Master's Thesis

The Excitation Spectrum of Bose Gases in the Gross–Pitaevskii Regime

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Abstract

Following the strategy in [28], as well as using ideas from [16] and [17], we prove the low-energy excitation spectrum for Bose gases in the Gross–Pitaevskii regime in the 3-dimensional box $[-1/2, 1/2]^3$ with periodic boundary conditions and provide an expansion of the ground state energy, extending previous results from [2, 17] to repulsive L^1 interaction potentials that are not necessarily compactly supported or spherically symmetric. Our main theorem is valid under the assumptions that there is Bose–Einstein condensation and that the leading order term of the ground state energy is $4\pi\mathfrak{a}N$, where $\mathfrak{a}$ denotes the scattering length of the interaction potential. It only applies if the interaction potential decays sufficiently fast.

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Contents

1 Introduction

The ideal gas has been well understood since the pioneering works of Bose and Einstein in 1924 [5, 9]. Einstein applied ideas used by Bose for photons to a gas of atoms and later predicted Bose–Einstein condensation [10]. Roughly speaking, Bose–Einstein condensation occurs when many bosons occupy the same quantum state at very low temperatures, which allows for effects like superfluidity.

The rigorous understanding of interacting Bose gases remains an important problem. In 1947, the theory of interacting Bose gases was first introduced by Bogoliubov [4], where he proposed an approximation of the low-lying excitation spectrum, which he used to justify Landau's criterion for superfluidity [18]. Namely, Bogoliubov discovered that the excitation spectrum behaves linearly in p for small momenta p, which predicts superfluidity by Landau's criterion.

The first experimental observations of Bose–Einstein condensation were made in 1995 [1, 7] and were awarded the Nobel prize in 2001. These experimental breakthroughs renewed the interest in formulas for the ground state of dilute Bose gases, like the ones used in the experiments. In 1957, Dyson proved the upper bound for the leading order term of the ground state energy of a hard-sphere gas [8]. The lower bound was proved in 1998 by Lieb and Yngvason [25]. The Gross–Pitaevskii equation for the ground state energy of trapped Bose gases [14, 15, 29] was rigorously justified by Lieb, Seiringer, and Yngvason in 2000 [24].

For the first time in 2011, the Bogoliubov excitation spectrum was justified by Seiringer for the homogeneous Bose gas on the torus in the mean-field regime [31]. Boccato, Brennecke, Cenatiempo, and Schlein extended this result to the Gross–Pitaevskii regime [2]. This result was further extended to trapped systems in $\mathbb{R}^3$ independently by Brennecke, Schlein, and Schraven [6] and Nam and Triay [28]. A shorter proof for the excitation spectrum of the homogeneous gas on the torus in the Gross–Pitaevskii regime for a more general class of interaction potentials was provided by Hainzl, Schlein, and Triay [17]. While the interaction range is long in the mean-field regime and the interaction strength is weak, in the Gross–Pitaevskii regime, the interaction range is short and the interaction strength is strong. Hence, the Gross–Pitaevskii regime is a model for dilute systems, such as the gases considered in the experiments [1, 7]. Because of the strong correlation between the particles at short distances, a subtle correction in Bogoliubov's theory is required, which makes the analysis more challenging.

We consider a Bose gas on the torus in the Gross–Pitaevskii regime, i.e., the scaling of the interaction potential of the Bosons is explicitly given by $N^2V(N\cdot)$. Following the approach in [28] and using ideas from [16] and [17], we extend the results obtained in [2, 17] to repulsive, not necessarily compactly supported or radial L^1 potentials. In the following section, we present the precise setting and the main theorem we intend to prove.

1.1 Bose gases in a box

We consider a Bose gas consisting of N particles in a box $\Lambda = [-1/2, 1/2]^3$ with periodic boundary conditions. The Hamilton operator acting on $\bigotimes_{\mathrm{sym}}^N L^2(\Lambda)$, the space of square integrable permutation symmetric functions of N variables $x_i \in \Lambda$, is given by

$$H_N = \sum_{i=1}^{N} -\Delta_i + \sum_{1 \leqslant i < j \leqslant N} V_N(x_i - x_j), \tag{1.1}$$

where $V_N \in L^1(\Lambda)$ is the periodization of $\widetilde{V}_N := N^2 \widetilde{V}(N\cdot)$ for some even function $0 \leqslant \widetilde{V} \in L^1(\mathbb{R}^3)$, i.e.,

$$V_N(x) = \sum_{z \in \mathbb{Z}^3} \widetilde{V}_N(x + z).$$

Since H_N is non-negative, it can be defined as self-adjoint using Friedrichs' method. We denote the periodization of $\widetilde{V}$ by V and we assume that $\widetilde{V}$ satisfies

$$\|\widetilde{V} \mathbb{1}_{B(0,R)^{\mathrm{c}}}\|_{L^1} \leqslant C R^{-\gamma} \tag{1.2}$$

for all $R > 0$, some $\gamma \geqslant 1$, and some constant C. This condition is satisfied, for example, if $\widetilde{V}$ decays at least like $|x|^{-3-\gamma}$.

We define the scattering length $\mathfrak{a} > 0$ of V through

$$8\pi\mathfrak{a} = \inf\Big\{ \int_{\mathbb{R}^3} (2|\nabla\varphi(x)|^2 + \widetilde{V}(x)|1 - \varphi(x)|^2)\,\mathrm{d}x : \varphi \in \dot{H}^1(\mathbb{R}^3) \Big\}, \tag{1.3}$$

where $\dot{H}^1(\mathbb{R}^3) := \{\varphi \in L^1_{\mathrm{loc}}(\mathbb{R}^3) : \|\nabla\varphi\|_{L^2} < \infty \text{ and } \lim_{|x|\to\infty} \varphi(x) = 0\}$ is the homogeneous Sobolev space. Here, $\lim_{|x|\to\infty} \varphi(x) = 0$ means that $\{x \in \mathbb{R}^3 : |\varphi(x)| > a\}$ has finite measure for all $a > 0$. $\dot{H}^1(\mathbb{R}^3)$ is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle_{\dot{H}^1} := \langle \nabla\cdot, \nabla\cdot \rangle_{L^2}$. A proof for the completeness of $\dot{H}^1(\mathbb{R}^3)$ can be found, for example, in [13, Lemma 2.84].

It was proved in [25] that, if $\widetilde{V}$ is *radial* and has *compact support*, the ground state energy $\lambda_1(H_N)$ of H_N satisfies

$$\lim_{N\to\infty} \frac{\lambda_1(H_N)}{N} = 4\pi\mathfrak{a}. \tag{1.4}$$

That means the leading order of the ground state energy of H_N depends only on the scattering length of V.

In [22] was proved that, for *radial* $\widetilde{V}$, the ground state Ψ_N exhibits complete Bose–Einstein condensation (BEC) in the zero-momentum state $u_0 \equiv 1$, i.e.,

$$\lim_{N\to\infty} \frac{\langle \Psi_N, a^*(u_0) a(u_0) \Psi_N \rangle}{N} = 1, \tag{1.5}$$

where a^* and a are the creation and annihilation operators on Fock space defined in Definition 2.5. This even holds for every normalized sequence Ψ_N of approximate ground states in the sense that $\langle \Psi_N, H_N \Psi_N \rangle = \lambda_1(H_N) + o(N)$.

In the following, we make the *assumptions* that for the chosen potential V (1.4) holds and we have BEC, i.e., for any normalized sequence Ψ_N, which satisfies $\langle \Psi_N, H_N \Psi_N \rangle = \lambda_1(H_N) + o(N)$, (1.5) holds. We need these assumptions to prove the optimal convergence rate for BEC in Chapter 6.

The main result we want to prove provides more information about the low-energy spectrum of H_N.

Theorem 1.1. *Let $0 \leqslant \kappa < 1/13$ such that (1.2) is satisfied for*

$$\gamma = \frac{1+\kappa}{1-2\kappa}.$$

Denote by $\lambda_1(H_N)$ the ground state energy of H_N with H_N defined as in (1.1). We have

$$\lambda_1(H_N) = 4\pi\mathfrak{a}(N-1) + e_\Lambda \mathfrak{a}^2 + \frac{1}{2} \sum_{p \in 2\pi\mathbb{Z}^3 \setminus \{0\}} \left(\sqrt{p^4 + 16\pi\mathfrak{a}p^2} - p^2 - 8\pi\mathfrak{a} + \frac{(8\pi\mathfrak{a})^2}{2p^2} \right) + \mathcal{O}(N^{-\kappa}),$$

where

$$e_\Lambda = 4\left(\int_\Lambda \frac{1}{p^2}\, \mathrm{d}p + \sum_{z \in \mathbb{Z}^3 \setminus \{0\}} \int_\Lambda \left(\frac{1}{(p+z)^2} - \frac{1}{z^2} \right) \mathrm{d}p \right).$$

Moreover, the spectrum of $H_N - \lambda_1(H_N)$ below a threshold $1 \leqslant \Theta \leqslant N^{\kappa/(1-\vartheta)}$ consists of eigenvalues of the form

$$\sum_{p \in 2\pi\mathbb{Z}^3 \setminus \{0\}} n_p \sqrt{p^4 + 16\pi\mathfrak{a}p^2} + \mathcal{O}(N^{-\kappa}\Theta^{1-\vartheta})$$

with $n_p \in \mathbb{N}_0$ for all $p \in 2\pi\mathbb{Z}^3 \setminus \{0\}$, where only finitely many n_p are non-zero. We can choose $0 \leqslant \vartheta < 1 - 12\kappa$ arbitrarily for $\gamma \in [6/5, 14/11)$ and

$$\vartheta = \frac{1-2\kappa}{3}$$

for $\gamma \in [1, 6/5)$.

Remark. We want to show the connection of the Gross–Pitaevskii regime to the thermodynamic limit. In 1957, Lee, Huang, and Yang [19] predicted that the ground state energy per unit volume of dilute Bose gases with density $\rho > 0$ interacting through a hard-sphere potential with scattering length $\mathfrak{a}_0 > 0$ is of the form

$$E_0 = 4\pi\mathfrak{a}_0\rho^2 \left(1 + \frac{128}{15\sqrt{\pi}} \sqrt{\rho\mathfrak{a}_0^3} \right) \tag{1.6}$$

and the low-lying eigenvalues are of the form

$$E_0 + \sum_{p \neq 0} n_p \sqrt{p^4 + 16\pi\mathfrak{a}_0\rho p^2}, \quad n_p \in \mathbb{N}_0, \tag{1.7}$$

up to errors in the dilute limit $\rho\mathfrak{a}_0^3 \to 0$. These formulas are expected to hold for a large class of repulsive interaction potentials with scattering length $\mathfrak{a}_0 > 0$. To describe such a system of n Bosons, a Hamilton operator of the form

$$H_{n,L} = \sum_{i=1}^{n} -\Delta_i + \sum_{1 \leqslant i < j \leqslant n} V(x_i - x_j)$$

is used, which acts on $\bigotimes_{\mathrm{sym}}^n L^2(\Lambda_L)$, with $\Lambda_L = [-L/2, L/2]^3$. One is interested in the thermodynamic limit $n \to \infty$, $nL^{-3} \to \rho$ of the eigenvalues of such an operator. To obtain lower bounds on the eigenvalues, Λ_L can be divided into smaller boxes with sidelength ℓ, which is done, for instance, in [12, 25] for the ground state energy and in [16] for the excitation spectrum. It suffices only to consider $\Lambda_\ell = [-\ell/2, \ell/2]^3$ with N particles and the Hamilton operator $H_{N,\ell}$ acting on $\bigotimes_{\mathrm{sym}}^N L^2(\Lambda_\ell)$. $H_{N,\ell}$ can be rescaled to obtain an operator on $\bigotimes_{\mathrm{sym}}^N L^2(\Lambda)$, namely

$$\ell^2 \mathcal{T}_\ell H_{N,\ell} \mathcal{T}_\ell^* = \sum_{i=1}^{N} -\Delta_i + \sum_{1 \leqslant i < j \leqslant N} V_\ell(x_i - x_j),$$

where we used the unitary rescaling transform $\mathcal{T}_\ell \Psi = \ell^{3N/2}\Psi(\ell \cdot)$. By choosing the Gross–Pitaevskii length scale (also called healing length)

$$\ell_{\mathrm{GP}} = \frac{1}{\sqrt{\rho\mathfrak{a}_0}},$$

the scaling of the interaction potential of this operator is the Gross–Pitaevskii scaling. In fact, we have $\rho = N\ell_{\mathrm{GP}}^{-3}$ and hence, $\ell_{\mathrm{GP}} = N\mathfrak{a}_0$.

We want to compare our result Theorem 1.1 with the formulas (1.6) and (1.7), see also [2] for more details. Using that the scattering length of V_N is given by $\mathfrak{a}_0 = \mathfrak{a}/N$ and the particle density in Λ is $\rho = N$, we notice that the ground state energy and the low-lying eigenvectors of H_N coincide with the Lee–Huang–Yang formulas up to an error $\mathcal{O}(1)$. Replacing the sum over $p \in 2\pi\mathbb{Z}^3 \setminus \{0\}$ in the formla for the ground state energy in Theorem 1.1 with $(2\pi)^{-3}$ times an integral over $\mathbb{R}^3$ yields

$$\begin{aligned}
&\frac{1}{2} \sum_{p \in 2\pi\mathbb{Z}^3 \setminus \{0\}} \left(\sqrt{p^4 + 16\pi\mathfrak{a}p^2} - p^2 - 8\pi\mathfrak{a} + \frac{(8\pi\mathfrak{a})^2}{2p^2} \right) \\
&\qquad \approx \frac{1}{(2\pi)^3} \frac{1}{2} \int_{\mathbb{R}^3} \left(\sqrt{p^4 + 16\pi\mathfrak{a}p^2} - p^2 - 8\pi\mathfrak{a} + \frac{(8\pi\mathfrak{a})^2}{2p^2} \right) \mathrm{d}p \\
&\qquad = 4\pi\mathfrak{a} \cdot \frac{128}{15\sqrt{\pi}} \sqrt{\mathfrak{a}^3} \\
&\qquad = 4\pi\mathfrak{a}_0\rho^2 \cdot \frac{128}{15\sqrt{\pi}} \sqrt{\rho\mathfrak{a}_0^3},
\end{aligned}$$

which is the second order term in (1.6). This and the term proportional to e_Λ, which arises due to the finiteness of Λ as a correction to the scattering length $\mathfrak{a}$, lead to an error term of order 1. To better control the boundary effects and make the Lee–Huang–Yang formulas visible, a larger length scale $\ell \gg \ell_{\mathrm{GP}}$ is necessary.

1.2 Proof strategy

We can write every $\Psi \in \bigotimes_{\text{sym}}^N L^2(\Lambda)$ as

$$\Psi = u_0^{\otimes N}\xi_0 + u_0^{\otimes N-1} \otimes_s \xi_1 + u_0^{\otimes N-2} \otimes_s \xi_2 + \cdots + \xi_N,$$

where $u_0 \equiv 1$ is the condensate, $\xi_n \in \left(\{u_0\}^\perp\right)^{\otimes_s n}$ for $n = 1, \ldots, N$, and $\xi_0 \in \mathbb{C}$. As in [28], we will start by conjugating H_N with the unitary operator $U\colon \bigotimes_{\text{sym}}^N L^2(\Lambda) \to \mathcal{F}_+^{\leqslant N}$, $\Psi \mapsto (\xi_0, \ldots, \xi_N)$, where $\mathcal{F}_+^{\leqslant N}$ is the truncated Fock space introduced in Section 2.3. The operator U is used to map the creation and annihilation operators $a^*(u_0)$ and $a(u_0)$ to $\sqrt{N - \mathcal{N}}$, where $\mathcal{N}$ is the number operator introduced in Section 2.2. Due to the complete BEC, $\sqrt{N - \mathcal{N}}$ is approximately equal to $\sqrt{N}$. Hence, U is a tool to rigorously implement the c-number substitution in Bogoliubov's approximation. Using the Fock space formalism, which we will recall in Section 2.2, we will obtain

$$UH_NU^* = \mathbb{1}_+^{\leqslant N}\mathcal{H}\mathbb{1}_+^{\leqslant N} \approx \mathbb{1}_+^{\leqslant N}\left(\frac{N-1}{2}\widehat{V}(0) + \mathrm{d}\Gamma(-\Delta) + H_2 + Q_2 + Q_3 + Q_4\right)\mathbb{1}_+^{\leqslant N},$$

with

$$\begin{aligned}
H_2 &= N\int_{\Lambda^2} V_{R,N}(x-y)a_x^*a_y \,\mathrm{d}x\,\mathrm{d}y,\\
Q_2 &= \frac{1}{2}\left(N - \mathcal{N} - \frac{1}{2}\right)\int_{\Lambda^2} V_N(x-y)a_xa_y\,\mathrm{d}x\,\mathrm{d}y + \text{h.c.},\\
Q_3 &= \sqrt{(N-\mathcal{N})_+}\int_{\Lambda^2} V_N(x-y)a_y^*a_xa_y\,\mathrm{d}x\,\mathrm{d}y + \text{h.c.},\\
Q_4 &= \frac{1}{2}\int_{\Lambda^2} V_N(x-y)a_x^*a_y^*a_xa_y\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Here, $\mathbb{1}_+^{\leqslant N}$ is the projection onto $\mathcal{F}_+^{\leqslant N}$ and $V_{R,N}$ is the periodization of $N^2(\widetilde{V}\mathbb{1}_{B(0,R)})(N\cdot)$. The transforms, which we will apply to $\mathcal{H}$ next, preserve $\mathcal{F}_+$, allowing us to estimate all error terms on $\mathcal{F}_+$.

First, we want to conjugate $\mathcal{H}$ with a quadratic transform T_1 as in [28]. To define T_1, we need the solution ω to the scattering equation

$$-\Delta\omega = \frac{1}{2}\widetilde{V}_R(1-\omega),$$

where $\widetilde{V}_R = \widetilde{V}\mathbb{1}_{B(0,R)}$. We define $\omega_{\ell,N} = \chi(\cdot/\ell)\omega(N\cdot)$, with $\chi \approx \mathbb{1}_{B(0,1)}$ and $RN^{-1} \lesssim \ell \lesssim 1$, and denote the periodization of $-N\omega_{\ell,N}(x-y)$ by $\tilde{s}_1(x,y)$. Now, we can find a quadratic transform T_1 such that

$$T_1^*a^*(g)T_1 = a^*\left(\sqrt{1+s_1^2}(g)\right) + a(s_1(\overline{g})), \quad \forall g \in L^2(\Lambda),$$

where $s_1 = Q^{\otimes 2}\tilde{s}_1$ with $Q = 1 - |u_0\rangle\langle u_0|$. The projection Q ensures that T_1 preserves $\mathcal{F}_+$. Conjugating with this transform renormalizes Q_2 by replacing the short-range potential

V_N by a long-range potential $\varepsilon_{\ell,N} = 2\Delta(\omega_{\ell,N} - \omega(N\cdot))$, and extracts the leading order term of the ground state energy. We will obtain

$$\begin{aligned} T_1^* \mathcal{H} T_1 \approx 4\pi\mathfrak{a}(N-1) &+ \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \frac{|N\hat{\varepsilon}_{\ell,N}(p)|^2}{4p^2}\,\mathrm{d}p + \mathrm{d}\Gamma(-\Delta) \\ &+ H_2^{(T_1)} + Q_2^{(T_1)} + Q_3^{(T_1)} + Q_4 + \mathcal{E}_2^{(T_1)}, \end{aligned}$$

with

$$\begin{aligned} H_2^{(T_1)} &= \mathrm{d}\Gamma(NV_{R,N}(x-y) + \widehat{V}_R(0) - 8\pi\mathfrak{a}_R), \\ Q_2^{(T_1)} &= \frac{N}{2}\int_{\Lambda^2} \sum_{z\in\mathbb{Z}^3} \varepsilon_{\ell,N}(x-y+z) a_x a_y \,\mathrm{d}x\,\mathrm{d}y + \mathrm{h.c.}, \\ Q_3^{(T_1)} &= \sqrt{N}\int_{\Lambda^2} V_{R,N}(x-y) a_y^* a_x a_y \,\mathrm{d}x\,\mathrm{d}y + \mathrm{h.c.}, \\ \mathcal{E}_2^{(T_1)} &= \frac{N}{2}\int_{\Lambda^2} (V_N - V_{R,N})(x-y)\Big(1 - \sum_{z\in\mathbb{Z}^3}\omega_{\ell,N}(x-y+z)\Big) a_x a_y \,\mathrm{d}x\,\mathrm{d}y + \mathrm{h.c.}, \end{aligned}$$

where $\mathfrak{a}_R$ is the scattering length of V_R. The error term $\mathcal{E}_2^{(T_1)}$ can only be bounded after applying the next transform.

With the second transform, we want to renormalize the diagonal quadratic term. We will use a cubic transform, i.e., a unitary operator of the form e^S where S is cubic in creation and annihilation operators. Similarly to [28] and [16], we choose

$$S = \theta_M K_c^* - K_c \theta_M,$$

where $\theta_M \approx \mathbb{1}^{\{\mathcal{N}\leqslant M\}}$, $1 \leqslant M \leqslant N$, and

$$K_c^* = \int_\Lambda a^*(Q_x) a^*(k_{c,x}) a(Q_x)\,\mathrm{d}x,$$

with $k_c = N^{-1/2} s_1$. Again, the projection Q ensures that T_c preserves $\mathcal{F}_+$. The cut-off θ_M allows us to determine the action of T_c by simple computations of commutators of the form $[A, S]$. S is chosen such that

$$[\mathrm{d}\Gamma(-\Delta) + Q_4, S] + Q_3^{(T_1)} \approx 0$$

and

$$H_2^{(T_1)} + \int_0^1\int_t^1 T_c^{-s}[Q_3^{(T_1)}, S] T_c^s \,\mathrm{d}s\,\mathrm{d}t \approx N\sum_p \hat{\varepsilon}_{\ell,N}(p) a_p^* a_p.$$

Conjugating with T_c yields

$$T_c^* T_1^* \mathcal{H} T_1 T_c \approx 4\pi\mathfrak{a}(N-1) + \mathbb{H}_{\mathrm{Bog}} + Q_4,$$

with

$$\mathbb{H}_{\mathrm{Bog}} = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \frac{|N\hat{\varepsilon}_{\ell,N}(p)|^2}{4p^2}\,\mathrm{d}p + \sum_{p\neq 0} p^2 a_p^* a_p + \frac{N}{2}\sum_{p\neq 0}\hat{\varepsilon}_{\ell,N}(p)(2a_p^* a_p + a_p^* a_{-p}^* + a_p a_{-p}).$$

This essentially puts us in the mean-field regime, so we can diagonalize this operator with a standard Bogoliubov transform T_2. After justifying that we can replace $N\hat{\varepsilon}_{\ell,N}(p)$ with $8\pi\mathfrak{a}$, we will obtain

$$T_2^* T_c^* T_1^* \mathcal{H} T_1 T_c T_2 \approx E_N + \mathrm{d}\Gamma(E_{\mathrm{Bog}}),$$

with

$$\begin{aligned}
\mathrm{d}\Gamma(E_{\mathrm{Bog}}) &= \sum_{p\in\Lambda^*\setminus\{0\}} \sqrt{p^4 + 16\pi\mathfrak{a}p^2} a_p^* a_p, \\
E_N &= 4\pi\mathfrak{a}(N-1) + e_\Lambda \mathfrak{a}^2 + \frac{1}{2}\sum_{p\in\Lambda^*\setminus\{0\}} \bigg(\sqrt{p^4 + 16\pi\mathfrak{a}p^2} - p^2 - 8\pi\mathfrak{a} + \frac{(8\pi\mathfrak{a})^2}{2p^2}\bigg), \\
e_\Lambda &= 4\bigg(\int_\Lambda \frac{1}{p^2}\,\mathrm{d}p + \sum_{z\in\mathbb{Z}^3\setminus\{0\}} \int_\Lambda \Big(\frac{1}{(p+z)^2} - \frac{1}{z^2}\Big)\,\mathrm{d}p\bigg).
\end{aligned}$$

Finally, the bound

$$UH_NU^* \geqslant 4\pi\mathfrak{a}N + C^{-1}\mathcal{N} - C$$

can be shown on $\mathcal{F}_+^{\leqslant N}$, which implies the optimal convergence rate for BEC. This bound follows from our analysis and the assumptions that the ground state energy of H_N is given by $4\pi\mathfrak{a}N + o(N)$ (1.4) and we have BEC (1.5). We will use this inequality in the proof of Theorem 1.1 to bound the error terms.

1.2.1 Organization of the proof

We will gather important definitions and lemmas in Chapter 2. More precisely, we will prove the existence and properties of a minimizer for the variational problem (1.3) in Section 2.1, we will recall the Fock space formalism in Section 2.2, and we will define U, recall its properties, and conjugate H_N with it in Section 2.3. In Section 2.4 we will recall some properties of general quadratic transforms, which will be useful for the definition of the quadratic transforms T_1 and T_2. In the remaining chapters, we will prove Theorem 1.1. The conjugations with T_1, T_c, and T_2 will be carried out in Chapters 3, 4, and 5, respectively. Finally, we will provide a proof for the optimal rate of BEC in Chapter 6, which will be used to conclude the proof of Theorem 1.1 in Chapter 7.

In Chapter A, we will provide proofs for some well-known results recalled in Sections 2.2, 2.3, and 2.4.

1.2.2 Notation

C always denotes a positive constant that only depends on V. We will often omit the integration variables if there is no ambiguity. For g defined on Λ^2, we will use the notations $g_x \coloneqq g(x,\cdot)$ and $g_y \coloneqq g(\cdot,y)$. We will use the following convention for the Fourier transform

$$\hat{g}(p) = \int g(x)e^{-ip\cdot x}\,\mathrm{d}x, \quad \forall g \in L^1.$$

2 Preliminaries

2.1 Truncated scattering solution

It was proven, for example in [23, Appendix C], that (1.3) has a unique real-valued minimizer $0 \leqslant \varphi \leqslant 1$, which is spherically symmetric if $0 \leqslant \widetilde{V} \in L^1(\mathbb{R}^3)$ is radial and compactly supported. A similar result for compactly supported $0 \leqslant \widetilde{V} \in L^\infty(\mathbb{R}^d)$ with $d \geqslant 3$, which is not necessarily radial, was obtained in [27, Theorem 6]. We will adapt the proof strategy in [27] to prove the following lemma.

Lemma 2.1. *Let $0 \leqslant \widetilde{V} \in L^1(\mathbb{R}^3)$ be even and denote $\widetilde{V}_R \coloneqq \widetilde{V}\mathbb{1}_{B(0,R)}$ for $R \geqslant 1$. The scattering length $\mathfrak{a}_R$ of $\widetilde{V}_R$ defined through*

$$8\pi\mathfrak{a}_R = \inf\Big\{\int_{\mathbb{R}^3} (2|\nabla\varphi(x)|^2 + \widetilde{V}_R(x)|1-\varphi(x)|^2)\,\mathrm{d}x : \varphi \in \dot{H}^1(\mathbb{R}^3)\Big\} \tag{2.1}$$

satisfies

$$8\pi|\mathfrak{a} - \mathfrak{a}_R| \leqslant \|\widetilde{V} - \widetilde{V}_R\|_{L^1}, \tag{2.2}$$

with $\mathfrak{a}$ as in (1.3). Moreover, (2.1) has a unique real-valued minimizer $\omega \in \dot{H}^1(\mathbb{R}^3)$, which is even, solves the scattering equation

$$-\Delta\omega = \frac{1}{2}\widetilde{V}_R(1-\omega) \tag{2.3}$$

in the sense of distributions, and satisfies the pointwise estimates

$$0 \leqslant \omega(x) \leqslant 1, \quad \omega(x) \leqslant \frac{C}{|x|+1}$$

for $x \in \mathbb{R}^3$, and

$$|\nabla\omega(x)| \leqslant \frac{C}{|x|^2}$$

for $|x| \geqslant 2R$, where C is a constant independent of R. Finally, we have

$$8\pi\mathfrak{a}_R = \int_{\mathbb{R}^3} \widetilde{V}_R(1-\omega). \tag{2.4}$$

We suppress the R-dependence of ω to simplify the notation.

Proof. Consider a minimizing sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ for the functional

$$\mathscr{E}[\varphi] = \int_{\mathbb{R}^3} (2|\nabla\varphi(x)|^2 + \widetilde{V}_R(x)|1-\varphi(x)|^2)\,\mathrm{d}x.$$

Since $\mathscr{E}[\varphi_n]$ is bounded, $\{\varphi_n\}_{n\in\mathbb{N}}$ is bounded in $\dot{H}^1(\mathbb{R}^3)$ and $\{\sqrt{\widetilde{V}_R}(1-\varphi_n)\}_{n\in\mathbb{N}}$ is bounded in $L^2(\mathbb{R}^3)$. By the Banach–Alaoglu theorem, $\varphi_n \rightharpoonup \omega$ weakly in $\dot{H}^1(\mathbb{R}^3)$ and $\sqrt{\widetilde{V}_R}(1-\varphi_n) \rightharpoonup \sqrt{\widetilde{V}_R}(1-\omega)$ weakly in $L^2(\mathbb{R}^3)$ up to subsequences. This shows that ω is a minimizer, by the weak lower semicontinuity of $\|\cdot\|_{\dot{H}^1}$ and $\|\cdot\|_{L^2}$.

Without loss of generality, we can assume that the ω is real-valued, since replacing it by $\mathrm{Re}(\omega)$ does not increase $\mathscr{E}$:

$$\mathscr{E}[\omega] = \mathscr{E}[\mathrm{Re}(\omega)] + \int_{\mathbb{R}^3} (2|\nabla \operatorname{Im}(\omega(x))|^2 + \widetilde{V}_R(x)|\operatorname{Im}(\omega(x))|^2)\,\mathrm{d}x \geqslant \mathscr{E}[\mathrm{Re}(\omega)].$$

We can also assume $0 \leqslant \omega \leqslant 1$, since we can replace ω by $\min(\max(\omega,0),1)$ without increasing $\mathscr{E}$.

Consider the functional

$$\mathscr{F}[\varphi] = \int_{\mathbb{R}^3} (2|\nabla\varphi(x)|^2 + \widetilde{V}(x)|1-\varphi(x)|^2)\,\mathrm{d}x.$$

Due to the inequality $\widetilde{V}_R \leqslant \widetilde{V}$, we have $\mathscr{E}[\varphi] \leqslant \mathscr{F}[\varphi]$ for all $\varphi \in \dot{H}^1(\mathbb{R}^3)$. Thus, by the definitions (1.3) and (2.1) of $\mathfrak{a}$ and $\mathfrak{a}_R$, respectively, we have $\mathfrak{a}_R \leqslant \mathfrak{a}$. Hence, using $8\pi\mathfrak{a}_R = \mathscr{E}[\omega]$ and the inequality $8\pi\mathfrak{a} \leqslant \mathscr{F}[\omega]$, we obtain

$$8\pi|\mathfrak{a} - \mathfrak{a}_R| \leqslant \mathscr{F}[\omega] - \mathscr{E}[\omega] = \int_{\mathbb{R}^3} (\widetilde{V} - \widetilde{V}_R)(x)|1-\omega(x)|^2\,\mathrm{d}x \leqslant \|\widetilde{V} - \widetilde{V}_R\|_{L^1}.$$

For the proof of (2.3), note that $\mathscr{E}[\omega] \leqslant \mathscr{E}[\omega + t\varphi]$ for $t \geqslant 0$ and $\varphi \in C_c^\infty(\mathbb{R}^3)$. We have

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}[\omega + t\varphi]\bigg|_{t=0} = 2\int_{\mathbb{R}^3} (2\nabla\omega\cdot\nabla\varphi + \widetilde{V}_R(\omega-1)\varphi)$$

and hence,

$$-\Delta\omega = \frac{1}{2}\widetilde{V}_R(1-\omega)$$

in the sense of distributions.

Now we can prove that ω is the unique real-valued minimizer of (2.1). Since we have $0 \leqslant \omega \leqslant 1$ and $\widetilde{V}_R$ is non-negative, (2.3) implies $\Delta\omega \leqslant 0$, i.e., ω is superharmonic. Thus, by [21, Theorem 9.4], we either have $\omega > 0$ or $\omega = 0$. Let us first assume $\omega = 0$. By (2.3) this can only be true if $\widetilde{V}_R = 0$. Since every minimizer of (2.1) solves the scattering equation (2.3), this implies that every minimizer of (2.1) is harmonic. By Harnack's inequality [21, Theorem 9.5] and the fact that every minimizer converges to zero and is bounded, we conclude that $\omega = 0$ is the unique minimizer of (2.1). Let us now assume $\omega > 0$. Assume there exists another real-valued minimizer $\widetilde{\omega}$. We define

$$h \coloneqq \frac{1}{\sqrt{2}}\sqrt{\omega^2 + \widetilde{\omega}^2}.$$

Observe that $h \in \dot{H}^1(\mathbb{R}^3)$ due to the convexity inequality [21, Theorem 7.8] and the fact that $\omega, \widetilde{\omega} \in \dot{H}^1(\mathbb{R}^3)$. We compute

$$\begin{aligned}
8\pi \mathfrak{a}_R \leqslant \mathscr{E}[h] &= \int_{\mathbb{R}^3} (2|\nabla h|^2 + \widetilde{V}_R |1-h|^2) \\
&\leqslant \int_{\mathbb{R}^3} \big(|\nabla \omega|^2 + |\nabla \widetilde{\omega}|^2 + \widetilde{V}_R (1 - \sqrt{2}\sqrt{\omega^2 + \widetilde{\omega}^2} + \frac{1}{2}(\omega^2 + \widetilde{\omega}^2))\big) \\
&\leqslant \int_{\mathbb{R}^3} \big(|\nabla \omega|^2 + |\nabla \widetilde{\omega}|^2 + \widetilde{V}_R (1 - \omega - \widetilde{\omega} + \frac{1}{2}(\omega^2 + \widetilde{\omega}^2))\big) \\
&= \frac{1}{2}\big(\mathscr{E}[\omega] + \mathscr{E}[\widetilde{\omega}]\big) = 8\pi \mathfrak{a}_R,
\end{aligned}$$

where we used the convexity inequality [21, Theorem 7.8] for the second inequality and the simple bound

$$x + y \leqslant \sqrt{2}\sqrt{x^2 + y^2}$$

for any $x, y \in \mathbb{R}$, for the third inequality. In particular, equality in the convexity inequality holds. Due to [21, Theorem 7.8] and the fact that $\omega > 0$, this implies $\widetilde{\omega} = c \cdot \omega$ for some constant c. Note that both ω and $\widetilde{\omega}$ solve the scattering equation (2.3). If we had $c \neq 1$, this would imply $\widetilde{V}_R = 0$, which is not possible, since in this case $\omega = 0$ would be the unique minimizer of (2.1). Therefore, ω is the unique real-valued minimizer of (2.1).

Since $\widetilde{V}_R$ is even, we observe that $\tilde{\omega} := \omega(-\cdot)$ also minimizes (2.1). Since ω is the unique real-valued minimizer, this implies that ω is an even function.

Let

$$G_y(x) := \frac{1}{4\pi} \frac{1}{|x-y|}$$

be the Green's function. Since ω is a nonnegative superharmonic function, i.e., $\omega \geqslant 0$ a.e. and $\Delta\omega = -\frac{1}{2}\widetilde{V}_R(1-\omega) \leqslant 0$ in the sense of distributions, by [21, Theorem 9.6], $\mu := -\Delta\omega$ is a positive Borel measure on $\mathbb{R}^3$ and

$$\tilde{\omega}(x) := \int_{\mathbb{R}^3} G_y(x)\, \mathrm{d}\mu(y) \tag{2.5}$$

is finite for a.e. $x \in \mathbb{R}^3$ and satisfies $\omega = \tilde{\omega} + C$ a.e. for a constant $C \geqslant 0$. Both ω and $\tilde{\omega}$ converge to 0 as $|x| \to \infty$ and are therefore equal. Since $\widetilde{V}_R$ is supported in $B(0,R)$, we have $\mu(A) = \mu(A \cap B(0,R))$ for every Borel measurable set A and thus, for $|x| \geqslant 2R$ we obtain

$$\begin{aligned}
\omega(x) &= \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|}\, \mathrm{d}\mu(y) \leqslant \frac{1}{|x|}\frac{1}{4\pi} \int_{B(0,R)} \frac{|x|}{|x|-|y|}\, \mathrm{d}\mu(y) \\
&\leqslant \frac{1}{|x|}\frac{1}{4\pi} \int_{B(0,R)} (\widetilde{V}_R(1-\omega))(y)\, \mathrm{d}y \leqslant \frac{1}{|x|}\frac{1}{4\pi} \|\widetilde{V}_R\|_{L^1} \leqslant \frac{1}{|x|}\frac{1}{4\pi} \|\widetilde{V}\|_{L^1},
\end{aligned}$$

where we used $\widetilde{V}_R(1-\omega) \leqslant \widetilde{V}_R$. This implies for $x \in \mathbb{R}^3$

$$\omega(x) \leqslant \frac{C}{|x|+1},$$

since we have $0 \leqslant \omega \leqslant 1$. From (2.5), we obtain for $|x| \geqslant 2R$

$$\nabla\omega(x) = -\frac{1}{8\pi}\int_{\mathbb{R}^3} \frac{(\widetilde{V}_R(1-\omega))(y)}{|x-y|^3}(x-y)\,\mathrm{d}y.$$

Similarly to before, this implies

$$|\nabla\omega(x)| \leqslant \frac{C}{|x|^2}$$

for $|x| \geqslant 2R$.

Let $0 \leqslant \chi \leqslant 1$ be smooth, radial, $\chi(x) = 1$ if $|x| \leqslant 1/2$ and $\chi(x) = 0$ if $|x| \geqslant 1$. Define $\omega_r := \chi(\cdot/r)\omega$. It can be shown (see, e.g., [13, proof of Lemma 2.81]) that $\omega_r \to \omega$ in $\dot{H}^1(\mathbb{R}^3)$. Since we have $\omega_r \in H^1(\mathbb{R}^3)$, we can find sequences $\{\varphi_n^{(r)}\}_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^3)$ such that $\varphi_n^{(r)} \to \omega_r$ in $H^1(\mathbb{R}^3)$. Hence, we can rewrite the scattering length

$$\begin{aligned}
8\pi\mathfrak{a}_R &= \int_{\mathbb{R}^3}(2|\nabla\omega|^2 + \widetilde{V}_R|1-\omega|^2) = \lim_{r\to\infty}\int_{\mathbb{R}^3}(2\nabla\omega\cdot\nabla\omega_r + \widetilde{V}_R|1-\omega|^2)\\
&= \lim_{r,n\to\infty}\int_{\mathbb{R}^3}(2\nabla\omega\cdot\nabla\varphi_n^{(r)} + \widetilde{V}_R|1-\omega|^2) = \lim_{r,n\to\infty}\int_{\mathbb{R}^3}(2\omega(-\Delta\varphi_n^{(r)}) + \widetilde{V}_R|1-\omega|^2)\\
&= \lim_{r,n\to\infty}\int_{\mathbb{R}^3}(\widetilde{V}_R(1-\omega)\varphi_n^{(r)} + \widetilde{V}_R|1-\omega|^2) = \lim_{r\to\infty}\int_{\mathbb{R}^3}(\widetilde{V}_R(1-\omega)\omega_r + \widetilde{V}_R|1-\omega|^2)\\
&= \int_{\mathbb{R}^3}(\widetilde{V}_R(1-\omega)\omega + \widetilde{V}_R|1-\omega|^2) = \int_{\mathbb{R}^3}\widetilde{V}_R(1-\omega),
\end{aligned}$$

where we used that ω minimizes (2.1) in the first equation and (2.3) in the fifth equation. The second to last equation follows by dominated convergence since $\omega_r \to \omega$ pointwise and $|\widetilde{V}_R(1-\omega)\omega_r| \leqslant \widetilde{V}_R \in L^1(\mathbb{R}^3)$. □

Note that by (1.2) and (2.2), we have

$$|\mathfrak{a} - \mathfrak{a}_R| \leqslant CR^{-\gamma}. \tag{2.6}$$

We will denote the periodization of $\widetilde{V}_R$ by V_R and the periodization of $\widetilde{V}_{R,N} := N^2\widetilde{V}_R(N\cdot)$ by $V_{R,N}$.

As in [28, Section 2.2], we want to use a modified version of ω with a cut-off. For

$$4RN^{-1} < \ell < 1/2,$$

we define

$$\omega_{\ell,N}(x) = \chi(x/\ell)\omega(Nx),$$

where $0 \leqslant \chi \leqslant 1$ is smooth, radial, $\chi(x) = 1$ if $|x| \leqslant 1/2$ and $\chi(x) = 0$ if $|x| \geqslant 1$. Using the notation $\omega_N = \omega(N\cdot)$ and $\chi_\ell = \chi(\cdot/\ell)$, we obtain

$$-2\Delta\omega_{\ell,N} = \widetilde{V}_{R,N}(1-\omega_N) - \varepsilon_{\ell,N} \tag{2.7}$$

in the sense of distributions, with

$$\varepsilon_{\ell,N} = 2\Delta(\omega_{\ell,N} - \omega_N) = 4\nabla\omega_N\cdot\nabla\chi_\ell + 2\omega_N\Delta\chi_\ell, \tag{2.8}$$

since $\Delta\omega_N \equiv 0$ outside of $B(0, RN^{-1}) \subset B(0, \ell/2)$. Note that $\varepsilon_{\ell,N} \in C_c^\infty(\mathbb{R}^3)$, since $\varepsilon_{\ell,N}$ has support in $\{\ell/2 \leqslant |x| \leqslant \ell\}$ and ω_N is harmonic, and hence smooth, outside of $B(0, \ell/2)$. Both $\omega_{\ell,N}$ and $\varepsilon_{\ell,N}$ are even and satisfy the following properties.

