1 \oplus g_2$ with $g_1 \in \mathrm{O}_{1,1}(\mathbb{R})$ preserving the wedge region $W_R^2 \subseteq \mathbb{R}^{1,1} = \mathrm{span}\{\mathbf{e}_0, \mathbf{e}_1\}$. As $g_1 W_R^2$ is a quarter plane bounded by light rays, it cannot be strictly smaller than W_R^2, hence $g_1 W_R^2 = W_R^2$, and finally $gW_R = W_R$. This completes the proof. $\square$

Lemma 2.24. *Let E be a finite-dimensional real vector space and $C \subseteq E$ be a closed convex cone. In the affine group $G := \mathrm{Aff}(E) \cong E \rtimes \mathrm{GL}(E)$, we then have*

$$S_C := \{g \in G \colon gC \subseteq C\} = C \rtimes \{g \in \mathrm{GL}(E) \colon gC \subseteq C\}. \tag{2.16}$$

If C has interior points, then $S_{C^\circ} = S_C$ and $C = \overline{C^\circ}$.

Proof. We write $g = (b, a)$ with $gx = b + ax$. Then $g.C \subseteq C$ implies $b = g.0 \in C$.

Moreover, for the recession cone

$$\lim(C) := \{x \in E \colon x + C \subseteq C\} = \{x \in E \colon (\exists c \in C)\, c + \mathbb{R}_+ x \subseteq C\}$$

([Ne99, Prop. V.1.6]) the relation $g.C \subseteq C$ implies

$$aC = \lim(b + aC) = \lim(g.C) \subseteq \lim(C) = C,$$

and this implies (2.16).

If C has interior points, then $g.C^\circ \subseteq C^\circ$ and $C = \overline{C^\circ}$ imply $g.C \subseteq C$, so that $S_{C^\circ} \subseteq S_C$. Conversely, $C + C^\circ \subseteq C^\circ$ implies that $S_C \subseteq S_{C^\circ}$. $\square$

2.5 Causal Lie groups

The most structured examples of causal homogeneous spaces are causal groups with a biinvariant causal structure.

Let G be a connected Lie group and $C_{\mathfrak{g}} \subseteq \mathfrak{g}$ be a pointed generating closed convex cone. Then $C_g := g.C_{\mathfrak{g}} \subseteq T_g(G)$ defines on G a left-invariant causal structure. These structures become more interesting if $C_{\mathfrak{g}}$ is also $\mathrm{Ad}(G)$-invariant, so that the action of $G \times G$ by $(g_1, g_2).g = g_1 g g_2^{-1}$ preserves the causal structure. [7] If $h_0 \in \mathfrak{g}$ is an Euler element, then $h := (h_0, h_0) \in \mathfrak{g}^{\oplus 2}$ is Euler as well. It generates the flow

$$\alpha_t(g) = \exp(th_0) g \exp(-th_0).$$

The corresponding vector field is

$$X_h^G(g) = \frac{d}{dt}\Big|_{t=0} \exp(th_0) g \exp(-th_0) = h_0.g - g.h_0 = g.(\mathrm{Ad}(g)^{-1} h_0 - h_0).$$

Therefore

$$W_G^+(h) = \{g \in G \colon \mathrm{Ad}(g)^{-1} h_0 - h_0 \in C_{\mathfrak{g}}^\circ\} = \{g \in G \colon \mathrm{Ad}(g) h_0 - h_0 \in -C_{\mathfrak{g}}^\circ\} \tag{2.17}$$

[7] That a G action on M preserves the causal structure $(C_m)_{m \in M}$ means that $g.C_m = C_{g.m}$ for $g \in G, m \in M$.

It is easy to see that this is an open subsemigroup of G, contained in the closed subsemigroup

$$S(h_0, C_{\mathfrak{g}}) := \{g \in G \colon h_0 - \mathrm{Ad}(g)h_0 \in C_{\mathfrak{g}}\}. \tag{2.18}$$

For the G-invariant order (causal structure) on $\mathfrak{g}$, defined by

$$x \leq_{C_{\mathfrak{g}}} y \quad \text{if} \quad y - x \in C_{\mathfrak{g}},$$

this means that

$$S(h_0, C_{\mathfrak{g}}) = \{g \in G \colon \mathrm{Ad}(g)h_0 \leq_{C_{\mathfrak{g}}} h_0\}.$$

We likewise have for the strict order, defined by

$$x <_{C_{\mathfrak{g}}} y \quad \text{if} \quad y - x \in C_{\mathfrak{g}}^\circ$$

that

$$W_G^+(h) = \{g \in G \colon \mathrm{Ad}(g)h_0 <_{C_{\mathfrak{g}}} h_0\}.$$

We consider the two pointed generating cones

$$C_\pm = \pm C_{\mathfrak{g}} \cap \mathfrak{g}_{\pm 1} \tag{2.19}$$

(cf. Lemma 2.17).

We claim that

$$\exp(C_+^\circ) G^{h_0} \exp(C_-^\circ) \subseteq W_G^+(h_0), \tag{2.20}$$

which, by passing to the closure, implies

$$\exp(C_+) G^{h_0} \exp(C_-) \subseteq S(h_0, C_{\mathfrak{g}}). \tag{2.21}$$

As the centralizer G^{h_0} of h_0 is obviously contained in $S(h_0, C_{\mathfrak{g}})$ and $\exp(C_+)G^h = G^h \exp(C_+)$, it suffices to show that $\exp(C_+^\circ)\exp(C_-^\circ) \subseteq W_G^+(h_0)$. For $x_\pm \in C_\pm^\circ$, this follows from

$$\begin{aligned} e^{\mathrm{ad}\, x_+} e^{\mathrm{ad}\, x_-} h - h &= e^{\mathrm{ad}\, x_+}(h + [x_-, h]) - h = e^{\mathrm{ad}\, x_+}(h + x_-) - h = [x_+, h] + e^{\mathrm{ad}\, x_+} x_- \\ &= -x_+ + e^{\mathrm{ad}\, x_+} x_- = e^{\mathrm{ad}\, x_+}(x_- - x_+) \in -e^{\mathrm{ad}\, x_+}(C_+^\circ - C_-^\circ) \subseteq -C_{\mathfrak{g}}^\circ \end{aligned} \tag{2.22}$$

(cf. the proof of Proposition 2.15). Here we used $-\tau_h(C_{\mathfrak{g}}) = C_{\mathfrak{g}}$ for the inclusion $C_+^\circ - C_-^\circ \subseteq C_{\mathfrak{g}}^\circ$ (Lemma 2.17(ii)).

Definition 2.25. Assume that $\tau_h^{\mathfrak{g}} = e^{\pi i\, \mathrm{ad}\, h}$ integrates to an automorphism τ_h of G. Using the complex Olshanski semigroup $S(iC_{\mathfrak{g}}) := G\,\mathrm{Exp}(iC_{\mathfrak{g}})$ (see [Ne99, §IX.1], [HN93, 3.20] and also [Ne22, §2.4] for a detailed discussion), [8] we define the subsemigroup $G_{\mathrm{KMS}} \subseteq G$ as the set of those elements $g \in G$ for which the orbit map

$$\alpha^g \colon \mathbb{R} \to G, \quad \alpha^g(t) = \alpha_t(g)$$

extends analytically to a map $\overline{\mathcal{S}_\pi} \to S(iC_{\mathfrak{g}})$ with $\alpha^g(\mathcal{S}_\pi) \subseteq S(iC_{\mathfrak{g}}^\circ)$, such that $\alpha^g(\pi i) = \tau_h(g)$.

[8] If G is simply connected and $\eta_G \colon G \to G_{\mathbb{C}}$ its universal complexification, then $S(iC_{\mathfrak{g}})$ is the simply connected covering of the subsemigroup $\eta_G(G)\exp(iC_{\mathfrak{g}}) \subseteq G_{\mathbb{C}}$, and if G is not simply connected, it is the quotient of $S_{\widetilde{G}}(iC_{\mathfrak{g}})$ by the kernel of the covering map $q_G \colon \widetilde{G} \to G$. The map $\mathrm{Exp} \colon iC_{\mathfrak{g}} \to S(iC_{\mathfrak{g}})$ is the map corresponding in this context to the exponential function $iC_{\mathfrak{g}} \to G_{\mathbb{C}}$ and $G \times C_{\mathfrak{g}} \to S(iC_{\mathfrak{g}}), (g, x) \mapsto g\,\mathrm{Exp}(ix)$ is a homeomorphism.

Theorem 2.26. *If G is simply connected, $h \in \mathfrak{g}$ an Euler element, and $C_\mathfrak{g} \subseteq \mathfrak{g}$ a pointed closed convex invariant cone with $-\tau_h^\mathfrak{g}(C_\mathfrak{g}) = C_\mathfrak{g}$, then*

$$S(h, C_\mathfrak{g}) = \exp(C_+)G^h \exp(C_-) = G^h \exp(C_+ + C_-), \tag{2.23}$$

the positivity domain

$$W_G^+(h) = \exp(C_+^\circ)G^h \exp(C_-^\circ)$$

is a subsemigroup, and

$$G_{\mathrm{KMS}} = \exp(C_+^\circ)G_e^h \exp(C_-^\circ) = G_e^h \exp(C_+^\circ + C_-^\circ) = S(h, C_\mathfrak{g})_e^\circ$$

is a connected component of $W_G^+(h)$.

Proof. The first two equalities in (2.23) are the Decomposition Theorem [Ne22, Thm. 2.16]. Further [Ne22, Thm. 2.21] shows that $S(h, C_\mathfrak{g})$ coincides with the set of all $g \in G$ for which α^g extends to a map $\overline{\mathcal{S}_\pi} \to S(iC_\mathfrak{g})$.

Next we show that the additional requirement that $\alpha^g(\mathcal{S}_\pi) \subseteq S(iC_\mathfrak{g}^\circ)$ specifies the open subset $G^h \exp(C_+^\circ + C_-^\circ) = S(h, C_\mathfrak{g})^\circ$. For $g = g_0 \exp(x_1 + x_{-1})$ with $x_{\pm 1} \in C_\pm$, we have

$$\alpha^g(z) = g_0 \operatorname{Exp}(e^z x_1 + e^{-z} x_{-1}).$$

For $z = a + ib$ with $0 < b < \pi$, we have for $x_{\pm 1} \in C_\pm^\circ$

$$\operatorname{Im}(e^z x_1 + e^{-z} x_{-1}) = \sin(y)(x_1 - x_{-1}) \in (C_+ + C_-)^\circ.$$

This shows that

$$G_e^h \exp(C_+^\circ + C_-^\circ) = S(h, C_\mathfrak{g})_e^\circ = \exp(C_+^\circ)G_e^h \exp(C_-^\circ) \subseteq G_{\mathrm{KMS}}.$$

If, conversely, $x_{\pm 1} \in C_\pm$ and $\alpha^g(\pi i/2) = g_0 \operatorname{Exp}(i(x_1 - x_{-1})) \in S(iC_\mathfrak{g}^\circ)$, then

$$x_1 - x_{-1} \in C_\mathfrak{g}^\circ \cap \mathfrak{g}^{-\tau_h} = C_+^\circ - C_-^\circ$$

(Lemma 2.17).

For $g = g_0 \exp(x_1 + x_{-1})$ we have $\tau_h(g) = \tau_h(g_0)\exp(-x_1 - x_{-1})$, so that we find for G_{KMS} the additional condition that $g_0 \in G^{\tau_h}$ (cf. [Ne22, Cor. 2.22]). □

Remark 2.27. For the antiholomorphic extension $\overline{\tau}_h$ of τ_h to the complex semigroup $S(iC_\mathfrak{g})$, the fixed point set

$$S(iC_\mathfrak{g})^{\overline{\tau}_h} = G^{\tau_h} \operatorname{Exp}(iC_\mathfrak{g}^{-\tau_h}) = G^{\tau_h} \operatorname{Exp}(i(C_+ - C_-)),$$

is a real Olshanski semigroup in the c-dual group G^c (with respect to τ_h) with Lie algebra $\mathfrak{g}^c = \mathfrak{g}_0 + i\mathfrak{g}^{-\tau_h}$. The invariance condition $-\tau_h^\mathfrak{g}(C_\mathfrak{g}) = C_\mathfrak{g}$ implies that $C_\mathfrak{g}^{-\tau_h} = C_+ - C_-$ has interior points (cf. Lemma 2.17).

Remark 2.28. In the context of causal Lie groups, specified by a pair $(G, C_\mathfrak{g})$ as above, $\mathfrak{g}$ may not contain an Euler element, but there may be an *Euler derivation* $D \in \operatorname{der}(\mathfrak{g})$, i.e., D is diagonalizable with eigenvalues contained in $\{-1, 0, 1\}$ (see Example 2.51 below, and Example 2.7 for the case where $G = E$ is a vector space). Then $\tau_D := e^{\pi i D}$ defines an involutive automorphism of $\mathfrak{g}$, and compatibility with the causal structure corresponds to the requirements

$$e^{\mathbb{R}D} C_\mathfrak{g} = C_\mathfrak{g} \quad \text{and} \quad -\tau_D C_\mathfrak{g} = C_\mathfrak{g}. \tag{2.24}$$

To implement a modular flow on G, we assume that all automorphisms $\alpha_t^{\mathfrak g} := e^{tD}$ of $\mathfrak g$ integrate to automorphisms α_t of G. Then $G^\flat := G \rtimes_\alpha \mathbb{R}$ is a Lie group acting by causal automorphisms on $M := G$, where $(g,0) \in G^\flat$ acts by left translation and $(0,t)$ by α_t. This action leaves the biinvariant cone field invariant, and the involution τ_D^G on G is *anti-causal*, i.e., flips the cone field into its negative. Now $h^\flat := (0,1) \in \mathfrak g^\flat$ is an Euler element, and for every $g = (g,0) \in G \subseteq G^\flat$, we have $\mathrm{Ad}(g)h^\flat - h^\flat \in \mathfrak g$. We may therefore consider the closed subsemigroup

$$S(h^\flat, C_{\mathfrak g}) := \{g \in G \colon h^\flat - \mathrm{Ad}(g)h^\flat \in C_{\mathfrak g}\}$$

and find the positivity domain

$$W_G^+(h^\flat) = \{g \in G \colon h^\flat - \mathrm{Ad}(g)h^\flat \in C_{\mathfrak g}^\circ\}.$$

With the same arguments as above, we also obtain with [Ne22]

$$W_G^+(h^\flat) = \exp(C_+^\circ)G^{h^\flat}\exp(C_-^\circ) = G^{h^\flat}\exp(C_+^\circ + C_-^\circ) = S(h^\flat, C_{\mathfrak g})^\circ. \tag{2.25}$$

Example 2.29. (a) Not every Euler element has a non-trivial positivity region. If $M = G$ is a causal Lie group with biinvariant cone field corresponding to $C_{\mathfrak g} \subseteq \mathfrak g$, on which $G \times G$-acts, then every Euler element $h_0 \in \mathfrak g$ specifies an Euler element $h := (h_0, 0) \in \mathfrak g^{\oplus 2}$, but the corresponding modular vector field is $X_h^G(g) = g.h$, and this is never contained in $C_g = g.C_{\mathfrak g}$ because $h \notin C_{\mathfrak g}$. This follows from the fact that h is hyperbolic and the semisimple Jordan components of elements in $C_{\mathfrak g}$ are elliptic ([NOe22]). We also note that $\tau_h = \tau_{h_0} \oplus \mathrm{id}_{\mathfrak g}$ does not commute with the flip, hence cannot be implemented on the symmetric space G in a natural way.
(b) For **left** invariant causal structure on a Lie group G, the cone $C \subseteq \mathfrak g \cong T_e(G)$ can be any pointed generating closed convex cone. Then $W_G^+(h) \neq \emptyset$ is equivalent to $h \in C^\circ$, and in this case $W_G^+(h) = G$, so that the situation is quite degenerate.

2.6 Causal flag manifolds

We have seen above that Euler elements $h \in \mathfrak g$ play a key role, and that we have to understand causal homogeneous spaces $M = G/H$ for which the positivity region $W_M^+(h)$ is non-empty, because otherwise we have no wedge regions for the Bisognano–Wichmann property. As the most well-behaved homogeneous spaces are symmetric spaces and flag manifolds, this is the class of manifolds for which we investigate this question first. In Physics, the most prominent example is the conformal compactification $(\mathbb{S}^1 \times \mathbb{S}^{d-1})/\{\pm\mathbf{1}\}$ of d-dimensional Minkowski space (Example 2.36).

Definition 2.30. To define flag manifolds for a connected semisimple Lie group, consider $x \in \mathfrak g$ such that $\mathrm{ad}\, x$ is diagonalizable, put

$$\mathfrak q_x = \sum_{\lambda \leq 0} \mathfrak g_\lambda(x) \quad \text{and} \quad Q_x := \{g \in G \colon \mathrm{Ad}(g)\mathfrak q_x = \mathfrak q_x\}.$$

Then Q_x is called a *parabolic subgroup* of G and G/Q_x the corresponding flag manifold.

For the description of the causal flag manifolds, we also need hermitian Lie algebras.

Definition 2.31. A simple Lie algebra $\mathfrak g$ with Cartan decomposition $\mathfrak g = \mathfrak k \oplus \mathfrak p$ is called *hermitian* if the center $\mathfrak z(\mathfrak k)$ of a maximal compactly embedded subalgebra $\mathfrak k$ is non-zero. For hermitian Lie algebras, the restricted root system $\Sigma = \Sigma(\mathfrak g, \mathfrak a)$, with respect to a maximal abelian subspace $\mathfrak a \subseteq \mathfrak p$, is either of type C_r or BC_r (cf. Harish Chandra's Theorem [Ne99, Thm. XII.1.14]), and we say that $\mathfrak g$ is *of tube type* if the restricted root system is of type C_r. The terminology comes from the fact that the corresponding hermitian symmetric space G/K is a tube domain, i.e., biholomorphic to $\mathrm{V}_+ + i\mathrm{V} \subseteq \mathrm{V}_{\mathbb C}$ for a real vector space V and an open convex cone $\mathrm{V}_+ \subseteq \mathrm{V}$.

Theorem 2.32. (Classification of causal flag manifolds, [Ne25]) *Let G be a connected semisimple Lie group and $Q \subseteq G$ be a parabolic subgroup such that $\mathfrak{q}$ contains no non-zero ideals of $\mathfrak{g}$. Suppose that the corresponding flag manifold G/Q carries a G-invariant causal structure. Then $\mathfrak{g}$ is a direct sum of hermitian simple ideals and there exists an Euler element $h \in \mathfrak{g}$ such that*

$$\mathfrak{q} = \mathfrak{q}_h = \mathfrak{g}_0(h) + \mathfrak{g}_{-1}(h).$$

If, conversely, this is the case, then G/Q_h is a causal flag manifold.

If $\mathfrak{g}$ is simple hermitian, then an Euler element h exists in $\mathfrak{g}$ if and only if $\mathfrak{g}$ is of tube type, and then they are all conjugate and h is symmetric (Proposition 2.46). We fix one and consider the corresponding causal flag manifold $M = G/Q_h$. The tangent space in the base point is

$$\mathfrak{g}/\mathfrak{q}_h \cong \mathfrak{g}_1(h),$$

and the causal structure on M is specified by the cone

$$C_+ = C_{\mathfrak{g}} \cap \mathfrak{g}_1(h),$$

where $C_{\mathfrak{g}}$ is a pointed generating closed convex $\mathrm{Ad}(G)$-invariant cone in $\mathfrak{g}$. We thus obtain an (up to sign) unique causal structure on M, i.e., any other cone $C'_{\mathfrak{g}}$ satisfies $C'_{\mathfrak{g}} \cap \mathfrak{g}_1(h) = C_+$ or $C'_{\mathfrak{g}} \cap \mathfrak{g}_1(h) = -C_+$ ([MNO23, §3.5]). This also follows from the fact that $\mathfrak{g}_1(h)$ only contains two $e^{\mathrm{ad}\,\mathfrak{g}_0}$-invariant non-trivial closed convex cones ([HNO94]).

On the open dense subset of M obtained by embedding $\mathfrak{g}_1$ via $\eta(x) := \exp(x)Q_h$, the vector field X_h^M is the **Euler vector field** on $\mathfrak{g}_1$, so that $\eta(C_+^\circ) \subseteq W_M^+(h)$, and we actually have that

$$W := W_M^+(h) = \eta(C_+^\circ) \tag{2.26}$$

([MN25, Lemma 2.7]).

Proposition 2.33. *The compression semigroup of $W \subseteq M = G/Q_h$ is*

$$S_W = \{g \in G \colon g.W \subseteq W\} = \exp(C_+)G^h \exp(C_-) \quad \textit{for} \quad C_\pm = \pm C_{\mathfrak{g}} \cap \mathfrak{g}_{\pm 1}(h). \tag{2.27}$$

Proof. As $G_1 = \exp(\mathfrak{g}_1)$ is abelian, the inclusions $\exp(C_+) \subseteq S_W$ and $G^h \subseteq S_W$ are obvious.

All elements $x \in C_+$ correspond to constant vector fields on the open subset $\eta(\mathfrak{g}_1) \subseteq M$, and since this subset is dense (Bruhat decomposition), we obtain

$$C_+ \subseteq C_M = \{y \in \mathfrak{g} \colon (\forall m \in M)\ X_y^M(m) \in C_m\}$$

(cf. (2.12)). As G acts effectively on M (the corresponding homomorphism $G \to \mathrm{Diff}(M)$ is injective), the closed convex $\mathrm{Ad}(G)$-invariant cone $C_M \subseteq \mathfrak{g}$ is pointed, and the preceding argument yields

$$C_{M,+} := C_M \cap \mathfrak{g}_1 = C_+.$$

As $C_M - C_M$ is an ideal of $\mathfrak{g}$ and $\mathfrak{g}$ is simple, the cone C_M is also generating, so that we also obtain $C_- = -C_M \cap \mathfrak{g}_{-1}$ by the discussion preceding the proposition. Thus $\exp(C_-) \subseteq S_W$ follows from Proposition 2.22. Putting everything together, we get

$$S_W \supseteq \exp(C_+)G^h \exp(C_-), \tag{2.28}$$

and the hard part is to verify equality in (2.27). This involves showing that the product set on the right is a subsemigroup (which is not easy to see) and that it actually coincides with S_W, by showing that it is maximal, hence equal to S_W. We refer to [Ne18, Lemma 3.7, Thm. 3.8] for more details and references. □

Problem 2.34. Theorem 2.32 describes all causal flag manifolds $M = G/Q_h$ for semisimple Lie groups, but it makes good sense to ask for a more general result:

(CF1) Let $x \in \mathfrak{g}$ be such that $\operatorname{ad} x$ is diagonalizable, put

$$\mathfrak{q}_x = \sum_{\lambda \leq 0} \mathfrak{g}_\lambda(x) \quad \text{and} \quad Q_x := \{g \in G \colon \operatorname{Ad}(g)\mathfrak{q}_x = \mathfrak{q}_x\}.$$

Show that, if $M = G/Q_x$ is causal, then x must be an Euler element (cf. [Ne25] for similar arguments). Note that $x \in \mathfrak{q}_x$ implies that $\mathfrak{q}_x$ is self-normalizing, so that $\mathbf{L}(Q_x) = \mathfrak{q}_x$.

(CF2) Assume that $h \in \mathfrak{g}$ is an Euler element. Determine those manifolds $M = Q/Q_h$ with an invariant causal structure on which G acts effectively.

Remark 2.35. (The affine case) Particular examples arise for Euler elements with $\mathfrak{g}_{-1} = \{0\}$. Then $M = G/Q_h = \eta(\mathfrak{g}_1) \cong \mathfrak{g}_1$ and we may assume that $G \cong \mathfrak{g}_1 \rtimes G_0$.

This covers the action of $\operatorname{Aff}(\mathbb{R})_e$ on $\mathbb{R}$ and of the Poincaré group on Minkowski space. More generally, we may start with a finite-dimensional real linear space E and a pointed generating convex cone $C \subseteq E$. We write $\operatorname{Aut}(C) \subseteq \operatorname{GL}(E)$ for its linear automorphism group, which is a closed subgroup. Then $G := E \rtimes \operatorname{Aut}(C)$ acts transitively on the affine causal manifold $M := E$, endowed with the constant cone field $C_m = C$ for $m \in M$. Further, $h := (0, \operatorname{id}_E)$ is an Euler element with $\mathfrak{g}_{-1} = \{0\}$, $\operatorname{Aut}(C) = G^h$ and $\mathfrak{g}_1 \cong E$. The corresponding positivity region is

$$W := W_M^+(h) = C^\circ,$$

and its compression semigroup is readily identified with

$$S_W = C \rtimes \operatorname{Aut}(C)$$

because $\operatorname{Aut}(C) \subseteq S_W$ (cf. also Lemma 2.24).

Lie algebra elements $(b, a) \in \mathfrak{g} = \mathfrak{g}_1 \rtimes \mathfrak{g}_0$ correspond to affine vector fields $X(x) = b + ax$, and such a vector field is positive on all of E if and only if $b + aE \subseteq C$, which is equivalent to $a = 0$ and $b \in C$. Therefore the invariant cone $C_M \subseteq \mathfrak{g}$ coincides with $C \subseteq \mathfrak{g}_1$.

Euclidean Jordan algebras

The causal flag manifolds of simple Lie groups are precisely the conformal compactifications of simple euclidean Jordan algebras.

The following table lists the simple hermitian Lie algebras of tube type, the only non-simple Lie algebra listed is $\mathfrak{so}_{2,2}(\mathbb{R}) \cong \mathfrak{so}_{1,2}(\mathbb{R})^{\oplus 2}$, corresponding to the non-simple Jordan algebra $\mathrm{V} = \mathbb{R}^{1,1} \cong \mathbb{R} \oplus \mathbb{R}$ (the Minkowski plane, decomposing in lightray coordinates).

Hermitian Lie algebra	$\mathfrak{g}$	$\mathfrak{sp}_{2r}(\mathbb{R})$	$\mathfrak{su}_{r,r}(\mathbb{C})$	$\mathfrak{so}^*(4r)$	$\mathfrak{e}_{7(-25)}$	$\mathfrak{so}_{2,d}(\mathbb{R})$
Euclidean Jordan algebra	V	$\operatorname{Sym}_r(\mathbb{R})$	$\operatorname{Herm}_r(\mathbb{C})$	$\operatorname{Herm}_r(\mathbb{H})$	$\operatorname{Herm}_3(\mathbb{O})$	$\mathbb{R}^{1,d-1}$
rank of V	rank V	r	r	r	3	2

Table 1: Hermitian Lie algebras of tube type and euclidean Jordan algebras

The corresponding flag manifolds M have interesting geometric interpretations. For $\mathfrak{g} = \mathfrak{so}_{2,d}(\mathbb{R})$, the manifold M is the isotropic quadric $Q = Q(\mathbb{R}^{2,d})$ in the real projective space $\mathbb{P}(\mathbb{R}^{2,d})$, and for

$$\Omega := \Omega_{2r} := \begin{pmatrix} 0 & \mathbf{1}_r \\ -\mathbf{1}_r & 0 \end{pmatrix} \in M_{2r}(\mathbb{K}), \quad \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H},$$

we obtain a uniform realization of the Lie algebras $\mathfrak{sp}_{2r}(\mathbb{R}), \mathfrak{u}_{r,r}(\mathbb{C})$ and $\mathfrak{so}^*(4r)$ as

$$\mathfrak{u}(\Omega, \mathbb{K}^{2r}) := \{x \in \mathfrak{gl}_{2r}(\mathbb{K}) \colon x^*\Omega + \Omega x = 0\}. \tag{2.29}$$

Then M is the space of maximal isotropic subspaces $L \subseteq \mathbb{K}^{2r}$ with respect to the skew-hermitian form $\beta(z, w) := z^*\Omega w$ on $\mathbb{K}^{2r}$

Example 2.36. (The Lorentzian case) For d-dimensional Minkowski space $\mathrm{V} = \mathbb{R}^{1,d-1}$, we realize the conformal completion M of V as the quadric

$$Q := Q(\mathbb{R}^{2,d}) := \{[\widetilde{v}] \in \mathbb{P}(\widetilde{\mathrm{V}}) \colon \widetilde{\beta}(\widetilde{v}, \widetilde{v}) = 0\}, \tag{2.30}$$

where $\widetilde{\beta}$ is the symmetric bilinear form on $\mathbb{R}^{2,d}$, given by

$$\widetilde{\beta}(x, y) = x_1y_1 + x_2y_2 - x_3y_3 - \cdots - x_{d+2}y_{d+2}.$$

The natural dense open embedding $\mathbb{R}^{1,d-1} \to Q$ is given by

$$\eta \colon \mathrm{V} \to Q, \quad \eta(v) := \Big[\frac{1 - \beta(v, v)}{2} : v : -\frac{1 + \beta(v, v)}{2}\Big] \in Q \subseteq \mathbb{P}(\mathbb{R}^{2,d}), \tag{2.31}$$

corresponding to the action of the translation group $(\mathrm{V}, +) \cong \mathfrak{g}_1(h)$ on Q (cf. [HN12, §17.4], [Ne25]).

2.7 Causal symmetric spaces

We start with some terminology and observations concerning symmetric spaces and symmetric Lie algebras (cf. [HÓ97]):

- A *symmetric Lie algebra* is a pair $(\mathfrak{g}, \tau)$, where $\mathfrak{g}$ is a finite-dimensional real Lie algebra and τ is an involutive automorphism of $\mathfrak{g}$. We write

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q} \quad \text{with} \quad \mathfrak{h} = \mathfrak{g}^\tau = \ker(\tau - \mathbf{1}) \quad \text{and} \quad \mathfrak{q} = \mathfrak{g}^{-\tau} = \ker(\tau + \mathbf{1}). \tag{2.32}$$

- A *symmetric space* is a homogeneous space of the form $M = G/H$, where $H \subseteq G^\tau$ is an open subgroup and $\tau \in \mathrm{Aut}(G)$ an involution. Then H contains the identity component $G^\tau_e := (G^\tau)_e$. We call the triple (G, τ, H) a *symmetric Lie group* because this triple specifies the symmetric space M.
- A *causal symmetric Lie algebra* is a triple $(\mathfrak{g}, \tau, C)$, where $(\mathfrak{g}, \tau)$ is a symmetric Lie algebra and $C \subseteq \mathfrak{q}$ is a pointed generating closed convex cone, invariant under the group $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}) := \langle e^{\mathrm{ad}\,\mathfrak{h}} \rangle \subseteq \mathrm{Aut}(\mathfrak{g})$. We call $(\mathfrak{g}, \tau, C)$
  - *compactly causal (cc)* if C is *elliptic* in the sense that, for $x \in C^\circ$ (the interior of C), the operator $\mathrm{ad}\,x$ is semisimple with purely imaginary spectrum.
  - *non-compactly causal (ncc)* if C is *hyperbolic* in the sense that, for $x \in C^\circ$, the operator $\mathrm{ad}\,x$ is diagonalizable.
- For a symmetric Lie algebra $(\mathfrak{g}, \tau)$, the pair $(\mathfrak{g}^c, \tau^c)$ with $\mathfrak{g}^c := \mathfrak{h} + i\mathfrak{q}$ and $\tau^c(x + iy) = x - iy$ is called the *c-dual symmetric Lie algebra*.

- A *modular causal symmetric Lie algebra* is a quadruple $(\mathfrak{g}, \tau, C, h)$, where $(\mathfrak{g}, \tau, C)$ is a causal symmetric Lie algebra, $h \in \mathfrak{g}^\tau$ is an Euler element, and the involution τ_h satisfies $\tau_h(C) = -C$.

Remark 2.37. (a) $(\mathfrak{g}, \tau, C)$ is non-compactly causal if and only if $(\mathfrak{g}^c, \tau^c, iC)$ is compactly causal. (b) $(\mathfrak{g}, \tau, C, h)$ is modular if and only if the c-dual quadruple $(\mathfrak{g}^c, \tau^c, iC, h)$ is modular.

Remark 2.38. If $C_\mathfrak{g} \subseteq \mathfrak{g}$ is a pointed generating invariant cone in $\mathfrak{g}$ and $h \in \mathfrak{g}$ an Euler element satisfying $-\tau_h(C_\mathfrak{g}) = C_\mathfrak{g}$, then there is a variety of associated causal symmetric Lie algebras:

(a) $(\mathfrak{g}^{\oplus 2}, \tau_{\text{flip}}, C, (h, h))$ with $C = \{(x, -x) \colon x \in C_\mathfrak{g}\}$ is a modular causal symmetric Lie algebra of group type (cf. Subsection 2.5).

(b) $(\mathfrak{g}_\mathbb{C}, \sigma, iC_\mathfrak{g}, h)$ is a modular non-compactly causal symmetric Lie algebra of complex type.

(c) $(\mathfrak{g}, \tau_h, C_+ - C_-, h)$ is a modular compactly causal symmetric Lie algebra of Cayley type. Note that $C_\mathfrak{g}^{-\tau_h} = C_+ - C_-$ by Lemma 2.17(ii).

(d) $(\mathfrak{g}, \tau_h, C_+ + C_-, h)$ is a modular non-compactly causal symmetric Lie algebra of Cayley type.

Note that

$$\kappa_h = e^{\frac{\pi i}{2} h} \colon \mathfrak{g}_\mathbb{C} \to \mathfrak{g}_\mathbb{C} \quad \text{satisfies} \quad \kappa_h(\mathfrak{g}) = \mathfrak{g}^c, \tag{2.33}$$

so that $(\mathfrak{g}, \tau_h) \cong (\mathfrak{g}^c, \tau_h)$ as symmetric Lie algebras. Moreover,

$$\kappa_h(C_\mathfrak{g}^{-\tau_h}) = \kappa_h(C_+ - C_-) = i(C_+ + C_-),$$

so that

$$(\mathfrak{g}, \tau_h, C_\mathfrak{g}^{-\tau_h}, h) = (\mathfrak{g}, \tau_h, C_+ - C_-, h) \cong (\mathfrak{g}^c, \tau_h, i(C_+ + C_-), h) \cong (\mathfrak{g}, \tau_h, C_+ + C_-, h)^c \tag{2.34}$$

as modular causal symmetric Lie algebras.

Remark 2.39. (Tangent spaces) If (G, τ, H) is a connected symmetric Lie group corresponding to the causal symmetric Lie algebra $(\mathfrak{g}, \tau, C)$, then we obtain on $\mathfrak{q}$ a constant causal structure defined by C that is invariant under the action of the semidirect product group $\mathfrak{q} \rtimes H$, where $H := G_e^\tau$ (cf. Remark 2.35). If, in addition, $h \in \mathfrak{h}$ is an Euler element with $\tau_h(C) = -C$, then the pair $(\mathfrak{q}, C)$ satisfies the assumptions (A1)-(A3) from Section 2.3.1. So $\mathfrak{q}$ is an affine causal symmetric space, and (2.11) in Subsection 2.3.1 implies that

$$W_\mathfrak{q}^+(h) = C_+^\circ \oplus \mathfrak{q}_0(h) \oplus C_-^\circ \quad \text{for} \quad C_\pm := \pm C \cap \mathfrak{q}_{\pm 1}(h).$$

Remark 2.40. (Lorentzian symmetric spaces) Important examples of causal symmetric spaces are those where causal structure comes from a Lorentzian form, for instance de Sitter space dS^d and anti-de Sitter space AdS^d (see Examples 1.15).

If $M_1 = G_1/H_1$ is a Lorentzian symmetric space and $M_2 = G_2/H_2$ is a Riemannian symmetric space, then the product $M = M_1 \times M_2$ is also Lorentzian. Important examples are

$$\mathrm{AdS}^p \times \mathbb{S}^q \quad \text{and} \quad \mathrm{dS}^p \times \mathrm{Hyp}^q$$

and the compact group $\mathrm{U}_n(\mathbb{C})$ carries biinvariant Lorentzian structures. We refer to [Ne25] for more details and conformal embeddings of these spaces for $p + q = d$ into $Q(\mathbb{R}^{2,d})$.

2.7.1 Causal symmetric spaces of group type

We assume first that $\mathfrak{g}$ is simple hermitian and that $h_0 \in \mathfrak{g}$ is an Euler element. Then any $\mathrm{Ad}(G)$-invariant closed convex pointed generating cone $C_{\mathfrak{g}} \subseteq \mathfrak{g}$ specifies a biinvariant causal structure on the group G, considered as a symmetric space on which $G \times G$ acts transitively. Then the Euler elements $h \in \mathfrak{g}^{\oplus 2}$ for which $W_G^+(h) \neq \emptyset$ are conjugate to $h = (h_0, h_0)$ for some Euler element $h_0 \in \mathfrak{g}$, and in this case

$$W_G^+(h) = \exp(C_+^\circ) G^h \exp(C_-^\circ) = S(h, C_{\mathfrak{g}})^\circ \tag{2.35}$$

follows from Theorem 2.26, cf. also (2.17) and (2.18). Note that $W_G^+(h)$ only depends on the cones $C_\pm$, hence is unique up to sign if $\mathfrak{g}$ is simple ([MNO23, §3.5]).

2.7.2 Modular compactly causal symmetric spaces

If $(\mathfrak{g}, \tau, C)$ is an irreducible compactly causal symmetric Lie algebra which is not of group type, then $\mathfrak{g}$ is simple hermitian ([NÓ23b, Prop. 2.13] and by c-duality[9]). If $\mathfrak{g}$ contains an Euler element, then $\mathfrak{g}$ is of tube type, $\mathrm{Ad}(G)$ acts transitively on $\mathcal{E}(\mathfrak{g})$ (Proposition 2.46) and there exist τ-fixed Euler elements (Corollary 2.57 in Appendix 2.8.6). Now the embedding

$$(\mathfrak{g}, \tau, C) \hookrightarrow (\mathfrak{g}^{\oplus 2}, \tau_{\mathrm{flip}}, \widetilde{C}), \quad x \mapsto (x, \tau(x)) \tag{2.36}$$

can be used to determine the positivity region $W_M^+(h)$ by using the results for spaces of group type.

On the global side, we consider the action of G on G by $g.x := gx\tau(g)^{-1}$, corresponding to the embedding (2.36). Then $M := G.e$ is the identity component in the fixed point set of the involution $g^\sharp := \tau(g)^{-1}$ and a symmetric space with symmetric Lie algebra $(\mathfrak{g}, \tau)$. If $C = C_{\mathfrak{g}} \cap \mathfrak{q}$, then we even have an embedding of causal symmetric spaces which is equivariant for the modular flow. This easily implies that

$$W_M^+(h) = W_G^+(h) \cap M = S(C_{\mathfrak{g}}, h)^\circ \cap M, \tag{2.37}$$

and

$$W = G_e^h.\exp(C_+^\circ + C_-^\circ) \quad \text{for} \quad C_\pm = \pm C_{\mathfrak{g}}^{-\tau} \cap \mathfrak{g}_{\pm 1}(h).$$

The compression semigroup of W is

$$S_W = G_W \exp(C_+ + C_-) \quad \text{with} \quad G_W = G_e^h H^h. \tag{2.38}$$

Furthermore, G_W is open in G^h ([NÓ23a, Thm. 9.1]). We refer to [NÓ23a] for details.

2.7.3 Non-compactly causal symmetric spaces

Irreducible non-compactly causal symmetric Lie algebras $(\mathfrak{g}, \tau, C)$ are c-dual to irreducible compactly causal ones. The dual $(\mathfrak{g}^c, \tau^c)$ is of group type if and only if $\mathfrak{g}$ is a complex simple Lie algebra (considered as a real one) and τ is antilinear, so that $\mathfrak{h} = \mathfrak{g}^\tau$ is a real form and $\mathfrak{g} \cong \mathfrak{h}_{\mathbb{C}}$. Then $(\mathfrak{g}^c, \tau^c) \cong (\mathfrak{h}^{\oplus 2}, \tau_{\mathrm{flip}})$. The existence of the causal structure implies that $\mathfrak{h}$ is hermitian, but these real forms are precisely those for which the corresponding conjugation τ is of the form $\theta\tau_h$, where $h \in \mathfrak{g}$ is an Euler element ([MNO23, Thm. 4.21]). So Euler elements in complex simple Lie algebras automatically determine causal symmetric Lie algebras of complex type.

This picture prevails for general simple Lie algebras $\mathfrak{g}$. Whenever $h \in \mathfrak{g}$ is an Euler element and θ a Cartan involution with $\theta(h) = -h$, then $\tau := \theta\tau_h$ is an involution of $\mathfrak{g}$. Further h is also Euler in

[9] The dual symmetric Lie algebra $(\mathfrak{g}^c, \tau^c, iC)$ is irreducible, non-complex and non-compactly causal. Hence $\mathfrak{g}^c$ is simple. Moreover $\tau^c = \tau_h \theta^c$ for a causal Euler element $h \in i\mathfrak{q} = \mathfrak{q}^c$. Then $\mathfrak{g}_0^c = \mathfrak{z}_{\mathfrak{g}^c}(h) = \mathfrak{h}_{\mathfrak{k}}^c \oplus \mathfrak{q}_{\mathfrak{p}}^c = \mathfrak{h}_{\mathfrak{k}} \oplus i\mathfrak{q}_{\mathfrak{k}}$ implies that $\mathfrak{z}_{\mathfrak{g}}(ih) = \mathfrak{k}$. So $ih \in \mathfrak{z}(\mathfrak{k})$ implies that $\mathfrak{g}$ is hermitian.

the complexification $\mathfrak{g}_\mathbb{C}$, on which the antilinear extension $\overline{\theta}$ of θ to $\mathfrak{g}_\mathbb{C}$ defines a Cartan involution. Then $\overline{\tau} := \overline{\theta}\tau_h$ is an antilinear extension of the involution $\tau = \theta\tau_h$ on $\mathfrak{g}$, and $\mathfrak{g}^c := (\mathfrak{g}_\mathbb{C})^{\overline{\tau}} = \mathfrak{h} + i\mathfrak{q}$ is a hermitian real form of $\mathfrak{g}_\mathbb{C}$ with $\mathfrak{z}(\mathfrak{k}^c) = \mathbb{R}ih$. For any invariant cone $C_{\mathfrak{g}^c} \subseteq \mathfrak{g}^c$ containing $-ih$, we then obtain by

$$C := iC_{\mathfrak{g}^c} \cap \mathfrak{q}$$

an $e^{\mathrm{ad}\,\mathfrak{h}}$-invariant cone in $\mathfrak{q}$ with $h \in C^\circ$. We have an embedding

$$(\mathfrak{g}, \tau, C) \hookrightarrow (\mathfrak{g}_\mathbb{C}, \overline{\tau}, iC_{\mathfrak{g}^c})$$

of causal symmetric Lie algebras of non-compact type, and we thus obtain a parametrization of irreducible non-compactly causal symmetric Lie algebras in terms of Euler elements:

Theorem 2.41. (Classification of irreducible non-compactly causal symmetric Lie algebras;[MNO23, Thm. 4.21]) *Let $\mathfrak{g}$ be a simple real Lie algebra and pick a Cartan involution θ with $\theta(h) = -h$. Then the assignment*

$$h \mapsto (\mathfrak{g}, \tau_h\theta, C)$$

described above defines a bijection from the set $\mathcal{E}(\mathfrak{g})/\operatorname{Inn}(\mathfrak{g})$ of conjugacy classes of Euler elements to the isomorphism classes of irreducible non-compactly causal symmetric Lie algebras with maximal $\operatorname{Inn}(\mathfrak{h})$-invariant cone.

Theorem 2.42. ([MNO24, Cor. 6.3]) *For an irreducible non-compactly causal symmetric space $M = G/H$ there exists a unique conjugacy class of Euler elements $\mathcal{O}_h \subseteq \mathfrak{g}$ for which $W_M^+(h) \neq \emptyset$. In particular $W_M^+(-h) = \emptyset$ if h is not symmetric.*

Let us assume for simplicity that $M = G/H$ is minimal, i.e., that all other causal symmetric spaces with the same triple $(\mathfrak{g}, \tau, C)$ are coverings of M (this is M_{ad} in the notation of Appendix 2.8.5). In addition, we assume that the causal structure is maximal, i.e., that $C \subseteq \mathfrak{q}$ is a maximal proper $\operatorname{Inn}(\mathfrak{h})$-invariant convex cone in $\mathfrak{q}$. We choose a Cartan involution θ commuting with τ. Let $\mathfrak{q}_\mathfrak{k} = \mathfrak{q} \cap \mathfrak{k}$ for a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with $h \in \mathfrak{q}_\mathfrak{p}$ and consider the domain

$$\Omega_{\mathfrak{q}_\mathfrak{k}} = \Big\{x \in \mathfrak{q}_\mathfrak{k} \colon r_{\mathrm{Spec}}(\mathrm{ad}\,x) < \frac{\pi}{2}\Big\},$$

where $r_{\mathrm{Spec}}(\mathrm{ad}\,x)$ is the spectral radius of $\mathrm{ad}\,x$. Then the connected component $W := W_M^+(h)_{eH}$ of the base point eH in the positivity domain $W_M^+(h)$ is the region

$$W = G_e^h \exp(\Omega_{\mathfrak{q}_\mathfrak{k}}).eH \tag{2.39}$$

([MNO24, Thm. 3.6] and (2.55) in Appendix 2.8.5). The semigroup S_W actually is a group, as we shall see in Theorem 4.35 below.

