# Standard Subspaces

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# Motivation and overview of minicourse

standard subspaces:

- inspired by von Neumann algebras in standard form, but of independent interest.
- natural setting of Tomita-Taksesaki modular theory.
- lots of applications: KMS states, quantum field theory, geometric modular action, split inclusions, relative entropies, ...
- single standard subspaces are quite simple, but inclusions/nets are very rich!

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contents:

1) From $^*$-algebras to standard subspaces
2) Some standard subspace theory
3) Inclusions of standard subspaces
4) The free current as an example of a net of standard subspaces
5) Standard subspaces and von Neumann algebras

# Standard Subspaces and Modular Theory

## general theory and applications to QFT

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**Abstract:** This is a preview of an upcoming monograph on standard subspaces and modular theory. Currently it consists of the first chapter on fundamentals of standard subspaces, and an appendix on Hardy spaces.

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cations in quantum physics, which are collected in Section 1.8. These applications include the local standard subspaces of a free quantum Klein-Gordon field, and the standard subspace entropy operator[illegible] via the cutting projection. We also mention the KMS states of a type I factor, which [illegible] of the operator algebraic structure of Chapter 2.

### Contents of Chapter 1

# 1) From *-algebras to standard subspaces

**abstract $(C)^*$-algebra $\mathcal{A}$**

**Representation on Hilbert space $\mathcal{H}$**

## abstract $(C)^*$-algebra $\mathcal{A}$

## Representation on Hilbert space $\mathcal{H}$

**$^*$-operation $\mathcal{A} \to \mathcal{A}, \quad A \mapsto A^*$**

- well-defined by definition of $^*$-algebra
- antilinear involution
- real fixed point space $\mathcal{A}_{\mathrm{sa}} = \{A = A^*\}$,

$$\mathcal{A}_{\mathrm{sa}} + i\mathcal{A}_{\mathrm{sa}} = \mathcal{A}, \qquad \mathcal{A}_{\mathrm{sa}} \cap i\mathcal{A}_{\mathrm{sa}} = \{0\}$$

- involution on $\mathcal{A}_{\mathrm{sa}} + i\mathcal{A}_{\mathrm{sa}}$:

$$A + iB \overset{*}{\longmapsto} A - iB$$

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## Representation on Hilbert space $\mathcal{H}$

Pick state $\omega$ on $\mathcal{A}$, or vector $\Omega \in \mathcal{H}$ (GNS)

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- $S^0_{\mathcal{A},\Omega}$ closable $\iff$ $H := \overline{\mathcal{A}_{\mathrm{sa}}\Omega}$ satisfies

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- Closure of $S^0_{\mathcal{A},\Omega}$ is

$$S_{\mathcal{A},\Omega} : H + iH \to H + iH, \quad h_1 + ih_2 \mapsto h_1 - ih_2$$

# Definition and first examples

$\mathcal{H}$: complex Hilbert space (always).

**Definition**

A **standard subspace** is a real subspace $H \subset \mathcal{H}$ such that

- $H$ is **closed**,
- $H$ is **separating:** $H \cap iH = \{0\}$,
- $H$ is **cyclic:** $H + iH \subset \mathcal{H}$ is dense.

**Some examples:**

(1) For a von Neumann algebra $\mathcal{M} \subset B(\mathcal{H})$ and a cyclic+separating vector $\Omega$,

$$H := \overline{\mathcal{M}_{\mathrm{sa}}\Omega}$$

is standard. Not all standard subspaces are like that.

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(2) $e^{i\alpha}\mathbb{R} \subset \mathbb{C}$ is standard ($\alpha \in \mathbb{R}$), and any 1d standard subspace looks like that.

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(3) Let $0 < \lambda \leq 1$. Then

$$H_\lambda := \left\{ \begin{pmatrix} \lambda z, & \bar{z} \end{pmatrix} : z \in \mathbb{C} \right\} \subset \mathbb{C}^2$$

is standard. Any 2d standard subspace is unitarily equivalent to one of those **(Exercise)**.