Lemma 2.2 (Properties of $\varepsilon_{\ell,N}$ and $\omega_{\ell,N}$)**.** *Let $\ell > 4RN^{-1}$. We have for $x \in \mathbb{R}^3$*

$$\begin{aligned} 0 \leqslant \omega_{\ell,N}(x) \leqslant \frac{C\mathbb{1}_{\{|x|\leqslant \ell\}}}{|Nx|+1}, \quad |\nabla\omega_{\ell,N}(x)| \leqslant |\nabla\omega_N(x)|\mathbb{1}_{\{|x|<2RN^{-1}\}} + \frac{CN\mathbb{1}_{\{|x|\leqslant\ell\}}}{|Nx|^2+1}, \\ N|\varepsilon_{\ell,N}(x)| \leqslant C\ell^{-3}\mathbb{1}_{\{\ell/2\leqslant|x|\leqslant\ell\}}. \end{aligned}$$

Moreover,

$$\begin{aligned} \|\omega_{\ell,N}\|_{L^1} &\leqslant CN^{-1}\ell^2, & \|\omega_{\ell,N}\|_{L^2} &\leqslant CN^{-1}\ell^{1/2}, \\ \|\nabla\omega_{\ell,N}\|_{L^1} &\leqslant CN^{-1}\ell, & \|\nabla\omega_N\|_{L^{3/2}(B(0,2RN^{-1}))} &\leqslant CN^{-1}, \\ \|N\varepsilon_{\ell,N}\|_{L^2} &\leqslant C\ell^{-3/2}, & \|N\Delta\varepsilon_{\ell,N}\|_{L^1} &\leqslant C\ell^{-2}, \end{aligned}$$

and

$$\int_{\mathbb{R}^3} N\varepsilon_{\ell,N} = 8\pi\mathfrak{a}_R.$$

Proof. We will use

$$0 \leqslant \omega(x) \leqslant \frac{C}{|x|+1} \quad \text{and} \quad |\nabla\omega(x)| \leqslant |\nabla\omega(x)|\mathbb{1}_{\{|x|<2R\}} + \frac{C}{|x|^2+1}$$

for $x \in \mathbb{R}^3$, from Lemma 2.1. Since $|\chi_\ell| \leqslant \mathbb{1}_{\{|x|\leqslant\ell\}}$ and $|\nabla\chi_\ell| \leqslant C\ell^{-1}\mathbb{1}_{\{\ell/2\leqslant|x|\leqslant\ell\}}$, we have

$$0 \leqslant \omega_{\ell,N}(x) \leqslant \frac{C\mathbb{1}_{\{|x|\leqslant\ell\}}}{|Nx|+1}$$

for $x \in \mathbb{R}^3$, and

$$\begin{aligned} |\nabla\omega_{\ell,N}(x)| &\leqslant \frac{C\ell^{-1}\mathbb{1}_{\{\ell/2\leqslant|x|\leqslant\ell\}}}{|Nx|+1} + \frac{CN\mathbb{1}_{\{|x|\leqslant\ell\}}}{|Nx|^2+1} \\ &\leqslant \frac{CN\mathbb{1}_{\{\ell/2\leqslant|x|\leqslant\ell\}}}{|Nx|^2+N\ell} + \frac{CN\mathbb{1}_{\{|x|\leqslant\ell\}}}{|Nx|^2+1} \leqslant \frac{CN\mathbb{1}_{\{|x|\leqslant\ell\}}}{|Nx|^2+1} \end{aligned}$$

for $|x| \geqslant 2RN^{-1}$. Using this, the pointwise bound on $\nabla\omega_{\ell,N}$ now follows from the fact that $\nabla\omega_{\ell,N} \equiv \nabla\omega_N$ on $\{|x| < 2RN^{-1}\} \subset \{|x| < \ell/2\}$. The pointwise bound on $\varepsilon_{\ell,N}$ follows from (2.8), the bounds $|\nabla\chi_\ell| \leqslant C\ell^{-1}\mathbb{1}_{\{\ell/2\leqslant|x|\leqslant\ell\}}$ and $|\Delta\chi_\ell| \leqslant C\ell^{-2}\mathbb{1}_{\{\ell/2\leqslant|x|\leqslant\ell\}}$, and the fact that

$$|\omega_N(x)| \leqslant \frac{C}{|Nx|} \quad \text{and} \quad |\nabla\omega_N(x)| \leqslant \frac{CN}{|Nx|^2} \tag{2.9}$$

on $\{\ell/2 \leqslant |x| \leqslant \ell\}$.

For the bound on $\|\omega_{\ell,N}\|_{L^1}$, we compute

$$\|\omega_{\ell,N}\|_{L^1} \leqslant \frac{C}{N}\int_{\mathbb{R}^3} \frac{\mathbb{1}_{\{|x|\leqslant \ell\}}}{|x|}\,\mathrm{d}x \leqslant CN^{-1}\ell^2.$$

Similar computations yield the bounds on $\|\omega_{\ell,N}\|_{L^2}$ and $\|N\varepsilon_{\ell,N}\|_{L^2}$. The bound on $\|\nabla\omega_{\ell,N}\|_{L^1}$ also follows similarly, in combination with

$$\|\nabla\omega_{\ell,N}\|_{L^1(B(0,2RN^{-1}))} = \|\nabla\omega_N\|_{L^1(B(0,2RN^{-1}))} = \|\nabla\omega\|_{L^1(B(0,2R))}N^{-2}.$$

We can bound $\|\nabla\omega_N\|_{L^{3/2}(B(0,2RN^{-1}))}$ by using

$$\|\nabla\omega_N\|_{L^{3/2}(B(0,2RN^{-1}))} = \|\nabla\omega\|_{L^{3/2}(B(0,2R))}N^{-1}.$$

For the last two bounds, we used that $\|\nabla\omega\|_{L^1(B(0,2R))} < \infty$ and $\|\nabla\omega\|_{L^{3/2}(B(0,2R))} < \infty$, which follows from $\nabla\omega \in L^2(\mathbb{R}^3)$. Recall that $\varepsilon_{\ell,N}$ is smooth. With (2.8) and the fact that ω_N is harmonic inside the support of $\varepsilon_{\ell,N}$, we can compute

$$\Delta\varepsilon_{\ell,N} = 8\nabla\omega_N\cdot\nabla(\Delta\chi_\ell) + 2\omega_N\Delta^2\chi_\ell = 8\nabla\omega_{N,\ell}\cdot\nabla(\Delta\chi_\ell) + 2\omega_{N,\ell}\Delta^2\chi_\ell.$$

We have the bounds $|\nabla(\Delta\chi_\ell)| \leqslant C\ell^{-3}\mathbb{1}_{\{\ell/2\leqslant|x|\leqslant\ell\}}$ and $|\Delta^2\chi_\ell| \leqslant C\ell^{-4}\mathbb{1}_{\{\ell/2\leqslant|x|\leqslant\ell\}}$. In combination with the bounds on $\|\omega_{\ell,N}\|_{L^1}$ and $\|\nabla\omega_{\ell,N}\|_{L^1}$, this implies $\|N\Delta\varepsilon_{\ell,N}\|_{L^1} \leqslant C\ell^{-2}$.

By (2.3), we have the scattering equation

$$-2\Delta\omega_N = \widetilde{V}_{R,N}(1-\omega_N)$$

in the sense of distributions. Using this, (2.8), the fact that $\chi_\ell \in C_c^\infty(\mathbb{R}^3)$, and (2.4), we obtain

$$\begin{aligned}\int_{\mathbb{R}^3} N\varepsilon_{\ell,N} &= N\int_{\mathbb{R}^3}(4\nabla\omega_N\cdot\nabla\chi_\ell + 2\omega_N\Delta\chi_\ell) = N\int_{\mathbb{R}^3} 2\omega_N(-\Delta\chi_\ell)\\ &= N\int_{\mathbb{R}^3}\widetilde{V}_{R,N}(1-\omega_N) = \int_{\mathbb{R}^3}\widetilde{V}_R(1-\omega) = 8\pi\mathfrak{a}_R,\end{aligned}$$

where we used that $\widetilde{V}_{R,N}(1-\omega_N)\chi_\ell = \widetilde{V}_{R,N}(1-\omega_N)$ by the choice $\ell > 4RN^{-1}$. □

2.2 Fock space formalism

In the following, let $\mathfrak{H}$ be a subspace of $L^2(\Lambda)$.

Definition 2.3. Let $n \in \mathbb{N}$. Then we define

$$\mathfrak{H}^n := \bigotimes_{\text{sym}}^n \mathfrak{H},$$

the space of permutation symmetric functions $\Psi \in \mathfrak{H}^{\otimes n}$ of n variables $x_i \in \Lambda$, i.e.,

$$\Psi(x_1,\ldots,x_n) = \Psi(x_{\sigma(1)},\ldots,x_{\sigma(n)})$$

for all permutations $\sigma \in S_n$. For $n = 0$ we define $\mathfrak{H}^n := \mathbb{C}$.

Definition 2.4 (Fock space)**.** We define the bosonic Fock space

$$\mathcal{F}(\mathfrak{H}) := \bigoplus_{n=0}^{\infty} \mathfrak{H}^n.$$

For $\xi = (\xi_n)_{n=0}^{\infty} \in \mathcal{F}(\mathfrak{H})$ we define the norm $\|\cdot\|_{\mathcal{F}(\mathfrak{H})}$ on $\mathcal{F}(\mathfrak{H})$ by

$$\|\xi\|_{\mathcal{F}(\mathfrak{H})}^2 = \sum_{n=0}^{\infty} \|\xi_n\|_{\mathfrak{H}^n}^2.$$

Definition 2.5 (Creation and annihilation operators)**.** Let $g \in \mathfrak{H}$. We define the creation operator $a^*(g)$ and the annihilation operator $a(g)$ on $\mathcal{F}(\mathfrak{H})$ by

$$(a^*(g)\Psi)(x_1, \dots, x_{n+1}) = \frac{1}{\sqrt{n+1}} \sum_{i=1}^{n+1} g(x_i) \Psi(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n),$$
$$(a(g)\Psi)(x_1, \dots, x_{n-1}) = \sqrt{n} \int_{\Lambda} \overline{g(x_n)} \Psi(x_1, \dots, x_n) \, \mathrm{d}x_n$$

for all $\Psi \in \mathfrak{H}^n$, $n \in \mathbb{N}$. Additionally, we define the operator valued distributions a_x^* and a_x with $x \in \Lambda$ such that

$$a^*(g) = \int_{\Lambda} g(x) a_x^* \, \mathrm{d}x, \quad a(g) = \int_{\Lambda} \overline{g(x)} a_x \, \mathrm{d}x, \quad \forall g \in \mathfrak{H}.$$

These operators satisfy the canonical commutation relations (CCR)

$$[a(f), a(g)] = [a^*(f), a^*(g)] = 0, \quad [a(f), a^*(g)] = \langle f, g \rangle, \quad \forall f, g \in \mathfrak{H},$$
$$[a_x, a_y] = [a_x^*, a_y^*] = 0, \quad [a_x, a_y^*] = \delta(x - y), \quad \forall x, y \in \Lambda.$$

Lemma 2.6 (Second quantization for one- and two-body operators)**.** *Let A be a self-adjoint operator on $\mathfrak{H}$ with domain $D(A)$. Then we define its second quantization as the operator on the bosonic Fock space $\mathcal{F}(\mathfrak{H})$*

$$\mathrm{d}\Gamma(A) := \bigoplus_{n=0}^{\infty} \Big(\sum_{i=1}^{n} A_i \Big) = 0 \oplus A \oplus (A \otimes 1 + 1 \otimes A) \oplus \cdots$$

with domain

$$\bigcup_{N=1}^{\infty} \bigoplus_{n=0}^{N} D(A_1 + \cdots + A_n),$$

where $D(A_1 + \cdots + A_n) \subset \mathfrak{H}^n$ is the domain of the operator $A_1 + \cdots + A_n$ on $\mathfrak{H}^n$. If we choose an orthonormal basis $\{u_n\}_{n \in \mathbb{N}}$ of $\mathfrak{H}$ such that $u_n \in D(A)$ for all $n \in \mathbb{N}$, we can also rewrite it as

$$\mathrm{d}\Gamma(A) = \int_{\Lambda} a_x^* A a_x \, \mathrm{d}x = \sum_{m,n=1}^{\infty} \langle u_m, A u_n \rangle a_m^* a_n,$$

where we used the notation $a_n^\sharp := a^\sharp(u_n)$ *with* $\sharp \in \{*, \cdot\}$*. Let* $W \in L^1(\mathbb{R}^3)$ *be an even, real-valued function. We think of* $W := W(x-y)$ *as a two-body translation invariant multiplication operator on* $\mathfrak{H} \otimes \mathfrak{H}$ *with domain* $D(W)$*. Its second quantization is defined as the operator on the bosonic Fock space* $\mathcal{F}(\mathfrak{H})$

$$\bigoplus_{n=0}^{\infty}\Big(\sum_{1\leqslant i<j\leqslant n} W_{ij}\Big) = 0 \oplus 0 \oplus W \oplus (W_{12} + W_{13} + W_{23}) \oplus \cdots$$

with domain

$$\bigcup_{N=1}^{\infty}\bigoplus_{n=0}^{N} D\Big(\sum_{1\leqslant i<j\leqslant n} W_{ij}\Big),$$

where $D(\sum_{1\leqslant i<j\leqslant n} W_{ij}) \subset \mathfrak{H}^n$ *is the domain of the operator* $\sum_{1\leqslant i<j\leqslant n} W_{ij}$ *on* $\mathfrak{H}^n$*. If we choose an orthonormal basis* $\{u_n\}_{n\in\mathbb{N}}$ *of* $\mathfrak{H}$ *such that* $u_m \otimes u_n \in D(W)$ *for all* $m, n \in \mathbb{N}$*, we can rewrite it as*

$$\frac{1}{2}\int_\Lambda\int_\Lambda W(x-y) a_x^* a_y^* a_x a_y \,\mathrm{d}x\,\mathrm{d}y = \frac{1}{2}\sum_{m,n,p,q=1}^{\infty} \langle u_m \otimes u_n, W u_p \otimes u_q\rangle a_m^* a_n^* a_p a_q.$$

We will provide a proof for Lemma 2.6 in Section A.1; a proof can also be found in [32, Lemmas 7.8 and 7.12] or [26, Section 5.2].

In particular, this lemma implies that the number operator $\mathcal{N} := \mathrm{d}\Gamma(1)$ can be rewritten as

$$\mathcal{N} = \int_\Lambda a_x^* a_x \,\mathrm{d}x = \sum_{n=1}^{\infty} a_n^* a_n$$

for any orthonormal basis $\{u_n\}_{n\in\mathbb{N}}$ of $\mathfrak{H}$.

2.3 The excitation Hamiltonian

We denote $\mathfrak{H} = L^2(\Lambda)$ and $\mathcal{F} = \mathcal{F}(\mathfrak{H})$. Let $\Lambda^* = 2\pi\mathbb{Z}^3$. We can choose $\{u_p\}_{p\in\Lambda^*}$ with $u_p(x) = e^{ip\cdot x}$ explicitly as an orthonormal basis of $L^2(\Lambda)$ and we define $a_p^\sharp := a^\sharp(u_p)$ with $\sharp \in \{*, \cdot\}$. Using the method of second quantization, we can write the Hamiltonian (1.1) as

$$\begin{aligned} H_N &= \sum_{p\in\Lambda^*} p^2 a_p^* a_p + \frac{1}{2}\sum_{p,q,r\in\Lambda^*} \widehat{V}_N(r) a_{p+r}^* a_q^* a_p a_{q+r} \\ &= \int_\Lambda a_x^*(-\Delta_x) a_x \,\mathrm{d}x + \frac{1}{2}\int_\Lambda\int_\Lambda V_N(x-y) a_x^* a_y^* a_x a_y \,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{2.10}$$

The right-hand side of (2.10) is an operator on $\mathcal{F}$, but we will always consider its restriction to the N-body sector, which is equal to (1.1). We will use the convention that the indices appearing in creation and annihilation operators are always non-zero. Let $Q = 1 - |u_0\rangle\langle u_0|$ and

$$\mathfrak{H}_+ = Q\mathfrak{H}, \quad \mathcal{F}_+ = \mathcal{F}(\mathfrak{H}_+), \quad \mathcal{F}_+^{\leqslant N} = \bigoplus_{n=0}^{N} \mathfrak{H}_+^n, \quad \mathbb{1}_+^{\leqslant N} = \mathbb{1}_{\mathcal{F}_+^{\leqslant N}}.$$

We will use the unitary transform $U\colon \mathfrak{H}^N \to \mathcal{F}_+^{\leqslant N}$ introduced in [20]. It can be defined by

$$U(\Psi) = \bigoplus_{k=0}^{N} Q^{\otimes k}\Big(\frac{a_0^{N-k}}{\sqrt{(N-k)!}}\Psi\Big)$$

for every $\Psi \in \mathfrak{H}^N$. Its adjoint is given by

$$U^*\Big(\bigoplus_{k=0}^{N}\xi_k\Big) = \sum_{k=0}^{N}\frac{(a_0^*)^{N-k}}{\sqrt{(N-k)!}}\xi_k$$

for all $\xi_k \in \mathfrak{H}_+^k$, $k = 0, \ldots, N$. We have the following identities on $\mathcal{F}_+^{\leqslant N}$:

$$\begin{aligned} Ua_0^*a_0U^* &= N - \mathcal{N}, \\ Ua^*(f)a_0U^* &= a^*(f)\sqrt{N-\mathcal{N}}, \\ Ua_0^*a(f)U^* &= \sqrt{N-\mathcal{N}}a(f), \\ Ua^*(f)a(g)U^* &= a^*(f)a(g) \end{aligned} \tag{2.11}$$

for all $f, g \in \mathfrak{H}_+$. A proof for these identities will be provided in Section A.2. Now we can prove the following lemma.

Lemma 2.7 (Operator bound on Fock space)**.** *The following operator identity holds on* $\mathcal{F}_+^{\leqslant N}$

$$UH_NU^* = \mathbb{1}_+^{\leqslant N}\mathcal{H}\mathbb{1}_+^{\leqslant N},$$

where

$$\mathcal{H} = \frac{N-1}{2}\widehat{V}(0) + \mathrm{d}\Gamma(-\Delta) + H_2 + Q_2 + Q_3 + Q_4 + \mathcal{E}^{(U)} \tag{2.12}$$

is an operator on the full Fock space $\mathcal{F}$ *with*

$$\begin{aligned} H_2 &= N\int_{\Lambda^2} V_{R,N}(x-y)a_x^*a_y \,\mathrm{d}x\,\mathrm{d}y, \\ Q_2 &= \frac{1}{2}\Big(N - \mathcal{N} - \frac{1}{2}\Big)\int_{\Lambda^2} V_N(x-y)a_xa_y\,\mathrm{d}x\,\mathrm{d}y + \text{h.c.}, \\ Q_3 &= \sqrt{(N-\mathcal{N})_+}\int_{\Lambda^2} V_N(x-y)a_y^*a_xa_y\,\mathrm{d}x\,\mathrm{d}y + \text{h.c.}, \\ Q_4 &= \frac{1}{2}\int_{\Lambda^2} V_N(x-y)a_x^*a_y^*a_xa_y\,\mathrm{d}x\,\mathrm{d}y, \end{aligned}$$

and $\mathcal{E}^{(U)}$ *satisfies the quadratic form estimate on* $\mathcal{F}$

$$\pm\mathcal{E}^{(U)} \leqslant CR^{-\gamma}\mathcal{N} + C\frac{\mathcal{N}^2}{N} + \varepsilon Q_4 + \varepsilon^{-1}C\frac{(\mathcal{N}+1)^2}{N^3} \tag{2.13}$$

for all $\varepsilon > 0$.

$\mathcal{H}$ is an operator on the full Fock space $\mathcal{F}$. In the following chapters, we will restrict it to $\mathcal{F}_+$ to simplify the notation.

Proof. Applying the identities (2.11), we can compute

$$
\begin{aligned}
UH_NU^* = \mathbb{1}_+^{\leqslant N}\Big(&\sum_p p^2 a_p^* a_p + \frac{N-1}{2}\widehat{V}(0) - \frac{\mathcal{N}(\mathcal{N}-1)}{2N}\widehat{V}(0) \\
&+ (N-\mathcal{N})\sum_p \widehat{V}_N(p) a_p^* a_p \\
&+ \frac{1}{2}\sqrt{(N-\mathcal{N})(N-\mathcal{N}-1)}\sum_p \widehat{V}_N(p) a_p a_{-p} + \text{h.c.} \\
&+ \sqrt{N-\mathcal{N}}\sum_{p,q} \widehat{V}_N(p) a_q^* a_p a_{q-p} + \text{h.c.} \\
&+ \frac{1}{2}\sum_{p,q,r} \widehat{V}_N(r) a_{p+r}^* a_q^* a_p a_{q+r}\Big)\mathbb{1}_+^{\leqslant N}.
\end{aligned}
\tag{2.14}
$$

Using this, we obtain $UH_NU^* = \mathbb{1}_+^{\leqslant N}\mathcal{H}\mathbb{1}_+^{\leqslant N}$ with $\mathcal{H}$ given by (2.12) and $\mathcal{E}^{(U)}$ by

$$
\begin{aligned}
\mathcal{E}^{(U)} = \mathbb{1}_+^{\leqslant N}\Big(&-\frac{\mathcal{N}(\mathcal{N}-1)}{2N}\widehat{V}(0) - \mathcal{N}\int_{\Lambda^2} V_N(x-y) a_x^* a_y \,\mathrm{d}x\,\mathrm{d}y \\
&+N\int_{\Lambda^2}(V_N - V_{R,N})(x-y) a_x^* a_y \,\mathrm{d}x\,\mathrm{d}y \\
&+\frac{1}{2}\Big(\sqrt{(N-\mathcal{N})(N-\mathcal{N}-1)_+} - N + \mathcal{N} + \frac{1}{2}\Big)\int_{\Lambda^2} V_N(x-y) a_x a_y \,\mathrm{d}x\,\mathrm{d}y + \text{h.c.}\Big)\mathbb{1}_+^{\leqslant N} \\
&=: \mathbb{1}_+^{\leqslant N}(\mathcal{E}^{(U,1)} + \mathcal{E}^{(U,2)} + \mathcal{E}^{(U,3)})\mathbb{1}_+^{\leqslant N}.
\end{aligned}
$$

Because of the projections, we only need to estimate $\mathcal{E}^{(U,1)}$, $\mathcal{E}^{(U,2)}$, and $\mathcal{E}^{(U,3)}$ on $\mathcal{F}_+^{\leqslant N}$. We obtain

$$
\pm\mathcal{E}^{(U,1)} \leqslant C\frac{\mathcal{N}^2}{N} + C\mathcal{N}\int_{\Lambda^2} V_N(x-y) a_x^* a_x \,\mathrm{d}x\,\mathrm{d}y \leqslant C\frac{\mathcal{N}^2}{N} + C\|V_N\|_{L^1}\mathcal{N}^2 \leqslant C\frac{\mathcal{N}^2}{N},
$$

where we used Cauchy–Schwarz and the fact that $V_N(x-y) = V_N(y-x)$ in the first inequality and $\|V_N\|_{L^1} \leqslant CN^{-1}$ in the last inequality.

Similarly, we have

$$
\pm\mathcal{E}^{(U,2)} \leqslant N\|V_N - V_{R,N}\|_{L^1}\mathcal{N} \leqslant CR^{-\gamma}\mathcal{N},
$$

where we used (1.2) in the last inequality.

For estimating $\mathcal{E}^{(U,3)}$ we use the elementary inequality

$$
\left|\sqrt{1-t}\sqrt{(1-t-N^{-1})_+} - 1 + t + \frac{1}{2N}\right| \leqslant C\Big(\frac{t}{N} + \frac{1}{N^2}\Big)
$$

for all $0 \leqslant t \leqslant 1$. We obtain for all $\varepsilon > 0$

$$
\begin{aligned}
\pm\mathcal{E}^{(U,3)} &\leqslant \varepsilon Q_4 + \varepsilon^{-1}N^2\|V_N\|_{L^1}\left(\sqrt{1-\frac{\mathcal{N}}{N}}\sqrt{\Big(1-\frac{\mathcal{N}+1}{N}\Big)_+} - 1 + \frac{\mathcal{N}}{N} + \frac{1}{2N}\right)^2 \\
&\leqslant \varepsilon Q_4 + C\varepsilon^{-1}\frac{(\mathcal{N}+1)^2}{N^3}
\end{aligned}
$$

on $\mathcal{F}_+^{\leqslant N}$, where we used Cauchy–Schwarz in the first inequality and $\|V_N\|_{L^1} \leqslant CN^{-1}$ in the second inequality. This completes the proof of the lemma. $\square$

2.4 General quadratic transforms

It is well-known, see, e.g., [32, Chapter 9], that for every real, symmetric Hilbert–Schmidt operator s on $\mathfrak{H}$, there exists a unitary transformation T on the Fock space $\mathcal{F}$ such that

$$T^* a^*(g) T = a^*(c(g)) + a(s(\overline{g})), \quad \forall g \in \mathfrak{H},$$

where $c = \sqrt{1+s^2}$.

We will need the following lemma about the conservation of the particle number.

Lemma 2.8. *The transformation T satisfies*

$$T^*(\mathcal{N}+1)^j T \leqslant \exp(C_j \|s\|_{\mathrm{HS}})(\mathcal{N}+1)^j, \quad \forall j \geqslant 1,$$

where C_j is a constant independent of s.

We will prove this lemma and construct T in Section A.3.

3 Quadratic renormalization

3.1 Definition of the first quadratic transform

Similarly to [16, Section 3], we define s_1, $\tilde{s}_1 \colon \Lambda^2 \to \mathbb{R}$ by

$$s_1 = Q^{\otimes 2}\tilde{s}_1, \quad \tilde{s}_1(x,y) = -\sum_{z\in\mathbb{Z}^3} N\omega_{\ell,N}(x-y+z), \tag{3.1}$$

with $\omega_{\ell,N}$ defined as in Section 2.1. We will also denote the operator with integral kernel $s_1(x,y)$ by s_1. We want to define the operators (here, 1 denotes the identity and s_1 denotes the operator, not its kernel)

$$c_1 \coloneqq \sqrt{1+s_1^2} \quad \text{and} \quad p_1 \coloneqq c_1 - 1 = \sqrt{1+s_1^2} - 1 = s_1^2\Big(1+\sqrt{1+s_1^2}\Big)^{-1}.$$

In the following lemma, we will show some useful properties of s_1, c_1, and p_1. In particular, we will prove that s_1 is bounded. Since s_1 is symmetric, it follows that s_1 is self-adjoint and bounded. Hence, c_1 and p_1 are well defined.

Lemma 3.1 (Properties of s_1)**.** *Assume that $4RN^{-1} < \ell < 1/2$. Then we have*

$$s_1(x,y) = \tilde{s}_1(x,y) + N\hat{\omega}_{\ell,N}(0)$$

and, in particular,

$$|s_1(x,y) - \tilde{s}_1(x,y)| \leqslant C\ell^2.$$

Moreover, there exist constants $C>0$, independent of N and ℓ, such that

$$\begin{aligned}
\|s_1\|_{\mathrm{op}} &\leqslant C\ell^2, \quad \|s_1\|_{L^2} + \|s_1\|_{L^\infty L^2} \leqslant C\ell^{1/2}, \quad \|s_1\|_{L^\infty} \leqslant CN,\\
\|\nabla s_1\|_{\mathrm{op}} &\leqslant C\ell,\\
\|p_1\|_{\mathrm{op}} &\leqslant C\ell^4, \quad \|p_1\|_{L^2} + \|p_1\|_{L^\infty L^2} \leqslant C\ell^{5/2},\\
\|\nabla p_1\|_{\mathrm{op}} &\leqslant C\ell^3, \quad \|\nabla_x p_1\|_{L^\infty L^2} \leqslant C\ell^{3/2},\\
\|c_1\|_{\mathrm{op}} &\leqslant C,
\end{aligned}$$

where we denoted $\|g\|_{L^\infty L^2} = \max(\|g\|_{L^\infty_1 L^2_2}, \|g\|_{L^\infty_2 L^2_1})$.

Proof. By writing $Q = 1-P$ with $P = |u_0\rangle\langle u_0|$, we obtain $s_1 = \tilde{s}_1 - (P\otimes 1)\tilde{s}_1 - (1\otimes P)\tilde{s}_1 + (P\otimes P)\tilde{s}_1$ and hence

$$\begin{aligned}
s_1(x,y) &= \tilde{s}_1(x,y) + \int_\Lambda \sum_{z\in\mathbb{Z}^3} N\omega_{\ell,N}(x-y+z)\,\mathrm{d}x + \int_\Lambda \sum_{z\in\mathbb{Z}^3} N\omega_{\ell,N}(x-y+z)\,\mathrm{d}y\\
&\quad - \int_\Lambda \sum_{z\in\mathbb{Z}^3} N\omega_{\ell,N}(x-y+z)\,\mathrm{d}x\,\mathrm{d}y\\
&= \tilde{s}_1(x,y) + N\hat{\omega}_{\ell,N}(0).
\end{aligned}$$

Here we used that $\omega_{\ell,N}$ is supported in Λ. Since we have $N\hat{\omega}_{\ell,N}(0) = N\|\omega_{\ell,N}\|_{L^1} \leqslant C\ell^2$ by Lemma 2.2, this implies

$$|s_1(x,y) - \tilde{s}_1(x,y)| \leqslant C\ell^2.$$

For the bound on $\|s_1\|_{\mathrm{op}}$, notice that the kernel of $\tilde{s}_1$ is a finite sum. In fact, all terms with $\max\{|z_i| : i = 1,2,3\} \geqslant 2$ are zero since $\omega_{\ell,N}$ is supported in Λ. Hence, we have

$$\|s_1\|_{\mathrm{op}} \leqslant \|\tilde{s}_1\|_{\mathrm{op}} \leqslant CN\|\omega_{\ell,N}\|_{L^1} \leqslant C\ell^2,$$

where we used Lemma 2.2 for the last inequality. Similarly, for the bound on $\|\nabla s_1\|_{\mathrm{op}}$, we have

$$\|\nabla s_1\|_{\mathrm{op}} = \|\nabla \tilde{s}_1\|_{\mathrm{op}} \leqslant CN\|\nabla\omega_{\ell,N}\|_{L^1} \leqslant C\ell,$$

where we used Lemma 2.2 again.

Since $\|s_1\|_{L^2} \leqslant \|s_1\|_{L^\infty L^2}$, we only need to bound $\|s_1\|_{L^\infty L^2}$. We have

$$\|s_1\|_{L^\infty L^2} \leqslant C\ell^2 + \|\tilde{s}_1\|_{L^\infty L^2} \leqslant C\ell^2 + CN\|\omega_{\ell,N}\|_{L^2} \leqslant C\ell^{1/2},$$

where we used Lemma 2.2 and $\ell \leqslant 1$.

Because the kernel of $\tilde{s}_1$ is a finite sum, we have

$$\|s_1\|_{L^\infty} \leqslant C\ell^2 + \|\tilde{s}_1\|_{L^\infty} \leqslant C\ell^2 + CN\|\omega_{\ell,N}\|_{L^\infty} \leqslant CN,$$

where we used that $\|\omega_{\ell,N}\|_{L^\infty} \leqslant C$ by Lemma 2.2 and $\ell \leqslant 1$.

Next, we want to prove the bounds for

$$p_1 = s_1^2\Big(1 + \sqrt{1 + s_1^2}\Big)^{-1} =: s_1^2 A.$$

Since $0 \leqslant A \leqslant 1$, we have

$$\begin{aligned}
\|p_1\|_{\mathrm{op}} &\leqslant C\|s_1\|_{\mathrm{op}}^2\|A\|_{\mathrm{op}} \leqslant C\ell^4, \\
\|\nabla p_1\|_{\mathrm{op}} &\leqslant C\|\nabla s_1\|_{\mathrm{op}}\|s_1\|_{\mathrm{op}}\|A\|_{\mathrm{op}} \leqslant C\ell^3.
\end{aligned}$$

Again, since $\|p_1\|_{L^2} \leqslant \|p_1\|_{L^\infty L^2}$, we only need to bound $\|p_1\|_{L^\infty L^2}$. For all $g \in L^2(\Lambda)$ and $x \in \Lambda$ we have

$$\begin{aligned}
\left|\int_\Lambda p_1(x,y)g(y)\,\mathrm{d}y\right| &= \left|\int_\Lambda s_1(x,y)(s_1 A g)(y)\,\mathrm{d}y\right| \\
&\leqslant \|s_1(x,\cdot)\|_{L^2}\|s_1\|_{\mathrm{op}}\|A\|_{\mathrm{op}}\|g\|_{L^2} \\
&\leqslant C\|s_1\|_{L^\infty L^2}\|s_1\|_{\mathrm{op}}\|g\|_{L^2} \\
&\leqslant C\ell^{5/2}\|g\|_{L^2}.
\end{aligned}$$

This implies $\|p_1\|_{L^\infty L^2} \leqslant C\ell^{5/2}$. The proof of $\|\nabla_x p_1\|_{L^\infty L^2} \leqslant C\ell^{3/2}$ is similar. Notice that $[\nabla, s_1] = 0$. Using this, we obtain for all $g \in L^2(\Lambda)$ and $x \in \Lambda$

$$\begin{aligned}\left|\int_\Lambda \nabla_x p_1(x,y) g(y)\,\mathrm{d}y\right| &= \left|\int_\Lambda s_1(x,y)(\nabla s_1 A g)(y)\,\mathrm{d}y\right| \\ &\leqslant \|s_1(x,\cdot)\|_{L^2}\|\nabla s_1\|_{\mathrm{op}}\|A\|_{\mathrm{op}}\|g\|_{L^2} \\ &\leqslant C\|s_1\|_{L^\infty L^2}\|\nabla s_1\|_{\mathrm{op}}\|g\|_{L^2} \\ &\leqslant C\ell^{3/2}\|g\|_{L^2}.\end{aligned}$$

This proves $\|\nabla_x p_1\|_{L^\infty L^2} \leqslant C\ell^{3/2}$.

Finally, since $c_1 = 1 + p_1$ and $\|p_1\|_{\mathrm{op}} \leqslant C\ell^4 \leqslant C$, we have $\|c_1\|_{\mathrm{op}} \leqslant C$. □

We want to use a quadratic transform similar to that in [28, Section 4]. Since $s_1(x,y)$ is in $L^2(\Lambda^2)$, s_1 is a Hilbert Schmidt operator on $L^2(\Lambda)$. Also, notice that $s_1(x,y)$ is real and symmetric. Hence, as remarked in Section 2.4, we can define the first quadratic transform T_1, that satisfies

$$T_1^* a^*(g) T_1 = a^*(c_1(g)) + a(s_1(\overline{g})), \quad \forall g \in \mathfrak{H}. \tag{3.2}$$

This operator is a unitary transform on $\mathcal{F}_+$ since s_1 is an operator on $\mathfrak{H}_+$. With this transformation, we want to renormalize Q_2 and extract contributions which allow us to reconstruct the leading order term of the ground state energy $4\pi\mathfrak{a}N$. In the remainder of this chapter, we will prove the following lemma.

Lemma 3.2. *Assume that $4RN^{-1} < \ell < 1/2$. We have*

$$\begin{aligned}T_1^*\mathcal{H}T_1 = 4\pi\mathfrak{a}(N-1) &+ \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{|N\hat{\varepsilon}_{\ell,N}(p)|^2}{4p^2}\,\mathrm{d}p + \mathrm{d}\Gamma(-\Delta) \\ &+ H_2^{(T_1)} + Q_2^{(T_1)} + Q_3^{(T_1)} + Q_4 + \mathcal{E}_2^{(T_1)} + \mathcal{E}^{(T_1)}\end{aligned} \tag{3.3}$$

on $\mathcal{F}_+$, with

$$\begin{aligned}H_2^{(T_1)} &= \mathrm{d}\Gamma\big(NV_{R,N}(x-y) + \widehat{V}_R(0) - 8\pi\mathfrak{a}_R\big), \\ Q_2^{(T_1)} &= \frac{N}{2}\int_{\Lambda^2}\sum_{z\in\mathbb{Z}^3}\varepsilon_{\ell,N}(x-y+z)a_x a_y\,\mathrm{d}x\,\mathrm{d}y + \mathrm{h.c.}, \\ Q_3^{(T_1)} &= \sqrt{N}\int_{\Lambda^2}V_{R,N}(x-y)a_y^* a_x a_y\,\mathrm{d}x\,\mathrm{d}y + \mathrm{h.c.}, \\ Q_4 &= \frac{1}{2}\int_{\Lambda^2}V_N(x-y)a_x^* a_y^* a_x a_y\,\mathrm{d}x\,\mathrm{d}y, \\ \mathcal{E}_2^{(T_1)} &= \frac{N}{2}\int_{\Lambda^2}(V_N - V_{R,N})(x-y)\Big(1 - \sum_{z\in\mathbb{Z}^3}\omega_{\ell,N}(x-y+z)\Big)a_x a_y\,\mathrm{d}x\,\mathrm{d}y + \mathrm{h.c.},\end{aligned}$$

and

$$\pm\mathcal{E}^{(T_1)} \leqslant CNR^{-\gamma} + C\ell^4\,\mathrm{d}\Gamma(-\Delta) + C\ell^{1/2}(\mathcal{N}+1) + C\ell^{1/2}\frac{(\mathcal{N}+1)^{3/2}}{N^{1/2}} \\ + \varepsilon Q_4 + \varepsilon^{-1}C\Big((R^{-\gamma}+\ell)(\mathcal{N}+1) + \frac{(\mathcal{N}+1)^2}{N}\Big) \tag{3.4}$$

for all $0<\varepsilon\leqslant 1$.

Remark.

- The condition $4RN^{-1}<\ell$ ensures that $\omega_{\ell,N}\equiv\omega_N$ on the support of $\widetilde{V}_{R,N}$. Since ℓ is bounded from above, $\|s_1\|_{L^2}$ is bounded by Lemma 3.1, which ensures that the powers of the particle number are conserved, see Lemma 2.8.
- Although $\mathcal{E}^{(T_1)}$ does not leave $\mathcal{F}_+$ invariant, we interpret the quadratic form estimate (3.4) as $\pm\langle\xi,\mathcal{E}^{(T_1)}\xi\rangle \leqslant \langle\xi,A\xi\rangle$ for all $\xi\in\mathcal{F}_+$, where A is the right-hand side of (3.4).

From (2.12) we have

$$T_1^*\mathcal{H}T_1 = \frac{N-1}{2}\widehat{V}(0) + T_1^*\,\mathrm{d}\Gamma(-\Delta)T_1 + T_1^*H_2T_1 \\ + T_1^*Q_2T_1 + T_1^*Q_3T_1 + T_1^*Q_4T_1 + T_1^*\mathcal{E}^{(U)}T_1. \tag{3.5}$$

In the following sections, we will estimate the terms on the right-hand side of (3.5). Then we will use these estimates to prove Lemma 3.2 in Section 3.7.

3.2 Estimating $T_1^*\,\mathrm{d}\Gamma(-\Delta)T_1$

Lemma 3.3. *Assume that* $4RN^{-1}<\ell<1/2$. *We have*

$$T_1^*\,\mathrm{d}\Gamma(-\Delta)T_1 = \mathrm{d}\Gamma(-\Delta) + \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{|N\hat{\varepsilon}_{\ell,N}(p)|^2}{4p^2}\,\mathrm{d}p + Q_2^{(T_1)} + \mathcal{E}^{(T_1)}_{\mathrm{d}\Gamma(-\Delta)} \\ + \frac{N}{2}\int_\Lambda (V_R(1-\omega)\omega)(x)\,\mathrm{d}x \\ - \frac{N}{2}\int_{\Lambda^2}\Big(V_{R,N}(x-y)\sum_{z\in\mathbb{Z}^3}(1-\omega_{\ell,N})(x-y+z)\Big)(a_x^*a_y^*+a_xa_y)\,\mathrm{d}x\,\mathrm{d}y$$

on $\mathcal{F}_+$, *with*

$$\pm\mathcal{E}^{(T_1)}_{\mathrm{d}\Gamma(-\Delta)} \leqslant C\ell^4\,\mathrm{d}\Gamma(-\Delta) + C\ell^2(\mathcal{N}+1).$$

Proof. Using $T_1^*a_x^*T_1 = a^*(c_{1,x}) + a(s_{1,x})$, the CCR, and $c_1 = 1+p_1$, we obtain

$$T_1^*\,\mathrm{d}\Gamma(-\Delta)T_1 = \int_\Lambda \nabla_x(a^*(c_{1,x})+a(s_{1,x}))\nabla_x(a(c_{1,x})+a^*(s_{1,x})) \\ = \int_\Lambda \nabla_x a^*(c_{1,x})\nabla_x a(c_{1,x}) + \int_\Lambda \nabla_x a^*(s_{1,x})\nabla_x a(s_{1,x})$$

$$
\begin{aligned}
&+\int_\Lambda (\nabla_x a^*(c_{1,x})\nabla_x a^*(s_{1,x}) + \nabla_x a(c_{1,x})\nabla_x a(s_{1,x})) + \|\nabla_x s_1\|_{L^2}^2 \\
&= \int_\Lambda \nabla_x a_x^* \nabla_x a_x + \int_\Lambda (-\Delta_x s_1)(x,y)(a_x^* a_y^* + a_x a_y) + \|\nabla_x s_1\|_{L^2}^2 \qquad (3.6)\\
&+\int_\Lambda (\nabla_x a_x^* a(\nabla_x p_{1,x}) + a^*(\nabla_x p_{1,x})\nabla_x a_x) \\
&+\int_\Lambda a^*(\nabla_x p_{1,x}) a(\nabla_x p_{1,x}) + \int_\Lambda a^*(\nabla_x s_{1,x}) a(\nabla_x s_{1,x}) \\
&+\int_\Lambda (a^*(\nabla_x p_{1,x}) a^*(\nabla_x s_{1,x}) + a(\nabla_x p_{1,x}) a(\nabla_x s_{1,x})).
\end{aligned}
$$

The terms in the last three lines of (3.6) are error terms. First, we want to bound them. Let $\xi \in \mathcal{F}_+$, then by Cauchy–Schwarz we have

$$
\begin{aligned}
\left|\left\langle \xi, \int_\Lambda \nabla_x a_x^* a(\nabla_x p_{1,x})\xi \right\rangle\right| &\leqslant \int_\Lambda \|\nabla_x a_x \xi\| \|a(\nabla_x p_{1,x})\xi\| \\
&\leqslant \left(\int_\Lambda \|\nabla_x a_x \xi\|^2\right)^{1/2} \left(\int_\Lambda \|a(\nabla_x p_{1,x})\xi\|^2\right)^{1/2} \\
&\leqslant \|\mathrm{d}\Gamma(-\Delta)^{1/2}\xi\| \|\nabla p_1\|_{\mathrm{op}} \|\mathcal{N}^{1/2}\xi\| \\
&\leqslant C\ell^4 \langle \xi, \mathrm{d}\Gamma(-\Delta)\xi\rangle + C\ell^2\langle \xi, \mathcal{N}\xi\rangle,
\end{aligned}
$$

where we used $\int_\Lambda a^*(\nabla_x p_{1,x}) a(\nabla_x p_{1,x}) \leqslant \|\nabla p_1\|_{\mathrm{op}}^2 \mathcal{N}$ and Lemma 3.1. Similarly, we have

$$
\begin{aligned}
\left|\left\langle \xi, \int_\Lambda a^*(\nabla_x p_{1,x}) a(\nabla_x p_{1,x})\xi \right\rangle\right| &\leqslant \|\nabla p_1\|_{\mathrm{op}}^2 \langle \xi, \mathcal{N}\xi\rangle \leqslant C\ell^6 \langle \xi, \mathcal{N}\xi\rangle, \\
\left|\left\langle \xi, \int_\Lambda a^*(\nabla_x s_{1,x}) a(\nabla_x s_{1,x})\xi \right\rangle\right| &\leqslant \|\nabla s_1\|_{\mathrm{op}}^2 \langle \xi, \mathcal{N}\xi\rangle \leqslant C\ell^2 \langle \xi, \mathcal{N}\xi\rangle, \\
\left|\left\langle \xi, \int_\Lambda a^*(\nabla_x p_{1,x}) a^*(\nabla_x s_{1,x})\xi \right\rangle\right| &\leqslant \int_\Lambda \|a(\nabla_x s_{1,x})\xi\| \|a^*(\nabla_x p_{1,x})\xi\| \\
&\leqslant \|\nabla s_1\|_{\mathrm{op}} \|\mathcal{N}^{1/2}\xi\| \|\nabla_x p_1\|_{L^\infty L^2} \|(\mathcal{N}+1)^{1/2}\xi\| \\
&\leqslant C\ell^{5/2} \langle \xi, (\mathcal{N}+1)\xi\rangle,
\end{aligned}
$$

for all $\xi \in \mathcal{F}_+$. Here we used $a(\nabla_x p_{1,x}) a^*(\nabla_x p_{1,x}) \leqslant \|\nabla_x p_1\|_{L^\infty L^2}^2 (\mathcal{N}+1)$ for the last bound.