2.7.4 Non-triviality of wedge regions

Wedge regions have been studied in detail for compactly and non-compactly causal symmetric spaces in [NÓ23a] and [NÓ23b, MNO24], respectively. For causal flag manifolds, we refer to [MN25], [Ne25] and Section 2.6. The case of general Lie groups is still poorly understood; but see [BN25] and [Oeh22, Oeh23]. We shall return to this topic below.

Problem 2.43. Let $h \in \mathfrak{g}$ be an Euler element and $M = G/H$ a causal homogeneous space.

(a) How can we determine effectively if $W_M^+(h) \neq \emptyset$? A sufficient condition is given in Proposition 2.15.

(b) If $-h = \mathrm{Ad}(g)h$ for some $g \in G$, then $W_M^+(-h) = g.W_M^+(h)$ is nonempty if $W_M^+(h) \neq \emptyset$. The converse is not true by Example 2.51, where $W_M^+(\pm h) \neq \emptyset$ but h is not symmetric. However, for irreducible non-compactly causal symmetric spaces it is true (Theorem 2.42). Is there a natural characterization of those cases where $W_M^+(\pm h) \neq \emptyset$?

(c) How are these conditions related to the existence of fixed points of the vector field X_h^M, i.e., to $\mathcal{O}_h \cap \mathfrak{h} \neq \emptyset$?

Example 2.44. In this context, the Euler element $h_1 \in \mathfrak{sl}_3(\mathbb{R})$ (cf. Example 2.9) is instructive. It is not symmetric; note that $-h_1 \in \mathcal{O}_{h_2} \neq \mathcal{O}_{h_1}$. The corresponding non-compactly causal symmetric space is

$$M = G.I_{1,2} = \{gI_{1,2}g^\top : g \in \mathrm{SL}_3(\mathbb{R})\} \subseteq \mathrm{Sym}_3(\mathbb{R}), \qquad I_{1,2} = \mathrm{diag}(1,-1,-1).$$

Then $I_{1,2} \in W_M^+(h_1) \neq \emptyset$, but $W_M^+(-h_1) = \emptyset$ and the vector field

$$X_{h_1}^M(x) = h_1 x + x h_1$$

has no zeros on $M \subseteq \mathrm{Sym}_3(\mathbb{R})$. In fact, if $X_{h_1}^M(x) = 0$, then x anticommutes with h_1. If $v \in \mathbb{R}^3$ is an h_1-eigenvector with $h_1 v = \lambda v$, it follows that $h_1 x v = -\lambda x v$; contradicting the fact that the eigenvalues of h_1 are $\frac{2}{3}$ and $-\frac{1}{3}$. We refer to [Ne25, Prop. 5.7] for a detailed discussion of this class of spaces and their modular flows.

2.8 Appendices to Section 2

2.8.1 Euler elements in simple Lie algebras

In this appendix, we present a classification of Euler elements in simple real Lie algebras, following [MN21]. As they correspond to 3-gradings, it can also be derived from [KA88]. We also reproduce the list of the 18 types from [Kan98, p. 600] and Kaneyuki's lecture notes [Kan00].

Let $\mathfrak{g}$ is a real semisimple Lie algebra. An involutive automorphism $\theta \in \mathrm{Aut}(\mathfrak{g})$ is called a *Cartan involution* if its eigenspaces

$$\mathfrak{k} := \mathfrak{g}^\theta = \{x \in \mathfrak{g} : \theta(x) = x\} \quad \text{and} \quad \mathfrak{p} := \mathfrak{g}^{-\theta} = \{x \in \mathfrak{g} : \theta(x) = -x\}$$

have the property that they are orthogonal with respect to the Cartan–Killing form $\kappa(x,y) = \mathrm{tr}(\mathrm{ad}\, x\, \mathrm{ad}\, y)$, which is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$. Then

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \tag{2.40}$$

is called a *Cartan decomposition.* Cartan involutions always exist and two such involutions are conjugate under the group $\mathrm{Inn}(\mathfrak{g})$ of inner automorphism, so they produce isomorphic decompositions ([HN12, Thm. 13.2.11]). The subalgebra $\mathfrak{k}$ is a maximal compactly embedded. An element $x \in \mathfrak{g}$ is elliptic if and only if its adjoint orbit $\mathcal{O}_x = \mathrm{Inn}(\mathfrak{g})x$ intersects $\mathfrak{k}$, and $x \in \mathfrak{g}$ is hyperbolic if and only if $\mathcal{O}_x \cap \mathfrak{p} \neq \emptyset$.

For the finer structure theory, we start with a Cartan involution θ and fix a maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$. As $\mathfrak{a}$ is abelian, $\mathrm{ad}\,\mathfrak{a}$ is a commuting set of diagonalizable operators, hence simultaneously diagonalizable. For a linear functional $0 \neq \alpha \in \mathfrak{a}^*$, the simultaneous eigenspaces

$$\mathfrak{g}_\alpha := \mathfrak{g}_\alpha(\mathfrak{a}) := \{y \in \mathfrak{g} : (\forall x \in \mathfrak{a})\ [x,y] = \alpha(x)y\}$$

are called *root spaces* and

$$\Sigma := \Sigma(\mathfrak{g}, \mathfrak{a}) := \{\alpha \in \mathfrak{a}^* \setminus \{0\} : \mathfrak{g}_\alpha \neq 0\}$$

is called the set of *restricted roots*. We pick a set

$$\Pi := \{\alpha_1, \dots, \alpha_n\} \subseteq \Sigma$$

of *simple roots*. This is a subset with the property that every root $\alpha \in \Sigma$ is a linear combination $\alpha = \sum_{j=1}^n n_j \alpha_j$, where the coefficients are either all in $\mathbb{Z}_{\geq 0}$ or in $\mathbb{Z}_{\leq 0}$. The convex cone

$$\Pi^\star := \{x \in \mathfrak{a} : (\forall \alpha \in \Pi)\ \alpha(x) \geq 0\}$$

is called the *closed positive (Weyl) chamber corresponding to* Π.

We have the *root space decomposition*

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a}, \quad \text{where} \quad \mathfrak{m} = \mathfrak{g}_0 \cap \mathfrak{k}.$$

Now $\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$, and for a non-zero element $x_\alpha \in \mathfrak{g}_\alpha$, the 3-dimensional subspace spanned by $x_\alpha, \theta(x_\alpha)$ and $[x_\alpha, \theta(x_\alpha)] \in \mathfrak{a}$ is a Lie subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. In particular, it contains a unique element $\alpha^\vee \in \mathfrak{a}$ with $\alpha(\alpha^\vee) = 2$. Then

$$r_\alpha : \mathfrak{a} \to \mathfrak{a}, \quad r_\alpha(x) := x - \alpha(x)\alpha^\vee$$

is a reflection, and the subgroup

$$\mathcal{W} := \langle r_\alpha : \alpha \in \Sigma \rangle \subseteq \mathrm{GL}(\mathfrak{a})$$

is called the *Weyl group*. Its action on $\mathfrak{a}$ provides a good description of the adjoint orbits of hyperbolic elements: Every hyperbolic element in $\mathfrak{g}$ is conjugate to a unique element in $\Pi^\star \subseteq \mathfrak{a}$, a fundamental domain for the G-action on the subset of hyperbolic elements in $\mathfrak{g}$. For $x \in \mathfrak{a}$, the intersection $\mathcal{O}_x \cap \mathfrak{a} = \mathcal{W}x$ is the Weyl group orbit ([KN96, Thm. III.10]).

From now on we assume that $\mathfrak{g}$ is simple. Then Σ is an irreducible root system, hence of one of the following types:

$$A_n, \qquad B_n, \qquad C_n, \qquad D_n, \qquad E_6, E_7, E_8, \quad F_4, \quad G_2 \quad \text{or} \quad BC_n, n \geq 1$$

(cf. [Bo90a]).

The adjoint orbit of an Euler element in $\mathfrak{g}$ contains a unique $h \in \Pi^\star$. For any Euler element $h \in \Pi^\star$, we have $\alpha(h) \in \{0, 1\}$ for $\alpha \in \Pi$ because the values of the roots on h are the eigenvalues of $\operatorname{ad} h$. If such an element exists, then the irreducible root system Σ must be reduced. Otherwise, for any root α with $2\alpha \in \Sigma$, we must have $\alpha(h) = 0$ because $\operatorname{ad} x$ has only three eigenvalues. As the set of such roots generates the same linear space as Σ, this leads to $h = 0$. This excludes the non-reduced simple root systems of type BC_n.

To see how many possibilities we have for Euler elements in $\mathfrak{a}$, we recall that Π is a linear basis of $\mathfrak{a}$, so that, for each $j \in \{1, \dots, n\}$, there exists a uniquely determined element

$$h_j \in \mathfrak{a}, \quad \text{satisfying} \quad \alpha_k(h_j) = \delta_{kj}. \tag{2.41}$$

The following theorem lists for each irreducible root system Σ the possible Euler elements in the positive chamber $\Pi^\star$. Since every adjoint orbit in $\mathcal{E}(\mathfrak{g})$ has a unique representative in $\Pi^\star$, this classifies the $\mathrm{Inn}(\mathfrak{g})$-orbits in $\mathcal{E}(\mathfrak{g})$ for any non-compact simple real Lie algebra. For **semisimple** Lie algebras $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$, an element $x = (x_1, \ldots, x_n)$ is an Euler element if and only if its components $x_j \in \mathfrak{g}_j$ are Euler elements, and its orbit is

$$\mathcal{O}_x = \mathcal{O}_{x_1} \times \cdots \times \mathcal{O}_{x_k}.$$

Therefore it suffices to consider simple Lie algebras, and for these the root system Σ is irreducible. As every complex simple Lie algebra $\mathfrak{g}$ is also a real simple Lie algebra, our discussion also covers complex Lie algebras.

Theorem 2.45. *Suppose that $\mathfrak{g}$ is a non-compact simple real Lie algebra, with restricted root system $\Sigma \subseteq \mathfrak{a}^*$ of type X_n. We follow the conventions of the tables in* [Bo90a] *for the classification of irreducible root systems and the enumeration of the simple roots $\alpha_1, \ldots, \alpha_n$. Then every Euler element $h \in \mathfrak{a}$ on which Π is non-negative is one of $h_1, \ldots, h_n$, and for every irreducible root system, the Euler elements among the h_j are the following:*

$$A_n : h_1, \ldots, h_n, \quad B_n : h_1, \quad C_n : h_n, \quad D_n : h_1, h_{n-1}, h_n, \quad E_6 : h_1, h_6, \quad E_7 : h_7. \tag{2.42}$$

For the root systems BC_n, E_8, F_4 and G_2 no Euler element exists (they have no 3-grading). The Euler elements with are symmetric in the sense that $-h \in \mathcal{O}_h = \mathrm{Inn}(\mathfrak{g})h$, are

$$A_{2n-1} : h_n, \quad B_n : h_1, \quad C_n : h_n, \quad D_n : h_1, \quad D_{2n} : h_{2n-1}, h_{2n}, \quad E_7 : h_7. \tag{2.43}$$

Proof. Writing the highest root in Σ with respect to the simple system Π as $\alpha_{\max} = \sum_{j=1}^n c_j \alpha_j$, we have $c_j \in \mathbb{Z}_{>0}$ for each j. If $h \in \Pi^\star$ is an Euler element, then $\Pi(h) \subseteq \{0, 1\}$, and $1 = \alpha_{\max}(h) = \sum_{j=1}^n c_j \alpha_j(h)$ implies that at most one value $\alpha_j(h)$ can be 1, and then the others are 0, i.e., $h = h_j$ for some $j \in \{1, \ldots, n\}$. Conversely, h_j is an Euler element if and only if $c_j = 1$. Consulting the tables on the irreducible root systems in [Bo90a], we obtain the Euler elements listed in (2.42).

To determine the symmetric ones, let $w_0 \in \mathcal{W}$ be the longest element of the Weyl group, which is uniquely determined by $w_0^*\Pi = -\Pi$ for the dual action of $\mathcal{W}$ on $\mathfrak{a}^*$. Then $h_j' := w_0(-h_j)$ is the Euler element in the positive chamber representing the orbit $\mathcal{O}_{-h_j}$. Therefore h_j is symmetric if and only if $-h_j \in \mathcal{W}h_j$, which is equivalent to $h_j' = h_j$. Using the description of w_0 and the root systems in [Bo90a], now leads to

$$A_{n-1} : h_j' = h_{n-j}, \quad B_n : h_1' = h_1, \quad C_n : h_n' = h_n, \tag{2.44}$$

$$D_n : h_1' = h_1, h_n' = \begin{cases} h_{n-1} & \text{for } n \text{ odd,} \\ h_n & \text{for } n \text{ even,} \end{cases} \tag{2.45}$$

$$E_6 : h_1' = h_6, \quad E_7 : h_7' = h_7. \tag{2.46}$$

Hence the symmetric Euler elements are those listed in (2.43). $\square$

There are many types of simple 3-graded Lie algebras that are neither complex nor hermitian of tube type; for instance the Lorentzian algebras $\mathfrak{so}_{1,n}(\mathbb{R})$. We refer to [Kan98, p. 600] or [Kan00]. for the list of all 18 types which is reproduced below in a different order. We identify $\mathfrak{so}^*(4n)$ with the Lie algebra $\mathfrak{u}_{2r}(\mathbb{H}, \Omega)$ of the isometry group of the non-degenerate skew-hermitian form on $\mathbb{H}^{2r}$ defined by the matrix $\Omega = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$.

	$\mathfrak{g}$	$\Sigma(\mathfrak{g},\mathfrak{a})$	h	$\mathfrak{g}_1(h)$
	Complex Lie algebras			
1	$\mathfrak{sl}_n(\mathbb{C})$	A_{n-1}	$h_j, 1 \leq j \leq n-1$	$M_{j,n-j}(\mathbb{C})$
2	$\mathfrak{sp}_{2n}(\mathbb{C})$	C_n	h_n	$\mathrm{Sym}_n(\mathbb{C})$
3a	$\mathfrak{so}_{2n+1}(\mathbb{C})$	B_n	h_1	$\mathbb{C}^{2n-1}$
3b	$\mathfrak{so}_{2n}(\mathbb{C})$	D_n	h_1	$\mathbb{C}^{2n-2}$
4	$\mathfrak{so}_{2n}(\mathbb{C})$	D_n	h_{n-1}, h_n	$\mathrm{Alt}_n(\mathbb{C})$
5	$\mathfrak{e}_6(\mathbb{C})$	E_6	$h_1 = h_6'$	$M_{1,2}(\mathbb{O})_{\mathbb{C}}$
6	$\mathfrak{e}_7(\mathbb{C})$	E_7	h_7	$\mathrm{Herm}_3(\mathbb{O})_{\mathbb{C}}$
	Hermitian Lie algebras			
7	$\mathfrak{su}_{n,n}(\mathbb{C})$	C_n	h_n	$\mathrm{Herm}_n(\mathbb{C})$
8	$\mathfrak{sp}_{2n}(\mathbb{R})$	C_n	h_n	$\mathrm{Sym}_n(\mathbb{R})$
9a	$\mathfrak{so}_{2,d}(\mathbb{R})$	C_2 $(2 < d)$	h_1	$\mathbb{R}^{1,d-1}$
10	$\mathfrak{so}^*(4n) \cong \mathfrak{u}_{2r}(\mathbb{H},\Omega)$	C_n	h_n	$\mathrm{Herm}_n(\mathbb{H})$
11	$\mathfrak{e}_{7(-25)}$	C_3	h_3	$\mathrm{Herm}_3(\mathbb{O})$
	Non-hermitian split forms			
12	$\mathfrak{sl}_n(\mathbb{R})$	A_{n-1}	$h_j, 1 \leq j \leq n-1$	$M_{j,n-j}(\mathbb{R})$
9b	$\mathfrak{so}_{n,n+1}(\mathbb{R})$	B_n	h_1	$\mathbb{R}^{2n-1}$
13	$\mathfrak{so}_{n,n}(\mathbb{R})$	D_n	h_{n-1}, h_n	$\mathrm{Alt}_n(\mathbb{R})$
14	$\mathfrak{e}_6(\mathbb{R})$	E_6	$h_1 = h_6'$	$M_{1,2}(\mathbb{O}_{\mathrm{split}})$
15	$\mathfrak{e}_7(\mathbb{R})$	E_7	h_7	$\mathrm{Herm}_3(\mathbb{O}_{\mathrm{split}})$
	Non-hermitian non-split forms			
16	$\mathfrak{sl}_n(\mathbb{H})$	A_{n-1}	$h_j, 1 \leq j \leq n-1$	$M_{j,n-j}(\mathbb{H})$
17	$\mathfrak{u}_{n,n}(\mathbb{H})$	C_n	h_n	$\mathrm{Aherm}_n(\mathbb{H})$
9c	$\mathfrak{so}_{p,q}(\mathbb{R}), 2 \neq p \neq q-1$	B_p $(p < q)$	h_1	$\mathbb{R}^{p+q-2}$
		D_p $(p = q)$		
18	$\mathfrak{e}_{6(-26)}$	A_2	h_1	$M_{1,2}(\mathbb{O})$

Table 2: Simple 3-graded Lie algebras

In our context, hermitian simple Lie algebras are of particular interest. We therefore collect some of their main properties in the following proposition.

Proposition 2.46. *For a simple real Lie algebra, the following assertions hold:*

(a) $\mathfrak{g}$ *is hermitian if and only if there exists a closed convex* $\mathrm{Inn}(\mathfrak{g})$*-invariant cone* $C_{\mathfrak{g}} \neq \{0\}, \mathfrak{g}$.

(b) *A simple hermitian Lie algebra contains an Euler element if and only if it is of tube type, and in this case* $\mathrm{Inn}(\mathfrak{g})$ *acts transitively on* $\mathcal{E}(\mathfrak{g})$.

Proof. (a) is a consequence of the Kostant–Vinberg Theorem (cf. [HÓ97, Lemma 2.5.1]).
(b) Since the restricted root system of a hermitian simple Lie algebra is of type C_r or BC_r (see [MNO23, §3.1] or Table 3 below), and the first case characterizes the algebras of tube type, the assertion follows from Theorem 2.45 because the root system C_r only permits one class of Euler elements. □

Remark 2.47. (a) As $h \in \mathfrak{a}$ implies $\theta(h) = -h$, the Cartan involution θ always maps h into $-h$, but this only implies that h is symmetric if $\theta \in \mathrm{Inn}(\mathfrak{g})$. This is the case if $\mathfrak{g}$ is hermitian, so that in these Lie algebras all Euler elements are symmetric (cf. Proposition 2.46).
(b) (Making Euler elements symmetric) If the Euler element $h \in \mathfrak{g}$ is not symmetric, we could still "make it symmetric" by doubling: In $\mathfrak{g}^{\oplus 2}$, the Euler element $h_d := (h, -h)$ has the property that

the flip involution $\tau_{\rm flip}(x,y) = (y,x)$ satisfies $\tau_{\rm flip}(h_d) = -h_d$. So we have at least $-h \in \mathrm{Aut}(\mathfrak{g}_d)h$ but not in $\mathrm{Inn}(\mathfrak{g})h$.

The classification of Euler elements requires some interpretation. So let us first see what it says about complex simple Lie algebras $\mathfrak{g}$. In (2.42) we see that, only if $\mathfrak{g}$ is not of type E_8, F_4 or G_2, the Lie algebra $\mathfrak{g}$ contains an Euler element. Euler elements correspond to 3-gradings of the root system and these in turn to hermitian real forms $\mathfrak{g}^\circ$, where $ih_j \in \mathfrak{z}(\mathfrak{k}^\circ)$ generates the center of a maximal compactly embedded subalgebra $\mathfrak{k}^\circ$ ([Ne99, Thm. A.V.1]). We thus obtain the following possibilities. In Table 3, we write $\mathfrak{g}^\circ$ for the hermitian real form, $\mathfrak{g}$ for the complex Lie algebra, Σ for its restricted root system, and h_j for the corresponding Euler element. These correspond to the cases (13)-(18) in Table 2:

$\mathfrak{g}^\circ$ (hermitian)	$\Sigma(\mathfrak{g}^\circ, \mathfrak{a}^\circ)$	$\mathfrak{g} = (\mathfrak{g}^\circ)_{\mathbb{C}}$	$\Sigma(\mathfrak{g}, \mathfrak{a})$	Euler element
$\mathfrak{su}_{p,q}(\mathbb{C}), 1 \leq p \leq q$	$BC_p (p < q)$, $C_p (p = q)$	$\mathfrak{sl}_{p+q}(\mathbb{C})$	A_{p+q-1}	h_p
$\mathfrak{so}_{2,d}(\mathbb{R}), d > 2$	C_2	$\mathfrak{so}_{2+d}(\mathbb{C})$	$B_{\frac{d+1}{2}}$, d odd	h_1
			$D_{1+\frac{d}{2}}$, d even	
$\mathfrak{sp}_{2n}(\mathbb{R})$	C_n	$\mathfrak{sp}_{2n}(\mathbb{C})$	C_n	h_n
$\mathfrak{so}^*(2n)$	$BC_{\lfloor \frac{n}{2} \rfloor} (n \text{ odd})$, $C_{\frac{n}{2}} (n \text{ even})$	$\mathfrak{so}_{2n}(\mathbb{C})$	D_n	h_{n-1}, h_n
$\mathfrak{e}_{6(-14)}$	BC_2	$\mathfrak{e}_6$	E_6	$h_1 = h_6'$
$\mathfrak{e}_{7(-25)}$	C_3	$\mathfrak{e}_7$	E_7	h_7

Table 3: Simple hermitian Lie algebras $\mathfrak{g}^\circ$ ($\mathfrak{g}$ as in (1)-(6) in Table 2)

Note that $\mathfrak{sl}_2(\mathbb{R}) \cong \mathfrak{so}_{2,1}(\mathbb{R}) \cong \mathfrak{su}_{1,1}(\mathbb{C})$. More exceptional isomorphisms are discussed in some detail in [HN12, §17].

In this correspondence, those hermitian simple Lie algebras corresponding to symmetric Euler elements are of particular interest. Comparing with the list of hermitian simple Lie algebras of tube type (cf. [FK94, p. 213]), we see that they correspond precisely to 3-gradings specified by symmetric Euler elements, as listed in (2.43). Since the Euler elements h_{n-1} and h_n for the root system of type D_n are conjugate under a diagram automorphism, they correspond to isomorphic hermitian real forms.

$\mathfrak{g}^\circ$ (hermitian)	$\Sigma(\mathfrak{g}^\circ, \mathfrak{a}^\circ)$	$\mathfrak{g} = (\mathfrak{g}^\circ)_{\mathbb{C}}$	$\Sigma(\mathfrak{g}, \mathfrak{a})$	symm. Euler element h
$\mathfrak{su}_{n,n}(\mathbb{C})$	C_n	$\mathfrak{sl}_{2n}(\mathbb{C})$	A_{2n-1}	h_n
$\mathfrak{so}_{2,d}(\mathbb{R}), d > 2$	C_2	$\mathfrak{so}_{2+d}(\mathbb{C})$	$B_{\frac{d+1}{2}}$, d odd	h_1
			$D_{1+\frac{d}{2}}$, d even	
$\mathfrak{sp}_{2n}(\mathbb{R})$	C_n	$\mathfrak{sp}_{2n}(\mathbb{C})$	C_n	h_n
$\mathfrak{so}^*(4n)$	C_n	$\mathfrak{so}_{4n}(\mathbb{C})$	D_{2n}	h_{2n-1}, h_{2n}
$\mathfrak{e}_{7(-25)}$	C_3	$\mathfrak{e}_7$	E_7	h_7

Table 4: Simple hermitian Lie algebras $\mathfrak{g}^\circ$ of tube type ((7)-(11) in Table 2)

2.8.2 Conjugacy classes of Euler elements in general Lie algebras

Remark 2.48. To understand Euler elements in general Lie algebra, it is instructive to consider abelian subalgebras $\mathfrak{a} \subseteq \mathfrak{g}$ which are maximal with respect to the property that $\mathrm{ad}\, \mathfrak{a}$ is diagonalizable. It follows from [KN96, Thm. III.3], applied to the symmetric Lie algebra $(\mathfrak{g}^{\oplus 2}, \tau_{\rm flip})$ that they are conjugate under $\mathrm{Inn}(\mathfrak{g})$. Moreover, there always exists an $\mathrm{ad}\, \mathfrak{a}$-invariant Levi complement $\mathfrak{s}$ ([KN96, Prop. I.2]), so that

$$\mathfrak{a} = \mathfrak{a}_{\mathfrak{r}} \oplus \mathfrak{a}_{\mathfrak{s}} \quad \text{for} \quad \mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}.$$

Then $[\mathfrak{a}_\mathfrak{r}, \mathfrak{s}] \subseteq \mathfrak{r} \cap \mathfrak{s} = \{0\}$. As $\mathfrak{g}$ is a nilpotent module of the ideal $[\mathfrak{g}, \mathfrak{r}]$, it further follows that

$$\mathfrak{a} \cap [\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{z}(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}],$$

so that

$$\mathfrak{a} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{a}_\mathfrak{r}^c \oplus \mathfrak{a}_\mathfrak{s}, \tag{2.47}$$

where $\mathfrak{a}_\mathfrak{r}^c \subseteq \mathfrak{a}_\mathfrak{r}$ is a complement of $\mathfrak{z}(\mathfrak{g})$.

Lemma 2.49. *For an Euler element $h \in \mathfrak{g}$, the following assertions hold:*

(a) *If $\mathfrak{g}$ is the Lie algebra of an algebraic group G with Levi decomposition $U \rtimes L$ (U unipotent and L reductive), then $\mathcal{O}_h$ intersects the Lie subalgebra $\mathfrak{z}(\mathfrak{g}) + \mathfrak{l}$, where $\mathfrak{l} = \mathbf{L}(L)$.*

(b) *If $h \in [\mathfrak{g}, \mathfrak{g}]$ is an Euler element contained in the commutator algebra, then $\mathcal{O}_h + \mathfrak{z}(\mathfrak{g})$ intersects every Levi complement.*

Proof. (a) Suppose that $\mathfrak{g} = \mathbf{L}(G)$ for a linear algebraic group G with Levi decomposition $G = U \rtimes L$, and the corresponding decomposition $\mathfrak{g} = \mathfrak{u} \rtimes \mathfrak{l}$. Then we have the Levi decomposition $\mathrm{Ad}(G) \cong \mathrm{Ad}(U) \rtimes \mathrm{Ad}(L)$ and $\exp(\mathbb{R}\,\mathrm{ad}\, h)$ is contained in a reductive subgroup of the Zariski closure of $\mathrm{Ad}(G)$, hence conjugate to a subgroup of $\mathrm{Ad}(L)$ ([Ho81, Prop. VIII.3.1]). This means that $\mathcal{O}_h + \mathfrak{z}(\mathfrak{g})$ intersects $\mathfrak{l}$.
(b) According to Remark 2.48, we may assume that $h \in \mathfrak{a}$, where $\mathfrak{a}$ is adapted to a Levi decomposition. Then (2.47) implies that

$$\mathfrak{a} \cap [\mathfrak{g}, \mathfrak{g}] \subseteq (\mathfrak{a} \cap [\mathfrak{g}, \mathfrak{r}]) + \mathfrak{s} \subseteq \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{s}.$$

Therefore $h \in [\mathfrak{g}, \mathfrak{g}]$ implies that $h \in \mathfrak{z}(\mathfrak{g}) + \mathfrak{s}$. Since all Levi complements are conjugate, (b) follows. ☐

Proposition 2.50. ([MN21, Prop. 3.2]) *The following assertions hold:*

(i) *An Euler element $h \in \mathfrak{g}$ is symmetric, if and only if h is contained in a Levi complement $\mathfrak{s}$ and h is a symmetric Euler element in $\mathfrak{s}$.*

(ii) *Let $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$ be a Levi decomposition.*

  (a) *If $h \in \mathfrak{g}$ is a symmetric Euler element, then $\mathcal{O}_h = \mathrm{Inn}(\mathfrak{g})(\mathcal{O}_h \cap \mathfrak{s}) = \mathcal{O}_{q(h)}$, where $q \colon \mathfrak{g} \to \mathfrak{s}$ is the projection map.*

  (b) *Two symmetric Euler elements are conjugate under $\mathrm{Inn}(\mathfrak{g})$ if and only if their images in $\mathfrak{s}$ are conjugate under $\mathrm{Inn}(\mathfrak{s})$.*

Proof. (i) As $\mathcal{O}_h \subseteq h + [\mathfrak{g}, \mathfrak{g}]$ follows from the invariance of the affine subspace $h + [\mathfrak{g}, \mathfrak{g}]$ under $\mathrm{Inn}(\mathfrak{g})$, the relation $-h \in \mathcal{O}_h$ implies $h \in [\mathfrak{g}, \mathfrak{g}]$. In view of Lemma 2.49(b), there exists a Levi complement $\mathfrak{s}$ with $h \in \mathfrak{z}(\mathfrak{g}) + \mathfrak{s}$. Then $\mathfrak{r}$ and $\mathfrak{s}$ are $\mathrm{ad}\, h$-invariant, so that the $\mathrm{ad}\, h$-eigenspaces of the restrictions satisfy

$$\mathfrak{r} = \mathfrak{r}_1(h) + \mathfrak{r}_0(h) + \mathfrak{r}_{-1}(h) \quad \text{and} \quad \mathfrak{s} = \mathfrak{s}_1(h) + \mathfrak{s}_0(h) + \mathfrak{s}_{-1}(h),$$

and define 3-gradings of $\mathfrak{r}$ and $\mathfrak{s}$. Further $\mathfrak{g}_{\pm 1}(h) \subseteq [h, \mathfrak{g}] \subseteq [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{s} = [\mathfrak{s}, \mathfrak{s}] \subseteq [\mathfrak{g}, \mathfrak{g}]$ imply that $\mathfrak{g} = \mathfrak{r}_0(h) + [\mathfrak{g}, \mathfrak{g}]$. As $[\mathfrak{g}, \mathfrak{g}]$ is an ideal and $\mathfrak{r}_0(h)$ a subalgebra of $\mathfrak{g}$, the subgroup $\mathrm{Inn}_\mathfrak{g}([\mathfrak{g}, \mathfrak{g}])$ of $\mathrm{Inn}(\mathfrak{g})$ is normal, and $\mathrm{Inn}(\mathfrak{g}) = \mathrm{Inn}_\mathfrak{g}([\mathfrak{g}, \mathfrak{g}])\,\mathrm{Inn}(\mathfrak{r}_0(h))$. As $\mathrm{Inn}(\mathfrak{r}_0(h))$ fixes h, this in turn shows that

$\mathcal{O}_h = \mathrm{Inn}_{\mathfrak g}([\mathfrak g,\mathfrak g])h = \mathrm{Inn}_{\mathfrak g}([\mathfrak g,\mathfrak r])\,\mathrm{Inn}_{\mathfrak g}(\mathfrak s)h$. Writing $h = h_z + h_s$ with $h_z \in \mathfrak z(\mathfrak g)$ and $h_s \in \mathcal{E}(\mathfrak s)$, we thus find $x \in [\mathfrak g, \mathfrak r]$ and $s \in \mathrm{Inn}_{\mathfrak g}(\mathfrak s)$ such that [10]

$$-h_z - h_s = -h = e^{\mathrm{ad}\,x} s.h = h_z + e^{\mathrm{ad}\,x} s.h_s.$$

Applying the Lie algebra homomorphism $q\colon \mathfrak g \to \mathfrak s$ to both sides, we derive from $q(h_z) = 0$ and $q \circ e^{\mathrm{ad}\,x} = q$ that $-h_s = s.h_s$, and therefore

$$e^{\mathrm{ad}\,x} h_s = h_s + 2h_z.$$

We conclude that the unipotent linear map $e^{\mathrm{ad}\,x}$ preserves the linear subspace $\mathbb{R}h_s + \mathbb{R}h_z$, and this implies that $\mathrm{ad}\,x = \log(e^{\mathrm{ad}\,x})$ also has this property. We thus arrive at

$$[h, x] = [h_s, x] \subseteq \mathbb{R}h_s + \mathbb{R}h_z \subseteq \mathfrak g_0(h),$$

so that we must have $x \in \mathfrak g_0(h) = \mathfrak g_0(h_s)$, which in turn leads to $0 = e^{\mathrm{ad}\,x} h_s - h_s = 2h_z$, i.e., $h = h_s \in \mathfrak s$.

To prove the second assertion of (i), we observe that the homomorphism $q\colon \mathfrak g \to \mathfrak s \cong \mathfrak g/\mathfrak r$ satisfies

$$q(\mathcal{O}_x) = \mathcal{O}^{\mathfrak s}_{q(x)} \quad \text{for} \quad x \in \mathfrak g. \tag{2.48}$$

Hence $q(\mathcal{E}_{\mathrm{sym}}(\mathfrak g)) \subseteq \mathcal{E}_{\mathrm{sym}}(\mathfrak s)$. If, conversely, $h \in \mathcal{E}_{\mathrm{sym}}(\mathfrak s)$, then we clearly have $-h \in \mathrm{Inn}_{\mathfrak g}(\mathfrak s)h \subseteq \mathrm{Inn}(\mathfrak g)h$, so that $h \in \mathcal{E}_{\mathrm{sym}}(\mathfrak g)$.
(ii)(a) As $\mathcal{O}_h$ intersects $\mathfrak s$ by (i), $q(\mathcal{O}_h) \cap \mathcal{O}_h \neq \emptyset$, and since $\mathrm{Inn}(\mathfrak s)$ acts transitively on $q(\mathcal{O}_h)$ by (2.48), we obtain $q(\mathcal{O}_h) \subseteq \mathcal{O}_h$ and thus $q(\mathcal{O}_h) = \mathcal{O}_h \cap \mathfrak s$. This further leads to

$$\mathcal{O}_h = \mathrm{Inn}(\mathfrak g)(\mathcal{O}_h \cap \mathfrak s) = \mathrm{Inn}(\mathfrak g)q(\mathcal{O}_h) = \mathrm{Inn}(\mathfrak g)\mathcal{O}^{\mathfrak s}_{q(h)} = \mathcal{O}_{q(h)}.$$

(ii)(b) follows immediately from (a). □

Proposition 2.50 reduces for a given Lie algebra $\mathfrak g$ the description of symmetric Euler elements up to conjugation by inner automorphisms to the case of simple Lie algebras.

It would be nice to have a classification of Euler elements in any Lie algebra $\mathfrak g$, but, due to the complexity of Levi decompositions $\mathfrak g = \mathfrak r \rtimes \mathfrak s$, this is not a well-posed problem. If $\mathfrak g$ is reductive, then the classification of Euler elements in $\mathfrak g$ follows immediately from the case of simple Lie algebras, which is described in Theorem 2.45. For symmetric Euler elements h, Proposition 2.50 below largely reduces the classification to the semisimple case, but then one has to describe the module structure of the radical. [11]

Example 2.51. (An example from symplectic geometry) A particularly interesting example which is neither semisimple nor solvable is the Lie algebra

$$\mathfrak g = \mathfrak{hcsp}(V, \omega) := \mathfrak{heis}(V, \omega) \rtimes \mathfrak{csp}(V, \omega),$$

where (V, ω) is a symplectic vector space, $\mathfrak{heis}(V, \omega) = \mathbb{R} \oplus V$ is the corresponding Heisenberg algebra with the bracket $[(z, v), (z', v')] = (\omega(v, v'), 0)$, and

$$\mathfrak{csp}(V, \omega) := \mathfrak{sp}(V, \omega) \oplus \mathbb{R}\,\mathrm{id}_V$$

[10] Here we use that the ideal $[\mathfrak g, \mathfrak r]$ is nilpotent, so that the exponential function of the corresponding group $\mathrm{Inn}_{\mathfrak g}([\mathfrak g, \mathfrak r])$ is surjective, see [HN12].

[11] The role of the symmetry of h for the existence of nets of real subspaces is still not yet well understood. It certainly plays an important role in specifying locality conditions (cf. Section 5.2). If h is not connected, one may be forced to also take non-connected causal manifolds M into consideration, resp., to replace G by a suitable non-connected group.

is the *conformal symplectic Lie algebra* of (V,ω). The hyperplane ideal $\mathfrak{j} := \mathfrak{heis}(V,\omega) \rtimes \mathfrak{sp}(V,\omega)$ (the *Jacobi algebra*) can be identified by the linear isomorphism

$$\varphi\colon \mathfrak{j} \to \mathrm{Pol}_{\leq 2}(V), \qquad \varphi(z,v,x)(\xi) := z + \omega(v,\xi) + \frac{1}{2}\omega(x\xi,\xi), \quad \xi \in V$$

with the Lie algebra of polynomials $\mathrm{Pol}_{\leq 2}(V)$ of degree ≤ 2 on V, endowed with the Poisson bracket ([Ne99, Prop. A.IV.15]). The set

$$C_{\mathfrak{g}} := \{f \in \mathrm{Pol}_{\leq 2}(V)\colon f \geq 0\}$$

is a pointed generating invariant cone in $\mathfrak{j}$. The element $h_0 := \mathrm{id}_V$ defines a derivation on $\mathfrak{j}$ by $(\mathrm{ad}\, h_0)(z,v,x) = (2z,v,0)$ for $z \in \mathbb{R}, v \in V, x \in \mathfrak{sp}(V,\omega)$. Any involution τ_V on V satisfying $\tau_V^*\omega = -\omega$ defines by

$$\widetilde{\tau}_V(z,v,x) := (-z, -\tau_V(v), \tau_V x \tau_V) \tag{2.49}$$

an involution on $\mathfrak{g}$ with $\widetilde{\tau}_V(h_0) = h_0$, and $-\widetilde{\tau}_V(C_{\mathfrak{g}}) = C_{\mathfrak{g}}$ follows from

$$\varphi(\widetilde{\tau}_V(z,v,x)) = -\varphi(z,v,x) \circ \tau_V.$$

Considering $h_{\mathfrak{s}} := \frac{1}{2}\tau_V$ as an element of $\mathfrak{sp}(V,\omega)$, the element

$$h := h_{\mathfrak{s}} + \tfrac{1}{2}\,\mathrm{id}_V \in \mathfrak{csp}(V,\omega)$$

is Euler in $\mathfrak{g}$. Writing $V = V_1 \oplus V_{-1}$ for the τ_V-eigenspace decomposition, we have

$$\mathfrak{g}_{-1} = 0 \oplus 0 \oplus \mathfrak{sp}(V,\omega)_{-1}, \quad \mathfrak{g}_0 = 0 \oplus V_{-1} \oplus \mathfrak{sp}(V,\omega)_0 \cong V_{-1} \rtimes \mathfrak{gl}(V_{-1}), \quad \mathfrak{g}_1 = \mathbb{R} \oplus V_1 \oplus \mathfrak{sp}(V,\omega)_1.$$

Note that

$$\tau_h = e^{\pi i\, \mathrm{ad}\, h} = \widetilde{\tau}_V. \tag{2.50}$$

Here $\mathfrak{g}_1$ can be identified with the space $\mathrm{Pol}_{\leq 2}(V_{-1})$ of polynomials of degree ≤ 2 on V_{-1} and

$$C_+ = C_{\mathfrak{g}} \cap \mathfrak{g}_1 = \{f \in \mathrm{Pol}_{\leq 2}(V_{-1})\colon f \geq 0\}.$$

This cone is invariant under the natural action of the affine group $G_0 \cong \mathrm{Aff}(V_{-1})_0 \cong V_{-1} \rtimes \mathrm{GL}(V_{-1})_0$ whose Lie algebra is $\mathfrak{g}_0$. We also note that

$$\mathfrak{g}_{-1} \cong \mathrm{Pol}_2(V_1) \quad \text{and} \quad C_- = -C_{\mathfrak{g}} \cap \mathfrak{g}_{-1} = \{f \in \mathrm{Pol}_2(V_1)\colon f \leq 0\},$$

so that C_- is also pointed and generating.

Note that h is not symmetric because $\dim \mathfrak{g}_1 \neq \dim \mathfrak{g}_{-1}$.

We also claim that the Lie algebra $\mathfrak{hsp}(V,\omega)$ contains **no Euler element**. In fact, as it is perfect, and $\mathfrak{heis}(V,\omega) \rtimes \mathfrak{sp}(V,\omega)$ is a Levi decomposition, it suffices by Lemma 2.49 to show that no Euler element of $\mathfrak{g}$ is contained in $\mathbb{R} \oplus \{0\} \oplus \mathfrak{sp}(V,\omega)$. Since all Euler elements h in the hermitian Lie algebra $\mathfrak{sp}(V,\omega)$ are conjugate (Proposition 2.46), it suffices to consider $h = h_{\mathfrak{s}} + (\lambda,0,0), \lambda \in \mathbb{R}$. As

$$\mathrm{Spec}(\mathrm{ad}\, h) = \mathrm{Spec}(\mathrm{ad}\, h_{\mathfrak{s}}) = \{\pm 1, \pm \tfrac{1}{2}, 0\},$$

h is not Euler in $\mathfrak{heis}(V,\omega) \rtimes \mathfrak{sp}(V,\omega)$.

2.8.3 Euler elements in small subalgebras

Lemma 2.52. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $h \in \mathcal{E}(\mathfrak{g})$ an Euler element. If h is not contained in the solvable radical* $\mathrm{rad}(\mathfrak{g})$*, then there exists a Lie subalgebra $\mathfrak{b} \subseteq \mathfrak{g}$ containing h such that*

(a) $\mathfrak{b} \cong \mathfrak{sl}_2(\mathbb{R})$ *if and only if h is symmetric, and*

(b) $\mathfrak{b} \cong \mathfrak{gl}_2(\mathbb{R})$ *if h is not symmetric.*

(c) *If h is symmetric, then* $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{b}) \cong \mathrm{PSL}_2(\mathbb{R})$.

(d) *If h is not symmetric and $\mathfrak{g}$ is simple, then* $\mathrm{Inn}_{\mathfrak{g}}([\mathfrak{b},\mathfrak{b}]) \cong \mathrm{SL}_2(\mathbb{R})$.