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(4) Let $I \subset \mathbb{R}$ be an interval and $f \mapsto \tilde{f}$ Fourier transform. Then

$$H(I) := \left\{ \tilde{f}|_{\mathbb{R}_+} \ : \ f \in C_c^\infty(\mathbb{R} \to \mathbb{R}), \ \mathrm{supp}(f) \subset I \right\}^- \subset L^2(\mathbb{R}_+, p\, dp)$$

is standard **(hard exercise)**. These standard subspaces generate a QFT (later).

# 2) Some standard subspace theory

# Standard subspaces are perfect for modular theory.

- $\mathrm{Inv}(\mathcal{H}) := \{$closed densely defined antilinear involutions $S : \mathrm{dom}(S) \to \mathrm{dom}(S)$ in $\mathcal{H}\}$

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- $\mathrm{Inv}(\mathcal{H}) := \{$closed densely defined antilinear involutions $S : \mathrm{dom}(S) \to \mathrm{dom}(S)$ in $\mathcal{H}\}$
- Given $H \subset \mathcal{H}$ standard, the **Tomita operator of $H$**

$$S_H : H + iH \to H + iH, \qquad h_1 + ih_2 \mapsto h_1 - ih_2,$$

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  lies in $\mathrm{Inv}(\mathcal{H})$.
- The fixed point space of $S \in \mathrm{Inv}(\mathcal{H})$, i.e. $H := \mathrm{Fix}(S) = \ker(S - 1)$, is standard.

This gives a bijection $\mathrm{Std}(\mathcal{H}) \longleftrightarrow \mathrm{Inv}(\mathcal{H})$.

**Properties of this bijection:**

1) equivariant under $\mathrm{AU}(\mathcal{H})$ (unitary and antiunitary operators), $S_{UH} = US_HU^*$.
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2) $K \subset H \iff S_K \subset S_H$. **Remark:** $\|S_H\| < \infty \implies$ no non-trivial inclusions in $H$!

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**Proposition**

*Let $H \in \mathrm{Std}(\mathcal{H})$. Then $S_H^* = S_{H'}$, with the symplectic complement*

$$H' = iH^{\perp_{\mathbb{R}}} = \{\xi \in \mathcal{H} : \mathrm{Im}\langle \xi, h\rangle = 0 \ \forall h \in H\}.$$

- *$H' \in \mathrm{Std}(\mathcal{H})$, and $H'' = H$.*

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- May have $H \cap H' \neq \{0\} \iff \overline{H + H'} \neq \mathcal{H}$, or $K \subsetneq H$ with $K' \cap H = \{0\}$.
- Call $Z(H) := H \cap H'$ the **center of $H$**, call $H$ **factorial** if $H \cap H' = \{0\}$ and **maximal abelian** if $H = H'$.

Maximal abelian standard subspaces are very simple: $H = \overline{\mathbb{R}\text{-span}\{e_n\}}$, with $(e_n)$ ONB of $\mathcal{H}$.

As any closed operator, Tomita operators have a polar decomposition,

$$\Delta_H := S_H^* S_H, \qquad S_H = J_H \Delta_H^{1/2}.$$

**Some consequences of $S_H$ being an antilinear involution:**

- $\ker \Delta_H = \{0\}$, $\Delta_H \geq 0$, $\Delta_H$ linear (modular operator).
- $J_H^* = J_H^{-1} = J_H$ antiunitary involution (modular conjugation)
- the **modular relation**

$$J_H \Delta_H^{1/2} = \Delta_H^{-1/2} J_H.$$

**Proof:** □

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**Further consequences:**

- The unitary one parameter group $\Delta_H^{it} = e^{it \log \Delta_H}$, $t \in \mathbb{R}$, exists (modular group)
- and commutes with the conjugation, $J_H \Delta_H^{it} = \Delta_H^{it} J_H$ (attention with the sign).
- Modular data of symplectic complement: $J_{H'} = J_H$, $\Delta_{H'} = \Delta_H^{-1}$ because $S_{H'} = (J_H \Delta_H^{1/2})^* = \Delta_H^{1/2} J_H = J_H \Delta_H^{-1/2}$.