The first term in (3.6) is $\mathrm{d}\Gamma(-\Delta)$. For the second term, notice that $s_1 - \tilde{s}_1$ is constant by Lemma 3.1. Hence, we have $-\Delta_x s_1 = -\Delta_x \tilde{s}_1$. Using the definition of $\tilde{s}_1$ (3.1) and the scattering equation (2.7), we obtain

$$
-\Delta_x \tilde{s}_1(x,y) = \sum_{z\in\mathbb{Z}^3} N\Delta_x \omega_{\ell,N}(x-y+z) = \frac{N}{2}\sum_{z\in\mathbb{Z}^3} (\varepsilon_{\ell,N} - \widetilde{V}_{R,N}(1-\omega_N))(x-y+z), \quad (3.7)
$$

so the second term in (3.6) is

$$\begin{aligned}
&Q_2^{(T_1)} - \frac{N}{2}\int_{\Lambda^2}\Big(\sum_{z\in\mathbb{Z}^3}(\widetilde{V}_{R,N}(1-\omega_N))(x-y+z)\Big)(a_x^*a_y^* + a_x a_y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad = Q_2^{(T_1)} - \frac{N}{2}\int_{\Lambda^2}\Big(V_{R,N}(x-y)\sum_{z\in\mathbb{Z}^3}(1-\omega_{\ell,N})(x-y+z)\Big)(a_x^*a_y^* + a_x a_y)\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

For the third term in (3.6), we use $\nabla_x s_1 = \nabla_x \tilde{s}_1$, the definition of $\tilde{s}_1$ (3.1), (3.7), and $\widetilde{V}_{R,N}\omega_{\ell,N} \equiv V_{R,N}\omega_N$ on Λ. Doing an integration by parts, we obtain

$$\begin{aligned}
\|\nabla_x s_1\|_{L^2}^2 &= \int_{\Lambda^2} \tilde{s}_1(x,y)(-\Delta_x \tilde{s}_1)(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= -\frac{N^2}{2}\int_\Lambda (\omega_{\ell,N}\varepsilon_{\ell,N} - V_{R,N}(1-\omega_N)\omega_N)(x)\,\mathrm{d}x \\
&= -\frac{N^2}{2}\int_\Lambda (\omega_{\ell,N}\varepsilon_{\ell,N})(x)\,\mathrm{d}x + \frac{N}{2}\int_\Lambda (V_R(1-\omega)\omega)(x)\,\mathrm{d}x \qquad (3.8)
\end{aligned}$$

Note that $\Delta\omega_N$ and $\omega_{\ell,N} - \omega_N$ have disjoint support and recall the definition (2.8) of $\varepsilon_{\ell,N}$. We have

$$\int_{\mathbb{R}^3} \omega_N \varepsilon_{\ell,N} = 2\int_{\mathbb{R}^3} \omega_N \Delta(\omega_{\ell,N} - \omega_N) = 2\int_{\mathbb{R}^3} \Delta\omega_N(\omega_{\ell,N} - \omega_N) = 0.$$

Using this and an integration by parts, we can rewrite the first term in (3.8)

$$\begin{aligned}
&-\frac{N^2}{2}\int_\Lambda (\omega_{\ell,N}\varepsilon_{\ell,N})(x)\,\mathrm{d}x = -\frac{N^2}{2}\int_{\mathbb{R}^3}((\omega_{\ell,N}-\omega_N)\varepsilon_{\ell,N})(x)\,\mathrm{d}x \\
&\quad = N^2\int_{\mathbb{R}^3}|\nabla(\omega_{\ell,N}-\omega_N)(x)|^2\,\mathrm{d}x = (2\pi)^{-3}N^2\int_{\mathbb{R}^3}|p(\hat{\omega}_{\ell,N}-\hat{\omega}_N)(p)|^2\,\mathrm{d}p \qquad (3.9) \\
&\quad = (2\pi)^{-3}N^2\int_{\mathbb{R}^3}\frac{|p^2(\hat{\omega}_{\ell,N}-\hat{\omega}_N)(p)|^2}{p^2}\,\mathrm{d}p = (2\pi)^{-3}\int_{\mathbb{R}^3}\frac{|N\hat{\varepsilon}_{\ell,N}(p)|^2}{4p^2}\,\mathrm{d}p,
\end{aligned}$$

which finishes the proof. □

3.3 Estimating $T_1^* H_2 T_1$

Recall

$$H_2 = N\int_{\Lambda^2} V_{R,N}(x-y)a_x^* a_y\,\mathrm{d}x\,\mathrm{d}y.$$

Lemma 3.4. *Assume that* $4RN^{-1} < \ell < 1/2$. *We have*

$$T_1^* H_2 T_1 = H_2 + \mathcal{E}_{H_2}^{(T_1)}$$

on $\mathcal{F}_+$*, with*

$$\pm\mathcal{E}_{H_2}^{(T_1)} \leqslant C\ell^{1/2}(\mathcal{N}+1).$$

Notice that $H_2 = \mathrm{d}\Gamma(NV_{R,N}(x-y))$, where $NV_{R,N}(x-y)$ is the kernel of a self-adjoint integral operator with $\|NV_{R,N}(x-y)\|_{\mathrm{op}} \leqslant \|V\|_{L^1} \leqslant C$. Hence, it suffices to prove the following lemma.

Lemma 3.5. *Assume that $4RN^{-1} < \ell < 1/2$. We have*

$$\pm(T_1^* \,\mathrm{d}\Gamma(A) T_1 - \mathrm{d}\Gamma(A)) \leqslant C\|A\|_{\mathrm{op}} \ell^{1/2}(\mathcal{N}+1)$$

on $\mathcal{F}$ for any bounded, self-adjoint operator A on $\mathfrak{H}$ with kernel $A(x,y)$.

Proof. Using (3.2), the CCR, and $c_1 = p_1 + 1$, we get

$$\begin{aligned}
T_1^* \,\mathrm{d}\Gamma(A) T_1 &= \int_{\Lambda^2} A(x,y)(a^*(c_{1,x}) + a(s_{1,x}))(a(c_{1,y}) + a^*(s_{1,y})) \\
&= \int_{\Lambda^3} s_1(u,x) A(x,y) s_1(y,u) + \mathrm{d}\Gamma(c_1 A c_1 + s_1 A s_1) \\
&\quad + \left(\int_{\Lambda^2} A(x,y) a^*(c_{1,x}) a^*(s_{1,y}) + \text{h.c.} \right) \\
&= \mathrm{d}\Gamma(A) + \mathrm{Tr}(s_1 A s_1) + \mathrm{d}\Gamma(p_1 A p_1 + p_1 A + A p_1 + s_1 A s_1) \\
&\quad + \left(\int_{\Lambda^2} (c_1 A s_1)(x,y) a_x^* a_y^* + \text{h.c.} \right).
\end{aligned} \tag{3.10}$$

We want to estimate the error terms on the right-hand side of (3.10) using the bounds in Lemma 3.1. Since $\|s_1\|_{\mathrm{HS}} = \|s_1\|_{L^2} \leqslant C\ell^{1/2}$, we have

$$|\mathrm{Tr}(s_1 A s_1)| \leqslant \|A\|_{\mathrm{op}} \|s_1\|_{\mathrm{HS}}^2 \leqslant C\|A\|_{\mathrm{op}} \ell.$$

Using $\|s_1\|_{\mathrm{op}} \leqslant C\ell^2$ and $\|p_1\|_{\mathrm{op}} \leqslant C\ell^4$, we obtain

$$\begin{aligned}
\pm \,\mathrm{d}\Gamma(p_1 A p_1 + p_1 A + A p_1 + s_1 A s_1) &\leqslant \mathcal{N} \|p_1 A p_1 + p_1 A + A p_1 + s_1 A s_1\|_{\mathrm{op}} \\
&\leqslant \mathcal{N} \|A\|_{\mathrm{op}} (2\|p_1\|_{\mathrm{op}} + \|p_1\|_{\mathrm{op}}^2 + \|s_1\|_{\mathrm{op}}^2) \\
&\leqslant C\mathcal{N} \|A\|_{\mathrm{op}} \ell^4.
\end{aligned}$$

By Lemma 3.1, we have $\|c_1\|_{\mathrm{op}} \leqslant C$ and thus

$$\|c_1 A s_1\|_{\mathrm{HS}} \leqslant \|c_1\|_{\mathrm{op}} \|A\|_{\mathrm{op}} \|s_1\|_{\mathrm{HS}} \leqslant C\|A\|_{\mathrm{op}} \ell^{1/2}.$$

Using this, we obtain

$$\pm \left(\int_{\Lambda^2} (c_1 A s_1)(x,y) a_x^* a_y^* + \text{h.c.} \right) \leqslant \|c_1 A s_1\|_{\mathrm{HS}} (\mathcal{N}+1) \leqslant C\|A\|_{\mathrm{op}} \ell^{1/2} (\mathcal{N}+1).$$

This holds since for a Hilbert-Schmidt operator B with kernel $B(x,y)$ we have

$$\begin{aligned}
\left\langle \xi, \int_{\Lambda^2} B(x,y) a_x^* a_y^* \xi \right\rangle &\leqslant \left(\int_{\Lambda} \|a_x \xi\|^2 \right)^{1/2} \left(\int_{\Lambda} \|a^*(B_x)\xi\|^2 \right)^{1/2} \\
&\leqslant \|\mathcal{N}^{1/2}\xi\| (\|B\|_{\mathrm{op}}^2 \|\mathcal{N}^{1/2}\xi\|^2 + \|B\|_{\mathrm{HS}}^2 \|\xi\|^2)^{1/2} \\
&\leqslant \|B\|_{\mathrm{HS}} \langle \xi, (\mathcal{N}+1)\xi \rangle
\end{aligned}$$

for all $\xi \in \mathcal{F}$, where we used Cauchy–Schwarz and $\|B\|_{\mathrm{op}} \leqslant \|B\|_{\mathrm{HS}}$. □

3.4 Estimating $T_1^* Q_2 T_1$

Recall

$$Q_2 = \frac{1}{2}\Big(N - \mathcal{N} - \frac{1}{2}\Big)\int_{\Lambda^2} V_N(x-y) a_x a_y \,\mathrm{d}x\,\mathrm{d}y + \text{h.c.}$$

Lemma 3.6. *Assume that $4RN^{-1} < \ell < 1/2$. We have*

$$\begin{aligned} T_1^* Q_2 T_1 &= \frac{N}{2}\int_{\Lambda^2} V_N(x-y)(a_x^* a_y^* + a_x a_y)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \Big(\mathcal{N} + \frac{1}{2} - N\Big)\int_\Lambda (V_R\omega)(x)\,\mathrm{d}x + \mathcal{E}_{Q_2}^{(T_1)} \end{aligned}$$

on $\mathcal{F}_+$, with

$$\pm\mathcal{E}_{Q_2}^{(T_1)} \leqslant R^{-\gamma}(\mathcal{N}+N) + C\ell^{1/2}(\mathcal{N}+1) + \varepsilon Q_4 + \varepsilon^{-1} C \frac{(\mathcal{N}+1)^2}{N}$$

for all $0 < \varepsilon \leqslant 1$.

Proof. Using $T_1^* a_x T_1 = a(c_{1,x}) + a^*(s_{1,x})$ and the CCR, we obtain

$$\begin{aligned} &T_1^* a_x a_y T_1 = \Big(a(c_{1,x}) + a^*(s_{1,x})\Big)\Big(a(c_{1,y}) + a^*(s_{1,y})\Big) \\ &= \Big(a(c_{1,x})a(c_{1,y}) + \langle c_{1,x}, s_{1,y}\rangle + a^*(s_{1,x})a(c_{1,y}) + a^*(s_{1,y})a(c_{1,x}) + a^*(s_{1,x})a^*(s_{1,y})\Big). \end{aligned}$$

With this, we can write

$$\begin{aligned} T_1^* Q_2 T_1 &= \frac{1}{2}\Big(1 - \frac{T_1^*\mathcal{N}T_1 + 1/2}{N}\Big)\int_{\Lambda^2} N V_N(x-y)\Big(a(c_{1,x})a(c_{1,y}) + \langle c_{1,x}, s_{1,y}\rangle \\ &\quad + a^*(s_{1,x})a(c_{1,y}) + a^*(s_{1,y})a(c_{1,x}) + a^*(s_{1,x})a^*(s_{1,y})\Big)\,\mathrm{d}x\,\mathrm{d}y + \text{h.c.} \end{aligned} \tag{3.11}$$

We will need the following lemma to bound the error terms.

Lemma 3.7. *Let $g, h \in L^2(\Lambda^2)$, $\sharp_1, \sharp_2 \in \{*, \cdot\}$, and $\xi', \xi \in \mathcal{F}_+$. Then we have*

$$\begin{aligned} &\left|\left\langle \xi', \int_{\Lambda^2} N V_N(x-y) a^{\sharp_1}(g_x) a^{\sharp_2}(h_y)\xi\right\rangle\right| \\ &\qquad \leqslant C\|V\|_{L^1}\|g\|_{L^2}\|h\|_{L^2}\|(\mathcal{N}+1)^{1/2}\xi'\|\|(\mathcal{N}+1)^{1/2}\xi\|, \end{aligned} \tag{3.12}$$

$$\left|\left\langle \xi', \int_{\Lambda^2} N V_N(x-y) a^{\sharp_1}(g_x) a_y \xi\right\rangle\right| \leqslant C\|V\|_{L^1}\|g\|_{L^2}\|(\mathcal{N}+1)^{1/2}\xi'\|\|(\mathcal{N}+1)^{1/2}\xi\|, \tag{3.13}$$

$$\left|\left\langle \xi', \int_{\Lambda^2} V_N(x-y) a_x a_y \xi\right\rangle\right| \leqslant CN^{-1/2}\|V\|_{L^1}^{1/2}\|\xi'\|\|Q_4^{1/2}\xi\|, \tag{3.14}$$

$$\left|\int_{\Lambda^2} N V_N(x-y)\langle p_{1,x}, s_{1,y}\rangle\right| \leqslant C\|V\|_{L^1}\|p_1\|_{L^2}\|s_1\|_{L^2}. \tag{3.15}$$

Proof of Lemma 3.7. We will denote $a^{\bar{\sharp}} = (a^\sharp)^*$. Let $\xi', \xi \in \mathcal{F}_+$. First, we want to prove (3.12). We have

$$\begin{aligned}
&\left|\left\langle \xi', \int_{\Lambda^2} NV_N(x-y) a^{\sharp_1}(g_x) a^{\sharp_2}(h_y)\xi \right\rangle\right| \\
&\qquad \leqslant \int_{\Lambda^2} NV_N(x-y)\|a^{\bar{\sharp}_1}(g_x)\xi'\|\|a^{\sharp_2}(h_y)\xi\| \\
&\qquad \leqslant \int_{\Lambda^2} NV_N(x-y)\|g_x\|_{L^2}\|(\mathcal{N}+1)^{1/2}\xi'\|\|h_y\|_{L^2}\|(\mathcal{N}+1)^{1/2}\xi\| \\
&\qquad \leqslant C\|V\|_{L^1}\|g\|_{L^2}\|h\|_{L^2}\|(\mathcal{N}+1)^{1/2}\xi'\|\|(\mathcal{N}+1)^{1/2}\xi\|.
\end{aligned}$$

Similarly, we obtain (3.13). Let $\eta > 0$, we have

$$\begin{aligned}
\left|\left\langle \xi', \int_{\Lambda^2} V_N(x-y) a_x a_y \xi \right\rangle\right| &\leqslant \int_{\Lambda^2} V_N(x-y)\|\xi'\|\|a_x a_y\xi\| \\
&\leqslant \eta^{-1}\int_{\Lambda^2} V_N(x-y)\|\xi'\|^2 + \eta\int_{\Lambda^2} V_N(x-y)\|a_x a_y \xi\|^2 \\
&\leqslant \eta^{-1}N^{-1}\|V\|_{L^1}\|\xi'\|^2 + C\eta\|Q_4^{1/2}\xi\|^2.
\end{aligned}$$

Optimizing in η shows (3.14). Using Hölder's inequality and $\langle p_{1,x}, s_{1,y}\rangle \leqslant \|p_{1,x}\|_{L^2}\|s_{1,y}\|_{L^2}$ yields (3.15). □

Using $c_1 = 1 + p_1$ and Lemma 3.7, we get from (3.11)

$$\begin{aligned}
T_1^* Q_2 T_1 = &\frac{1}{2}\int_{\Lambda^2} NV_N(x-y)(a_x^* a_y^* + a_x a_y) \\
&+ \int_{\Lambda^2} NV_N(x-y) s_1(x,y) \\
&- (T_1^*\mathcal{N}T_1 + 1/2)\int_{\Lambda^2} V_N(x-y)s_1(x,y) + \mathcal{E},
\end{aligned} \tag{3.16}$$

with

$$\begin{aligned}
|\langle \xi', \mathcal{E}\xi\rangle| \leqslant\; &C(\|p_1\|_{L^2}^2 + \|p_1\|_{L^2} + \|s_1\|_{L^2} + \|p_1\|_{L^2}\|s_1\|_{L^2} + \|s_1\|_{L^2}^2) \\
&\times (\|(\mathcal{N}+1)^{1/2}\xi'\|\|(\mathcal{N}+1)^{1/2}\xi\| + N^{-1}\|(T_1^*\mathcal{N}T_1+1)\xi'\|\|(\mathcal{N}+1)\xi\|) \\
&+ CN^{-1/2}\|(T_1^*\mathcal{N}T_1+1)\xi'\|\|Q_4^{1/2}\xi\| \\
\leqslant\; &C\ell^{1/2}\|(\mathcal{N}+1)^{1/2}\xi'\|\|(\mathcal{N}+1)^{1/2}\xi\| + C\ell^{1/2}N^{-1}\|(\mathcal{N}+1)\xi'\|\|(\mathcal{N}+1)\xi\| \\
&+ CN^{-1/2}\|(\mathcal{N}+1)\xi'\|\|Q_4^{1/2}\xi\|
\end{aligned}$$

for all $\xi', \xi \in \mathcal{F}_+$. Here we used Lemma 3.1 to bound $\|p_1\|_{L^2}, \|s_1\|_{L^2}$ and Lemma 2.8 to bound $T_1^*\mathcal{N}T_1 \leqslant C(\mathcal{N}+1)$. This implies

$$\pm\mathcal{E} \leqslant C\ell^{1/2}(\mathcal{N}+1) + C\ell^{1/2}N^{-1}(\mathcal{N}+1)^2 + \varepsilon Q_4 + \varepsilon^{-1}CN^{-1}(\mathcal{N}+1)^2, \quad \forall\varepsilon > 0.$$

Now we want to deal with the second term in (3.16). By Lemma 3.1 we have $\|s_1\|_{L^\infty} \leqslant CN$ and $|s_1(x,y) - \tilde{s}_1(x,y)| \leqslant C\ell^2$. Therefore, we obtain

$$\left| \int_{\Lambda^2} N(V_N - V_{R,N})(x-y)s_1(x,y) \right| \leqslant CN^2 \|V_N - V_{R,N}\|_{L^1} \leqslant CNR^{-\gamma}$$

and, using $\|NV_{R,N}\|_{L^1} \leqslant C$,

$$\begin{aligned}\int_{\Lambda^2} NV_N(x-y)s_1(x,y) &= -\int_{\Lambda^2} \sum_{z\in\mathbb{Z}^3} N^2 V_{R,N}(x-y)\omega_{\ell,N}(x-y+z)\,\mathrm{d}x\,\mathrm{d}y + \mathcal{E} \\ &= -\int_{\Lambda} N^2(V_{R,N}\omega_N)(x)\,\mathrm{d}x + \mathcal{E} = -\int_{\Lambda} N(V_R\omega)(x)\,\mathrm{d}x + \mathcal{E},\end{aligned}$$

with

$$\pm\mathcal{E} \leqslant C(NR^{-\gamma} + \ell^2).$$

Similarly for the third term in (3.16), we have

$$\begin{aligned}-(T_1^*\mathcal{N}T_1 + 1/2)\int_{\Lambda^2} V_N(x-y)s_1(x,y) &= -(T_1^*\mathcal{N}T_1 + 1/2)\int_{\Lambda^2} V_{R,N}(x-y)\tilde{s}_1(x,y) + \mathcal{E} \\ &= (T_1^*\mathcal{N}T_1 + 1/2)\int_{\Lambda}(V_R\omega)(x)\,\mathrm{d}x + \mathcal{E},\end{aligned}$$

with

$$\pm\mathcal{E} \leqslant CR^{-\gamma}(\mathcal{N}+1) + C\ell^2 N^{-1}(\mathcal{N}+1),$$

where we used Lemma 2.8 to bound $T_1^*\mathcal{N}T_1 + 1/2 \leqslant C(\mathcal{N}+1)$. By definition, we have $\mathcal{N} = \mathrm{d}\Gamma(1)$, so we can apply Lemma 3.5 to obtain

$$\pm(T_1^*\mathcal{N}T_1 - \mathcal{N}) \leqslant C\ell^{1/2}(\mathcal{N}+1).$$

Hence, using $\|V_R\omega\|_{L^1} \leqslant C$, we get

$$-(T_1^*\mathcal{N}T_1 + 1/2)\int_{\Lambda^2} V_N(x-y)s_1(x,y) = (\mathcal{N} + 1/2)\int_{\Lambda}(V_R\omega)(x)\,\mathrm{d}x + \mathcal{E},$$

with

$$\pm\mathcal{E} \leqslant CR^{-\gamma}(\mathcal{N}+1) + C\ell^{1/2}(\mathcal{N}+1). \qquad \square$$

3.5 Estimating $T_1^* Q_3 T_1$

Recall

$$Q_3 = \sqrt{(N-\mathcal{N})_+}\int_{\Lambda^2} V_N(x-y)a_y^* a_x a_y\,\mathrm{d}x\,\mathrm{d}y + \text{h.c.}$$

Lemma 3.8. *Assume that $4RN^{-1} < \ell < 1/2$. We have*

$$T_1^* Q_3 T_1 = \sqrt{N}\int_{\Lambda^2} V_{R,N}(x-y)a_y^* a_x a_y\,\mathrm{d}x\,\mathrm{d}y + \text{h.c.} + \mathcal{E}_{Q_3}^{(T_1)} = Q_3^{(T_1)} + \mathcal{E}_{Q_3}^{(T_1)}$$

on $\mathcal{F}_+$, with

$$\pm\mathcal{E}_{Q_3}^{(T_1)} \leqslant C\ell^{1/2}\frac{(\mathcal{N}+1)^{3/2}}{N^{1/2}} + \varepsilon Q_4 + \varepsilon^{-1}C\Big((R^{-\gamma}+\ell)(\mathcal{N}+1) + \frac{(\mathcal{N}+1)^2}{N}\Big)$$

for all $0 < \varepsilon \leqslant 1$.

Proof. We will use the notation $\mathcal{N}_{T_1} = T_1^*\mathcal{N}T_1$. Using $T_1^* a_x^* T_1 = a^*(c_{1,x}) + a(s_{1,x})$ and the CCR, we obtain

$$\begin{aligned} T_1^* Q_3 T_1 &= \sqrt{(N-\mathcal{N}_{T_1})_+}\int_{\Lambda^2} V_N(x-y)(a^*(c_{1,y}) + a(s_{1,y}))(a(c_{1,x}) + a^*(s_{1,x})) \\ &\qquad \times (a(c_{1,y}) + a^*(s_{1,y}))\,\mathrm{d}x\,\mathrm{d}y + \text{h.c.} \\ &= \sqrt{(N-\mathcal{N}_{T_1})_+}\int_{\Lambda^2} V_N(x-y)(a^*(c_{1,y}) + a(s_{1,y}))\Big(a(c_{1,x})a(c_{1,y}) + \langle c_{1,x}, s_{1,y}\rangle \\ &\qquad + a^*(s_{1,x})a(c_{1,y}) + a^*(s_{1,y})a(c_{1,x}) + a^*(s_{1,x})a^*(s_{1,y})\Big)\,\mathrm{d}x\,\mathrm{d}y + \text{h.c.} \end{aligned} \tag{3.17}$$

We will need the following lemma to bound the error terms.

Lemma 3.9. *Let $g, h \in L^2(\Lambda^2)$, $\sharp_1, \sharp_2 \in \{*, \cdot\}$, and $\xi', \xi \in \mathcal{F}_+$. Then we have*

$$\begin{aligned} &\Big|\Big\langle \xi', \int_{\Lambda^2} V_N(x-y) a^*(c_{1,y}) a^{\sharp_1}(g_x) a_y \xi\Big\rangle\Big| \\ &\qquad \leqslant CN^{-1}\|V\|_{L^1}\|g\|_{L^\infty L^2}\|(\mathcal{N}+1)^{3/4}\xi'\|\|(\mathcal{N}+1)^{3/4}\xi\|, \end{aligned} \tag{3.18}$$

$$\begin{aligned} &\Big|\Big\langle \xi', \int_{\Lambda^2} V_N(x-y) a^*(c_{1,y}) a_x a(h_y) \xi\Big\rangle\Big| \\ &\qquad \leqslant CN^{-1}\|V\|_{L^1}\|h\|_{L^\infty L^2}\|(\mathcal{N}+1)^{3/4}\xi'\|\|(\mathcal{N}+1)^{3/4}\xi\|, \end{aligned} \tag{3.19}$$

$$\begin{aligned} &\Big|\Big\langle \xi', \int_{\Lambda^2} V_N(x-y) a^*(c_{1,y}) a^{\sharp_1}(g_x) a^{\sharp_2}(h_y) \xi\Big\rangle\Big| \\ &\qquad \leqslant CN^{-1}\|V\|_{L^1}\|g\|_{L^\infty L^2}\|h\|_{L^2}\|(\mathcal{N}+1)^{3/4}\xi'\|\|(\mathcal{N}+1)^{3/4}\xi\|, \end{aligned} \tag{3.20}$$

$$\begin{aligned} &\Big|\Big\langle \xi', \int_{\Lambda^2} V_N(x-y) a(s_{1,y}) a^{\sharp_1}(g_x) a_y \xi\Big\rangle\Big| \\ &\qquad \leqslant CN^{-1}\|V\|_{L^1}\|g\|_{L^2}\|(\mathcal{N}+1)^{3/4}\xi'\|\|(\mathcal{N}+1)^{3/4}\xi\|, \end{aligned} \tag{3.21}$$

$$\begin{aligned} &\Big|\Big\langle \xi', \int_{\Lambda^2} V_N(x-y) a(s_{1,y}) a_x a(h_y) \xi\Big\rangle\Big| \\ &\qquad \leqslant CN^{-1}\|V\|_{L^1}\|h\|_{L^2}\|(\mathcal{N}+1)^{3/4}\xi'\|\|(\mathcal{N}+1)^{3/4}\xi\|, \end{aligned} \tag{3.22}$$

$$\begin{aligned} &\Big|\Big\langle \xi', \int_{\Lambda^2} V_N(x-y) a(s_{1,y}) a^{\sharp_1}(g_x) a^{\sharp_2}(h_y) \xi\Big\rangle\Big| \\ &\qquad \leqslant CN^{-1}\|V\|_{L^1}\|g\|_{L^2}\|h\|_{L^2}\|(\mathcal{N}+1)^{3/4}\xi'\|\|(\mathcal{N}+1)^{3/4}\xi\|, \end{aligned} \tag{3.23}$$

$$\begin{aligned} &\Big|\Big\langle \xi', \int_{\Lambda^2} V_N(x-y) a^{\sharp_1}(g_y) a_x a_y \xi\Big\rangle\Big| \\ &\qquad \leqslant CN^{-1/2}\|V\|_{L^1}^{1/2}\|g\|_{L^2}\|(\mathcal{N}+1)^{1/2}\xi'\|\|Q_4^{1/2}\xi\|, \end{aligned} \tag{3.24}$$

$$
\begin{aligned}
&\left|\left\langle \xi', \int_{\Lambda^2} V_N(x-y)\langle p_{1,x}, s_{1,y}\rangle (a^*(c_{1,y}) + a(s_{1,y}))\xi \right\rangle\right| \\
&\qquad\qquad \leqslant C\ell^3 N^{-1}\|V\|_{L^1}\|(\mathcal{N}+1)^{1/2}\xi'\|\|\xi\|.
\end{aligned}
\tag{3.25}
$$

Proof of Lemma 3.9. Let $\xi', \xi \in \mathcal{F}_+$. First, we want to prove (3.18).

$$
\begin{aligned}
\left|\left\langle \xi', \int_{\Lambda^2} V_N(x-y) a^*(c_{1,y}) a^{\sharp_1}(g_x) a_y \xi \right\rangle\right| &\leqslant \int_{\Lambda^2} V_N(x-y)\|a(c_{1,y})\xi'\|\|a^{\sharp_1}(g_x)a_y\xi\| \\
&\leqslant \int_{\Lambda^2} V_N(x-y)\|g_x\|_{L^2}\|a(c_{1,y})\xi'\|\|a_y(\mathcal{N}+1)^{1/2}\xi\| \\
&\leqslant \eta^{-1}\int_{\Lambda^2} V_N(x-y)\|a(c_{1,y})\xi'\|^2 + \eta\|g\|^2_{L^\infty L^2}\int_{\Lambda^2} V_N(x-y)\|a_y(\mathcal{N}+1)^{1/2}\xi\|^2 \\
&\leqslant \eta^{-1}N^{-1}\|V\|_{L^1}\|c_1\|^2_{\mathrm{op}}\|\mathcal{N}^{1/2}\xi'\|^2 + \eta N^{-1}\|g\|^2_{L^\infty L^2}\|V\|_{L^1}\|(\mathcal{N}+1)\xi\|^2 \\
&\leqslant CN^{-1}\|V\|_{L^1}\|g\|_{L^\infty L^2}\|\mathcal{N}^{1/2}\xi'\|\|(\mathcal{N}+1)\xi\|,
\end{aligned}
$$

where we optimized over η in the last inequality and used that $\|c_1\|_{\mathrm{op}} \leqslant C$ by Lemma 3.1. Changing state $\xi' \to (\mathcal{N}+1)^{1/4}$ and $\xi \to (\mathcal{N}+1)^{-1/4}$ and using $(\mathcal{N}+1)a_x = a_x\mathcal{N}$ proves (3.18). The bounds (3.19)–(3.23) can be shown similarly, but for (3.21)–(3.23) we use $\|s_1\|_{L^2} \leqslant C\ell^{1/2} \leqslant C$ instead of $\|c_1\|_{\mathrm{op}} \leqslant C$.

Recall the notation $a^{\bar{\sharp}} = (a^\sharp)^*$. For (3.24) we have

$$
\begin{aligned}
&\left|\left\langle \xi', \int_{\Lambda^2} V_N(x-y) a^{\sharp_1}(g_y) a_x a_y \xi \right\rangle\right| \\
&\qquad \leqslant \eta^{-1}\int_{\Lambda^2} V_N(x-y)\|a^{\bar{\sharp}_1}(g_y)\xi'\|^2 + \eta\int_{\Lambda^2} V_N(x-y)\|a_x a_y\xi\|^2 \\
&\qquad \leqslant \eta^{-1}N^{-1}\|V\|_{L^1}\|g\|^2_{L^2}\|(\mathcal{N}+1)^{1/2}\xi'\|^2 + \eta\|Q_4^{1/2}\xi\|^2 \\
&\qquad \leqslant CN^{-1/2}\|V\|^{1/2}_{L^1}\|g\|_{L^2}\|(\mathcal{N}+1)^{1/2}\xi'\|\|Q_4^{1/2}\xi\|,
\end{aligned}
$$

where we optimized over η in the last inequality.

For (3.25) we have

$$
\begin{aligned}
&\left|\left\langle \xi', \int_{\Lambda^2} V_N(x-y)\langle p_{1,x}, s_{1,y}\rangle (a^*(c_{1,y}) + a(s_{1,y}))\xi \right\rangle\right| \\
&\qquad \leqslant \int_{\Lambda^2} V_N(x-y)\|p_{1,x}\|_{L^2}\|s_{1,y}\|_{L^2}(\|a(c_{1,y})\xi'\| + \|a^*(s_{1,y})\xi'\|)\|\xi\| \\
&\qquad \leqslant \eta\int_{\Lambda^2} V_N(x-y)(\|a(c_{1,y})\xi'\|^2 + \|a^*(s_{1,y})\xi'\|^2) \\
&\qquad\quad + \eta^{-1}\int_{\Lambda^2} V_N(x-y)\|p_{1,x}\|^2_{L^2}\|s_{1,y}\|^2_{L^2}\|\xi\|^2 \\
&\qquad \leqslant C\eta N^{-1}\|V\|_{L^1}\|(\mathcal{N}+1)^{1/2}\xi'\|^2 + \eta^{-1}N^{-1}\|V\|_{L^1}\|p_1\|^2_{L^\infty L^2}\|s_1\|^2_{L^2}\|\xi\|^2 \\
&\qquad \leqslant C\ell^3 N^{-1}\|V\|_{L^1}\|(\mathcal{N}+1)^{1/2}\xi'\|\|\xi\|,
\end{aligned}
$$

where we optimized over η and used the bounds $\|s_1\|_{L^2} \leqslant C\ell^{1/2} \leqslant C$, $\|c_1\|_{\mathrm{op}} \leqslant C$, and $\|p_1\|_{L^\infty L^2}\|s_1\|_{L^2} \leqslant C\ell^3$ from Lemma 3.1. □

Using $c_1 = 1 + p_1$ and Lemma 3.9, we obtain from (3.17)

$$\begin{aligned} T_1^* Q_3 T_1 &= \Big(\sqrt{(N-\mathcal{N}_{T_1})_+} \int_{\Lambda^2} V_N(x-y) a_y^* a_x a_y \,\mathrm{d}x\,\mathrm{d}y + \text{h.c.} \Big) \\ &\quad + \Big(\sqrt{(N-\mathcal{N}_{T_1})_+} \int_{\Lambda^2} V_N(x-y) s_1(x,y) (a^*(c_{1,y}) + a(s_{1,y})) + \text{h.c.} \Big) + \mathcal{E}, \end{aligned} \tag{3.26}$$

where for all $\xi', \xi \in \mathcal{F}_+$ we have

$$\begin{aligned} |\langle \xi', \mathcal{E}\xi\rangle| &\leqslant C\ell^{1/2} N^{-1} \|(\mathcal{N}+1)^{3/4} \sqrt{(N-\mathcal{N}_{T_1})_+} \xi'\| \|(\mathcal{N}+1)^{3/4}\xi\| \\ &\quad + C\ell^{1/2} N^{-1/2} \|(\mathcal{N}+1)^{1/2} \sqrt{(N-\mathcal{N}_{T_1})_+}\xi'\| \|Q_4^{1/2}\xi\| \\ &\leqslant C\ell^{1/2} N^{-1/2} \|(\mathcal{N}+1)^{3/4}\xi'\| \|(\mathcal{N}+1)^{3/4}\xi\| \\ &\quad + C\ell^{1/2} \|(\mathcal{N}+1)^{1/2}\xi'\| \|Q_4^{1/2}\xi\|. \end{aligned}$$

Here we used the bound $(N - \mathcal{N}_{T_1})_+ \leqslant N$ and

$$\|s_1\|_{L^\infty L^2}, \|p_1\|_{L^\infty L^2}, \|s_1\|_{L^2}, \|p_1\|_{L^2} \leqslant C\ell^{1/2}$$

from Lemma 3.1.

To extract the main contribution from the first term in (3.26), we write

$$\sqrt{N} - \sqrt{(N-\mathcal{N}_{T_1})_+} = \mathbb{1}^{\{\mathcal{N}_{T_1} \leqslant N\}} \Big(\sqrt{N} - \sqrt{(N-\mathcal{N}_{T_1})_+} \Big) + \mathbb{1}^{\{\mathcal{N}_{T_1} > N\}} \sqrt{N}.$$

By Lemma 2.8, we have

$$\begin{aligned} &\Big| \Big\langle \xi', \mathbb{1}^{\{\mathcal{N}_{T_1} \leqslant N\}} \Big(\sqrt{N} - \sqrt{(N-\mathcal{N}_{T_1})_+} \Big) \int_{\Lambda^2} V_N(x-y) a_y^* a_x a_y \,\mathrm{d}x\,\mathrm{d}y \xi \Big\rangle \Big| \\ &\qquad \leqslant \varepsilon^{-1} N^{-1} \|V\|_{L^1} \Big\| \mathcal{N}^{1/2} \Big(\sqrt{N} - \sqrt{(N-\mathcal{N}_{T_1})_+} \Big) \mathbb{1}^{\{\mathcal{N}_{T_1} \leqslant N\}} \xi' \Big\|^2 + \varepsilon \|Q_4^{1/2}\xi\|^2 \\ &\qquad \leqslant \varepsilon^{-1} C \Big\| \frac{\mathcal{N}_{T_1}}{\sqrt{N}} \xi' \Big\|^2 + \varepsilon \|Q_4^{1/2}\xi\|^2 \\ &\qquad \leqslant \varepsilon^{-1} C N^{-1} \|(\mathcal{N}+1)\xi'\|^2 + \varepsilon \|Q_4^{1/2}\xi\|^2 \end{aligned}$$

and

$$\begin{aligned} &\Big| \Big\langle \xi', \mathbb{1}^{\{\mathcal{N}_{T_1} > N\}} \sqrt{N} \int_{\Lambda^2} V_N(x-y) a_y^* a_x a_y \,\mathrm{d}x\,\mathrm{d}y \xi \Big\rangle \Big| \\ &\qquad \leqslant \varepsilon^{-1} \|V\|_{L^1} \|\mathcal{N}^{1/2} \mathbb{1}^{\{\mathcal{N}_{T_1} > N\}} \xi'\|^2 + \varepsilon \|Q_4^{1/2}\xi\|^2 \\ &\qquad \leqslant \varepsilon^{-1} C N^{-1} \|(\mathcal{N}+1)\xi'\|^2 + \varepsilon \|Q_4^{1/2}\xi\|^2 \end{aligned}$$

for all $\xi', \xi \in \mathcal{F}_+$ and $\varepsilon > 0$. Here we used

$$\mathbb{1}^{\{\mathcal{N}_{T_1} \leqslant N\}} \mathcal{N} \mathbb{1}^{\{\mathcal{N}_{T_1} \leqslant N\}} \leqslant CN, \qquad \sqrt{N} - \sqrt{(N-\mathcal{N}_{T_1})_+} \leqslant C \frac{\mathcal{N}_{T_1}}{\sqrt{N}},$$

and

$$\mathbb{1}^{\{\mathcal{N}_{T_1}>N\}}\mathcal{N}\mathbb{1}^{\{\mathcal{N}_{T_1}>N\}} \leqslant C\mathbb{1}^{\{\mathcal{N}_{T_1}>N\}}(\mathcal{N}_{T_1}+1) \leqslant CN^{-1}(\mathcal{N}_{T_1}+1)^2 \leqslant CN^{-1}(\mathcal{N}+1)^2.$$

Finally, using $0 \leqslant V_N - V_{R,N} \leqslant V_N$, we have

$$\begin{aligned}
&\left|\left\langle \xi', \sqrt{N}\int_{\Lambda^2}(V_N - V_{R,N})(x-y)a_y^* a_x a_y \, \mathrm{d}x\,\mathrm{d}y\xi\right\rangle\right| \\
&\quad \leqslant \varepsilon^{-1}\|V - V_R\|_{L^1}\|\mathcal{N}^{1/2}\xi'\|^2 + \varepsilon\|Q_4^{1/2}\xi\|^2 \\
&\quad \leqslant \varepsilon^{-1}CR^{-\gamma}\|\mathcal{N}^{1/2}\xi'\|^2 + \varepsilon\|Q_4^{1/2}\xi\|^2
\end{aligned}$$

for all $\xi', \xi \in \mathcal{F}_+$ and $\varepsilon > 0$.

Now we want to deal with the second term in (3.26). First, using

$$|s_1(x,y) - \tilde{s}_1(x,y)| \leqslant C\ell^2, \quad \|c_1\|_{\mathrm{op}} \leqslant C, \quad \|s_1\|_{L^\infty L^2} \leqslant C\ell^{1/2} \leqslant C$$

from Lemma 3.1, we obtain

$$\begin{aligned}
&\left|\left\langle \xi', \sqrt{(N-\mathcal{N}_{T_1})_+}\int_{\Lambda^2} V_N(x-y)(s_1(x,y) - \tilde{s}_1(x,y))(a^*(c_{1,y}) + a(s_{1,y}))\xi\right\rangle\right| \\
&\quad \leqslant C\ell^2 N^{-1}\|V\|_{L^1}(\|c_1\|_{\mathrm{op}} + \|s_1\|_{L^\infty L^2})\|(\mathcal{N}+1)^{1/2}\sqrt{(N-\mathcal{N}_{T_1})_+}\xi'\|\|\xi\| \\
&\quad \leqslant C\ell^2 N^{-1/2}\|(\mathcal{N}+1)^{1/2}\xi'\|\|\xi\|.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\sqrt{(N-\mathcal{N}_{T_1})_+}\int_{\Lambda^2} V_N(x-y)s_1(x,y)(a^*(c_{1,y}) + a(s_{1,y})) + \text{h.c.} \\
&\quad = \sqrt{(N-\mathcal{N}_{T_1})_+}\int_{\Lambda^2} V_N(x-y)\tilde{s}_1(x,y)(a^*(c_{1,y}) + a(s_{1,y})) + \text{h.c.} + \mathcal{E},
\end{aligned}$$

with

$$\pm\mathcal{E} \leqslant C\ell^2 N^{-1/2}(\mathcal{N}+1)^{1/2}.$$

Notice that we have

$$\begin{aligned}
&\left\langle \xi', T_1\Big(-\sqrt{(N-\mathcal{N}_{T_1})_+}\int_{\Lambda^2} V_N(x-y)\tilde{s}_1(x,y)(a^*(c_{1,y}) + a(s_{1,y}))\,\mathrm{d}x\,\mathrm{d}y\Big)T_1^*\xi\right\rangle \\
&\quad = \left\langle \xi', \Big(-\sqrt{(N-\mathcal{N})_+}\int_{\Lambda^2} V_N(x-y)\tilde{s}_1(x,y)a_y^*\,\mathrm{d}x\,\mathrm{d}y\Big)\xi\right\rangle \\
&\quad = \left\langle \xi', \Big(-\sqrt{(N-\mathcal{N})_+}N\int_{\Lambda^2} V_N(x)\sum_{z\in\mathbb{Z}^3}\omega_{\ell,N}(x+z)a_y^*\,\mathrm{d}x\,\mathrm{d}y\Big)\xi\right\rangle \\
&\quad = \left\langle \xi', \Big(-\sqrt{(N-\mathcal{N})_+}N\int_{\Lambda} V_N(x)\sum_{z\in\mathbb{Z}^3}\omega_{\ell,N}(x+z)\,\mathrm{d}x\Big)a_0^*\xi\right\rangle = 0
\end{aligned}$$

for all $\xi', \xi \in \mathcal{F}_+$. Since T_1 is a unitary transform on $\mathcal{F}_+$, this implies

$$\left\langle \xi', \Big(-\sqrt{(N-\mathcal{N}_{T_1})_+}\int_{\Lambda^2} V_N(x-y)\tilde{s}_1(x,y)(a^*(c_{1,y}) + a(s_{1,y}))\,\mathrm{d}x\,\mathrm{d}y + \text{h.c.}\Big)\xi\right\rangle = 0$$

for all $\xi', \xi \in \mathcal{F}_+$. This finishes the proof of Lemma 3.8. $\square$

3.6 Estimating $T_1^* Q_4 T_1$

Recall

$$Q_4 = \frac{1}{2}\int_{\Lambda^2} V_N(x-y) a_x^* a_y^* a_x a_y \,\mathrm{d}x\,\mathrm{d}y.$$

Lemma 3.10. *Assume that $4RN^{-1} < \ell < 1/2$. We have*

$$\begin{aligned} T_1^* Q_4 T_1 = Q_4 &- \frac{N}{2}\int_{\Lambda^2}\Big(V_N(x-y)\sum_{z\in\mathbb{Z}^3}\omega_{\ell,N}(x-y+z)\Big)(a_x^* a_y^* + a_x a_y)\,\mathrm{d}x\,\mathrm{d}y \\ &+ \frac{N}{2}\int_{\Lambda}(V_R\omega^2)(x)\,\mathrm{d}x + \mathcal{E}_{Q_4}^{(T_1)} \end{aligned}$$

on $\mathcal{F}_+$, with

$$\pm\mathcal{E}_{Q_4}^{(T_1)} \leqslant CNR^{-\gamma} + C\ell^{1/2}(\mathcal{N}+1) + \varepsilon Q_4 + \varepsilon^{-1}C\ell^{1/2}\frac{(\mathcal{N}+1)^2}{N}$$

for all $0 < \varepsilon \leqslant 1$.