Proof. (a) If $h \in \mathfrak{b} \cong \mathfrak{sl}_2(\mathbb{R})$, then h is symmetric because all Euler elements in $\mathfrak{sl}_2(\mathbb{R})$ are symmetric by Example 2.8. If, conversely, h is symmetric, then Proposition 2.50 implies that h is contained in a Levi complements $\mathfrak{s}$. Therefore [MN21, Thm. 3.13] implies that h is contained in an $\mathfrak{sl}_2$-subalgebra.
(b) Suppose that h is not symmetric and pick a maximal abelian hyperbolic subspace $\mathfrak{a} \subseteq \mathfrak{g}$ containing h. With [KN96, Prop. I.2] we find an $\mathfrak{a}$-invariant Levi complement $\mathfrak{s} \subseteq \mathfrak{g}$. Then $\mathfrak{a}_{\mathfrak{s}} := \mathfrak{a} \cap \mathfrak{s}$ is maximal hyperbolic in $\mathfrak{s}$ and $\mathfrak{a} = \mathfrak{a}_{\mathfrak{s}} + \mathfrak{z}_{\mathfrak{a}}(\mathfrak{s})$. As h is not contained in $\mathrm{rad}(\mathfrak{g})$, there exists a root $\alpha \in \Delta(\mathfrak{s}, \mathfrak{a})$ with $\alpha(h) = 1$ and root vectors $x_\alpha \in \mathfrak{s}_\alpha$ and $y_\alpha \in \mathfrak{s}_{-\alpha}$ with $h_\alpha := [x_\alpha, y_\alpha] \neq 0$. We stress that $x_\alpha \in \mathfrak{s}_1(h)$. We use that

$$[x_\alpha, y_\alpha] = \kappa(x_\alpha, y_\alpha) a_\alpha,$$

where $a_\alpha \in \mathfrak{a}$ is the unique element with $\alpha(a) = \kappa(a_\alpha, a)$ for all $a \in \mathfrak{a}$, and that the Cartan–Killing form κ induces a dual pairing $\mathfrak{s}_\alpha \times \mathfrak{s}_{-\alpha} \to \mathbb{R}$. Then

$$\mathfrak{b}_\alpha := \mathbb{R}x_\alpha + \mathbb{R}y_\alpha + \mathbb{R}h_\alpha \cong \mathfrak{sl}_2(\mathbb{R})$$

and $[h, \mathfrak{b}_\alpha] \subseteq \mathfrak{b}_\alpha$. Hence $\mathfrak{b} := \mathbb{R}h + \mathfrak{b}_\alpha$ is a Lie subalgebra of $\mathfrak{g}$. As h is not symmetric, $h \notin \mathfrak{b}_\alpha$, and therefore $\mathfrak{b} \cong \mathfrak{gl}_2(\mathbb{R})$.
(c) If h is symmetric and $\mathfrak{b} = [\mathfrak{b}, \mathfrak{b}] \cong \mathfrak{sl}_2(\mathbb{R})$ as in (a), then the fact that $\mathfrak{b}$ contains an Euler element of $\mathfrak{g}$ implies that all simple $\mathfrak{b}$-submodules of $\mathfrak{g}$ are either trivial of isomorphic to the adjoint representation of $\mathfrak{sl}_2(\mathbb{R})$ (consider eigenspaces of $\mathrm{ad}\, h$). This implies that $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{b}) \cong \mathrm{PSL}_2(\mathbb{R})$.
(d) Suppose that $\mathfrak{g}$ is simple. If h is not symmetric, then the Weyl group reflection s_α corresponding to the root α from above satisfies

$$s_\alpha(h) = h - \alpha(h)\alpha^\vee = h - \alpha^\vee.$$

As h is not contained in $\mathbb{R}\alpha^\vee \subseteq \mathfrak{b}_\alpha$, we have $s_\alpha(h) \notin \mathbb{R}h$.

The simplicity of $\mathfrak{g}$ ensures that the root system $\Delta = \Delta(\mathfrak{g}, \mathfrak{a})$ is irreducible and 3-graded by $h \in \mathfrak{a}$. Therefore

$$\Delta_0 := \{\alpha \in \Delta \colon \alpha(h) = 0\}$$

spans a hyperplane in $\mathfrak{a}^*$, which coincides with $h^\perp$, and thus $\mathbb{R}h = \Delta_0^\perp$ by duality. Since $s_\alpha(h)$ is not contained in $\mathbb{R}h$, there exists a $\beta \in \Delta_0$ with $\beta(s_\alpha(h)) \neq 0$. Now $\beta(h) = 0$ implies

$$0 \neq \beta(s_\alpha(h)) = -\beta(\alpha^\vee).$$

As $s_\alpha(h)$ is an Euler element, we obtain $|\beta(\alpha^\vee)| = 1$. Therefore the central element $e^{\pi i\, \mathrm{ad}\, \alpha^\vee}$ of $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{b}_\alpha)$ acts non-trivially, and this implies that $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{b}_\alpha) \cong \mathrm{SL}_2(\mathbb{R})$ because it is a linear Lie group with non-trivial center ([HN12, Ex. 9.5.18]). $\square$

2.8.4 The Brunetti–Guido–Longo (BGL) net

Here we recall a construction we introduced in [MN21] that generalizes the algebraic construction of free fields for AQFT models presented in [BGL02]. We refer to [MN21] for a detailed discussion of this construction; see also Exercise 1.5.

Definition 2.53. For an involutive automorphism σ of G, we write $G_\sigma := G \rtimes \{\mathrm{id}_G, \sigma\}$ for the corresponding group extension.

The set

$$\mathcal{G} := \mathcal{G}(G_\sigma) := \{(h,\tau) \in \mathfrak{g} \times G\sigma \colon \tau^2 = e, \mathrm{Ad}(\tau)h = h\}$$

is called the *abstract wedge space of* G_σ. An element $(h,\tau) \in \mathcal{G}$ is called an *Euler couple* if $h \in \mathcal{E}(\mathfrak{g})$ and

$$\mathrm{Ad}(\tau) = \tau_h. \tag{2.51}$$

Then τ is called an *Euler involution* on G. We write $\mathcal{G}_E \subseteq \mathcal{G}$ for the subset of Euler couples.
(c) Consider the homomorphism $\varepsilon \colon G_\sigma \to \{\pm 1\}$, defined by $\ker \varepsilon = G$. On $\mathfrak{g}$ we consider the *twisted adjoint action* of G_σ which changes the sign on odd group elements:

$$\mathrm{Ad}^\varepsilon \colon G_\sigma \to \mathrm{Aut}(\mathfrak{g}), \qquad \mathrm{Ad}^\varepsilon(g) := \varepsilon(g)\,\mathrm{Ad}(g). \tag{2.52}$$

It extends to an action of G_σ on $\mathcal{G}$ by

$$g.(h,\tau) := (\mathrm{Ad}^\varepsilon(g)h, g\tau g^{-1}). \tag{2.53}$$

(d) (Duality operation) The notion of a "causal complement" is defined on the abstract wedge space as follows: For $W = (h,\tau) \in \mathcal{G}$, we define the *dual wedge* by

$$W' := (-h,\tau) = \tau.W.$$

Note that $(W')' = W$ and $(gW)' = gW'$ for $g \in G$ by (2.53). This relation fits the geometric interpretation in the context of wedge domains in spacetime manifolds.

Definition 2.54. If $(U,\mathcal{H})$ is an antiunitary representation of G_σ, then we obtain a standard subspace $\mathsf{H}_U(W)$, determined for $W = (h,\tau) \in \mathcal{G}$ by the couple of operators (cf. Proposition 1.5):

$$J_{\mathsf{H}_U(W)} = U(\tau) \quad \text{and} \quad \Delta_{\mathsf{H}_U(W)} = e^{2\pi i \cdot \partial U(h)}, \tag{2.54}$$

and thus a G-equivariant map $\mathsf{H}_U \colon \mathcal{G} \to \mathrm{Stand}(\mathcal{H})$ (cf. Exercise 1.11). This is the so-called *Brunetti–Guido–Longo (BGL) net*

$$\mathsf{H}_U^{\mathrm{BGL}} \colon \mathcal{G}(G_\sigma) \to \mathrm{Stand}(\mathcal{H}).$$

2.8.5 Wedge regions in non-compactly causal symmetric spaces

In this appendix, we put some of the results from [MNO24] into the context in which they are used here.

As above, G denotes a connected simple Lie group, $h \in \mathfrak{g}$ is an Euler element, $\tau = \theta\tau_h$ for a Cartan involution θ satisfying $\theta(h) = -h$ and $M = G/H$ is a corresponding non-compactly causal symmetric space, where the causal structure is specified by a maximal $\mathrm{Ad}(H)$-invariant closed convex cone $C \subseteq \mathfrak{q}$ satisfying $h \in C^\circ$ (cf. [MNO23, Thm. 4.21]).

First we consider the "minimal" space associated to the triple $(\mathfrak{g}, \tau, C)$. It is obtained as

$$M_{\mathrm{ad}} := G_{\mathrm{ad}}/H_{\mathrm{ad}},$$

where

$$G_{\mathrm{ad}} := \mathrm{Ad}(G) = \mathrm{Inn}(\mathfrak{g}) \quad \text{and} \quad H_{\mathrm{ad}} := K^h_{\mathrm{ad}} \exp(\mathfrak{h}_{\mathfrak{p}}) \subseteq G^\tau_{\mathrm{ad}}$$

(see [MNO23, Rem. 4.20(b)] for more details). In this space, the positivity domain $W^+_{M_{\mathrm{ad}}}(h)$ is connected by [MNO24, Thm. 7.1]. Further, [MNO24, Thm. 8.2, Prop. 8.3] imply that the positivity domain is connected and given by

$$W^+_{M_{\mathrm{ad}}}(h) = G^h_e \exp(\Omega_{\mathfrak{q}_{\mathfrak{k}}}).eH_{\mathrm{ad}}. \tag{2.55}$$

By [MNO23, Rem. 4.20(a)] (see also Subsection 2.7.3), we have $H = H_K \exp(\mathfrak{h}_{\mathfrak{p}})$ with $H_K \subseteq K^h$, so that $\mathrm{Ad}(H) \subseteq H_{\mathrm{ad}}$. Therefore

$$q\colon M \to M_{\mathrm{ad}}, \quad gH \mapsto \mathrm{Ad}(g)H_{\mathrm{ad}} \in M_{\mathrm{ad}}$$

defines a covering of causal symmetric spaces. The stabilizer in G of the base point in M_{ad} is the subgroup

$$H^\sharp := \mathrm{Ad}^{-1}(H_{\mathrm{ad}}) = K^h \exp(\mathfrak{h}_{\mathfrak{p}})$$

because $Z(G) = \ker(\mathrm{Ad}) \subseteq K^h$. Note that $H^\sharp$ need not be contained in G^τ because τ may act non-trivially on K^h (cf. Remark 2.55). So we may consider M_{ad} as the homogeneous G-space

$$M_{\mathrm{ad}} \cong G/H^\sharp.$$

As q is a G-equivariant covering of causal manifolds,

$$\begin{aligned} W^+_M(h) &= q^{-1}(W^+_{M_{\mathrm{ad}}}(h)) = q^{-1}(G^h_e \exp(\Omega_{\mathfrak{q}_{\mathfrak{k}}}).eH_{\mathrm{ad}}) = G^h_e \exp(\Omega_{\mathfrak{q}_{\mathfrak{k}}})H^\sharp.eH \\ &= G^h_e \exp(\Omega_{\mathfrak{q}_{\mathfrak{k}}})K^h.eH = G^h_e K^h \exp(\Omega_{\mathfrak{q}_{\mathfrak{k}}}).eH = G^h \exp(\Omega_{\mathfrak{q}_{\mathfrak{k}}}).eH, \end{aligned}$$

and the inverse image under the map $q_M\colon G \to G/H = M$ is therefore given by

$$\begin{aligned} q_M^{-1}(W^+_M(h)) &= G^h \exp(\Omega_{\mathfrak{q}_{\mathfrak{k}}})H^\sharp = G^h \exp(\Omega_{\mathfrak{q}_{\mathfrak{k}}})K^h \exp(\mathfrak{h}_{\mathfrak{p}}) \\ &= G^h K^h \exp(\Omega_{\mathfrak{q}_{\mathfrak{k}}}) \exp(\mathfrak{h}_{\mathfrak{p}}) = G^h \exp(\Omega_{\mathfrak{q}_{\mathfrak{k}}}) \exp(\mathfrak{h}_{\mathfrak{p}}). \end{aligned}$$

Next we recall from [MNO24, Cor. 8.4] that the map

$$G^h_e \times_{K^h_e} \Omega_{\mathfrak{q}_{\mathfrak{k}}} \to W^+_{M_{\mathrm{ad}}}(h), \quad [g, x] \mapsto g \exp(x) H_{\mathrm{ad}} \tag{2.56}$$

is a diffeomorphism. Therefore $W^+_{M_{\mathrm{ad}}}(h)$ is an affine bundle over the Riemannian symmetric space G^h_e/K^h_e, hence contractible and therefore simply connected. So its inverse image $W^+_M(h)$ in M is a union of open connected components, all of which are mapped diffeomorphically onto $W^+_{M_{\mathrm{ad}}}(h)$ by q_M, and the group $\pi_0(K^h) \cong K^h/K^h_e$ acts transitively on the set of connected components. It follows in particular that the diffeomorphism (2.56) lifts to a diffeomorphism

$$G^h_e \times_{K^h_e} \Omega_{\mathfrak{q}_{\mathfrak{k}}} \to W := W^+_M(h)_{eH}, \quad [g, x] \mapsto g \exp(x) H. \tag{2.57}$$

Remark 2.55. (The possibilities for H) For $m \in \mathbb{N} \cup \{\infty\}$, let G_m be a connected Lie group with Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ and $|Z(G_m)| = m$. For $m \in \mathbb{N}$ this means that $Z(G_m) \cong \mathbb{Z}/m\mathbb{Z}$ and G_m is an m-fold covering of $\mathrm{Ad}(G_m) \cong \mathrm{PSL}_2(\mathbb{R}) \cong G_1$. Note that $G_2 \cong \mathrm{SL}_2(\mathbb{R})$. Further $G_\infty \cong \widetilde{\mathrm{SL}}_2(\mathbb{R})$ is simply connected with $Z(G_\infty) \cong \mathbb{Z}$.

We consider the Cartan involution $\theta(x) = -x^\top$, the Euler element

$$h = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad z_{\mathfrak{k}} = \frac{1}{2}\begin{pmatrix} 0 & \\ -1 & 0 \end{pmatrix} \in \mathfrak{k} = \mathfrak{so}_2(\mathbb{R}),$$

which satisfies $e^{2\pi z_{\mathfrak{k}}} = -\mathbf{1}$. Then

$$K = \exp(\mathbb{R}z), \quad Z(G_m) = \exp(2\pi\mathbb{Z}z_{\mathfrak{k}}), \quad \text{and} \quad \tau_h(\exp tz_{\mathfrak{k}}) = \tau(\exp tz_{\mathfrak{k}}) = \exp(-tz_{\mathfrak{k}})$$

because $\tau = \theta\tau_h$. We conclude that

$$K^\tau = \{e\} \quad \text{if} \quad m = \infty \quad \text{and} \quad K^\tau = \{e, \exp(m\pi z_{\mathfrak{k}})\} \quad \text{otherwise.}$$

For $m = \infty$, $H = G_m^\tau$ is connected. For $m \in \mathbb{N}$, the group $G_m^\tau = K^\tau \exp(\mathfrak{h})$ has two connected components, but if m is odd, then K^τ does not fix the Euler element $h \in C^\circ$. Therefore only $H := \exp(\mathfrak{h})$ leads to a causal symmetric space G_m/H. If m is even, then H can be either $(G_m)_e^\tau$ or G_m^τ.

In $G_1 \cong \mathrm{PSL}_2(\mathbb{R})$, the subgroup H corresponds to $\mathrm{SO}_{1,1}(\mathbb{R})_e$ and the non-compactly causal symmetric space $G_1/H \cong \mathrm{dS}^2$ is the 2-dimensional de Sitter space.

The universal covering $\widetilde{\mathrm{dS}}^2$ is obtained for $m = \infty$, $G_\infty = \widetilde{\mathrm{SL}}_2(\mathbb{R})$ and then $H = \exp(\mathfrak{h})$ is connected. All other coverings of dS^2 are obtained as G_m/H for $H = \exp(\mathfrak{h})$.

2.8.6 Modular structures on reductive compactly causal symmetric spaces

In [NÓ23a] positivity regions of modular flows have been studied in modular compactly causal symmetric spaces, because the existence of an Euler element in $\mathfrak{g}$ already implies the existence of a modular structure (Corollary 2.57), and this is needed for wedge regions and positivity regions to be defined.

The following observation follows from [Oeh22b, Prop. 3.12].

Proposition 2.56. *Let $\mathfrak{g}$ be simple hermitian, $h \in \mathfrak{g}$ an Euler element, and $\mathrm{V} := \mathfrak{g}_1(h)$ the corresponding euclidean Jordan algebra. For every involutive automorphism $\alpha \in \mathrm{Aut}(\mathrm{V})$, there exists a unique automorphism $\sigma_\alpha \in \mathrm{Aut}(\mathfrak{g})$ with $\sigma_\alpha|_{\mathrm{V}} = \alpha$, and then $(\mathfrak{g}, \tau_h\sigma_\alpha, C_{\mathfrak{g}}^{-\tau_h\sigma_\alpha}, h)$ is modular compactly causal. Conversely, every simple modular compactly causal Lie algebra is of this form.*

Corollary 2.57. *Let $(\mathfrak{g}, \tau, C)$ be simple compactly causal and $h \in \mathfrak{g}$ an Euler element. Then $\mathcal{O}_h \cap \mathfrak{h} \neq \emptyset$.*

Proof. Since $\mathcal{E}(\mathfrak{g}) = \mathcal{O}_h$, the assertion follows from Proposition 2.56, which asserts that τ fixes some Euler element k with $\tau = \tau_k\sigma_\alpha$. □

Proposition 2.58. (Modular structures on reductive compactly causal symmetric Lie algebras) *Let $(\mathfrak{g}, \tau, C)$ be an effective reductive compactly causal symmetric Lie algebra with $C^\circ \cap [\mathfrak{g}, \mathfrak{g}] \neq \emptyset$. If $\mathfrak{g}$ contains a non-central Euler element, then there exist an Euler element $h' \in \mathfrak{q} = \mathfrak{g}^\tau$ and a cone $C' \subseteq C$ such that $(\mathfrak{g}, \tau, C', h')$ is a modular causal symmetric Lie algebra.*

Proof. (cf. [Ne25, Lemma 3.3]) (a) First we use the Extension Theorem [NÓ23a, Thm. 2.4] to find a pointed generating $\mathrm{Inn}(\mathfrak{g})$-invariant cone $C_{\mathfrak{g}}$ in $\mathfrak{g}$ with $-\tau(C_{\mathfrak{g}}) = C_{\mathfrak{g}}$ and $C = C_{\mathfrak{g}} \cap \mathfrak{q}$. It follows in particular that $\mathfrak{g}$ is quasihermitian, i.e., its simple ideals are either compact or hermitian. We write $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}_h \oplus \mathfrak{u}$ with $\mathfrak{u}$ compact semisimple and $\mathfrak{g}_h$ a sum of hermitian simple ideals. Projecting along the compact semisimple ideal $p_{\mathfrak{u}} \colon \mathfrak{g} \to \mathfrak{z}(\mathfrak{g}) + \mathfrak{g}_h$ (the fixed point projection of the compact group $\mathrm{Inn}(\mathfrak{u})$), it follows that

$$C_{\mathfrak{g}}^\circ \cap (\mathfrak{z}(\mathfrak{g}) + \mathfrak{g}_h) = p_{\mathfrak{u}}(C_{\mathfrak{g}}^\circ) \neq \emptyset$$

(cf. Lemma E.1) and likewise

$$C_{\mathfrak{g}}^\circ \cap \mathfrak{g}_h = p_{\mathfrak{u}}(C_{\mathfrak{g}}^\circ \cap [\mathfrak{g}, \mathfrak{g}]) \neq \emptyset.$$

Here we use that our assumption implies that

$$\emptyset \neq C^\circ \cap [\mathfrak{g}, \mathfrak{g}] = C_{\mathfrak{g}}^\circ \cap \mathfrak{q} \cap [\mathfrak{g}, \mathfrak{g}]. \tag{2.58}$$

(b) Let $h_1 \in \mathfrak{g}$ be an Euler element. Then the ideal $\mathfrak{g}_1 \trianglelefteq \mathfrak{g}$ generated by $[h_1, \mathfrak{g}]$ has trivial center and contains no compact ideal, hence only simple hermitian ones with an Euler element, so that they are of tube type. The τ-invariant ideal $\mathfrak{g}_2 := \mathfrak{g}_1 + \tau(\mathfrak{g}_1)$ also has only simple hermitian tube type ideals. We may thus replace h_1 by an Euler element $h_2 \in [\mathfrak{g}, \mathfrak{g}]$ generating the ideal $\mathfrak{g}_2$.
(c) Let $\mathfrak{j} \trianglelefteq \mathfrak{g}_2$ be a minimal τ-invariant ideal. Then either $\mathfrak{j}$ is simple or a sum of two simple ideals exchanged by τ. In the latter case, $\mathfrak{j} \cong \mathfrak{b} \oplus \mathfrak{b}$ with τ acting by $\tau(a, b) = (b, a)$. Any generating Euler element in $\mathfrak{j}$ has non-zero components, and all these are conjugate under inner automorphisms (Proposition 2.50). So the projection of h_2 to $\mathfrak{j}$ is conjugate to an element of the form $(x, x) \in \mathfrak{j}^\tau$. If $\mathfrak{j}$ is simple, then $\mathfrak{h} = \mathfrak{g}^\tau$ contains an Euler element by Proposition 2.56. Putting these results on minimal invariant ideals together, we see that h_2 is conjugate to an element of $\mathfrak{g}^\tau$, i.e., $\mathfrak{g}^\tau$ contains an Euler element h_3 generating $\mathfrak{g}_2$.
(d) The involution $\tau_3 := \tau_{h_3}$ commutes with τ. Next we observe that $\mathfrak{g}^{-\tau_3} \subseteq \mathfrak{g}_2$ is contained in a sum of hermitian simple ideals. Therefore [NÓ23a, Prop. 2.7(d)] implies that the cones $C_{\mathfrak{g}}^{\min}$ and $C_{\mathfrak{g}}^{\max}$ are $-\tau_3$-invariant and

$$(C_{\mathfrak{g}}^{\max})^{-\tau_3} = (C_{\mathfrak{g}}^{\min})^{-\tau_3} = C_{\mathfrak{g}}^{-\tau_3}.$$

As $\mathfrak{g}_2$ intersects the interior of $C_{\mathfrak{g}}$ by (2.58), and the cone $C_{\mathfrak{g}}^{\min} \subseteq \mathfrak{g}_2$ is generating, it follows with (Lemma E.1) that

$$\emptyset \neq (C \cap \mathfrak{g}_2^{-\tau_3})^\circ = (C_{\mathfrak{g}} \cap \mathfrak{g}_2^{-\tau_3})^\circ = C_{\mathfrak{g}}^\circ \cap \mathfrak{g}_2^{-\tau_3}.$$

Now

$$C' := C \cap (-\tau_3(C)) \subseteq \mathfrak{q}$$

is an $\mathrm{Inn}(\mathfrak{h})$-invariant pointed cone in $\mathfrak{q}$. As it contains $C_{\mathfrak{g}} \cap \mathfrak{g}_2^{-\tau_3} \cap \mathfrak{q} = C \cap \mathfrak{g}_2^{-\tau_3}$, hence interior points of $C_{\mathfrak{g}}$, it has non-trivial interior. Therefore $(\mathfrak{g}, \tau, C', h_3)$ is modular. $\square$

2.9 Exercises for Section 2

Exercise 2.1. Let $h \in \mathfrak{sl}_n(\mathbb{R})$. Show that h is an Euler element if and only if h is diagonalizable with 2 eigenvalues λ, μ satisfying $\lambda - \mu = 1$.

Exercise 2.2. Describe the conjugacy classes of Euler elements in the Lie algebras $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R}), \mathfrak{gl}_n(\mathbb{R})$ and $\mathfrak{so}_{1,n}(\mathbb{R})$ up to conjugation.

Exercise 2.3. The Cayley transform $C \colon \mathbb{R} \to \mathbb{S}^1, C(x) = \frac{i-x}{i+x}$ has a natural interpretation in terms of the stereographic projection. Show that, projecting the point $1 + 2ix$ on the tangent line through $1 \in \mathbb{S}^1 \subseteq \mathbb{C}$ with the center $-1 \in \mathbb{S}^1$ onto the circle yields $C(x)$.

Exercise 2.4. We consider the following linear bijection

$$\varphi\colon \mathbb{R}^3 \to \mathfrak{sl}_2(\mathbb{R}), \quad x = (x_0, x_1, x_2) \mapsto \widetilde{x} := \frac{1}{2}\begin{pmatrix} x_1 & -x_0 - x_2 \\ x_0 - x_2 & -x_1 \end{pmatrix},$$

and

$$\sigma_0 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ \sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Show that

(a) $\varphi^{-1}(X) = (-2\operatorname{Tr}(X\sigma_0), 2\operatorname{Tr}(X\sigma_2), -2\operatorname{Tr}(X\sigma_1))$.

(b) The Lorentz form $x^2 = x_0^2 - x_1^2 - x_2^2$ on $\mathbb{R}^3$ corresponds to the determinant by $x^2 = 4\det\widetilde{x}$. In particular, $x \in \mathrm{dS}^2$ if and only if $\det\widetilde{x} = -\frac{1}{4}$.

(c) Show that

$$\Lambda\colon\ \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SO}_{1,2}(\mathbb{R})_e, \quad \Lambda(g) = \varphi^{-1} \circ \mathrm{Ad}(g) \circ \varphi$$

defines a 2-fold covering with kernel $Z(\mathrm{SL}_2(\mathbb{R})) = \{\pm\mathbf{1}\}$.

(d) The one-parameter groups $\lambda_{\sigma_i}(t) = \exp(\sigma_i t) \in \mathrm{SL}_2(\mathbb{R}), i = 1, 2$, are lifts of Lorentz boosts and $r(\theta) = \exp(-\sigma_0\theta)$ is the one-parameter group lifting the space rotations

$$\Lambda(r(\theta)) = R(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}. \tag{2.59}$$

3 Analytic continuation of orbit maps and crown domains

In this section, we turn to constructions of nets for a given antiunitary representation $(U, \mathcal{H})$ of $G_{\tau_h} = G \rtimes \{\mathrm{id}_G, \tau_h\}$. This specifies in particular a standard subspace $\mathtt{V} = \mathtt{V}(h, U)$ by

$$\Delta_{\mathtt{V}} = e^{2\pi i \partial U(h)} \quad \text{and} \quad J_{\mathtt{V}} = U(\tau_h).$$

We shall assume (for simplicity) that $G \subseteq G_{\mathbb{C}}$ and consider domains $\Xi \subseteq G_{\mathbb{C}}$, so that analytic extension to Ξ of orbit maps $U^v\colon G \to \mathcal{H}, g \mapsto U(g)v$ and their boundary values provide real subspaces $\mathtt{E} \subseteq \mathcal{H}^{-\infty}$ of distribution vectors. Then

$$\mathsf{H}^G_{\mathtt{E}}(\mathcal{O}) := \overline{\mathrm{span}_{\mathbb{R}}\{U^{-\infty}(\varphi)\mathtt{E}\colon \varphi \in C^\infty_c(\mathcal{O}, \mathbb{R})\}},$$

leads a net of real subspaces on G satisfying (Iso), (Cov), (RS) and (BW).

The motivation for the introduction of the domain Ξ is that its boundary contains a manifold M, a suitable coset of G, such that we have analytic extensions of orbit maps

$$U^v\colon \Xi \cup M \to \mathcal{H}^{-\infty},$$

and these map a suitable subset $W^M \subseteq M$ (actually $W^M \subseteq M_{\mathrm{KMS}}$ in the sense of Appendix 3.4.4) into $\mathcal{H}^{-\infty}_{\mathrm{KMS}}$. One then obtains nets with $\mathsf{H}^G_{\mathtt{E}}(W^G) = \mathtt{V}$ for the inverse image W^G of W^M under the orbit map.

For semisimple groups, there are canonical candidates for Ξ, obtained from the crown of the Riemannian symmetric space G/K, but for general Lie groups the situation is more complicated and suitable candidates have to be determined by other means (see [BN25] for some very first steps).

3.1 Crown domains in Lie groups

We consider the following setting:

- G is a connected Lie group whose universal complexification $\eta_G\colon G \to G_{\mathbb{C}}$ is **injective.** [12] Then $G_{\mathbb{C}}$ is a complex Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and there exists an antiholomorphic involutive automorphisms σ of $G_{\mathbb{C}}$ with $G \cong (G_{\mathbb{C}}^{\sigma})_e$.
- $h \in \mathfrak{g}$ is an Euler element for which the associated involution $\tau_h^{\mathfrak{g}} = e^{\pi i \operatorname{ad} h}$ of $\mathfrak{g}$ integrates to an involutive automorphism τ_h of G. By the universal property of $G_{\mathbb{C}}$, it extends to a holomorphic involution on $G_{\mathbb{C}}$, denoted τ_h.

We consider the antiholomorphic involution $\overline{\tau}_h := \sigma \circ \tau_h = \tau_h \circ \sigma$ of $G_{\mathbb{C}}$. Then

$$G^c := (G_{\mathbb{C}}^{\overline{\tau}_h})_e = (G_{\mathbb{C}})_e^{\overline{\tau}_h}$$

is a connected subgroup of $G_{\mathbb{C}}$ with Lie algebra

$$\mathfrak{g}^c = \mathfrak{g}_0(h) + i(\mathfrak{g}_1(h) + \mathfrak{g}_{-1}(h)).$$

For

$$\zeta := \exp\Big(-\frac{\pi i}{2}h\Big) \in G_{\mathbb{C}}, \tag{3.1}$$

we have

$$G^c = \zeta^{-1}G\zeta \quad \text{ and } \quad G\zeta = \zeta G^c. \tag{3.2}$$

Remark 3.1. For a connected Lie group G, the natural map $\eta_G\colon G \to G_{\mathbb{C}}$ to its universal complexification can be quite pathological in the sense that $\ker(\eta_G)$ need not be discrete ([HN12]). This is not the case if G is simply connected. Then $\ker(\eta_G)$ is discrete and $G_{\mathbb{C}}$ is the simply connected Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$.

If $\eta_G\colon G \to G_{\mathbb{C}}$ has discrete kernel, then the setting in this section applies to the closed subgroup $\eta_G(G) \subseteq G_{\mathbb{C}}$.

To construct "crown domains" on which G acts freely, a natural idea is to look for a covering manifold $\widehat{\Xi} \to \Xi$.

We consider $\Xi \subseteq G_{\mathbb{C}}$ as an $\eta_G(G)$-principal bundle. There is an obstruction for lifting its structure group $\eta_G(G)$ to its covering group G that is contained in $H^2(\Xi/\eta_G(G), \ker(\eta_G))$ ([NWW13]). We shall return to this issue in Remark 3.19 in Section 3.3 below.

We now present an axiomatic specification of domains in $G_{\mathbb{C}}$ to which orbit maps of J-fixed vectors in antiunitary representations may extend in such a way that boundary values lead to nets of real subspaces on G.

Definition 3.2. A (G,h)-*crown domain in* $G_{\mathbb{C}}$ is an open subset $\Xi \subseteq G_{\mathbb{C}}$ with the following properties:

(Cr1) $G\Xi = \Xi$ and $\overline{\tau}_h(\Xi) = \Xi$, i.e., Ξ is invariant under the action of G_{τ_h} by (anti-)holomorphic maps of $G_{\mathbb{C}}$.

(Cr2) $\exp(\mathcal{S}_{\pm\pi/2}) \subseteq \Xi$ and

(Cr3) $\zeta = \exp\big(-\frac{\pi i}{2}h\big) \in \partial\Xi$.

[12] This assumption is made for convenience and can be overcome with some more technical effort; see Subsection 3.3 for the semisimple case.

(Cr4) There exists a connected open subset $W^G \subseteq M := G\zeta$ (called a *wedge region*) such that

$$\exp(\mathbb{R}h)W^M = W^M \quad \text{and} \quad \exp(\mathcal{S}_\pi h)W^M \subseteq \Xi.$$

Remark 3.3. (a) (Cr4) implies that $W^G := W^M \cdot \zeta^{-1} \subseteq G$ is a domain with $\exp(\mathbb{R}h)W^G = W^G$, and

$$\exp(\mathcal{S}_\pi h)W^G \subseteq \Xi \cdot \zeta^{-1}.$$

(b) Note that $\zeta^{-1}W^M = \exp(\frac{\pi i}{2}h)W^M \subseteq \Xi^{\overline{\tau}_h}$ follows from $\zeta^{-1}W^M \subseteq \zeta^{-1}G\zeta = G^c \subseteq (G_\mathbb{C})^{\overline{\tau}_h}$. As a consequence

$$W^M \subseteq \zeta\Xi^{\overline{\tau}_h}. \tag{3.3}$$

Example 3.4. For $G = \mathbb{R}$ and $h = 1$ (a basis element in $\mathfrak{g} = \mathbb{R}$), the above conditions are only satisfied for the strip

$$\Xi = \mathcal{S}_{\pm\pi/2} \subseteq \mathbb{C} = G_\mathbb{C} \quad \text{and} \quad \overline{\tau}_h(z) = \overline{z}.$$

In this case, $M = W^M = \mathbb{R} - \frac{\pi i}{2}$.

Given a domain $\Xi \subseteq G_\mathbb{C}$ satisfying (Cr1-4), and an antiunitary representation $(U, \mathcal{H})$ of G_{τ_h}, we write

$$\mathcal{H}^\omega(\Xi) \subseteq \mathcal{H}$$

for the subspace of those analytic vectors, whose orbit map extends to Ξ. That the non-triviality of this space imposes serious restrictions on Ξ follows in particular from the discussion in the last section of [BN23], where we have seen in particular that for the group $G = \mathrm{Aff}(\mathbb{R})_e \cong \mathbb{R} \rtimes \mathbb{R}_+$, the domain must be contained in $\mathbb{C} \rtimes \mathbb{C}_r$, where $\mathbb{C}_r$ is the open right half-plane. So one has to understand the boundary behavior of the extended orbit maps on the domain Ξ. Let

$$\mathcal{H}^J_{\mathrm{temp}} \subseteq \mathcal{H}^J = \mathrm{Fix}(J) \quad \text{for} \quad J = U(\tau_h)$$

be the dense real linear subspace of $\mathcal{H}^J$, consisting of those vectors v for which the orbit map $U^v_h(t) = U(\exp th)v$ extends to the open strip

$$\mathcal{S}_{\pm\pi/2} := \{z \in \mathbb{C} \colon |\operatorname{Im} z| < \pi/2\} \subseteq \mathbb{C} \tag{3.4}$$

and the limit

$$\beta^+(v) := \lim_{t \to \pi/2} U^v_h(-it) \tag{3.5}$$

exists in the subspace $\mathcal{H}^{-\infty}(U_h) \subseteq \mathcal{H}^{-\infty}$ of distribution vectors of the one-parameter group U_h (see Appendix C, Theorem 3.32). [13]

These boundary values are actually contained in the space $\mathcal{H}^{-\infty}_{\mathrm{KMS}}$ (see Appendix 3.4.2), consisting of those distribution vectors α whose orbit map $U^{-\infty,\alpha}_h \colon \mathbb{R} \to \mathcal{H}^{-\infty}$ extends analytically to the closed strip $\overline{\mathcal{S}_\pi}$ such that

$$U^{-\infty,\alpha}_h(\pi i) = J\alpha.$$

Using Theorem 3.24, it then follows that smearing with test functions on $\mathbb{R}$ maps α into $\mathtt{V} = \mathtt{V}(h, U)$. Therefore any real linear subspace

$$\mathtt{F} \subseteq \mathcal{H}^\omega(\Xi) \cap \mathcal{H}^J_{\mathrm{temp}}$$

which is G-cyclic in the sense that $U(G)\mathtt{F}$ spans a dense subspace of $\mathcal{H}$, leads to a real subspace

$$\mathtt{E} := \beta^+(\mathtt{F}) \subseteq \mathcal{H}^{-\infty}, \tag{3.6}$$

and from this space we construct a net of real subspaces on G as follows.

[13] The notation $\mathcal{H}_{\mathrm{temp}}$ refers to the "temperedness" of the boundary values, which in the classical context corresponds to tempered distributions.

Definition 3.5. Let $\mathtt{E} \subseteq \mathcal{H}^{-\infty}$ be a real linear subspace. Then, for each $\varphi \in C_c^\infty(G, \mathbb{C})$, the operator

$$U^{-\infty}(\varphi) = \int_G \varphi(g) U^{-\infty}(g)\, dg$$

maps $\mathcal{H}^{-\infty}$ into $\mathcal{H}$, because it is an adjoint of a continuous operator $U(\varphi^*) \colon \mathcal{H} \to \mathcal{H}^\infty$. To an open subset $\mathcal{O} \subseteq G$, we may thus associate the closed real subspace

$$\mathsf{H}^G_{\mathtt{E}}(\mathcal{O}) := \overline{\mathrm{span}_{\mathbb{R}} U^{-\infty}(C_c^\infty(\mathcal{O}, \mathbb{R}))\mathtt{E}}, \tag{3.7}$$

where the closure is taken with respect to the topology of $\mathcal{H}$.

Remark 3.6. It is obvious that the net $\mathsf{H}^G_{\mathtt{E}}$ satisfies (Iso). To see that (Cov) also holds, observe that the left-invariance of the Haar measure dg on G yields

$$U^{-\infty}(g) U^{-\infty}(\varphi) = U^{-\infty}(\delta_g * \varphi),$$

where $(\delta_g * \varphi)(x) = \varphi(g^{-1}x)$ is the left translate of φ.

Remark 3.7. One may also consider subspaces $\mathtt{E} \subseteq \mathcal{H}$, but the key advantage of working with the larger space $\mathcal{H}^{-\infty}$ of distribution vectors is that it contains finite-dimensional subspaces invariant under ad-diagonalizable elements and non-compact subgroups. For finite-dimensional subspaces of $\mathcal{H}$, this is excluded by Moore's Theorem if $\ker U$ is discrete ([Mo80]).

Theorem 3.8. (Construction Theorem for nets of real subspaces) *Let $(U, \mathcal{H})$ be an antiunitary representation of $G_{\tau_h} := G \rtimes \{\mathbf{1}, \tau_h\}$ and*

$$\mathtt{F} \subseteq \mathcal{H}^J_{\mathrm{temp}} \cap \mathcal{H}^\omega(\Xi)$$

be a G-cyclic subspace of $\mathcal{H}$, i.e., $U(G)\mathtt{F}$ is total in $\mathcal{H}$. We consider the linear subspace

$$\mathtt{E} = \beta^+(\mathtt{F}) \subseteq \mathcal{H}^{-\infty}.$$

Then the net $\mathsf{H}^G_{\mathtt{E}}$ on G satisfies (Iso), (Cov), (RS) *and* (BW)*, in the sense that*

$$\mathsf{H}^G_{\mathtt{E}}(W^G) = \mathtt{V} \quad \textit{for} \quad W^G = W^M \zeta^{-1} = \{g \in G \colon g.\zeta \in W^M\},$$

where $\zeta = \exp\big(-\frac{\pi i}{2} h\big)$.

Proof. (Outline) The Reeh–Schlieder property follows from [BN25, Thm. 2.15]. One has to show that, for $\emptyset \neq \mathcal{O}$, we have $\mathsf{H}^G_{\mathtt{E}}(\mathcal{O})^\perp = \{0\}$. This is derived from the fact that, for $\alpha \in \mathtt{E}$, the orbit map $U^{-\infty,\alpha} \colon G \to \mathcal{H}^{-\infty}$ extends to a holomorphic map $\Xi \to \mathcal{H}^{-\infty}$, so that it is determined by its boundary values on every open subset of G.

For (BW), it suffices to show that $U^{-\infty}(W^G)\mathtt{E} \subseteq \mathcal{H}^{-\infty}_{\mathrm{KMS}}$ (Proposition 3.35). Then Proposition 3.25 yields $\mathsf{H}^G_{\mathtt{E}}(W^G) \subseteq \mathtt{V}$, and by (RS), $\mathsf{H}^G_{\mathtt{E}}(W^G)$ is cyclic, so that (Cov) and $\exp(\mathbb{R}h) W^G = W^G$ lead to equality (Proposition 1.20). □

Example 3.9. We consider the 2-dimensional affine group of the real line $G = \mathrm{Aff}(\mathbb{R})_e \cong \mathbb{R} \rtimes \mathbb{R}$ with $\mathfrak{g} = \mathbb{R}x \rtimes \mathbb{R}h$, $[h, x] = x$ and $\tau(b, a) = (-b, a)$. Here a pair $(b, c) \in G$ acts on $\mathbb{R}$ by the affine map

$$(b, c).x = b + e^c x.$$

Then

$$G_{\mathbb{C}} \cong \mathbb{C} \rtimes_\alpha \mathbb{C} \quad \text{with} \quad \alpha_z(w) = e^z w \tag{3.8}$$

acts on $\mathbb{C}$ by the same formula, and $\eta_G \colon G \to G_{\mathbb{C}}$ is the inclusion. The antiholomorphic extension of τ_h to $G_{\mathbb{C}}$, is given by

$$\overline{\tau}_h(b,c) = (-\overline{b}, \overline{c}) \quad \text{with} \quad G_{\mathbb{C}}^{\overline{\tau}_h} = i\mathbb{R} \rtimes \mathbb{R}.$$

(a) First, we consider in $G_{\mathbb{C}}$ the domain

$$\Xi := \mathbb{C} \times \mathcal{S}_{\pm\pi/2} \quad \text{with} \quad \Xi^{\overline{\tau}_h} = i\mathbb{R} \times \mathbb{R} = G^c.$$

For $\zeta = (0, -\pi i/2) = \exp(-\frac{\pi i}{2} h) \in \mathbb{C} \rtimes \mathbb{C} = G_{\mathbb{C}}$ we then have

$$G.\zeta = \mathbb{R} \times (\mathbb{R} - \pi i/2) \subseteq \partial\Xi,$$

and

$$W := \zeta \Xi^{\overline{\tau}_h} = (0, -\pi i/2).(i\mathbb{R} \times \mathbb{R}) = \mathbb{R} \times (\mathbb{R} - \pi i/2) = G.(0, -\pi i/2).$$

Based on results in [BN23], we show in [BN25] that there are irreducible unitary representations $(U, \mathcal{H})$ of G for which $\mathcal{H}^\omega(\Xi)$ is dense and $\partial U(1,0)$ is injective but

$$\mathcal{H}^\omega(\Xi) \cap \mathcal{H}^J_{\mathrm{temp}} = \{0\}.$$

Therefore this domain Ξ is too large for our purposes.
(b) A natural strategy, to find good smaller domains, is inspired by the crown domains of Riemannian symmetric spaces (see Subsection 3.3 below). For the upper half-plane $\mathbb{C}_+$, considered as a real 2-dimensional homogeneous space of G via the orbit map $(a,c) \mapsto (a,c).i = a + e^c i$, we consider the "complexification"

$$\eta_{\mathbb{C}_+} \colon \mathbb{C}_+ \to \mathbb{C}_+ \times \mathbb{C}_- \subseteq \mathbb{C}^2, \quad \eta_{\mathbb{C}_+}(z) = (z, \overline{z}).$$

The complex Lie group $G_{\mathbb{C}}$ acts naturally on $\mathbb{C} \times \mathbb{C}$ by the diagonal action with respect to the canonical action on $\mathbb{C}$ by affine maps.

The $G_{\mathbb{C}}$-orbits in $\mathbb{C}^2$ are:

- The diagonal $\Delta_{\mathbb{C}} \subseteq \mathbb{C}^2$, which is one-dimensional complex.
- Its complement $\mathbb{C}^2 \setminus \Delta_{\mathbb{C}}$, which is 2-dimensional complex.

We consider the complex manifold $\Xi_{\mathbb{C}_+} := \mathbb{C}_+ \times \mathbb{C}_-$ as a crown domain of $\mathbb{C}_+ \cong \eta_{\mathbb{C}_+}(\mathbb{C}_+)$. It is invariant under the real group $G = \mathbb{R} \rtimes \mathbb{R}_+$. As $\mathbb{C}_+ = G.i$, we obtain the corresponding crown domain in $G_{\mathbb{C}}$ as:

$$\begin{aligned}\Xi &:= \{g \in G_{\mathbb{C}} \colon g.\eta_{\mathbb{C}_+}(i) \in \mathbb{C}_+ \times \mathbb{C}_-\} = \{(b,c) \in G_{\mathbb{C}} \colon b \pm e^c i \in \mathbb{C}_\pm\} \\ &= \{(b,c) \in G_{\mathbb{C}} \colon e^c i \pm b \in \mathbb{C}_+\} = \{(b,c) \in G_{\mathbb{C}} \colon \operatorname{Re} e^c > 0, |\operatorname{Im} b| < \operatorname{Re} e^c\}.\end{aligned}$$

The boundary of $\mathbb{C}_+ \times \mathbb{C}_-$ is the totally real submanifold $M := \mathbb{R}^2$. The G-orbits in $M = \mathbb{R}^2$ are:

- The diagonal $\Delta_{\mathbb{R}} \subseteq \mathbb{R}^2$, which is one-dimensional.
- The two regions $\mathbb{R}^2_> := \{(x,y) \in \mathbb{R}^2 \colon x > y\}$ and $\mathbb{R}^2_< := \{(x,y) \in \mathbb{R}^2 \colon x < y\}$.