**Theorem (Tomita-Takesaki Theorem for standard subspaces)**

*Let* $H \in \mathrm{Std}(\mathcal{H})$. *Then*

$$\Delta_H^{it} H = H, \quad t \in \mathbb{R}, \qquad J_H H = H'.$$

**Proof:**

$\square$

Any standard subspace $H$ has an intrinsic unitary one-parameter group of automorphisms, and a conjugation mapping it to its symplectic complement $H'$.

So far: $H \to \Delta_H^{it}, J_H$. Also the converse works:

**Lemma**

*Let $U(t) = e^{itP}$ a strongly continuous unitary one-parameter group and $J$ an antiunitary involution with $JU(t) = U(t)J$. Then*

$$H := \ker(Je^{P/2} - 1)$$

*is a standard subspace, with $J_H = J$ and $\Delta_H = e^P$.*

**Example:**

- $\mathcal{H} = L^2(\mathbb{R}, d\theta)$. Group $(U(t)f)(\theta) = f(\theta - t)$, conjugation $(Jf)(\theta) = \overline{f(\theta)}$.

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  $\rightsquigarrow H \in \mathrm{Std}(\mathcal{H})$.

**Exercise:** Write down $H$ explicitly. Hint: Fourier transform.

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**Theorem (The KMS condition characterizes the modular group)**

*Let $H \in \mathrm{Std}(\mathcal{H})$ and $U(t)$ a one-par. group with $U(t)H = H$. Then $U(t) = \Delta_H^{it}$ iff for all $h_1, h_2 \in H$,*

$$f(t) := \langle h_1, U(t)h_2 \rangle$$

*has a continuous extension to $\{t \in \mathbb{C} : -1 \leq \mathrm{Im}(t) \leq 0\}$, analytic in $-1 < \mathrm{Im}(t) < 0$, and*

$$f(t - i) = \overline{f(t)}, \qquad t \in \mathbb{R}.$$

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**Corollary**

*Let $K \subset H$ be standard subspaces such that $\Delta_H^{it} K \subset K$, $t \in \mathbb{R}$. Then $K = H$.*

**Proof:**

How to characterize the modular conjugation?

## How to characterize the modular conjugation?

**Proposition (Reflection Positivity characterizes the modular conjugation)**

*Let $H \in \mathrm{Std}(\mathcal{H})$ and $J$ an anti-unitary involution. Then $J = J_H$ iff $JH \supset H'$ and*

$$\langle h, Jh \rangle \geq 0, \qquad h \in H.$$

# Classification of standard subspaces

$$
\begin{aligned}
\mathrm{Std}(\mathcal{H}) &\longleftrightarrow \mathrm{Inv}(\mathcal{H}) \\
&\longleftrightarrow (U(t), J : [U(t), J] = 0) \\
&\stackrel{\sim}{\longleftrightarrow} (A = A^*, A \cong -A)
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**Proposition**

*$H, K \in \mathrm{Std}(\mathcal{H})$ unitarily equivalent $\iff$ $\Delta_H = U\Delta_K U^{-1}$ for some $U \in \mathcal{U}(\mathcal{H})$.*

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**Theorem**

*Let $V$ be a real Hilbert space, and $U(t)$ a strongly continuous one-parameter group of orthogonal transformations. Then there exists uniquely (up to unitary equivalence) a complex Hilbert space $\mathcal{H}$, a standard subspace $H \in \mathrm{Std}(\mathcal{H})$, and a real orthogonal map $\Phi : V \to \mathcal{H}$ such that*

$$\Phi(V) = H, \qquad \Phi \circ U(t) \circ \Phi^{-1} = \Delta_H^{it}.$$

# Recap Lecture 1

- Standard subspace: closed real subspace $H \subset \mathcal{H}$ s.t. $H \cap iH = \{0\}$ and $H + iH \subset \mathcal{H}$ dense.
- Standard subspaces $H$ are in natural bijection with closed antilinear densely defined involutions $S_H$ (Tomita operators)

$$S_H : H + iH \to H + iH, \qquad h_1 + ih_2 \mapsto h_1 - ih_2.$$

- Polar decomposition $S_H = J_H \Delta_H^{1/2}$ yields one-parameter group $\Delta_H^{it}$, $t \in \mathbb{R}$ (the modular group), and antiunitary involution $J_H$, s.t.

$$J_H \Delta_H^{1/2} = \Delta_H^{-1/2} J_H, \qquad \Delta_H^{it} H = H, \quad t \in \mathbb{R}, \qquad J_H H = H'.$$

- Here $H'$ is the symplectic complement of $H$. We have $S_H^* = S_{H'}$.
- The modular group is characterized by an analytic extension and symmetry property (KMS condition.)