Proof. Using $T_1^* a_x^* T_1 = a^*(c_{1,x}) + a(s_{1,x})$ and the CCR, we obtain

$$\begin{aligned} T_1^* Q_4 T_1 &= \frac{1}{2}\int_{\Lambda^2} V_N(x-y)(a^*(c_{1,x}) + a(s_{1,x}))(a^*(c_{1,y}) + a(s_{1,y})) \\ &\quad\times (a(c_{1,x}) + a^*(s_{1,x}))(a(c_{1,y}) + a^*(s_{1,y}))\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{1}{2}\int_{\Lambda^2} V_N(x-y)\Big(a^*(c_{1,x})a^*(c_{1,y}) + \langle s_{1,x}, c_{1,y}\rangle + a^*(c_{1,x})a(s_{1,y}) \\ &\quad + a^*(c_{1,y})a(s_{1,x}) + a(s_{1,x})a(s_{1,y})\Big)\Big(a(c_{1,x})a(c_{1,y}) + \langle c_{1,x}, s_{1,y}\rangle \\ &\quad + a^*(s_{1,y})a(c_{1,x}) + a^*(s_{1,x})a(c_{1,y}) + a^*(s_{1,x})a^*(s_{1,y})\Big)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{3.27}$$

Writing $c_1(x,y) = \delta_{x,y} + p_1(x,y)$, we can expand the above expression and obtain a sum of quartic, quadratic, and constant terms. The quartic terms are of the form

$$\frac{1}{2}\int_{\Lambda^2} V_N(x-y) a^{\sharp_1}(g_{1,x}) a^{\sharp_2}(g_{2,y}) a^{\sharp_3}(g_{3,x}) a^{\sharp_4}(g_{4,y})$$

with $g_i \in \{\delta_{x,y}\} \cup L^2(\Lambda^2)$ and $\sharp_i \in \{*, \cdot\}$ with the condition that $(\sharp_1, \sharp_2, \sharp_3, \sharp_4)$ is in normal order if at least one of the g_i is $\delta_{x,y}$. Among these terms, only the term with $g_i = \delta_{x,y}$ for all $i \in \{1,2,3,4\}$ contributes; all the other terms are errors. In the following lemma, we will prove the bounds for the error terms.

Lemma 3.11. *Let $g_i \in L^2(\Lambda^2)$. Then we have for all $\varepsilon > 0$ on $\mathcal{F}_+$*

$$\begin{aligned}
&\pm\Big(\int_{\Lambda^2} V_N(x-y)a^{\sharp_1}(g_{1,x})a^{\sharp_2}(g_{2,y})a^{\sharp_3}(g_{3,x})a^{\sharp_4}(g_{4,y}) + \text{h.c.}\Big) \\
&\qquad \leqslant CN^{-1}\|V\|_{L^1}\|g_1\|_{L^2}\|g_2\|_{L^\infty L^2}\|g_3\|_{L^2}\|g_4\|_{L^\infty L^2}(\mathcal{N}+1)^2,
\end{aligned} \tag{3.28}$$

$$\begin{aligned}
&\pm\Big(\int_{\Lambda^2} V_N(x-y)a^*_x a^{\sharp_2}(g_{2,y})a^{\sharp_3}(g_{3,x})a^{\sharp_4}(g_{4,y}) + \text{h.c.}\Big) \\
&\qquad \leqslant CN^{-1}\|V\|_{L^1}\|g_2\|_{L^\infty L^2}\|g_3\|_{L^2}\|g_4\|_{L^\infty L^2}(\mathcal{N}+1)^2,
\end{aligned} \tag{3.29}$$

$$\begin{aligned}
&\pm\Big(\int_{\Lambda^2} V_N(x-y)a^*_x a^*_y a^{\sharp_3}(g_{3,x})a^{\sharp_4}(g_{4,y}) + \text{h.c.}\Big) \\
&\qquad \leqslant \varepsilon Q_4 + \varepsilon^{-1}CN^{-1}\|V\|_{L^1}\|g_3\|^2_{L^2}\|g_4\|^2_{L^\infty L^2}(\mathcal{N}+1)^2,
\end{aligned} \tag{3.30}$$

$$\begin{aligned}
&\pm\Big(\int_{\Lambda^2} V_N(x-y)a^*_x a^{\sharp_2}(g_{2,y})a^{\sharp_3}(g_{3,x})a_y + \text{h.c.}\Big) \\
&\qquad \leqslant CN^{-1}\|V\|_{L^1}\|g_2\|_{L^\infty L^2}\|g_3\|_{L^\infty L^2}(\mathcal{N}+1)^2,
\end{aligned} \tag{3.31}$$

$$\begin{aligned}
&\pm\Big(\int_{\Lambda^2} V_N(x-y)a^*_x a^{\sharp_2}(g_{2,y})a^{\sharp_4}(g_{4,y})a_x + \text{h.c.}\Big) \\
&\qquad \leqslant CN^{-1}\|V\|_{L^1}\|g_2\|_{L^\infty L^2}\|g_4\|_{L^\infty L^2}(\mathcal{N}+1)^2,
\end{aligned} \tag{3.32}$$

$$\begin{aligned}
&\pm\Big(\int_{\Lambda^2} V_N(x-y)a^*_x a^*_y a^{\sharp_3}(g_{3,x})a_y + \text{h.c.}\Big) \\
&\qquad \leqslant \varepsilon Q_4 + \varepsilon^{-1}CN^{-1}\|V\|_{L^1}\|g_3\|^2_{L^\infty L^2}(\mathcal{N}+1)^2,
\end{aligned} \tag{3.33}$$

$$\begin{aligned}
&\pm\Big(\int_{\Lambda^2} V_N(x-y)a^*_x a^*_y a^{\sharp_4}(g_{4,y})a_x + \text{h.c.}\Big) \\
&\qquad \leqslant \varepsilon Q_4 + \varepsilon^{-1}CN^{-1}\|V\|_{L^1}\|g_4\|^2_{L^\infty L^2}(\mathcal{N}+1)^2.
\end{aligned} \tag{3.34}$$

Proof of Lemma 3.11. Recall the notation $a^{\bar{\sharp}} = (a^\sharp)^*$. Let $\xi', \xi \in \mathcal{F}_+$. First, we want to prove (3.28). We have

$$\begin{aligned}
&\Big|\Big\langle \xi', \int_{\Lambda^2} V_N(x-y)a^{\sharp_1}(g_{1,x})a^{\sharp_2}(g_{2,y})a^{\sharp_3}(g_{3,x})a^{\sharp_4}(g_{4,y})\xi\Big\rangle\Big| \\
&\qquad \leqslant \int_{\Lambda^2} V_N(x-y)\|a^{\bar{\sharp}_2}(g_{2,y})a^{\bar{\sharp}_1}(g_{1,x})\xi'\|\|a^{\sharp_3}(g_{3,x})a^{\sharp_4}(g_{4,y})\xi\| \\
&\qquad \leqslant C\int_{\Lambda^2} V_N(x-y)\|g_{1,x}\|_{L^2}\|g_{2,y}\|_{L^2}\|(\mathcal{N}+1)\xi'\|\|g_{3,x}\|_{L^2}\|g_{4,y}\|_{L^2}\|(\mathcal{N}+1)\xi\| \\
&\qquad \leqslant CN^{-1}\|V\|_{L^1}\|g_1\|_{L^2}\|g_2\|_{L^\infty L^2}\|g_3\|_{L^2}\|g_4\|_{L^\infty L^2}\|(\mathcal{N}+1)\xi'\|\|(\mathcal{N}+1)\xi\|.
\end{aligned}$$

The proof for (3.29) is similar. We obtain

$$\begin{aligned}
&\Big|\Big\langle \xi', \int_{\Lambda^2} V_N(x-y)a^*_x a^{\sharp_2}(g_{2,y})a^{\sharp_3}(g_{3,x})a^{\sharp_4}(g_{4,y})\xi\Big\rangle\Big| \\
&\qquad \leqslant \int_{\Lambda^2} V_N(x-y)\|a^{\bar{\sharp}_2}(g_{2,y})a_x\xi'\|\|a^{\sharp_3}(g_{3,x})a^{\sharp_4}(g_{4,y})\xi\| \\
&\qquad \leqslant C\int_{\Lambda^2} V_N(x-y)\|g_{2,y}\|_{L^2}\|(\mathcal{N}+1)^{1/2}a_x\xi'\|\|g_{3,x}\|_{L^2}\|g_{4,y}\|_{L^2}\|(\mathcal{N}+1)\xi\| \\
&\qquad \leqslant CN^{-1}\|V\|_{L^1}\|g_2\|_{L^\infty L^2}\|g_3\|_{L^2}\|g_4\|_{L^\infty L^2}\|(\mathcal{N}+1)\xi'\|\|(\mathcal{N}+1)\xi\|.
\end{aligned}$$

Let $\varepsilon > 0$. We have

$$\begin{aligned}
&\left|\left\langle \xi', \int_{\Lambda^2} V_N(x-y) a_x^* a_y^* a^{\sharp_3}(g_{3,x}) a^{\sharp_4}(g_{4,y})\xi \right\rangle\right| \\
&\qquad \leqslant \varepsilon \int_{\Lambda^2} V_N(x-y)\|a_x a_y \xi'\|^2 + \varepsilon^{-1}\int_{\Lambda^2} V_N(x-y)\|a^{\sharp_3}(g_{3,x})a^{\sharp_4}(g_{4,y})\xi\|^2 \\
&\qquad \leqslant C\varepsilon\|Q_4^{1/2}\xi'\|^2 + C\varepsilon^{-1}\int_{\Lambda^2} V_N(x-y)\|g_{3,x}\|_{L^2}^2\|g_{4,y}\|_{L^2}^2\|(\mathcal{N}+1)\xi\|^2 \\
&\qquad \leqslant C\varepsilon\|Q_4^{1/2}\xi'\|^2 + C\varepsilon^{-1}N^{-1}\|V\|_{L^1}\|g_3\|_{L^2}^2\|g_4\|_{L^\infty L^2}^2\|(\mathcal{N}+1)\xi\|^2,
\end{aligned}$$

this proves (3.30). Now we prove (3.31),

$$\begin{aligned}
&\left|\left\langle \xi', \int_{\Lambda^2} V_N(x-y) a_x^* a^{\sharp_2}(g_{2,y}) a^{\sharp_3}(g_{3,x}) a_y\xi \right\rangle\right| \\
&\qquad \leqslant \int_{\Lambda^2} V_N(x-y)\|a^{\bar{\sharp}_2}(g_{2,y})a_x\xi'\|\|a^{\sharp_3}(g_{3,x})a_y\xi\| \\
&\qquad \leqslant C\int_{\Lambda^2} V_N(x-y)\|g_{2,y}\|_{L^2}\|g_{3,x}\|_{L^2}\|(\mathcal{N}+1)^{1/2}a_x\xi'\|\|(\mathcal{N}+1)^{1/2}a_y\xi\| \\
&\qquad \leqslant CN^{-1}\|V\|_{L^1}\|g_2\|_{L^\infty L^2}\|g_3\|_{L^\infty L^2}\|(\mathcal{N}+1)\xi'\|\|(\mathcal{N}+1)\xi\|.
\end{aligned}$$

The estimate (3.32) can be proven similarly. Let us now prove (3.33). For all $\varepsilon > 0$ we have

$$\begin{aligned}
&\left|\left\langle \xi', \int_{\Lambda^2} V_N(x-y) a_x^* a_y^* a^{\sharp_3}(g_{3,x}) a_y\xi \right\rangle\right| \\
&\qquad \leqslant \varepsilon \int_{\Lambda^2} V_N(x-y)\|a_x a_y \xi'\|^2 + \varepsilon^{-1}\int_{\Lambda^2} V_N(x-y)\|a^{\sharp_3}(g_{3,x})a_y\xi\|^2 \\
&\qquad \leqslant C\varepsilon\|Q_4^{1/2}\xi'\|^2 + C\varepsilon^{-1}\int_{\Lambda^2} V_N(x-y)\|g_{3,x}\|_{L^2}^2\|(\mathcal{N}+1)^{1/2}a_y\xi\|^2 \\
&\qquad \leqslant C\varepsilon\|Q_4^{1/2}\xi'\|^2 + C\varepsilon^{-1}N^{-1}\|V\|_{L^1}\|g_3\|_{L^\infty L^2}^2\|(\mathcal{N}+1)\xi\|^2.
\end{aligned}$$

Similarly, we obtain the estimate (3.34). □

The quadratic terms in (3.27) are either of the form

$$\frac{1}{2}\int_{\Lambda^2} V_N(x-y)s_1(x,y)a^{\sharp_1}(g_{1,x})a^{\sharp_2}(g_{2,y})$$

or of the form

$$\frac{1}{2}\int_{\Lambda^2} V_N(x-y)\langle p_{1,x}, s_{1,y}\rangle a^{\sharp_1}(g_{1,x})a^{\sharp_2}(g_{2,y})$$

with $g_i \in \{\delta_{x,y}\} \cup L^2(\Lambda^2)$ and $\sharp_i \in \{*, \cdot\}$ with the condition that $(\sharp_1, \sharp_2)$ is in normal order if at least one of the g_i is $\delta_{x,y}$. The constant terms in (3.27) are of the form

$$\frac{1}{2}\int_{\Lambda^2} V_N(x-y)g_1(x,y)g_2(x,y)$$

with $g_i \in L^\infty(\Lambda^2)$. Only the quadratic terms of the first form with $g_i = \delta_{x,y}$ for $i = 1,2$ and the constant term with $g_i = s_1$ for $i = 1,2$ contribute. Using

$$\begin{aligned}
&\|s_1\|_{L^\infty} \leqslant CN, \quad \|\langle p_{1,x}, s_{1,y}\rangle\|_{L^\infty} \leqslant C\|p_1\|_{L^\infty L^2}\|s_1\|_{L^\infty L^2} \leqslant C\ell^3, \\
&\|s_1\|_{L^2} \leqslant C\ell^{1/2}, \quad \|p_1\|_{L^2} \leqslant C\ell^{5/2}
\end{aligned}$$

from Lemma 3.1 and the following lemma, we can bound all the other quadratic and constant terms.

Lemma 3.12. *Let $g_i \in L^2(\Lambda^2)$, $h \in L^\infty(\Lambda^2)$, $\sharp_1, \sharp_2 \in \{*, \cdot\}$, and $\xi', \xi \in \mathcal{F}_+$. Then we have*

$$\begin{aligned}
&\left|\left\langle \xi', \int_{\Lambda^2} V_N(x-y)h(x,y)a^{\sharp_1}(g_{1,x})a^{\sharp_2}(g_{2,y})\xi\right\rangle\right| \\
&\qquad \leqslant CN^{-1}\|V\|_{L^1}\|h\|_{L^\infty}\|g_1\|_{L^2}\|g_2\|_{L^2}\|(\mathcal{N}+1)^{1/2}\xi'\|\|(\mathcal{N}+1)^{1/2}\xi\|,
\end{aligned} \tag{3.35}$$

$$\begin{aligned}
&\left|\left\langle \xi', \int_{\Lambda^2} V_N(x-y)h(x,y)a^{\sharp_1}(g_{1,x})a_y\xi\right\rangle\right| \\
&\qquad \leqslant CN^{-1}\|V\|_{L^1}\|h\|_{L^\infty}\|g_1\|_{L^2}\|(\mathcal{N}+1)^{1/2}\xi'\|\|(\mathcal{N}+1)^{1/2}\xi\|,
\end{aligned} \tag{3.36}$$

$$\left|\left\langle \xi', \int_{\Lambda^2} V_N(x-y)h(x,y)a_xa_y\xi\right\rangle\right| \leqslant CN^{-1/2}\|V\|_{L^1}^{1/2}\|h\|_{L^\infty}\|\xi'\|\|Q_4^{1/2}\xi\|, \tag{3.37}$$

$$\left|\int_{\Lambda^2} V_N(x-y)g_1(x,y)g_2(x,y)\right| \leqslant CN^{-1}\|V\|_{L^1}\|g_1\|_{L^\infty}\|g_2\|_{L^\infty}. \tag{3.38}$$

The proof of Lemma 3.12 is similar to that of Lemma 3.7, and we will omit it.

By Lemma 3.1 we have

$$|s_1(x,y) - \tilde{s}_1(x,y)| \leqslant C\ell^2$$

and therefore

$$\begin{aligned}
&\frac{1}{2}\int_{\Lambda^2} V_N(x-y)s_1(x,y)(a_x^*a_y^* + a_xa_y)\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad = -\frac{N}{2}\int_{\Lambda^2}\Big(V_N(x-y)\sum_{z\in\mathbb{Z}^3}\omega_{\ell,N}(x-y+z)\Big)(a_x^*a_y^* + a_xa_y)\,\mathrm{d}x\,\mathrm{d}y + \mathcal{E},
\end{aligned}$$

with

$$\pm\mathcal{E} \leqslant \varepsilon^{-1}C\ell^4N^{-1} + \varepsilon Q_4$$

for all $\varepsilon > 0$. Here we used (3.37) to get the bound for $\mathcal{E}$. For the constant term, we have

$$\left|\int_{\Lambda^2}(V_N - V_{R,N})(x-y)s_1(x,y)^2\,\mathrm{d}x\,\mathrm{d}y\right| \leqslant C\|N^2(V_N - V_{R,N})\|_{L^1} \leqslant CNR^{-\gamma}$$

and

$$\left|\int_{\Lambda^2} V_{R,N}(x-y)(s_1 - \tilde{s}_1)(x,y)^2\,\mathrm{d}x\,\mathrm{d}y\right| \leqslant C\|NV_{R,N}\|_{L^1}\ell^2 \leqslant C\ell^2,$$

where we used $\|s_1\|_{L^\infty} \leqslant CN$ and

$$|s_1(x,y)^2 - \tilde{s}_1(x,y)^2| \leqslant |s_1(x,y) - \tilde{s}_1(x,y)|(|s_1(x,y)| + |\tilde{s}_1(x,y)|) \leqslant C\ell^2N$$

from Lemma 3.1. Thus, we obtain

$$\begin{aligned}\frac{1}{2}\int_{\Lambda^2} V_N(x-y)s_1(x,y)^2\,\mathrm{d}x\,\mathrm{d}y &= \frac{1}{2}\int_{\Lambda^2} V_{R,N}(x-y)\tilde{s}_1(x,y)^2\,\mathrm{d}x\,\mathrm{d}y+\mathcal{E}\\ &= \frac{N^2}{2}\int_{\Lambda^2} V_{R,N}(x-y)\Big(\sum_{z\in\mathbb{Z}^3}\omega_{\ell,N}(x-y+z)^2\Big)\,\mathrm{d}x\,\mathrm{d}y+\mathcal{E}\\ &= \frac{N^2}{2}\int_{\Lambda}(V_{R,N}\omega_N^2)(x)\,\mathrm{d}x+\mathcal{E} = \frac{N}{2}\int_{\Lambda}(V_R\omega^2)(x)\,\mathrm{d}x+\mathcal{E},\end{aligned}$$

with

$$\pm\mathcal{E}\leqslant C(NR^{-\gamma}+\ell^2).$$

Finally, using this, Lemma 3.11, and Lemma 3.12 we obtain

$$\begin{aligned}T_1^*Q_4T_1 = Q_4 &- \frac{N}{2}\int_{\Lambda^2}\Big(V_N(x-y)\sum_{z\in\mathbb{Z}^3}\omega_{\ell,N}(x-y+z)\Big)(a_x^*a_y^*+a_xa_y)\,\mathrm{d}x\,\mathrm{d}y\\ &+ \frac{N}{2}\int_{\Lambda}(V_R\omega^2)(x)\,\mathrm{d}x+\mathcal{E},\end{aligned}$$

with

$$\pm\mathcal{E}\leqslant CNR^{-\gamma}+C\ell^{1/2}(\mathcal{N}+1)+\varepsilon Q_4+\varepsilon^{-1}C\ell^{1/2}N^{-1}(\mathcal{N}+1)^2.$$

This concludes the proof of Lemma 3.10. □

3.7 Proof of Lemma 3.2

Proof. Inserting Lemmas 3.3, 3.4, 3.6, 3.8, and 3.10 in (3.5), we obtain

$$\begin{aligned}T_1^*\mathcal{H}T_1 = \frac{N-1}{2}\widehat{V}(0) &- \frac{N-1}{2}\int_{\Lambda}(V_R\omega)(x)\,\mathrm{d}x+\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{|N\hat{\varepsilon}_{\ell,N}(p)|^2}{4p^2}\,\mathrm{d}p\\ &+\mathrm{d}\Gamma(-\Delta)+Q_2^{(T_1)}+H_2+\mathcal{N}\int_{\Lambda}(V_R\omega)(x)\,\mathrm{d}x+Q_3^{(T_1)}+Q_4\\ &+\mathcal{E}_2^{(T_1)}+\mathcal{E}_1^{(T_1)}+T_1^*\mathcal{E}^{(U)}T_1,\end{aligned}$$

with

$$\mathcal{E}_2^{(T_1)} = \frac{N}{2}\int_{\Lambda^2}\Big((V_N-V_{R,N})(x-y)\sum_{z\in\mathbb{Z}^3}(1-\omega_{\ell,N})(x-y+z)\Big)(a_x^*a_y^*+a_xa_y)\,\mathrm{d}x\,\mathrm{d}y$$

and

$$\begin{aligned}\pm\mathcal{E}_1^{(T_1)}\leqslant CNR^{-\gamma}&+C\ell^4\,\mathrm{d}\Gamma(-\Delta)+C\ell^{1/2}(\mathcal{N}+1)+C\ell^{1/2}\frac{(\mathcal{N}+1)^{3/2}}{N^{1/2}}\\ &+\varepsilon Q_4+\varepsilon^{-1}C\Big((R^{-\gamma}+\ell)(\mathcal{N}+1)+\frac{(\mathcal{N}+1)^2}{N}\Big)\end{aligned}$$

for all $0 < \varepsilon \leqslant 1$. The constant term equals $4\pi\mathfrak{a}(N-1)$ up to an error. Indeed, using (1.2), (2.4), and (2.6), we obtain

$$\begin{aligned}
&\left|\frac{N-1}{2}\widehat{V}(0) - \frac{N-1}{2}\int_\Lambda (V_R\omega)(x)\,\mathrm{d}x - 4\pi\mathfrak{a}(N-1)\right| \\
&\qquad\leqslant \frac{N-1}{2}\|V - V_R\|_{L^1} + \left|\frac{N-1}{2}\int_{\mathbb{R}^3}(\widetilde{V}_R(1-\omega))(x)\,\mathrm{d}x - 4\pi\mathfrak{a}_R(N-1)\right| \\
&\qquad\quad + 4\pi|\mathfrak{a}_R - \mathfrak{a}|(N-1) \\
&\qquad\leqslant CNR^{-\gamma}.
\end{aligned}$$

Moreover, we have

$$H_2 + \mathcal{N}\int_\Lambda (V_R\omega)(x)\,\mathrm{d}x = H_2 + (\widehat{V}_R(0) - 8\pi\mathfrak{a}_R)\mathcal{N} = H_2^{(T_1)}.$$

Lemma 3.10 implies

$$T_1^* Q_4 T_1 \leqslant C\left(N + Q_4 + \ell^{1/2}(\mathcal{N}+1) + \ell^{1/2}\frac{(\mathcal{N}+1)^2}{N}\right)$$

by choosing $\varepsilon = 1$ and noticing that

$$\begin{aligned}
\pm\frac{N}{2}\int_{\Lambda^2}\left(V_N(x-y)\sum_{z\in\mathbb{Z}^3}\omega_{\ell,N}(x-y+z)\right)(a_x^* a_y^* + a_x a_y)\,\mathrm{d}x\,\mathrm{d}y &\leqslant \frac{N^2}{2}\int_\Lambda V_N(x)\,\mathrm{d}x + Q_4 \\
&\leqslant CN + Q_4.
\end{aligned}$$

With this, Lemma 2.8 and the bound (2.13) for $\mathcal{E}^{(U)}$, we have for all $\varepsilon > 0$

$$\begin{aligned}
\pm\, T_1^*\mathcal{E}^{(U)}T_1 &\leqslant CR^{-\gamma}(\mathcal{N}+1) + C\frac{(\mathcal{N}+1)^2}{N} \\
&\qquad + \varepsilon\left(N + Q_4 + \ell^{1/2}(\mathcal{N}+1) + \ell^{1/2}\frac{(\mathcal{N}+1)^2}{N}\right) + \varepsilon^{-1}C\frac{(\mathcal{N}+1)^2}{N^3}.
\end{aligned}$$

Replacing ε by $N^{-2}\varepsilon$, we obtain the bound

$$\begin{aligned}
\pm\, T_1^*\mathcal{E}^{(U)}T_1 &\leqslant CR^{-\gamma}(\mathcal{N}+1) \\
&\qquad + \varepsilon\left(N^{-1} + N^{-2}Q_4 + \ell^{1/2}\frac{(\mathcal{N}+1)}{N^2} + \ell^{1/2}\frac{(\mathcal{N}+1)^2}{N^3}\right) + \varepsilon^{-1}C\frac{(\mathcal{N}+1)^2}{N}
\end{aligned}$$

for all $0 < \varepsilon \leqslant 1$. Gathering all the error terms and using $N^{-1}, \ell, \varepsilon \leqslant 1$ to simplify them concludes the proof. $\square$

4 Cubic renormalization

4.1 Definition of the cubic transform

We follow the strategy used in [16, Section 5] and [28, Section 5]. Recall the definition (3.1) of s_1 and let us define the operator $k_c \coloneqq N^{-1/2}s_1$. For $M \geqslant 50$ we define the function $\theta_M \colon \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\theta_M(x) = \begin{cases} 1 & x \leqslant \frac{M}{2} + 10, \\ \frac{1}{2}\left(\frac{4x-3M}{40-M} + 1\right) & \frac{M}{2} + 10 \leqslant x \leqslant M - 10, \\ 0 & x \geqslant M - 10. \end{cases}$$

For $j \in [0, 10]$ it satisfies $\theta_M(x \pm j) = 1$ when $x \leqslant M/2$, $\theta_M(x \pm j) = 0$ when $x \geqslant M$, and

$$\|\theta_M - \theta_M(\cdot \pm j)\|_{L^\infty} \leqslant CM^{-1}. \tag{4.1}$$

We will use the notation $\theta_{M,\pm j} = \theta_M(\cdot \pm j)$ for $j \geqslant 0$. In the following, we will write $\theta_M = \theta_M(\mathcal{N})$ and similarly $\theta_{M,\pm j} = \theta_{M,\pm j}(\mathcal{N})$.

Recall $Q = 1 - |u_0\rangle\langle u_0|$ and let $Q(x, y)$ be its integral kernel. We define

$$K_c^* = \int_\Lambda q_x^* a^*(k_{c,x}) q_x \,\mathrm{d}x, \qquad K_c = \int_\Lambda q_x^* a(k_{c,x}) q_x \,\mathrm{d}x,$$

where

$$q_x = \int_\Lambda Q(x, y) a_y \,\mathrm{d}y = a(Q_x).$$

We use q_x instead of a_x in the definitions of K_c^* and K_c to ensure they leave $\mathcal{F}_+$ invariant. These operators satisfy the following commutation relations with the usual creation and annihilation operators

$$[q_x, a_y^*] = \delta_{x,y} - 1, \quad \forall x, y \in \Lambda.$$

Since $q_x|\xi\rangle = a_x|\xi\rangle$ for all $\xi \in \mathcal{F}_+$, q_x may be replaced by a_x in all normal ordered expressions on $\mathcal{F}_+$.

Now we want to prove that $i\theta_M K_c^* + iK_c\theta_M$ is self-adjoint. In fact, it is symmetric and also bounded, since we have

$$\|\theta_M K_c^*\|_{\mathrm{op}} \leqslant \|\theta_M(\mathcal{N}+1)^{3/2}\|_{\mathrm{op}} \|(\mathcal{N}+1)^{-3/2} K_c^*\|_{\mathrm{op}} \leqslant CM^{3/2}\|k_c\|_{L^\infty L^2} < \infty,$$

where we used $\|k_c\|_{L^\infty L^2} \leqslant N^{-1/2}\|s_1\|_{L^\infty L^2} \leqslant CN^{-1/2}\ell^{1/2}$ from Lemma 3.1. Therefore, $i\theta_M K_c^* + iK_c\theta_M$ is self-adjoint and hence, we can define the cubic transform

$$T_c = \exp\left(\theta_M \int_\Lambda q_x^* a^*(k_{c,x}) q_x \,\mathrm{d}x - \text{h.c.}\right) = \exp(\theta_M K_c^* - K_c\theta_M).$$

This transformation renormalizes $Q_3^{(T_1)}$ on the sector with $\mathcal{N} \lesssim M$. More precisely, it is chosen such that $[\mathrm{d}\Gamma(-\Delta) + Q_4, \theta_M K_c^* - K_c \theta_M] + Q_3^{(T_1)} \approx 0$ and it extracts contributions that allow us to renormalize $H_2^{(T_1)}$. Due to the cutoff, it suffices to compute commutators of the form $[A, \theta_M K_c^* - K_c \theta_M]$ to determine the action of the transform.

Lemma 4.1. *Assume that $4RN^{-1} < \ell < 1/2$ and that $MN^{-1} \leqslant \ell$. We have*

$$T_c^* T_1^* \mathcal{H} T_1 T_c = 4\pi\mathfrak{a}(N-1) + \mathbb{H}_{\mathrm{Bog}} + Q_4 + \mathcal{E}^{(T_c)} \tag{4.2}$$

on $\mathcal{F}_+$, with

$$\mathbb{H}_{\mathrm{Bog}} = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{|N\hat{\varepsilon}_{\ell,N}(p)|^2}{4p^2} \,\mathrm{d}p + \sum_{p\neq 0} p^2 a_p^* a_p + \frac{N}{2} \sum_{p \neq 0} \hat{\varepsilon}_{\ell,N}(p)(2a_p^* a_p + a_p^* a_{-p}^* + a_p a_{-p}),$$

$$Q_4 = \frac{1}{2} \int_{\Lambda^2} V_N(x-y) a_x^* a_y^* a_x a_y \,\mathrm{d}x\,\mathrm{d}y,$$

and

$$\begin{aligned} \pm\mathcal{E}^{(T_c)} \leqslant\ & CNR^{-\gamma} + CM^{1/2}N^{-1/2}(\mathrm{d}\Gamma(-\Delta) + Q_4 + \ell^{-1}(\mathcal{N}+1)) \\ & + C\ell^2 \,\mathrm{d}\Gamma(-\Delta) + C\ell^4 Q_4 + C\ell^{1/2}(\mathcal{N}+1) + C\ell^{1/2}\frac{(\mathcal{N}+1)^{3/2}}{N^{1/2}} \\ & + \frac{1}{4}Q_4 + \varepsilon(Q_4 + \mathcal{N} + 1) \\ & + \varepsilon^{-1} C\Big((R^{-\gamma} + \ell)(\mathcal{N}+1) + \frac{(\mathcal{N}+1)^2}{N} + C_j \frac{(\mathcal{N}+1)^{j+1}}{M^j}\Big) \end{aligned} \tag{4.3}$$

for all $j \geqslant 1$ and $0 < \varepsilon \leqslant 1$.

For $t \in \mathbb{R}$, we define the unitary operator on $\mathcal{F}$:

$$T_c^t = \exp(t(\theta_M K_c^* - K_c \theta_M)).$$

We will need the following bound on the particle number. The proof can also be found in [28].

Lemma 4.2. *Assume that $4RN^{-1} < \ell < 1/2$, $M \leqslant N$, and $t \in [-1,1]$. On $\mathcal{F}$ we have*

$$T_c^{-t}(\mathcal{N}+1)^j T_c^t \leqslant C_j(\mathcal{N}+1)^j, \quad \forall j \geqslant 1,$$

where C_j is a constant independent of N.

Proof. We prove the bound for $t \in [0,1]$, the proof for $t \in [-1,0)$ is similar. Recall that $\theta_M K_c^* - K_c \theta_M$ is bounded with $\|\theta_M K_c^* - K_c \theta_M\|_{\mathrm{op}} \leqslant CM^{3/2}\|k_c\|_{L^\infty L^2}$ and note that it satisfies $\mathbb{1}^{\{\mathcal{N}>M\}}(\theta_M K_c^* - K_c \theta_M) = 0$. Hence, T_c^t leaves the domain of $(\mathcal{N}+1)^j$ invariant for all $j \geqslant 1$. In fact, let $\xi \in \mathcal{F}$ with $\|(\mathcal{N}+1)^j \xi\| < \infty$, then

$$
\begin{aligned}
\|(\mathcal{N}+1)^j T_c^t \xi\| &= \left\| \sum_{m=0}^{\infty} (\mathcal{N}+1)^j \frac{t^m(\theta_M K_c^* - K_c\theta_M)^m}{m!} \xi \right\| \\
&= \left\| (\mathcal{N}+1)^j \mathbb{1}^{\{\mathcal{N}>M\}} \xi + \sum_{m=0}^{\infty} (\mathcal{N}+1)^j \mathbb{1}^{\{\mathcal{N}\leqslant M\}} \frac{t^m(\theta_M K_c^* - K_c\theta_M)^m}{m!} \xi \right\| \\
&\leqslant \left\| (\mathcal{N}+1)^j \mathbb{1}^{\{\mathcal{N}>M\}} \xi \right\| + \sum_{m=0}^{\infty} \left\| (\mathcal{N}+1)^j \mathbb{1}^{\{\mathcal{N}\leqslant M\}} \right\|_{\mathrm{op}} \left\| \frac{t^m(\theta_M K_c^* - K_c\theta_M)^m}{m!} \right\|_{\mathrm{op}} \|\xi\| \\
&\leqslant \|(\mathcal{N}+1)^j \xi\| + (M+1)^j e^{CtM^{3/2}\|k_c\|_{L^\infty L^2}} \|\xi\| < \infty.
\end{aligned}
$$

We can improve this bound with a Grönwall argument. By the Duhamel formula, for all $0 \leqslant t \leqslant 1$ and $j \geqslant 1$ we have

$$
\begin{aligned}
T_c^{-t}(\mathcal{N}+1)^j T_c^t &= (\mathcal{N}+1)^j + \int_0^t T_c^{-s}[(\mathcal{N}+1)^j, \theta_M K_c^* - K_c\theta_M] T_c^s \,\mathrm{d}s \\
&= (\mathcal{N}+1)^j + \int_0^t T_c^{-s}\Big(\theta_M K_c^*((\mathcal{N}+2)^j - (\mathcal{N}+1)^j) + \text{h.c.}\Big) T_c^s \,\mathrm{d}s,
\end{aligned}
$$

where we used $\mathcal{N}K_c^* = K_c^*(\mathcal{N}+1)$ and $\mathcal{N}\theta_M = \theta_M\mathcal{N}$. Using $\|(\mathcal{N}+1)^{-3/2}K_c^*\|_{\mathrm{op}} \leqslant C\|k_c\|_{L^\infty L^2} \leqslant CN^{-1/2}\ell^{1/2}$, $(\mathcal{N}+2)^j - (\mathcal{N}+1)^j \leqslant C_j(\mathcal{N}+1)^{j-1}$, and that $\theta_M = 0$ outside of $\{\mathcal{N} \leqslant M\}$, we obtain

$$
\begin{aligned}
T_c^{-t}(\mathcal{N}+1)^j T_c^t &\leqslant (\mathcal{N}+1)^j + C_j N^{-1/2}\ell^{1/2} \int_0^t T_c^{-s} \mathbb{1}^{\{\mathcal{N}\leqslant M\}} (\mathcal{N}+1)^{j+1/2} T_c^s \,\mathrm{d}s \\
&\leqslant (\mathcal{N}+1)^j + C_j M^{1/2} N^{-1/2} \int_0^t T_c^{-s}(\mathcal{N}+1)^j T_c^s \,\mathrm{d}s.
\end{aligned}
$$

Let $\xi \in \mathcal{F}$ such that $\langle \xi, (\mathcal{N}+1)^j \xi \rangle < \infty$. Due to the bound from above, $u(s) = \langle \xi, T_c^{-s}(\mathcal{N}+1)^j T_c^s \xi \rangle$ is finite and we have

$$
u(t) \leqslant \langle \xi, (\mathcal{N}+1)^j \xi \rangle + C_j \int_0^t u(s)\,\mathrm{d}s,
$$

where we used $M^{1/2}N^{-1/2} \leqslant 1$. Applying the Grönwall lemma concludes the proof. □

From (3.3) we obtain

$$
\begin{aligned}
T_c^* T_1^* \mathcal{H} T_1 T_c = 4\pi\mathfrak{a}(N-1) &+ \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{|N\hat{\varepsilon}_{\ell,N}(p)|^2}{4p^2} \,\mathrm{d}p + T_c^* \,\mathrm{d}\Gamma(-\Delta) T_c \\
&+ T_c^* H_2^{(T_1)} T_c + T_c^*(Q_2^{(T_1)} + \mathcal{E}_2^{(T_1)}) T_c + T_c^* Q_3^{(T_1)} T_c + T_c^* Q_4 T_c + T_c^* \mathcal{E}^{(T_1)} T_c. \quad (4.4)
\end{aligned}
$$

In the following sections, we will estimate the terms on the right-hand side of (4.4). Then we will use these estimates to prove Lemma 4.1 in Section 4.6.

4.2 Estimates involving $\mathrm{d}\Gamma(-\Delta)$ and Q_4

In this section, we want to estimate the terms $T_c^* \,\mathrm{d}\Gamma(-\Delta)T_c$ and $T_c^* Q_4 T_c$.

Lemma 4.3. *Assume that* $4RN^{-1} < \ell < 1/2$, $MN^{-1} \leqslant \ell$, $NR^{-\gamma} \leqslant C$, *and* $t \in [0,1]$. *We have on* $\mathcal{F}_+$

$$T_c^{-t}\,\mathrm{d}\Gamma(-\Delta)T_c^t \leqslant C(\mathrm{d}\Gamma(-\Delta) + Q_4 + 1), \tag{4.5}$$
$$T_c^{-t} Q_4 T_c^t \leqslant C(Q_4 + \mathcal{N} + 1 + MN^{-1}\,\mathrm{d}\Gamma(-\Delta)). \tag{4.6}$$

To prove Lemma 4.3, we will need some a priori estimates on the commutators $[\mathrm{d}\Gamma(-\Delta), K_c^* - K_c]$ and $[Q_4, K_c^* - K_c]$, which we will show in the following two lemmas.

Lemma 4.4. *Assume that* $4RN^{-1} < \ell < 1/2$. *We have on* $\mathcal{F}_+^{\leqslant M}$

$$[\mathrm{d}\Gamma(-\Delta), K_c^*] = \sqrt{N}\int_{\Lambda^2}\Big(\sum_{z\in\mathbb{Z}^3}(\widetilde{V}_{R,N}(\omega_N - 1))(x-y+z)\Big)a_x^* a_y^* a_x \,\mathrm{d}x\,\mathrm{d}y + \mathcal{E}^{(T_c)}_{\mathrm{d}\Gamma(-\Delta)},$$

where $\mathcal{E}^{(T_c)}_{\mathrm{d}\Gamma(-\Delta)}$ *satisfies the bounds*

$$\pm\mathcal{E}^{(T_c)}_{\mathrm{d}\Gamma(-\Delta)} + \text{h.c.} \leqslant CM^{1/2}N^{-1/2}(\mathrm{d}\Gamma(-\Delta) + \ell^{-1}\mathcal{N})$$

and

$$\pm\mathcal{E}^{(T_c)}_{\mathrm{d}\Gamma(-\Delta)} + \text{h.c.} \leqslant CM^{1/2}N^{-1/2}\ell^{-1/2}\,\mathrm{d}\Gamma(-\Delta).$$

Lemma 4.5. *Assume that* $4RN^{-1} < \ell < 1/2$. *We have on* $\mathcal{F}_+^{\leqslant M}$

$$[Q_4, K_c^*] = -\sqrt{N}\int_{\Lambda^2}\Big(\sum_{z\in\mathbb{Z}^3}(\widetilde{V}_{R,N}\omega_N)(x-y+z)\Big)a_x^* a_y^* a_x\,\mathrm{d}x\,\mathrm{d}y + \mathcal{E}^{(T_c)}_{Q_4},$$

with

$$\pm\mathcal{E}^{(T_c)}_{Q_4} + \text{h.c.} \leqslant CMN^{-1}(\mathrm{d}\Gamma(-\Delta) + Q_4 + \ell^{1/2}) + \varepsilon Q_4 + \varepsilon^{-1}CR^{-\gamma}\mathcal{N}$$

for all $0 < \varepsilon \leqslant 1$.

Proof of Lemma 4.4. First, we note that

$$\begin{aligned} k_c(x,y) &= -\sqrt{N}\sum_{z\in\mathbb{Z}^3}\omega_{\ell,N}(x-y+z) + \sqrt{N}\hat{\omega}_{\ell,N}(0) = -\sqrt{N}\sum_{p\in\Lambda^*\setminus\{0\}}\hat{\omega}_{\ell,N}(p)e^{ip\cdot(y-x)} \\ &= -\sqrt{N}\sum_{p\in\Lambda^*\setminus\{0\}}\hat{\omega}_{\ell,N}(p)u_p(y)u_{-p}(x), \end{aligned}$$

due to $k_c = N^{-1/2}s_1$, the definition of s_1 (3.1), Lemma 3.1, and the fact that $\omega_{\ell,N}$ is even. Hence, we can write

$$K_c^* = \int_\Lambda q_x^* a^*(k_{c,x})q_x\,\mathrm{d}x = -\sqrt{N}\sum_{p,q\in\Lambda^*\setminus\{0\}}\hat{\omega}_{\ell,N}(p)a_q^* a_p^* a_{p+q}.$$

We compute

$$
\begin{aligned}
[\mathrm{d}\Gamma(-\Delta), K_c^*] &= -\sqrt{N} \sum_{p,q\in\Lambda^*\setminus\{0\}} \hat{\omega}_{\ell,N}(p)(q^2+p^2-(p+q)^2) a_q^* a_p^* a_{p+q} \\
&= -2\sqrt{N} \sum_{p,q\in\Lambda^*\setminus\{0\}} p^2 \hat{\omega}_{\ell,N}(p) a_q^* a_p^* a_{p+q} - 2\sqrt{N} \sum_{p,q\in\Lambda^*\setminus\{0\}} ip\hat{\omega}_{\ell,N}(p) a_q^* a_p^* a(\nabla u_{p+q}) \\
&= 2\int_{\Lambda^2} (-\Delta_x k_c)(x,y) q_x^* a_y^* q_x - 2\int_{\Lambda^2} (\nabla_x k_c)(x,y) q_x^* a_y^* \nabla_x q_x
\end{aligned}
$$

and, using the definition of $\varepsilon_{\ell,N}$ (2.8),

$$
-\Delta_x k_c(x,y) = \sqrt{N} \sum_{z\in\mathbb{Z}^3} (\Delta_x \omega_N)(x-y+z) + \frac{\sqrt{N}}{2} \sum_{z\in\mathbb{Z}^3} \varepsilon_{\ell,N}(x-y+z).
$$

On $\mathcal{F}_+^{\leqslant M}$ we can replace q_x with a_x and obtain

$$
\begin{aligned}
[\mathrm{d}\Gamma(-\Delta), K_c^*] &= 2\sqrt{N} \int_{\Lambda^2} \sum_{z\in\mathbb{Z}^3} (\Delta_x \omega_N)(x-y+z) a_x^* a_y^* a_x \\
&\quad + \sqrt{N} \int_{\Lambda^2} \sum_{z\in\mathbb{Z}^3} \varepsilon_{\ell,N}(x-y+z) a_x^* a_y^* a_x - 2\int_{\Lambda^2} (\nabla_x k_c)(x,y) a_x^* a_y^* \nabla_x a_x \\
&=: 2\sqrt{N} \int_{\Lambda^2} \sum_{z\in\mathbb{Z}^3} (\Delta_x \omega_N)(x-y+z) a_x^* a_y^* a_x + G_1 + G_2.
\end{aligned}
$$

Using the scattering equation (2.3) in the first term yields the correct main term in Lemma 4.4. We conclude the proof by estimating the error terms G_1 and G_2.