For $(x,y) \in \mathbb{R}^2$, we have $e^{ith}.(x,y) = (e^{it}x, e^{it}y)$, so that

$$e^{ith}.(x,y) \in \mathbb{C}_+ \times \mathbb{C}_- \quad \text{for} \quad 0 < t < \pi$$

is equivalent to $x > 0$ and $y < 0$. This specifies the wedge region

$$W := \mathbb{R}_+ \times \mathbb{R}_- \subseteq M = \mathbb{R}^2.$$

It turns out that the domain Ξ behaves much better than the one from (a). This is due to the fact that the irreducible unitary representations of G extend to unitary representations of $\mathrm{PSL}_2(\mathbb{R})$, for which $\mathbb{C}_+$ is a Riemannian symmetric space. Therefore the results outlined in Subsection 3.3 below for the semisimple case (Theorem 3.17) imply corresponding results for G. We refer to [BN25] for details and a discussion of more general solvable Lie groups.

3.2 Push forwards to homogeneous spaces

Definition 3.10. On a homogeneous space $M = G/H$ with the projection map $q_M \colon G \to M$, we now obtain a "push-forward net"

$$\mathsf{H}^M_{\mathtt{E}}(\mathcal{O}) := ((q_M)_*\mathsf{H}^G_{\mathtt{E}})(\mathcal{O}) = \mathsf{H}^G_{\mathtt{E}}(q_M^{-1}(\mathcal{O})). \tag{3.9}$$

The so-obtained net on M thus corresponds to the restriction of the net $\mathsf{H}^G_{\mathtt{E}}$ indexed by open subsets of G, to those open subsets $\mathcal{O} \subseteq G$ which are H-right invariant in the sense that $\mathcal{O} = \mathcal{O}H$; these are the inverse images of open subsets of M under q_M.

Remark 3.11. (a) If $\mathtt{E}$ is invariant under $U^{-\infty}(H)$, then Lemma 3.22(a) in Appendix 3.4.1 implies that $\mathsf{H}^G_{\mathtt{E}}(\mathcal{O}) = \mathsf{H}^G_{\mathtt{E}}(\mathcal{O}H)$ for any open subset $\mathcal{O} \subseteq G$, so that $\mathsf{H}^G_{\mathtt{E}}$ can be recovered from the net $\mathsf{H}^M_{\mathtt{E}}$ on M by $\mathsf{H}^G_{\mathtt{E}}(\mathcal{O}) = \mathsf{H}^G_{\mathtt{E}}(\mathcal{O}H) = \mathsf{H}^M_{\mathtt{E}}(q_M(\mathcal{O}))$.

(b) We have already seen in Remark 3.6 that the net $\mathsf{H}^G_{\mathtt{E}}$ and hence also $\mathsf{H}^M_{\mathtt{E}}$ satisfy (Iso) and (Cov). Further, the net $\mathsf{H}^M_{\mathtt{E}}$ inherits (RS) from $\mathsf{H}^G_{\mathtt{E}}$. If (BW) holds for $\mathsf{H}^G_{\mathtt{E}}$ and the wedge region $W^G \subseteq G$ in the sense that $\mathsf{H}^G_{\mathtt{E}}(W^G) = \mathtt{V}$, then it holds for its image in G/H if $\mathtt{E}$ **is H-invariant**, which implies with $W^M = q_M(W^G)$ that

$$\mathsf{H}^G_{\mathtt{E}}(W^G) = \mathsf{H}^G_{\mathtt{E}}(W^G H) = \mathsf{H}^G_{\mathtt{E}}(q_M^{-1}(W^M)) = \mathsf{H}^M_{\mathtt{E}}(W^M)$$

(Lemma 3.21 in Appendix 3.4.1).

(c) If $\mathtt{E}$ is not H-invariant, then the situation is more complicated. We may enlarge $\mathtt{E}$ to the closed subspace $\widehat{\mathtt{E}}$ of $\mathcal{H}^{-\infty}$ generated by $U^{-\infty}(H)\mathtt{E}$, but then it is not clear if this still has the form $\beta^+(\widehat{\mathtt{F}})$ for some $\widehat{\mathtt{F}} \subseteq \mathcal{H}^\omega(\Xi) \cap \mathcal{H}^J_{\mathrm{temp}}$.

3.3 Crown domains for semisimple groups

To apply Theorem 3.8, one first has to specify a domain Ξ satisfying (Cr1-4), and then one has to find for an antiunitary representation $(U, \mathcal{H})$ subspaces $\mathtt{F} \subseteq \mathcal{H}^J_{\mathrm{temp}} \cap \mathcal{H}^\omega(\Xi)$. We now explain how this can be done if G is a connected **semisimple real Lie group**. In this case, the Euler element $h \in \mathfrak{g}$ specifies non-compactly causal symmetric spaces $M = G/H$ with $\tau = \theta\tau_h$ (Theorem 2.41), and for these, the subspaces $\mathtt{E} = \beta^+(\mathtt{F})$ is automatically H-invariant, so that we obtain nets on M by (3.9).

The precise context is the following (cf. [FNÓ23]):

- G is a connected semisimple Lie group
- $\eta_G\colon G \to G_{\mathbb{C}}$ is the universal complexification of G; its kernel is discrete.
- $h \in \mathfrak{g}$ is an Euler element.
- θ is a Cartan involution on G and its Lie algebra $\mathfrak{g}$, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the eigenspace decomposition, and we assume that $\theta(h) = -h$.
- $K := G^\theta$ is the group of fixed points of a Cartan involution θ, so that $\mathrm{Ad}(K) \subseteq \mathrm{Ad}(G)$ is maximally compact.
- $(\mathfrak{g}, \tau)$ is non-compactly causal, i.e., $\tau = \tau_h\theta$, $\tau_h = e^{\pi i \,\mathrm{ad}\, h}|_{\mathfrak{g}}$, and the τ-eigenspace decomposition is denoted $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ (cf. Theorem 2.41). We also write
$$\mathfrak{h}_{\mathfrak{k}} = \mathfrak{h} \cap \mathfrak{k}, \quad \mathfrak{h}_{\mathfrak{p}} = \mathfrak{h} \cap \mathfrak{p}, \quad \mathfrak{q}_{\mathfrak{k}} = \mathfrak{q} \cap \mathfrak{k}, \quad \mathfrak{q}_{\mathfrak{p}} = \mathfrak{q} \cap \mathfrak{p}.$$
- $C \subseteq \mathfrak{q}$ is the **maximal** $\mathrm{Inn}(\mathfrak{h})$-invariant cone containing h (cf. [MNO23, §3]).
- τ defines an involution on G and $H \subseteq G^\tau$ is an open θ-invariant subgroup with $\mathrm{Ad}(H)C = C$. In view of [MNO23, Cor. 4.6], this is equivalent to $H_K := H \cap K$ fixing h. Polar decomposition then yields $H = H_K \exp(\mathfrak{h}_{\mathfrak{p}}) = H^h \exp(\mathfrak{h}_{\mathfrak{p}})$. If G is given, this means that $H_{\min} \subseteq H \subseteq H_{\max}$, where $H_{\min} = G^\tau_e$ is connected and $H_{\max} = K^{\tau,h} \exp(\mathfrak{h}_{\mathfrak{p}})$ (cf. Subsection 2.7.3 and Appendix 2.8.5).
- $M = G/H$ is the corresponding non-compactly causal symmetric space (cf. Subsection 2.7). The simply connected covering $\widetilde{M}$ is $\widetilde{G}/\widetilde{G}^\tau$, where $\widetilde{G}^\tau$ is connected ([Lo69]).
- $M_r := G/K$ is the Riemannian symmetric space of G, and if $K_{\mathbb{C}} \subseteq G_{\mathbb{C}}$ is the integral subgroup corresponding to $\mathfrak{k}_{\mathbb{C}}$, then $M_{\mathbb{C}} := G_{\mathbb{C}}/K_{\mathbb{C}}$ is a complex homogeneous space containing M_r as the G-orbit of the base point.
- $G_{\tau_h} := G \rtimes \{\mathbf{1}, \tau_h\}$ is the corresponding extended Lie group. Here we use that $\tau_h = \theta\tau$, as an involution of G.
- $(U, \mathcal{H})$ an **irreducible** antiunitary representation of G_{τ_h}. In particular, $J := U(\tau_h)$ is a conjugation on $\mathcal{H}$ and $U(G) \subseteq \mathrm{U}(\mathcal{H})$.

Lemma 3.12. *The automorphism* $\zeta := e^{-\frac{\pi i}{2}\,\mathrm{ad}\, h} \in \mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$ *satisfies*
$$\zeta(\mathfrak{h}_{\mathfrak{k}} + i\mathfrak{q}_{\mathfrak{k}}) = \mathfrak{h}, \tag{3.10}$$
hence in particular $\zeta(\mathfrak{k}_{\mathbb{C}}) = \mathfrak{h}_{\mathbb{C}}$.

Proof. As $\tau_h = \tau\theta$, we have $\mathfrak{g}_0(h) = \mathfrak{g}^{\tau_h} = \mathfrak{h}_{\mathfrak{k}} \oplus \mathfrak{q}_{\mathfrak{p}}$ and $\mathfrak{g}^{-\tau_h} = \mathfrak{h}_{\mathfrak{p}} \oplus \mathfrak{q}_{\mathfrak{k}} = \mathfrak{g}_1(h) \oplus \mathfrak{g}_{-1}(h)$. As $\theta(h) = -h$, we have $\theta(\mathfrak{g}_1(h)) = \mathfrak{g}_{-1}(h)$. So $\mathfrak{q}_{\mathfrak{k}} = \{x + \theta(x)\colon x \in \mathfrak{g}_1(h)\}$. This shows that
$$\zeta(\mathfrak{q}_{\mathfrak{k}}) = \{i(x - \theta(x))\colon x \in \mathfrak{g}_1(h)\} = i\mathfrak{h}_{\mathfrak{p}},$$
and therefore $\zeta(\mathfrak{h}_{\mathfrak{k}} + i\mathfrak{q}_{\mathfrak{k}})) = \mathfrak{h}_{\mathfrak{k}} + \mathfrak{h}_{\mathfrak{p}} = \mathfrak{h}$, which entails $\zeta(\mathfrak{k}_{\mathbb{C}}) = \mathfrak{h}_{\mathbb{C}}$. □

Let $\Omega_{\mathfrak{p}} \subseteq \mathfrak{p}$ consists of all elements for which the spectral radius of $\mathrm{ad}\, x$ is smaller than $\frac{\pi}{2}$, i.e.,
$$\Omega_{\mathfrak{p}} = \mathrm{Ad}(K)\Omega_{\mathfrak{a}} \quad \text{with} \quad \Omega_{\mathfrak{a}} = \Big\{x \in \mathfrak{a}\colon (\forall \alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}))\ |\alpha(x)| < \frac{\pi}{2}\Big\}.$$
Note that $th \in \Omega_{\mathfrak{p}}$ if and only if $|t| < \pi/2$.

Remark 3.13. (a) The domain

$$\Xi_{G/K} := G.\exp(i\Omega_{\mathfrak{p}})K_{\mathbb{C}} \subseteq G_{\mathbb{C}}/K_{\mathbb{C}}$$

is called the *crown of the Riemannian symmetric space* G/K, realized naturally in $G_{\mathbb{C}}/K_{\mathbb{C}}$. This set has a natural fiber bundle structure $\Xi_{G/K} \cong G \times_K i\Omega_{\mathfrak{p}} \twoheadrightarrow G/K$. Further $\overline{\tau}_h$ induces the antiholomorphic involution $[g,x] \mapsto [\tau_h(g), -\tau_h(x)]$, and as

$$(G/K)^{\tau_h} = \exp(\mathfrak{p}^{\tau_h}).eK = \exp(\mathfrak{q}_{\mathfrak{p}}).eK = G^h.eK,$$

the fiber bundle structure shows that the fixed point set is

$$\Xi_{G/K}^{\overline{\tau}_h} = G^h_e \exp(i\Omega_{\mathfrak{p}}^{-\tau_h}).eK_{\mathbb{C}} = G^h_e \exp(i\Omega_{\mathfrak{h}_{\mathfrak{p}}}).eK_{\mathbb{C}}.$$

The argument given in [MNO24, §8] for the adjoint group applies also in this, slightly more general, context because the crown domains are the same.
(b) Gindikin and Krötz analyze in [GK02] the distinguished boundary of $\Xi_{G/K}$ and find, up to coverings, all irreducible non-compactly causal symmetric spaces.

The idea is rather simple. Let us assume that $G \subseteq G_{\mathbb{C}}$. Then $M_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$ is a complex homogeneous space containing $M_r \cong G/K$ as a totally real submanifold.

Then $\Xi_{G/K} = G\exp(i\Omega_{\mathfrak{p}})K_{\mathbb{C}} \subseteq M_{\mathbb{C}}$, and

$$o_M := \exp\Big(-\frac{\pi i}{2}h\Big).eK_{\mathbb{C}} = \zeta.eK_{\mathbb{C}} \in M_{\mathbb{C}}$$

is a point whose stabilizer in G is

$$H_1 := (\zeta K_{\mathbb{C}}\zeta^{-1}) \cap G.$$

As the holomorphic involutions $\theta_{\mathbb{C}}, \tau_{h,\mathbb{C}}$ and $\tau_{\mathbb{C}}$ on $G_{\mathbb{C}}$ satisfy

$$\tau_{\mathbb{C}}(g) = \zeta\theta_{\mathbb{C}}(\zeta^{-1}g\zeta)\zeta^{-1} = \zeta^2\theta_{\mathbb{C}}(g)\zeta^{-2} = \tau_{h,\mathbb{C}}\theta_{\mathbb{C}}(g),$$

we have $H_1 \subseteq G^\tau$, so that

$$M_1 := G.o_M \cong G/H_1$$

is a symmetric space corresponding to the symmetric Lie algebra $(\mathfrak{g}, \tau)$. However, it may not be causal because $\mathrm{Ad}(H_1)$-invariant pointed convex cones in $\mathfrak{q}$ may not exist, i.e., $H_{1,K}$ may not fix h ([MNO23, Rem. 4.20(a)]).

Assume, in addition, that $G_{\mathbb{C}}$ **is simply connected**. Then the subgroup $G_{\mathbb{C}}^{\theta_{\mathbb{C}}}$ is connected, hence equal to $K_{\mathbb{C}}$. Let $\overline{\tau} \in \mathrm{Aut}(G_{\mathbb{C}})$ be the antiholomorphic involution for which $G^c := (G_{\mathbb{C}})^{\overline{\tau}}$ is the connected subgroup with Lie algebra $\mathfrak{g}^c = \mathfrak{h} + i\mathfrak{q}$. Then

$$G^\tau = G \cap G^c$$

and therefore

$$K^\tau = K \cap G^c \subseteq K^c,$$

where K^c is a connected Lie group with Lie algebra

$$\mathfrak{k}^c = \mathfrak{h}_{\mathfrak{k}} + i\mathfrak{q}_{\mathfrak{p}} = \mathfrak{z}_{\mathfrak{g}^c}(h).$$

As a consequence, the connected group K^c is contained in $G^h_{\mathbb{C}}$, so that also $K^\tau \subseteq K^h$. This shows that $H := G^\tau$ leaves a cone in $\mathfrak{q}$ invariant, and thus G/H is a non-compactly causal symmetric space ([MNO23, Rem. 4.20(a)]). Moreover, $H_1 = G^\tau_{\mathbb{C}} \cap G = G^\tau$ implies that $M_1 \cong G/H$.

(c) If $G_{\mathbb{C}}$ is not necessarily simply connected and $K_{\mathbb{C}} = (G_{\mathbb{C}}^{\theta_{\mathbb{C}}})_e$, then $H_{\mathbb{C}} = (G_{\mathbb{C}}^{\tau_{\mathbb{C}}})_e$. For $G^c := (G_{\mathbb{C}}^{\overline{\tau}})_e$ and $H := G^c \cap G$, we then have $H \subseteq G_{\mathbb{C}}^{\overline{\tau}} \cap G = G^\tau$, and with the polar decomposition $H = H_K \exp(\mathfrak{h}_{\mathfrak{p}})$, and the compactness of the subgroup

$$H_K = H \cap K = G^c \cap K$$

of G^c, it follows that $H_K \subseteq K^c = (G^c)^h$, i.e., H_K fixes h, so that G/H is causal ([MNO23, Rem. 4.20(a)]). However, $H_{\mathbb{C}} \cap G$ need not be contained in H.

Proposition 3.14. *If $G \subseteq G_{\mathbb{C}}$ and $M_{\mathbb{C}} := G_{\mathbb{C}}/K_{\mathbb{C}}$, then the domain*

$$\Xi_{G_{\mathbb{C}}} := G \exp(i\Omega_{\mathfrak{p}}) K_{\mathbb{C}} \subseteq G_{\mathbb{C}}$$

is an open connected subset satisfying (Cr1-4) *with*

$$W^G := G_e^h \exp(\Omega_{\mathfrak{q}_{\mathfrak{k}}}) \exp(\mathfrak{h}_{\mathfrak{p}}).$$

Proof. Properties (Cr2) and (Cr3) are obvious from the construction, and so is the G-invariance. Further $\overline{\tau}_h(i\Omega_{\mathfrak{p}}) = -i\Omega_{\mathfrak{p}} = i\Omega_{\mathfrak{p}}$ and $\overline{\tau}_h(K_{\mathbb{C}}) = K_{\mathbb{C}}$ imply (Cr1). As in Remark 3.13(b) above, we have the G-orbit $M_1 := G.o_M \cong G/H_1$ in $\partial\Xi_{G/K}$. We consider the connected open subset

$$W_1 := G_e^h \exp(\Omega_{\mathfrak{q}_{\mathfrak{k}}}).o_M \subseteq M_1$$

(cf. (2.39)). Its inverse image under the orbit map

$$q_{M_1} \colon G \to M_1 = G/H_1, \quad g \mapsto g.o_M$$

is

$$G_e^h \exp(\Omega_{\mathfrak{q}_{\mathfrak{k}}}) H_1 = G_e^h H_{1,K} \exp(\Omega_{\mathfrak{q}_{\mathfrak{k}}}) \exp(\mathfrak{h}_{\mathfrak{p}}) \subseteq G^{\tau_h} \exp(\Omega_{\mathfrak{q}_{\mathfrak{k}}}) \exp(\mathfrak{h}_{\mathfrak{p}}).$$

Its identity component is

$$W^G := G_e^h \exp(\Omega_{\mathfrak{q}_{\mathfrak{k}}}) \exp(\mathfrak{h}_{\mathfrak{p}}).$$

Now

$$\exp(\mathcal{S}_\pi h) W_1 \subseteq \Xi_{G/K}$$

follows with the same argument as in [MNO24, Thm. 8.2], and this implies that

$$\exp(\mathcal{S}_\pi h) W^G \subseteq \Xi_{G_{\mathbb{C}}}.$$

Therefore (C4) is also satisfied. □

Example 3.15. For de Sitter space $M = \mathrm{dS}^d \subseteq \mathrm{V} := \mathbb{R}^{1,d}$ and the Lorentzian forms $x^2 = x_0^2 - \mathbf{x}^2$ on $\mathbb{R}^{1,d}$, a natural complexification is the complex sphere

$$M_{\mathbb{C}} := \{z = (z_0, \mathbf{z}) \in \mathbb{C}^{1+d} \colon z_0^2 - \mathbf{z}^2 = -1\}.$$

It contains $M = M_{\mathbb{C}} \cap \mathbb{R}^{1,d}$ and also the Riemannian symmetric spaces

$$\mathbb{H}_\pm := \{(iy_0, i\mathbf{y}) \colon y_0^2 - \mathbf{y}^2 = 1, \pm y_0 > 0\} \cong \mathrm{SO}_{1,d}(\mathbb{R})_e / \mathrm{SO}_d(\mathbb{R}).$$

Here $G = \mathrm{SO}_{1,d}(\mathbb{R})_e \subseteq G_{\mathbb{C}} = \mathrm{SO}_{1,d}(\mathbb{C}) \cong \mathrm{SO}_{1+d}(\mathbb{C})$ and $K = \mathrm{SO}_d(\mathbb{R}) \subseteq K_{\mathbb{C}} = \mathrm{SO}_d(\mathbb{C})$. The crown domains of the hyperbolic spaces $\mathbb{H}_\pm \cong G/K$ are the intersections with tube domains $\mathrm{V} \pm i\mathrm{V}_+$:

$$\Xi_\pm := M_{\mathbb{C}} \cap (\mathrm{V} \pm i\mathrm{V}_+).$$

For both domains,

$$\mathrm{dS}^d = \{(x_0, \mathbf{x}) \in \mathbb{R}^{1,d} \colon x_0^2 - \mathbf{x}^2 = -1\} \subseteq \partial_{M_\mathbb{C}} \Xi_\pm.$$

For the Euler element given by the Lorentz boost

$$h.(x_0, x_1, \ldots, x_{d-1}) = (x_1, x_0, 0, \ldots, 0),$$

we have

$$\zeta.x = \exp\Big(-\frac{\pi i}{2} h\Big).x = (-ix_1, -ix_0, x_2, \ldots, x_d),$$

so that $\zeta.i\mathbf{e}_0 = \mathbf{e}_1 \in \mathrm{dS}^d$.

We also note that, for $\mathrm{V} \subseteq \partial(\mathrm{V} + i\mathrm{V}_+)$ and $C := \overline{\mathrm{V}_+}$, the set of KMS-points is

$$\mathrm{V}_{\mathrm{KMS}} = C_+^\circ + \mathrm{V}_0 + C_-^\circ = W_\mathrm{V}^+(h), \quad \text{where} \quad C_\pm = \mathbb{R}_{\geq 0}(\mathbf{e}_1 \pm \mathbf{e}_0)$$

(cf. Examples 3.31). Accordingly,

$$\mathrm{dS}^d_{\mathrm{KMS}} = \mathrm{V}_{\mathrm{KMS}} \cap \mathrm{dS}^d = W^+_{\mathrm{dS}^d}(h).$$

Example 3.16. For $G = \mathrm{SL}_n(\mathbb{R})$ and the Euler element

$$h_q := \frac{1}{n}\begin{pmatrix} q\mathbf{1}_p & 0 \\ 0 & -p\mathbf{1}_q \end{pmatrix} \in \mathfrak{sl}_n(\mathbb{R})$$

from (2.5), the corresponding involution is

$$\tau(x) = -I_{p,q} x^\top I_{p,q} \quad \text{for} \quad p + q = n.$$

Therefore $G^\tau = \mathrm{SO}_{p,q}(\mathbb{R})$ and, for the action of G on $\mathrm{Sym}_n(\mathbb{R})$, we have

$$M := G.I_{p,q} = \{gI_{p,q}g^\top \colon g \in \mathrm{SL}_n(\mathbb{R})\}.$$

This space carries a causal structure for which $M \hookrightarrow (\mathrm{Sym}_n(\mathbb{R}), \mathrm{Sym}_n(\mathbb{R})_+)$ becomes an embedding of causal manifolds.

Here $M_r := G.I_n \cong G/K$ is the corresponding Riemannian symmetric space. For

$$\zeta := \exp\Big(-\frac{\pi i}{2} h_q\Big) \quad \text{we have} \quad \zeta.I_n = \exp(-\pi i h_p) = e^{-\pi i q/n}\mathbf{1}_p \oplus e^{\pi i p/n}\mathbf{1}_q = e^{-\pi i q/n} I_{p,q},$$

so that $G.(\zeta.I_n) \cong G.I_{p,q} \cong M$.

Theorem 3.17. *Let $(U, \mathcal{H})$ be an irreducible antiunitary representation of $G_{\tau_h} = G \rtimes \{\mathbf{1}, \tau_h\}$, let $\mathcal{F}$ be a finite-dimensional subspace invariant under K and J, and $\mathtt{F} := \mathcal{F}^J$. Then $\mathtt{E} = \beta^+(\mathtt{F}) \subseteq \mathcal{H}^{-\infty}$ with β^+ from* (3.5) *and the net $\mathsf{H}^M_\mathtt{E}$ from* (3.9) *on the non-compactly causal symmetric space $M = G/H$ satisfies* (Iso), (Cov), (RS) *and* (BW), *where $W = W_M^+(h)_{eH}$ is the connected component of the positivity domain of h on M, containing the base point.*

Note that $\tau_h(K) = K$ implies that J leaves the dense subspace $\mathcal{H}^{[K]}$ of K-finite vectors invariant. Therefore J-invariant finite-dimensional K-invariant subspaces exist in abundance.

Proof. (Sketch; assuming that $G \subseteq G_{\mathbb{C}}$, [FNÓ23, Thm. 8]). First, Simon's Growth Theorem 3.34 implies that

$$\mathcal{F} \subseteq \bigcap_{x \in \Omega_{\mathfrak{p}}} \mathcal{D}(e^{i\partial U(x)}). \tag{3.11}$$

With [FNÓ23, Prop. 6] we now see that the map $G \times_K \mathcal{F} \to \mathcal{H}, [g,v] \mapsto U(g)v$ extends to a holomorphic map

$$\Psi_{\mathcal{F}} \colon \mathbb{F} = (G \times i\Omega_{\mathfrak{p}}) \times_K \mathcal{F} \to \mathcal{H}, \quad [g, ix, v] \mapsto U(g)e^{i\partial U(x)}v, \tag{3.12}$$

where $\mathbb{F}$ carries the structure of a G-equivariant holomorphic vector bundle over the complex manifold $\Xi_{G/K} \cong G \times_K i\Omega_{\mathfrak{p}}$ ([FNÓ23, Prop. 5]).

Assuming that $G \subseteq G_{\mathbb{C}}$, this implies in particular that $\mathcal{F} \subseteq \mathcal{H}^{\omega}(\Xi_{G_{\mathbb{C}}})$, so that the assumptions of Theorem 3.8 are satisfied. Further, Theorem 3.32 implies that $\mathtt{F} := \mathcal{F}^J \subseteq \mathcal{H}^J_{\mathrm{temp}}$, so that, for $v \in \mathcal{F}$, the limit

$$\beta^+(v) := \lim_{t \to \pi/2} e^{-it\partial U(h)}v$$

exists in the space $\mathcal{H}^{-\infty}$ of distribution vectors, endowed with the weak-$*$ topology. Natural equivariance properties (Proposition 3.35 in Appendix 3.4.1) then imply that

$$\mathtt{E} := \beta^+(\mathtt{F}) \subseteq \mathcal{H}^{-\infty} \tag{3.13}$$

is a finite-dimensional H**-invariant** subspace, and the net $\mathsf{H}^M_{\mathtt{E}} = (q_M)_*\mathsf{H}^G_{\mathtt{E}}$ on $M = G/H$, defined as in (3.9), also satisfies (RS) and (BW). □

Remark 3.18. ("Independence" of the net from the choice of H) In the context of Theorem 3.17, the real subspace $\mathtt{E} \subseteq \mathcal{H}^{-\infty}$ is invariant under $U^{-\infty}(H)$. For any open subset $\mathcal{O}_G \subseteq G$ we therefore have

$$\mathsf{H}^G_{\mathtt{E}}(\mathcal{O}) = \mathsf{H}^G_{\mathtt{E}}(\mathcal{O}H)$$

by Lemma 3.22. Hence the inclusions $H_{\min} \subseteq H \subseteq H_{\max}$ (see the context list at the beginning of this subsection) imply that

$$\mathsf{H}^G_{\mathtt{E}}(\mathcal{O}) = \mathsf{H}^G_{\mathtt{E}}(\mathcal{O}H) = \mathsf{H}^G_{\mathtt{E}}(\mathcal{O}H_{\max}).$$

Here we use that the real subspace $\mathtt{E} \subseteq \mathcal{H}^{-\infty}$ is invariant under $H_{\max}$ because Proposition 3.35(c) also applies to $H_{\max}$. For the covering

$$q_m \colon G/H \to M_{\min} := G/H_{\max}$$

it therefore follows that the net $\mathsf{H}^M_{\mathtt{E}}$ on M can be recovered from its pushforward $\mathsf{H}^{M_{\min}}_{\mathtt{E}}$ to $M_{\min}$ because

$$\mathsf{H}^M_{\mathtt{E}}(\mathcal{O}) = \mathsf{H}^M_{\mathtt{E}}(q_m^{-1}(q_m(\mathcal{O}))) = \mathsf{H}^{M_{\min}}_{\mathtt{E}}(q_m(\mathcal{O}))$$

for any open subset $\mathcal{O} \subseteq M$.

Remark 3.19. To deal with the case where η_G is not injective, we may assume that G is simply connected, which implies that its universal complexification $G_{\mathbb{C}}$ is also simply connected.

We first consider the crown domain of the Riemannian symmetric space G/K:

$$\Xi_{G/K} := G \times_K i\Omega_{\mathfrak{p}} = (G \times i\Omega_{\mathfrak{p}})/\sim \quad \text{with} \quad (g, ix) \sim (gk, \mathrm{Ad}(k)^{-1}ix), k \in K. \tag{3.14}$$

The complex structure on this domain is determined by the requirement that the map

$$q \colon \Xi_{G/K} \to G_{\mathbb{C}}^{-\theta} = \{g \in G_{\mathbb{C}} \colon \theta(g) = g^{-1}\}, \quad q([g, ix]) \mapsto g.\exp(2ix) = g\exp(2ix)\theta(g)^{-1},$$

which is a covering of an open subset of the complex symmetric space $G_{\mathbb{C}}^{-\theta_{\mathbb{C}}}$, is holomorphic. This domain is biholomorphic to $\Xi_{G/K}$, and we may thus consider $\Xi_{G/K}$ as a domain in $G_{\mathbb{C}}/K_{\mathbb{C}}$. Let $q_{M_{\mathbb{C}}} \colon G_{\mathbb{C}} \to G_{\mathbb{C}}/K_{\mathbb{C}}$ denote the quotient map and

$$\Xi_{G_{\mathbb{C}}} := q_{M_{\mathbb{C}}}^{-1}(\Xi_{G/K}) = G\exp(i\Omega_{\mathfrak{p}})K_{\mathbb{C}}.$$

This is an open subset of $G_{\mathbb{C}}$ that is right $K_{\mathbb{C}}$-invariant, so that $\Xi_{G_{\mathbb{C}}}$ is a $K_{\mathbb{C}}$-principal bundle over $\Xi_{G/K}$. As $\Xi_{G/K}$ is contractible (it is an affine bundle with convex fibers over the contractible space G/K), the natural homomorphism $\pi_1(K_{\mathbb{C}}) \to \pi_1(\Xi_{G_{\mathbb{C}}})$ is an isomorphism by the long exact homotopy sequence for fiber bundles. We conclude that the simply connected covering $\widetilde{\Xi}_{G_{\mathbb{C}}}$ is a holomorphic $\widetilde{K}_{\mathbb{C}}$-principal bundle over $\Xi_{G/K}$.

As G is simply connected, the G-action lifts naturally to an action on $\widetilde{\Xi}_{G_{\mathbb{C}}}$. To see that this action is free, note that the group of deck transformations is

$$\pi_1(K_{\mathbb{C}}) \cong \pi_1(\eta(K)) \cong \pi_1(\eta(G)) = \ker \eta_G.$$

Comparing the action on the base point, it follows that the subgroup $\ker \eta_G \subseteq G$ acts faithfully by deck transformations, and so does G.

3.4 Appendices to Section 3

3.4.1 Tools for nets of real subspaces

Lemma 3.20. *Let $\mathcal{O} \subseteq G$ be open and $\varphi \in C_c^\infty(\mathcal{O})$. We further assume that $(\mathcal{O}_j)_{j\in J}$ is an open cover of $\mathcal{O}$. Then there exist $j_1, \ldots, j_k \in J$ and $\varphi_\ell \in C_c^\infty(\mathcal{O}_{j_\ell})$ such that $\varphi = \varphi_1 + \cdots + \varphi_k$.*

Proof. Let $\mathcal{O} \subseteq G$ be open and $\varphi \in C_c^\infty(\mathcal{O})$. We further assume that $(\mathcal{O}_j)_{j\in J}$ is an open cover of $\mathcal{O}$. Then $(\mathcal{O}_j)_{j\in J}$ also in an open cover of $\mathrm{supp}(\varphi)$, and there exist $j_1, \ldots, j_k \in J$ such that

$$\mathrm{supp}(\varphi) \subseteq \mathcal{O}_{j_1} \cup \cdots \cup \mathcal{O}_{j_k}.$$

Then

$$G \setminus \mathrm{supp}(\varphi), \quad \mathcal{O}_{j_1}, \ldots, \mathcal{O}_{j_k}$$

is an open cover of G. Let $\chi_0, \ldots, \chi_k$ be a subordinated partition of unity. Then $\varphi = \sum_{j=1}^k \varphi_j$, where $\varphi_j := \chi_j \varphi$ satisfies $\mathrm{supp}(\varphi_j) \subseteq \mathcal{O}_j$. □

Lemma 3.21. (Fragmentation Lemma) *For $\emptyset \neq \mathcal{O} \subseteq G$ open, the following assertions hold:*

(a) *If $H \subseteq G$ is a closed subgroup, then*

  (i) *every test function $\varphi \in C_c^\infty(\mathcal{O}H, \mathbb{R})$ is a finite sum of test functions of the form*

$$\psi \circ \rho_p \colon G \to \mathbb{C}, \quad g \mapsto \psi(gp), \quad \psi \in C_c^\infty(\mathcal{O}, \mathbb{R}), p \in H.$$

  (ii) *every test function $\varphi \in C_c^\infty(H\mathcal{O}, \mathbb{R})$ is a finite sum of test functions of the form*

$$\psi \circ \lambda_p \colon G \to \mathbb{C}, \quad g \mapsto \psi(pg), \quad \psi \in C_c^\infty(\mathcal{O}, \mathbb{R}), p \in H.$$

(b) *Every $\varphi \in C_c^\infty(G, \mathbb{R})$ is a finite sum $\sum_{j=1}^n \varphi_j \circ \lambda_{g_j}$ with $\varphi_j \in C_c^\infty(\mathcal{O}, \mathbb{R})$ and $g_j \in G$.*

Proof. (a)(i) The family $(\mathcal{O}p)_{p\in H}$ is an open cover of the compact subset $\mathrm{supp}(\varphi)$, so that Lemma 3.20 implies that $\varphi = \sum_{j=1}^n \varphi_j$ with $\mathrm{supp}(\varphi_j) \subseteq \mathcal{O}p_j$. Then $\psi_j := \varphi_j \circ \rho_{p_j} \in C_c^\infty(\mathcal{O},\mathbb{R})$ and $\varphi = \sum_{j=1}^n \psi_j \circ \rho_{p_j^{-1}}$.
(a)(ii) and (b) are proved along the same lines. For (b), we use the open cover $(g\mathcal{O})_{g\in G}$ of the group G. $\square$

Lemma 3.22. *Let $(U,\mathcal{H})$ be a unitary representation of G, let $\mathtt{E} \subseteq \mathcal{H}^{-\infty}$ be a real linear subspace, $H \subseteq G$ a closed subgroup and $\emptyset \neq \mathcal{O} \subseteq G$. Then the following assertions hold:*

(a) $\mathsf{H}_{\mathtt{E}}^G(\mathcal{O}H) = \mathsf{H}_{\mathtt{E}}^G(\mathcal{O})$ *if $\mathtt{E}$ is H-invariant.*

(b) $\mathsf{H}_{\mathtt{E}}^G(H\mathcal{O})$ *is the closed real span of $U(H)\mathsf{H}_{\mathtt{E}}^G(\mathcal{O})$.*

(c) *The real subspace spanned by $U(G)\mathsf{H}_{\mathtt{E}}^G(\mathcal{O})$ is dense in $\mathsf{H}_{\mathtt{E}}^G(G)$.*

Proof. (a) The inclusion $\mathsf{H}_{\mathtt{E}}^G(\mathcal{O}) \subseteq \mathsf{H}_{\mathtt{E}}^G(\mathcal{O}H)$ is trivial. Conversely, for $\varphi = \psi \circ \rho_p$, $\psi \in C_c^\infty(\mathcal{O})$ and $p \in H$, we obtain with (C.7)

$$U^{-\infty}(\varphi)\mathtt{E} = U^{-\infty}(\psi \circ \rho_p)\mathtt{E} = \Delta_G(p)^{-1}U^{-\infty}(\psi)U^{-\infty}(p^{-1})\mathtt{E} = U^{-\infty}(\psi)\mathtt{E} \subseteq \mathtt{E}.$$

Hence the assertion follows from Lemma 3.21(a).
(b) From Remark 3.6 we know that $U(p)\mathsf{H}_{\mathtt{E}}^G(\mathcal{O}) = \mathsf{H}_{\mathtt{E}}^G(p\mathcal{O}) \subseteq \mathsf{H}_{\mathtt{E}}^G(H\mathcal{O})$ for $p \in H$. Now the assertion follows from Lemma 3.21(b).
(c) is an immediate consequence of (b), applied with $H = G$. $\square$

3.4.2 KMS vectors for 1-parameter groups

In this subsection, we discuss some general tools concerning holomorphic extensions of orbit maps of one-parameter groups on locally convex spaces to strips in the complex plane. They are instrumental in formulating Kubo–Martin–Schwinger (KMS) boundary conditions that are related to the construction of standard subspaces.

Definition 3.23. Let $(U_t)_{t\in\mathbb{R}}$ be a one-parameter subgroup of $\mathrm{GL}(\mathcal{Y})$ for a topological vector space $\mathcal{Y}$ and J an antilinear operator on $\mathcal{Y}$, commuting with $(U_t)_{t\in\mathbb{R}}$.

We write $\mathcal{Y}_{\mathrm{KMS}}$ for the subspace of those $y \in \mathcal{Y}$, whose orbit map $U^v \colon \mathbb{R} \to \mathcal{Y}, t \mapsto U_t v$ extends to a continuous map on $\overline{\mathcal{S}}_{0,\pi} := \mathbb{R} + i[0,\pi]$, holomorphic on the interior $\mathcal{S}_{0,\pi}$, such that[14]

$$U^v(\pi i + t) = JU^v(t) = JU_t v \quad \text{for} \quad t \in \mathbb{R}. \tag{3.15}$$

We call the elements of this space *KMS vectors* (with respect to U and J).

In [BN23] we study for an antiunitary representation $(U,\mathcal{H})$ of G_{τ_h} the space $\mathcal{H}^{-\infty}_{\mathrm{KMS}} := (\mathcal{H}^{-\infty})_{\mathrm{KMS}}$ of distribution vectors (see Appendix C for details), on which we have the one-parameter group $U^{-\infty}(\exp th)$ generated by the Euler element h and the action of the conjugation $J^{-\infty} = U^{-\infty}(\tau_h)$.

In combination with the following Proposition 3.25, the following theorem is the key tool to verify that nets of real subspaces satisfy the Bisognano–Wichmann property $\mathsf{H}(W) \subseteq \mathtt{V}$.

Theorem 3.24. *Let $(U,\mathcal{H})$ be an antiunitary representation of G_{τ_h} and $\mathtt{V} \subseteq \mathcal{H}$ the standard subspace specified by $\Delta_{\mathtt{V}} = e^{2\pi i \cdot \partial U(h)}$ and $J_{\mathtt{V}} = U(\tau_h)$. Then the following assertions hold:*

(a) $(\mathcal{H}^{-\infty})_{\mathrm{KMS}}$ *is a weak-$*$-closed subspace of $\mathcal{H}^{-\infty}$.*

[14] By equivariance, it actually suffices that $U^v(\pi i) = Jv$.

(b) $\mathcal{H}^{-\infty}_{\mathrm{KMS}} \cap \mathcal{H} = \mathtt{V}$.

(c) $\mathtt{V}$ *is dense in* $\mathcal{H}^{-\infty}_{\mathrm{KMS}}$.

Proof. (a) is [BN23, Thm. 6.2], (b) is [BN23, Thm. 6.4], and (c) is [BN23, Thm. 6.5]. $\square$

The following proposition is useful to verify that $\mathsf{H}^G_{\mathtt{E}}(W^G) \subseteq \mathtt{V}$.

Proposition 3.25. *Let $(U, \mathcal{H})$ be an antiunitary representation of G_{τ_h} and $\mathtt{V} = \mathtt{V}(h, U)$ the corresponding standard subspace. For an open subset $\mathcal{O} \subseteq G$ and a real subspace $\mathtt{E} \subseteq \mathcal{H}^{-\infty}$, the following are equivalent:*

(a) $\mathsf{H}^G_{\mathtt{E}}(\mathcal{O}) \subseteq \mathtt{V}$.

(b) *For all $\varphi \in C^\infty_c(\mathcal{O}, \mathbb{R})$ we have $U^{-\infty}(\varphi)\mathtt{E} \subseteq \mathtt{V}$.*

(c) *For all $\varphi \in C^\infty_c(\mathcal{O}, \mathbb{R})$ we have $U^{-\infty}(\varphi)\mathtt{E} \subseteq \mathcal{H}^{-\infty}_{\mathrm{KMS}}$.*

(d) *$U^{-\infty}(g)\mathtt{E} \subseteq \mathcal{H}^{-\infty}_{\mathrm{KMS}}$ for every $g \in \mathcal{O}$.*

To show that $\mathsf{H}^G_{\mathtt{E}}(W^G) \subseteq \mathtt{V}$, we thus need to show that $U^{-\infty}(W^G)\mathtt{E} \subseteq \mathcal{H}^{-\infty}_{\mathrm{KMS}}$.

Proof. ([FNÓ23, Prop. 9]) It is clear that (a) implies (b) by the definition of $\mathsf{H}^G_{\mathtt{E}}(\mathcal{O})$. Further, (b) implies (c) because $\mathtt{V} \subseteq \mathcal{H}^{-\infty}_{\mathrm{KMS}}$ (Theorem 3.24(b)).

For the implication (c) $\Rightarrow$ (d), let $(\delta_n)_{n\in\mathbb{N}}$ be a δ-sequence in $C^\infty_c(G, \mathbb{R})$. Then $U(\delta_n)\xi \to \xi$ in $\mathcal{H}^\infty$ and hence also in $\mathcal{H}^{-\infty}$. It follows in particular that

$$U^{-\infty}(\delta_n * \delta_g)\eta = U^{-\infty}(\delta_n)U^{-\infty}(g)\eta \to U^{-\infty}(g)\eta \quad \text{for} \quad \eta \in \mathcal{H}^{-\infty}.$$

Hence the closedness of $\mathcal{H}^{-\infty}_{\mathrm{KMS}}$ (Theorem 3.24(a)), shows that (c) implies (d). Here we use that $\delta_n * \delta_g \in C^\infty_c(\mathcal{O}, \mathbb{R})$ for $g \in \mathcal{O}$ if n is sufficiently large.

As the G-orbit maps in $\mathcal{H}^{-\infty}$ are continuous and $\mathcal{H}^{-\infty}_{\mathrm{KMS}}$ is closed, hence stable under integrals over compact subsets and $U^{-\infty}(C^\infty_c(\mathcal{O}, \mathbb{R}))\mathcal{H}^{-\infty} \subset \mathcal{H}^\infty$, we see that (d) implies (a). $\square$

3.4.3 Standard subspaces and J-fixed points

In this subsection, we derive a characterization of the elements of a standard subspace $\mathtt{V}$ specified by the pair (Δ, J) in terms of analytic continuation of orbit maps of the unitary one-parameter group $(\Delta^{it})_{t\in\mathbb{R}}$ and the real space $\mathcal{H}^J$.

In the terminology of Appendix 3.4.2, the following proposition asserts that

$$\mathtt{V} = \mathcal{H}_{\mathrm{KMS}}.$$

Proposition 3.26. *Let $\mathtt{V} \subseteq \mathcal{H}$ be a standard subspace with modular objects (Δ, J). For $\xi \in \mathcal{H}$, we consider the orbit map $\alpha^\xi \colon \mathbb{R} \to \mathcal{H}, t \mapsto \Delta^{-it/2\pi}\xi$. Then the following are equivalent:*

(i) $\xi \in \mathtt{V}$.