# 3) Inclusions of standard subspaces

So far we looked at a single standard subspace $H \in \mathrm{Std}(\mathcal{H})$.

- Standard subspaces behave like little brothers of von Neumann algebras:
  - center $\longleftrightarrow H \cap H'$
  - maximal abelian vNA $\longleftrightarrow$ maximal abelian standard subspace $H = H'$
  - factor $\longleftrightarrow$ factorial standard subspace $H \cap H' = \{0\}$
  - commutant $\longleftrightarrow$ symplectic complement
  - double commutant $\longleftrightarrow H = H''$

  We can classify (factorial) standard subspaces quite easily.

So far we looked at a single standard subspace $H \in \mathrm{Std}(\mathcal{H})$.

- Standard subspaces behave like little brothers of von Neumann algebras:
  - center $\longleftrightarrow H \cap H'$
  - maximal abelian vNA $\longleftrightarrow$ maximal abelian standard subspace $H = H'$
  - factor $\longleftrightarrow$ factorial standard subspace $H \cap H' = \{0\}$
  - commutant $\longleftrightarrow$ symplectic complement
  - double commutant $\longleftrightarrow H = H''$

  We can classify (factorial) standard subspaces quite easily.
- Inspired by subfactors, one may look at inclusions of standard subspaces:

$$K \subset H, \qquad K, H \in \mathrm{Std}(\mathcal{H})$$

- We have already seen: If $K \subset H$ and $\|S_H\| < \infty$, then $K = H$. Let us prove the converse.

**Lemma (Figliolini-Guido '00)**

*If $K \in \mathrm{Std}(\mathcal{H})$ has unbounded Tomita operator $S_K$, there exists $H \in \mathrm{Std}(\mathcal{H})$ such that $K \subsetneq H$.*

**Proof idea:** Find $v \in \mathcal{H}$, $v \notin K + iK = \mathrm{dom}\,\Delta_K^{1/2}$ (possible because $\Delta_H^{1/2}$ is unbounded selfadjoint). Define $H := K + \mathbb{R}v \supsetneq K$. cyclic: ✔ separating: **(Exercise)**

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- Any extension $K \subset H$ defines an extension

$$S_K^* S_H \supset S_H^* S_H = \Delta_H,$$

with

$$\mathrm{Fix}(S_K^* S_H) = (K' \cap H) + i(K' \cap H).$$

The extension $S_K^* S_H$ detects the relative "commutant". **[Correa da Silva-L '25]**

- Problem: $S_K^* S_H$ is not closable. This gets subtle and needs more tools ... ask Andreas!

# Inclusions and one-parameter groups

Here we rather consider inclusions with additional symmetries.

- Inspired by quantum physics, we consider a unitary one-parameter group $T(x), x \in \mathbb{R}$, with positive generator, e.g. $T(x) = e^{iPx}, P \geq 0$.

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**Lemma**

*Let $H \in \mathrm{Std}(\mathcal{H})$ and $T$ a positively generated one-parameter group. Then*

$$T(x)H \subset H \quad \forall x \in \mathbb{R} \implies P = 0.$$

**Proof:** The assumption implies $T(x)H = H$ for all $x$. Hence

$$e^{iPx} = T(x) = J_H T(x) J_H = e^{-iJ_H P J_H x} \implies 0 \leq P = -J_H P J_H \leq 0.$$

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**Theorem (Borchers ’92, Florig ’98)**

*Let $H \in \mathrm{Std}(\mathcal{H})$, $T(x)$ positively generated one-parameter group, $T(x)H \subset H$, $x \geq 0$. Then*

$$\Delta_H^{it} T(x) \Delta_H^{-it} = T(e^{-2\pi t} x),$$
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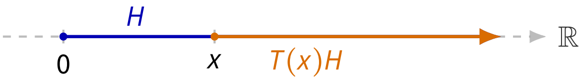