To bound G_1, we need the estimate

$$
\mathrm{d}\Gamma(\Phi) \leqslant C\|\Phi\|_{L^{3/2}}\, \mathrm{d}\Gamma(-\Delta), \quad \forall \Phi \in L^{3/2}(\Lambda)
$$

on $\mathcal{F}_+$. In fact, since we have

$$
|\langle g, \Phi g\rangle| = \left| \int_\Lambda \Phi(x)|g(x)|^2 \,\mathrm{d}x \right| \leqslant \|\Phi\|_{L^{3/2}} \|g\|_{L^6}^2 \leqslant C\|\Phi\|_{L^{3/2}} (\|g\|_{L^2}^2 + \|\nabla g\|_{L^2}^2),
$$

for all $g \in H^1(\Lambda)$, by the Hölder and Sobolev inequalities, and $\mathcal{N} \leqslant \mathrm{d}\Gamma(-\Delta)$ on $\mathcal{F}_+$, we obtain

$$
\mathrm{d}\Gamma(\Phi) \leqslant C\|\Phi\|_{L^{3/2}} \int_\Lambda a_x^*(1-\Delta_x) a_x \,\mathrm{d}x \leqslant C\|\Phi\|_{L^{3/2}}\, \mathrm{d}\Gamma(-\Delta)
$$

on $\mathcal{F}_+$. We want to apply this bound to $\Phi(x) := \varepsilon_{\ell,N}(x-y+z)$ for fixed y and z. By Lemma 2.2 we have

$$
|\varepsilon_{\ell,N}(x-y+z)| \leqslant CN^{-1}\ell^{-3} \mathbb{1}_{\{|x-y+z|\leqslant \ell\}},
$$

which we will use to bound the L^p–norms of $\varepsilon_{\ell,N}$. Now we can estimate for all $\xi',\xi \in \mathcal{F}_+^{\leqslant M}$

$$\begin{aligned}
&\Big|\Big\langle \xi', \sqrt{N}\int_{\Lambda^2}\sum_{z\in\mathbb{Z}^3}\varepsilon_{\ell,N}(x-y+z)a_x^*a_y^*a_x\xi\Big\rangle\Big| \\
&\quad\leqslant \eta\sqrt{N}\int_{\Lambda^2}\sum_{z\in\mathbb{Z}^3}|\varepsilon_{\ell,N}(x-y+z)|\|a_xa_y\xi'\|^2+\eta^{-1}\sqrt{N}\int_{\Lambda^2}\sum_{z\in\mathbb{Z}^3}|\varepsilon_{\ell,N}(x-y+z)|\|a_x\xi\|^2 \\
&\quad\leqslant \eta C\sqrt{N}\|\varepsilon_{\ell,N}\|_{L^{3/2}}M\|\mathrm{d}\Gamma(-\Delta)^{1/2}\xi'\|^2+\eta^{-1}C\sqrt{N}\|\varepsilon_{\ell,N}\|_{L^1}\|\mathcal{N}^{1/2}\xi\|^2 \\
&\quad\leqslant \eta CN^{-1/2}\ell^{-1}M\|\mathrm{d}\Gamma(-\Delta)^{1/2}\xi'\|^2+\eta^{-1}CN^{-1/2}\|\mathcal{N}^{1/2}\xi\|^2 \\
&\quad\leqslant CM^{1/2}N^{-1/2}\ell^{-1/2}\|\mathrm{d}\Gamma(-\Delta)^{1/2}\xi'\|\|\mathcal{N}^{1/2}\xi\|,
\end{aligned}$$

where we used Cauchy–Schwarz and the fact that the sum over $z\in\mathbb{Z}^3$ is finite since only terms with $|z|\leqslant 3$ contribute. In the last inequality, we optimized over $\eta>0$.

For G_2, we first note that for $x,y\in\Lambda$

$$\begin{aligned}
|\nabla_x k_c(x,y)| &\leqslant \sqrt{N}\sum_{z\in\mathbb{Z}^3}|\nabla_x\omega_{\ell,N}(x-y+z)| \\
&\leqslant \sqrt{N}\sum_{z\in\mathbb{Z}^3}|\nabla\omega_N(x-y+z)|\mathbb{1}_{\{|x-y+z|<2RN^{-1}\}}+\frac{CN\mathbb{1}_{|x-y+z|\leqslant\ell}}{|N(x-y+z)|^2+1} \\
&\leqslant \sum_{|z|\leqslant 3}\sqrt{N}|\nabla\omega_N(x-y+z)|\mathbb{1}_{\{|x-y+z|<2RN^{-1}\}}+\frac{1}{\sqrt{N}}\frac{C}{|x-y+z|^2}
\end{aligned}$$

by Lemma 2.2. For all $f\in H^1(\Lambda^2)$, $z\in\mathbb{Z}^3$, and a.e. $y\in\Lambda$, we want to use the Hardy inequality

$$\int_\Lambda \frac{|f(x,y)|^2}{|x-y+z|^2}\,\mathrm{d}x\leqslant C\|f(\cdot,y)\|^2_{H^1(\Lambda)} \tag{4.7}$$

and the inequality

$$\begin{aligned}
&\int_\Lambda|\nabla\omega_N(x-y+z)|\mathbb{1}_{\{|x-y+z|<2RN^{-1}\}}|f(x,y)|^2\,\mathrm{d}x \\
&\qquad\qquad\leqslant \|\nabla\omega_N\|_{L^{3/2}(B(0,2RN^{-1}))}\|f(\cdot,y)\|^2_{L^6(\Lambda)}\leqslant CN^{-1}\|f(\cdot,y)\|^2_{H^1(\Lambda)},
\end{aligned}$$

which follows from the Sobolev embedding $H^1(\Lambda)\subset L^6(\Lambda)$ and Lemma 2.2. These inequalities imply

$$|\nabla_x k_c(x,y)|\leqslant CN^{-1/2}(1-\Delta_x).$$

Now we can estimate on $\mathcal{F}_+^{\leqslant M}$

$$\begin{aligned}
&\pm\int_{\Lambda^2}(\nabla_x k_c)(x,y)a_x^*a_y^*\nabla_x a_x+\mathrm{h.c.} \\
&\quad\leqslant \eta\int_{\Lambda^2}|\nabla_x k_c(x,y)|a_x^*a_y^*a_ya_x+\eta^{-1}\int_{\Lambda^2}|\nabla_x k_c(x,y)|\nabla_x a_x^*\nabla_x a_x \\
&\quad\leqslant \eta C\int_{\Lambda^2}N^{-1/2}a_x^*a_y^*(1-\Delta_x)a_ya_x+\eta^{-1}\int_{\Lambda^2}N^{-1/2}|\nabla_x s_1(x,y)|\nabla_x a_x^*\nabla_x a_x
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \eta C N^{-1/2}\mathcal{N}\,\mathrm{d}\Gamma(-\Delta) + \eta^{-1}C\sqrt{N}\|\nabla\omega_{\ell,N}\|_{L^1}\int_\Lambda \nabla_x a_x^*\nabla_x a_x\\
&\leqslant \eta C N^{-1/2}\mathcal{N}\,\mathrm{d}\Gamma(-\Delta) + \eta^{-1}C N^{-1/2}\ell\,\mathrm{d}\Gamma(-\Delta)\\
&\leqslant C M^{1/2}N^{-1/2}\ell^{1/2}\,\mathrm{d}\Gamma(-\Delta),
\end{aligned}
$$

where we chose $\eta = \ell^{1/2}M^{-1/2}$ and used $\|\nabla\omega_{\ell,N}\|_{L^1} \leqslant CN^{-1}\ell$ from Lemma 2.2. $\square$

Proof of Lemma 4.5. Since we have $q_x = a_x$ on $\mathcal{F}_+^{\leqslant M}$ in all normal ordered expressions, we can compute

$$
\begin{aligned}
[Q_4, K_c^*] &= -\int_{\Lambda^2} V_N(x-y)\Big(\sqrt{N}\sum_{z\in\mathbb{Z}^3}\omega_{\ell,N}(x-y+z) - \sqrt{N}\hat{\omega}_{\ell,N}(0)\Big)a_x^*a_y^*a_x\\
&\quad - \int_{\Lambda^2} V_N(x-y)a_x^*a_y^*a(k_{c,x}) + \int_{\Lambda^3} V_N(x-y)a_x^*a_y^*a_v^*k_c(v,y)a_va_x\\
&\quad + \int_{\Lambda^3} V_N(x-y)a_x^*a_v^*a^*(k_{c,v})a_xa_y - \int_{\Lambda^3} V_N(x-y)a_x^*a_y^*a^*(k_{c,v})a_va_x\\
&=: -\sqrt{N}\int_{\Lambda^2} V_N(x-y)\Big(\sum_{z\in\mathbb{Z}^3}\omega_{\ell,N}(x-y+z)\Big)a_x^*a_y^*a_x + \sum_{i=1}^5 I_i.
\end{aligned}
$$

The first term is the main contribution and can be written as

$$
\begin{aligned}
&-\sqrt{N}\int_{\Lambda^2} V_N(x-y)\Big(\sum_{z\in\mathbb{Z}^3}\omega_{\ell,N}(x-y+z)\Big)a_x^*a_y^*a_x\\
&\qquad\qquad = -\sqrt{N}\int_{\Lambda^2}\Big(\sum_{z\in\mathbb{Z}^3}(\widetilde{V}_{R,N}\omega_N)(x-y+z)\Big)a_x^*a_y^*a_x + \mathcal{E},
\end{aligned}
$$

with

$$
\begin{aligned}
\pm\,\mathcal{E} &= \pm\sqrt{N}\int_{\Lambda^2}(V_{R,N}-V_N)(x-y)\Big(\sum_{z\in\mathbb{Z}^3}\omega_{\ell,N}(x-y+z)\Big)a_x^*a_y^*a_x\\
&\leqslant \varepsilon Q_4 + \varepsilon^{-1}\|N(V_{R,N}-V_N)\|_{L^1}\mathcal{N} \leqslant \varepsilon Q_4 + \varepsilon^{-1}CR^{-\gamma}\mathcal{N}
\end{aligned}
$$

for all $0<\varepsilon\leqslant 1$, where we used $|V_{R,N}-V_N|\leqslant V_N$. It remains to bound the other terms. For I_1, recall the bound $\hat{\omega}_{\ell,N}(0) = \|\omega_{\ell,N}\|_{L^1} \leqslant CN^{-1}\ell^2$ from Lemma 2.2. With this we can bound on $\mathcal{F}_+^{\leqslant M}$

$$
\pm I_1 + \text{h.c.} \leqslant \eta N^{-1}\ell^4 Q_4 + \eta^{-1}\|V_N\|_{L^1}\mathcal{N} \leqslant CN^{-1}\ell^2(Q_4 + \mathrm{d}\Gamma(-\Delta)),
$$

where we used $\mathcal{N}\leqslant \mathrm{d}\Gamma(-\Delta)$ and optimized over $\eta>0$. For I_2, we obtain on $\mathcal{F}_+^{\leqslant M}$

$$
\pm I_2 + \text{h.c.} \leqslant \eta Q_4 + \eta^{-1}\|V_N\|_{L^1}\|k_c\|_{\text{op}}^2\mathcal{N} \leqslant CN^{-1}\ell^2(Q_4 + \mathrm{d}\Gamma(-\Delta)),
$$

where used $\|k_c\|_{\text{op}} = N^{-1/2}\|s_1\|_{\text{op}} \leqslant CN^{-1/2}\ell^2$ from Lemma 3.1, $\mathcal{N} \leqslant \mathrm{d}\Gamma(-\Delta)$, and optimized over $\eta > 0$. We want to bound I_3 next. By Lemma 2.2, we have

$$\begin{aligned}|k_c(v,y) - \sqrt{N}\hat{\omega}_{\ell,N}(0)|^2 &\leqslant N \sum_{z\in\mathbb{Z}^3} |\omega_{\ell,N}(v-y+z)|^2 \\ &\leqslant N \sum_{z\in\mathbb{Z}^3} \left(\frac{C\mathbb{1}_{|v-y+z|\leqslant\ell}}{|N(v-y+z)|+1} \right)^2 \leqslant \frac{1}{N} \sum_{|z|\leqslant 3} \frac{C}{|v-y+z|^2}.\end{aligned}$$

Using Cauchy–Schwarz, the bound $N\hat{\omega}_{\ell,N}(0)^2 \leqslant CN^{-1}\ell^4 \leqslant C$, which follows from Lemma 2.2, and the Hardy inequality (4.7), we obtain on $\mathcal{F}_+^{\leqslant M}$

$$\begin{aligned}\pm I_3 + \text{h.c.} &= \pm \int_{\Lambda^3} V_N(x-y) a_x^* a_y^* a_v^* (k_c(v,y) - \sqrt{N}\hat{\omega}_{\ell,N}(0) + \sqrt{N}\hat{\omega}_{\ell,N}(0)) a_v a_x + \text{h.c.} \\ &\leqslant \eta C \int_{\Lambda^3} V_N(x-y) a_x^* a_y^* a_v^* a_v a_y a_x \\ &\quad + \eta^{-1} C \int_{\Lambda^3} V_N(x-y) \big(|k_c(v,y) - \sqrt{N}\hat{\omega}_{\ell,N}(0)|^2 + N\hat{\omega}_{\ell,N}(0)^2 \big) a_x^* a_v^* a_v a_x \\ &\leqslant \eta C \mathcal{N} Q_4 + \eta^{-1} C N^{-1} \int_{\Lambda^3} V_N(x-y) \Big(\sum_{|z|\leqslant 3} \frac{1}{|v-y+z|^2} + N\hat{\omega}_{\ell,N}(0)^2 \Big) a_x^* a_v^* a_v a_x \\ &\leqslant \eta C M Q_4 + \eta^{-1} C N^{-1} \|V_N\|_{L^1} \int_{\Lambda^2} a_x^* a_v^* (1-\Delta_v) a_v a_x \\ &\leqslant \eta C M Q_4 + \eta^{-1} C N^{-2} \mathcal{N}\, \mathrm{d}\Gamma(-\Delta) \leqslant CMN^{-1} Q_4 + CMN^{-1}\, \mathrm{d}\Gamma(-\Delta),\end{aligned}$$

where we chose $\eta = N^{-1}$. The term I_4 can be bounded on $\mathcal{F}_+^{\leqslant M}$ by

$$\begin{aligned}\pm I_4 + \text{h.c.} &= \pm \int_{\Lambda^3} V_N(x-y) a_x^* a_v^* a^*(k_{c,v}) a_x a_y + \text{h.c.} \\ &\leqslant \eta \|V_N\|_{L^1} \|k_c\|_{L^\infty L^2}^2 \mathcal{N}^3 + \eta^{-1} Q_4 \leqslant CMN^{-1}\ell^{1/2}(Q_4 + \mathrm{d}\Gamma(-\Delta)),\end{aligned}$$

where we used $\|k_c\|_{L^\infty L^2} = N^{-1/2}\|s_1\|_{L^\infty L^2} \leqslant CN^{-1/2}\ell^{1/2}$ from Lemma 3.1, $\mathcal{N} \leqslant \mathrm{d}\Gamma(-\Delta)$, and optimized over $\eta > 0$. Similarly, we have on $\mathcal{F}_+^{\leqslant M}$

$$\begin{aligned}\pm I_5 + \text{h.c.} &= \mp \int_{\Lambda^3} V_N(x-y) a_x^* a_y^* a^*(k_{c,v}) a_v a_x + \text{h.c.} \\ &\leqslant \eta Q_4 + \eta^{-1} \|V_N\|_{L^1} \|k_c\|_{L^\infty L^2}^2 (\mathcal{N}+1)^3 \\ &\leqslant CMN^{-1}\ell^{1/2}(Q_4 + \mathrm{d}\Gamma(-\Delta) + 1).\end{aligned}$$

□

Proof of Lemma 4.3. First, we want to show

$$T_c^{-t}(Q_4 + \mathrm{d}\Gamma(-\Delta))T_c^t \leqslant C(Q_4 + \mathrm{d}\Gamma(-\Delta) + 1) \tag{4.8}$$

for $t \in [0,1]$. Using the Duhamel formula, Lemma 4.4 and Lemma 4.5, Cauchy–Schwarz with $V_{R,N} \leqslant V_N$ and $\|NV_N\|_{L^1} \leqslant C$, Lemma 4.2, $\mathcal{N} \leqslant \mathrm{d}\Gamma(-\Delta)$ on $\mathcal{F}_+$, $MN^{-1} \leqslant \ell \leqslant 1$,

$\theta_M \leqslant 1$, $R^{-\gamma} \leqslant 1$, and choosing $\varepsilon = 1$, we obtain

$$
\begin{aligned}
T_c^{-t}(Q_4 + \mathrm{d}\Gamma(-\Delta))T_c^t - (Q_4 + \mathrm{d}\Gamma(-\Delta)) &= \int_0^t T_c^{-s}[Q_4 + \mathrm{d}\Gamma(-\Delta), \theta_M K_c^* - K_c\theta_M]T_c^s \,\mathrm{d}s \\
&= -\int_0^t T_c^{-s}\theta_M\Big(\sqrt{N}\int_{\Lambda^2} V_{R,N}(x-y)a_x^* a_y^* a_x \,\mathrm{d}x\,\mathrm{d}y - \mathcal{E}_{\mathrm{d}\Gamma(-\Delta)}^{(T_c)} - \mathcal{E}_{Q_4}^{(T_c)} + \mathrm{h.c.}\Big)T_c^s \,\mathrm{d}s \\
&\leqslant \int_0^t T_c^{-s}(Q_4 + C\mathcal{N})T_c^s \,\mathrm{d}s + C\int_0^t T_c^{-s}(Q_4 + \mathrm{d}\Gamma(-\Delta) + 1)T_c^s \,\mathrm{d}s \\
&\leqslant C\int_0^t T_c^{-s}(Q_4 + \mathrm{d}\Gamma(-\Delta))T_c^s \,\mathrm{d}s + C.
\end{aligned}
$$

Applying the Grönwall lemma shows (4.8). This immediately implies the bound (4.5) for the kinetic operator $\mathrm{d}\Gamma(-\Delta)$. Repeating this argument for Q_4, choosing $\varepsilon = MN^{-1}$ in Lemma 4.5, and additionally using $0 \leqslant \omega_N \leqslant 1$, $M^{-1}NR^{-\gamma} \leqslant C$, and (4.8), we have

$$
\begin{aligned}
T_c^{-t}Q_4T_c^t - Q_4 &= \int_0^t T_c^{-s}[Q_4, \theta_M K_c^* - K_c\theta_M]T_c^s \,\mathrm{d}s \\
&\leqslant \int_0^t T_c^{-s}(Q_4 + C\mathcal{N} + (\mathcal{E}_{Q_4}^{(T_c)} + \mathrm{h.c.}))T_c^s \,\mathrm{d}s \\
&\leqslant \int_0^t T_c^{-s}Q_4T_c^s \,\mathrm{d}s + CMN^{-1}\int_0^t T_c^{-s}(Q_4 + \mathrm{d}\Gamma(-\Delta) + \ell^{1/2})T_c^s \,\mathrm{d}s + C(\mathcal{N}+1) \\
&\leqslant \int_0^t T_c^{-s}Q_4T_c^s \,\mathrm{d}s + CMN^{-1}\big(Q_4 + \mathrm{d}\Gamma(-\Delta) + 1\big) + C(\mathcal{N}+1) \\
&\leqslant \int_0^t T_c^{-s}Q_4T_c^s \,\mathrm{d}s + C\big(Q_4 + MN^{-1}\,\mathrm{d}\Gamma(-\Delta) + \mathcal{N} + 1\big).
\end{aligned}
$$

Applying the Grönwall lemma again, we obtain (4.6). □

4.3 Estimating $T_c^* H_2^{(T_1)} T_c$

Lemma 4.6. *Assume that $4RN^{-1} < \ell < 1/2$, $MN^{-1} \leqslant \ell$, and $t \in [0,1]$. We have*

$$T_c^{-t}H_2^{(T_1)}T_c^t = H_2^{(T_1)} + \mathcal{E}$$

on $\mathcal{F}_+$, with

$$\pm\mathcal{E} \leqslant CM^{1/2}N^{-1/2}\ell^{1/2}(\mathcal{N}+1).$$

Since we have $H_2^{(T_1)} = \mathrm{d}\Gamma\big(NV_{R,N}(x-y) + \widehat{V}_R(0) - 8\pi\mathfrak{a}_R\big)$ and $\|NV_{R,N}(x-y) + \widehat{V}_R(0) - 8\pi\mathfrak{a}_R\|_{\mathrm{op}} \leqslant C$, it suffices to prove the following lemma.

Lemma 4.7. *Assume that $4RN^{-1} < \ell < 1/2$, $MN^{-1} \leqslant \ell$, and $t \in [0,1]$. Let A be a bounded, self-adjoint operator on $\mathfrak{H}$ with kernel $A(x,y)$. We have*

$$\pm\big(T_c^{-t}\,\mathrm{d}\Gamma(A)T_c^t - \mathrm{d}\Gamma(A)\big) \leqslant C\|A\|_{\mathrm{op}}M^{1/2}N^{-1/2}\ell^{1/2}(\mathcal{N}+1)$$

on $\mathcal{F}_+$.

Proof. We use the Duhamel formula

$$T_c^* \, \mathrm{d}\Gamma(A) T_c - \mathrm{d}\Gamma(A) = \int_0^t T_c^{-s} [\mathrm{d}\Gamma(A), \theta_M K_c^* - K_c \theta_M] T_c^s \, \mathrm{d}s$$

and calculate

$$[\mathrm{d}\Gamma(A), \theta_M K_c^* - K_c \theta_M] =: \theta_M \sum_{i=1}^{5} I_i + \text{h.c.},$$

where

$$\begin{aligned}
I_1 &= \int_{\Lambda^3} A(x,v) k_c(y,v) a_x^* a_y^* a_y = \int_\Lambda a^*((Ak_c)_y) a_y^* a_y, \\
I_2 &= \int_{\Lambda^3} A(x,y) k_c(y,v) a_x^* a_v^* a_y = \int_\Lambda a^*(A_y) a^*(k_{c,y}) a_y, \\
I_3 &= -\int_{\Lambda^3} A(x,y) k_c(x,v) a_x^* a_v^* a_y = -\int_\Lambda a_x^* a^*(k_{c,x}) a(A_x), \\
I_4 &= -\int_{\Lambda^4} A(x,y) k_c(v,z) a_x^* a_z^* a_v = -\int_{\Lambda^2} a^*(A_y) a^*(k_{c,v}) a_v, \\
I_5 &= \int_{\Lambda^4} A(x,y) k_c(v,z) a_v^* a_z^* a_y = \int_{\Lambda^2} a_v^* a^*(k_{c,v}) a(A_x).
\end{aligned}$$

Because of θ_M, it suffices to bound these terms on $\mathcal{F}_+^{\leqslant M}$. We will use $k_c = N^{-1/2} s_1$ and the inequalities $\|s_1\|_{L^\infty L^2} \leqslant C\ell^{1/2}$ and $\|s_1\|_{\mathrm{op}} \leqslant C\ell^2$ from Lemma 3.1. For $\xi', \xi \in \mathcal{F}_+^{\leqslant M}$, we have

$$\begin{aligned}
|\langle \xi', I_1 \xi \rangle| &\leqslant \int_\Lambda \|(\mathcal{N}+1)^{-1/4} a_y a((Ak_c)_y) \xi'\| \|(\mathcal{N}+1)^{1/4} a_y \xi\| \\
&\leqslant \eta \sup_{y \in \Lambda} \|(Ak_c)_y\|_{L^2}^2 \|(\mathcal{N}+1)^{3/4} \xi'\|^2 + \eta^{-1} \|(\mathcal{N}+1)^{3/4} \xi\|^2 \\
&\leqslant C \|A\|_{\mathrm{op}} M^{1/2} N^{-1/2} \|s_1\|_{L^\infty L^2} \|(\mathcal{N}+1)^{1/2} \xi'\| \|(\mathcal{N}+1)^{1/2} \xi\| \\
&\leqslant C \|A\|_{\mathrm{op}} M^{1/2} N^{-1/2} \ell^{1/2} \|(\mathcal{N}+1)^{1/2} \xi'\| \|(\mathcal{N}+1)^{1/2} \xi\|,
\end{aligned}$$

$$\begin{aligned}
|\langle \xi', I_2 \xi \rangle| &\leqslant \int_\Lambda \|(\mathcal{N}+1)^{-1/4} a(k_{c,y}) a(A_y) \xi'\| \|(\mathcal{N}+1)^{1/4} a_y \xi\| \\
&\leqslant \eta \|A\|_{\mathrm{op}}^2 \sup_{y \in \Lambda} \|k_{c,y}\|_{L^2}^2 \|(\mathcal{N}+1)^{3/4} \xi'\|^2 + \eta^{-1} \|(\mathcal{N}+1)^{3/4} \xi\|^2 \\
&\leqslant C \|A\|_{\mathrm{op}} M^{1/2} N^{-1/2} \ell^{1/2} \|(\mathcal{N}+1)^{1/2} \xi'\| \|(\mathcal{N}+1)^{1/2} \xi\|,
\end{aligned}$$

$$\begin{aligned}
|\langle \xi', I_3 \xi \rangle| &\leqslant \int_\Lambda \|(\mathcal{N}+1)^{-1/4} a(k_{c,x}) a_x \xi'\| \|(\mathcal{N}+1)^{1/4} a(A_x) \xi\| \\
&\leqslant \eta \sup_{x \in \Lambda} \|k_{c,x}\|_{L^2}^2 \|(\mathcal{N}+1)^{3/4} \xi'\|^2 + \eta^{-1} \|A\|_{\mathrm{op}}^2 \|(\mathcal{N}+1)^{3/4} \xi\|^2 \\
&\leqslant C \|A\|_{\mathrm{op}} M^{1/2} N^{-1/2} \ell^{1/2} \|(\mathcal{N}+1)^{1/2} \xi'\| \|(\mathcal{N}+1)^{1/2} \xi\|,
\end{aligned}$$

$$
\begin{aligned}
|\langle \xi', I_4 \xi\rangle| &\leqslant \int_{\Lambda^2} \|(\mathcal{N}+1)^{-1/4} a(k_{c,v}) a(A_y)\xi'\| \|(\mathcal{N}+1)^{1/4} a_v \xi\| \\
&\leqslant \eta \|A\|_{\mathrm{op}}^2 \|k_c\|_{\mathrm{op}}^2 \|(\mathcal{N}+1)^{3/4}\xi'\|^2 + \eta^{-1}\|(\mathcal{N}+1)^{3/4}\xi\|^2 \\
&\leqslant C\|A\|_{\mathrm{op}} M^{1/2} N^{-1/2} \ell^2 \|(\mathcal{N}+1)^{1/2}\xi'\| \|(\mathcal{N}+1)^{1/2}\xi\|,
\end{aligned}
$$

and

$$
\begin{aligned}
|\langle \xi', I_5 \xi\rangle| &\leqslant \int_{\Lambda^2} \|(\mathcal{N}+1)^{-1/4} a(k_{c,v}) a_v \xi'\| \|(\mathcal{N}+1)^{1/4} a(A_x) \xi\| \\
&\leqslant \eta \sup_{v\in\Lambda} \|k_{c,v}\|_{L^2}^2 \|(\mathcal{N}+1)^{3/4}\xi'\|^2 + \eta^{-1}\|A\|_{\mathrm{op}}^2 \|(\mathcal{N}+1)^{3/4}\xi\|^2 \\
&\leqslant C\|A\|_{\mathrm{op}} M^{1/2} N^{-1/2} \ell^{1/2} \|(\mathcal{N}+1)^{1/2}\xi'\| \|(\mathcal{N}+1)^{1/2}\xi\|,
\end{aligned}
$$

where we optimized over $\eta > 0$ in each inequality. Using Lemma 4.2 finishes the proof. □

4.4 Estimating $T_c^*(Q_2^{(T_1)} + \mathcal{E}_2^{(T_1)})T_c$

Recall

$$
\begin{aligned}
Q_2^{(T_1)} &= \frac{N}{2} \int_{\Lambda^2} \sum_{z\in\mathbb{Z}^3} \varepsilon_{\ell,N}(x-y+z) a_x^* a_y^* \, \mathrm{d}x \, \mathrm{d}y + \mathrm{h.c.}, \\
\mathcal{E}_2^{(T_1)} &= \frac{N}{2} \int_{\Lambda^2} (V_N - V_{R,N})(x-y) \Big(1 - \sum_{z\in\mathbb{Z}^3} \omega_{\ell,N}(x-y+z)\Big) a_x^* a_y^* \, \mathrm{d}x \, \mathrm{d}y + \mathrm{h.c.}
\end{aligned}
$$

Lemma 4.8. *Assume that $4RN^{-1} < \ell < 1/2$ and that $MN^{-1} \leqslant \ell$. We have*

$$
T_c^*(Q_2^{(T_1)} + \mathcal{E}_2^{(T_1)})T_c = Q_2^{(T_1)} + \mathcal{E}
$$

on $\mathcal{F}_+$, with

$$
\begin{aligned}
\pm \mathcal{E} \leqslant CM^{1/2}N^{-1/2}\ell^{-1}(\mathcal{N}+1) + CNR^{-\gamma} + \frac{1}{4}Q_4 \\
+ \varepsilon(\mathcal{N} + 1 + MN^{-1}\,\mathrm{d}\Gamma(-\Delta)) + \varepsilon^{-1} C\ell R^{-\gamma}(\mathcal{N}+1)
\end{aligned}
$$

for all $0 < \varepsilon \leqslant 1$.

Proof. By the Duhamel formula, we have

$$
T_c^* Q_2^{(T_1)} T_c - Q_2^{(T_1)} = \int_0^1 T_c^{-t} [Q_2^{(T_1)}, \theta_M K_c^* - K_c \theta_M] T_c^t \, \mathrm{d}t.
$$

By Lemma 4.2, it is enough to show $\pm[Q_2^{(T_1)}, \theta_M K_c^* - K_c\theta_M] \leqslant CM^{1/2}N^{-1/2}\ell^{-1}(\mathcal{N}+1)$.

On $\mathcal{F}_+$ we have

$$
\begin{aligned}
[Q_2^{(T_1)}, \theta_M K_c^* - K_c\theta_M] &= \frac{N}{2}\int_{\Lambda^2}\sum_{z\in\mathbb{Z}^3}\varepsilon_{\ell,N}(x-y+z)[a_x^*a_y^*, \theta_M K_c^* - K_c\theta_M] + \text{h.c.}\\
&= \Big(\theta_M \frac{N}{2}\int_{\Lambda^2}\sum_{z\in\mathbb{Z}^3}\varepsilon_{\ell,N}(x-y+z)[a_x^*a_y^*, K_c^*]\\
&\quad - \theta_{M,-1}\frac{N}{2}\int_{\Lambda^2}\sum_{z\in\mathbb{Z}^3}\varepsilon_{\ell,N}(x-y+z)[a_x^*a_y^*, K_c]\\
&\quad + (\theta_{M,-2} - \theta_M)\frac{N}{2}\int_{\Lambda^2}\sum_{z\in\mathbb{Z}^3}\varepsilon_{\ell,N}(x-y+z)a_x^*a_y^*K_c^*\\
&\quad - \frac{N}{2}\int_{\Lambda^2}\sum_{z\in\mathbb{Z}^3}\varepsilon_{\ell,N}(x-y+z)a_x^*a_y^*K_c(\theta_M - \theta_{M,+2})\Big) + \text{h.c.}
\end{aligned}
\tag{4.9}
$$

Let us bound the first two terms in (4.9). Disregarding θ_M, we have on $\mathcal{F}_+$

$$
\begin{aligned}
&\frac{N}{2}\int_{\Lambda^2}\sum_{z\in\mathbb{Z}^3}\varepsilon_{\ell,N}(x-y+z)[a_x^*a_y^*, K_c^*]\\
&= -N\int_{\Lambda^3}\sum_{z\in\mathbb{Z}^3}\varepsilon_{\ell,N}(x-y+z)k_c(x,v)a_x^*a_y^*a_v^* + N\int_{\Lambda^4}\sum_{z\in\mathbb{Z}^3}\varepsilon_{\ell,N}(x-y+z)k_c(t,v)a_x^*a_v^*a_t^*\\
&= -N\int_{\Lambda^2}\sum_{z\in\mathbb{Z}^3}\varepsilon_{\ell,N}(x-y+z)a_x^*a_y^*a^*(k_{c,x}) + 8\pi\mathfrak{a}_R\int_{\Lambda^2}a_x^*a_v^*a^*(k_{c,v}) =: I_1 + I_2
\end{aligned}
$$

and

$$
\begin{aligned}
&-\frac{N}{2}\int_{\Lambda^2}\sum_{z\in\mathbb{Z}^3}\varepsilon_{\ell,N}(x-y+z)[a_x^*a_y^*, K_c]\\
&= N\int_{\Lambda^3}\sum_{z\in\mathbb{Z}^3}\varepsilon_{\ell,N}(x-y+z)k_c(x,v)a_x^*a_v^*a_v\\
&\quad + N\int_{\Lambda^3}\sum_{z\in\mathbb{Z}^3}\varepsilon_{\ell,N}(x-y+z)k_c(x,v)a_x^*a_y^*a_v - N\int_{\Lambda^4}\sum_{z\in\mathbb{Z}^3}\varepsilon_{\ell,N}(x-y+z)k_c(t,v)a_x^*a_t^*a_v\\
&\quad + N\int_{\Lambda^2}\sum_{z\in\mathbb{Z}^3}\varepsilon_{\ell,N}(x-y+z)k_c(x,y)a_x^* - N\int_{\Lambda^3}\sum_{z\in\mathbb{Z}^3}\varepsilon_{\ell,N}(x-y+z)k_c(x,v)a_v^*\\
&= 8\pi\mathfrak{a}_R\int_{\Lambda^2}k_c(x,v)a_x^*a_v^*a_v + N\int_{\Lambda^2}\sum_{z\in\mathbb{Z}^3}\varepsilon_{\ell,N}(x-y+z)a_x^*a_y^*a(k_{c,x})\\
&\quad - 8\pi\mathfrak{a}_R\int_{\Lambda^2}a_x^*a^*(k_{c,v})a_v + N\int_{\Lambda^2}\sum_{z\in\mathbb{Z}^3}\varepsilon_{\ell,N}(x-y+z)k_c(x,y)a_x^* - 8\pi\mathfrak{a}_R\int_{\Lambda}a^*(k_{c,x})\\
&=: \sum_{i=3}^{7} I_i,
\end{aligned}
$$

where we used that $q_x = a_x$ on $\mathcal{F}_+$ in all normal ordered expressions. Thanks to θ_M, it is enough to bound the above terms on $\mathcal{F}_+^{\leqslant M}$.

By Lemma 2.2, we have $\|N\varepsilon_{\ell,N}\|_{L^2} \leqslant C\ell^{-3/2}$ and by Lemma 3.1, we have $\|s_1\|_{L^2} + \|s_1\|_{L^\infty L^2} \leqslant C\ell^{1/2}$. Using this we can bound all I_i. We have on $\mathcal{F}_+^{\leqslant M}$

$$\begin{aligned}
\pm I_1 + \text{h.c.} &\leqslant \eta \int_{\Lambda^2} a_x^* a_y^* a_y a_x + \eta^{-1} C \|N\varepsilon_{\ell,N}\|_{L^2}^2 \|k_c\|_{L^\infty L^2}^2 (\mathcal{N}+1) \\
&\leqslant \eta C \mathcal{N}^2 + \eta^{-1} C N^{-1} \ell^{-3} \|s_1\|_{L^\infty L^2}^2 (\mathcal{N}+1) \\
&\leqslant \eta C M \mathcal{N} + \eta^{-1} C N^{-1} \ell^{-2} (\mathcal{N}+1) \\
&\leqslant C M^{1/2} N^{-1/2} \ell^{-1} (\mathcal{N}+1),
\end{aligned}$$

where we optimized over $\eta > 0$. Similarly, we obtain

$$\begin{aligned}
\pm I_2 + \text{h.c.} &\leqslant C M^{1/2} N^{-1/2} \ell^{1/2} (\mathcal{N}+1), \\
\pm I_3 + \text{h.c.} &\leqslant C M^{1/2} N^{-1/2} \ell^{1/2} \mathcal{N}, \\
\pm I_4 + \text{h.c.} &\leqslant C M^{1/2} N^{-1/2} \ell^{-1} \mathcal{N}, \\
\pm I_5 + \text{h.c.} &\leqslant C M^{1/2} N^{-1/2} \ell^{1/2} \mathcal{N}.
\end{aligned}$$

For I_6, we have on $\mathcal{F}_+^{\leqslant M}$

$$\pm I_6 + \text{h.c.} \leqslant \eta \|N\varepsilon_{\ell,N}\|_{L^2}^2 \mathcal{N} + \eta^{-1} \|k_c\|_{L^2}^2 \leqslant \eta C \ell^{-3} \mathcal{N} + \eta^{-1} C N^{-1} \ell \leqslant C M^{1/2} N^{-1/2} \ell^{-1},$$

where we optimized over $\eta > 0$. Similarly, we obtain

$$\pm I_7 + \text{h.c.} \leqslant C M^{1/2} N^{-1/2} \ell^{1/2}.$$

Now we want to bound the last two terms in (4.9). We have on $\mathcal{F}_+$

$$\begin{aligned}
&\pm (\theta_{M,-2} - \theta_M) \frac{N}{2} \int_{\Lambda^2} \sum_{z \in \mathbb{Z}^3} \varepsilon_{\ell,N}(x-y+z) a_x^* a_y^* K_c^* + \text{h.c.} \\
&\quad \leqslant \eta \int_{\Lambda^2} (\theta_{M,-2} - \theta_M)^2 a_x^* a_y^* a_y a_x + \eta^{-1} \|N\varepsilon_{\ell,N}\|_{L^2}^2 \mathbb{1}^{\{\mathcal{N} \leqslant M\}} K_c K_c^* \\
&\quad \leqslant \eta C \|\theta_{M,-2} - \theta_M\|_{\text{op}}^2 \mathbb{1}^{\{\mathcal{N} \leqslant M\}} \mathcal{N}^2 + \eta^{-1} C \ell^{-3} M^2 \mathcal{N} K_c (\mathcal{N}+1)^{-3} K_c^* \\
&\quad \leqslant \eta C M^{-2} \mathbb{1}^{\{\mathcal{N} \leqslant M\}} \mathcal{N}^2 + \eta^{-1} C \ell^{-3} M^2 \|(\mathcal{N}+1)^{3/2} K_c^*\|_{\text{op}}^2 \mathcal{N} \\
&\quad \leqslant \eta C M^{-1} \mathcal{N} + \eta^{-1} C \ell^{-2} M^2 N^{-1} \mathcal{N} \\
&\quad \leqslant C M^{1/2} N^{-1/2} \ell^{-1} \mathcal{N},
\end{aligned}$$

where we used the bounds $\|(\mathcal{N}+1)^{3/2} K_c^*\|_{\text{op}} \leqslant C \|k_c\|_{L^\infty L^2} \leqslant C N^{-1/2} \|s_1\|_{L^\infty L^2} \leqslant C N^{-1/2} \ell^{1/2}$ and (4.1), and optimized over $\eta > 0$. Similarly, we have on $\mathcal{F}_+$

$$\pm \frac{N}{2} \int_{\Lambda^2} \sum_{z \in \mathbb{Z}^3} \varepsilon_{\ell,N}(x-y+z) a_x^* a_y^* K_c (\theta_M - \theta_{M,+2}) + \text{h.c.} \leqslant C M^{1/2} N^{-1/2} \ell^{-1} \mathcal{N}.$$

We proceed similarly for $\mathcal{E}_2^{(T_1)}$. By the Duhamel formula, we have

$$T_c^* \mathcal{E}_2^{(T_1)} T_c - \mathcal{E}_2^{(T_1)} = \int_0^1 T_c^{-t} [\mathcal{E}_2^{(T_1)}, \theta_M K_c^* - K_c \theta_M] T_c^t \, \mathrm{d}t.$$

To bound $[\mathcal{E}_2^{(T_1)}, \theta_M K_c^* - K_c\theta_M]$, we expand it as in (4.9). Using $\|N(V_N - V_{R,N})\|_{L^1} \leqslant CR^{-\gamma}$ from (1.2) and $\|k_c\|_{L^\infty} = N^{-1/2}\|s_1\|_{L^\infty} \leqslant CN^{1/2}$ from Lemma 3.1, we obtain

$$\pm[\mathcal{E}_2^{(T_1)}, \theta_M K_c^* - K_c\theta_M] \leqslant CM^{1/2}N^{1/2}R^{-\gamma} + \frac{\varepsilon}{8}Q_4 + \varepsilon^{-1}C\ell R^{-\gamma}(\mathcal{N}+1)$$

for all $0 < \varepsilon \leqslant 1$. The proof of this bound is similar to the one above, and we skip the details. Applying (4.6) and Lemma 4.2, we have

$$\begin{aligned}\pm\,(T_c^*\mathcal{E}_2^{(T_1)}T_c - \mathcal{E}_2^{(T_1)}) &\leqslant CM^{1/2}N^{1/2}R^{-\gamma} \\ &\quad + \frac{\varepsilon}{8}(Q_4 + \mathcal{N} + 1 + MN^{-1}\,\mathrm{d}\Gamma(-\Delta)) + \varepsilon^{-1}C\ell R^{-\gamma}(\mathcal{N}+1)\end{aligned}$$

for all $0 < \varepsilon \leqslant 1$. It remains to bound $\mathcal{E}_2^{(T_1)}$. Using $V_N - V_{R,N} \leqslant V_N$ and

$$\left|1 - \sum_{z\in\mathbb{Z}^3}\omega_{\ell,N}(x-y+z)\right| \leqslant 1,$$

we obtain

$$\pm\mathcal{E}_2^{(T_1)} \leqslant \frac{1}{8}Q_4 + C\|N^2(V_N - V_{R,N})\|_{L^1} \leqslant \frac{1}{8}Q_4 + CNR^{-\gamma}.$$

Gathering all the bounds, we conclude the proof. □

4.5 Estimating $T_c^*Q_3^{(T_1)}T_c$

Recall

$$Q_3^{(T_1)} = \sqrt{N}\int_{\Lambda^2} V_{R,N}(x-y)a_y^*a_x^*a_y\,\mathrm{d}x\,\mathrm{d}y + \text{h.c.}$$

Using the Duhamel formula, we can expand

$$T_c^*Q_3^{(T_1)}T_c = \int_0^1\int_t^1 T_c^{-s}[Q_3^{(T_1)}, \theta_M K_c^* - K_c\theta_M]T_c^s\,\mathrm{d}s\,\mathrm{d}t + \int_0^1 T_c^{-t}Q_3^{(T_1)}T_c^t\,\mathrm{d}t. \tag{4.10}$$

In the next lemma, we will compute the commutator in the first integral.