(ii) *$\xi \in \mathcal{D}(\Delta^{1/2})$ with $\Delta^{1/2}\xi = J\xi$.*

(iii) *The orbit map $\alpha^\xi \colon \mathbb{R} \to \mathcal{H}$ extends to a continuous map $\overline{\mathcal{S}_\pi} \to \mathcal{H}$ which is holomorphic on $\mathcal{S}_\pi$ and satisfies $\alpha^\xi(\pi i) = J\xi$.*

(iv) *There exists an element $\eta \in \mathcal{H}^J$ whose orbit map α^η extends to a continuous map $\overline{\mathcal{S}_{\pm\pi/2}} \to \mathcal{H}$ which is holomorphic on the interior and satisfies $\alpha^\eta(-\pi i/2) = \xi$.*

Proof. The equivalence of (i) and (ii) follows from the definition of Δ and J.
(ii) $\Rightarrow$ (iii): If $\xi \in \mathcal{D}(\Delta^{1/2})$, then $\xi \in \mathcal{D}(\Delta^z)$ for $0 \leq \operatorname{Re} z \leq 1/2$, so that the map

$$f \colon \overline{\mathcal{S}_\pi} \to \mathcal{H}, \quad f(z) := \Delta^{-\frac{iz}{2\pi}}\xi$$

is defined. Let P denote the spectral measure of the selfadjoint operator

$$H := -\frac{1}{2\pi}\log\Delta \quad \text{ and let } \quad P^\xi = \langle \xi, P(\cdot)\xi\rangle$$

denote the corresponding positive measure on $\mathbb{R}$ defined by $\xi \in \mathcal{H}$. Then [NÓ18, Lemma A.2.5] shows that

$$\mathcal{L}(P^\xi)(2\pi) = \int_{\mathbb{R}} e^{-2\pi\lambda}\, dP^\xi(\lambda) < \infty.$$

This implies that the kernel

$$\langle f(w), f(z)\rangle = \langle \Delta^{-\frac{iw}{2\pi}}\xi, \Delta^{-\frac{iz}{2\pi}}\xi\rangle = \langle \xi, \Delta^{-\frac{i(z-\overline{w})}{2\pi}}\xi\rangle = \langle \xi, e^{(z-\overline{w})iH}\xi\rangle = \mathcal{L}(P^\xi)\Big(\frac{z-\overline{w}}{i}\Big)$$

is continuous on $\overline{\mathcal{S}_\pi} \times \overline{\mathcal{S}_\pi}$ by the Dominated Convergence Theorem, holomorphic in z, and antiholomorphic in w on the interior ([Ne99, Prop. V.4.6]). This implies (iii) because it shows that f is holomorphic on $\mathcal{S}_\pi$ ([Ne99, Lemma A.III.1]) and continuous on $\overline{\mathcal{S}_\pi}$ (Exercise 1.1).
(iii) $\Rightarrow$ (iv): For $\alpha^\xi \colon \overline{\mathcal{S}_\pi} \to \mathcal{H}$ as in (iii), we have

$$J\alpha^\xi(z) = \alpha^\xi(\pi i + \overline{z}) \tag{3.16}$$

by analytic continuation, so that

$$\eta := \alpha^\xi(\pi i/2) \in \mathcal{H}^J \quad \text{ with } \quad \alpha^\eta(z) = \alpha^\xi\Big(z + \frac{\pi i}{2}\Big).$$

(iv) $\Rightarrow$ (ii): We abbreviate $\mathcal{S} := \mathcal{S}_{\pm\pi/2}$. The kernel $K(z,w) := \langle \alpha^\eta(w), \alpha^\eta(z)\rangle$ is continuous on $\overline{\mathcal{S}} \times \overline{\mathcal{S}}$ and holomorphic in z and antiholomorphic in w on $\mathcal{S}$. It also satisfies $K(z+t,w) = K(z,w-t)$ for $t \in \mathbb{R}$. Hence there exists a continuous function φ on $\overline{\mathcal{S}}$, holomorphic on $\mathcal{S}$, such that

$$K(z,w) = \varphi\Big(\frac{z-\overline{w}}{2}\Big).$$

For $t \in \mathbb{R}$, we then have $\varphi(t) = \langle \eta, \alpha^\eta(2t)\rangle = \int_{\mathbb{R}} e^{2it\lambda}\, dP^\eta(\lambda)$, so that [NÓ18, Lemma A.2.5] yields $\mathcal{L}(P^\eta)(\pm\pi) < \infty$ and $\eta \in \mathcal{D}(\Delta^{\pm 1/4})$. This implies that $\alpha^\eta(z) = \Delta^{-iz/2\pi}\eta$ for $z \in \overline{\mathcal{S}}$.

From $\xi = \alpha^\eta(-\pi i/2) = \Delta^{-1/4}\eta$ we derive that

$$\alpha^\xi(z) = \alpha^\eta\Big(z - \frac{\pi i}{2}\Big) = \Delta^{-iz/2\pi}\xi \quad \text{ for } \quad z \in \overline{\mathcal{S}_\pi}.$$

Further, $J\eta = \eta$ implies

$$J\alpha^\xi(z) = J\alpha^\eta\Big(z - \frac{\pi i}{2}\Big) = \alpha^\eta\Big(\overline{z} + \frac{\pi i}{2}\Big) = \alpha^\xi(\pi i + \overline{z}).$$

For $z = 0$, we obtain in particular $J\xi = \alpha^\xi(\pi i) = \Delta^{1/2}\xi$. □

3.4.4 The geometric KMS condition

On the geometric side, KMS conditions can be modeled as follows. We consider a connected complex manifold Ξ, endowed with a smooth action

$$\sigma\colon \mathbb{R}^\times \times \Xi \to \Xi, \quad (r,m) \mapsto r.m =: \sigma_r(m) =: \sigma^m(r)$$

for which the diffeomorphisms σ_r are holomorphic for $r > 0$ and antiholomorphic for $r < 0$. In particular, $\tau_\Xi := \sigma_{-1}$ is an antiholomorphic involution of Ξ. We further assume that Ξ is an open domain in a larger complex manifold and that the boundary $\partial\Xi$ contains a real submanifold M with the property that, for every fixed point $m \in \Xi^{\tau_\Xi}$, the orbit map $\mathbb{R} \to \Xi, t \mapsto \sigma^m(e^t)$ extends to a holomorphic map $\sigma^m\colon \mathcal{S}_{\pm\pi/2} \to \Xi$ which further extends to a continuous map

$$\sigma^m\colon \overline{\mathcal{S}_{\pm\pi/2}} \to \Xi \cup M \quad \text{with} \quad \sigma^m(\pm i\pi/2) \in M. \tag{3.17}$$

Definition 3.27. We then write

$$W_{\mathrm{KMS}} \subseteq M$$

for the set of all elements whose orbit map $\sigma^m\colon \mathbb{R} \to M$ extends analytically to a continuous map $\overline{\mathcal{S}_\pi} \to \Xi \cup M$, analytic on $\mathcal{S}_\pi$, such that

$$\sigma^m(\pi i) = \tau_\Xi(m).$$

Examples 3.28. (Domains in $\mathbb{C}$) In one-dimension we have the following standard examples of simply connected proper domains in $\mathbb{C}$ with their natural $\mathbb{R}^\times$-actions.
(a) (Strips) On the strip $\mathcal{S}_\pi = \{z \in \mathbb{C}\colon 0 < \operatorname{Im} z < \pi\}$ we have the antiholomorphic involution $\tau_{\mathcal{S}_\pi}(z) = \pi i + \overline{z}$ with fixed point set

$$\mathcal{S}_\pi^{\tau_{\mathcal{S}_\pi}} = \Big\{ z \in \mathcal{S}_\pi\colon \operatorname{Im} z = \frac{\pi}{2}\Big\}.$$

The group $\mathbb{R}_+^\times$ acts by translations via $\sigma_{e^t}(z) = z + t$, $M := \mathbb{R} \cup (\pi i + \mathbb{R}) = \partial\mathcal{S}_\pi$ is a real submanifold, and for $\operatorname{Im} z = \pi/2$, the orbit map $\sigma^z(t)$ extends to the closure of the strip $\mathcal{S}_{\pm\frac{\pi}{2}}$ with $\sigma^z\big(\pm\frac{\pi i}{2}\big) = z \pm \frac{\pi i}{2} \in M$.
(b) (Upper half-plane) On the upper half-plane $\mathbb{C}_+ = \{z \in \mathbb{C}\colon \operatorname{Im} z > 0\}$, we have the antiholomorphic involution $\tau_{\mathbb{C}_+}(z) = -\overline{z}$ and the action of $\mathbb{R}_+^\times$ by dilations $\sigma_r(z) = rz$. Here $M := \mathbb{R} = \partial\mathbb{C}_+$ is a real submanifold, and for $z = iy$, $y > 0$, the orbit map $\sigma^z(t) = e^t z$ extends to the closure of the strip $\mathcal{S}_{\pm\frac{\pi}{2}}$ with $\sigma^z\big(\pm\frac{\pi i}{2}\big) = \pm i(iy) = \mp y$.
(c) (Unit disc) On the unit disc $\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$ we have the antiholomorphic involution $\tau_{\mathbb{D}}(z) = \overline{z}$ and the action of $\mathbb{R}_+^\times \cong \mathrm{SO}_{1,1}(\mathbb{R})_0$ by the fractional linear maps

$$\sigma_t(z) = \frac{\cosh(t/2)z + \sinh(t/2)}{\sinh(t/2)z + \cosh(t/2)}. \tag{3.18}$$

Here $M := \mathbb{S}^1 = \partial\mathbb{D}$ is a real submanifold, and for $z \in \mathbb{D} \cap \mathbb{R}$, the orbit map $\sigma^z(t)$ extends to the closure of the strip $\mathcal{S}_{\pm\pi/2}$ with

$$\sigma^z(\pm\pi i/2) = \frac{\cos(\pi/4)z \pm i\sin(\pi/4)}{\pm i\sin(\pi/4)z + \cos(\pi/4)} = \frac{z \pm i}{\pm iz + 1} = \mp i \cdot \frac{z \pm i}{z \mp i}.$$

The biholomorphic maps

$$\mathrm{Exp}\colon \mathcal{S}_\pi \to \mathbb{C}_+, \quad z \mapsto e^z \quad \text{and} \quad \mathrm{Cay}\colon \mathbb{C}_+ \to \mathbb{D}, \quad \mathrm{Cay}(z) := \frac{z-i}{z+i} \tag{3.19}$$

are equivariant for the described $\mathbb{R}^\times$-actions on the respective domains.

Lemma 3.29. *For a proper simply connected domain $\Omega \subseteq \mathbb{C}$, two antiholomorphic involutions on Ω are conjugate under the group* $\mathrm{Aut}(\Omega)$ *of biholomorphic automorphisms. In particular, they have fixed points.*

Proof. ([ANS22]) By the Riemann Mapping Theorem, we may assume that $\Omega = \mathbb{D}$ is the unit disc. Let $\sigma \colon \mathbb{D} \to \mathbb{D}$ be an antiholomorphic involution. Then σ is an isometry for the hyperbolic metric. Therefore the midpoint of 0 and $\sigma(0)$ is fixed by σ. Conjugating by a suitable automorphism of $\mathbb{D}$, we may therefore assume that $\sigma(0) = 0$. Then $\psi(z) := \sigma(\overline{z})$ is a holomorphic automorphism fixing 0, hence of the form $\psi(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$, so that $\sigma(z) = e^{i\theta}\overline{z} = \gamma(\overline{\gamma^{-1}(z)})$ for $\gamma(z) = e^{i\theta/2}z$. $\square$

Proposition 3.30. *Up to automorphisms of $\mathbb{R}^\times$, any antiholomorphic $\mathbb{R}^\times$-action on a proper simply connected domain $\mathcal{O} \subseteq \mathbb{C}$ is equivalent to the one in* Examples 3.28(a)-(c).

Proof. Up to conjugation with biholomorphic maps, we may assume that $\sigma_{-1}(z) = \overline{z}$ on $\mathcal{O} = \mathbb{D}$ (Lemma 3.29). Now we simply observe that the centralizer of σ_{-1} in the group $\mathrm{PSU}_{1,1}(\mathbb{C}) \cong \mathrm{Aut}(\mathbb{D})$ is $\mathrm{PSO}_{1,1}(\mathbb{R})$, and, up to automorphisms of $\mathbb{R}^\times$, this leads to the action in (3.18). [15] $\square$

Examples 3.31. (Examples of KMS domains) (a) If $G = E \rtimes_\alpha \mathbb{R}$ as in Example 2.7, then $\Xi := E + iC^\circ$ is a tube domain in the complex vector space $E_\mathbb{C}$ with $E \subseteq \partial\Xi$, and Theorem 2.26 implies that

$$E_{\mathrm{KMS}} = C_+^\circ \oplus E_0 \oplus C_-^\circ,$$

which in this case can be verified easily.
(b) For a causal Lie group G and the complex semigroup $\Xi = S(iC_\mathfrak{g}^\circ)$, we obtain from Theorem 2.26 that

$$G_{\mathrm{KMS}} = \exp(C_+^\circ) G_e^h \exp(C_-^\circ).$$

(c) For a non-compactly causal symmetric space $M = G/H$, realized in the boundary of a complex crown domain $\Xi \subseteq G_\mathbb{C}/K_\mathbb{C}$ as the orbit of $o_M := \exp\big(-\frac{\pi i}{2}h\big).K_\mathbb{C}$, we also have

$$M_{\mathrm{KMS}} = W_M^+(h)_{eH} = G_e^h \exp(\Omega_{\mathfrak{q}_\mathfrak{k}}).eH$$

([MNO24, Thm. 8.2]).

3.4.5 Boundary values for one-parameter groups

In this section, we collect some useful facts on boundary values of analytically extended orbit maps of unitary one-parameter groups $(U_t)_{t\in\mathbb{R}}$ and a conjugation J, commuting with U. The main point is to identify the subspace $\mathcal{H}^{-\infty}_{\mathrm{KMS}}$ of distribution vectors, satisfying the KMS condition (cf. Definition 3.23). with elements of a real subspace $\mathcal{H}^J_{\mathrm{temp}}$, specified in terms of the spectral measure P of U.

Let P be the uniquely determined spectral measure on $\mathbb{R}$ for which

$$U_t = \int_\mathbb{R} e^{itx}\, dP(x), \quad \text{resp.} \quad U_t = e^{itA},\ t \in \mathbb{R}, \quad \text{with} \quad A = \int_\mathbb{R} p\, dP(p).$$

For $v \in \mathcal{H}$, we thus obtain finite measures $P^v := \langle v, P(\cdot)v\rangle$, and we define

$$\mathcal{H}^J_{\mathrm{temp}} := \{v \in \mathcal{H}^J \colon e^{\pi p}\, dP^v(p) \ \text{ tempered}\} = \{v \in \mathcal{H}^J \colon e^{-\pi p}\, dP^v(p) \ \text{ tempered}\}. \tag{3.20}$$

[15] The automorphisms of the group $\mathbb{R}^\times$ have the form $\varphi(x) = \mathrm{sgn}(x)|x|^\lambda$, $\lambda \in \mathbb{R}$.

The equality of both spaces on the right follows from the symmetry of the measures P^v, which is a consequence of $Jv = v$. For the positive selfadjoint operator $\Delta := e^{-2\pi A}$, we have $J\Delta J = \Delta^{-1}$, so that $J\mathcal{D}(\Delta^{1/4}) = \mathcal{D}(\Delta^{-1/4})$ implies that

$$\mathcal{D}(\Delta^{1/4}) \cap \mathcal{H}^J = \mathcal{D}(\Delta^{-1/4}) \cap \mathcal{H}^J = \Big\{ v \in \mathcal{H}^J \colon \int_{\mathbb{R}} e^{\pm\pi p}\, dP^v(p) < \infty \Big\} \subseteq \mathcal{H}^J_{\mathrm{temp}}.$$

Theorem 3.32. ([FNÓ24, Thm. 6.1]) *For $v \in \mathcal{H}^J \cap \bigcap_{|t|<\pi/2} \mathcal{D}(e^{tA})$, the following are equivalent:*

(a) $v \in \mathcal{H}^J_{\mathrm{temp}}$.

(b) *The limits $\beta^\pm(v) := \lim_{t\to\pm\pi/2} e^{-tA}v$ exist in $\mathcal{H}^{-\infty}(U)$.*

(c) *There exist $C, N > 0$ such that $\|e^{\pm tA}v\|^2 \leq C\big(\frac{\pi}{2} - |t|\big)^{-N}$ for $|t| < \pi/2$.*

Proof. (a) $\Leftrightarrow$ (b): From [FNÓ23, Prop. 4], we recall that the temperedness of the measure ν_v, given by $d\nu_v(p) := e^{\pi p}\, dP^v(p)$ is equivalent to the existence of $C, N > 0$ with

$$\int_{\mathbb{R}} e^{(\pi - t)p}\, dP^v(p) \leq Ct^{-N} \quad \text{for} \quad 0 \leq t < \pi.$$

Further, [NÓ15, Lemma 10.7] shows that this condition is equivalent to the function $e^{\pi p/2}$ to define a distribution vector for the canonical multiplication representation on $L^2(\mathbb{R}, P^v)$. This representation is equivalent to the subrepresentation of $(U, \mathcal{H})$, generated by v, where the constant function 1 corresponds to v.
(b) $\Rightarrow$ (c): If $\lim_{t\to\pi/2} e^{tA}v$ exist in $\mathcal{H}^{-\infty}(U)$, then [NÓ15, Lemma 10.7], applied to the cyclic subrepresentation generated by v, implies that the measure ν_v is tempered. Then the argument from above implies the existence of $C, N > 0$ with

$$\|e^{tA}v\|^2 = \int_{\mathbb{R}} e^{2tx}\, dP^v(x) \leq C\Big(\frac{\pi}{2} - t\Big)^{-N} \quad \text{for} \quad |t| < \pi/2. \tag{3.21}$$

If $\lim_{t\to-\pi/2} e^{tA}v$ also exists in $\mathcal{H}^{-\infty}(U)$, then the same argument applies again and we obtain (c).
(c) $\Rightarrow$ (a): With the leftmost equality in (3.21), we see that (c) implies that the measures $d\nu_v(x) := e^{\pm\pi x}\, dP^v(x)$ are tempered ([FNÓ23, Prop. 4]). Here we use that the measure P^v is symmetric because $Jv = v$. □

Proposition 3.33. *The map β^+ defines a bijection $\beta^+ \colon \mathcal{H}^J_{\mathrm{temp}} \to \mathcal{H}^{-\infty}_{\mathrm{KMS}}$.*

Proof. (a) Let $v \in \mathcal{H}^J_{\mathrm{temp}}$. First we show that $\beta^+(v) \in \mathcal{H}^{-\infty}_{\mathrm{KMS}}$. To this end, note that, for a real-valued test function $\varphi \in C^\infty_c(\mathbb{R}, \mathbb{R})$, we have $JU(\varphi) = U(\varphi)J$. For $v \in \mathcal{H}^J_{\mathrm{temp}}$ we therefore have $w := U(\varphi)v \in \mathcal{H}^J$. Moreover,

$$dP^w(x) = |\widehat{\varphi}(x)|^2 dP^v(x) \quad \text{with} \quad \widehat{\varphi}(x) = \int_{\mathbb{R}} e^{itx}\varphi(t)\, dt,$$

where $\widehat{\varphi}$ is a Schwartz function, which even implies that the measure

$$e^{\pi x}\, dP^w(x) = e^{\pi x}|\widehat{\varphi}(x)|^2\, dP^v(x)$$

is finite, and thus $w \in \mathcal{D}(\Delta^{1/4}) \cap \mathcal{H}^J$. This implies with [NÓØ21, Prop. 2.1] that

$$U^{-\infty}(\varphi)\beta^+(v) = \beta^+(U(\varphi)v) = \Delta^{1/4}w \in \mathtt{V}.$$

From Proposition 3.25, we derive for $G = \mathbb{R}$ that

$$\mathcal{H}^{-\infty}_{\mathrm{KMS}} = \{\alpha \in \mathcal{H}^{-\infty} \colon (\forall \varphi \in C^\infty_c(\mathbb{R}, \mathbb{R}))\, U^{-\infty}(\varphi)\alpha \in \mathtt{V}\}.$$

Hence the above argument implies that $\beta^+(v) \in \mathcal{H}^{-\infty}_{\mathrm{KMS}}$.
(b) To see that β^+ is injective, we assume that $\beta^+(v) = 0$. Then the above argument implies that $U(\varphi)v \in \mathcal{H}^J \cap \mathcal{D}(\Delta^{1/4})$ vanishes for every $\varphi \in C^\infty_c(\mathbb{R}, \mathbb{R})$ because $\Delta^{1/4}$ is injective. Using an approximate identity in this space, $v = 0$ follows.
(c) To see that β^+ is surjective, let $\gamma \in \mathcal{H}^{-\infty}_{\mathrm{KMS}}$. Replacing $\mathcal{H}$ by the cyclic subrepresentation generated by γ, resp., the subspace $U^{-\infty}(C^\infty_c(\mathbb{R}, \mathbb{C}))\gamma \subseteq \mathcal{H}$, we may w.l.o.g. assume that $\mathcal{H} = L^2(\mathbb{R}, \nu)$ for a positive Borel measure, where the constant function 1 corresponds to γ. Hence the measure ν on $\mathbb{R}$ is tempered ([NÓ15, Lemma 10.7]). Then, for $z = x + iy \in \mathcal{S}_\pi$, the analytic continuation of the orbit map of $\gamma = 1$ takes the form

$$U^\gamma \colon \overline{\mathcal{S}_\pi} \to L^2(\mathbb{R}, \nu)^{-\infty}, \quad U^\gamma(z)(p) = e^{izp} = e^{ixp} e^{-yp}.$$

Therefore all measures $e^{-yp}\, d\nu(p), 0 \leq y \leq \pi$, are tempered. It follows in particular that they are actually finite for $0 < y < p$. Hence $v(p) := e^{-\pi p/2}$ is an L^2-function, and $v = U^\gamma(\pi i/2)$ implies that $Jv = v$. As a consequence, the measure $dP^v(p) = e^{-\pi p}\, d\nu(p)$ is finite and $e^{\pi p}\, dP^v(p) = d\nu(p)$ is tempered, so that $v \in \mathcal{H}^J_{\mathrm{temp}}$. Therefore $\beta^+(v) = 1$ shows that β^+ is surjective. $\square$

For $v, w \in \mathcal{H}^J_{\mathrm{temp}}$, we consider the complex-valued measure

$$P^{v,w}(E) := \langle v, P(E)w\rangle, \quad E \subseteq \mathbb{R}.$$

Then

$$\overline{P^{v,w}(E)} = \overline{\langle v, P(E)w\rangle} = \langle w, P(E)v\rangle = P^{w,v}(E) \tag{3.22}$$

and the relation $JP(E)J = P(-E)$ implies that

$$P^{v,w}(E) = \langle Jv, P(E)Jw\rangle = \langle Jv, JP(-E)w\rangle = \langle P(-E)w, v\rangle = P^{w,v}(-E) = \overline{P^{v,w}(-E)}. \tag{3.23}$$

In particular, the measures $P^{v,v}$ are symmetric and positive.

We obtain on the strip $\mathcal{S}_{\pm\pi}$ the holomorphic function

$$\varphi^{v,w}(z) := \widehat{P^{v,w}}(z) = \int_{\mathbb{R}} e^{izp}\, dP^{v,w}(p),$$

and the temperedness of the measures $e^{\pm\pi p}\, dP^{v,w}(p)$ implies that this function has boundary values that are tempered distributions on $\pm\pi i + \mathbb{R}$. For $t \in \mathbb{R}$, we have $\varphi^{v,w}(t) = \langle v, U_t w\rangle$. Hence

$$\varphi^{w,v}(-t) = \overline{\varphi^{v,w}(t)} = \langle U_t w, v\rangle = \langle U_t Jw, Jv\rangle = \langle JU_t w, Jv\rangle = \langle v, U_t w\rangle = \varphi^{v,w}(t),$$

and therefore

$$\overline{\varphi^{v,w}(z)} = \varphi^{w,v}(-\overline{z}) = \varphi^{v,w}(\overline{z}) \quad \text{for} \quad z \in \mathcal{S}_{\pm\pi}. \tag{3.24}$$

For $\alpha := \beta^+(v)$ and $\gamma := \beta^+(w)$ the distribution

$$D_{\alpha,\gamma}(\xi) := \gamma(U^{-\infty}(\xi)\alpha)$$

can be represented by the boundary values of a holomorphic function

$$D_{\alpha,\gamma}(x) = \lim_{t\to\pi/2} \langle U_x e^{tA} v, e^{tA} w\rangle = \lim_{t\to\pi/2} \int_{\mathbb{R}} e^{(2t-ix)p}\, dP^{v,w}(p) = \varphi^{v,w}(-\pi i - x) = \varphi^{w,v}(\pi i + x).$$

3.4.6 Simon's Growth Theorem

The following result is [Si24, Thm. 3.2.6], where, in addition, we use [FNÓ23, Thm. 3] for the existence of the limit in the smaller subspace $\mathcal{H}^{-\infty}(\partial U(x)) \subseteq \mathcal{H}^{-\infty}$. This result generalizes the extension results by Krötz and Stanton [KSt04] by removing the linearity condition on the group G.

Theorem 3.34. (Simon's Growth Theorem) *Let G be a connected semisimple Lie group with Cartan decomposition $G = K \exp \mathfrak{p}$ and $(\pi, \mathcal{H})$ be an irreducible unitary representation of G. Then there exist for every K-finite vector $v \in \mathcal{H}$ constants $C, n > 0$ such that, for every $x \in \mathfrak{p}$ with spectral radius $r_{\mathrm{Spec}}(\mathrm{ad}\, x) < \pi/2$, we have*

$$\|e^{i\partial U(x)} v\| \leq C\Big(\frac{\pi}{2} - r_{\mathrm{Spec}}(\mathrm{ad}\, x)\Big)^{-n}.$$

In particular, $\lim_{t \to \frac{\pi}{2}-} e^{it\partial U(h)} v$ exists in $\mathcal{H}^{-\infty}(U_h)$ for $h \in \mathfrak{p}$ with $r_{\mathrm{Spec}}(\mathrm{ad}\, h) = 1$ and $U_h(t) = U(\exp th)$.

The last statement uses the equivalence of (b) and (c) in Theorem 3.32.

The following proposition supplements Section 3.3. It neither requires that η_G is injective nor that $G_{\mathbb{C}}$ is simply connected. The main information is contained in (c).

Proposition 3.35. *Let G be a connected semisimple Lie group with Cartan decomposition $G = K \exp \mathfrak{p}$ and $h \in \mathfrak{p}$ an Euler element. Consider an irreducible anti-unitary representation $(\pi, \mathcal{H})$ of G_{τ_h} and put $U_h(t) = \exp(th)$ for $t \in \mathbb{R}$. Then the limits*

$$\beta^{\pm}(v) := \lim_{t \to \pi/2} e^{\mp it\partial U(h)} v$$

exists in $\mathcal{H}^{-\infty}(U_h)$ for every K-finite vector $v \in \mathcal{H}^{[K]}$ and the following assertions hold:

(a) *The maps $\beta^{\pm} \colon \mathcal{H}^{[K]} \to \mathcal{H}^{-\infty}(U_h) \subseteq \mathcal{H}^{-\infty}$ are injective.*

(b) *We have the intertwining relation*

$$\beta^{\pm} \circ \mathtt{d}U(x) = \mathtt{d}U^{-\infty}(\zeta^{\pm 1}(x)) \circ \beta^{\pm} \colon \mathcal{H}^{[K]} \to \mathcal{H}^{-\infty} \quad \text{for} \quad x \in \mathfrak{g}_{\mathbb{C}}.$$

(c) *If $\mathcal{F} \subseteq \mathcal{H}^{[K]}$ is finite-dimensional and J-invariant, then the finite-dimensional real subspaces $\beta^{\pm}(\mathtt{F}) = \beta^{\pm}(\mathcal{F}^J) \subseteq \mathcal{H}^{-\infty}$ is $U^{-\infty}(H)$-invariant for $H = K^h \exp(\mathfrak{p}^{-\tau_h})$. We further have*

$$J\beta^{\pm}(v) = \beta^{\mp}(v) \quad \text{for} \quad v \in \mathcal{H}^{[K]}.$$

Proof. (cf. [FNÓ23, Prop. 7]) For $|t| < \frac{\pi}{2}$, we have $th \in \Omega_{\mathfrak{p}}$, so that the existence of the limits follows from Theorem 3.34. Hence the existence of the limits $\beta^{\pm}(v)$ in the weak-$*$-topology on $\mathcal{H}^{-\omega}(U_h)$ follows from Theorem 3.34, combined with Theorem 3.32.
(a) (cf. [GKÓ04, Thm. 2.1.3]) Suppose that $\beta^+(v) = 0$. As $v \in \mathcal{H}^{[K]}$ is contained in $\mathcal{D}(e^{ti\partial U(h)})$ for $|t| < \frac{\pi}{2}$, the function

$$f \colon \mathbb{R} \to \mathbb{C}, \quad f(t) := \langle v, e^{t\partial U(h)} v\rangle$$

extends analytically to the strip $\mathcal{S}_{\pm\pi/2}$. Our assumption implies that

$$f\big(-\tfrac{\pi i}{2} + t\big) = \beta^+(v)(e^{t\partial U(h)} v) = 0 \quad \text{for} \quad t \in \mathbb{R},$$

so that $f = 0$ by analytic continuation, and thus $0 = f(0) = \|v\|^2$ leads to $v = 0$.

(b) For a K-finite vector v, we have

$$\begin{aligned}\mathtt{d}U^{-\infty}(\zeta^{\pm 1}(x))\beta^{\pm}(v) &= \lim_{t\to\pi/2}\mathtt{d}U(\zeta^{\pm 1}(x))e^{\mp it\partial U(h)}v = \lim_{t\to\pi/2}e^{\mp it\partial U(h)}\mathtt{d}U(e^{\pm it\,\mathrm{ad}\,h}\zeta^{\pm 1}(x))v\\ &= \beta^{\pm}\big(\mathtt{d}U(e^{\pm i\frac{\pi}{2}\,\mathrm{ad}\,h}\zeta^{\pm 1}(x))v\big) = \beta^{\pm}(\mathtt{d}U(x)v).\end{aligned}$$

Here we use that $t\mapsto \mathtt{d}U\big(e^{it\,\mathrm{ad}\,h}\zeta(x)\big)v$ is a continuous curve in a finite-dimensional subspace.
(c) As $Ji\partial U(h)J = -i\partial U(h)$, we have $J\beta^{\pm}(v) = \beta^{\mp}(Jv)$ for $v\in\mathcal{H}^{[K]}$. The relation

$$J\partial U(z)J = \partial U(\tau_h(\overline{z})) \tag{3.25}$$

shows that, on $\mathcal{H}^{[K]}$, the operator $\mathtt{d}U(z)$ for $z\in\mathfrak{h}_{\mathfrak{k}}+i\mathfrak{q}_{\mathfrak{k}}$, commutes with J. By (b), $\beta^{\pm}$ intertwines these operators with $\mathtt{d}U^{-\infty}(\mathfrak{h})$. Hence the subspaces $\beta^{\pm}(\mathtt{F})$ are $\mathtt{d}U^{-\infty}(\mathfrak{h})$-invariant. The subspace $\mathtt{F}=\mathcal{F}^J\subseteq\mathcal{F}$ is invariant under the subgroup K^{τ_h}. As $H = H_K\exp(\mathfrak{h}_{\mathfrak{p}})$ with $H_K\subseteq K^{\tau}=K^{\tau_h}$ and $H_K\subseteq K^h$ ([MNO23, Lemma 4.11]), the K^h-equivariance of $\beta^{\pm}$ entails that the $\mathtt{d}U^{-\infty}(\mathfrak{h})$-invariant subspaces $\beta^{\pm}(\mathtt{F})$ are invariant under $U^{-\infty}(H)$. □

4 Constructing nets of real subspaces

Let G be a connected Lie group, $h\in\mathfrak{g}$ an Euler element, and suppose that the involution $\tau_h^{\mathfrak{g}} = e^{\pi i\,\mathrm{ad}\,h}$ on $\mathfrak{g}$ integrates to an involution τ_h on G, so that we can form the semidirect product G_{τ_h}.

We also fix a homogeneous space $M = G/H$, in which we consider an open subset W invariant under the one-parameter group $\exp(\mathbb{R}h)$. We call the translates $(gW)_{g\in G}$ of W *wedge regions*. At the outset, we do not assume any specific properties of W, but Lemma 4.2 below will indicate which properties good choices of W should have.

We consider an antiunitary representation $(U,\mathcal{H})$ of G_{τ_h} and the canonical standard subspace $\mathtt{V}=\mathtt{V}(h,U)\subseteq\mathcal{H}$, specified by $\Delta_{\mathtt{V}} = e^{2\pi i\partial U(h)}$ and $J_{\mathtt{V}} = U(\tau_h)$ (cf. The Euler Element Theorem 2.3).

4.1 Minimal and maximal nets

We associate to the open subset $W\subseteq M = G/H$ and the antiunitary representation $(U,\mathcal{H})$ of G_{τ_h} the two nets $\mathsf{H}_M^{\min}$ and $\mathsf{H}_M^{\max}$, defined on open subsets of M by

$$\mathsf{H}_M^{\max}(\mathcal{O}) := \bigcap_{g\in G,\mathcal{O}\subseteq gW} U(g)\mathtt{V} \quad\text{and}\quad \mathsf{H}_M^{\min}(\mathcal{O}) := \overline{\sum_{g\in G, gW\subseteq\mathcal{O}} U(g)\mathtt{V}}. \tag{4.1}$$

We call $\mathsf{H}_M^{\max}$ the *maximal net* and $\mathsf{H}_M^{\min}$ the *minimal net*, which is justified by Lemma 4.7 below. By construction, these nets are isotone and covariant, and it is easy to see that they assign $\mathtt{V}$ to $W\subseteq M$ if and only if

$$S_W = \{g\in G\colon g.W\subseteq W\}\subseteq S_{\mathtt{V}} = \{g\in G\colon U(g)\mathtt{V}\subseteq\mathtt{V}\} \tag{4.2}$$

(cf. Lemma 4.2). Any other properties of these nets require a more detailed analysis.

Remark 4.1. (a) If there exists no $g\in G$ with $\mathcal{O}\subseteq gW$, i.e., $\mathcal{O}$ is not contained in any wedge region, then $\mathsf{H}_M^{\max}(\mathcal{O}) = \mathcal{H}$ (the empty intersection). We likewise get $\mathsf{H}_M^{\min}(\mathcal{O}) := \{0\}$ (the empty sum) if there exists no $g\in G$ with $gW\subseteq\mathcal{O}$, i.e., $\mathcal{O}$ contains no wedge region.
(b) If $\emptyset\neq W\neq M$, then we have in particular

$$\mathsf{H}_M^{\min}(\emptyset) = \{0\}\subseteq\mathsf{H}_M^{\max}(\emptyset) = \bigcap_{g\in G}U(g)\mathtt{V} \quad\text{and}\quad \mathsf{H}_M^{\min}(M) = \overline{\sum_{g\in G}U(g)\mathtt{V}}\subseteq\mathsf{H}_M^{\max}(M) = \mathcal{H}.$$

The following lemma is rather elementary. It only uses Proposition 1.20 to verify the equality of standard subspaces.

Lemma 4.2. *The following assertions hold:*

(a) *The nets* $\mathsf{H}_M^{\max}$ *and* $\mathsf{H}_M^{\min}$ *on* M *satisfy* (Iso) *and* (Cov).

(b) *The set of all open subsets* $\mathcal{O} \subseteq M$ *for which* $\mathsf{H}_M^{\max}(\mathcal{O})$ *is cyclic is* G*-invariant.*

(c) *The following are equivalent:*

(i) $S_W := \{g \in G \colon gW \subseteq W\} \subseteq S_{\mathtt{V}}$.

(ii) $\mathsf{H}_M^{\max}(W) = \mathtt{V}$.

(iii) $\mathsf{H}_M^{\max}(W)$ *is standard.*

(iv) $\mathsf{H}_M^{\max}(W)$ *is cyclic.*

(v) $\mathsf{H}_M^{\min}(W) = \mathtt{V}$.

(vi) $\mathsf{H}_M^{\min}(W)$ *is standard.*

(vii) $\mathsf{H}_M^{\min}(W)$ *is separating.*

Proof. (a) Isotony is clear and covariance of the maximal net follows from

$$\mathsf{H}_M^{\max}(g_0\mathcal{O}) = \bigcap_{g_0\mathcal{O}\subseteq gW} U(g)\mathtt{V} = U(g_0) \bigcap_{g_0\mathcal{O}\subseteq gW} U(g_0^{-1}g)\mathtt{V} = U(g_0)\mathsf{H}_M^{\max}(\mathcal{O}).$$

The argument for the minimal net is similar.
(b) follows from covariance.
(c) (i) $\Leftrightarrow$ (ii): Clearly, $\mathsf{H}_M^{\max}(W) \subseteq \mathtt{V}$, and equality holds if and only if $W \subseteq gW$ implies $U(g)\mathtt{V} \supseteq \mathtt{V}$, which is equivalent to $S_W^{-1} \subseteq S_{\mathtt{V}}^{-1}$, and this is equivalent to (i).

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are trivial.

(iv) $\Rightarrow$ (ii): By covariance and $\exp(\mathbb{R}h).W = W$, the subspace $\mathsf{H}_M^{\max}(W) \subseteq \mathtt{V}$ is invariant under the modular group $U(\exp \mathbb{R}h)$ of $\mathtt{V}$. If $\mathsf{H}_M^{\max}(W)$ is cyclic, then Proposition 1.20 implies $\mathsf{H}_M^{\max}(W) = \mathtt{V}$.

(i) $\Leftrightarrow$ (v) is obvious.

(v) $\Rightarrow$ (vi) $\Rightarrow$ (vii) are trivial.

(vii) $\Rightarrow$ (v): By covariance and $\exp(\mathbb{R}h).W = W$, the subspace $\mathsf{H}_M^{\min}(W) \supseteq \mathtt{V}$ is invariant under the modular group $U(\exp \mathbb{R}h)$ of $\mathtt{V}$. If $\mathsf{H}_M^{\min}(W)$ is separating, then Proposition 1.20 implies $\mathsf{H}_M^{\min}(W) = \mathtt{V}$. □

The following lemma is a consequence of the naturality of the minimal and the maximal net.

Lemma 4.3. *The cyclicity of a subspace* $\mathsf{H}_M^{\max}(\mathcal{O})$ *is inherited by subrepresentations, direct sums, direct integrals and finite tensor products.*

Proof. We use that

$$\mathsf{H}_M^{\max}(\mathcal{O}) = \mathtt{V}_A := \bigcap_{g\in A} U(g)\mathtt{V} \quad \text{for} \quad A := \{g \in G \colon g^{-1}\mathcal{O} \subseteq W\}. \tag{4.3}$$

For a direct sum representation $U = U_1 \oplus U_2$ we have $\mathtt{V} = \mathtt{V}_1 \oplus \mathtt{V}_2$, which leads to

$$\mathtt{V}_A = \mathtt{V}_{1,A} \oplus \mathtt{V}_{2,A} \tag{4.4}$$

because $U(g)^{-1}(v_1, v_2) \in \mathtt{V}$ is equivalent to $U_j(g)^{-1}v_j \in \mathtt{V}_j$ for $j = 1, 2$. We thus obtain

$$\mathsf{H}_M^{\max}(\mathcal{O}) = \mathsf{H}_{M,1}^{\max}(\mathcal{O}) \oplus \mathsf{H}_{M,2}^{\max}(\mathcal{O}).$$

This proves that cyclicity of $\mathsf{H}_M^{\max}(\mathcal{O})$ is inherited by subrepresentations and direct sums. For finite tensor products, the assertion follows from Lemma 4.28 in Appendix 4.5.2. If $U = \int_X^\oplus U_x \, d\mu(x)$ is a direct integral, then (4.3) and Lemma D.3(a) imply that

$$\mathsf{H}_M^{\max}(\mathcal{O}) = \int_X^\oplus \mathsf{H}_{M,x}^{\max}(\mathcal{O}) \, d\mu(x) \tag{4.5}$$

for direct integrals. So Lemma D.1 shows that $\mathsf{H}_M^{\max}(\mathcal{O})$ is cyclic if every $\mathsf{H}_{M,x}^{\max}(\mathcal{O})$ is cyclic in $\mathcal{H}_x$. $\square$

Remark 4.4. If we write

$$\mathcal{O}^\wedge := \Big(\bigcap_{gW \supseteq \mathcal{O}} gW\Big)^\circ \supseteq \mathcal{O} \quad \text{and} \quad \mathcal{O}^\vee := \bigcup_{gW \subseteq \mathcal{O}} gW \subseteq \mathcal{O},$$

then $\mathcal{O}^\wedge$ and $\mathcal{O}^\vee$ are open subsets satisfying $(\mathcal{O}^\wedge)^\wedge = \mathcal{O}^\wedge$, $(\mathcal{O}^\vee)^\vee = \mathcal{O}^\vee$, and

$$\mathsf{H}_M^{\max}(\mathcal{O}^\wedge) = \mathsf{H}_M^{\max}(\mathcal{O}) \quad \text{and} \quad \mathsf{H}_M^{\min}(\mathcal{O}^\vee) = \mathsf{H}_M^{\min}(\mathcal{O}). \tag{4.6}$$

So, effectively, the maximal net "lives" on all open subsets $\mathcal{O}$ satisfying $\mathcal{O} = \mathcal{O}^\wedge$ (interiors of intersections of wedge regions) and the minimal net on those open subsets satisfying $\mathcal{O} = \mathcal{O}^\vee$ (unions of wedge regions)

Remark 4.5. (The case where S_W is a group) If the semigroup S_W is a group, i.e., $S_W = G_W$ and $\ker(U)$ is discrete, then the inclusion $S_W \subseteq S_{\mathtt{V}}$ is equivalent to

$$G_W \subseteq G_{\mathtt{V}} = G^{h,J} = \{g \in G^h \colon JU(g)J = U(g)\} \tag{4.7}$$

(cf. Exercise 1.11). In the context of causal homogeneous spaces, the definition of W as a connected component of $W_M^+(h)$ (Definition 2.12) implies that $\exp(\mathbb{R}h) \subseteq G_e^h \subseteq G_W$, and we have in many concrete examples that $G_W \subseteq G^h$, and always $\mathbf{L}(G_W) = \mathfrak{g}^h$ (Proposition 2.22). However, $U(G_W)$ need not commute with J, so that (4.7) may fail. Examples arise already for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$; see Remark 4.6 below.

Remark 4.6. If $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$, then $G_{\mathrm{ad}} = \mathrm{PSL}_2(\mathbb{R}) \cong \mathrm{SO}_{1,2}(\mathbb{R})_e$, and $H_{\mathrm{ad}} = \exp(\mathbb{R}h)$, so that $G_{\mathrm{ad}}/H_{\mathrm{ad}} \cong \mathrm{dS}^2$ (Example 2.21). If $Z(G)$ is non-trivial, then the connected components of $W_M^+(h)$ can be labeled by the elements of $Z(G)$ because this subgroup acts non-trivially on $M = G/H$, leaving the positivity region $W_M^+(h)$ invariant. In any irreducible representation $(U, \mathcal{H})$ we have $U(Z(G)) \subseteq \mathbb{T}$, but this subgroup preserves the standard subspace $\mathtt{V}$ if and only if it is contained in $\{\pm 1\}$.

The following lemma justifies the terminology "minimal" and "maximal".

Lemma 4.7. *Let $(U, \mathcal{H})$ be an antiunitary representation of G_{τ_h} and H a net of real subspaces on open subsets of M, satisfying* (Iso), (Cov) *and $\mathsf{H}(W) = \mathtt{V}$ with respect to $h \in \mathfrak{g}$ and $W \subseteq M$. Then*

$$\mathsf{H}_M^{\min}(\mathcal{O}) \subseteq \mathsf{H}(\mathcal{O}) \subseteq \mathsf{H}_M^{\max}(\mathcal{O}) \qquad \textit{for } \mathcal{O} \subseteq M \textit{ open},$$

and equality holds for all domains of the form $\mathcal{O} = g.W$, $g \in G$ (wedge regions in M).