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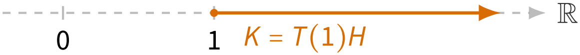
- The modular group of $H$ acts on the translation group $T(x)$ by scaling.
- We obtain an (anti-)unitary representation of the affine group $\mathrm{Aff}(\mathbb{R})$: group elements $(\pm e^t, x) : a \mapsto \pm e^t a + x$, representation

$$U(e^t, x) = T(x)\Delta_H^{-it/2\pi}, \qquad U(-1, 0) = J_H.$$

In the situation of Borchers' Theorem ($T(x)$ positively generated, $T(x)H \subset H$ for $x \geq 0$), consider

$$K := T(1)H \subset H$$



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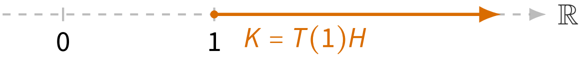


We have

$$\begin{aligned}
\Delta_H^{it} K &= \Delta_H^{it} T(1)H \\
&= T(e^{-2\pi t})\Delta_H^{it} H \\
&= T(e^{-2\pi t})H \\
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&\subset T(1)H \qquad \text{for } t \leq 0 \\
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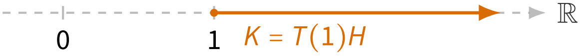


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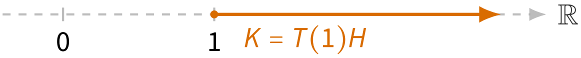


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$$\Delta_H^{it} K \subset K, \qquad t \leq 0.$$

- Recall that $K \subset H$ and $\Delta_H^{it} K \subset K$ for **all** $t \in \mathbb{R}$ implies $K = H$. But **half-sided** modular inclusions exist.

In the situation of Borchers' Theorem, the translational symmetry $T(x)$ was put in by hand. Can we recover it from modular data? We have

$$\Delta_{T(1)H}^{it}\Delta_H^{-it} = T(1)\Delta_H^{it}T(-1)\Delta_H^{-it} = T(1-e^{-2\pi t}).$$

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**Theorem (Wiesbrock 93, Araki-Zsido 05)**

*Let $K \subset H$ be half-sided modular. Then there exists a positively generated one-parameter group $T$ such that*

$$T(1-e^{-2\pi t}) = \Delta_K^{it}\Delta_H^{-it}, \qquad t \in \mathbb{R}.$$

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- This means that half-sided modular inclusions are the same as standard subspaces with half-sided positively generated translation symmetry.
- The modular groups generate the affine group $\mathrm{Aff}(\mathbb{R})$! No symmetry has to be imposed by hand.

# 4) The free current as an example of a net of standard subspaces

This will be an example of a system of standard subspaces showcasing half-sided modular inclusions and much more (all main properties of a quantum field theory).

**Philosophy:**

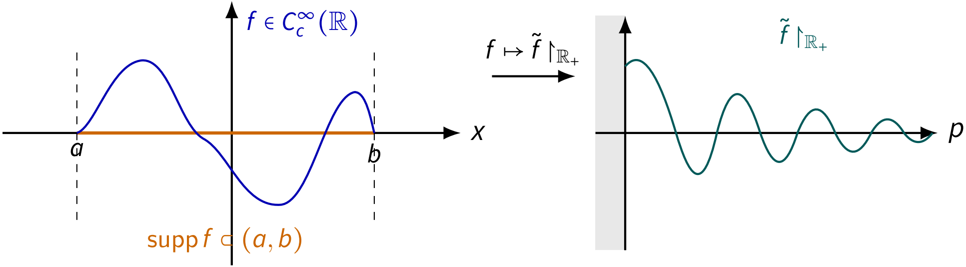
Let us start with a family of real subspaces indexed by intervals $I \subset \mathbb{R}$:

$$L(I) := C_c^\infty(I \to \mathbb{R}) \subset L^2(\mathbb{R}, dx).$$

- separating but not cyclic for $I \neq \mathbb{R}$.