Lemma 4.9. *Assume that $4RN^{-1} < \ell < 1/2$ and that $MN^{-1} \leqslant \ell$. We have*

$$[Q_3^{(T_1)}, \theta_M K_c^* - K_c\theta_M] = -2N\int_{\Lambda^2}\Bigg(\sum_{z\in\mathbb{Z}^3}(\widetilde{V}_{R,N}\omega_N)(x-y+z)\Bigg)(a_y^*a_y + a_x^*a_y) + \mathcal{E}$$

on $\mathcal{F}_+$, with

$$\pm\mathcal{E} \leqslant CMN^{-1}\,\mathrm{d}\Gamma(-\Delta) + CM^{1/2}N^{-1/2}\ell^{1/2}(Q_4 + \mathcal{N} + 1) + C2^j\frac{\mathcal{N}^{j+1}}{M^j}$$

for all $j \geqslant 1$.

Proof. First, we compute the commutator $[Q_3^{(T_1)}, \theta_M K_c^* - K_c\theta_M]$, using that $q_x = a_x$ on $\mathcal{F}_+$ in all normal ordered expressions and $k_c = N^{-1/2}s_1$. We obtain

$$\begin{aligned}
&[Q_3^{(T_1)}, \theta_M K_c^* - K_c\theta_M] \\
&\quad = \theta_M\sqrt{N}\int_{\Lambda^2} V_{R,N}(x-y)[a_y^*a_x^*a_y, K_c^*] - \theta_{M,+1}\sqrt{N}\int_{\Lambda^2} V_{R,N}(x-y)[a_y^*a_x^*a_y, K_c] \\
&\qquad + \sqrt{N}\int_{\Lambda^2} V_{R,N}(x-y)a_y^*a_x^*a_y K_c^*(\theta_{M,+1} - \theta_{M,+2}) \\
&\qquad + \sqrt{N}\int_{\Lambda^2} V_{R,N}(x-y)a_y^*a_x^*a_y K_c(\theta_{M,+1} - \theta_M) + \text{h.c.} \\
&\quad = \theta_{M,+1}\int_{\Lambda^2} V_{R,N}(x-y)s_1(x,y)(a_y^*a_y + a_x^*a_y) + \text{h.c.} \\
&\qquad + \Big\{\theta_{M,+1}\Big(-\int_{\Lambda^2} V_{R,N}(x-y)(a^*(s_{1,y}) + a^*(s_{1,x}))a_y\Big) + \text{h.c.}\Big\} \\
&\qquad + \Big\{\theta_{M,+1}\Big(\int_{\Lambda^2} V_{R,N}(x-y)a_x^*a_y^*a(s_{1,x})a_y + \int_{\Lambda^3} V_{R,N}(x-y)a_x^*a_y^*a(s_{1,v})a_v\Big) \\
&\qquad\quad + \theta_M\Big(\int_{\Lambda^3} V_{R,N}(x-y)a_x^*a_y^*a_v^*s_1(v,y)a_v - \int_{\Lambda^3} V_{R,N}(x-y)a_x^*a_y^*a^*(s_{1,v})a_v \\
&\qquad\quad - \int_{\Lambda^2} V_{R,N}(x-y)a_x^*a_y^*a^*(s_{1,x})a_y\Big) + \text{h.c.}\Big\} \\
&\qquad + \Big\{\theta_M\int_{\Lambda^3} V_{R,N}(x-y)(a_y^* + a_x^*)a_v^*a^*(s_{1,v})a_y \\
&\qquad\quad - \theta_{M,+1}\int_{\Lambda^3} V_{R,N}(x-y)(a_y^* + a_x^*)a_v^*a(s_{1,v})a_y + \text{h.c.}\Big\} \\
&\qquad + \Big\{\theta_{M,+1}\int_{\Lambda^3} V_{R,N}(x-y)a_v^*(a_y^*s_1(v,x) + a_x^*s_1(v,y))a_y a_v + \text{h.c.}\Big\} \\
&\qquad + \Big\{\sqrt{N}\int_{\Lambda^2} V_{R,N}(x-y)a_y^*a_x^*a_y K_c^*(\theta_{M,+1} - \theta_{M,+2}) + \text{h.c.}\Big\} \\
&\qquad + \Big\{\sqrt{N}\int_{\Lambda^2} V_{R,N}(x-y)a_y^*a_x^*a_y K_c(\theta_{M,+1} - \theta_M) + \text{h.c.}\Big\} \\
&\quad =: 2\int_{\Lambda^2} V_{R,N}(x-y)s_1(x,y)(a_y^*a_y + a_x^*a_y) + \sum_{i=1}^{7} I_i,
\end{aligned}$$

where

$$I_1 = (\theta_{M,+1} - 1)\int_{\Lambda^2} V_{R,N}(x-y)s_1(x,y)(a_y^*a_y + a_x^*a_y) + \text{h.c.}$$

and the other I_i are the terms in curly brackets. By Lemma 3.1 and Lemma 2.2, we have

$$\begin{aligned}
&\pm\int_{\Lambda^2}\Big(V_{R,N}(x-y)s_1(x,y) + N\sum_{z\in\mathbb{Z}^3}(\widetilde{V}_{R,N}\omega_N)(x-y+z)\Big)(a_y^*a_y + a_x^*a_y) \\
&= \pm\int_{\Lambda^2} V_{R,N}(x-y)N\hat{\omega}_{\ell,N}(0)(a_y^*a_y + a_x^*a_y) \leqslant C\|V_{R,N}\|_{L^1}\|N\omega_{\ell,N}\|_{L^1}\mathcal{N} \leqslant CN^{-1}\ell^2\mathcal{N}.
\end{aligned}$$

Hence, it just remains to estimate the error terms I_i. For I_1 we use

$$0 \leqslant (1-\theta_{M,+1}) \leqslant \mathbb{1}^{\{\mathcal{N}>M/2\}}.$$

Applying Cauchy–Schwarz and Lemma 3.1, we obtain

$$\begin{aligned}
\pm I_1 &\leqslant \mathbb{1}^{\{\mathcal{N}>M/2\}} \|V_{R,N}\|_{L^1} \|s_1\|_{L^\infty} \mathcal{N} \\
&\quad + \|V_{R,N}\|_{L^1} \|s_1\|_{L^\infty} \Big(\int_\Lambda (\theta_{M,+1}-1) a_x^* a_x (\theta_{M,+1}-1) + \int_\Lambda \mathbb{1}^{\{\mathcal{N}>M/2\}} a_y^* a_y \mathbb{1}^{\{\mathcal{N}>M/2\}} \Big) \\
&\leqslant C \mathbb{1}^{\{\mathcal{N}>M/2\}} \|V_{R,N}\|_{L^1} \|s_1\|_{L^\infty} \mathcal{N} \\
&\leqslant C 2^j \frac{\mathcal{N}^{j+1}}{M^j}
\end{aligned}$$

for all $j \geqslant 1$. Similarly, using $0 \leqslant \theta_M, \theta_{M,+1} \leqslant \mathbb{1}^{\{\mathcal{N}\leqslant M\}}$ and Lemma 3.1, we can bound the other error terms. We obtain

$$\pm I_2 \leqslant C \|V_{R,N}\|_{L^1} \|s_1\|_{L^2} \mathcal{N} \leqslant C N^{-1} \ell^{1/2} \mathcal{N},$$

$$\begin{aligned}
\pm I_3 &\leqslant \eta_1 C Q_4 + \eta_1^{-1} C \|V_{R,N}\|_{L^1} \|s_1\|^2_{L^\infty L^2} \mathbb{1}^{\{\mathcal{N}\leqslant M\}} (\mathcal{N}+1)^2 \\
&\quad + \eta_2 C \mathbb{1}^{\{\mathcal{N}\leqslant M\}} \mathcal{N} Q_4 + \eta_2^{-1} C \|V_{R,N}\|_{L^1} \|s_1\|^2_{L^\infty L^2} \mathcal{N} \\
&\leqslant C M^{1/2} N^{-1/2} \ell^{1/2} (Q_4 + \mathcal{N}),
\end{aligned}$$

where we optimized over $\eta_1, \eta_2 > 0$, and

$$\begin{aligned}
\pm I_4 &\leqslant \eta C \mathbb{1}^{\{\mathcal{N}\leqslant M\}} \|V_{R,N}\|_{L^1} \mathcal{N}^2 + \eta^{-1} C \mathbb{1}^{\{\mathcal{N}\leqslant M\}} \|V_{R,N}\|_{L^1} \|s_1\|^2_{L^2} (\mathcal{N}+1)^2 \\
&\leqslant C M N^{-1} \ell^{1/2} (\mathcal{N}+1),
\end{aligned}$$

where we optimized over $\eta > 0$. By Lemma 2.2, we have

$$\begin{aligned}
|s_1(v,y)|^2 &\leqslant N^2 \sum_{z\in\mathbb{Z}^3} |\omega_{\ell,N}(v-y+z)|^2 \\
&\leqslant N^2 \sum_{z\in\mathbb{Z}^3} \Big(\frac{C \mathbb{1}_{|v-y+z|\leqslant \ell}}{|N(v-y+z)|+1} \Big)^2 \leqslant \sum_{|z|\leqslant 3} \frac{C}{|v-y+z|^2}.
\end{aligned}$$

Using this, the Hardy inequality (4.7), and $\mathcal{N} \leqslant \mathrm{d}\Gamma(-\Delta)$, we obtain

$$\begin{aligned}
\pm I_5 &\leqslant C \mathbb{1}^{\{\mathcal{N}\leqslant M\}} \int_{\Lambda^3} V_{R,N}(x-y) \Big(\sum_{|z|\leqslant 3} \frac{1}{|v-y+z|^2} \Big) a_y^* a_v^* a_v a_y \\
&\quad + C \mathbb{1}^{\{\mathcal{N}\leqslant M\}} \int_{\Lambda^3} V_{R,N}(x-y) a_y^* a_v^* a_v a_y \\
&\leqslant C \|V_{R,N}\|_{L^1} \mathbb{1}^{\{\mathcal{N}\leqslant M\}} \mathcal{N} \, \mathrm{d}\Gamma(-\Delta) + C \mathbb{1}^{\{\mathcal{N}\leqslant M\}} \|V_{R,N}\|_{L^1} \mathcal{N}^2 \\
&\leqslant C M N^{-1} \, \mathrm{d}\Gamma(-\Delta).
\end{aligned}$$

Next, we want to bound I_6 and I_7. Using (4.1) and $\|(\mathcal{N}+1)^{-3/2}K_c^*\|_{\mathrm{op}} \leqslant C\|k_c\|_{L^\infty L^2} \leqslant CN^{-1/2}\ell^{1/2}$, we have

$$
\begin{aligned}
\pm I_6 &\leqslant \eta Q_4 + \eta^{-1}N\|V_{R,N}\|_{L^1}\int_\Lambda (\theta_{M,+1}-\theta_{M,+2})K_c a_y^* a_y K_c^*(\theta_{M,+1}-\theta_{M,+2}) \\
&\leqslant \eta Q_4 + \eta^{-1}C(\mathcal{N}+1)^2(\theta_{M,+1}-\theta_{M,+2})K_c(\mathcal{N}+1)^{-3}K_c^*(\theta_{M,+1}-\theta_{M,+2})(\mathcal{N}+1)^2 \\
&\leqslant \eta Q_4 + \eta^{-1}CM^{-2}\|(\mathcal{N}+1)^{-3/2}K_c^*\|_{\mathrm{op}}^2 \mathbb{1}^{\{\mathcal{N}\leqslant M\}}(\mathcal{N}+1)^4 \\
&\leqslant \eta Q_4 + \eta^{-1}CMN^{-1}\ell(\mathcal{N}+1) \\
&\leqslant CM^{1/2}N^{-1/2}\ell^{1/2}(Q_4+\mathcal{N}+1),
\end{aligned}
$$

where we optimized over $\eta > 0$. Similarly, we obtain

$$
\pm I_7 \leqslant CM^{1/2}N^{-1/2}\ell^{1/2}(Q_4+\mathcal{N}+1).
$$

Gathering all the bounds and using $M^{1/2}N^{-1/2} \leqslant 1$, we conclude the proof. □

Now, we can compute the first integral in (4.10).

Lemma 4.10. *Assume that $4RN^{-1} < \ell < 1/2$ and that $MN^{-1} \leqslant \ell$. We have*

$$
\begin{aligned}
&\int_0^1\int_t^1 T_c^{-s}[Q_3^{(T_1)}, \theta_M K_c^* - K_c\theta_M]T_c^s \,\mathrm{d}s\,\mathrm{d}t \\
&\qquad\qquad = -\,\mathrm{d}\Gamma\Big(\int_\Lambda (V_R\omega)(x)\,\mathrm{d}x + N\sum_{z\in\mathbb{Z}^3}(\widetilde{V}_{R,N}\omega_N)(x-y+z)\Big) + \mathcal{E}_{Q_3}^{(T_c)}
\end{aligned}
$$

on $\mathcal{F}_+$, with

$$
\pm\mathcal{E}_{Q_3}^{(T_c)} \leqslant CMN^{-1}\,\mathrm{d}\Gamma(-\Delta) + CM^{1/2}N^{-1/2}\ell^{1/2}(Q_4+\mathcal{N}+1) + C_j\frac{(\mathcal{N}+1)^{j+1}}{M^j}
$$

for all $j \geqslant 1$.

Proof. By Lemma 4.9, we have

$$
[Q_3^{(T_1)}, \theta_M K_c^* - K_c\theta_M] = -2N\int_{\Lambda^2}\Big(\sum_{z\in\mathbb{Z}^3}(\widetilde{V}_{R,N}\omega_N)(x-y+z)\Big)(a_y^*a_y + a_x^*a_y) + \mathcal{E},
$$

with

$$
\pm\mathcal{E} \leqslant CMN^{-1}\,\mathrm{d}\Gamma(-\Delta) + CM^{1/2}N^{-1/2}\ell^{1/2}(Q_4+\mathcal{N}+1) + C2^j\frac{\mathcal{N}^{j+1}}{M^j}.
$$

Since

$$
\begin{aligned}
&-2N\int_{\Lambda^2}\Big(\sum_{z\in\mathbb{Z}^3}(\widetilde{V}_{R,N}\omega_N)(x-y+z)\Big)(a_y^*a_y + a_x^*a_y) \\
&\qquad\qquad = -2\,\mathrm{d}\Gamma\Big(\int_\Lambda (V_R\omega)(x)\,\mathrm{d}x + N\sum_{z\in\mathbb{Z}^3}(\widetilde{V}_{R,N}\omega_N)(x-y+z)\Big)
\end{aligned}
$$

and

$$\Big\| \int_\Lambda (V_R\omega)(x)\,\mathrm{d}x + N \sum_{z\in\mathbb{Z}^3} (\widetilde{V}_{R,N}\omega_N)(x-y+z) \Big\|_{\mathrm{op}} \leqslant 2\|V_R\omega\|_{L^1} \leqslant C,$$

we can apply Lemma 4.7 and we obtain

$$\begin{aligned}
&-2N\int_0^1\int_t^1 T_c^{-s}\int_{\Lambda^2}\Big(\sum_{z\in\mathbb{Z}^3}(\widetilde{V}_{R,N}\omega_N)(x-y+z)\Big)(a_y^*a_y+a_x^*a_y)T_c^s\,\mathrm{d}s\,\mathrm{d}t \\
&\qquad\qquad = -N\int_{\Lambda^2}\Big(\sum_{z\in\mathbb{Z}^3}(\widetilde{V}_{R,N}\omega_N)(x-y+z)\Big)(a_y^*a_y+a_x^*a_y)+\tilde{\mathcal{E}},
\end{aligned}$$

with

$$\pm\tilde{\mathcal{E}} \leqslant CM^{1/2}N^{-1/2}\ell^{1/2}(\mathcal{N}+1).$$

It remains to estimate the error term

$$\int_0^1\int_t^1 T_c^{-s}\mathcal{E}T_c^s\,\mathrm{d}s\,\mathrm{d}t.$$

Using Lemma 4.3, Lemma 4.2, and $MN^{-1}\leqslant\ell\leqslant 1$, we get

$$\begin{aligned}
\pm\int_0^1\int_t^1 T_c^{-s}\mathcal{E}T_c^s\,\mathrm{d}s\,\mathrm{d}t &\leqslant CMN^{-1}(\mathrm{d}\Gamma(-\Delta)+Q_4+1) \\
&\quad + CM^{1/2}N^{-1/2}\ell^{1/2}(Q_4+\mathcal{N}+1+MN^{-1}\,\mathrm{d}\Gamma(-\Delta)) \\
&\quad + C_j\frac{(\mathcal{N}+1)^{j+1}}{M^j} \\
&\leqslant CMN^{-1}\,\mathrm{d}\Gamma(-\Delta)+CM^{1/2}N^{-1/2}\ell^{1/2}(Q_4+\mathcal{N}+1) \\
&\quad + C_j\frac{(\mathcal{N}+1)^{j+1}}{M^j}
\end{aligned}$$

for all $j\geqslant 1$. □

4.6 Proof of Lemma 4.1

Proof. Applying the Duhamel formula to (4.4) and using (4.10) and Lemma 4.10, we obtain

$$\begin{aligned}
T_c^*T_1^*\mathcal{H}T_1T_c &= 4\pi\mathfrak{a}(N-1)+\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{|N\hat{\varepsilon}_{\ell,N}(p)|^2}{4p^2}\,\mathrm{d}p+\mathrm{d}\Gamma(-\Delta)+Q_4 \\
&\quad + \int_0^1 T_c^{-t}\Big([\mathrm{d}\Gamma(-\Delta)+Q_4,\theta_M K_c^*-K_c\theta_M]+Q_3^{(T_1)}\Big)T_c^t\,\mathrm{d}t \\
&\quad - \mathrm{d}\Gamma\Big(\int_\Lambda (V_R\omega)(x)\,\mathrm{d}x+N\sum_{z\in\mathbb{Z}^3}(\widetilde{V}_{R,N}\omega_N)(x-y+z)\Big) \\
&\quad + T_c^*H_2^{(T_1)}T_c+T_c^*(Q_2^{(T_1)}+\mathcal{E}_2^{(T_1)})T_c+T_c^*\mathcal{E}^{(T_1)}T_c+\mathcal{E}_{Q_3}^{(T_c)},
\end{aligned}$$

with

$$\pm\mathcal{E}_{Q_3}^{(T_c)} \leqslant CMN^{-1}\,\mathrm{d}\Gamma(-\Delta) + CM^{1/2}N^{-1/2}\ell^{1/2}(Q_4+\mathcal{N}+1) + C_j\frac{(\mathcal{N}+1)^{j+1}}{M^j}$$

for all $j \geqslant 1$. Inserting Lemmas 4.4, 4.5, 4.6, and 4.8, we have

$$\begin{aligned}
T_c^*T_1^*\mathcal{H}T_1T_c = {} & 4\pi\mathfrak{a}(N-1) + \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{|N\hat{\varepsilon}_{\ell,N}(p)|^2}{4p^2}\,\mathrm{d}p + \mathrm{d}\Gamma(-\Delta) + Q_4 \\
& - \mathrm{d}\Gamma\Big(\int_\Lambda (V_R\omega)(x)\,\mathrm{d}x + N\sum_{z\in\mathbb{Z}^3}(\widetilde{V}_{R,N}\omega_N)(x-y+z)\Big) \\
& + \mathrm{d}\Gamma\big(NV_{R,N}(x-y) + \widehat{V}_R(0) - 8\pi\mathfrak{a}_R\big) \\
& + \frac{N}{2}\int_{\Lambda^2}\sum_{z\in\mathbb{Z}^3}\varepsilon_{\ell,N}(x-y+z)a_x^*a_y^*\,\mathrm{d}x\,\mathrm{d}y + \text{h.c.} \\
& + \int_0^1 T_c^{-t}\Big((1-\theta_M)\sqrt{N}\int_{\Lambda^2}V_{R,N}(x-y)a_y^*a_x^*a_y\,\mathrm{d}x\,\mathrm{d}y + \text{h.c.}\Big)T_c^t\,\mathrm{d}t \\
& + \int_0^1 T_c^{-t}\Big(\theta_M(\mathcal{E}_{\mathrm{d}\Gamma(-\Delta)}^{(T_c)} + \mathcal{E}_{Q_4}^{(T_c)}) + \text{h.c.}\Big)T_c^t\,\mathrm{d}t + T_c^*\mathcal{E}^{(T_1)}T_c + \mathcal{E}_1^{(T_c)},
\end{aligned}$$

where the last two lines are error terms and

$$\begin{aligned}
\pm\,\mathcal{E}_1^{(T_c)} \leqslant {} & CM^{1/2}N^{-1/2}\ell^{-1}(\mathcal{N}+1) + CMN^{-1}\,\mathrm{d}\Gamma(-\Delta) \\
& + CM^{1/2}N^{-1/2}\ell^{1/2}Q_4 + C_j\frac{(\mathcal{N}+1)^{j+1}}{M^j} + CNR^{-\gamma} + \frac{1}{4}Q_4 \\
& + \varepsilon(\mathcal{N}+1) + \varepsilon^{-1}C\ell R^{-\gamma}(\mathcal{N}+1)
\end{aligned}$$

for all $j \geqslant 1$ and $0 < \varepsilon \leqslant 1$. We can rewrite

$$\mathrm{d}\Gamma(-\Delta) = \sum_p p^2 a_p^* a_p$$

and

$$\frac{N}{2}\int_{\Lambda^2}\sum_{z\in\mathbb{Z}^3}\varepsilon_{\ell,N}(x-y+z)a_x^*a_y^*\,\mathrm{d}x\,\mathrm{d}y + \text{h.c.} = \frac{N}{2}\sum_p \hat{\varepsilon}_{\ell,N}(p)(a_p^*a_{-p}^* + a_pa_{-p}).$$

Using (2.4), we obtain

$$\begin{aligned}
\mathrm{d}\Gamma\big(NV_{R,N}(x-y) + \widehat{V}_R(0) - 8\pi\mathfrak{a}_R\big) - \mathrm{d}\Gamma\Big(\int_\Lambda (V_R\omega)(x)\,\mathrm{d}x + N\sum_{z\in\mathbb{Z}^3}(\widetilde{V}_{R,N}\omega_N)(x-y+z)\Big) \\
= \mathrm{d}\Gamma\Big(N\sum_{z\in\mathbb{Z}^3}(\widetilde{V}_{R,N}(1-\omega_N))(x-y+z)\Big)
\end{aligned}$$

We want to replace this with

$$\mathrm{d}\Gamma\Big(N\sum_{z\in\mathbb{Z}^3}\varepsilon_{\ell,N}(x-y+z)\Big) = N\sum_p \hat{\varepsilon}_{\ell,N}(p)a_p^*a_p.$$

For this purpose we want to bound $\mathrm{d}\Gamma(\Phi(x-y))$, with

$$\Phi(x) = N \sum_{z\in\mathbb{Z}^3} (\widetilde{V}_{R,N}(1-\omega_N) - \varepsilon_{\ell,N})(x+z).$$

Note that we have $\widehat{\Phi}(0) = 0$ by using (2.4) and Lemma 2.2, and $\nabla\widehat{\Phi}(0) = 0$ since $\widetilde{V}_{R,N}(1-\omega_N)$ and $\varepsilon_{\ell,N}$ are even. Hence, using $\mathrm{supp}(\widetilde{V}_{R,N}) \cup \mathrm{supp}(\varepsilon_{\ell,N}) \subset \{|x| \leqslant \ell\}$, we have

$$|\widehat{\Phi}(p)| = |\widehat{\Phi}(p) - \widehat{\Phi}(0) - p\cdot\nabla\widehat{\Phi}(0)| \leqslant C\||\cdot|^2\Phi\|_{L^1} p^2 \leqslant C\|\Phi\|_{L^1}\ell^2 p^2 \leqslant C\ell^2 p^2.$$

Using this, we obtain for $g \in H^1(\Lambda)$

$$\left|\int_{\Lambda^2} \overline{g(x)}\Phi(x-y)g(y)\,\mathrm{d}x\,\mathrm{d}y\right| = \left\|\sum_p \overline{\hat{g}(p)}\widehat{\Phi}(p)\hat{g}(p)\right\| \leqslant C\ell^2\left\|\sum_p p^2\overline{\hat{g}(p)}\hat{g}(p)\right\| \leqslant C\ell^2\|g\|_{H^1}^2.$$

This shows $\Phi(x-y) \leqslant C\ell^2(1-\Delta)$ and hence, we have on $\mathcal{F}_+$

$$\pm\,\mathrm{d}\Gamma(\Phi(x-y)) \leqslant C\ell^2\,\mathrm{d}\Gamma(-\Delta).$$

It remains to bound the error terms. We use $0 \leqslant (1-\theta_M) \leqslant \mathbb{1}^{\{\mathcal{N}>M/2\}}$ and $V_{R,N} \leqslant V_N$ to obtain

$$\begin{aligned}\pm\,(1-\theta_M)\sqrt{N}&\int_{\Lambda^2} V_{R,N}(x-y)a_y^*a_x^*a_y\,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant \varepsilon Q_4 + \varepsilon^{-1}N\|V_N\|_{L^1}\mathbb{1}^{\{\mathcal{N}>M/2\}}\mathcal{N} \leqslant \varepsilon Q_4 + \varepsilon^{-1}C_j\frac{\mathcal{N}^{j+1}}{M^j}.\end{aligned}$$

for all $j \geqslant 1$ and $\varepsilon > 0$. Using Lemma 4.2, Lemma 4.3, the bounds in Lemma 4.4 and Lemma 4.5, and (3.4), we have

$$\begin{aligned}&\int_0^1 T_c^{-t}\Big((1-\theta_M)\sqrt{N}\int_{\Lambda^2} V_{R,N}(x-y)a_y^*a_x^*a_y\,\mathrm{d}x\,\mathrm{d}y\Big)T_c^t\,\mathrm{d}t \\ &\qquad + \int_0^1 T_c^{-t}\Big(\theta_M(\mathcal{E}^{(T_c)}_{\mathrm{d}\Gamma(-\Delta)} + \mathcal{E}^{(T_c)}_{Q_4}) + \mathrm{h.c.}\Big)T_c^t\,\mathrm{d}t + T_c^*\mathcal{E}^{(T_1)}T_c \\ &\quad\leqslant CNR^{-\gamma} + CM^{1/2}N^{-1/2}(\mathrm{d}\Gamma(-\Delta) + Q_4 + \ell^{-1}(\mathcal{N}+1)) \\ &\qquad + C\ell^4(\mathrm{d}\Gamma(-\Delta)+Q_4) + C\ell^{1/2}(\mathcal{N}+1) + C\ell^{1/2}\frac{(\mathcal{N}+1)^{3/2}}{N^{1/2}} \\ &\qquad + \varepsilon(Q_4+\mathcal{N}+1) + \varepsilon^{-1}C\Big((R^{-\gamma}+\ell)(\mathcal{N}+1) + \frac{(\mathcal{N}+1)^2}{N} + C_j\frac{(\mathcal{N}+1)^{j+1}}{M^j}\Big)\end{aligned}$$

for all $j \geqslant 1$ and $0 < \varepsilon \leqslant 1$. We simplified the estimate using $MN^{-1} \leqslant \ell \leqslant 1$.

Collecting all the error bounds, we conclude the proof. □

5 Diagonalization

In this chapter, we will use a transform similar to that in [16, Section 6]. We want to diagonalize the Bogoliubov Hamiltonian from Lemma 4.1

$$\mathbb{H}_{\mathrm{Bog}} = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \frac{|N\hat{\varepsilon}_{\ell,N}(p)|^2}{4p^2}\,\mathrm{d}p + \sum_{p\neq 0} p^2 a_p^* a_p + \frac{N}{2}\sum_{p\neq 0}\hat{\varepsilon}_{\ell,N}(p)(2a_p^*a_p + a_p^*a_{-p}^* + a_p a_{-p}).$$

As remarked in Section 2.4, by choosing

$$s = \sum_{p\neq 0}\sinh(\tau_p)|u_p\rangle\langle u_p|,$$

we can define a quadratic transform on $\mathcal{F}_+$

$$T_2 = \exp\bigg(\frac{1}{2}\sum_{p\neq 0}\tau_p(a_p^*a_{-p}^* - a_p a_{-p})\bigg) =: \exp(K_2),$$

that satisfies

$$T_2^* a_p^* T_2 = \cosh(\tau_p)a_p^* + \sinh(\tau_p)a_{-p}, \tag{5.1}$$

where

$$\tau_p = \sinh^{-1}(\nu_p), \quad \nu_p = -\operatorname{sgn}(B_p)\sqrt{\frac{1}{2}\bigg(\frac{A_p}{\sqrt{A_p^2 - B_p^2}} - 1\bigg)},$$
$$A_p = p^2 + N\hat{\varepsilon}_{\ell,N}(p), \quad B_p = N\hat{\varepsilon}_{\ell,N}(p),$$

for $p \in \Lambda^* \setminus \{0\}$, if we can show that s is a Hilbert–Schmidt operator, which is equivalent to $\sum_p |\nu_p|^2 < \infty$. First, note that ν_p is well-defined for ℓ small enough. We have $\nabla\hat{\varepsilon}_{\ell,N}(0) = 0$, since $\varepsilon_{\ell,N}$ is even, and, by Lemma 2.2, $N|\varepsilon_{\ell,N}(x)| \leqslant C\ell^{-3}\mathbb{1}_{\{|x|\leqslant \ell\}}$ and $N\hat{\varepsilon}_{\ell,N}(0) = 8\pi\mathfrak{a}_R$. Hence, we obtain $\|N\Delta\hat{\varepsilon}_{\ell,N}\|_{L^\infty} \leqslant \||\cdot|^2 N\varepsilon_{\ell,N}\|_{L^1} \leqslant C\ell^2$ and

$$\begin{aligned}|N\hat{\varepsilon}_{\ell,N}(p) - 8\pi\mathfrak{a}_R| &= |N\hat{\varepsilon}_{\ell,N}(p) - N\hat{\varepsilon}_{\ell,N}(0) - p\cdot N\nabla\hat{\varepsilon}_{\ell,N}(0)| \\ &\leqslant C\min(1, \||\cdot|^2 N\varepsilon_{\ell,N}\|_{L^1}p^2) \leqslant C\min(1, \ell^2 p^2).\end{aligned} \tag{5.2}$$

Thus, $A_p > |B_p|$ and ν_p is well-defined for all p, if ℓ is small enough. From (5.2), we have $p^4 + 2p^2 B_p \geqslant Cp^4$. Thus, using $|\sqrt{1+x} - 1 - x/2| \leqslant Cx^2$ for $-1 \leqslant x \leqslant 1$, we obtain

$$|\nu_p|^2 = \frac{1}{2}\frac{p^2 + B_p - p^2\sqrt{1 + 2p^{-2}B_p}}{\sqrt{p^4 + 2p^2 B_p}} \leqslant \frac{Cp^{-2}B_p^2}{\sqrt{p^4 + 2p^2 B_p}} \leqslant Cp^{-4}B_p^2 \leqslant Cp^{-4}, \tag{5.3}$$

which shows $\sum_p |\nu_p|^2 < \infty$. Using (5.1), we can compute

$$
\begin{aligned}
T_2^*\Big(\mathbb{H}_{\mathrm{Bog}} - \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \frac{|N\hat{\varepsilon}_{\ell,N}(p)|^2}{4p^2}\,\mathrm{d}p\Big)T_2 &= \sum_{p\neq 0}(A_p \sinh(\tau_p)^2 + B_p\cosh(\tau_p)\sinh(\tau_p)) \\
&+ \sum_{p\neq 0}(A_p(\cosh(\tau_p)^2 + \sinh(\tau_p)^2) + 2B_p\cosh(\tau_p)\sinh(\tau_p))a_p^* a_p \\
&+ \sum_{p\neq 0}\Big(A_p\cosh(\tau_p)\sinh(\tau_p) + \frac{B_p}{2}(\cosh(\tau_p)^2+\sinh(\tau_p)^2)\Big)(a_p^* a_{-p}^* + a_p a_{-p}).
\end{aligned}
$$

We have

$$
\begin{aligned}
\cosh(\tau_p)^2 + \sinh(\tau_p)^2 = 1 + 2\sinh(\tau_p)^2 = 1 + \frac{A_p - \sqrt{A_p^2 - B_p^2}}{\sqrt{A_p^2-B_p^2}} &= \frac{A_p}{\sqrt{A_p^2-B_p^2}}, \\
\cosh(\tau_p)\sinh(\tau_p) = \sinh(\tau_p)\sqrt{\sinh(\tau_p)^2+1} &= \nu_p\sqrt{\nu_p^2+1} \\
&= -\operatorname{sgn}(B_p)\sqrt{\frac{1}{2}\Big(\frac{A_p}{\sqrt{A_p^2-B_p^2}} - 1\Big)}\sqrt{\frac{1}{2}\Big(\frac{A_p}{\sqrt{A_p^2-B_p^2}} + 1\Big)} \\
&= -\frac{\operatorname{sgn}(B_p)}{2}\sqrt{\frac{A_p^2}{A_p^2-B_p^2} - 1} = -\frac{1}{2}\frac{B_p}{\sqrt{A_p^2-B_p^2}}, \\
\sinh(\tau_p)^2 = \frac{1}{2}\frac{A_p - \sqrt{A_p^2-B_p^2}}{\sqrt{A_p^2-B_p^2}}&.
\end{aligned}
$$

With these identities, we obtain

$$
\begin{aligned}
T_2^*\mathbb{H}_{\mathrm{Bog}}T_2 &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{|N\hat{\varepsilon}_{\ell,N}(p)|^2}{4p^2}\,\mathrm{d}p + \frac{1}{2}\sum_{p\neq 0}\Big(\sqrt{A_p^2-B_p^2} - A_p\Big) + \sum_{p\neq 0}\sqrt{A_p^2-B_p^2}a_p^* a_p \\
&= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{|N\hat{\varepsilon}_{\ell,N}(p)|^2}{4p^2}\,\mathrm{d}p + \frac{1}{2}\sum_{p\neq 0}\Big(\sqrt{p^4 + 2N\hat{\varepsilon}_{\ell,N}(p)p^2} - p^2 - N\hat{\varepsilon}_{\ell,N}(p)\Big) \\
&\quad + \sum_{p\neq 0}\sqrt{p^4 + 2N\hat{\varepsilon}_{\ell,N}(p)p^2}a_p^* a_p \\
&=: \Big(\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{|N\hat{\varepsilon}_{\ell,N}(p)|^2}{4p^2}\,\mathrm{d}p - \sum_{p\neq 0}\frac{|N\hat{\varepsilon}_{\ell,N}(p)|^2}{4p^2}\Big) + e_{\mathrm{Bog}} + \sum_{p\neq 0} e_p a_p^* a_p. \quad (5.4)
\end{aligned}
$$

The main result in this chapter is the following lemma.

Lemma 5.1. *Assume that $4RN^{-1} < \ell < 1/2$, $MN^{-1} \leqslant \ell$, and ℓ is small enough. We have*

$$
T_2^* T_c^* T_1^* \mathcal{H} T_1 T_c T_2 = E_N + \mathrm{d}\Gamma(E_{\mathrm{Bog}}) + T_2^* Q_4 T_2 + \mathcal{E}^{(T_2)}
$$

on $\mathcal{F}_+$, with Q_4 as in Lemma 4.1,

$$\mathrm{d}\Gamma(E_{\mathrm{Bog}}) = \sum_{p\in\Lambda^*\setminus\{0\}} \sqrt{p^4+16\pi\mathfrak{a}p^2}a_p^*a_p,$$

$$E_N = 4\pi\mathfrak{a}(N-1) + e_\Lambda \mathfrak{a}^2 + \frac{1}{2}\sum_{p\in\Lambda^*\setminus\{0\}}\left(\sqrt{p^4+16\pi\mathfrak{a}p^2} - p^2 - 8\pi\mathfrak{a} + \frac{(8\pi\mathfrak{a})^2}{2p^2}\right),$$

$$e_\Lambda = 4\left(\int_\Lambda \frac{1}{p^2}\,\mathrm{d}p + \sum_{z\in\mathbb{Z}^3\setminus\{0\}}\int_\Lambda\left(\frac{1}{(p+z)^2} - \frac{1}{z^2}\right)\mathrm{d}p\right),$$

and

$$\begin{aligned}\pm\mathcal{E}^{(T_2)} \leqslant{}& CNR^{-\gamma} + CM^{1/2}N^{-1/2}(\mathrm{d}\Gamma(-\Delta) + T_2^*Q_4T_2 + \ell^{-1}(\mathcal{N}+1))\\ &+ C\ell^2\,\mathrm{d}\Gamma(-\Delta) + C\ell^4T_2^*Q_4T_2 + C\ell^{1/2}(\mathcal{N}+1) + C\ell^{1/2}\frac{(\mathcal{N}+1)^{3/2}}{N^{1/2}}\\ &+ \frac{1}{4}T_2^*Q_4T_2 + \varepsilon(T_2^*Q_4T_2 + \mathcal{N}+1)\\ &+ \varepsilon^{-1}C\left((R^{-\gamma}+\ell)(\mathcal{N}+1) + \frac{(\mathcal{N}+1)^2}{N} + C_j\frac{(\mathcal{N}+1)^{j+1}}{M^j}\right)\end{aligned} \tag{5.5}$$

for all $j\geqslant 1$ and $0<\varepsilon\leqslant 1$.

For the proof of Lemma 5.1 we need the following lemma, which essentially shows that we can replace $N\hat{\varepsilon}_{\ell,N}(p)$ with $8\pi\mathfrak{a}$ in (5.4).

Lemma 5.2. *Assume that $4RN^{-1}<\ell<1/2$, $MN^{-1}\leqslant\ell$, and ℓ is small enough. We have the pointwise bound*

$$\left|e_p - \sqrt{p^4+16\pi\mathfrak{a}p^2}\right| \leqslant C\ell^2p^2 + CR^{-\gamma}. \tag{5.6}$$

Moreover, we have

$$\left|e_{\mathrm{Bog}} - \frac{1}{2}\sum_{p\in\Lambda^*\setminus\{0\}}\left(\sqrt{p^4+16\pi\mathfrak{a}p^2} - p^2 - 8\pi\mathfrak{a} + \frac{(8\pi\mathfrak{a})^2}{2p^2}\right)\right| \leqslant C\ell + CR^{-\gamma}, \tag{5.7}$$

$$\left|\left(\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{|N\hat{\varepsilon}_{\ell,N}(p)|^2}{4p^2}\,\mathrm{d}p - \sum_{p\in\Lambda^*\setminus\{0\}}\frac{|N\hat{\varepsilon}_{\ell,N}(p)|^2}{4p^2}\right) - e_\Lambda\mathfrak{a}^2\right| \leqslant C\ell^{1/2} + CR^{-\gamma}. \tag{5.8}$$

We will need the following bounds on the number operator, the kinetic term $\mathrm{d}\Gamma(-\Delta)$, and the term Q_4.

Lemma 5.3. *Assume that $4RN^{-1}<\ell<1/2$, $MN^{-1}\leqslant\ell$, and ℓ is small enough. We have*

$$T_2^{-t}(\mathcal{N}+1)^jT_2^t \leqslant C_j(\mathcal{N}+1)^j, \tag{5.9}$$

$$T_2^*\,\mathrm{d}\Gamma(-\Delta)T_2 \leqslant C(\mathrm{d}\Gamma(-\Delta)+\ell^{-1}), \tag{5.10}$$

$$T_2^*Q_4T_2 \leqslant CQ_4 + CN^{-1}\ell^{-2} + CN^{-1}(\mathcal{N}+1)^2, \tag{5.11}$$

for all $j\geqslant 1$ and $t\in[0,1]$.

We begin by proving Lemma 5.2 and Lemma 5.3. At the end of the chapter, we will prove Lemma 5.1.