Proof. First we show that the three properties (Iso), (Cov) and $\mathsf{H}(W) = \mathtt{V}$ of the net H imply that $S_W \subseteq S_{\mathtt{V}}$. In fact, $g.W \subseteq W$ implies

$$U(g)\mathtt{V} = U(g)\mathsf{H}(W) \overset{\text{(Cov)}}{=} \mathsf{H}(g.W) \overset{\text{(Iso)}}{\subseteq} \mathsf{H}(W) = \mathtt{V}.$$

From Lemma 4.2(c) we thus obtain $\mathsf{H}_M^{\max}(W) = \mathsf{H}_M^{\min}(W) = \mathtt{V}$. Hence

$$\mathsf{H}(gW) = U(g)\mathtt{V} = \mathsf{H}_M^{\max}(gW) = \mathsf{H}_M^{\min}(gW)$$

by covariance for any $g \in G$ (Lemma 4.2(a)). By (Iso), $\mathcal{O} \subseteq gW$ implies $\mathsf{H}(\mathcal{O}) \subseteq \mathsf{H}(gW) = U(g)\mathtt{V}$, so that $\mathsf{H}(\mathcal{O}) \subseteq \mathsf{H}_M^{\max}(\mathcal{O})$. Likewise, $gW \subseteq \mathcal{O}$ implies $U(g)\mathtt{V} = \mathsf{H}(gW) \subseteq \mathsf{H}(\mathcal{O})$, and thus $\mathsf{H}_M^{\min}(\mathcal{O}) \subseteq \mathsf{H}(\mathcal{O})$. ☐

Remark 4.8. The construction of the minimal and the maximal net can also be carried out on G itself with respect to $W^G = q_M^{-1}(W)$. It then makes sense to compare both nets and their properties.

For $\mathcal{O} \subseteq M$, the relation $q_M^{-1}(\mathcal{O}) \subseteq gW^G$ is equivalent to $\mathcal{O} \subseteq gW$, so that

$$(q_M)_* \mathsf{H}_G^{\max} = \mathsf{H}_M^{\max}.$$

Likewise, $q_M^{-1}(\mathcal{O}) \supseteq gW^G$ is equivalent to $\mathcal{O} \subseteq gW$, which shows that

$$(q_M)_* \mathsf{H}_G^{\min} = \mathsf{H}_M^{\min}.$$

If, however, $W^G \subseteq G$ is not the full inverse image of $W \subseteq M$, then these relations may fail.

4.2 The endomorphism semigroup of a standard subspace

To describe the semigroup $S_{\mathtt{V}}$, we need the *positive cone*

$$C_U := \{x \in \mathfrak{g} \colon -i \cdot \partial U(x) \geq 0\}, \qquad \partial U(x) = \frac{d}{dt}\Big|_{t=0} U(\exp tx) \tag{4.8}$$

of a unitary representation U. It is a closed, convex, $\mathrm{Ad}(G)$-invariant cone in $\mathfrak{g}$ ([Ne99, Prop. X.1.5]).

The key point of the identity

$$S(h, C_{\mathfrak{g}}) = \exp(C_+) G^h \exp(C_-)$$

in Theorem 2.26 is that it provides two different perspectives on the same subsemigroup of G, and this is instrumental for the descriptions of the semigroups $S_{\mathtt{V}}$. To see this connection, let us first consider an antiunitary representation $(U, \mathcal{H})$ with discrete kernel for a semidirect product G_{τ_h}. We consider the standard subspace $\mathtt{V} := \mathtt{V}(h, U) \subseteq \mathcal{H}$ from (0.1) and Definition 2.54. The Monotonicity Theorem [Ne22, Thm. 3.3] asserts that

$$S_{\mathtt{V}} \subseteq S(h, C_U). \tag{4.9}$$

Its proof is based on the fact that, for two standard subspaces $\mathtt{V}_1 \subseteq \mathtt{V}_2$, we have $\log \Delta_{\mathtt{V}_2} \leq \log \Delta_{\mathtt{V}_1}$ in the sense of quadratic forms. Since these selfadjoint operators are typically not semibounded, the order relation requires some explanation, provided in an appendix to [Ne22]. Put differently, the Monotonicity Theorem asserts that the well-defined G-equivariant map

$$\mathcal{O}_{\mathtt{V}} = U(G)\mathtt{V} \cong G/G_{\mathtt{V}} \to \mathcal{O}_h \cong G/G^0, \quad U(g)\mathtt{V} \mapsto \mathrm{Ad}(g)h$$

is monotone with respect to the $C_{\mathfrak{g}}$-order on $\mathfrak{g}$ (cf. Section 2.5), hence the name.

Theorem 4.9. ([Ne22, Thm. 3.4]) *If $(U,\mathcal{H})$ is an antiunitary representation of G_{τ_h} with discrete kernel, then*

$$S_{\mathtt{V}} = \exp(C_+) G_{\mathtt{V}} \exp(C_-) \quad \textit{for} \quad C_{\mathfrak{g}} = C_U.$$

The Borchers-Wiesbrock Theorem 4.21 in Appendix 4.5.1 immediately shows that $\exp(C_+) \subseteq S_{\mathtt{V}}$. Applying it again with $-h$ and $\mathtt{V}' = \mathtt{V}(-h, U)$, we also get $\exp(C_-) \subseteq S_{\mathtt{V}}$. Therefore the main point is to show that the right-hand side in Theorem 4.9 is a semigroup and that $S_{\mathtt{V}}$ is not larger (cf. Section 2.5).

Example 4.10. (Poincaré group) In Quantum Field Theory on Minkowski space, the natural symmetry group is the proper Poincaré group $P(d) \cong \mathbb{R}^{1,d-1} \rtimes \mathrm{O}_{1,d-1}(\mathbb{R})^\uparrow$ acting by causal isometries on d-dimensional Minkowski space $M := \mathbb{R}^{1,d-1}$. Its Lie algebra is $\mathfrak{g} := \mathfrak{p}(d) \cong \mathbb{R}^{1,d-1} \rtimes \mathfrak{so}_{1,d-1}(\mathbb{R})$ and the closed forward light cone

$$C_{\mathfrak{g}} := \{(x_0, \mathbf{x}) \in \mathbb{R}^{1,d-1} \colon x_0 \geq 0, x_0^2 \geq \mathbf{x}^2\} \tag{4.10}$$

is a pointed invariant cone in $\mathfrak{p}(d)$. The generator $h \in \mathfrak{so}_{1,d-1}(\mathbb{R})$ of the Lorentz boost on the (x_0, x_1)-plane

$$h(x_0, x_1, \ldots, x_{d-1}) = (x_1, x_0, 0, \ldots, 0)$$

is an Euler element and $\tau_h = e^{\pi i \operatorname{ad} h}$ defines an involution on $\mathfrak{g}$, acting on the ideal $\mathbb{R}^{1,d-1}$ (Minkowski space) by

$$\tau_M(x_0, x_1, \ldots, x_{d-1}) = (-x_0, -x_1, x_2, \ldots, x_{d-1}).$$

We apply the results in this section to $G := P(d)_e \cong \mathbb{R}^{1,d-1} \rtimes \mathrm{SO}_{1,d-1}(\mathbb{R})_e$. A unitary representation $(U,\mathcal{H})$ of G is called a *positive energy representation* if $C_{\mathfrak{g}} \subseteq C_U$. If $\ker(U)$ is discrete, then C_U is pointed, and $C_{\mathfrak{g}} = C_U$ follows from the fact that this is, up to sign, the only non-zero pointed invariant cone in the Lie algebra $\mathfrak{g} = \mathfrak{p}(d)$ for $d > 2$; for $d = 2$ there are four pointed invariant cones which are quarter planes.

Here $\mathfrak{g}_0 = \ker(\operatorname{ad} h)$ is the centralizer of the Lorentz boost:

$$\mathfrak{g}_0 = (\{(0,0)\} \times \mathbb{R}^{d-2}) \rtimes (\mathfrak{so}_{1,1}(\mathbb{R}) \oplus \mathfrak{so}_{d-2}(\mathbb{R})) \cong (\mathbb{R}^{d-2} \rtimes \mathfrak{so}_{d-2}(\mathbb{R})) \oplus \mathbb{R}h,$$

and,

$$C_+ = C_{\mathfrak{g}} \cap \mathfrak{g}_1 = \mathbb{R}_{\geq 0}(\mathbf{e}_1 + \mathbf{e}_0) \quad \text{and} \quad C_- = -C_{\mathfrak{g}} \cap \mathfrak{g}_{-1} = \mathbb{R}_{\geq 0}(\mathbf{e}_1 - \mathbf{e}_0). \tag{4.11}$$

The subsemigroup

$$S(h, C_{\mathfrak{g}}) = \{g \in G \colon h - \mathrm{Ad}(g)h \in C_{\mathfrak{g}}\}$$

is easy to determine. The relation $\mathrm{Ad}(g)h - h \in \mathbb{R}^d$ implies that $g = (v, \ell)$ with $\mathrm{Ad}(\ell)h = h$, and then $\mathrm{Ad}(g)h = \mathrm{Ad}(v, \mathbf{1})h = -hv \in -C_{\mathfrak{g}}$ is equivalent to $hv \in C_{\mathfrak{g}}$, which specifies the closure $\overline{W_R}$ of the standard right wedge

$$W_R = \{x \in \mathbb{R}^{1,d-1} \colon x_1 > |x_0|\}.$$

The two cones $C_\pm$ generate a proper Lie subalgebra of $\mathfrak{g}$. We therefore obtain with Lemma 2.23

$$S(h, C_{\mathfrak{g}}) = \overline{W_R} \rtimes \big(\mathrm{SO}_{1,1}(\mathbb{R})^\uparrow \times \mathrm{SO}_{d-2}(\mathbb{R})\big) \overset{2.23}{=} \{g \in G \colon gW_R \subseteq W_R\} = S_{W_R},$$

where $\mathrm{SO}_{1,1}(\mathbb{R})^\uparrow = \exp(\mathbb{R}h)$. For any antiunitary positive energy representation of G_{τ_h}, the semigroup $S_{\mathtt{V}}$ corresponding to the standard subspace $\mathtt{V} = \mathtt{V}(h, U)$ is given by

$$S_{\mathtt{V}} = S(h, C_{\mathfrak{g}}) = S_{W_R}. \tag{4.12}$$

In fact, (4.9) implies $S_{\mathtt{V}} \subseteq S(h, C_{\mathfrak{g}})$, and since $S(h, C_{\mathfrak{g}}) = S_{W_R} = \exp(C_+)G_{W_R}\exp(C_-)$ and the group G_{W_R} is connected, hence contained in $G^{h,\tau_h} \subseteq G_{\mathtt{V}}$, so that (4.12) follows.

For the simply connected covering group $\widetilde{G}$, we obtain the same picture because the involution τ_h acts trivially on the covering group $\widetilde{G}^h$ of G^h.

Example 4.11. (Conformal groups $\mathrm{SO}_{2,d}(\mathbb{R})$) The Lie algebra of the conformal group $G := \mathrm{SO}_{2,d}(\mathbb{R})_e$ of Minkowski space is $\mathfrak{g} = \mathfrak{so}_{2,d}(\mathbb{R})$, which contains the Poincaré algebra as those elements corresponding to affine vector fields on $\mathrm{V} := \mathbb{R}^{1,d-1}$. For $d \geq 3$ it is a simple hermitian Lie algebra. It contains many Euler elements h, but they are all mutually conjugate (Proposition 2.46). One arises from the element $h = \mathrm{id}_{\mathrm{V}}$ corresponding to the Euler vector field on V. Then $\mathfrak{g}_j(h)$, $j = -1, 0, 1$, are spaces of vector fields on V which are linear (for $j = 0$), constant (for $j = 1$) and quadratic (for $j = -1$).

Another important example is the element $h_1 \in \mathfrak{so}_{1,d-1}(\mathbb{R}) \subseteq \mathfrak{so}_{2,d-1}(\mathbb{R})$ corresponding to a Lorentz boost in the Poincaré algebra (see Example 4.10).

We consider the minimal invariant cone $C_{\mathfrak{g}} \subseteq \mathfrak{g}$ which intersects V in the positive light cone $C_+ \subseteq \mathrm{V}$. We obtain a complete description of the corresponding semigroups by

$$S_{\mathtt{V}} = \exp(C_+)G_{\mathtt{V}}\exp(C_-),$$

and here these semigroups have interior points because $C_\pm$ generate the subspaces $\mathfrak{g}_{\pm 1}$.

Example 4.12. Another interesting example which is neither semisimple nor an affine group is given by the Lie algebra

$$\mathfrak{g} = \mathfrak{hcsp}(V, \omega) := \mathfrak{heis}(V, \omega) \rtimes \mathfrak{csp}(V, \omega)$$

from Example 2.51. Now we turn to the corresponding group and one of its irreducible unitary representations. Choosing a symplectic basis, we obtain an isomorphism with

$$V \cong V_{-1} \oplus V_1 \cong \mathbb{R}^n \oplus \mathbb{R}^n$$

with the canonical symplectic form specified by $\omega((q,0),(0,p)) = \langle q, p\rangle$ and $\tau_V(q,p) = (-q,p)$. Let $\mathrm{Mp}_{2n}(\mathbb{R})$ denote the *metaplectic group*, which is the unique non-trivial double cover of $\mathrm{Sp}_{2n}(\mathbb{R})$. We consider the group

$$G := \mathrm{Heis}(\mathbb{R}^{2n}) \rtimes_\alpha (\mathbb{R}^\times_+ \times \mathrm{Mp}_{2n}(\mathbb{R})),$$

where $\mathbb{R}^\times$ acts on $\mathrm{Heis}(\mathbb{R}^{2n}) = \mathbb{R} \times \mathbb{R}^{2n}$ by $\alpha_r(z, v) = (r^2 z, rv)$. Its Lie algebra is $\mathfrak{g} = \mathfrak{hcsp}(V, \omega)$. Then

$$\mathcal{H} := L^2\Big(\mathbb{R}^\times_+, \frac{d\lambda}{\lambda}; L^2(\mathbb{R}^n)\Big) \cong L^2\Big(\mathbb{R}^\times_+ \times \mathbb{R}^n, \frac{d\lambda}{\lambda} \otimes dx\Big),$$

carries an irreducible unitary representation of G, where $L^2(\mathbb{R}^n) \cong L^2(V_{-1})$ carries the oscillator representation U_0 of $\mathrm{Heis}(\mathbb{R}^{2n}) \rtimes \mathrm{Mp}_{2n}(\mathbb{R})$. The Heisenberg group $\mathrm{Heis}(\mathbb{R}^{2n})$ is represented on $\mathcal{H}$ by

$$(U(z,0,0)f)(\lambda, x) = e^{i\lambda^2 z} f(\lambda, x), \tag{4.13}$$

$$(U(0,q,0)f)(\lambda, x) = e^{i\lambda\langle q, x\rangle} f(\lambda, x), \tag{4.14}$$

$$(U(0,0,p)f)(\lambda, x) = f(\lambda, x - \lambda p). \tag{4.15}$$

The group $\mathrm{Mp}_{2n}(\mathbb{R})$ acts by the metaplectic representation on $L^2(\mathbb{R}^n)$ via

$$(U(g)f)(\lambda, \cdot) := U_0(g) f(\lambda, \cdot),$$

independently of λ. The one-parameter group $\mathbb{R}_+^\times = \exp(\mathbb{R}h_0)$ acts by

$$(U'(r)f)(\lambda, x) := f(r\lambda, x) \quad \text{for} \quad r > 0.$$

We also note that we have a conjugation J on $\mathcal{H}$ defined by

$$(Jf)(\lambda, x) := \overline{f(\lambda, -x)} \quad \text{satisfying} \quad JU(g)J = U(\tau_G(g)),$$

where τ_G induces on $\mathfrak{g}$ the involution $e^{\pi i \operatorname{ad} h} = (-\tau_V)^{\sim}$(cf. Example 2.51).

The positive cone $C_U \subseteq \mathfrak{g}$ is the same as the one of the metaplectic representation. It intersects $\mathfrak{sp}(V, \omega)$ in its unique invariant cone of non-negative polynomials of degree 2 on V. This implies that $(C_U)_- = C_-$. To determine $(C_U)_+ = C_U \cap \mathfrak{g}_1$, we observe that $\mathfrak{g}_1$ acts on $L^2(\mathbb{R}^n) \cong L^2(V_-)$ by multiplication operators. This shows that we also have $(C_U)_+ = C_+$, so that we can determine the semigroup $S_{\mathtt{V}}$ for the standard subspace $\mathtt{V} = \mathtt{V}(h, U)$. It takes the form

$$S_{\mathtt{V}} = \exp(C_+) G_{\mathtt{V}} \exp(C_-),$$

where $G_{\mathtt{V}} = G^h$ is a double cover of $\mathrm{Aff}(\mathbb{R}^n)_e$, its inverse image in $\mathrm{Mp}_{2n}(\mathbb{R})$.

4.3 Causal symmetric spaces

The following theorem follows from the Localization Theorem 4.33:

Theorem 4.13. *If $M = G/H$ is a semisimple non-compactly causal symmetric space and $(U, \mathcal{H})$ an antiunitary representation of G, then the net $\mathsf{H}_M^{\max}$ satisfies* (Iso), (Cov), (RS) *and* (BW).

Proof. In this case, $S_W = G_W = G_e^h H^h \subseteq G^h$ is a group by Theorem 4.35. Since $\tau = \tau_h \theta$ coincides on K with τ_h, we further have $H^h \subseteq K^{h,\tau} \subseteq K^{\tau_h}$, so that $S_W \subseteq G^{h,\tau_h} \subseteq G_{\mathtt{V}}$. Therefore Lemma 4.2 shows that $\mathsf{H}_M^{\max}$ satisfies (BW), and (RS) follows from Theorem 4.33. □

Theorem 4.14. *If $M = G/H$ is an irreducible compactly causal symmetric space and $(U, \mathcal{H})$ an antiunitary representation of G, then the net $\mathsf{H}^{\max}$ satisfies* (Iso), (Cov), (RS) *and* (BW) *if and only if*

- *the positive spectrum condition $C_+ \subseteq C_U$ is satisfied, and*
- *$U(G_W)$ commutes with J, i.e., $\tau_h(g)g^{-1} \in \ker U$ for $g \in H^h$.*

Proof. From [NÓ23a, Thm. 9.1] it follows that

$$S_W = G_e^h H^h \exp(C_+ + C_-) \quad \text{for} \quad C_\pm = \pm C_{\mathfrak{g}} \cap \mathfrak{g}_{\pm 1}, \quad C = C_{\mathfrak{g}} \cap \mathfrak{q}.$$

Therefore $S_W \subseteq S_{\mathtt{V}}$ is equivalent to

$$C_\pm \subseteq \pm C_U \quad \text{and} \quad U(g)J = JU(g) \quad \text{for} \quad g \in H^h.$$

The first condition implies that $C_U \neq \{0\}$. As $\ker U$ is discrete and $\mathfrak{g}$ is simple or a sum of two simple ideals, C_U is a pointed generating invariant cone with $C_{U,\pm} = C_\pm$. Conversely, $C_\pm \subseteq \pm C_U$ implies that $C_{U,\pm} = C_\pm = (C_{\mathfrak{g}})_\pm$ ([NÓ23a, Prop. 2.7(d)]). □

4.4 Causal flag manifolds

Let $M = G/Q_h$ be an irreducible causal flag manifold. The results in this section can be found in [MN25]. The fundamental group $\pi_1(M)$ is isomorphic to $\mathbb{Z}$ ([Wig98, Thm. 1.1]), so that there exists for every $k \in \mathbb{N} \cup \{\infty\}$ a k-fold covering M_k, where M_∞ is simply connected.

- M_∞ is a simple space-time manifold in the sense of Mack/de Riese [MdR07]. It carries a global causal order (no closed causal curves).
- $M_k, k < \infty$, has closed causal curves, hence no global causal order.
- The open embedding $\iota_M \colon \mathrm{V} = \mathfrak{g}_1 \hookrightarrow M$ of the euclidean Jordan algebra V lifts to open embedding $\iota_{M_k} \colon E \hookrightarrow M_k$.
- In M_k the canonical wedge region is

  $$W_{M_k} := \iota_{M_k}(C_+^\circ) \subseteq M_k.$$

  It is a connected component of the positivity domain $W_{M_k}^+(h)$ of the Euler vector field $X_{M_k}^h$ on M_k. In M_k the positivity domain $W_{M_k}^+(h)$ has k connected components.

Examples 4.15. (a) For Minkowski space $\mathrm{V} = \mathbb{R}^{1,d-1}$, the conformal completion

$$M \cong (\mathbb{S}^1 \times \mathbb{S}^{d-1})/\{\pm\mathbf{1}\} \subseteq \mathbb{P}(\mathbb{R}^{2,d})$$

is the isotropic quadric in $\mathbb{P}(\mathbb{R}^{2,d})$ on which $G = \mathrm{SO}_{2,d}(\mathbb{R})_e$ acts. In this case,

$$M_\infty \cong \mathbb{R} \times \mathbb{S}^{d-1}.$$

(b) For the euclidean Jordan algebra $\mathrm{V} = \mathrm{Herm}_r(\mathbb{C})$, we have $M \cong \mathrm{U}_r(\mathbb{C})$, on which $G = \mathrm{SU}_{r,r}(\mathbb{C})$ acts by fractional linear transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.z = (az+b)(cz+d)^{-1}.$$

Here

$$\widetilde{\mathrm{U}}_r(\mathbb{C}) \cong \mathbb{R} \times \mathrm{SU}_r(\mathbb{C}).$$

(c) For the euclidean Jordan algebra $\mathrm{V} = \mathrm{Sym}_r(\mathbb{R})$, the conformal compactification is the space M of Lagrangian subspaces in the symplectic vector space $(\mathbb{R}^{2r}, \omega)$, on which $G = \mathrm{Sp}_{2r}(\mathbb{R})$ acts naturally. Here $M_\infty \cong \mathbb{R} \times (\mathrm{SU}_r(\mathbb{C})/\mathrm{SO}_r(\mathbb{R}))$.

To formulate existence criteria for nets on the M_k, we observe that the simply connected covering group $\widetilde{G}$ acts on every $M_k, k \in \mathbb{N} \cup \{\infty\}$. The centralizer $\widetilde{G}^h$ of h in this group satisfies

$$\pi_0(\widetilde{G}^h) \cong \pi_1(M) \cong \mathbb{Z}.$$

We pick $g_h \in \widetilde{G}^h$ so that its connected component generates $\pi_0(\widetilde{G}^h)$ and

$$\tau_h(g_h) = g_h^{-1}. \tag{4.16}$$

This element can be chosen to be central in an $\widetilde{\mathrm{SL}}_2(\mathbb{R})$-subgroup with $g_h^2 \in Z(\widetilde{G})$.

Theorem 4.16. (Existence of nets) *For an antiunitary representation* $(U, \mathcal{H})$ *of* $\widetilde{G}_{\tau_h}$*, a net* H *on open subsets of* M_k *satisfying* (Iso), (Cov), (RS) *and* (BW) *exists if and only if*

- U *satisfies the positive energy condition* $C_+ \subseteq C_U$.
- $g_h^{2k} \in \ker U$ *for* $k < \infty$*, and no condition for* $k = \infty$.

Proof. (Sketch; see [MN25]) In view of Lemmas 4.2 and 4.7, the existence of a net H satisfying (Iso), (Cov) and (BW) is equivalent to $S_{W_{M_k}} \subseteq S_{\mathtt{V}}$. Now

$$S_{W_{M_k}} = \widetilde{G}_{W_{M_k}} \exp(C_+ + C_-) \quad \text{and} \quad S_{\mathtt{V}} = \widetilde{G}_{\mathtt{V}} \exp(C_+^U + C_-^U) \ \text{ for } \ C_\pm^U = \pm C_U \cap \mathfrak{g}_{\pm 1}(h)$$

reduces the problem to the inclusion

$$\widetilde{G}_{W_{M_k}} \subseteq \widetilde{G}_{\mathtt{V}}$$

and the positive energy condition

$$C_+ \subseteq C_U.$$

As $\widetilde{G}_{W_{M_k}} \subseteq \widetilde{G}_{W_M} = \widetilde{G}^h$ commutes with the Euler element h, the inclusion $\widetilde{G}_{W_{M_k}} \subseteq \widetilde{G}_{\mathtt{V}}$ is equivalent to $\{g\tau_h(g)^{-1} \colon g \in \widetilde{G}_{W_{M_k}}\} = g_h^{2k\mathbb{Z}} \subseteq \ker U$.

So it only remains to verify the Reeh–Schlieder condition. We refer to [MN25] for details. □

Theorem 4.17. (Existence and uniqueness of additive nets) *For every antiunitary representation* $(U, \mathcal{H})$ *of* $\widetilde{G}$ *satisfying the positive energy condition* $C_+ \subseteq C_U$*, there exists a unique additive net* H *on open subsets of* M_∞ *satisfying* (Iso), (Cov), (RS) *and* (BW).
On M_k *such nets exist if and only if, in addition,* $g_h^{2k} \subseteq \ker U$.

Proof. **Uniqueness:** On every M_k, the wedge regions form a basis for the topology. Every additive covariant net H satisfying (BW) thus satisfies

$$\mathsf{H}(\mathcal{O}) = \mathsf{H}\Bigg(\bigcup_{g.W_{M_k} \subseteq \mathcal{O}} g.W_{M_k}\Bigg) = \overline{\sum_{g.W_{M_k} \subseteq \mathcal{O}} U(g)\mathsf{H}(W_{M_k})} = \overline{\sum_{g.W_{M_k} \subseteq \mathcal{O}} U(g)\mathtt{V}},$$

so that H is determined by the representation U via $\mathsf{H}(W_{M_k}) = \mathtt{V} = \mathtt{V}(h, U)$.
Existence: The argument for existence builds on nets for irreducible representations ([NÓ21]) and direct integral techniques. □

The preceding theorem extends results by Brunetti, Guido and Longo [BGL93] for the Jordan algebra $\mathrm{V} = \mathbb{R}^{1,d-1}$ and the group $G = \mathrm{SO}_{2,d}(\mathbb{R})_e$.

4.4.1 Locality

Locality conditions concern open G-invariant subsets $\mathcal{D}_{\mathrm{loc}} \subseteq M \times M$. Here are some relevant facts:

- $M \times M$ contains a unique open G-orbit $\mathcal{D}^*$.
- $M_\infty \times M_\infty$ contains infinitely many open $\widetilde{G}$-orbits $(\mathcal{D}_n^*)_{n \in \mathbb{Z}}$, permuted the group $\pi_1(M) \cong \mathbb{Z}$ acting by deck transformations.
- $M_k \times M_k$ contains k open $\widetilde{G}$-orbits $\mathcal{D}_n^*$, $n \in \mathbb{Z}/k\mathbb{Z}$, permuted by deck transformations of $\mathrm{Deck}(M_k) \cong \pi_1(M)/\pi_1(M_k) \cong \mathbb{Z}/k\mathbb{Z}$.

Let $g_h \in \widetilde{G}^h$ be as above and pick $z_{\mathfrak{k}} \in \mathfrak{z}(\mathfrak{k})$ such that $\theta := \exp(\pi \operatorname{ad} z_{\mathfrak{k}})$ is a Cartan involution.

Theorem 4.18. (Locality properties of the nets) *For the unique additive net associated to the positive energy representation $(U, \mathcal{H})$ of $\widetilde{G}_{\tau_h}$ on M_k, $k \in \mathbb{N} \cup \{\infty\}$, the following are equivalent:*

(a) *The locality condition for $\mathcal{D}_n^*$ (n-locality with $n \in \mathbb{Z}/k\mathbb{Z}$):*

$$\mathcal{O}_1 \times \mathcal{O}_2 \subseteq \mathcal{D}_n^* \quad \Rightarrow \quad \mathsf{H}(\mathcal{O}_1) \subseteq \mathsf{H}(\mathcal{O}_2)'$$

(b) $\mathsf{H}(g_h^n.W_{M_\infty}) \subseteq \mathsf{H}(W'_{M_\infty})'$ *for the dual wedge* $W'_{M_\infty} := \theta.W_{M_\infty}$.

(c) $\exp(2\pi z_{\mathfrak{k}}) g_h^{2n} \in \ker U$. [16]

4.4.2 The massless spin 0 representation on Minkowski space

We consider Minkowski space $\mathrm{V} = \mathbb{R}^{1,d-1}$ and its conformal compactification M. On $\widetilde{G} = \widetilde{\mathrm{SO}}_{2,d}(\mathbb{R})_e$ the "minimal" positive energy representation $(U, \mathcal{H})$ is the extension of the Poincaré representation corresponding to *massless spin 0-particles.*

It depends on the dimension d, to which quotient group of $\widetilde{G}$ the representation U descends, and on which covering of M the net can be implemented. We have the following properties (cf. [BGL93]):

- $d - 2 \in 4\mathbb{Z}$: U is defined on the adjoint group $\mathrm{SO}_{2,d}(\mathbb{R})_e/\{\pm\mathbf{1}\}$ and the net lives on M.
- $d \in 4\mathbb{Z}$: U is defined on $\mathrm{SO}_{2,d}(\mathbb{R})_e$ with $U(-1) = -\mathbf{1}$ and here also the net lives on M.
- d odd: U is defined on a 2-fold covering of $\mathrm{SO}_{2,d}(\mathbb{R})_e$ and the net lives on on $M_2 \cong \mathbb{S}^1 \times \mathbb{S}^{d-1}$.

Remark 4.19. The n-locality condition on M_2 (for $n = 0, 1$) is $(n+1)d \in 2\mathbb{Z}$.

- For d even, the net therefore is 0- and 1-local, which corresponds to spacelike and timelike locality on Minkowski space.
- For d odd, it is only 1-local, which corresponds to spacelike locality on Minkowski space.

These locality conditions relate to support properties of the fundamental solutions of the Klein–Gordon equation (Huygens' Principle). We refer to [MN25] for details.

4.5 Appendices to Section 4

4.5.1 Standard pairs

Definition 4.20. *Standard pairs* $(U, \mathtt{V})$ consist of a standard subspace $\mathtt{V}$ of a complex Hilbert space $\mathcal{H}$ and a unitary one-parameter group $(U_t)_{t \in \mathbb{R}}$ on $\mathcal{H}$ such that $U_t \mathtt{V} \subseteq \mathtt{V}$ for $t \geq 0$ and $U_t = e^{itH}$ with $H \geq 0$.

Theorem 4.21. (Borchers–Wiesbrock Theorem) *Any standard pair defines an antiunitary positive energy representation of* $\mathrm{Aff}(\mathbb{R}) \cong \mathbb{R} \rtimes \mathbb{R}^\times$ *by*

$$U(b, e^s) := U_b \Delta_{\mathtt{V}}^{-is/2\pi} \quad \text{and} \quad U(0, -1) := J_{\mathtt{V}}. \tag{4.17}$$

Conversely, all these representations define standard pairs.

[16] Note that $\ker U \subseteq G$ is central.

Proof. This is the Borchers–Wiesbrock Theorem ([NÓ17, Thm. 3.18], see also [Bo92], [Wi93]). $\square$

Proposition 4.22. *Consider a Lie group $G_\sigma = G \rtimes \{\mathbf{1}, \sigma\}$, where $\sigma \in \mathrm{Aut}(G)$ is an involution. Let $(U, \mathcal{H})$ be an antiunitary representation of G_σ. Suppose that $(\mathtt{V}, U^j)$, $j = 1, 2$, are standard pairs for which there exists a graded homomorphism $\gamma \colon \mathbb{R}^\times \to G$ and $x_1, x_2 \in \mathfrak{g}$ such that*

$$J_{\mathtt{V}} = U(\gamma(-1)), \quad \Delta_{\mathtt{V}}^{-it/2\pi} = U(\gamma(e^t)), \quad \text{and} \quad U^j(t) = U(\exp tx_j), \quad t \in \mathbb{R}, j = 1, 2.$$

Then the unitary one-parameter groups U^1 and U^2 commute.

Proof. The positive cone $C_U \subseteq \mathfrak{g}$ of the representation U is a closed convex $\mathrm{Ad}(G)$-invariant cone. As we may w.l.o.g. assume that U is injective, the cone C_U is pointed.

Writing $\Delta_{\mathtt{V}}^{-it/2\pi} = U(\exp th)$ and $U_t^j = U(\exp tx_j)$ with $h, x_1, x_2 \in \mathfrak{g}$, we have $[h, x_j] = x_j$ for $j = 1, 2$ and $x_1, x_2 \in C_U$ by (4.17). If

$$\mathfrak{g}_\lambda(h) = \ker(\mathrm{ad}\, h - \lambda\mathbf{1})$$

is the λ-eigenspace of $\mathrm{ad}\, h$ in $\mathfrak{g}$, then $[\mathfrak{g}_\lambda(h), \mathfrak{g}_\mu(h)] \subseteq \mathfrak{g}_{\lambda+\mu}(h)$, so that $\mathfrak{g}_+ := \sum_{\lambda>0} \mathfrak{g}_\lambda(h)$ is a nilpotent Lie algebra. Therefore $\mathfrak{n} := (C_U \cap \mathfrak{g}_+) - (C_U \cap \mathfrak{g}_+)$ is a nilpotent Lie algebra generated by the pointed invariant cone $C_U \cap \mathfrak{g}_+$. By [Ne99, Ex. VII.3.21], $\mathfrak{n}$ is abelian. Finally $x_j \in C_U \cap \mathfrak{g}_1(h) \subseteq \mathfrak{n}$ implies that $[x_1, x_2] = 0$. $\square$

One may expect that one-parameter groups U^1 and U^2, for which $(\mathtt{V}, U^j)$ form a standard pair, commute. By Proposition 4.22 this is true if they both come from an antiunitary representation of a finite-dimensional Lie group. The following example shows that this is not true in general, not even if the two one-parameter groups are conjugate under the stabilizer group $\mathrm{U}(\mathcal{H})_{\mathtt{V}}$.

Example 4.23. On $L^2(\mathbb{R})$ we consider the selfadjoint operators

$$(Qf)(x) = xf(x) \quad \text{and} \quad (Pf)(x) = if'(x),$$

satisfying the canonical commutation relations $[P, Q] = i\mathbf{1}$. For both operators, the Schwartz space $\mathcal{S}(\mathbb{R}) \subseteq L^2(\mathbb{R})$ is a core. Actually it is the space of smooth vectors for the representation of the 3-dimensional Heisenberg group generated by the corresponding unitary one-parameter groups

$$(e^{itQ}f)(x) = e^{itx}f(x) \quad \text{and} \quad (e^{itP}f)(x) = f(x - t).$$

Since e^{ix^3} is a smooth function for which all derivatives grow at most polynomially, it defines a continuous linear operator on $\mathcal{S}(\mathbb{R})$ ([Tr67, Thm. 25.5]). Therefore the unitary operator $T := e^{iQ^3}$ maps $\mathcal{S}(\mathbb{R})$ continuously onto itself, and

$$\widetilde{P} := TPT^{-1} = e^{iQ^3} P e^{-iQ^3}$$

is a selfadjoint operator for which $\mathcal{S}(\mathbb{R})$ is a core. For $f \in \mathcal{S}(\mathbb{R})$, we obtain

$$(\widetilde{P}f)(x) = ie^{ix^3} \frac{d}{dx} e^{-ix^3} f(x) = i(-i3x^2 f(x) + f'(x)),$$

so that $\widetilde{P} = P + 3Q^2$ on $\mathcal{S}(\mathbb{R})$.

The two selfadjoint operators Q and e^P are the infinitesimal generators of the irreducible antiunitary representation of $\mathrm{Aff}(\mathbb{R}) = \mathbb{R} \rtimes \mathbb{R}^\times$, given by

$$U(b, e^t) = e^{ibe^P} e^{itQ} \quad \text{and} \quad (U(0, -1)f)(x) = \overline{f(-x)}.$$

Accordingly, the pair (Δ, J) with

$$\Delta = e^{-2\pi Q} \quad \text{and} \quad J = U(0, -1)$$

specifies a standard subspace $\mathtt{V}$ which combines with $U^1_t := e^{ite^P}$ to an irreducible standard pair $(\mathtt{V}, U^1)$. The unitary operator T commutes with Δ and with J because $JQJ = -Q$, so that $T(\mathtt{V}) = \mathtt{V}$. Therefore the unitary one-parameter group $U^2_t := e^{iQ^3}U^1_t e^{-iQ^3} = e^{ite^{\tilde{P}}}$ also defines a standard pair $(\mathtt{V}, U^2)$. These two one-parameter groups do not commute because otherwise the selfadjoint operators P and $P + 3Q^2$ would commute in the strong sense, hence in particular on their core $\mathcal{S}(\mathbb{R})$.

4.5.2 Regularity of unitary representations

Definition 4.24. We call an antiunitary representation $(U, \mathcal{H})$ of G_{τ_h} *regular with respect to* h, or *h-regular*, if there exists an e-neighborhood $N \subseteq G$ such that $\mathtt{V}_N = \bigcap_{g \in N} U(g)\mathtt{V}$ is cyclic. Replacing N by its interior, we may always assume that N is open.

A key motivation for [BN25] was the "regularity conjecture" from [MN24]. It asserts that, for any Euler element $h \in \mathfrak{g}$ any antiunitary representation $(U, \mathcal{H})$ of G_{τ_h} is h-regular. This conjecture holds for connected reductive groups by Corollary 4.34 and for several specific classes of groups and representations (see [MN24] for details).

Lemma 4.25. *For an antiunitary representation $(U, \mathcal{H})$ of G_{τ_h}, the following assertions hold:*

(a) *If $U = U_1 \oplus U_2$ is a direct sum, then U is h-regular if and only if U_1 and U_2 are h-regular.*

(b) *If U is h-regular, then every subrepresentation is h-regular.*

(c) *Assume that G has at most countably many connected components and let $U = \int_X^\oplus U_m \, d\mu(m)$ be an antiunitary direct integral representation of G_{τ_h}, then U is regular if and only if there exists an e-neighborhood $N \subseteq G$ such that, for μ-almost every $m \in X$, the subspace $\mathtt{V}_{m,N}$ is cyclic.*

Proof. (a) If $U \cong U_1 \oplus U_2$, then (4.4) implies that $\mathtt{V}_N = \mathtt{V}_{1,N} \oplus \mathtt{V}_{2,N}$ for every e-neighborhood $N \subseteq G$. In particular, $\mathtt{V}_N$ is cyclic if and only if $\mathtt{V}_{1,N}$ and $\mathtt{V}_{2,N}$ are.
(b) follows immediately from (a).
(c) Applying Lemma D.3(b) to $A := N$, we obtain (c). □

Note that the following theorem does not require any assumption concerning the irreducibility of the representation. Although its proof draws heavily from [FNÓ23], which deals with irreducible representations, Proposition 4.26 is a convenient tool to reduce to this situation.

Proposition 4.26. *Assume that G has at most countably many connected components and that $A \subseteq G$ is a subset. Then the following are equivalent:*

(a) *For all antiunitary representations $(U, \mathcal{H})$ of G_{τ_h}, the subspace $\mathtt{V}_A := \bigcap_{g \in A} U(g)\mathtt{V}$ is cyclic.*

(b) *For all irreducible antiunitary representations $(U, \mathcal{H})$ of G_{τ_h}, the subspace $\mathtt{V}_A$ is cyclic.*

(c) *For all irreducible unitary representations $(U, \mathcal{H})$ of G, the subspace $\widetilde{\mathtt{V}}_A$ is cyclic in $\widetilde{\mathcal{H}}$, where $\widetilde{\mathtt{V}} := \mathtt{V}(h, \widetilde{U})$.*

(d) (Characterization in terms of unitary representations) *For all unitary representations* $(U, \mathcal{H})$ *of* G*, the subspace* $\widetilde{\mathtt{V}}_A$ *is cyclic in* $\widetilde{\mathcal{H}}$

Proposition 4.27. (Localizability implies regularity) *Let* $\emptyset \neq \mathcal{O} \subseteq W \subseteq M$ *be open subsets such that* $N := \{g \in G \colon g^{-1}\mathcal{O} \subseteq W\}$ *is an* e*-neighborhood. If* $(U, \mathcal{H})$ *is an antiunitary representation for which* $\mathsf{H}_M^{\max}(W) = \mathtt{V}$ *and* $\mathsf{H}_M^{\max}(\mathcal{O})$ *is cyclic, then it is regular.*

Proof. By assumption $\mathsf{H}_M^{\max}(\mathcal{O})$ is cyclic, and

$$\mathsf{H}_M^{\max}(\mathcal{O}) \subseteq \bigcap_{g \in N} \mathsf{H}_M^{\max}(gW) = \bigcap_{g \in N} U(g)\mathsf{H}_M^{\max}(W) = \bigcap_{g \in N} U(g)\mathtt{V} = \mathtt{V}_N.$$

It follows that $\mathtt{V}_N$ is cyclic. □

Lemma 4.28. *Suppose that* $(U, \mathcal{H}) = \otimes_{j=1}^n (U_j, \mathcal{H}_j)$ *is a tensor product of antiunitary representations of* G_{τ_h}*. Then the standard subspace* $\mathtt{V} = \mathtt{V}(h, U)$ *is a tensor product*

$$\mathtt{V} = \mathtt{V}_1 \otimes \cdots \otimes \mathtt{V}_n,$$

and for every non-empty subset $A \subseteq G$ *the subset* $\mathtt{V}_A := \bigcap_{g \in A} U(g)\mathtt{V}$ *satisfies*

$$\mathtt{V}_A \supseteq \mathtt{V}_{1,A} \otimes \cdots \otimes \mathtt{V}_{n,A}. \tag{4.18}$$

Proof. We have $\xi \in \mathtt{V}_A$ if and only if $U(A)^{-1}\xi \subseteq \mathtt{V}$. This shows that any $\xi = \xi_1 \otimes \cdots \otimes \xi_n$ with $\xi_j \in \mathtt{V}_{j,A}$ is contained in $\mathtt{V}_A$, which is (4.18). □

Regularity for a suitable wedge region

Theorem 4.29. *Let* $(U, \mathcal{H})$ *be an antiunitary representation of* G_{τ_h} *and* $\mathtt{V} = \mathtt{V}(h, U) \subseteq \mathcal{H}$ *the corresponding standard subspace. Then there exists a net* $(\mathsf{H}(\mathcal{O}))_{\mathcal{O} \subseteq G}$ *on open subsets of* G *satisfying* (Iso), (Cov), (RS) *and* (BW) *for some open subset* $W \subseteq G$ *if and only if* U *is* h*-regular, i.e.,* $\mathtt{V}_N$ *is cyclic for some* e*-neighborhood* $N \subseteq G$*.*

Proof. "⇒": If a net H with the asserted properties exists, then $\mathtt{V} = \mathsf{H}(W)$, and for any relatively compact open subset $\mathcal{O} \subseteq W$ there exists an identity neighborhood $N \subseteq G$ with $N\mathcal{O} \subseteq W$. Then, for all $g^{-1} \in N$, we have

$$U(g)^{-1}\mathsf{H}(\mathcal{O}) = \mathsf{H}(g^{-1}.\mathcal{O}) \subseteq \mathsf{H}(W) = \mathtt{V}, \quad \text{hence} \quad \mathsf{H}(\mathcal{O}) \subseteq \mathtt{V}_N.$$

Now (RS) implies that U is h-regular.

"⇐": Assume that $\mathtt{V}_N$ is cyclic for an e-neighborhood N. Pick an open e-neighborhood $N_1 \subseteq N$ with $N_1 N_1^{-1} \subseteq N$. Then

$$W := \exp(\mathbb{R}h)N_1$$

is an open subset of G. We consider the net $\mathsf{H} := \mathsf{H}_{G,U,W}^{\max}$, defined by

$$\mathsf{H}(\mathcal{O}) = \bigcap_{g \in G, \mathcal{O} \subseteq gW} U(g)\mathtt{V}.$$

This net satisfies (Iso) and (Cov) by Lemma 4.2.