Change representation by Fourier transform:

$$H(I) := \overline{\{\tilde{f}|_{\mathbb{R}_+} : f \in L(I)\}} \subset L^2(\mathbb{R}_+, p\,dp)$$




**Note:** $f \mapsto \tilde{f}|_{\mathbb{R}_+}$ is injective because reality of $f$ implies $\tilde{f}(-p) = \overline{\tilde{f}(p)}$.

## Some symmetry considerations:

- May shift functions $f \mapsto f(\cdot - x)$ in $C_c^\infty(\mathbb{R} \to \mathbb{R})$
- In $L^2(\mathbb{R}, dx)$, this is a unitary one-parameter group with non-positive generator.
- In $\mathcal{H} = L^2(\mathbb{R}_+, p\, dp)$, this is transported to the one-parameter group

$$(T(x)\psi)(p) = e^{ipx}\psi(x),$$

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$$(T(x)\psi)(p) = e^{ipx}\psi(x),$$

  with positive generator $(P\psi)(p) = p\psi(p)$.
- $\mathcal{H} = L^2(\mathbb{R}_+, p\, dp)$ carries an (anti-)unitary **irreducible** representation of $\mathrm{Aff}(\mathbb{R})$, namely

$$(U(e^t, x)\psi)(p) = e^{ipx}\, e^t \psi(e^t p),$$
$$(U(j)\psi)(p) = -\overline{\psi(p)}, \qquad j := (-1, 0) \quad \text{reflection}$$

- Up to unitary equivalence, this is the unique irreducible positive energy representation (translation group has a positive generator) of $\mathrm{Aff}(\mathbb{R})$.

In the following, $I \subset \mathbb{R}$ denotes intervals (possibly unbounded), $I \neq \varnothing$, $I \neq \mathbb{R}$.

**Theorem**

*The real subspaces* $H(I) := \overline{\{\tilde{f}|_{\mathbb{R}_+} \,:\, \mathrm{supp} f \subset I\}} \subset \mathcal{H}$ *have the following properties:*

a) *The map* $I \mapsto H(I)$ *is inclusion preserving (a "net", or "isotonous")*

b) *The net is* $\mathrm{Aff}(\mathbb{R})$*-covariant:*

$$U(\gamma)H(I) = H(\gamma I), \qquad \gamma \in \mathrm{Aff}(\mathbb{R}).$$

c) *The net is local:*

$$I_1 \cap I_2 = \varnothing \implies H(I_1) \subset H(I_2)'.$$

d) $H(I)$ *is standard.*

e) $H([1,\infty)) \subset H([0,\infty)) =: H_+$ *is a half-sided modular inclusion, with*

$$\Delta_{H_+}^{it} = U(e^{-2\pi t}, 0), \qquad J_{H_+} = U(j).$$

Let us prove the locality c):

$$I_1 \cap I_2 = \varnothing \implies H(I_1) \subset H(I_2)'.$$

This follows by pulling back the symplectic form $\operatorname{Im}\langle \cdot, \cdot \rangle$ to $C_c^\infty(\mathbb{R} \to \mathbb{R})$:

$\square$

Let us prove the standardness d). This is done by what is called a “Reeh-Schlieder argument” which shows the importance of positive energy.

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**Proof that** $H(I)$ **is cyclic.** We have to show that $H(I) + iH(I)$ is dense, i.e. that only $\xi = 0$ satisfies $0 = \langle \xi, h \rangle$, $h \in H(I)$.

**Proof that** $H(I)$ **is separating.**

$\square$

Let us prove the half-sided modular inclusion and $\Delta_{+}^{it}) = U(e^{-2\pi t}, 0)$ in e).

This is the simplest example of "geometric modular action" and the "Bisognano-Wichmann Theorem".

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- We also have $U(e^{-2\pi t}, 0) T(x) U(e^{2\pi t}, 0) = T(e^{-2\pi t} x)$ (representation of $\mathrm{Aff}(\mathbb{R})$).

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- Furthermore, $U(e^{2\pi t}, 0)$ and $\Delta_+^{is}$ commute for all $t, s$ because $[0,\infty)$ is dilation invariant.
- irreducibility of $U$ $\Longrightarrow$ $\Delta_+^{it} = e^{i\alpha t}U(e^{-2\pi t}, 0)$ for some $\alpha \in \mathbb{R}$
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- Standardness of $H_+$ and $e^{i\alpha t} H_+ = H_+$ force $\alpha = 0$.
- Argument for $J_+ = U(j)$ is similar.