Proof of Lemma 5.2. First, we prove (5.6). Using the inequality $|\sqrt{1+x}-\sqrt{1+y}| \leqslant C|x-y|$ for $x, y \geqslant -1/2$ and the bounds (5.2) and (2.6), we obtain

$$\left|e_p - \sqrt{p^4 + 16\pi\mathfrak{a}p^2}\right| \leqslant C|N\hat{\varepsilon}_{\ell,N}(p) - 8\pi\mathfrak{a}_R| + C|\mathfrak{a}_R - \mathfrak{a}| \leqslant C\ell^2 p^2 + CR^{-\gamma}.$$

Next, let us prove (5.7). We have

$$v(x) = \sqrt{1+2x} - 1 - x + \frac{x^2}{2} = \mathcal{O}(x^3), \quad v'(x) = \mathcal{O}(x^2),$$

and $|v(x) - v(y)| = |v'(\theta x + (1-\theta)y)||x-y|$ for some $\theta = \theta(x,y) \in [0,1]$. Applying this to

$$x = N\hat{\varepsilon}_{\ell,N}(p)p^{-2}, \quad y = 8\pi\mathfrak{a}p^{-2},$$

and using the bounds (5.2) and (2.6), we obtain

$$\begin{aligned} p^2\left|v(N\hat{\varepsilon}_{\ell,N}(p)p^{-2}) - v(8\pi\mathfrak{a}p^{-2})\right| &\leqslant p^2\left(Cp^{-4}|N\hat{\varepsilon}_{\ell,N}(p)p^{-2} - 8\pi\mathfrak{a}p^{-2}|\right) \\ &\leqslant C\frac{|N\hat{\varepsilon}_{\ell,N}(p) - 8\pi\mathfrak{a}_R| + |8\pi\mathfrak{a}_R - 8\pi\mathfrak{a}|}{p^4} \leqslant C\frac{\min(1,\ell^2p^2) + R^{-\gamma}}{p^4}. \end{aligned}$$

Hence, we have

$$\begin{aligned} &\left|e_{\mathrm{Bog}} - \frac{1}{2}\sum_{p\in\Lambda^*\setminus\{0\}}\left(\sqrt{p^4 + 16\pi\mathfrak{a}p^2} - p^2 - 8\pi\mathfrak{a} + \frac{(8\pi\mathfrak{a})^2}{2p^2}\right)\right| \\ &\leqslant C\sum_{p\in\Lambda^*\setminus\{0\}}\frac{\min(1,\ell^2p^2)+R^{-\gamma}}{p^4} \leqslant C\sum_{0<|p|\leqslant\ell^{-1}}\frac{\ell^2}{p^2} + C\sum_{|p|>\ell^{-1}}\frac{1}{p^4} + CR^{-\gamma} \leqslant C\ell + CR^{-\gamma}. \end{aligned}$$

Finally, we prove (5.8). We compute

$$\begin{aligned} &\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{|N\hat{\varepsilon}_{\ell,N}(p)|^2}{4p^2}\,\mathrm{d}p - \sum_{p\in\Lambda^*\setminus\{0\}}\frac{|N\hat{\varepsilon}_{\ell,N}(p)|^2}{4p^2} \\ &\quad = \int_{\mathbb{R}^3}\frac{|N\hat{\varepsilon}_{\ell,N}(2\pi p)|^2}{4(2\pi p)^2}\,\mathrm{d}p - \sum_{z\in\mathbb{Z}^3\setminus\{0\}}\frac{|N\hat{\varepsilon}_{\ell,N}(2\pi z)|^2}{4(2\pi z)^2} \\ &\quad = \int_{\Lambda}\frac{|N\hat{\varepsilon}_{\ell,N}(2\pi p)|^2}{4(2\pi p)^2}\,\mathrm{d}p + \sum_{z\in\mathbb{Z}^3\setminus\{0\}}\left(\int_{\Lambda+z}\frac{|N\hat{\varepsilon}_{\ell,N}(2\pi p)|^2}{4(2\pi p)^2} - \frac{|N\hat{\varepsilon}_{\ell,N}(2\pi z)|^2}{4(2\pi z)^2}\,\mathrm{d}p\right). \end{aligned}$$

In the first term we can replace $N\hat{\varepsilon}_{\ell,N}(2\pi p)$ with $8\pi\mathfrak{a}_R$, up to an error term, by using the bound $\|N\hat{\varepsilon}_{\ell,N}\|_{L^\infty} \leqslant \|N\varepsilon_{\ell,N}\|_{L^1} \leqslant C$ and (5.2). We find

$$\left|\int_{\Lambda}\frac{|N\hat{\varepsilon}_{\ell,N}(2\pi p)|^2}{4(2\pi p)^2} - \frac{|8\pi\mathfrak{a}_R|^2}{4(2\pi p)^2}\,\mathrm{d}p\right| \leqslant C\int_{\Lambda}\frac{|N\hat{\varepsilon}_{\ell,N}(2\pi p) - 8\pi\mathfrak{a}_R|}{4(2\pi p)^2}\,\mathrm{d}p \leqslant C\ell^2.$$

For the second term, we separate the sum into terms with $0 < |z| \leqslant \ell^{-1/2}$ and terms with $|z| > \ell^{-1/2}$. The sum over $|z| > \ell^{-1/2}$ is an error term and can be bounded in the following way. First, we compute for $|z| > \ell^{-1/2}$

$$\begin{aligned}
&\int_{\Lambda+z} \frac{|N\hat{\varepsilon}_{\ell,N}(2\pi p)|^2}{4(2\pi p)^2} - \frac{|N\hat{\varepsilon}_{\ell,N}(2\pi z)|^2}{4(2\pi z)^2}\,\mathrm{d}p \\
&\quad= \frac{1}{16\pi^2}\int_\Lambda \Bigg(\frac{z^2\big(|N\hat{\varepsilon}_{\ell,N}(2\pi(p+z))|^2 - |N\hat{\varepsilon}_{\ell,N}(2\pi z)|^2\big)}{(p+z)^2z^2} - \frac{2p\cdot z|N\hat{\varepsilon}_{\ell,N}(2\pi z)|^2}{(p+z)^2z^2} \\
&\qquad - \frac{p^2|N\hat{\varepsilon}_{\ell,N}(2\pi z)|^2}{(p+z)^2z^2}\Bigg)\,\mathrm{d}p \\
&\quad= \frac{1}{16\pi^2}\int_\Lambda \Bigg(\Bigg\{ \frac{z^2\big(N\hat{\varepsilon}_{\ell,N}(2\pi(p+z)) - N\hat{\varepsilon}_{\ell,N}(2\pi z) - p\cdot N\nabla\hat{\varepsilon}_{\ell,N}(2\pi z)\big)}{(p+z)^2z^2} \\
&\qquad\quad \times\big(N\hat{\varepsilon}_{\ell,N}(2\pi(p+z)) + N\hat{\varepsilon}_{\ell,N}(2\pi z)\big)\Bigg\} \\
&\qquad + \Bigg\{ \frac{p\cdot N\nabla\hat{\varepsilon}_{\ell,N}(2\pi z)(N\hat{\varepsilon}_{\ell,N}(2\pi(p+z)) + N\hat{\varepsilon}_{\ell,N}(2\pi z))}{(p+z)^2z^2}\Bigg\} \\
&\qquad - \Bigg\{ \frac{2p\cdot z|N\hat{\varepsilon}_{\ell,N}(2\pi z)|^2}{(p+z)^2z^2}\Bigg\} - \Bigg\{ \frac{p^2|N\hat{\varepsilon}_{\ell,N}(2\pi z)|^2}{(p+z)^2z^2}\Bigg\}\Bigg)\,\mathrm{d}p \\
&\quad=: \frac{1}{16\pi^2}\big(G_1(z) + G_2(z) + G_3(z) + G_4(z)\big).
\end{aligned}$$

To bound $G_1(z)$, we use

$$\begin{aligned}
&|N\hat{\varepsilon}_{\ell,N}(2\pi(p+z)) - N\hat{\varepsilon}_{\ell,N}(2\pi z) - p\cdot N\nabla\hat{\varepsilon}_{\ell,N}(2\pi z)||N\hat{\varepsilon}_{\ell,N}(2\pi(p+z)) + N\hat{\varepsilon}_{\ell,N}(2\pi z)| \\
&\qquad\leqslant C\|N\Delta\hat{\varepsilon}_{\ell,N}\|_{L^\infty}\||\cdot|^2N\hat{\varepsilon}_{\ell,N}\|_{L^\infty}p^2\big((p+z)^{-2} + z^{-2}\big) \\
&\qquad\leqslant Cp^2\big((p+z)^{-2} + z^{-2}\big),
\end{aligned}$$

which follows from a Taylor expansion, the bound $\|N\Delta\hat{\varepsilon}_{\ell,N}\|_{L^\infty} \leqslant \||\cdot|^2N\varepsilon_{\ell,N}\|_{L^1} \leqslant C\ell^2$, and the bound $\||\cdot|^2N\hat{\varepsilon}_{\ell,N}\|_{L^\infty} \leqslant \|N\Delta\varepsilon_{\ell,N}\|_{L^1} \leqslant C\ell^{-2}$ from Lemma 2.2. Thus, we have

$$|G_1(z)| \leqslant Cz^{-4}.$$

Next, we notice that $G_2(z) = -G_2(-z)$, since $\hat{\varepsilon}_{\ell,N}$ is an even function. From this symmetry, we deduce

$$\sum_{|z|>\ell^{-1/2}} G_2(z) = \sum_{\substack{z\in(\mathbb{N}\times\mathbb{Z}^2)\cup(\{0\}\times\mathbb{N}\times\mathbb{Z})\cup(\{0\}\times\{0\}\times\mathbb{N}),\\ |z|>\ell^{-1/2}}} G_2(z) + G_2(-z) = 0.$$

For $G_3(z)$, we use $\|N\hat{\varepsilon}_{\ell,N}\|_{L^\infty} \leqslant \|N\varepsilon_{\ell,N}\|_{L^1} \leqslant C$ and

$$\int_\Lambda p\cdot z\,\mathrm{d}p = 0$$

to obtain

$$|G_3(z)| = \left| \int_\Lambda \frac{2p\cdot z|N\hat{\varepsilon}_{\ell,N}(2\pi z)|^2}{z^4} - \frac{2p\cdot z|N\hat{\varepsilon}_{\ell,N}(2\pi z)|^2}{(p+z)^2 z^2}\,\mathrm{d}p \right|$$
$$\leqslant C\int_\Lambda \frac{2|p\cdot z|^2 + |p\cdot z|p^2}{(p+z)^2 z^4}\,\mathrm{d}p \leqslant Cz^{-4}.$$

Finally, we use $\|N\hat{\varepsilon}_{\ell,N}\|_{L^\infty} \leqslant \|N\varepsilon_{\ell,N}\|_{L^1} \leqslant C$ to bound

$$|G_4(z)| \leqslant Cz^{-4}.$$

In conclusion, we have

$$\left| \sum_{|z|>\ell^{-1/2}} \left(\int_{\Lambda+z} \frac{|N\hat{\varepsilon}_{\ell,N}(2\pi p)|^2}{4(2\pi p)^2} - \frac{|N\hat{\varepsilon}_{\ell,N}(2\pi z)|^2}{4(2\pi z)^2}\,\mathrm{d}p \right) \right| \leqslant C \sum_{|z|>\ell^{-1/2}} \frac{1}{z^4} \leqslant C\ell^{1/2}.$$

For the sum over $0 < |z| \leqslant \ell^{-1/2}$, we write

$$\begin{aligned}
&\int_{\Lambda+z} \frac{|N\hat{\varepsilon}_{\ell,N}(2\pi p)|^2}{4(2\pi p)^2} - \frac{|N\hat{\varepsilon}_{\ell,N}(2\pi z)|^2}{4(2\pi z)^2}\,\mathrm{d}p \\
&\quad = \left\{ \int_{\Lambda+z} \frac{|N\hat{\varepsilon}_{\ell,N}(2\pi p)|^2}{4(2\pi p)^2} - \frac{|8\pi\mathfrak{a}_R|^2}{4(2\pi p)^2}\,\mathrm{d}p \right\} + \left\{ \frac{|8\pi\mathfrak{a}_R|^2}{4(2\pi z)^2} - \frac{|N\hat{\varepsilon}_{\ell,N}(2\pi z)|^2}{4(2\pi z)^2} \right\} \\
&\qquad + \left\{ \int_{\Lambda+z} \frac{|8\pi\mathfrak{a}_R|^2}{4(2\pi p)^2} - \frac{|8\pi\mathfrak{a}_R|^2}{4(2\pi z)^2}\,\mathrm{d}p \right\} \\
&\quad =: I_1(z) + I_2(z) + I_3(z).
\end{aligned}$$

Again, by using the bound $\|N\hat{\varepsilon}_{\ell,N}\|_{L^\infty} \leqslant \|N\varepsilon_{\ell,N}\|_{L^1} \leqslant C$ and (5.2), we find

$$|I_1(z)| \leqslant \int_{\Lambda+z} \frac{|N\hat{\varepsilon}_{\ell,N}(2\pi p) - 8\pi\mathfrak{a}_R|}{4(2\pi p)^2}\,\mathrm{d}p \leqslant C\ell^2$$

and similarly,

$$|I_2(z)| \leqslant C\ell^2.$$

This shows the bound

$$\sum_{0<|z|\leqslant\ell^{-1/2}} |I_1(z) + I_2(z)| \leqslant \sum_{0<|z|\leqslant\ell^{-1/2}} C\ell^2 \leqslant C\ell^{1/2}.$$

Similarly to $G_3(z)$ and $G_4(z)$, we can bound

$$|I_3(z)| \leqslant Cz^{-4}.$$

This implies

$$\left| \sum_{|z|>\ell^{-1/2}} I_3(z) \right| \leqslant C\ell^{1/2}.$$

Summing up all the terms, we obtain

$$
\begin{aligned}
&\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{|N\hat{\varepsilon}_{\ell,N}(p)|^2}{4p^2}\,\mathrm{d}p-\sum_{p\in\Lambda^*\setminus\{0\}}\frac{|N\hat{\varepsilon}_{\ell,N}(p)|^2}{4p^2}\\
&\qquad\qquad=4\mathfrak{a}_R^2\left(\int_\Lambda\frac{1}{p^2}\,\mathrm{d}p+\sum_{z\in\mathbb{Z}^3\setminus\{0\}}\int_\Lambda\left(\frac{1}{(p+z)^2}-\frac{1}{z^2}\right)\mathrm{d}p\right)+\mathcal{E}=e_\Lambda\mathfrak{a}_R^2+\mathcal{E},
\end{aligned}
$$

with

$$|\mathcal{E}|\leqslant C\ell^{1/2}.$$

By (2.6) we have $|\mathfrak{a}^2-\mathfrak{a}_R^2|\leqslant C|\mathfrak{a}-\mathfrak{a}_R|\leqslant CR^{-\gamma}$, which finishes the proof. □

Proof of Lemma 5.3. The first bound (5.9) follows from Lemma 2.8.

For the proof of (5.10), we compute

$$
\begin{aligned}
T^*\,\mathrm{d}\Gamma(-\Delta)T&=T^*\Big(\sum_{p\neq0}p^2a_p^*a_p\Big)T\\
&=\sum_{p\neq0}p^2\big(\cosh(\tau_p)a_p^*+\sinh(\tau_p)a_{-p}\big)\big(\cosh(\tau_p)a_p+\sinh(\tau_p)a_{-p}^*\big)\\
&=\sum_{p\neq0}p^2\big(\cosh(\tau_p)^2a_p^*a_p+\cosh(\tau_p)\sinh(\tau_p)(a_p^*a_{-p}^*+a_{-p}a_p)\\
&\qquad+\sinh(\tau_p)^2a_{-p}^*a_{-p}+\sinh(\tau_p)^2\big)\\
&\leqslant\sum_{p\neq0}p^2(1+2\sinh(\tau_p)^2)a_p^*a_p+\sum_{p\neq0}p^2\big(\cosh(\tau_p)^2a_p^*a_p+\sinh(\tau_p)^2a_{-p}a_{-p}^*\big)\\
&\qquad+\sum_{p\neq0}p^2\sinh(\tau_p)^2\\
&=2\sum_{p\neq0}p^2(1+2\nu_p^2)a_p^*a_p+2\sum_{p\neq0}p^2\nu_p^2.
\end{aligned}
$$

Using (5.3) and $B_p=N\hat{\varepsilon}_{\ell,N}(p)$, we obtain

$$T^*\,\mathrm{d}\Gamma(-\Delta)T\leqslant C\,\mathrm{d}\Gamma(-\Delta)+C\sum_{p\neq0}\frac{|N\hat{\varepsilon}_{\ell,N}(p)|^2}{p^2}.$$

To estimate the last term, we use that by (5.8), we have

$$\left|\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{|N\hat{\varepsilon}_{\ell,N}(p)|^2}{4p^2}\,\mathrm{d}p-\sum_{p\neq0}\frac{|N\hat{\varepsilon}_{\ell,N}(p)|^2}{4p^2}\right|\leqslant C.$$

By using (3.9) and Lemma 2.2, we can bound the integral. We obtain

$$\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{|N\hat{\varepsilon}_{\ell,N}(p)|^2}{4p^2}\,\mathrm{d}p=-\frac{N^2}{2}\int_\Lambda(\omega_{\ell,N}\varepsilon_{\ell,N})(x)\,\mathrm{d}x\leqslant C\ell^{-3}\int_{\mathbb{R}^3}\frac{\mathbb{1}_{\{|x|\leqslant\ell\}}}{|x|}\,\mathrm{d}x\leqslant C\ell^{-1}.$$

This finishes the proof of (5.10).

Finally, we prove (5.11). Using the Duhamel formula, we have

$$T^*Q_4T = Q_4 + \int_0^1 e^{-tK_2}[Q_4, K_2]e^{tK_2}\,\mathrm{d}t.$$

Computing the commutator yields

$$[Q_4, K_2] = \frac{1}{2}\sum_{p,q} \widehat{V}_N(p-q)\tau_q a_p^* a_{-p}^* + \sum_{p,q,r} \widehat{V}_N(r)\tau_p a_{p+r}^* a_{-p}^* a_q^* a_{q+r} + \text{h.c.}$$

Rewriting this in position space, we find the bound

$$\begin{aligned}
[Q_4, K_2] &= \frac{1}{2}\int_{\Lambda^2} V_N(x-y)\check{\tau}(x-y)a_x^* a_y^* + \int_{\Lambda^3} V_N(x-y)\check{\tau}(x-v)a_x^* a_y^* a_v^* a_y + \text{h.c.}\\
&\leqslant Q_4 + \int_{\Lambda^2} V_N(x-y)|\check{\tau}(x-y)|^2 + \int_{\Lambda^3} V_N(x-y)a_x^* a_y^* a_v^* (\mathcal{N}+1)^{-1} a_v a_y a_x\\
&\quad + \int_{\Lambda^3} V_N(x-y)|\check{\tau}(x-v)|^2 a_y^* (\mathcal{N}+1) a_y\\
&\leqslant CQ_4 + \|V_N\|_{L^1}\|\tau\|_1^2 + \|V_N\|_{L^1}\|\tau\|_2^2(\mathcal{N}+1)^2.
\end{aligned}$$

Using $|\tau_p| \leqslant |\nu_p|$, (5.3), and Lemma 2.2, we obtain

$$\begin{aligned}
\|\tau\|_1 \leqslant \|\nu\|_1 &\leqslant \sum_{p\neq 0}\frac{|N\hat{\varepsilon}_{\ell,N}(p)|}{p^2} \leqslant C\sum_{p\neq 0}\frac{\min(1, p^{-2}\|N\Delta\varepsilon_{\ell,N}\|_{L^1})}{p^2}\\
&\leqslant C\sum_{p\neq 0}\frac{\min(1, p^{-2}\ell^{-2})}{p^2} \leqslant C\sum_{0<|p|\leqslant \ell^{-1}}\frac{1}{p^2} + C\sum_{|p|>\ell^{-1}}\frac{\ell^{-2}}{p^4} \leqslant C\ell^{-1}
\end{aligned}$$

and

$$\|\tau\|_2^2 \leqslant \|\nu\|_2^2 \leqslant \sum_{p\neq 0}\frac{1}{p^4} \leqslant C.$$

Hence, applying (5.9), we have

$$\begin{aligned}
T^*Q_4T &= Q_4 + C\int_0^1 e^{-tK_2}(Q_4 + N^{-1}\ell^{-2} + N^{-1}(\mathcal{N}+1)^2)e^{tK_2}\,\mathrm{d}t\\
&\qquad \leqslant Q_4 + CN^{-1}\ell^{-2} + CN^{-1}(\mathcal{N}+1)^2 + C\int_0^1 e^{-tK_2}Q_4e^{tK_2}\,\mathrm{d}t.
\end{aligned}$$

The bound (5.11) now follows from the Grönwall lemma. □

Proof of Lemma 5.1. Applying Lemma 5.2 to (5.4), we obtain from Lemma 4.1

$$T_2^*T_c^*T_1^*\mathcal{H}T_1T_cT_2 = E_N + \mathrm{d}\Gamma(E_{\mathrm{Bog}}) + T_2^*Q_4T_2 + T_2^*\mathcal{E}^{(T_c)}T_2 + \mathcal{E},$$

with

$$\pm\mathcal{E} \leqslant C\ell^{1/2} + C\ell^2\,\mathrm{d}\Gamma(-\Delta) + CR^{-\gamma}(\mathcal{N}+1).$$

Using (4.3) and Lemma 5.3, we have

$$
\begin{aligned}
T_2^* \mathcal{E}^{(T_c)} T_2 \leqslant\, & CNR^{-\gamma} + CM^{1/2}N^{-1/2}\big(\mathrm{d}\Gamma(-\Delta) + T_2^* Q_4 T_2 + \ell^{-1}(\mathcal{N}+1)\big) \\
& + C\ell^2\, \mathrm{d}\Gamma(-\Delta) + C\ell^4 T_2^* Q_4 T_2 + C\ell^{1/2}(\mathcal{N}+1) + C\ell^{1/2}\frac{(\mathcal{N}+1)^{3/2}}{N^{1/2}} \\
& + \frac{1}{4} T_2^* Q_4 T_2 + \varepsilon\big(T_2^* Q_4 T_2 + \mathcal{N} + 1\big) \\
& + \varepsilon^{-1} C\left((R^{-\gamma} + \ell)(\mathcal{N}+1) + \frac{(\mathcal{N}+1)^2}{N} + C_j \frac{(\mathcal{N}+1)^{j+1}}{M^j}\right)
\end{aligned}
$$

for all $j \geqslant 1$ and $0 < \varepsilon \leqslant 1$. We simplified the bound using $MN^{-1} \leqslant \ell \leqslant 1$. $\square$

6 Optimal BEC

In this chapter, we will prove the following lemma. The proof is similar to that of [28, Lemma 31] and [17, Proposition 20] and uses a localization technique from [20].

Lemma 6.1. *On the truncated Fock space $\mathcal{F}_+^{\leqslant N}$, we have*

$$UH_NU^* = \mathbb{1}_+^{\leqslant N}\mathcal{H}\mathbb{1}_+^{\leqslant N} \geqslant 4\pi\mathfrak{a}N + C^{-1}\mathcal{N} - C,$$

for some constant $C > 0$ independent of N.

Proof. We introduce the notation $\mathcal{N}_+ = \mathcal{N} - a_0^*a_0$. Recall that for any normalized sequence Ψ_N of approximate ground states such that

$$\lim_{N\to\infty} \frac{\langle \Psi_N, H_N\Psi_N\rangle}{N} = 4\pi\mathfrak{a},$$

we made the assumption that (1.5) holds, which is equivalent to

$$\lim_{N\to\infty} \frac{\langle \Psi_N, \mathcal{N}_+\Psi_N\rangle}{N} = 0. \tag{6.1}$$

Let $f, g\colon \mathbb{R} \to [0,1]$ be smooth such that $f(x)^2 + g(x)^2 = 1$ for all $x \in \mathbb{R}$, $f(x) = 1$ for $x \leqslant 1/2$, and $f(x) = 0$ for $x \geqslant 1$. For $M_0 \geqslant 1$ we define $f_{M_0} \coloneqq f(\mathcal{N}/M_0)$ and $g_{M_0} \coloneqq g(\mathcal{N}/M_0)$. We have

$$\mathcal{H} = f_{M_0}\mathcal{H}f_{M_0} + g_{M_0}\mathcal{H}g_{M_0} + \mathcal{E}_{M_0},$$

where

$$\mathcal{E}_{M_0} = \frac{1}{2}([f_{M_0},[f_{M_0},\mathcal{H}]] + [g_{M_0},[g_{M_0},\mathcal{H}]]). \tag{6.2}$$

From (2.14), we easily obtain on $\mathcal{F}_+^{\leqslant N}$ with $h = f, g$

$$\begin{aligned}
&[h_{M_0},[h_{M_0},\mathcal{H}]] \\
&\quad = \frac{1}{2}\big(h((\mathcal{N}+2)/M_0) - h(\mathcal{N}/M_0)\big)^2\sqrt{(N-\mathcal{N})(N-\mathcal{N}-1)}\int_{\Lambda^2} V_N(x-y)a_xa_y + \text{h.c.} \\
&\qquad + \big(h((\mathcal{N}+1)/M_0) - h(\mathcal{N}/M_0)\big)^2\sqrt{N-\mathcal{N}}\int_{\Lambda^2} V_N(x-y)a_y^*a_xa_y + \text{h.c.},
\end{aligned}$$

which implies

$$\pm\mathcal{E}_{M_0} \leqslant CM_0^{-2}(Q_4 + N). \tag{6.3}$$

We can assume that $\gamma = 1$ and we choose $R = N\ell/5$, $M_0 = \delta_\alpha N$, and $M = \delta_\beta N$ for $\delta_\alpha, \delta_\beta > 0$ fixed and to be chosen later. Denoting $\mathcal{U} \coloneqq T_1 T_c T_2$, we obtain from Lemma 5.1 on $\mathcal{F}_+$

$$\mathcal{U}^* \mathcal{H} \mathcal{U} = E_N + \mathrm{d}\Gamma(E_{\mathrm{Bog}}) + T_2^* Q_4 T_2 + \mathcal{E}^{(T_2)} \geqslant 4\pi\mathfrak{a}N - C + \mathrm{d}\Gamma(-\Delta) + T_2^* Q_4 T_2 + \mathcal{E}^{(T_2)},$$

where

$$\begin{aligned}
\pm\mathcal{E}^{(T_2)} \leqslant{}& C\ell^{-1} + C\delta_\beta^{1/2}(\mathrm{d}\Gamma(-\Delta) + T_2^* Q_4 T_2 + \ell^{-1}(\mathcal{N}+1)) \\
&+ C\ell^2 \,\mathrm{d}\Gamma(-\Delta) + C\ell^4 T_2^* Q_4 T_2 + C\ell^{1/2}(\mathcal{N}+1) + C\ell^{1/2}\frac{(\mathcal{N}+1)^{3/2}}{N^{1/2}} \\
&+ \frac{1}{4} T_2^* Q_4 T_2 + \varepsilon(T_2^* Q_4 T_2 + \mathcal{N} + 1) \\
&+ \varepsilon^{-1} C\Big((N^{-1}\ell^{-1} + \ell)(\mathcal{N}+1) + \frac{(\mathcal{N}+1)^2}{N} + C_j \delta_\beta^{-j}\frac{(\mathcal{N}+1)^{j+1}}{N^j}\Big)
\end{aligned}$$

for all $j \geqslant 1$ and $0 < \varepsilon \leqslant 1$. By choosing $j = 1$, $\varepsilon = 1/4$, $\delta_\beta = \ell^4$, and ℓ sufficiently small and fixed and using $\mathcal{N} \leqslant \mathrm{d}\Gamma(-\Delta)$ on $\mathcal{F}_+$, for N large enough, we can absorb all terms proportional to $\mathcal{N}$ and $\mathrm{d}\Gamma(-\Delta)$ into $\mathrm{d}\Gamma(-\Delta)$ and all terms proportional to $T_2^* Q_4 T_2$ into $T_2^* Q_4 T_2$. Hence, we have on $\mathcal{F}_+$

$$\mathcal{U}^* \mathcal{H} \mathcal{U} \geqslant 4\pi\mathfrak{a}N + \frac{1}{4}(\mathrm{d}\Gamma(-\Delta) + T_2^* Q_4 T_2) - C\Big(\ell^{-1} + \ell^{1/2}\frac{(\mathcal{N}+1)^{3/2}}{N^{1/2}} + \ell^{-4}\frac{(\mathcal{N}+1)^2}{N}\Big).$$

We denote $\mathcal{N}_\mathcal{U} \coloneqq \mathcal{U}^* \mathcal{N} \mathcal{U}$. From the bounds in Lemma 2.8 and Lemma 4.2 follows

$$(\mathcal{N}+1)^j \leqslant C_j(\mathcal{N}_\mathcal{U}+1)^j \leqslant C_j'(\mathcal{N}+1)^j$$

for all $j \geqslant 1$. With this and the bounds $\mathrm{d}\Gamma(-\Delta) \geqslant \mathcal{N}$ and $Q_4 \geqslant 0$ on $\mathcal{F}_+$, we obtain

$$\begin{aligned}
\mathcal{U}^* f_{M_0}(\mathcal{H} - 4\pi\mathfrak{a}N) f_{M_0} \mathcal{U} &\geqslant f(\mathcal{N}_\mathcal{U}/M_0)\mathcal{U}^*(\mathcal{H} - 4\pi\mathfrak{a}N)\mathcal{U} f(\mathcal{N}_\mathcal{U}/M_0) \\
&\geqslant f(\mathcal{N}_\mathcal{U}/M_0)\Big(\frac{1}{4}(\mathrm{d}\Gamma(-\Delta) + T_2^* Q_4 T_2) \\
&\qquad - C\Big(\ell^{-1} + \ell^{1/2}\frac{(\mathcal{N}+1)^{3/2}}{N^{1/2}} + \ell^{-4}\frac{(\mathcal{N}+1)^2}{N}\Big)\Big) f(\mathcal{N}_\mathcal{U}/M_0) \\
&\geqslant \frac{1}{4} f(\mathcal{N}_\mathcal{U}/M_0)\mathcal{N} f(\mathcal{N}_\mathcal{U}/M_0) \\
&\qquad - C f(\mathcal{N}_\mathcal{U}/M_0)^2\Big(\ell^{-1} + \ell^{1/2}\frac{(\mathcal{N}_\mathcal{U}+1)^{3/2}}{N^{1/2}} + \ell^{-4}\frac{(\mathcal{N}_\mathcal{U}+1)^2}{N}\Big) \\
&\geqslant C f(\mathcal{N}_\mathcal{U}/M_0)^2(\mathcal{N}_\mathcal{U} - \ell^{-1} - (\ell^{1/2} M_0^{1/2} N^{-1/2} + \ell^{-4} M_0 N^{-1})\mathcal{N}_\mathcal{U}) \\
&\geqslant C f(\mathcal{N}_\mathcal{U}/M_0)^2((1 - \ell^{1/2}\delta_\alpha^{1/2} - \ell^{-4}\delta_\alpha)\mathcal{N}_\mathcal{U} - \ell^{-1}).
\end{aligned}$$

Choosing δ_α sufficiently small and fixed and conjugating back by $\mathcal{U}$, we have

$$f_{M_0}(\mathcal{H} - 4\pi\mathfrak{a}N) f_{M_0} \geqslant f_{M_0}^2(C^{-1}\mathcal{N} - C).$$

By an argument from [3, Proposition 6.1], we obtain

$$\mathbb{1}_+^{\leqslant N} g_{M_0}(\mathcal{H} - 4\pi\mathfrak{a}N) g_{M_0} \mathbb{1}_+^{\leqslant N} \geqslant C \mathbb{1}_+^{\leqslant N} g_{M_0}^2 N \geqslant C \mathbb{1}_+^{\leqslant N} g_{M_0}^2 \mathcal{N}.$$

Since, by (1.4), we have

$$\frac{1}{N} \mathbb{1}_+^{\leqslant N} g_{M_0}(\mathcal{H} - 4\pi\mathfrak{a}N) g_{M_0} \mathbb{1}_+^{\leqslant N} \geqslant \mathbb{1}_+^{\leqslant N} g_{M_0}^2 \left(\frac{\lambda_1(H_N)}{N} - 4\pi\mathfrak{a} \right) \xrightarrow{N\to\infty} 0,$$

we would otherwise find a normalized sequence $\{\xi_N\} \subset \operatorname{supp}(g_{M_0} \mathbb{1}_+^{\leqslant N}) \subset \{\mathcal{N}_+ \geqslant M_0/2\}$ such that (up to a subsequence)

$$\lim_{N\to\infty} \frac{1}{N} \langle \xi_N, (\mathcal{H} - 4\pi\mathfrak{a}N)\xi_N \rangle = 0.$$

This would contradict (6.1).

In conclusion, we have

$$\mathbb{1}_+^{\leqslant N} \mathcal{H} \mathbb{1}_+^{\leqslant N} \geqslant \mathbb{1}_+^{\leqslant N} (4\pi\mathfrak{a}N + C^{-1}\mathcal{N} - C - C\delta_\alpha^{-2} N^{-2} Q_4) \mathbb{1}_+^{\leqslant N}.$$

From (2.12), we can derive the straightforward bound

$$\mathbb{1}_+^{\leqslant N} \mathcal{H} \mathbb{1}_+^{\leqslant N} \geqslant \mathbb{1}_+^{\leqslant N} \left(\frac{1}{2} Q_4 - CN \right) \mathbb{1}_+^{\leqslant N}. \tag{6.4}$$

Dividing the second bound by N and adding it to the first, we obtain

$$\mathbb{1}_+^{\leqslant N} \mathcal{H} \mathbb{1}_+^{\leqslant N} \geqslant \mathbb{1}_+^{\leqslant N} (4\pi\mathfrak{a}N + C^{-1}\mathcal{N} - C) \mathbb{1}_+^{\leqslant N}.$$

This finishes the proof. $\square$

7 Proof of Theorem 1.1

We will use the notations $\mathcal{U} = T_1 T_c T_2$, $\widetilde{H}_N = H_N - E_N$ and $\widetilde{\mathcal{H}} = \mathcal{H} - E_N$ with E_N as in Lemma 5.1. The main result in this chapter is the following lemma. The proof is similar to the ones of [28, Lemma 32] and [17, Theorem 1].

Lemma 7.1. *Let $0 \leqslant \kappa < 1/13$ such that (1.2) is satisfied for*

$$\gamma = \frac{1+\kappa}{1-2\kappa}.$$

The eigenvalues $\lambda_1(\widetilde{H}_N) \leqslant \lambda_2(\widetilde{H}_N) \leqslant \cdots \leqslant \Theta$, with $1 \leqslant \Theta \leqslant N^{\kappa/(1-\vartheta)}$, of $\widetilde{H}_N$ on $\mathfrak{H}^N$ satisfy

$$\lambda_L(\widetilde{H}_N) = \lambda_L(\mathrm{d}\Gamma(E_{\mathrm{Bog}})) + \mathcal{O}(N^{-\kappa}\Theta^{1-\vartheta}),$$

where $0 \leqslant \vartheta < 1 - 12\kappa$ is arbitrarily chosen for $\gamma \in [6/5, 14/11)$ and

$$\vartheta = \frac{1-2\kappa}{3}$$

for $\gamma \in [1, 6/5)$. Here, $\mathrm{d}\Gamma(E_{\mathrm{Bog}})$ is understood as an operator on $\mathcal{F}_+$.

Proof. As in Chapter 6, we denote $R = N\ell/5$, $M_0 = \delta_\alpha N$, and $M = \delta_\beta N$, but with different parameters δ_α and δ_β. Let $1 \leqslant \Theta \leqslant N^{\kappa/(1-\vartheta)}$. We differentiate between two cases.

Case 1: $\gamma \in [6/5, 14/11)$. We choose $0 < \eta \leqslant (1 - 13\kappa)/2$ and

$$\vartheta = 1 - \frac{12\kappa}{1-2\eta} \in [1/13, 1 - 12\kappa).$$

Case 2: $\gamma \in [1, 6/5)$. We choose

$$\eta = \frac{6-5\gamma}{2\gamma} \quad \text{and} \quad \vartheta = 1 - \frac{12\kappa}{1-2\eta} = \frac{1-2\kappa}{3}.$$

In both cases, let

$$\delta_\alpha = N^{(-2-4\kappa-\eta)/5}\Theta^{-2(1+2\vartheta)/5}, \qquad \delta_\beta = N^{(-2-4\kappa+4\eta)/5}\Theta^{-2(1+2\vartheta)/5},$$
$$\ell = N^{-2\kappa}\Theta^{-2\vartheta}, \qquad \varepsilon = N^{-\kappa}\Theta^{-\vartheta},$$

and $j \geqslant 1$ such that

$$\eta \geqslant \frac{3\kappa}{j}.$$

The proof consists of two parts. In the first, we prove the lower bound on $\lambda_L(\widetilde{H}_N)$ and in the second, the upper bound on $\lambda_L(\widetilde{H}_N)$.

Proof of the lower bound on $\lambda_L(\widetilde{H}_N)$. We use the same localization technique as in Chapter 6. With f_{M_0} and g_{M_0} defined as in Chapter 6, we can write

$$\widetilde{\mathcal{H}} = f_{M_0}\widetilde{\mathcal{H}}f_{M_0} + g_{M_0}\widetilde{\mathcal{H}}g_{M_0} + \mathcal{E}_{M_0},$$

where $\mathcal{E}_{M_0}$ is given in (6.2). We denote by $Z \subset \mathfrak{H}^N$ the space generated by the first L eigenfunctions of $\widetilde{H}_N$. We define $Y \coloneqq UZ \subset \mathcal{F}_+^{\leqslant N}$ and let P_Y be the orthogonal projection onto Y. Since U is unitary, we have $\dim(Y) = \dim(Z) = L$. With the min-max principle, we obtain the quadratic form estimate

$$\lambda_L(\widetilde{H}_N) \geqslant P_Y U \widetilde{H}_N U^* P_Y = P_Y \widetilde{\mathcal{H}} P_Y = P_Y (f_{M_0}\widetilde{\mathcal{H}}f_{M_0} + g_{M_0}\widetilde{\mathcal{H}}g_{M_0} + \mathcal{E}_{M_0}) P_Y. \qquad (7.1)$$

We want to bound $f_{M_0}\widetilde{\mathcal{H}}f_{M_0}$, $g_{M_0}\widetilde{\mathcal{H}}g_{M_0}$, and $\mathcal{E}_{M_0}$ next.

By Lemma 5.1, we have

$$\mathcal{U}^*\widetilde{\mathcal{H}}_N\mathcal{U} = \mathrm{d}\Gamma(E_{\mathrm{Bog}}) + T_2^* Q_4 T_2 + \mathcal{E}^{(T_2)}. \qquad (7.2)$$

To bound $\mathcal{E}^{(T_2)}$ from below, we want to use the bound (5.5), the fact that $\mathrm{Ran}(f_{M_0}) \subset \mathcal{F}_+^{\leqslant M_0}$, and the bound $\mathcal{N} \leqslant \mathrm{d}\Gamma(-\Delta) \leqslant \mathrm{d}\Gamma(E_{\mathrm{Bog}})$ on $\mathcal{F}_+$. Since we have $\mathcal{N} \leqslant CM_0$ on $\mathcal{U}^*\mathcal{F}_+^{\leqslant M_0}$ by Lemma 2.8 and Lemma 4.2, with the choice of the parameters above, we find the bound

$$\mathcal{E}^{(T_2)} \geqslant -CN^{-\kappa}\Theta^{-\vartheta}\,\mathrm{d}\Gamma(E_{\mathrm{Bog}}) - CN^{-\kappa}\Theta^{1-\vartheta} - \left(\frac{1}{4} + CN^{-\kappa}\Theta^{-\vartheta}\right)T_2^* Q_4 T_2 \qquad (7.3)$$

on $\mathcal{U}^*\mathcal{F}_+^{\leqslant M_0}$. In both cases, we used $R^{-\gamma} \leqslant N^{-1} \leqslant \delta_\alpha = \delta_\beta N^{-\eta} \leqslant \delta_\beta \leqslant \ell = \varepsilon^2$ and

$$\begin{aligned} 1 - \gamma + 2\gamma\kappa &= -\kappa, \\ \frac{-1-2\kappa+2\eta}{5} + 2\kappa &= -\kappa + \left(\frac{13\kappa - 1 + 2\eta}{5}\right), \\ N^{(13\kappa-1+2\eta)/5}\Theta^{-(1+2\vartheta)/5}\Theta^{2\vartheta} &\leqslant \Theta^{(1-13\vartheta)/5}\Theta^{(8\vartheta-1)/5} = \Theta^{-\vartheta}, \\ N^{\kappa - j\eta}\Theta^{\vartheta} &\leqslant N^{-2\kappa}\Theta^{\vartheta} \leqslant N^{-\kappa}\Theta^{-\vartheta}, \end{aligned}$$

where the inequality in the second to last line is due to $\Theta^{(1-\vartheta)/\kappa} \leqslant N$. We also used

$$2\gamma\vartheta \leqslant \frac{3(1-12\kappa)}{1-2\kappa} - \vartheta \leqslant 1 - \vartheta,$$

which follows from the inequalities $\vartheta \leqslant 1 - 12\kappa$ and $\kappa \geqslant 1/17$, in case 1 and

$$2\gamma\vartheta = \frac{2+2\kappa}{3} = 1 - \vartheta$$

in case 2.

Thus, for large N, using $Q_4 \geqslant 0$, the lower bound on $\mathcal{F}_+$

$$f_{M_0}\widetilde{\mathcal{H}}f_{M_0} \geqslant (1 - CN^{-\kappa}\Theta^{-\vartheta})f_{M_0}\mathcal{U}\,\mathrm{d}\Gamma(E_{\mathrm{Bog}})\mathcal{U}^* f_{M_0} - CN^{-\kappa}\Theta^{1-\vartheta} \tag{7.4}$$

follows from (7.2) and (7.3).

Using Lemma 6.1 and the fact that $\mathrm{supp}(g_{M_0}) \subset \{\mathcal{N} \geqslant M_0/2\}$, we obtain for large N on $Y \subset \mathcal{F}_+^{\leqslant N}$

$$g_{M_0}\widetilde{\mathcal{H}}g_{M_0} \geqslant g_{M_0}(C^{-1}\mathcal{N} - C)g_{M_0} \geqslant g_{M_0}(C^{-1}M_0 - C)g_{M_0} \geqslant 0. \tag{7.5}$$

From the bounds (6.3) and (6.4), the fact that $Y \subset \mathcal{F}_+^{\leqslant N}$, the bound $\mathcal{H} \leqslant \lambda_1(H_N) + \lambda_L(\widetilde{H}_N)$ on Y, and the inequality $\Theta^{(1-\vartheta)/\kappa} \leqslant N$, we have on $\mathcal{F}_+$

$$\begin{aligned} P_Y\mathcal{E}_{M_0}P_Y &\geqslant -CM_0^{-2}P_Y(Q_4 + N)P_Y \geqslant -CM_0^{-2}P_Y(\mathcal{H} + N)P_Y \\ &\geqslant -CM_0^{-2}(\lambda_1(H_N) + \lambda_L(\widetilde{H}_N) + N) \\ &\geqslant -CM_0^{-2}N = -CN^{(-1+8\kappa+2\eta)/5}\Theta^{(4+8\vartheta)/5} \geqslant -CN^{-\kappa}\Theta^{1-\vartheta}. \end{aligned} \tag{7.6}$$

Inserting (7.4), (7.5), and (7.6) into (7.1), we obtain

$$\lambda_L(\widetilde{H}_N) \geqslant (1 - CN^{-\kappa}\Theta^{-\vartheta})P_Y f_{M_0}\mathcal{U}\,\mathrm{d}\Gamma(E_{\mathrm{Bog}})\mathcal{U}^* f_{M_0}P_Y - CN^{-\kappa}\Theta^{1-\vartheta}.$$

Testing against $\xi \in Y$ and taking the supremum, we find

$$\lambda_L(\widetilde{H}_N) \geqslant (1 - CN^{-\kappa}\Theta^{-\vartheta}) \sup_{0\neq\xi\in Y} \frac{\langle f_{M_0}\xi, \mathcal{U}\,\mathrm{d}\Gamma(E_{\mathrm{Bog}})\mathcal{U}^* f_{M_0}\xi\rangle}{\|\xi\|^2} - CN^{-\kappa}\Theta^{1-\vartheta}.$$

For $0 \neq \xi \in Y$, we have

$$\begin{aligned} \|g_{M_0}\xi\|^2 &\leqslant 2\langle\xi, (\mathcal{N}/M_0)\xi\rangle \leqslant C\langle\xi, (U\widetilde{H}_N U^* + C)\xi\rangle M_0^{-1} \\ &\leqslant C\|\xi\|^2(\lambda_L(\widetilde{H}_N) + C)M_0^{-1} \leqslant C\|\xi\|^2\Theta N^{(-3+4\kappa+\eta)/5}\Theta^{2(1+2\vartheta)/5} \\ &\leqslant C\|\xi\|^2 N^{-16/65} \leqslant C\|\xi\|^2 N^{-\kappa}\Theta^{-\vartheta}, \end{aligned}$$

where we used $g_{M_0} \leqslant \mathbb{1}^{\{\mathcal{N}\geqslant M_0/2\}}$ in the first, Lemma 6.1 in the second, and the bounds $\kappa < \frac{1}{13}$, $\eta \leqslant \frac{1}{2}$, $\vartheta \leqslant \frac{1}{3}$, and $1 \leqslant \Theta \leqslant N^{\kappa/(1-\vartheta)}$ in the fifth and sixth inequality. This implies

$$\frac{\|f_{M_0}\xi\|^2}{\|\xi\|^2} = 1 - \frac{\|g_{M_0}\xi\|^2}{\|\xi\|^2} \geqslant 1 - CN^{-\kappa}\Theta^{-\vartheta}.$$

Thus, using that $\mathcal{U}^*$ is unitary, we obtain

$$\begin{aligned} \lambda_L(\widetilde{H}_N) &\geqslant (1 - CN^{-\kappa}\Theta^{-\vartheta}) \sup_{0\neq\xi\in Y} \frac{\langle f_{M_0}\xi, \mathcal{U}\,\mathrm{d}\Gamma(E_{\mathrm{Bog}})\mathcal{U}^* f_{M_0}\xi\rangle}{\|f_{M_0}\xi\|^2} - CN^{-\kappa}\Theta^{1-\vartheta} \\ &= (1 - CN^{-\kappa}\Theta^{-\vartheta}) \sup_{0\neq\xi\in\mathcal{U}^* f_{M_0}Y} \frac{\langle\xi, \mathrm{d}\Gamma(E_{\mathrm{Bog}})\xi\rangle}{\|\xi\|^2} - CN^{-\kappa}\Theta^{1-\vartheta}. \end{aligned} \tag{7.7}$$

We want to show that for N large enough, we have $\dim(\mathcal{U}^* f_{M_0} Y) = \dim(f_{M_0} Y) = L$. It is enough to show that f_{M_0} is injective on Y. By contradiction, we assume there exists $\xi \in Y$ with $\|\xi\| = 1$ such that $f_{M_0}\xi = 0$. Using $\xi = (f_{M_0}^2 + g_{M_0}^2)\xi = g_{M_0}^2\xi$, the bounds $g_{M_0}^2 \leqslant \mathbb{1}^{\{\mathcal{N} \leqslant M_0/2\}} \leqslant 2(\mathcal{N}/M_0)$ and $U\widetilde{H}_N U^* \leqslant \lambda_L(\widetilde{H}_N) \leqslant \Theta$ on Y, and Lemma 6.1, we obtain

$$M_0 \leqslant 2\langle \xi, \mathcal{N}\xi\rangle \leqslant C^{-1}\langle \xi, U\widetilde{H}_N U^*\xi\rangle + C \leqslant C^{-1}\Theta + C,$$

which leads to a contradiction, since we have $M_0 \gg \Theta$. Thus, by the min-max principle, we have

$$\sup_{0\neq\xi\in\mathcal{U}^* f_{M_0} Y} \frac{\langle \xi, \mathrm{d}\Gamma(E_{\mathrm{Bog}})\xi\rangle}{\|\xi\|^2} \geqslant \lambda_L(\mathrm{d}\Gamma(E_{\mathrm{Bog}})).$$

Inserting this into (7.7), we obtain

$$\lambda_L(\widetilde{H}_N) \geqslant (1 - CN^{-\kappa}\Theta^{-\vartheta})\lambda_L(\mathrm{d}\Gamma(E_{\mathrm{Bog}})) - CN^{-\kappa}\Theta^{1-\vartheta}.$$

From this and the bound $\lambda_L(\widetilde{H}_N) \leqslant \Theta$, we get $\lambda_L(\mathrm{d}\Gamma(E_{\mathrm{Bog}})) \leqslant C\Theta$ and hence,

$$\lambda_L(\widetilde{H}_N) \geqslant \lambda_L(\mathrm{d}\Gamma(E_{\mathrm{Bog}})) + \mathcal{O}(N^{-\kappa}\Theta^{1-\vartheta}).$$

Proof of the upper bound on $\lambda_L(\widetilde{H}_N)$. For $L \geqslant 1$, the eigenvalue $\lambda_L(\mathrm{d}\Gamma(E_{\mathrm{Bog}}))$ is associated with a normalized eigenfunction of the form

$$\xi_L = \prod_{j=1}^{k} \frac{(a_{p_j}^*)^{n_j}}{\sqrt{n_j!}}\Omega,$$

for some $p_j \in \Lambda^* \setminus \{0\}$ and $k, n_j \in \mathbb{N}$. Here, $\Omega = (1, 0, 0, \dots) \in \mathcal{F}$ is the vacuum. Let $Y = \mathrm{span}\{\xi_n\}_{1\leqslant n\leqslant L}$ be the space generated by the first L eigenfunctions of $\mathrm{d}\Gamma(E_{\mathrm{Bog}})$ and denote by P_Y the orthogonal projection onto Y. By Lemma 2.8 and Lemma 4.2, we have

$$\begin{aligned}P_Y\mathcal{U}^*\mathcal{N}\mathcal{U}P_Y \leqslant CP_Y(\mathcal{N}+1)P_Y \leqslant CP_Y(\mathrm{d}\Gamma(E_{\mathrm{Bog}}) + 1)P_Y& \\ \leqslant C\big(\lambda_L(\mathrm{d}\Gamma(E_{\mathrm{Bog}})) + 1\big) \leqslant C\Theta \leqslant N&,\end{aligned}$$

for N large enough. Here we used $\lambda_L(\mathrm{d}\Gamma(E_{\mathrm{Bog}})) \leqslant C\Theta$, which follows from the lower bound on $\lambda_L(\widetilde{H}_N)$. Since $\mathcal{U}$ is a unitary operator on $\mathcal{F}_+$, this shows that $\mathcal{U}Y \subset \mathcal{F}_+^{\leqslant N}$ and we have $\dim \mathcal{U}Y = \dim Y = L$. Thus, by the min-max principle and (7.2), we obtain

$$\begin{aligned}\lambda_L(\widetilde{H}_N) &\leqslant \sup_{0\neq\xi\in\mathcal{U}Y} \frac{\langle \xi, U\widetilde{H}_N U^*\xi\rangle}{\|\xi\|^2} = \sup_{0\neq\xi\in Y} \frac{\langle \xi, \mathcal{U}^*\widetilde{\mathcal{H}}_N\mathcal{U}\xi\rangle}{\|\xi\|^2} \\ &\leqslant \sup_{0\neq\xi\in Y} \frac{\langle \xi, \mathrm{d}\Gamma(E_{\mathrm{Bog}})\xi\rangle}{\|\xi\|^2} + \sup_{0\neq\xi\in Y} \frac{\langle \xi, (T_2^*Q_4T_2 + \mathcal{E}^{(T_2)})\xi\rangle}{\|\xi\|^2} \\ &= \lambda_L(\mathrm{d}\Gamma(E_{\mathrm{Bog}})) + \sup_{0\neq\xi\in Y} \frac{\langle \xi, (T_2^*Q_4T_2 + \mathcal{E}^{(T_2)})\xi\rangle}{\|\xi\|^2}.\end{aligned} \tag{7.8}$$

It remains to bound $T_2^* Q_4 T_2$ and $\mathcal{E}^{(T_2)}$ on Y.