We now verify the Reeh–Schlieder property (RS). So let $\emptyset \neq \mathcal{O} \subseteq G$ be an open subset. By (Iso) and (Cov), it suffices to show that $\mathsf{H}(\mathcal{O})$ is cyclic if $\mathcal{O} \subseteq N_1$. Then $\mathcal{O} \subseteq gW = g\exp(\mathbb{R}h)N_1$ implies

$$g \in \mathcal{O}N_1^{-1}\exp(\mathbb{R}h) \subseteq N_1 N_1^{-1}\exp(\mathbb{R}h) \subseteq N\exp(\mathbb{R}h),$$

so that

$$\mathsf{H}(\mathcal{O}) \supseteq \bigcap_{g\in N\exp(\mathbb{R}h)} U(g)\mathtt{V} = \bigcap_{g\in N} U(g)\mathtt{V} = \mathtt{V}_N$$

implies that $\mathsf{H}(\mathcal{O})$ is cyclic. This proves (RS). It follows in particular that $\mathsf{H}(W)$ is cyclic, so that Lemma 4.2(c) implies $\mathsf{H}(W) = \mathtt{V}$. Therefore (BW) is also satisfied. □

Remark 4.30. Note that $v \in \mathsf{H}^{\max}(\mathcal{O})$ is equivalent to

$$g^{-1}\mathcal{O} \subseteq W \quad \Rightarrow \quad U(g)^{-1}v \in \mathtt{V}.$$

If $\mathcal{O} \subseteq W$ is relatively compact, this condition holds for g in an e-neighborhood. Therefore $\mathsf{H}^{\max}(\mathcal{O})$ consists of vectors $v \in \mathcal{H}$ whose orbit map $U^v\colon G \to \mathcal{H}$ maps an identity neighborhood into $\mathtt{V}$ (cf. Proposition 3.25(d)). Put differently, the subset $(U^v)^{-1}(\mathtt{V}) \subseteq G$ has interior points.

Remark 4.31. (a) Suppose that $v \in \mathtt{V} \cap \mathcal{H}^\omega$ is an analytic vector and $U(N)v \subseteq \mathtt{V}$ holds for an identity neighborhood $N \subseteq G$, then uniqueness of analytic continuation implies $U(G)v \subseteq \mathtt{V}$, i.e., $v \in \mathtt{V}_G$.

If, in addition, v is G-cyclic, then $\mathtt{V}_G$ is a cyclic real subspace, so that its invariance under the modular group of $\mathtt{V}$ implies with Proposition 1.20 that $\mathtt{V} = \mathtt{V}_G$, i.e., that $\mathtt{V}$ is G-invariant. If U has discrete kernel, this implies that $h \in \mathfrak{z}(\mathfrak{g})$. Hence τ_h is trivial and therefore $J_{\mathtt{V}}$ commutes with G. Therefore $\mathcal{H}^{J_{\mathtt{V}}}$ is a real orthogonal representation of G, and U is the complexification, considered as a representation of G_{τ_h} on $\mathcal{H} \cong (\mathcal{H}^{J_{\mathtt{V}}})_{\mathbb{C}}$. This is the context where $\partial U(h)$ and $J_{\mathtt{V}}$ commute with G_{τ_h}.
(b) Another perspective on (a) is that the cyclic representation generated by any $v \in \mathcal{H}^\omega \cap \mathtt{V}_N$ is such that $\partial U(h)$ and $J_{\mathtt{V}}$ commute with G. So v is fixed by the normal subgroup B with Lie algebra

$$\mathfrak{b} := \mathfrak{g}_1 + [\mathfrak{g}_1, \mathfrak{g}_{-1}] + \mathfrak{g}_{-1}.$$

4.5.3 Regularity for reductive Lie groups

From now on we assume that $\mathfrak{g}$ is reductive and that G is a corresponding connected Lie group. We choose an involution θ on $\mathfrak{g}$ in such a way that it fixes the center pointwise and restricts to a Cartan involution on the semisimple Lie algebra $[\mathfrak{g}, \mathfrak{g}]$. Then the corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ satisfies $\mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{k}$. We write $K := G^\theta$ for the subgroup of θ-fixed points in G.

For an Euler element $h \in \mathfrak{g}$, we write $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where $\mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{g}_1$, $h = h_2 + h_z \in \mathfrak{g}_2 \oplus \mathfrak{z}(\mathfrak{g})$, and $\mathfrak{g}_2$ is minimal, i.e., $\mathfrak{g}_2$ is the ideal generated by the projection h_2 of h to the commutator algebra. We consider the involution τ on $\mathfrak{g}$ with

$$\tau|_{\mathfrak{g}_1} = \mathrm{id}_{\mathfrak{g}_1} \quad \text{and} \quad \tau|_{\mathfrak{g}_2} = \tau_h\theta.$$

We **assume** that τ integrates to an involutive automorphism τ^G of G. We write $\mathfrak{h} := \mathfrak{g}^\tau$ and $\mathfrak{q} := \mathfrak{g}^{-\tau} \subseteq \mathfrak{g}_2$ for the τ-eigenspaces in $\mathfrak{g}$. Then there exists in $\mathfrak{q}$ a unique maximal pointed generating $e^{\mathrm{ad}\,\mathfrak{h}}$-invariant cone C containing h_2 in its interior ([MNO23, Thm. 4.21] deals with minimal cones, but these cones determine each other by duality). We choose an open θ-invariant subgroup $H \subseteq G^\tau$, satisfying $\mathrm{Ad}(H)C = C$. This is always the case for $H = G^\tau_e$ (the minimal choice). By [MNO23, Cor. 4.6], $\mathrm{Ad}(H)C = C$ is equivalent to $H_K = H \cap K$ fixing h. Then

$$M = G/H \cong G_2/H_2 \quad \text{for} \quad H_2 := G_2 \cap H \tag{4.19}$$

is called the corresponding non-compactly causal symmetric space (cf. Section 2.7.3). The normal subgroups $G_1 \subseteq H$ acts trivially on M. The homogeneous space M carries a G-invariant causal structure, represented by the field $(C_m)_{m\in M}$ of closed convex cones $C_m \subseteq T_m(M)$, which is uniquely determined by $C_{eH} = C \subseteq \mathfrak{q} \cong T_{eH}(M)$ (Subsection 2.3). By construction, $eH \in W_M^+(h)$, and we write

$$W := W_M^+(h)_{eH} \tag{4.20}$$

for the connected component of the base point in $W_M^+(h)$.

Definition 4.32. We say that the (anti-)unitary representation $(U, \mathcal{H})$ of G_{τ_h} is (h, W)-*localizable* in those open subsets $\mathcal{O} \subseteq M$ for which $\mathsf{H}^{\max}(\mathcal{O})$ is cyclic.

Theorem 4.33. (Localization for reductive groups) *If $\mathfrak{g}$ is reductive and $(U, \mathcal{H})$ is an antiunitary representation of G_{τ_h}, then the canonical net $\mathsf{H}_M^{\max}$ on the non-compactly causal symmetric space $M = G/H$ from* (4.19) *and W from* (4.20) *satisfies* (Iso), (Cov), (RS) *and* (BW).

Proof. As the standard subspace $\mathtt{V}$ is invariant under $G_1 \subseteq G^{h,\tau_h}$ and G_1 acts trivially on M, the real subspaces $\mathsf{H}_M^{\max}(\mathcal{O})$ only depend on $U|_{G_2}$. We may therefore assume that $G = G_2$, i.e., that G is semisimple and that $\mathfrak{g}_0(h)$ contains no non-zero ideal.

In view of Lemma 4.2(c), assertion (a) follows from (b), applied to $\mathcal{O} = W$. So it suffices to verify (b). By Proposition 4.26, we may further assume that $(U, \mathcal{H})$ is irreducible.

Then (h, W)-localizability in the family of all non-empty open subsets of M follows from Theorem 3.17. It provides a net $\mathsf{H}_{\mathtt{E}}^M$ satisfying (Iso), (Cov), (RS) and (BW), and this net satisfies $\mathsf{H}_{\mathtt{E}}^M(\mathcal{O}) \subseteq \mathsf{H}^{\max}(\mathcal{O})$ for each $\mathcal{O} \subseteq M$ (Lemma 4.7). □

Corollary 4.34. (Regularity for reductive groups) *Let G be a connected reductive Lie group. Then there exists an e-neighborhood $N \subseteq G$ such that for every separable antiunitary representation $(U, \mathcal{H})$ of G_{τ_h} and $\mathtt{V} = \mathtt{V}(h, U)$, the real subspace $\mathtt{V}_N = \bigcap_{g\in N} U(g)\mathtt{V}$ is cyclic. In particular, $(U, \mathcal{H})$ is h-regular.*

Proof. Let $\mathcal{O} \subseteq W \subseteq M = G/H$ (with M as in Theorem 4.33) be an open subset whose closure $\overline{\mathcal{O}}$ is relatively compact. In Theorem 4.33 we have seen that $\mathsf{H}_M^{\max}(\mathcal{O})$ is cyclic. Further

$$N := \{g \in G \colon g\mathcal{O} \subseteq W\} \supseteq \{g \in G \colon g\overline{\mathcal{O}} \subseteq W\}$$

is an e-neighborhood because $\overline{\mathcal{O}} \subseteq W$ is compact. Therefore the h-regularity of $(U, \mathcal{H})$ follows from Proposition 4.27. □

Theorem 4.35. (Triviality of the semigroups of wedge regions in ncc symmetric spaces) *If G is a connected reductive Lie group and $M = G/H$ a corresponding non-compactly causal symmetric space as in* (4.19)*, with causal Euler element h, and the maximal causal structure, then the following assertions hold:*

(a) $S_W = G_W = \{g \in G^h \colon g.W = W\}$.

(b) $S_W = G^h$ *if $\mathfrak{g}$ is simple and* $Z(G) = \{e\}$.

Proof. (a) First we apply the Localization Theorem 4.33 to a unitary representation with discrete kernel and $C_U = \{0\}$; for instance a principal series representation. Then $S_{\mathtt{V}} = G_{\mathtt{V}} \subseteq G^h$ by Theorem 4.9. As $W \subseteq W_M^+(h)$ is a connected component and G^h preserves $W_M^+(h)$, we obtain $S_W \subseteq G^h$, and thus (a) follows.
(b) follows from $G_W = G^h$ in this case ([MNO24, Prop. 7.3]). □

4.6 Localizability for the Poincaré group

The following result is well-known in AQFT ([BGL02, Thm. 4.7]). Below (cf. [MN24]) we derive it naturally in the context of our theory for general Lie groups. It connects regularity, resp., localizability with the positive energy condition.

Theorem 4.36. (Localization for the Poincaré group) *Let $(U, \mathcal{H})$ be an (anti-)unitary representation of the proper Poincaré group $\mathcal{P}_+ = \mathbb{R}^{1,d} \rtimes \mathcal{L}_+$ (identified with $\mathcal{P}_{\tau_h}$) and consider the standard boost h and the corresponding Rindler wedge $W_R \subseteq \mathbb{R}^{1,d}$. Then $(U, \mathcal{H})$ is (h, W_R)-localizable in the set of all spacelike open cones if and only if it is a positive energy representation, i.e.,*

$$C_U \supseteq \overline{V_+} := \{(x_0, \mathbf{x})\colon x_0 \geq 0, x_0^2 \geq \mathbf{x}^2\}. \tag{4.21}$$

These representations are regular.

Note that $\mathrm{Ad}(\mathcal{P}_+^\uparrow)$ acts transitively on the set $\mathcal{E}(\mathfrak{p})$ of Euler elements, so that the choice of a specific Euler element h is inessential.

Proof. First we show that the positive energy condition is necessary for localizability in spacelike cones. In fact, the localizability condition implies in particular that $\mathsf{H}(W_R)$ is cyclic, so that Lemma 4.2 implies $S_{W_R} \subseteq S_{\mathtt{V}}$. As a consequence, $\mathbf{e}_1 + \mathbf{e}_0 \in C_U$, and thus $\overline{V_+} \subseteq C_U$ by Lorentz invariance of C_U. Therefore $(U, \mathcal{H})$ is a positive energy representation.

Now we assume that $(U, \mathcal{H})$ is a positive energy representation. For the standard boost, we have $h \in \mathfrak{l} \cong \mathfrak{so}_{1,d}(\mathbb{R})$, and the restriction $(U|_{L_+}, \mathcal{H})$ is (h, W)-localizable in the family of all non-empty open subsets of dS^d, where $W = W_R \cap \mathrm{dS}^d$ is the canonical wedge region (Theorem 4.33).

Next we recall from [NÓ17, Lemma 4.12] that

$$S_{W_R} = \{g \in \mathcal{P}_+^\uparrow\colon gW_R \subseteq W_R\} = \overline{W_R} \rtimes \mathrm{SO}_{1,d}(\mathbb{R})^\uparrow_{W_R},$$

where

$$\mathrm{SO}_{1,d}(\mathbb{R})^\uparrow_{W_R} = \mathrm{SO}_{1,1}(\mathbb{R})^\uparrow \times \mathrm{SO}_{d-2}(\mathbb{R})$$

is connected, hence coincides with L_e^h. It follows that

$$S_{W_R} = G_e^h \exp(\mathbb{R}_{\geq 0}(\mathbf{e}_0 + \mathbf{e}_1)) \exp(\mathbb{R}_{\geq 0}(-\mathbf{e}_0 + \mathbf{e}_1)).$$

Let us assume that $(U, \mathcal{H})$ is a positive energy representation, i.e., that $C_U \supseteq \overline{V_+}$ (cf. (4.21)). Then

$$C_\pm = \mathbb{R}_{\geq 0}(\mathbf{e}_1 \pm \mathbf{e}_0) \subseteq \overline{W_R}, \quad \text{so that} \quad S_{W_R} \subseteq S_{\mathtt{V}}.$$

By Lemma 4.2(c), the net $\mathsf{H}^{\max}$ satisfies $\mathsf{H}^{\max}(W_R) = \mathtt{V}$.

Now suppose that $\mathcal{C} \subseteq W_R$ is a spacelike cone, so that

$$\mathcal{C} = \mathbb{R}_+(\mathcal{C} \cap \mathrm{dS}^d),$$

where $\mathcal{C} \cap \mathrm{dS}^d$ is an open subset of the wedge region $W = W_R \cap \mathrm{dS}^d$ in de Sitter space. For $g^{-1} = (v, \ell) \in \mathcal{P}_+^\uparrow$, the condition $\mathcal{C} \subseteq g.W_R$ is equivalent to

$$g^{-1}.\mathcal{C} = v + \ell.\mathcal{C} \subseteq W_R,$$

which in turn means that $v \in \overline{W_R}$ and $\ell.\mathcal{C} \subseteq W_R$. Then

$$U(g)\mathtt{V} = U(\ell)^{-1}U(v)^{-1}\mathtt{V} \supseteq U(\ell)^{-1}\mathtt{V}$$

follows from $\overline{W_R} \subseteq S_{\mathtt{V}}$, and therefore

$$\begin{aligned}\mathsf{H}^{\max}(\mathcal{C}) &= \bigcap_{\mathcal{C}\subseteq g.W_R} U(g)\mathtt{V} \supseteq \bigcap_{\mathcal{C}\subseteq \ell^{-1}.W_R} U(\ell)^{-1}\mathtt{V} \\ &= \bigcap_{\mathcal{C}\cap \mathrm{dS}^d\subseteq \ell^{-1}.(W_R\cap \mathrm{dS}^d)} U(\ell)^{-1}\mathtt{V} = \mathsf{H}^{\max}_{U|_L}(\mathcal{C}\cap \mathrm{dS}^d).\end{aligned}$$

We conclude that, on spacelike cones with vertex in 0, the net $\mathsf{H}^{\max}$ coincides with the net $\mathsf{H}^{\max}_{U|_L}$ on de Sitter space. As the latter net has the Reeh–Schlieder property by Theorem 4.33, and all spacelike cones can be translated to one with vertex 0, localization in spacelike cones follows.

Finally we show that $(U,\mathcal{H})$ is regular. For $v \in W_R$ and a pointed spacelike cone C with $v + C \subseteq W$, there exists an e-neighborhood $N \subseteq G$ with $v + C \subseteq g.W$ for all $g \in N$. This implies that $\mathsf{H}^{\max}(v + C) \subseteq \mathtt{V}_N$, so that $(U,\mathcal{H})$ is regular. □

Definition 4.37. (a) (Causal complement) Let $M = \mathbb{R}^{1,d}$ be Minkowski space. Its causal structure allows us to define the *causal complement (or the spacelike complement)* of an open subset $\mathcal{O} \subset M$ by

$$\mathcal{O}' = \{x \in M : (\forall y \in \mathcal{O})\,(y - x)^2 < 0\}^\circ. \tag{4.22}$$

This is the interior of the set of all the points that cannot be reached from E with a timelike or lightlike curve.
(b) (Spacelike cones) In Minkowski space $\mathbb{R}^{1,d}$, we call an open subset $\mathcal{O}$ *spacelike* if $x_0^2 < \mathbf{x}^2$ holds for all $(x_0,\mathbf{x}) \in \mathcal{O}$. A spacelike open subset is called a *spacelike (convex) cone* if, in addition, it is a (convex) cone. (c) (Double cone) A *double cone* is, up to Poincaré covariance, the causal closure

$$\mathbb{B}_r'' = (r\mathbf{e}_0 - V_+) \cap (-r\mathbf{e}_0 + V_+)$$

of an open ball of the time zero hyper-plane $\mathbb{B}_r = \{(0,\mathbf{x}) \in \mathbb{R}^{1,d} : \mathbf{x}^2 < r^2\}$.

Remark 4.38. Infinite helicity representations $(U,\mathcal{H})$ of $\mathcal{P}_+$ in $\mathbb{R}^{1,d}$ are **not** localizable in double cones (Definition 4.37). Let $\mathtt{V} = \mathtt{V}(h,U)$ for h as in Example 1.15. In [LMR16, Thm. 6.1] it is shown that, if $\mathcal{O} \subseteq \mathbb{R}^{1,d}$ is a double cone, then

$$\mathsf{H}^{\max}(\mathcal{O}) = \bigcap_{\mathcal{O}\subseteq g.W_R} U(g)\mathtt{V} = \{0\}. \tag{4.23}$$

The argument to conclude (4.23) can be sketched as follows. Infinite spin representations are massless representations, i.e., the support of the spectral measure of the space-time translation group is

$$\partial V_+ = \{(x_0,\mathbf{x}) \in \mathbb{R}^{1,d} : x_0^2 - \mathbf{x}^2 = 0, x_0 \geq 0\}.$$

Covariant nets of standard subspaces on double cones in massless representations are also dilation covariant in the sense that the representation of $\mathcal{P}_+$ extends to the Poincaré and dilation group $\mathbb{R}^{1,d} \rtimes (\mathbb{R}^+ \times \mathcal{L})$, and that the net is also covariant under this larger group, cf. [LMR16, Prop. 5.4]. When $d = 3$, this follows from the fact that due to the Huygens Principle, one can associate by additivity a standard subspace to the forward lightcone $\mathsf{H}(V_+) = \overline{\sum_{\mathcal{O}\subset V_+} \mathsf{H}(\mathcal{O})}$ (sum over all double cones in V_+) and the modular group of $\mathsf{H}(V_+)$ is geometrically implemented by the dilation group. As massless infinite helicity representations are not dilation covariant, it follows that they do not permit localization in double cones. Properties of the free wave equation permit to extend this argument to any space dimension $d \geq 2$ including even space dimensions, and the Huygens Principle fails ([LMR16, Sect. 8.2]). However, in Theorem 4.36, we recover in our general setting the result contained in [BGL02, Thm. 4.7] that all positive energy representations of $\mathcal{P}_+$ are localizable in spacelike cones.

5 Perspectives

5.1 Additivity

In this subsection, we take a closer look at the additivity condition (Add) for nets of real subspaces. We show in particular that the nets $\mathsf{H}_{\mathsf{E}}^{M}$ are always additive. For causal flag manifolds, this implies already that nets satisfying (Iso), (Cov), (BW) and (Add) are uniquely determined by the representation $(U, \mathcal{H})$ of G_{τ_h} (cf. [MN25]).

Proposition 5.1. *For a real subspace $\mathsf{E} \subseteq \mathcal{H}^{-\infty}$, the net $\mathsf{H}_{\mathsf{E}}^{M}$ is additive.*

Proof. Let $\mathcal{O} \subseteq M$ be open and $(\mathcal{O}_j)_{j\in J}$ an open covering of $\mathcal{O}$. We write $q_M \colon G \to M = G/H$ for the quotient map and consider some $\varphi \in C_c^\infty(q_M^{-1}(\mathcal{O}))$.

The open subsets $q_M^{-1}(\mathcal{O}_j)$ form an open cover of $q_M^{-1}(\mathcal{O})$. Therefore Lemma 3.20 implies the existence of $j_1, \ldots, j_k$ and test functions φ_ℓ, supported in $q_M^{-1}(\mathcal{O}_{j_\ell})$, such that $\varphi = \varphi_1 + \cdots + \varphi_k$. Now

$$U^{-\infty}(\varphi)\mathsf{E} \subseteq \sum_{\ell=1}^{k} U^{-\infty}(\varphi_\ell)\mathsf{E} \subseteq \sum_{\ell=1}^{k} \mathsf{H}_{\mathsf{E}}^{M}(\mathcal{O}_{j_\ell}) \subseteq \sum_{j\in J} \mathsf{H}_{\mathsf{E}}^{M}(\mathcal{O}_j)$$

implies that $\mathsf{H}_{\mathsf{E}}^{M}(\mathcal{O}) \subseteq \sum_{j\in J} \mathsf{H}_{\mathsf{E}}^{M}(\mathcal{O}_j)$, and additivity thus follows from isotony. □

5.2 Locality

Definition 5.2. Let H be a net of real subspaces on $M = G/H$ that is isotone and covariant. In $M \times M$, we defined the *locality set of* H by

$$\mathcal{L}_{\mathsf{H}} = \bigcup_{\mathsf{H}(\mathcal{O}_1) \subseteq \mathsf{H}(\mathcal{O}_2)'} \mathcal{O}_1 \times \mathcal{O}_2 \subseteq M \times M.$$

By definition, this is an open subset, and (Cov) implies that it is G-invariant. Moreover, it is symmetric in the sense that $(x, y) \in \mathcal{L}_{\mathsf{H}}$ implies $(y, x) \in \mathcal{L}_{\mathsf{H}}$.

The subset $\mathcal{L}_{\mathsf{H}} \subseteq M \times M$ completely encodes the locality properties of the net H in a G-invariant subset of the set of pairs in M. To connect locality properties of a net H with the given structures on M therefore reduces to comparing $\mathcal{L}_{\mathsf{H}}$ with the given geometric data.

Lemma 5.3. *If H is additive and $\mathcal{O}_1, \mathcal{O}_2$ are open subsets of M with $\mathcal{O}_1 \times \mathcal{O}_2 \subseteq \mathcal{L}_{\mathsf{H}}$, then*

$$\mathsf{H}(\mathcal{O}_1) \subseteq \mathsf{H}(\mathcal{O}_2)'.$$

This corresponds to the locality condition (Loc) in Section 1.

Proof. Since H is additive, it is also isotone, and the real subspace $\mathsf{H}(\mathcal{O}_2)$ is generated by the subspaces $\mathsf{H}(\mathcal{C})$, where $\mathcal{C} \subseteq \mathcal{O}_2$ is a relatively compact open subset of $\mathcal{O}_2$. So it suffices to show that $\mathsf{H}(\mathcal{O}_1) \subseteq \mathsf{H}(\mathcal{C})'$ for such subsets.

For any $(x, y) \in \mathcal{O}_1 \times \overline{\mathcal{C}} \subseteq \mathcal{L}_{\mathsf{H}}$, there exist open subsets $\mathcal{O}_x^y, \mathcal{O}_y^x \subseteq M$ with

$$x \in \mathcal{O}_x^y \subseteq \mathcal{O}_1,\ y \in \mathcal{O}_y^x \subseteq \mathcal{O}_2 \quad \text{and} \quad \mathsf{H}(\mathcal{O}_x^y) \subseteq \mathsf{H}(\mathcal{O}_y^x)'.$$

Then, for each $x \in \mathcal{O}_1$, the sets $(\mathcal{O}_y^x)_{y\in\overline{\mathcal{C}}}$ form an open covering of the compact subset $\overline{\mathcal{C}} \subseteq \mathcal{O}_2$, so that there exist finitely many points $y_1, \ldots, y_n \in \mathcal{O}_2$ with

$$\mathcal{C} \subseteq \mathcal{O}_{y_1}^x \cup \cdots \cup \mathcal{O}_{y_n}^x.$$

Then

$$\mathcal{O}_x := \mathcal{O}_x^{y_1} \cap \cdots \cap \mathcal{O}_x^{y_n} \subseteq \mathcal{O}_1$$

is an open neighborhood of x for which $\mathsf{H}(\mathcal{O}_x) \subseteq \mathsf{H}(\mathcal{O}_{y_j}^x)'$ for $j = 1, \ldots, n$. Additivity of H thus implies $\mathsf{H}(\mathcal{O}_x) \subseteq \mathsf{H}(\mathcal{C})'$. Finally, we observe that the $\mathcal{O}_x$ form an open cover of $\mathcal{O}_1$, so that additivity further implies that $\mathsf{H}(\mathcal{O}_1) \subseteq \mathsf{H}(\mathcal{C})'$. $\square$

Remark 5.4. If W and W' are open subsets of M with $\mathsf{H}(W) = \mathtt{V}$ and $\mathsf{H}(W') = \mathtt{V}'$, then we have

$$\mathcal{L}_W := G.(W \times W') \cup G.(W' \times W) \subseteq \mathcal{L}_{\mathsf{H}}.$$

If H is additive, it follows from Lemma 5.3 that $\mathcal{O}_1 \times \mathcal{O}_2 \subseteq \mathcal{L}_W$ implies $\mathsf{H}(\mathcal{O}_1) \subseteq \mathsf{H}(\mathcal{O}_2)'$.

Examples 5.5. (a) If $M = \mathbb{R}^{1,d-1}$ is Minkowski space and $W = W_R$ is the Rindler wedge, then $W' = -W_R$ and

$$\mathcal{L}_W = G.(W \times W') \subseteq M \times M$$

is the set of spacelike pairs. For an open subset $\mathcal{O} \subseteq M$, the maximal open subset $\mathcal{O}'$ satisfying

$$\mathcal{O} \times \mathcal{O}' \subseteq \mathcal{L}_W$$

is called the *causal complement* of $\mathcal{O}$.

The same picture prevails for de Sitter space $\mathrm{dS}^d \subseteq \mathbb{R}^{1,d}$.

(b) For $M = \mathbb{S}^1$, a causal flag manifold of $G = \mathrm{SL}_2(\mathbb{R})$, the wedge regions are open non-dense intervals $W \subseteq \mathbb{S}^1$ (Example 2.20). If $W = W_M^+(h)$, then $W' = W_M^+(-h)$ is the interior of the complement of W, and

$$\mathcal{L}_W = G.(W \times W') = M^2 \setminus \Delta_M.$$

So $(x, y) \in \mathcal{L}_W$ if and only if $x \neq y$.

(c) In the non-compactly causal symmetric space $M = \mathrm{SL}_4(\mathbb{R})/\mathrm{SO}_{2,2}(\mathbb{R})$, not all acausal pairs are contained in $\mathcal{L}_W$ and $M \times M$ contains several open acausal G-orbits.

Remark 5.6. In the context of abstract wedges, represented by elements of the set

$$\mathcal{G}(G_{\tau_h}) := \{(x, \tau) \in \mathfrak{g} \times G\tau_h \colon \mathrm{Ad}(\tau)x = x, \tau^2 = e\}$$

(cf. Exercise 1.5), there is a natural complementation map

$$(x, \tau) \mapsto (x, \tau)' := (-x, \tau).$$

In this context, it is a natural question if $(h, \tau_h)' = (-h, \tau_h)$ is contained in the G-orbit of (h, τ_h). This is equivalent to the symmetry of h and the additional condition that there exists a $g \in G^{\tau_h}$ with $\mathrm{Ad}(g)h = -h$.

If this is the case and $(U, \mathcal{H})$ is an antiunitary representation of G_{τ_h}, then $\mathtt{V}' = U(g)\mathtt{V}$. For any net H satisfying (Cov) and (BW), this implies that

$$W \times g.W \subseteq \mathcal{L}_{\mathsf{H}}.$$

We refer to [MN25] for a detailed discussion of locality properties of nets of causal flag manifolds and to [MN21] for a discussion of twisted locality conditions.

5.3 Representations of Lie supergroups

Lie supergroups and their unitary representations arise naturally in Physics in connection with supersymmetry (cf. [Gu75]). It would be interesting to extend the theory developed in these notes to this context, where the Euler element $h \in \mathfrak{g}$ is supposed to be an even element.

Definition 5.7. A *Lie supergroup* is a pair $(G, \mathfrak{g})$, where $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ is a finite-dimensional Lie superalgebra and G is a real Lie group with Lie algebra $\mathfrak{g}_{\overline{0}}$, acting smoothly by automorphisms on $\mathfrak{g}$ via $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$ in such a way that the action on $\mathfrak{g}_{\overline{0}}$ is the adjoint action of G.

Definition 5.8. A *unitary representation* of a Lie supergroup $(G, \mathfrak{g})$ is a pair (U, β), where $(U, \mathcal{H})$ is a unitary representation of the Lie group G on a graded Hilbert space $\mathcal{H} = \mathcal{H}_{\overline{0}} \oplus \mathcal{H}_{\overline{1}}$, preserving the grading, and

$$\beta\colon \mathfrak{g} \to \mathrm{End}(\mathcal{H}^\infty)$$

is a representation of the Lie superalgebra $\mathfrak{g}$ on the space of smooth vectors of U ([NS11]).

Problem 5.9. One can associate to each real subspace $\mathtt{E} \subseteq \mathcal{H}^{-\infty}$ (distribution vectors for U) the closed real subspaces $\mathsf{H}^G_{\mathtt{E}}(\mathcal{O})$ generated by

$$\beta(U(\mathfrak{g}))U^{-\infty}(C^\infty_c(\mathcal{O}, \mathbb{R}))\mathtt{E}.$$

Does this construction lead to nets that are compatible with fermionic nets in AQFT? Possibly one can develop a "supersymmetric" variant of the theory described in these notes.

Here are some relevant structures and observations.

Definition 5.10. (The $*$-monoid associated to a Lie supergroup) The anti-linear map

$$\mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}\ ,\ x \mapsto x^*, \quad \text{defined by} \quad x^* := \begin{cases} -x & \text{if } x \in \mathfrak{g}_{\overline{0}}\,, \\ -\sqrt{-1}\,x & \text{if } x \in \mathfrak{g}_{\overline{1}}\,. \end{cases}$$

is an anti-automorphism. It extends to an anti-linear anti-automorphism

$$U(\mathfrak{g}_{\mathbb{C}}) \to U(\mathfrak{g}_{\mathbb{C}})\ ,\ D \mapsto D^* \tag{5.1}$$

in a canonical way. Consider the monoid S with underlying set $G \times U(\mathfrak{g}_{\mathbb{C}})$ and multiplication

$$(D_1, g_1)(D_2, g_2) = (D_1 g_1 \cdots D_2, g_1 g_2)$$

where $g \cdot D$ denotes the adjoint action of $g \in G$ on $D \in U(\mathfrak{g}_{\mathbb{C}})$. The neutral element of S is $1_{\mathsf{S}} := (1_{U(\mathfrak{g}_{\mathbb{C}})}, e)$. The map

$$\mathsf{S} \to \mathsf{S}\,,\ s \mapsto s^* \quad \text{defined by} \quad (D, g)^* := (g^{-1} \cdot D^*, g^{-1})$$

is an involution of S.

Recall that $U(\mathfrak{g}_{\mathbb{C}})$ is an associative superalgebra. An element $(D, g) \in \mathsf{S}$ is called *odd* (resp. *even*) if D is an odd (resp. even) element of $U(\mathfrak{g}_{\mathbb{C}})$.

Replacing in this construction the elements $g \in G$ by compactly supported smooth functions on G, one can even construct a graded $*$-algebra $(C^\infty_c(\mathcal{G}), *)$ in such a way that every unitary $\mathcal{G}$-representation integrated to a $*$-representation of $C^\infty_c(\mathcal{G})$ by bounded operators. This way, one even obtains "supergroup C^*-algebras". We refer to [NS16] for details.

Remark 5.11. From the table in [NS11, §2.5] we get some information on which finite-dimensional simple Lie superalgebras $\mathfrak{g}$ have non-trivial unitary representations. According to [NS11, §6], for any unitary representation of $\mathcal{G} = (G, \mathfrak{g})$, we must have the inclusion

$$\mathrm{Cone}(\mathfrak{g}) = \mathrm{cone}\{[x, x] \colon x \in \mathfrak{g}_{\overline{1}}\} \subseteq C_U = \{x \in \mathfrak{g}_{\overline{0}} \colon -i\partial U(x) \geq 0\}.$$

For a unitary representation with discrete kernel, the cone C_U is pointed, so that the pointedness of the cone generated by the brackets of odd elements is necessary for the existence of non-trivial unitary representations. Accordingly, [NS11, Thm. 6.2.1] compiles a negative list of simple Lie algebras for which this is not the case.

5.4 The geometric structure on M

Let $(U, \mathcal{H})$ be an antiunitary representation of G_{τ_h} and $\mathtt{V} := \mathtt{V}(h, U) \subseteq \mathcal{H}$ be the canonical standard subspace obtained from the BGL construction (Exercise 1.5) resp., (0.1). Let $\mathtt{E} \subseteq \mathcal{H}^{-\infty}$ be a finite-dimensional linear subspace, invariant under the subgroup $H \subseteq G$ and $M := G/H$. Then we obtain a net $\mathsf{H}_{\mathtt{E}}^M$ on M, satisfying (Iso) and (Cov).

Then

$$W_{\mathtt{E}} := \{gH \in M \colon U^{-\infty}(g)\mathtt{E} \subseteq \mathcal{H}_{\mathrm{KMS}}^{-\infty}\}^\circ$$

specifies an open subset of M that deserves to be called the *wedge region associated to* $\mathtt{E}$, but it may be empty; depending on the subspace $\mathtt{E}$.

If $W_{\mathtt{E}} \neq \emptyset$, then Proposition 3.25 implies that

$$\mathsf{H}_{\mathtt{E}}^M(W_{\mathtt{E}}) \subseteq \mathtt{V} \quad \text{and} \quad \exp(\mathbb{R}h)W_{\mathtt{E}} \subseteq W_{\mathtt{E}}.$$

For $\xi \in \mathtt{V}^\infty := \mathtt{V} \cap \mathcal{H}^\infty$, we have

$$\langle \xi, i\partial U(h)\xi\rangle = \frac{d}{dt}\Big|_{t=0} \langle \xi, e^{it\partial U(h)}\xi\rangle = \frac{d}{dt}\Big|_{t=0} \langle \xi, \Delta_{\mathtt{V}}^{t/2\pi}\xi\rangle \leq 0 \tag{5.2}$$

because the convex function $t \mapsto \langle \xi, \Delta_{\mathtt{V}}^{t/2\pi}\xi\rangle$ on $[0, \pi]$ takes its minimal value in $t = \pi$ and has a local minimum in $t = 0$. Here convexity follows from the Spectral Theorem, which implies that it is a Laplace transform, and $\xi \in \mathtt{V} = \mathrm{Fix}(J_{\mathtt{V}}\Delta_{\mathtt{V}}^{1/2})$ implies that it is an even function.

For $\alpha \in \mathtt{E}$ and $\varphi \in C_c^\infty(W_{\mathtt{E}}, \mathbb{R})$, we have $U(\varphi)\alpha \in \mathtt{V}^\infty$, so that we get

$$\langle U(\varphi)\alpha, i\partial U(h)U(\varphi)\alpha\rangle \leq 0.$$

For $\varphi \to \delta_g$, $g \in W_{\mathtt{E}}$, we obtain

$$\langle \alpha, i\partial U(\mathrm{Ad}(g)^{-1}h)\alpha\rangle \leq 0$$

in the sense of distributions on G. Maybe these inequalities can be related to the generalized positive energy conditions appearing in [JN24].

Remark 5.12. In this context, it becomes apparent that the closed convex cone $C(W_{\mathtt{E}}) \subseteq \mathfrak{g}$, generated by

$$\mathrm{Ad}(g)^{-1}h, \quad g \in W_{\mathtt{E}},$$

plays an important role. It is clearly invariant under $e^{\mathbb{R}\,\mathrm{ad}\,h}$, so that

$$C(W_{\mathtt{E}}) \subseteq C(W_{\mathtt{E}})_+ + \mathfrak{g}_0(h) - C(W_{\mathtt{E}})_- \quad \text{for} \quad C(W_{\mathtt{E}})_\pm := \pm C(W_{\mathtt{E}}) \cap \mathfrak{g}_{\pm 1}(h) \subseteq C(W_{\mathtt{E}})$$

(cf. Lemma 2.17).

Here is an alternative approach.

Proposition 5.13. *Consider an antiunitary representation $(U, \mathcal{H})$ of G_{τ_h} and the corresponding standard subspace $\mathtt{V} := \mathtt{V}(h, U)$. Assume that the net H on open subsets of $M = G/H$ satisfies* (Iso) *and* (Cov)*. Suppose that there exists an open subset $\emptyset \neq \mathcal{O} \subseteq M$ for which $\mathsf{H}(\mathcal{O})$ is cyclic and contained in $\mathtt{V}$. Then* (BW) *holds for the open subset $W := \exp(\mathbb{R}h).\mathcal{O}$.*

Proof. Clearly, W is an $\exp(\mathbb{R}h)$-invariant open subset of M and (Cov) and (Iso) imply that $\mathsf{H}(W) \subseteq \mathtt{V}$ in an $U(\exp \mathbb{R}h)$-invariant subspace. As $\mathsf{H}(W)$ contains $\mathsf{H}(\mathcal{O})$, it is cyclic, so that $\mathsf{H}(W) = \mathtt{V}$ follows from Proposition 1.20. □

Corollary 5.14. *Assume that the net H on open subsets of $M = G/H$ satisfies* (Iso), (Cov), (RS) *and* (Add) *and that there exists an open subset $\mathcal{O} \subseteq M$ such that $\mathsf{H}(\mathcal{O}) \subseteq \mathtt{V} = \mathtt{V}(h, U)$. Then the union W^{H} of all such open subsets is non-empty, open,* $\exp(\mathbb{R}h)$*-invariant, and satisfies*

$$\mathsf{H}(W^{\mathsf{H}}) = \mathtt{V}.$$

Problem 5.15. Compare W with $W_M^+(h)$ for causal homogeneous spaces $M = G/H$.

5.5 Classification of nets of real subspaces

We expect that there are various contexts in which nets could be classified. Specifically, the (BW) property determines the net for a given antiunitary representation $(U, \mathcal{H})$ of G_{τ_h} on all wedge regions $g.W$, $g \in G$, in M.

For causal flag manifolds, this fact already implies that a net satisfying (Iso), (Cov), (RS), (BW) and (Add) is uniquely determined by the antiunitary representation $(U, \mathcal{H})$ of G_{τ_h} (see [MN25] for details).

Problem 5.16. Consider $G := \mathrm{Aff}(\mathbb{R})_e$ with the non-symmetric Euler element $h = (0, 1)$ (Example 2.18). Here the intervals (x, ∞), $x \in \mathbb{R}$, are natural wedge regions in $M = \mathbb{R}$. Given an antiunitary representation $(U, \mathcal{H})$ of $G_{\tau_h} = \mathrm{Aff}(\mathbb{R})$, is it possible to classify all nets on open subsets of $\mathbb{R}$ that satisfy the (BW) condition? Here additivity and locality conditions certainly help to reduce the problem.

For instance, if H is additive, then it is easy to see that the whole net is determined by the real subspace $\mathsf{H}((0, 1))$, assigned to the open unit interval $(0, 1)$. So one has to determine which real subspaces arise in such nets.

A The category of W^*-algebras

By the Gelfand–Naimark Theorem, C^*-algebras can be characterized as closed $*$-subalgebras of some $B(\mathcal{H})$, $\mathcal{H}$ a complex Hilbert space. On the other hand, we have defined von Neumann algebras directly as $*$-subalgebras $\mathcal{M} \subseteq B(\mathcal{H})$ satisfying $\mathcal{M}'' = \mathcal{M}$. So they are in particular closed with respect to the weak-$*$-topology on $B(\mathcal{H})$, specified by the subspace $B_1(\mathcal{H}) \subseteq B(\mathcal{H})^*$ of trace class operators. As $B(\mathcal{H}) \cong B_1(\mathcal{H})^*$, the duality theory of Banach spaces easily implies that $\mathcal{M} \cong Q^*$ for $Q := B_1(\mathcal{H})/\mathcal{M}^\perp$. Hence every von Neumann algebra has a *predual*.

This observation can be used to specify von Neumann algebras axiomatically, independently of an embedding in some $B(\mathcal{H})$.

Definition A.1. A C^*-algebra $\mathcal{M}$ is called a W^**-algebras* if it has a predual, i.e., there exists a closed subspace $\mathcal{M}_* \subseteq \mathcal{M}^*$ with $\mathcal{M} \cong (\mathcal{M}_*)^*$ as Banach spaces.

This approach has been pursued by S. Sakai, and his monograph [Sa71] is an excellent reference. [Sa71, Cor. 1.13.3] asserts in particular that W^*-algebras have a unique predual $\mathcal{M}_*$. Its elements are called *normal linear functionals*, they are the continuous linear functionals for the $\sigma(\mathcal{M}, \mathcal{M}_*)$-topology on $\mathcal{M}$, i.e., the coarsest topology for which all functionals in $\mathcal{M}_*$ are continuous. Any normal selfadjoint functional is a difference of two positive ones, and the positive normal functionals φ can also be characterized by the property that, for every uniformly bounded increasing directed subset $(x_j)_{j\in J}$ of $\mathcal{M}$, we have

$$\varphi(\sup x_j) = \sup \varphi(x_j)$$

([Sa71, Thm. 1.13.2]).

We also note that W^*-algebras always have an identity, which can be derived from the Krein–Milman Theorem because it ensures the existence of extreme points in the unit ball of $\mathcal{M}$ ([Sa71, §1.7]).

Examples A.2. (a) For every complex Hilbert space $\mathcal{H}$, the full operator algebra $B(\mathcal{H})$ is a W^*-algebra with predual $B(\mathcal{H})_* = B_1(\mathcal{H})$ (trace class operators).
(b) For every σ-finite measures space $(X, \mathfrak{S}, \mu)$, the Banach algebra $L^\infty(X, \mathfrak{S}, \mu)$ is a commutative W^*-algebra with $L^\infty(X, \mathfrak{S}, \mu)_* \cong L^1(X, \mathfrak{S}, \mu)$.

The same holds for ℓ^∞-direct sums (whose preduals are ℓ^1-direct sums), and all commutative W^*-algebras are such sums. More intrinsically, they can be described as the space $L^\infty_{\rm loc}(X, \mathfrak{S}, \mu)$ of bounded, locally measurable functions on a semi-finite measure space $(X, \mathfrak{S}, \mu)$. Here *semi-finite* means that every $E \in \mathfrak{S}$ with $\mu(E) = \infty$ contains a measurable subset of finite positive measure, and *locally measurable* means measurable on every measurable subset of finite measure.

Definition A.3. A *morphism of W^*-algebras* is a complex linear $*$-algebra morphism $\pi \colon \mathcal{M} \to \mathcal{N}$ with $\pi^*\mathcal{N}_* \subseteq \mathcal{M}_*$, i.e., pullbacks of normal functionals are normal. We call these algebra morphisms *normal*. For every complex Hilbert space $\mathcal{H}$, a *normal representation* $(\pi, \mathcal{H})$ of $\mathcal{M}$ is a normal morphism $\pi \colon \mathcal{M} \to B(\mathcal{H})$.

Remark A.4. (a) For normal states, the GNS construction produces a normal representation.
(b) This is more generally true for semi-finite weights. A *weight* $\omega \colon \mathcal{M}_+ \to [0, \infty]$ is an additive, positively homogeneous function. It is called *normal* if it is compatible with bounded sup's. A weight w on $\mathcal{M}$ is called *semi-finite* if the set

$$\{M \in \mathcal{M}_+ \mid w(M) < \infty\}$$

generates a $*$-algebra which is $\sigma(\mathcal{M}, \mathcal{M}_*)$–dense in $\mathcal{M}$.

The GNS construction and the Tomita–Takesaki Theorem extend to normal weights and faithful normal semi-finite weights always exist ([Bla06, III.2.2.26]). Normal semi-finite weights are sums (in the sense of summability of general families) of normal positive forms (cf. [Haa75b]). As a consequence, any von Neumann algebra $\mathcal{M}$ has a standard form representation (cf. [BGN20], [Bla06], [BGN20, §3.1]).

Remark A.5. (a) Any σ-finite measure is semi-finite. If X is a set, then the counting measure

$$\mu \colon \mathbb{P}(X) \to \mathbb{N}_0 \cup \{\infty\}, \quad \mu(E) := |E|$$

is semi-finite. It is σ-finite if and only if X is countable.
(b) If $(X_j, \mathfrak{S}_j, \mu_j)_{j\in J}$ are semi-finite measure spaces, and we put

$$X := \dot{\bigcup}_{j\in J} X_j, \quad \mathfrak{S} := \{E \subseteq X \colon (\forall j \in J)\, E \cap X_j \in \mathfrak{S}_j\} \quad \text{and} \quad \mu(E) := \sum_{j\in J} \mu_j(E \cap X_j),$$

then $\mathfrak{S}$ is a σ-algebra on X, μ is a measure, and $(X, \mathfrak{S}, \mu)$ is a semi-finite measure space. Exercise A.3 shows that, conversely, up to sets of measure zero, any semi-finite measure space is such a direct sum of finite measure spaces.

Exercises for Appendix A

Exercise A.1. (Direct sums of von Neumann algebras) Let $\mathcal{M}_j \subseteq B(\mathcal{H}_j)$ be a family of von Neumann algebras, $\mathcal{H} := \widehat{\bigoplus}_{j\in J}\mathcal{H}_j$ the Hilbert space direct sum of the $\mathcal{H}_j$ and

$$\mathcal{M} := \overline{\bigoplus}_{j\in J}\mathcal{M}_j := \Big\{(M_j)_{j\in J} \in \prod_{j\in J}\mathcal{M}_j \colon \sup_{j\in J}\|M_j\| < \infty\Big\}$$

the ℓ^∞-direct sum of the von Neumann algebras $\mathcal{M}_j$ with the norm $\|M\| := \sup_{j\in J}\|M_j\|$. Show that $\mathcal{M}$ can be realized in a natural way as a von Neumann algebra on $\mathcal{H}$.

Exercise A.2. (Separability and σ-finiteness) Let $(X, \mathfrak{S}, \mu)$ be a measure space. Show that:

(a) If $f \in L^p(X, \mu)$, $1 \leq p < \infty$, then the measurable subset $\{f \neq 0\}$ of X is σ-finite.

(b) If $\mathcal{H} \subseteq L^2(X, \mu)$ is a separable Hilbert subspace, then there exists a σ-finite measurable subset $X_0 \subseteq X$ with the property that each $f \in \mathcal{H}$ vanishes μ-almost everywhere on $X_0^c = X \setminus X_0$.

Exercise A.3. Let $(X, \mathfrak{S}, \mu)$ be a measure space. Show that there exist measurable subsets $X_j \subseteq X$, $j \in J$, of finite measure such that

$$L^2(X, \mu) \cong \widehat{\bigoplus}_{j\in J} L^2(X_j, \mu|_{X_j}).$$

Hint: Use Zorn's Lemma to find a maximal family $(X_j)_{j\in J}$ of measurable subsets of finite positive measure of X for which $\mu(X_j \cap X_k) = 0$ for $j \neq k$. Conclude that the corresponding subspaces $L^2(X_j, \mu|_{X_j})$ of $L^2(X, \mu)$ are mutually orthogonal and that the intersection of their orthogonal complements is trivial. For the latter argument, use Exercise A.2(a).

B From unitary to antiunitary representations

Lemma B.1. (The antiunitary extension) *Let $(U, \mathcal{H})$ be a unitary representation of G and write $\overline{\mathcal{H}}$ for the Hilbert space $\mathcal{H}$, endowed with the opposite complex structure. Then the following assertions hold:*

(a) *On $\widetilde{\mathcal{H}} := \mathcal{H} \oplus \overline{\mathcal{H}}$ we obtain by $\widetilde{U}(g) := U(g) \oplus U(\tau_h(g))$ a unitary representation which extends by $\widetilde{U}(\tau_h)(v, w) := \widetilde{J}(v, w) := (w, v)$ to an antiunitary representation of G_{τ_h}. The corresponding standard subspace $\widetilde{\mathtt{V}} := \mathtt{V}(h, \widetilde{U})$ coincides with the graph*

$$\widetilde{\mathtt{V}} = \Gamma(\Delta^{1/2}), \tag{B.1}$$

and its modular operator is $\widetilde{\Delta} := \Delta \oplus \Delta^{-1}$.

(b) *If U extends to an antiunitary representation of G_{τ_h} by $J = U(\tau_h)$, then the following assertions hold:*

(1) $\Phi\colon \mathcal{H}^{\oplus 2} \to \widetilde{\mathcal{H}}, \Phi(v,w) = (v, Jw)$ *is a unitary intertwiner of* $\widetilde{U}$ *and the antiunitary representation* $U^\sharp$ *of* G_{τ_h} *on* $\mathcal{H}^{\oplus 2}$*, given by*

$$U^\sharp|_G = U^{\oplus 2} \quad \text{and} \quad U^\sharp(\tau_h)(v,w) := J^\sharp(v,w) := (Jw, Jv).$$

(2) *The standard subspace* $\mathtt{V}^\sharp := \mathtt{V}(h, U^\sharp)$ *coincides with the graph* $\Gamma(T_{\mathtt{V}})$ *of the Tomita operator* $T_{\mathtt{V}} = J\Delta^{1/2}$ *of* $\mathtt{V}$.

(3) *The antiunitary representation* $\widetilde{U}$ *is equivalent to the antiunitary representation* $U^{\oplus 2}$ *of* G_{τ_h} *on* $\mathcal{H}^{\oplus 2}$.

(4) *If* $A \subseteq G$ *is a subset, then* $\widetilde{\mathtt{V}}_A$ *is cyclic in* $\widetilde{\mathcal{H}}$ *if and only if* $\mathtt{V}_A$ *is cyclic in* $\mathcal{H}$.

Proof. ([MN24]) (a) The first assertion is a direct verification (cf. [NÓ17, Lemma 2.10]). Since

$$\widetilde{\Delta} = e^{2\pi i \partial \widetilde{U}(h)} = \Delta \oplus \Delta^{-1},$$

the description of the standard subspace $\widetilde{\mathtt{V}} = \mathrm{Fix}(\widetilde{J}\widetilde{\Delta}^{1/2})$ follows immediately.
(b) (1) Clearly, Φ is a complex linear isometry that intertwines the antiunitary representation $\widetilde{U}$ with the antiunitary representation $U^\sharp$.

(2) As $\Delta^\sharp = \Phi^{-1}\widetilde{\Delta}\Phi = \Delta \oplus \Delta$, the relation

$$(v,w) = J^\sharp(\Delta^\sharp)^{1/2}(v,w) = (J\Delta^{1/2}w, J\Delta^{1/2}v) = (T_{\mathtt{V}}w, T_{\mathtt{V}}v)$$

is equivalent to $w = T_{\mathtt{V}}v$. Hence $\mathtt{V}^\sharp = \Gamma(T_{\mathtt{V}})$.

(3) As the restrictions of $U^{\oplus 2}$ and $U^\sharp$ to G coincide, [NÓ17, Thm. 2.11] implies their equivalence as antiunitary representations. However, in the present concrete case, it is easy to see an intertwining operator. The matrix

$$A := \frac{1}{2}\begin{pmatrix} (1+i)\mathbf{1} & (1-i)\mathbf{1} \\ (1-i)\mathbf{1} & (1+i)\mathbf{1} \end{pmatrix} \quad \text{with} \quad A^2 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}$$

defines a unitary operator on $\mathcal{H}^{\oplus 2}$ commuting with $U^\sharp(G)$. It satisfies $J^{\oplus 2}AJ^{\oplus 2} = A^* = A^{-1}$, so that

$$AJ^{\oplus 2}A^{-1} = A^2 J^{\oplus 2} = J^\sharp.$$

(4) If $U|_G$ extends to an antiunitary representation U of G_{τ_h} on $\mathcal{H}$, then (3) implies that $\widetilde{U} \cong U^{\oplus 2}$, and any equivalence $\Psi\colon (\widetilde{U}, \widetilde{\mathcal{H}}) \to (U^{\oplus 2}, \mathcal{H}^{\oplus 2})$ maps $\widetilde{\mathtt{V}}_A$ to $(\mathtt{V} \oplus \mathtt{V})_A = \mathtt{V}_A \oplus \mathtt{V}_A$. Therefore $\widetilde{\mathtt{V}}_A$ is cyclic if and only if $\mathtt{V}_A$ is cyclic in $\mathcal{H}$. $\square$

The following definition extends the classical type of irreducible complex representations to the case where the involution on G is non-trivial. For a unitary representation $(U, \mathcal{H})$, we write $(\overline{U}, \overline{\mathcal{H}})$ for the canonical unitary representation on the complex conjugate space $\overline{\mathcal{H}}$ by $\overline{U}(g) = U(g)$. We observe that, for an antiunitary representation $(U, \mathcal{H})$ of G_{τ_h}, its *commutant*

$$U(G_{\tau_h})' = \{A \in B(\mathcal{H})\colon (\forall g \in G_{\tau_h})\, AU(g) = U(g)A\} = \{A \in U(G)'\colon U(\tau_h)A = AU(\tau_h)\}$$

is only a real subalgebra of $B(\mathcal{H})$ because some $U(g)$ are antilinear.

Definition B.2. ([NÓ17, Def. 2.12]) Let $(U, \mathcal{H})$ be an irreducible unitary representation of G. We say that U is (with respect to τ_h), of

- *real type* if there exists an antiunitary involution J on $\mathcal{H}$ such that $U^\sharp(\tau_h) := J$ extends U to an antiunitary representation $U^\sharp$ of G_{τ_h} on $\mathcal{H}$, i.e., $JU(g)J = U(\tau_h(g))$ for $g \in G$. Then the commutant of $U^\sharp(G_{\tau_h})$ is $\mathbb{R}$.
- *quaternionic type* if there exists an antiunitary complex structure I on $\mathcal{H}$ satisfying $IU(g)I^{-1} = U(\tau_h(g))$ for $g \in G$. Then $\overline{U} \circ \tau_h \cong U$, U has no extension on the same space, and the antiunitary representation $(\widetilde{U}, \widetilde{\mathcal{H}})$ of G_{τ_h} with $\widetilde{U}|_G \cong U \oplus (\overline{U} \circ \tau_h)$ is irreducible with commutant $\mathbb{H}$.
- *complex type* if $\overline{U} \circ \tau_h \not\cong U$. This is equivalent to the non-existence of $V \in \mathrm{AU}(\mathcal{H})$ such that $U(\tau_h(g)) = VU(g)V^{-1}$ for all $g \in G$, i.e., to the non-existence of an antiunitary extension of U to G_{τ_h} on $\mathcal{H}$. Then $(\widetilde{U}, \widetilde{\mathcal{H}})$ is an irreducible antiunitary representation of G_{τ_h} with commutant $\mathbb{C}$.

Remark B.3. (Antiunitary tensor products) Let $G = G_1 \times G_2$ be a product of type I groups and τ an involutive automorphism of G preserving both factors, i.e., $\tau = \tau_1 \times \tau_2$. We want to describe irreducible antiunitary representations $(U, \mathcal{H})$ of the group $G_\tau = G \rtimes \{\mathrm{id}_G, \tau\}$ using [NÓ17, Thm. 2.11(d)].
(a) The first possibility is that $U|_G$ is irreducible, so that $U(G)' \cong \mathbb{R}$. Then

$$(U|_G, \mathcal{H}) \cong (U_1, \mathcal{H}_1) \otimes (U_2, \mathcal{H}_2)$$

with irreducible unitary representations $(U_j, \mathcal{H}_j)$ of G_j both extending to antiunitary representations $U_j^\sharp$ of G_j. Hence both U_1 and U_2 are of real type.
(b) The second possibility is that $U|_G$ is reducible with $U(G)' \cong \mathbb{C}$ or $\mathbb{H}$, so that

$$U|_G \cong V \oplus (\overline{V} \circ \tau),$$

where $(V, \mathcal{K})$ is an irreducible unitary representation of G of complex or quaternionic type. Now $V = U_1 \otimes U_2$, and thus

$$\mathcal{H} \cong (\mathcal{H}_1 \otimes \mathcal{H}_2) \oplus (\overline{\mathcal{H}}_1 \otimes \overline{\mathcal{H}}_2), \quad U|_G \cong (U_1 \otimes U_2) \oplus (\overline{U_1} \circ \tau_1 \otimes \overline{U_2} \circ \tau_2).$$

If U_j is of complex type, then $\overline{U_j} \circ \tau_j \not\cong U_j$ implies that V is of complex type. If both U_1 and U_2 are of quaternionic type, then $\overline{U_j} \circ \tau_j \cong U_j$ for $j = 1, 2$ implies $\overline{V} \circ \tau \cong V$, so that V is of quaternionic type.

C Smooth and analytic vectors

In this appendix, we collect some material on distribution vectors and hyperfunction vectors of unitary representations $U \colon G \to \mathrm{U}(\mathcal{H})$ that we use in this paper.

C.1 The integrated representation

Definition C.1. Let G be a Lie group. We fix a left-invariant Haar measure μ_G on G and we often write dg for $d\mu_G(g)$. This measure defines on $L^1(G) := L^1(G, \mu_G)$ the structure of a Banach-$*$-algebra by the *convolution product* and

$$(\varphi * \psi)(x) = \int_G \varphi(g)\psi(g^{-1}x)\, d\mu_G(g), \quad \text{and} \quad \varphi^*(g) = \overline{\varphi(g^{-1})}\Delta_G(g)^{-1} \tag{C.1}$$

is the involution, where $\Delta_G : G \to \mathbb{R}_+$ is the *modular function* determined by

$$\int_G \varphi(y)\, d\mu_G(y) = \int_G \varphi(y^{-1})\Delta_G(y)^{-1}\, d\mu_G(y) \quad \text{and}$$
$$\Delta_G(x)\int_G \varphi(yx)\, d\mu_G(y) = \int_G \varphi(y)\, d\mu_G(y) \quad \text{for} \quad \varphi \in C_c(G).$$

We put

$$\varphi^\vee(g) = \varphi(g^{-1}) \cdot \Delta_G(g)^{-1} \quad \text{so that} \quad \int_G \varphi(g)\, d\mu_G(g) = \int_G \varphi^\vee(g)\, d\mu_G(g). \tag{C.2}$$

The formulas above show that we have two isometric actions of G on $L^1(G)$, given by

$$(\lambda_g f)(x) = f(g^{-1}x) \quad \text{and} \quad (\rho_g f)(x) = f(xg)\Delta_G(g). \tag{C.3}$$

Note that

$$(\lambda_g f)^* = \rho_g f^* \quad \text{and} \quad (\lambda_g f)^\vee = \rho_g f^\vee. \tag{C.4}$$

Now let $(U, \mathcal{H})$ be a continuous unitary representation of the Lie group G, i.e., a homomorphism $U : G \to \mathrm{U}(\mathcal{H}), g \mapsto U(g)$ such that, for each $\eta \in \mathcal{H}$, the orbit map $U^\eta(g) = U(g)\eta$ is continuous. For $\varphi \in L^1(G)$ the operator-valued integral

$$U(\varphi) := \int_G \varphi(g)U(g)\, dg$$

exists and is uniquely determined by

$$\langle \eta, U(\varphi)\zeta\rangle = \int_G \varphi(g)\langle \eta, U(g)\zeta\rangle\, dg \quad \text{for} \quad \eta, \zeta \in \mathcal{H}. \tag{C.5}$$

Then $\|U(\varphi)\| \leq \|\varphi\|_1$ and the so-obtained continuous linear map $L^1(G) \to B(\mathcal{H})$ is a representation of the Banach-$*$-algebra $L^1(G)$, i.e., $U(\varphi * \psi) = U(\varphi)U(\psi)$ and $U(\varphi^*) = U(\varphi)^*$. We also note that

$$U(g)U(\varphi) = U(\lambda_g\varphi) \quad \text{and} \quad U(\varphi)U(g) = U(\rho_g^{-1}\varphi) \quad \text{for} \quad g \in G, \varphi \in L^1(G). \tag{C.6}$$

For $\varphi_g(x) := \varphi(xg)$, we then have $\varphi_g = \Delta_G(g)^{-1}\rho_g\varphi$ by (C.3), and thus by (C.6)

$$U(\varphi_g) = \Delta_G(g)^{-1}U(\varphi)U(g^{-1}) \quad \text{for} \quad g \in G. \tag{C.7}$$

C.2 The space of smooth vectors and its dual

A *smooth vector* is an element $\eta \in \mathcal{H}$ for which the orbit map $U^\eta : G \to \mathcal{H}, g \mapsto U(g)\eta$ is smooth. We write $\mathcal{H}^\infty = \mathcal{H}^\infty(U)$ for the space of smooth vectors. It carries the *derived representation* $\mathtt{d}U$ of the Lie algebra $\mathfrak{g}$ given by

$$\mathtt{d}U(x)\eta = \lim_{t\to 0} \frac{U(\exp tx)\eta - \eta}{t}. \tag{C.8}$$

For $x \in \mathfrak{g}$, we write $\partial U(x)$ for the infinitesimal generator of the one-parameter group $U(\exp tx)$, so that $U(\exp tx) = e^{t\partial U(x)}$. As $\mathcal{H}^\infty$ is dense and $U(G)$-invariant, $\partial U(x)$ is the closure of $\mathtt{d}U(x)$ ([RS73, Thm. VIII.10]).

We extend this representation to a homomorphism $\mathtt{d}U\colon \mathcal{U}(\mathfrak{g}) \to \mathrm{End}(\mathcal{H}^\infty)$, where $\mathcal{U}(\mathfrak{g})$ is the complex enveloping algebra of $\mathfrak{g}$. This algebra carries an involution $D \mapsto D^*$ determined uniquely by $x^* = -x$ for $x \in \mathfrak{g}$. For $D \in \mathcal{U}(\mathfrak{g})$, we obtain a seminorm on $\mathcal{H}^\infty$ by

$$p_D(\eta) = \|\mathtt{d}U(D)\eta\| \quad \text{for} \quad \eta \in \mathcal{H}^\infty.$$

These seminorms define a topology on $\mathcal{H}^\infty$ which turns the injection

$$\eta\colon \mathcal{H}^\infty \to \mathcal{H}^{\mathcal{U}(\mathfrak{g}_{\mathbb{C}})}, \quad \xi \mapsto (\mathtt{d}U(D)\xi)_{D\in\mathcal{U}(\mathfrak{g}_{\mathbb{C}})} \tag{C.9}$$

into a topological embedding, where the right-hand side carries the product topology (cf. [Mag92, 3.19]). It turns $\mathcal{H}^\infty$ into a complete locally convex space for which the linear operators $\mathtt{d}U(D)$, $D \in \mathcal{U}(\mathfrak{g})$, are continuous. Since $\mathcal{U}(\mathfrak{g})$ has a countable basis, countably many such seminorms already determine the topology, so that $\mathcal{H}^\infty$ is metrizable. As it is also complete, it is a Fréchet space. We also observe that the inclusion $\mathcal{H}^\infty \hookrightarrow \mathcal{H}$ is continuous.

The space $\mathcal{H}^\infty$ of smooth vectors is G-invariant and we denote the corresponding representation by U^∞. We thus obtain a smooth action of G on this Fréchet space ([Ne10]). We have the intertwining relation

$$\mathtt{d}U(\mathrm{Ad}(g)x) = U(g)\mathtt{d}U(x)U(g)^{-1} \quad \text{for} \quad g \in G, x \in \mathfrak{g}.$$

If $\varphi \in C_c^\infty(G)$ and $\xi \in \mathcal{H}$, then $U(\varphi)\xi \in \mathcal{H}^\infty$ and differentiation under the integral sign shows that

$$\mathtt{d}U(x)U(\varphi)\xi := U(-x^R\varphi)\xi, \quad \text{where} \quad (x^R\varphi)(g) = \frac{d}{dt}\Big|_{t=0}\varphi((\exp tx)g). \tag{C.10}$$

A sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $C_c^\infty(G)$ is called a *δ-sequence* if $\int_G \varphi_n(g)\,dg = 1$ for every $n \in \mathbb{N}$ and, for every e-neighborhood $U \subseteq G$, we have $\mathrm{supp}(\varphi_n) \subseteq U$ if n is sufficiently large. If $(\varphi_n)_{n\in\mathbb{N}}$ is a δ-sequence, then $U(\varphi_n)\xi \to \xi$, so that $\mathcal{H}^\infty$ is dense in $\mathcal{H}$.

We write $\mathcal{H}^{-\infty}$ for the space of continuous anti-linear functionals on $\mathcal{H}^\infty$. Its elements are called *distribution vectors*. The group G, $\mathcal{U}(\mathfrak{g})$ and $C_c^\infty(G)$ act on $\eta \in \mathcal{H}^{-\infty}$ by

- $(U^{-\infty}(g)\eta)(\xi) := \eta(U(g^{-1})\xi)$, $g \in G, \xi \in \mathcal{H}^\infty$.
  If $U\colon G \to \mathrm{AU}(\mathcal{H})$ is an antiunitary representation and $U(g)$ is antiunitary, then we have to modify this definition slightly by $(U^{-\infty}(g)\eta)(\xi) := \overline{\eta(U(g^{-1})\xi)}$.
- $(\mathtt{d}U^{-\infty}(D)\eta)(\xi) := \eta(\mathtt{d}U(D^*)\xi)$, $D \in \mathcal{U}(\mathfrak{g}), \xi \in \mathcal{H}^\infty$.
- $U^{-\infty}(\varphi)\eta = \eta \circ U^\infty(\varphi^*)$, $\varphi \in C_c^\infty(G)$.

We have natural G-equivariant linear embeddings

$$\mathcal{H}^\infty \hookrightarrow \mathcal{H} \xrightarrow{\xi \mapsto \langle\cdot,\xi\rangle} \mathcal{H}^{-\infty}. \tag{C.11}$$

It is an important feature of (C.11) that the representation of $\mathcal{U}(\mathfrak{g})$ on $\mathcal{H}^{-\infty}$ provides an embedding of the whole Hilbert space $\mathcal{H}$ into a larger space on which the Lie algebra acts. The following lemma shows that, $\mathcal{H}^\infty$ is the maximal $\mathfrak{g}$-invariant subspace of $\mathcal{H} \subseteq \mathcal{H}^{-\infty}$ and that the subspace $\mathcal{H}$ generates $\mathcal{H}^{-\infty}$ as a $\mathfrak{g}$-module.

Lemma C.2. *The following assertions hold:*

(a) $\mathcal{H}^\infty = \{\xi \in \mathcal{H} \subseteq \mathcal{H}^{-\infty} \colon (\forall D \in \mathcal{U}(\mathfrak{g}))\ \mathtt{d}U^{-\infty}(D)\xi \in \mathcal{H}\}$.

(b) $\mathcal{H}^{-\infty} = \operatorname{span}\big(\mathtt{d}U^{-\infty}(\mathcal{U}(\mathfrak{g}))\mathcal{H}\big)$.

Proof. (a) This follows by combining [Oeh20, Prop. A.1], asserting that

$$\mathcal{D}(\partial U(x)) = \{\xi \in \mathcal{H} \colon \mathtt{d}U^{-\infty}(x)\xi \in \mathcal{H}\},$$

with the fact that

$$\mathcal{H}^\infty = \bigcap\{\mathcal{D}(\partial U(x_1)\cdots\partial U(x_n)) \colon n \in \mathbb{N}, x_1, \ldots, x_n \in \mathfrak{g}\}$$

([Ne10, Lemma 3.4]).
(b) Let $\eta \in \mathcal{H}^{-\infty}$ and consider $\mathcal{H}^\infty$ as a subspace of the topological product $\mathcal{H}^{\mathcal{U}(\mathfrak{g})}$. By the Hahn-Banach extension theorem, η extends to a continuous antilinear functional $\widetilde{\eta}$ on $\mathcal{H}^{\mathcal{U}(\mathfrak{g})}$. Since the dual of a direct product is the direct sum of the dual spaces, there exist $D_1, \ldots, D_n \in \mathcal{U}(\mathfrak{g})$ and $\xi_1, \ldots, \xi_n \in \mathcal{H}$, such that

$$\eta(\xi) = \sum_{j=1}^n \langle \mathtt{d}U(D_j)\xi, \xi_j\rangle = \sum_{j=1}^n \langle \xi, \mathtt{d}U^{-\infty}(D_j^*)\xi_j\rangle \quad \text{for} \quad \xi \in \mathcal{H}^\infty,$$

which means that $\eta = \sum_{j=1}^n \mathtt{d}U^{-\infty}(D_j^*)\xi_j$. □

For each $\varphi \in C_c^\infty(G)$, the map $U(\varphi) \colon \mathcal{H} \to \mathcal{H}^\infty$ is continuous, so that its adjoint defines a weak-$*$-continuous map $U^{-\infty}(\varphi^*) \colon \mathcal{H}^{-\infty} \to \mathcal{H}$. We actually have $U^{-\infty}(\varphi)\mathcal{H}^{-\infty} \subseteq \mathcal{H}^\infty$ as a consequence of the Dixmier–Malliavin Theorem [DM78, Thm. 3.1], which asserts that every $\varphi \in C_c^\infty(G)$ can be written as a finite sum of functions of the form $\varphi_1 * \varphi_2$ with $\varphi_j \in C_c^\infty(G)$.

C.3 The space of analytic vectors and its dual

In this subsection, we briefly discuss the space of analytic vectors of a unitary representation of a Lie group. Let $(U, \mathcal{H})$ be a unitary representation of the connected real Lie group G. We write

$$\mathcal{H}^\omega = \mathcal{H}^\omega(U) \subseteq \mathcal{H}$$

for the space of *analytic vectors*, i.e., those $\xi \in \mathcal{H}$ for which the orbit map $U^\xi \colon G \to \mathcal{H}, g \mapsto U(g)\xi$, is analytic.

To endow $\mathcal{H}^\omega$ with a locally convex topology, we specify subspaces $\mathcal{H}^\omega_V$ by open convex 0-neighborhoods $V \subseteq \mathfrak{g}$ as follows. Let $\eta_G \colon G \to G_\mathbb{C}$ denote the universal complexification of G and assume that η_G has discrete kernel (this is always the case if G is semisimple or 1-connected). We assume that V is so small that the map

$$\eta_{G,V} \colon G_V := G \times V \to G_\mathbb{C}, \quad (g, x) \mapsto \eta_G(g)\exp(ix) \tag{C.12}$$

is a covering. Then we endow G_V with the unique complex manifold structure for which $\eta_{G,V}$ is holomorphic.

We now write $\mathcal{H}^\omega_V$ for the set of those analytic vectors ξ for which the orbit map $U^\xi \colon G \to \mathcal{H}$ extends to a holomorphic map

$$U^\xi_V \colon G_V \to \mathcal{H}.$$

As any such extension is G-equivariant by uniqueness of analytic continuation, it must have the form

$$U^\xi_V(g, x) = U(g)e^{i\partial U(x)}\xi \quad \text{for} \quad g \in G, x \in V, \tag{C.13}$$

so that $\mathcal{H}^\omega_V \subseteq \bigcap_{x \in V} \mathcal{D}(e^{i\partial U(x)})$.

The following lemma ([FNÓ23, Lemma 1]) shows that we even have equality.

Lemma C.3. *If $V \subseteq \mathfrak{g}$ is an open convex* 0*-neighborhood for which* (C.12) *is a covering, then* $\mathcal{H}^\omega_V = \bigcap_{x\in V} \mathcal{D}(e^{i\partial U(x)})$.

Proof. It remains to show that each $\xi \in \bigcap_{x\in V} \mathcal{D}(e^{i\partial U(x)})$ is contained in $\mathcal{H}^\omega_V$. For that, we first observe that the holomorphy of the functions $z \mapsto e^{iz\partial U(x)}v$ on a neighborhood of the closed unit disc in $\mathbb{C}$ implies that the $\mathcal{H}$-valued power series

$$f_\xi(x) := \sum_{n=0}^\infty \frac{i^n}{n!}\partial U(x)^n\xi$$

converges for each $x \in V$. Further, [Go69, Thm. 1.1] implies that $\xi \in \mathcal{H}^\infty$, so that the functions $x \mapsto \partial U(x)^n\xi = \mathtt{d}U(x)^n\xi$ are homogeneous $\mathcal{H}$-valued polynomials (cf. [BS71]). Thus [BS71, Thm. 5.2] shows that the above series defines an analytic function $f_\xi\colon V \to \mathcal{H}$. It follows in particular that ξ is an analytic vector, and the map

$$U^\xi_V\colon G_V \to \mathcal{H}, \quad (g,x) \mapsto U^\xi(g,x) := U(g)e^{i\partial U(x)}\xi$$

is defined. It is clearly equivariant. We claim that it is holomorphic. As it is locally bounded, it suffices to show that, for each $\eta \in \mathcal{H}^\omega$, the function

$$f\colon G_V \to \mathbb{C}, \quad f(g,x) := \langle \eta, U^\xi(g,x)\rangle$$

is holomorphic ([Ne99, Cor. A.III.3]). As

$$f(g,x) = \langle U(g)^{-1}\eta, e^{i\partial U(x)}\xi\rangle$$

and the orbit map of η is analytic, f is real analytic. Therefore it suffices to show that it is holomorphic on some 0-neighborhood. This follows from the fact that it is G-equivariant and coincides on some 0-neighborhood with the local holomorphic extension of the orbit map of ξ. Here we use that, for $x, y \in \mathfrak{g}$ sufficiently small, the holomorphic extension U^ξ of the ξ-orbit map satisfies

$$U^\xi(\exp(x * iy)) = U(\exp x)U^\xi(\exp iy) = U(\exp x)f_\xi(y) = U^\xi_V(\exp x, y),$$

where $a * b = a + b + \frac{1}{2}[a,b] + \cdots$ denotes the Baker–Campbell–Hausdorff series. $\square$

We topologize the space $\mathcal{H}^\omega_V$ by identifying it with $\mathcal{O}(G_V,\mathcal{H})^G$, the Fréchet space of G-equivariant holomorphic maps $F\colon G_V \to \mathcal{H}$, endowed with the Fréchet topology of uniform convergence on compact subsets. Now $\mathcal{H}^\omega = \bigcup_V \mathcal{H}^\omega_V$, and we topologize $\mathcal{H}^\omega$ as the locally convex direct limit of the Fréchet spaces $\mathcal{H}^\omega_V$. If the universal complexification $\eta_G\colon G \to G_\mathbb{C}$ is injective, it is easy to see that we thus obtain the same topology as in [GKS11]. Note that, for any monotone basis $(V_n)_{n\in\mathbb{N}}$ of convex 0-neighborhoods in $\mathfrak{g}$, we then have

$$\mathcal{H}^\omega \cong \varinjlim \mathcal{H}^\omega_{V_n},$$

so that $\mathcal{H}^\omega$ is a countable locally convex limit of Fréchet spaces. As the evaluation maps

$$\mathcal{O}(G_V,\mathcal{H})^G \to \mathcal{H}, \quad F \mapsto F(e,0)$$

are continuous, the inclusion $\iota\colon \mathcal{H}^\omega \to \mathcal{H}$ is continuous.

We write $\mathcal{H}^{-\omega}$ for the space of continuous antilinear functionals $\eta\colon \mathcal{H}^\omega \to \mathbb{C}$ (called *hyperfunction vectors*) and

$$\langle\cdot,\cdot\rangle\colon \mathcal{H}^\omega \times \mathcal{H}^{-\omega} \to \mathbb{C}$$

for the natural sesquilinear pairing that is linear in the second argument. We endow $\mathcal{H}^{-\omega}$ with the weak-$*$-topology. We then have natural continuous inclusions

$$\mathcal{H}^{\omega} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}^{-\omega}.$$

Our specification of the topology on $\mathcal{H}^{\omega}$ differs from the one [GKS11] because we do not want to assume that the universal complexification $\eta_G \colon G \to G_{\mathbb{C}}$ is injective, but both constructions define the same topology. Moreover, the arguments in [GKS11] apply with minor changes to general Lie groups.

We actually have the following chain of complex linear embeddings

$$\mathcal{H}^{\omega} \subseteq \mathcal{H}^{\infty} \subseteq \mathcal{H} \subseteq \mathcal{H}^{-\infty} \subseteq \mathcal{H}^{-\omega}, \tag{C.14}$$

where all inclusions are continuous and G acts on all spaces by representations denoted U^{ω}, U^{∞}, U, $U^{-\infty}$, and $U^{-\omega}$, respectively. All of the three above representations can be integrated to the convolution algebra $C_c^{\infty}(G) := C_c^{\infty}(G, \mathbb{C})$ of test functions, for instance

$$U^{-\infty}(\varphi) := \int_G \varphi(g) U^{-\infty}(g)\, dg, \tag{C.15}$$

where dg stands for a left Haar measure on G.

D Direct integral techniques

Here we collect some material from [MN24] and [BN25]. We refer to [BR87] for the basics on direct integrals.

Let $\mathcal{H} = \int_X^{\oplus} \mathcal{H}_m \, d\mu(m)$ be a direct integral of Hilbert spaces on a standard measure space (X, μ). We call a closed real subspace $\mathsf{H} \subseteq \mathcal{H}$ *decomposable* if it is of the form

$$\mathsf{H} = \int_X^{\oplus} \mathsf{H}_m \, d\mu(m),$$

where $(\mathsf{H}_m)_{m \in X}$ is a measurable field of closed real subspaces. Now let $(\mathsf{H}^k)_{k \in K}$ be an at most countable family of decomposable real subspaces. Then we have ([MT19, Lemma B.3]):

(DI1) $\mathsf{H}' = \int_X^{\oplus} \mathsf{H}_m' \, d\mu(m)$.

(DI2) $\bigcap_{k \in K} \mathsf{H}^k = \int_X^{\oplus} \bigcap_{k \in K} \mathsf{H}_m^k \, d\mu(m)$.

(DI3) $\overline{\sum_k \mathsf{H}^k} = \int_X^{\oplus} \overline{\sum_k \mathsf{H}_m^k} \, d\mu(m)$.

Lemma D.1. *The subspace H is cyclic/separating/standard if and only if μ-almost all H_m have this property.*

Proof. (a) First we deal with the separating property. By (DI2) we have

$$\mathsf{H} \cap i\mathsf{H} = \int_X^{\oplus} (\mathsf{H}_m \cap i\mathsf{H}_m) \, d\mu(m),$$

and this space is trivial if and only if μ-almost all spaces $\mathsf{H}_m \cap i\mathsf{H}_m$ are trivial, which means that H_m is separating.
(b) The subspace H is cyclic if and only if H' is separating. By (DI1) and (a) this means that μ-almost all H_m' are separating, i.e., that H_m is cyclic.
(c) By (a) and (b) H is standard if and only if μ-almost all H_m are cyclic and separating, i.e., standard. $\square$

Lemma D.2. *For a countable family* $(\mathsf{H}^k)_{k\in K}$ *of decomposable cyclic closed real subspaces, the intersection* $\mathsf{V} := \bigcap_{k\in K} \mathsf{H}^k$ *is cyclic if and only if, for* μ*-almost every* $m \in X$*, the subspace* $\mathsf{V}_m := \bigcap_{k\in K} \mathsf{H}^k_m$ *is cyclic.*

Proof. By (DI2), we have $\mathsf{V} = \int_X^\oplus \mathsf{V}_m \, d\mu(m)$, so that the assertion follows from Lemma D.1. $\square$

For a direct integral

$$(U, \mathcal{H}) = \int_X^\oplus (U_m, \mathcal{H}_m) \, d\mu(m)$$

of antiunitary representations of G_{τ_h}, the canonical standard subspace $\mathsf{V} = \mathsf{V}(h, U) \subseteq \mathcal{H}$ from (0.1) is specified by the decomposable operator $J\Delta^{1/2} = U(\tau_h)e^{\pi i \, \partial U(h)}$, hence decomposable:

$$\mathsf{V} = \int_X^\oplus \mathsf{V}_m \, d\mu(m). \tag{D.1}$$

Lemma D.3. *Assume that* G *has at most countably many components. For any subset* $A \subseteq G$ *and a real subspace* $\mathsf{H} \subseteq \mathcal{H}$*, we put*

$$\mathsf{H}_A := \bigcap_{g\in A} U(g)\mathsf{H}. \tag{D.2}$$

Then the following assertions hold:

(a) *If* H *is decomposable, then* $\mathsf{H}_A = \int_X^\oplus \mathsf{H}_{m,A} \, d\mu(m)$*.*

(b) H_A *is cyclic if and only if* μ*-almost all* $\mathsf{H}_{m,A}$ *are cyclic.*

Proof. (a) As G has at most countably many components, it carries a separable metric. Hence there exists a countable subset $B \subseteq A$ which is dense in A. For $\xi \in \mathcal{H}$, we have

$$\xi \in \mathsf{H}_A \quad \text{if and only if} \quad U(A)^{-1}\xi \subseteq \mathsf{H}.$$

Now the closedness of H and the density of B in A show that this is equivalent to $U(B)^{-1}\xi \subseteq \mathsf{H}$, i.e., to $\xi \in \mathsf{H}_B$. This shows that $\mathsf{H}_A = \mathsf{H}_B$. We likewise obtain $\mathsf{H}_{m,A} = \mathsf{H}_{m,B}$ for every $m \in X$. Hence the assertion follows by applying (DI2) to $\mathsf{H}_B = \mathsf{H}_A$.
(b) follows from (a) and Lemma D.1. $\square$

Lemma D.4. *Let* $\mathcal{H} = \int_X^\oplus \mathcal{H}_x d\mu(x)$*, a direct integral von Neumann algebra* $\mathcal{A} = \int_X^\oplus \mathcal{A}_x d\mu(x)$ *and a strongly continuous, unitary, direct integral representation of a Lie group* G *with countable many connected components,* $(U, \mathcal{H}) = \int_X^\oplus (U_x, \mathcal{H}_x) d\mu(x)$*. Then, for any subset* $N \subset G$*, we have*

$$\bigcap_{g\in N} \mathcal{A}_g = \int_X^\oplus \bigcap_{g\in N} (\mathcal{A}_g)_x d\mu(x)$$

where $\mathcal{A}_g = U(g)\mathcal{A}U(g)^*$*.*

Proof. As G has at most countably many components, it carries a separable metric. Hence there exists a countable subset $N_0 \subseteq N$ which is dense in N. For $A \in B(\mathcal{H})$, the map

$$F \colon G \to B(\mathcal{H}), \quad F(g) = U(g)AU(g)^*,$$

is weak operator continuous, so that the set of all $g \in G$ with $F(g) \in \bigcap_{g\in N_0} \mathcal{A}_g$ is a closed subset, hence contains N. We conclude that

$$\bigcap_{g\in N_0} \mathcal{A}_g = \bigcap_{g\in N} \mathcal{A}_g.$$

We likewise obtain for every $x \in X$ the relation

$$\bigcap_{g\in N_0} \mathcal{A}_{x,g} = \bigcap_{g\in N} \mathcal{A}_{x,g} \quad \text{for} \quad \mathcal{A}_{x,g} = U_x(g)\mathcal{A}_x U_x(g)^*.$$

From [BR87, Prop. 4.4.6(b)] we thus obtain

$$\bigcap_{g\in N} \mathcal{A}_g = \bigcap_{g\in N_0} \mathcal{A}_g = \int_X^\oplus \bigcap_{g\in N_0} \mathcal{A}_{x,g}\, d\mu(x) = \int_X^\oplus \bigcap_{g\in N} \mathcal{A}_{x,g}\, d\mu(x).$$

Finally, we observe that, for every $g \in G$

$$\mathcal{A}_g = \int_X^\oplus (\mathcal{A}_g)_x\, d\mu(x) = \int_X^\oplus \mathcal{A}_{x,g}\, d\mu(x)$$

follows by the uniqueness of the direct integral decomposition. □

Tools to verify additivity

Definition D.5. We call a net H on open subsets of M *additive* if $\mathcal{O} = \bigcup_{j\in J} \mathcal{O}_j$ implies $\mathsf{H}(\mathcal{O}) = \overline{\sum_{j\in J} \mathsf{H}(\mathcal{O}_j)}$. We call it *countably additive*, it this relation holds for countable index sets.

Lemma D.6. *If M has a countable basis for its topology, then every countably additive net on open subsets of M is additive.*

Proof. Let $(\mathcal{O}_j)_{j\in J}$ be a family of open subsets of M. Further, let $\mathfrak{B}$ be a countable basis for the topology of M. Then each $\mathcal{O}_j$ is the union of the countable set $\mathfrak{B}_j$ of basis elements contained in $\mathcal{O}_j$, and therefore

$$\mathcal{O} = \bigcup\{\mathcal{B} \colon \mathcal{B} \in \mathfrak{B}_{\mathcal{O}}\}, \quad \mathfrak{B}_{\mathcal{O}} := \cup_{j\in J}\mathfrak{B}_j,$$

where $\mathfrak{B}_{\mathcal{O}}$ is countable. Countable additivity thus implies that

$$\mathsf{H}(\mathcal{O}) = \overline{\sum_{\mathcal{B}\in\mathfrak{B}_{\mathcal{O}}} \mathsf{H}(\mathcal{B})} = \overline{\sum_{j\in J}\sum_{\mathcal{B}\in\mathfrak{B}_j} \mathsf{H}(\mathcal{B})} = \overline{\sum_{j\in J} \mathsf{H}(\mathcal{O}_j)}.$$

Therefore H is additive. □

Remark D.7. Every additive net is isotone because $\mathcal{O}_1 \subseteq \mathcal{O}_2$ implies $\mathcal{O}_2 = \mathcal{O}_1 \cup \mathcal{O}_2$, so that additivity entails

$$\mathsf{H}(\mathcal{O}_2) = \overline{\mathsf{H}(\mathcal{O}_1) + \mathsf{H}(\mathcal{O}_2)} \supseteq \mathsf{H}(\mathcal{O}_1).$$

Lemma D.8. *If $\mathsf{H}(\mathcal{O})_{\mathcal{O}\subseteq M}$ is a net on open subsets of the second countable space M, each subspace $\mathsf{H}(\mathcal{O})$ is decomposable as*

$$\mathsf{H}(\mathcal{O}) = \int_X^\oplus \mathsf{H}_x(\mathcal{O})\, d\mu(x),$$

and μ-almost all the nets $(\mathsf{H}_x)_{x\in X}$ are additive, then H is additive.

Proof. In view of Lemma D.6, it suffices to show that H is countably additive. So let $\mathcal{O} = \bigcup_{j\in J} \mathcal{O}_j$ with a countable index set J. Then (DI3) and the additivity of the nets H_x imply that

$$\overline{\sum_{j\in J} \mathsf{H}(\mathcal{O}_j)} = \int_X \overline{\sum_{j\in J} \mathsf{H}_x(\mathcal{O}_j)}\, d\mu(x) = \int_X \mathsf{H}_x(\mathcal{O})\, d\mu(x) = \mathsf{H}(\mathcal{O}). \qquad \square$$

E Some facts on convex cones

Lemma E.1. ([MNO23, Lemma B.1]) *Let E be a finite-dimensional real vector space, $C \subseteq E$ a closed convex cone and $E_1 \subseteq E$ a linear subspace. If the interior C° of C intersects E_1, then $C^\circ \cap E_1$ coincides with the relative interior C_1° of the cone $C_1 := C \cap E_1$ in E_1.*

Lemma E.2. *Let V be a finite-dimensional real vector space, $A \in \mathrm{End}(V)$ diagonalizable, and let $C \subseteq V$ be a closed convex cone invariant under $e^{\mathbb{R}A}$. Let $\lambda_{\min}$ and $\lambda_{\max}$ be the minimal/maximal eigenvalues of A. For an eigenvalue λ of A we write $V_\lambda(A)$ for the corresponding eigenspace and $p_\lambda \colon V \to V_\lambda(A)$ for the projection along all other eigenspaces. Then*

$$p_{\lambda_{\min}}(C) = C \cap V_{\lambda_{\min}}(A) \quad \text{ and } \quad p_{\lambda_{\max}}(C) = C \cap V_{\lambda_{\max}}(A).$$

If A has only two eigenvalues, it follows that $C = p_{\lambda_{\min}}(C) \oplus p_{\lambda_{\max}}(C)$.

Proof. Since we can replace A by $-A$, it suffices to verify the second assertion. So let $v \in C$ and write it as a sum $v = \sum_\lambda v_\lambda$ of A-eigenvectors. Then

$$v_{\lambda_{\max}} = \lim_{t\to\infty} e^{-t\lambda_{\max}} e^{tA} v \in C$$

implies that $p_{\lambda_{\max}}(C) \subseteq C \cap V_{\lambda_{\max}}(A)$, and the other inclusion is trivial. $\square$

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