By the bound (5.11) and using $\mathcal{N} \leqslant \mathrm{d}\Gamma(E_{\mathrm{Bog}}) \leqslant C\Theta$ on Y, we have

$$P_Y T_2^* Q_4 T_2 P_Y \leqslant C P_Y Q_4 P_Y + CN^{-1}\ell^{-2} + CN^{-1}\Theta^2.$$

From [11, Lemma 5.3], we obtain

$$V_N(x-y) \leqslant CN^{-1}\|V\|_{L^1}(1-\Delta_x)(1-\Delta_y).$$

Therefore, we find the bound

$$Q_4 \leqslant CN^{-1}\,\mathrm{d}\Gamma(-\Delta)^2.$$

With $\mathrm{d}\Gamma(-\Delta) \leqslant \mathrm{d}\Gamma(E_{\mathrm{Bog}}) \leqslant C\Theta$ on Y, we conclude

$$P_Y T_2^* Q_4 T_2 P_Y \leqslant CN^{-1}(\ell^{-2} + \Theta^2) \leqslant CN^{4\kappa-1}\Theta^{4\vartheta} \leqslant CN^{-\kappa}. \tag{7.9}$$

Using (5.5), $\mathrm{Ran}(P_Y) \subset \mathcal{F}_+^{\leqslant C\Theta}$, the bound $\mathcal{N} \leqslant \mathrm{d}\Gamma(-\Delta) \leqslant \mathrm{d}\Gamma(E_{\mathrm{Bog}}) \leqslant C\Theta$ on Y, and (7.9), we obtain similarly to (7.3)

$$P_Y \mathcal{E}^{(T_2)} P_Y \leqslant CN^{-\kappa}\Theta^{1-\vartheta} + \left(\frac{1}{4} + CN^{-\kappa}\Theta^{-\vartheta}\right) P_Y T_2^* Q_4 T_2 P_Y \leqslant CN^{-\kappa}\Theta^{1-\vartheta}. \tag{7.10}$$

In conclusion, inserting (7.9) and (7.10) into (7.8), we find the desired upper bound

$$\lambda_L(\widetilde{H}_N) \leqslant \lambda_L(\mathrm{d}\Gamma(E_{\mathrm{Bog}})) + \mathcal{O}(N^{-\kappa}\Theta^{1-\vartheta}).$$

This finishes the proof of Lemma 7.1. □

To conclude the proof of Theorem 1.1, we note that $\lambda_1(\mathrm{d}\Gamma(E_{\mathrm{Bog}})) = 0$. Hence, by Lemma 7.1, we have $\lambda_1(H_N) = E_N + \lambda_1(\widetilde{H}_N) = E_N + \mathcal{O}(N^{-\kappa})$ for any admissible $0 \leqslant \kappa < 1/13$, which shows the first part of Theorem 1.1. This, in combination with Lemma 7.1, also implies the second part of the theorem, since the spectrum of $\widetilde{H}_N$ below a threshold $\Theta \leqslant N^{\kappa/(1-\vartheta)}$ consists of eigenvalues of the form

$$\sum_{p \in 2\pi\mathbb{Z}^3 \setminus \{0\}} n_p \sqrt{p^4 + 16\pi\mathfrak{a}p^2} + \mathcal{O}(N^{-\kappa}\Theta^{1-\vartheta})$$

with $n_p \in \mathbb{N}_0$ for all $p \in 2\pi\mathbb{Z}^3 \setminus \{0\}$, where only finitely many n_p are non-zero, and we have

$$\lambda_L(H_N - \lambda_1(H_N)) = \lambda_L(\widetilde{H}_N) + \mathcal{O}(N^{-\kappa})$$

for the L-th lowest eigenvalue $\lambda_L(H_N - \lambda_1(H_N))$ of $H_N - \lambda_1(H_N)$ below Θ.

A Appendix

We will prove some well-known results mentioned in Section 2.2, Section 2.3, and Section 2.4.

A.1 Proof of Lemma 2.6

We follow the proof in [26, Section 5.2].

Let A be a self-adjoint operator on $\mathfrak{H}$ with domain $D(A)$ and choose an orthonormal basis $\{u_n\}_{n\in\mathbb{N}}$ of $\mathfrak{H}$ with $u_n \in D(A)$ for all $n \in \mathbb{N}$. We denote $a_n^\sharp := a^\sharp(u_n)$ for $\sharp \in \{*, \cdot\}$. For the first part of the lemma, it suffices to prove

$$\sum_{i=1}^{N} A_i \Psi_N = \sum_{m,n=1}^{\infty} \langle u_m, A u_n \rangle a_m^* a_n \Psi_N = \int_\Lambda a_x^* A a_x \Psi_N \, \mathrm{d}x \tag{A.1}$$

for $\Psi_N \in D(A_1 + \cdots + A_N) \subset \mathfrak{H}^N$. We start by proving the first equality in (A.1). We compute

$$(a_m^* a_n \Psi_N)(x_1, \ldots, x_N) = \sum_{i=1}^{N} u_m(x_i) \int_\Lambda \overline{u_n(x)} \Psi_N(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N, x) \, \mathrm{d}x.$$

Thus, we have

$$\begin{aligned}
&\sum_{m,n=1}^{\infty} \langle u_m, A u_n \rangle (a_m^* a_n \Psi_N)(x_1, \ldots, x_N) \\
&\quad = \sum_{m,n=1}^{\infty} \langle u_m, A u_n \rangle \sum_{i=1}^{N} u_m(x_i) \int_\Lambda \overline{u_n(x)} \Psi_N(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N, x) \, \mathrm{d}x \\
&\quad = \sum_{i=1}^{N} \sum_{m=1}^{\infty} u_m(x_i) \Bigg(\sum_{n=1}^{\infty} \langle A u_m, u_n \rangle \langle u_n, \Psi_N(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N, \cdot) \rangle \Bigg) \\
&\quad = \sum_{i=1}^{N} \sum_{m=1}^{\infty} u_m(x_i) \langle A u_m, \Psi_N(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N, \cdot) \rangle \\
&\quad = \sum_{i=1}^{N} \sum_{m=1}^{\infty} u_m(x_i) \langle u_m, A \Psi_N(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N, \cdot) \rangle \\
&\quad = \sum_{i=1}^{N} A_i \Psi_N(x_1, \ldots, x_N).
\end{aligned}$$

The second equality in (A.1) follows from

$$
\begin{aligned}
\sum_{m,n=1}^{\infty} \langle u_m, A u_n\rangle a_m^* a_n \Psi_N &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \langle A u_m, u_n\rangle \int_\Lambda \overline{u_n(y)} a_m^* a_y \Psi_N \, \mathrm{d}y \\
&= \sum_{m=1}^{\infty} \int_\Lambda \overline{A u_m(y)} a_m^* a_y \Psi_N \, \mathrm{d}y \\
&= \int_\Lambda a_x^* \sum_{m=1}^{\infty} \Big(\int_\Lambda \overline{u_m(y)} A a_y \Psi_N \, \mathrm{d}y \Big) u_m(x) \, \mathrm{d}x \\
&= \int_\Lambda a_x^* A a_x \Psi_N \, \mathrm{d}x.
\end{aligned}
$$

Let $W \in L^1(\mathbb{R}^3)$ be an even, real-valued function. We think of $W \coloneqq W(x-y)$ as a two-body translation invariant multiplication operator on $\mathfrak{H} \otimes \mathfrak{H}$ with domain $D(W)$. Choose an orthonormal basis $\{u_n\}_{n\in\mathbb{N}}$ of $\mathfrak{H}$ with $u_m \otimes u_n \in D(W)$ for all $n, m \in \mathbb{N}$. For the second part of the lemma, it suffices to prove

$$
\begin{aligned}
2\Big\langle \Phi_N, \sum_{1\leqslant i<j\leqslant N} W_{ij} \Psi_N \Big\rangle &= \Big\langle \Phi_N, \sum_{m,n,p,q=1}^{\infty} \langle u_m \otimes u_n, W u_p \otimes u_q\rangle a_m^* a_n^* a_p a_q \Psi_N \Big\rangle \\
&= \Big\langle \Phi_N, \int_{\Lambda^2} W(x-y) a_x^* a_y^* a_x a_y \Psi_N \, \mathrm{d}x \, \mathrm{d}y \Big\rangle
\end{aligned}
\tag{A.2}
$$

for $\Phi_N, \Psi_N \in D(\sum_{1\leqslant i<j\leqslant N} W_{ij}) \subset \mathfrak{H}^N$. To prove the first equality in (A.2), we compute

$$
(a_p a_q \Psi_N)(x_1, \ldots, x_N) = \sqrt{N(N-1)} \int_{\Lambda^2} \overline{u_p(y) u_q(z)} \Psi_N(x_1, \ldots, x_{N-2}, y, z) \, \mathrm{d}y \, \mathrm{d}z.
$$

Therefore, using that $\{u_m \otimes u_n\}_{m,n\in\mathbb{N}}$ is an orthonormal basis for $\mathfrak{H} \otimes \mathfrak{H}$, we have

$$
\begin{aligned}
&\Big\langle \Phi_N, \sum_{m,n,p,q=1}^{\infty} \langle u_m \otimes u_n, W u_p \otimes u_q\rangle a_m^* a_n^* a_p a_q \Psi_N \Big\rangle \\
&\quad = \sum_{m,n,p,q=1}^{\infty} \langle u_m \otimes u_n, W u_p \otimes u_q\rangle \langle a_m a_n \Phi_N, a_p a_q \Psi_N\rangle \\
&\quad = N(N-1) \sum_{m,n,p,q=1}^{\infty} \langle u_m \otimes u_n, W u_p \otimes u_q\rangle \\
&\qquad \times \int_{\Lambda^{N-2}} \Big(\int_{\Lambda^2} u_m(y) u_n(z) \overline{\Phi_N(x_1, \ldots, x_{N-2}, y, z)} \, \mathrm{d}y \, \mathrm{d}z \Big) \\
&\qquad \times \Big(\int_{\Lambda^2} \overline{u_p(y) u_q(z)} \Psi_N(x_1, \ldots, x_{N-2}, y, z) \, \mathrm{d}y \, \mathrm{d}z \Big) \, \mathrm{d}x_1 \ldots \mathrm{d}x_{N-2} \\
&\quad = N(N-1) \sum_{p,q=1}^{\infty} \int_{\Lambda^{N-2}} \Big(\int_{\Lambda^2} W(y-z) u_p(y) u_q(z) \overline{\Phi_N(x_1, \ldots, x_{N-2}, y, z)} \, \mathrm{d}y \, \mathrm{d}z \Big)
\end{aligned}
$$

$$\times \left(\int_{\Lambda^2} \overline{u_p(y)u_q(z)} \Psi_N(x_1, \ldots, x_{N-2}, y, z) \, \mathrm{d}y \, \mathrm{d}z \right) \mathrm{d}x_1 \ldots \mathrm{d}x_{N-2}$$

$$= N(N-1) \int_{\Lambda^N} \overline{\Phi_N(x_1, \ldots, x_{N-2}, y, z)} W(y-z) \Psi_N(x_1, \ldots, x_{N-2}, y, z)$$

$$= N(N-1) \langle \Phi_N, W_{N-1,N} \Psi_N \rangle = 2 \left\langle \Phi_N, \sum_{1 \leqslant i < j \leqslant N} W_{ij} \Psi_N \right\rangle.$$

For the second equality in (A.2), again using that $\{u_m \otimes u_n\}_{m,n \in \mathbb{N}}$ is an orthonormal basis for $\mathfrak{H} \otimes \mathfrak{H}$, we compute

$$\left\langle \Phi_N, \sum_{m,n,p,q=1}^{\infty} \langle u_m \otimes u_n, W u_p \otimes u_q \rangle a_m^* a_n^* a_p a_q \Psi_N \right\rangle$$

$$= \sum_{m,n,p,q=1}^{\infty} \langle u_m \otimes u_n, W u_p \otimes u_q \rangle \langle a_m a_n \Phi_N, a_p a_q \Psi_N \rangle$$

$$= \sum_{m,n,p,q=1}^{\infty} \langle u_m \otimes u_n, W u_p \otimes u_q \rangle \int_{\Lambda^2} u_m(x) u_n(y) \langle a_x a_y \Phi_N, a_p a_q \Psi_N \rangle \, \mathrm{d}x \, \mathrm{d}y$$

$$= \sum_{p,q=1}^{\infty} \int_{\Lambda^2} W(x-y) u_p(x) u_q(y) \langle a_x a_y \Phi_N, a_p a_q \Psi_N \rangle \, \mathrm{d}x \, \mathrm{d}y$$

$$= \int_{\Lambda^2} \left\langle a_x a_y \Phi_N, W(x-y) \sum_{p,q=1}^{\infty} \left(\int_{\Lambda^2} \overline{u_p(z) u_q(v)} a_z a_v \Psi_N \, \mathrm{d}z \, \mathrm{d}v \right) u_p(x) u_q(y) \right\rangle \mathrm{d}x \, \mathrm{d}y$$

$$= \int_{\Lambda^2} \langle a_x a_y \Phi_N, W(x-y) a_x a_y \Psi_N \rangle \, \mathrm{d}x \, \mathrm{d}y$$

$$= \left\langle \Phi_N, \int_{\Lambda^2} W(x-y) a_x^* a_y^* a_x a_y \Psi_N \, \mathrm{d}x \, \mathrm{d}y \right\rangle$$

This finishes the proof of Lemma 2.6. $\square$

A.2 Action of the unitary transform U

In this section, we want to prove the identities in (2.11). We start by proving

$$U a^*(f) a(g) U^* = a^*(f) a(g)$$

for $f, g \in \mathfrak{H}_+$. Let $\xi = (\xi_0, \ldots, \xi_N) \in \mathcal{F}_+^{\leqslant N}$. Using the CCR, we have

$$U a^*(f) a(g) U^* \xi = U a^*(f) a(g) \sum_{k=0}^{N} \frac{(a_0^*)^{N-k}}{\sqrt{(N-k)!}} \xi_k$$

$$= U \sum_{k=0}^{N} \frac{(a_0^*)^{N-k}}{\sqrt{(N-k)!}} a^*(f) a(g) \xi_k$$

$$= \bigoplus_{j=0}^{N} Q^{\otimes j} \frac{a_0^{N-j}}{\sqrt{(N-j)!}} \sum_{k=0}^{N} \frac{(a_0^*)^{N-k}}{\sqrt{(N-k)!}} a^*(f) a(g) \xi_k$$

$$
\begin{aligned}
&= \bigoplus_{k=0}^{N} Q^{\otimes k} \frac{a_0^{N-k}(a_0^*)^{N-k}}{(N-k)!} a^*(f)a(g)\xi_k \\
&= \bigoplus_{k=0}^{N} a^*(f)a(g)\xi_k = a^*(f)a(g)\xi.
\end{aligned}
$$

Since $\mathcal{N} \equiv N$ on $\mathfrak{H}^N$, we can deduce

$$
Ua_0^* a_0 U^* = N - \sum_{p \in \Lambda^* \setminus \{0\}} U a_p^* a_p U^* = N - \sum_{p \in \Lambda^* \setminus \{0\}} a_p^* a_p = N - \mathcal{N}
$$

on $\mathcal{F}_+^{\leqslant N}$.

A similar computation as before yields

$$
\begin{aligned}
Ua^*(f)a_0 U^* \xi &= Ua^*(f)a_0 \sum_{k=0}^{N} \frac{(a_0^*)^{N-k}}{\sqrt{(N-k)!}} \xi_k \\
&= Ua_0 \sum_{k=0}^{N} \frac{(a_0^*)^{N-k}}{\sqrt{(N-k)!}} a^*(f)\xi_k \\
&= \bigoplus_{j=0}^{N} Q^{\otimes j} \frac{a_0^{N-j+1}}{\sqrt{(N-j)!}} \sum_{k=0}^{N} \frac{(a_0^*)^{N-k}}{\sqrt{(N-k)!}} a^*(f)\xi_k \\
&= \bigoplus_{k=0}^{N} Q^{\otimes k+1} \frac{a_0^{N-k}(a_0^*)^{N-k}}{\sqrt{(N-k-1)!(N-k)!}} a^*(f)\xi_k \\
&= \bigoplus_{k=0}^{N} a^*(f)\sqrt{N-k}\,\xi_k \\
&= a^*(f)\sqrt{N-\mathcal{N}}\,\xi
\end{aligned}
$$

for $f \in \mathfrak{H}_+$ and $\xi = (\xi_0, \ldots, \xi_N) \in \mathcal{F}_+^{\leqslant N}$.

Finally, we have

$$
Ua_0^* a(g) U^* = \left(Ua^*(g)a_0 U^*\right)^* = \left(a^*(g)\sqrt{N-\mathcal{N}}\right)^* = \sqrt{N-\mathcal{N}}\,a(g)
$$

on $\mathcal{F}_+^{\leqslant N}$ for $g \in \mathfrak{H}_+$.

A.3 Construction of the quadratic transform

Our construction for the quadratic transform in Section 2.4 is similar to the one in [26, Section 6.3]. We start by proving the Duhamel formula.

Lemma A.1 (Duhamel formula)**.** *Let $\mathscr{H}$ be a Hilbert space and $A\colon D(A) \to \mathscr{H}$ a densely defined operator on $\mathscr{H}$, such that iA is self-adjoint. Let $B\colon D(B) \to \mathscr{H}$ be another* *densely defined operator on $\mathscr{H}$ and let $D \subset D(A)$ such that $e^{tA}(D) \cup Ae^{tA}(D) \subset D(B)$ and $Be^{tA}(D) \subset D(A)$ for $t \in [0,1]$. Then, we have for $x, y \in D$*

$$\langle y, (e^{-A}Be^{A} - B)x\rangle = \int_0^1 \langle y, e^{-tA}[B,A]e^{tA}x\rangle \,\mathrm{d}t.$$

Proof. We first show $e^{tA}x \in D(A)$ for all $x \in D(A)$ and $t \in \mathbb{R}$. Since iA is self-adjoint, by the spectral theorem, we can find a measure space (Ω, μ), a measurable function $a\colon \Omega \to \mathbb{R}$, and a unitary operator $U\colon \mathscr{H} \to L^2(\Omega,\mu)$, such that $UiAU^* = M_a$ and $UD(A) = D(M_a) = \{u \in L^2 : au \in L^2\}$. Since e^{tA} is defined by $Ue^{tA}U^* = M_{e^{-ita}}$ and we have $|e^{-ita}u| = |u|$ for $u \in D(M_a)$, we can conclude that $e^{-ita}u \in D(M_a)$ for all $u \in D(M_a)$ and hence, $e^{tA}x \in D(A)$ for all $x \in D(A)$.

We define $f(t) = \langle e^{tA}y, Be^{tA}x\rangle$ for $x, y \in D$. Since $D \subset D(A)$, we have by Stone's theorem

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{tA}x = \lim_{s\to 0}\frac{e^{(t+s)A}x - e^{tA}x}{s} = Ae^{tA}x,$$

where we mean strong convergence. We compute

$$\begin{aligned}
f'(t) &= \langle \frac{\mathrm{d}}{\mathrm{d}t}e^{tA}y, Be^{tA}x\rangle + \lim_{s\to 0}\langle e^{tA}y, \frac{Be^{(t+s)A}x - Be^{tA}x}{s}\rangle \\
&= \langle Ae^{tA}y, Be^{tA}x\rangle + \lim_{s\to 0}\langle B^*e^{tA}y, \frac{e^{(t+s)A}x - e^{tA}x}{s}\rangle \\
&= \langle Ae^{tA}y, Be^{tA}x\rangle + \langle B^*e^{tA}y, \frac{\mathrm{d}}{\mathrm{d}t}e^{tA}x\rangle \\
&= \langle Ae^{tA}y, Be^{tA}x\rangle + \langle B^*e^{tA}y, Ae^{tA}x\rangle \\
&= \langle y, e^{-tA}[B,A]e^{tA}x\rangle,
\end{aligned}$$

where we used $A^* = -A$. Applying the fundamental theorem of calculus, we conclude

$$\langle y, (e^{-A}Be^{A} - B)x\rangle = f(1) - f(0) = \int_0^1 f'(t)\,\mathrm{d}t = \int_0^1 \langle y, e^{-tA}[B,A]e^{tA}x\rangle\,\mathrm{d}t. \qquad \square$$

Remark. Let $\mathscr{H} = \mathcal{F}(\mathfrak{H})$ for a closed subspace $\mathfrak{H} \subset L^2(\Lambda)$ and consider a relatively $\mathcal{N}$-bounded operator A on $\mathscr{H}$, i.e., we have $D(\mathcal{N}) \subset D(A)$ and constants $C_1, C_2 > 0$ exist, such that for all $\xi \in D(\mathcal{N})$

$$\|A\xi\| \leqslant C_1\|\mathcal{N}\xi\| + C_2\|\xi\|.$$

Let B be an operator with $\mathcal{F}^{\leqslant N}(\mathfrak{H}) \subset D(B)$ for all $N \in \mathbb{N}$, where we used the definition $\mathcal{F}^{\leqslant N}(\mathfrak{H}) = \bigoplus_{n=0}^{N}\mathfrak{H}^n$ for $N \in \mathbb{N}$. Additionally, assume there exists $M \in \mathbb{N}$ such that

$$A\big(\mathcal{F}^{\leqslant N}(\mathfrak{H})\big) \cup B\big(\mathcal{F}^{\leqslant N}(\mathfrak{H})\big) \subset \mathcal{F}^{\leqslant N+M}(\mathfrak{H})$$

for every $N \in \mathbb{N}$. In this case, we can choose

$$D = \bigcup_{N=1}^{\infty}\mathcal{F}^{\leqslant N}(\mathfrak{H}). \tag{A.3}$$

We will verify this for the operators considered in the following.

Let s be a real, symmetric Hilbert–Schmidt operator on the Hilbert space $\mathfrak{H} = L^2(\Lambda)$. We want to find a unitary transform T on the Fock space $\mathcal{F}(\mathfrak{H})$ such that

$$T^* a^*(g) T = a^*(c(g)) + a(s(\overline{g})), \quad \forall g \in \mathfrak{H}, \tag{A.4}$$

where $c = \sqrt{1+s^2}$. For that purpose, we define the symmetric operator

$$iK = \frac{i}{2} \int_{\Lambda^2} k(x,y)(a_x^* a_y^* - a_x a_y)\,\mathrm{d}x\,\mathrm{d}y =: iK^* - iK^\circ,$$

where $k = \sinh^{-1}(s)$, on the domain $D(\mathcal{N})$. To justify that iK is well-defined on $D(\mathcal{N})$, let $\xi' \in \mathcal{F}(\mathfrak{H})$ and $\xi \in D(\mathcal{N})$ and note that, using Cauchy–Schwarz, we have

$$\begin{aligned}
|\langle \xi', iK^\circ \xi\rangle| &\leqslant \frac{1}{2}\int_{\Lambda^2} |\langle k(x,y)\xi', a_x a_y \xi\rangle| \\
&\leqslant C \int_{\Lambda^2} \|k(x,y)\xi'\| \|a_x a_y \xi\| \\
&\leqslant \eta C \|k\|_{\mathrm{HS}}^2 \|\xi'\|^2 + \eta^{-1} C \|\mathcal{N}\xi\|^2 \\
&\leqslant C\|k\|_{\mathrm{HS}} \|\xi'\| \|\mathcal{N}\xi\|,
\end{aligned}$$

where we optimized over $\eta > 0$. This implies the bound $\|iK^\circ \xi\| \leqslant C\|k\|_{\mathrm{HS}}\|\mathcal{N}\xi\|$. Similarly, we can show $\|iK^*\xi\| \leqslant C\|k\|_{\mathrm{HS}}\|(\mathcal{N}+1)\xi\|$. Hence, iK is well-defined on $D(\mathcal{N})$ and we have

$$\|iK\xi\| \leqslant C\|k\|_{\mathrm{HS}}\|(\mathcal{N}+1)\xi\|. \tag{A.5}$$

We want to use [30, Theorem X.36] to show that iK is essentially self-adjoint. Note that $\mathcal{N}+1$ is a self-adjoint operator with $\mathcal{N}+1 \geqslant 1$. We have to verify that for all $\xi', \xi \in D((\mathcal{N}+1)^{1/2})$ we have

$$|\langle \xi', \xi\rangle_{Q(iK)}| \leqslant C\|(\mathcal{N}+1)^{1/2}\xi'\| \|(\mathcal{N}+1)^{1/2}\xi\| \tag{A.6}$$

and for all $\xi', \xi \in D((\mathcal{N}+1)^{3/2})$ we have

$$|\langle (\mathcal{N}+1)\xi', \xi\rangle_{Q(iK)} - \langle \xi', (\mathcal{N}+1)\xi\rangle_{Q(iK)}| \leqslant C\|(\mathcal{N}+1)^{1/2}\xi'\| \|(\mathcal{N}+1)^{1/2}\xi\|, \tag{A.7}$$

where $\langle \cdot, \cdot \rangle_{Q(iK)}$ denotes the quadratic form associated with iK. Let $\xi', \xi \in D(\mathcal{N})$. We compute, using Cauchy–Schwarz,

$$\begin{aligned}
|\langle \xi', iK^\circ \xi\rangle| &= \left| \langle (\mathcal{N}+1)^{1/2}\xi', \frac{i}{2}\int_{\Lambda^2} k(x,y)(\mathcal{N}+1)^{-1/2} a_x a_y \xi\rangle \right| \\
&\leqslant C \int_{\Lambda^2} \|k(x,y)(\mathcal{N}+1)^{1/2}\xi'\| \|(\mathcal{N}+1)^{-1/2} a_x a_y \xi\| \\
&\leqslant \eta C\|k\|_{\mathrm{HS}}^2 \|(\mathcal{N}+1)^{1/2}\xi'\|^2 + \eta^{-1} C\|(\mathcal{N}+1)^{1/2}\xi\|^2 \\
&\leqslant C\|k\|_{\mathrm{HS}} \|(\mathcal{N}+1)^{1/2}\xi'\| \|(\mathcal{N}+1)^{1/2}\xi\|,
\end{aligned}$$

where we optimized over $\eta > 0$. This implies (A.6). Using

$$[(\mathcal{N}+1), iK] = 2(iK^* + iK^\circ),$$

this also immediately implies (A.7). Thus, by [30, Theorem X.36], iK is essentially self-adjoint on $D(\mathcal{N})$ and therefore, we can define the unitary transform $T = \exp(\overline{K}) = \exp(-i(\overline{iK}))$ by the functional calculus. In the following, we will write K instead of $\overline{K}$ to simplify notation.

Let $N \in \mathbb{N}$ and $\xi \in \mathcal{F}^{\leqslant N}(\mathfrak{H})$. From (A.5), we see that K is bounded on $\mathcal{F}^{\leqslant M}(\mathfrak{H})$ for every $M \in \mathbb{N}$. Hence, we can write for $j \geqslant 0$ and $t > 0$ small enough

$$\begin{aligned}
\|\mathcal{N}^j e^{tK}\xi\| &= \left\| \sum_{m=0}^{\infty} \mathcal{N}^j \frac{t^m K^m}{m!}\xi \right\| \\
&= \left\| \sum_{m=0}^{\infty} \mathcal{N}^j \mathbb{1}^{\{\mathcal{N} \leqslant N+2m\}} \frac{t^m K^m}{m!} \mathbb{1}^{\{\mathcal{N} \leqslant N\}}\xi \right\| \\
&\leqslant \sum_{m=0}^{\infty} \left\| \mathcal{N}^j \mathbb{1}^{\{\mathcal{N} \leqslant N+2m\}} \right\|_{\mathrm{op}} \left\| \frac{t^m K^m}{m!} \mathbb{1}^{\{\mathcal{N} \leqslant N\}} \right\|_{\mathrm{op}} \|\xi\| \\
&\leqslant \sum_{m=0}^{\infty} (N+2m)^j (tC\|k\|_{\mathrm{HS}})^m \prod_{n=1}^{m} \frac{N+2n}{n} \|\xi\| \\
&\leqslant \sum_{m=0}^{\infty} (N+2m)^j (tC\|k\|_{\mathrm{HS}})^m (N+2)^N \prod_{n=N}^{\max(N,m)} \frac{N+2n}{n} \|\xi\| \\
&\leqslant \sum_{m=0}^{\infty} (N+2m)^j (tC\|k\|_{\mathrm{HS}})^m (N+2)^N \|\xi\| < \infty.
\end{aligned} \tag{A.8}$$

The expression on the right-hand side is finite if $|t| < (C\|k\|_{\mathrm{HS}})^{-1}$. This shows $e^{tK}(D) \subset D(\mathcal{N}^j)$ for $|t| < (C\|k\|_{\mathrm{HS}})^{-1}$ and all $j \geqslant 0$, with D as in (A.3). The following lemma implies that we even have $e^{tK}(D) \subset D$ for $|t| < (C\|k\|_{\mathrm{HS}})^{-1}$.

Lemma A.2. *Let $0 \leqslant t < (C\|k\|_{\mathrm{HS}})^{-1}$. On D, we have the bound*

$$e^{-tK}(\mathcal{N}+1)^j e^{tK} \leqslant \exp(C_j\|k\|_{\mathrm{HS}})(\mathcal{N}+1)^j, \quad \forall j \geqslant 1,$$

where C_j is a constant independent of s.

Proof. To simplify notation, we will write $A \leqslant B$ for $\langle \xi, A\xi \rangle \leqslant \langle \xi, B\xi \rangle$ for $\xi \in D$ and $A = B$ for $\langle \xi', A\xi \rangle = \langle \xi', B\xi \rangle$ for $\xi', \xi \in D$ and operators A and B, that are well-defined on D, in the following. Let $j \geqslant 1$. Since we have $e^{rK}(D) \subset D(\mathcal{N}^{j+1})$, $K(D(\mathcal{N}^{j+1})) \subset D(\mathcal{N}^j)$, and $(\mathcal{N}+1)^j(D(\mathcal{N}^{j+1})) \subset D(\mathcal{N})$ for $r \in [0,t]$, Lemma A.1 can be applied. We obtain

$$\begin{aligned}
e^{-tK}(\mathcal{N}+1)^j e^{tK} &= (\mathcal{N}+1)^j + \int_0^t e^{-rK}[(\mathcal{N}+1)^j, K^* - K^\circ]e^{rK}\,\mathrm{d}r \\
&= (\mathcal{N}+1)^j + \int_0^t e^{-rK}\Big(K^*((\mathcal{N}+3)^j - (\mathcal{N}+1)^j) + \mathrm{h.c.}\Big)e^{rK}\,\mathrm{d}r,
\end{aligned}$$

where we used $\mathcal{N}K^* = K^*(\mathcal{N}+2)$. From (A.5) we have $\|K(\mathcal{N}+1)^{-1}\|_{\mathrm{op}} \leqslant C\|k\|_{\mathrm{HS}}$. Using this and the bound $(\mathcal{N}+3)^j - (\mathcal{N}+1)^j \leqslant C_j(\mathcal{N}+1)^{j-1}$, we obtain

$$e^{-tK}(\mathcal{N}+1)^j e^{tK} \leqslant (\mathcal{N}+1)^j + C_j\|k\|_{\mathrm{HS}} \int_0^t e^{-rK}(\mathcal{N}+1)^j e^{rK}\,\mathrm{d}r.$$

Defining $u(r) = \langle \xi, e^{-rK}(\mathcal{N}+1)^j e^{rK}\xi\rangle$ for $\xi \in D$, we have

$$u(t) \leqslant \langle \xi, (\mathcal{N}+1)^j \xi\rangle + C_j \|k\|_{\mathrm{HS}} \int_0^t u(r)\,\mathrm{d}r.$$

Since $u(r)$ is finite for all $r \in [0,t]$ by (A.8), we can apply the Grönwall lemma and obtain

$$\langle \xi, e^{-tK}(\mathcal{N}+1)^j e^{tK}\xi\rangle \leqslant \exp(C_j\|k\|_{\mathrm{HS}})\langle \xi, (\mathcal{N}+1)^j\xi\rangle. \qquad \square$$

Since this lemma implies $e^{tK}(D) \subset D$ for $0 \leqslant t < (C\|k\|_{\mathrm{HS}})^{-1}$, we obtain the bound

$$e^{-ntK}(\mathcal{N}+1)^j e^{ntK} \leqslant \exp(nC_j\|k\|_{\mathrm{HS}})(\mathcal{N}+1)^j \tag{A.9}$$

on D by iteration, for all $j \geqslant 1$ and $n \in \mathbb{N}$. Thus, we have $e^{tK}(D) \subset D$ for all $0 \leqslant t < \infty$. In particular, this allows us to repeat the proof of Lemma A.2 for $t = 1$, which, using $\|k\|_{\mathrm{HS}} \leqslant \|s\|_{\mathrm{HS}}$, shows Lemma 2.8.

We can now prove (A.4). Let $g \in \mathfrak{H}$. First, let us compute

$$\begin{aligned}[a^*(g), K] = [a^*(g), K^* - K^\circ] = -[a^*(g), K^\circ] = -\frac{1}{2}\int_{\Lambda^2} k(x,y)[a^*(g), a_x a_y]\\
= \frac{1}{2}\int_{\Lambda^2} k(x,y)g(x)a_y + \frac{1}{2}\int_{\Lambda^2} k(x,y)g(y)a_x = \int_{\Lambda^2} k(x,y)g(y)a_x = a(k(\overline{g})),\end{aligned}$$

where we used the fact that k is real and symmetric. Similarly, we obtain

$$[a(g), K] = a^*(k(\overline{g})).$$

We want to justify (A.4) by repeatedly using the Duhamel formula from Lemma A.1. Let $\xi', \xi \in D$. Note that creation and annihilation operators satisfy the properties of B in Lemma A.1 and K satisfies the requirements for A, since iK is self-adjoint and $K(D) \cup e^{tK}(D) \subset D$ holds for $t \in [0,1]$. Hence, by Lemma A.1, we have

$$\begin{aligned}\langle \xi', T^* a^*(g) T\xi\rangle &= \langle \xi', a^*(g)\xi\rangle + \int_0^1 \langle \xi', e^{-tK}[a^*(g), K]e^{tK}\xi\rangle\,\mathrm{d}t\\
&= \langle \xi', a^*(g)\xi\rangle + \langle \xi', a(k(\overline{g}))\xi\rangle + \int_0^1\int_0^{t_0} \langle \xi', e^{-t_1 K} a^*(k^{(2)}(g))e^{t_1K}\xi\rangle\,\mathrm{d}t_1\,\mathrm{d}t_0.\end{aligned}$$

By iterating, we obtain for every $m \in \mathbb{N}$

$$\begin{aligned}\langle \xi', T^* a^*(g) T\xi\rangle = \langle \xi', a^*\Bigg(\Bigg(\sum_{n=0}^m \frac{k^{(2n)}}{(2n)!}\Bigg)(g)\Bigg)\xi\rangle + \langle \xi', a\Bigg(\Bigg(\sum_{n=0}^m \frac{k^{(2n+1)}}{(2n+1)!}\Bigg)(\overline{g})\Bigg)\xi\rangle\\
+ \int_0^1\int_0^{t_0}\cdots\int_0^{t_{2m}} \langle \xi', e^{-t_{2m+1}K}a^*\big(k^{(2m+2)}(g)\big)e^{t_{2m+1}K}\xi\rangle\,\mathrm{d}t_{2m+1}\ldots\mathrm{d}t_1\,\mathrm{d}t_0.\end{aligned} \tag{A.10}$$

We need the following inequalities to show that the right-hand side converges as $m \to \infty$. Let $\xi \in D(\mathcal{N}^{1/2})$ and write $\xi = (\xi_0, \xi_1, \dots)$, with $\xi_n \in \mathfrak{H}^n$. We have

$$\begin{aligned}
\|a(g)\xi\|^2 &= \sum_{n=1}^{\infty} \|a(g)\xi_n\|^2_{\mathfrak{H}^{n-1}} = \sum_{n=1}^{\infty} \int_{\Lambda^{n-1}} \left| \sqrt{n} \int_{\Lambda} \overline{g(x_n)} \xi_n(x, x_n) \, \mathrm{d}x_n \right|^2 \mathrm{d}x \\
&\leqslant \sum_{n=1}^{\infty} \|g\|^2_{\mathfrak{H}} \|\sqrt{n}\xi_n\|^2_{\mathfrak{H}^n} \leqslant \|g\|^2_{\mathfrak{H}} \sum_{n=1}^{\infty} \|\mathcal{N}^{1/2}\xi_n\|^2_{\mathfrak{H}^n} \\
&= \|g\|^2_{\mathfrak{H}} \sum_{n=0}^{\infty} \|\mathcal{N}^{1/2}\xi_n\|^2_{\mathfrak{H}^n} = \|g\|^2_{\mathfrak{H}} \|\mathcal{N}^{1/2}\xi\|^2.
\end{aligned}$$

Similarly, using

$$a(g)a^*(g) = \|g\|^2_{\mathfrak{H}} + a^*(g)a(g),$$

we obtain

$$\|a^*(g)\xi\| \leqslant \|g\|_{\mathfrak{H}} \|(\mathcal{N}+1)^{1/2}\xi\|.$$

In particular, these inequalities imply that $g \mapsto a^\sharp(g)\xi$ is continuous for $\xi \in D(\mathcal{N}^{1/2})$ and $\sharp \in \{\cdot, *\}$. Using the Taylor series for cosh and sinh, shows that the first two terms on the right-hand side of (A.10) converge. It remains to prove that the last term in (A.10) vanishes as $m \to \infty$. We compute

$$\begin{aligned}
&\left| \int_0^1 \int_0^{t_0} \cdots \int_0^{t_{2m}} \langle \xi', e^{-t_{2m+1}K} a^*(k^{(2m+2)}(g)) e^{t_{2m+1}K} \xi \rangle \, \mathrm{d}t_{2m+1} \dots \mathrm{d}t_1 \, \mathrm{d}t_0 \right| \\
&\qquad \leqslant \int_0^1 \int_0^{t_0} \cdots \int_0^{t_{2m}} \|e^{t_{2m+1}K}\xi'\| \|a^*(k^{(2m+2)}(g)) e^{t_{2m+1}K}\xi\| \, \mathrm{d}t_{2m+1} \dots \mathrm{d}t_1 \, \mathrm{d}t_0 \\
&\qquad \leqslant \int_0^1 \int_0^{t_0} \cdots \int_0^{t_{2m}} \|\xi'\| \|k^{(2m+2)}(g)\|_{\mathfrak{H}} \|(\mathcal{N}+1)^{1/2} e^{t_{2m+1}K}\xi\| \, \mathrm{d}t_{2m+1} \dots \mathrm{d}t_1 \, \mathrm{d}t_0 \\
&\qquad \leqslant C_{\xi',\xi} \|g\|_{\mathfrak{H}} \frac{\|k\|^{2m+2}_{\mathrm{op}}}{(2m+2)!} \xrightarrow{m\to\infty} 0,
\end{aligned}$$

where $C_{\xi',\xi} > 0$ is a constant depending on ξ' and ξ. Here we used that $\|(\mathcal{N}+1)^{1/2} e^{t_{2m+1}K}\xi\|$ can be bounded uniformly in $t_{2m+1} \in [0,1]$ by (A.9), since we chose $\xi \in D$. In conclusion, letting $m \to \infty$ in (A.10), we obtain

$$\langle \xi', T^* a^*(g) T \xi \rangle = \langle \xi', (a^*(\cosh(k)(g)) + a(\sinh(k)(\overline{g})))\xi \rangle = \langle \xi', (a^*(c(g)) + a(s(\overline{g})))\xi \rangle$$

for all $g \in \mathfrak{H}$ and $\xi', \xi \in D$.

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