# Recap Lecture 2

- Inclusions $K \subset H$ of standard subspaces are subtle, analogous to subfactors.
- Interesting special type of incluion: half-sided modular inclusions $K \subset H$ such that
$$\Delta_H^{it} K \subset K, \qquad t \leq 0.$$
- Leads to unexpected symmetries: affine group $\mathrm{Aff}(\mathbb{R})$ represented by combinations of modular unitaries and conjugations. In particular, translations $a \mapsto a + x$ are positively generated (Theorems by Borchers and Wiesbrock/Araki-Zsido).
- The basic building block is given by the unique positive energy irreducible representation of $\mathrm{Aff}(I)$, involving the net of standard subspaces
$$I \mapsto H(I) = \overline{\{\tilde{f}|_{\mathbb{R}_+} \,:\, \mathrm{supp} f \subset I\}} \subset L^2(\mathbb{R}_+, p\,dp).$$
- Have seen: $I \mapsto H(I)$ is inclusion preserving, covariant under $\mathrm{Aff}(\mathbb{R})$, consists of standard subspaces, local ($I_1 \subset \mathbb{R} \backslash I_2 \implies H(I_1) \subset H(I_2)'$), and $H([1,\infty)) \subset H([0,\infty))$ is half-sided modular.

Still to do: Verify claims about modular data of $H := H([0,\infty))$, namely

$$\Delta_H^{it} = U(e^{-2\pi t}, 0) \quad \text{(dilations)}, \qquad J_H = U(-1, 0) \quad \text{(reflection)}.$$

**Plan for today:**

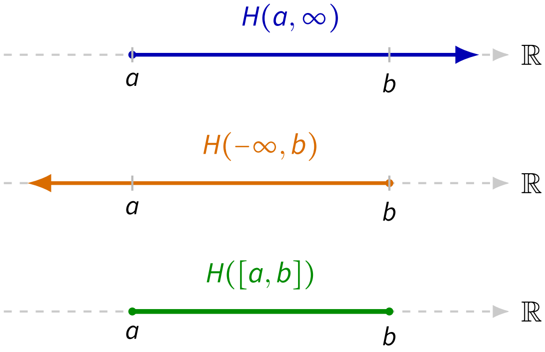
- Sketch idea of proof of that (see slides for more details)
- Take you back to von Neumann algebras

In this example, the whole net $I \mapsto H(I)$ is generated from the data

$$H := H([0,\infty)), \qquad T(x) = U(1,x)$$

because

$$H([a,b]) = T(a)H \cap T(b)H'.$$



# 5) Standard subspaces and von Neumann algebras

## Let's go back to the beginning: Von Neumann algebras vs. standard subspaces.

Fixing $\mathcal{H}$ and $\Omega \in \mathcal{H}\backslash\{0\}$, we have a map

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Nonetheless, there are ways of constructing von Neumann algebras from standard subspaces.

- These constructions will start from a standard subspace $H \subset \mathcal{H}$ and construct a von Neumann algebra $\mathcal{M}(H)$ and a cyclic separating vector $\Omega$ on a larger Hilbert space,
- such that the modular theory of $(\mathcal{M}(H), \Omega)$ can be read off from $H$.

The simplest version of this construction uses full (free) Fock spaces.

- Given a Hilbert space $\mathcal{H}$, consider the Hilbert tensor algebra (full Fock space)

$$\mathcal{F}_0(\mathcal{H}) := \bigoplus_{n\geq 0} \mathcal{H}^{\otimes n}, \qquad \mathcal{H}^{\otimes 0} := \mathbb{C}$$

- Distinguished vector: $\Omega := 1 \oplus 0 \oplus 0 \ldots$.
- Distinguished operators: left/right tensor multiplication,

$$a_L^*(\xi)\Psi := \xi \otimes \Psi, \qquad a_R^*(\xi)\Psi := \Psi \otimes \xi, \qquad \xi \in \mathcal{H}, \quad \Psi \in \mathcal{F}_0(\mathcal{H}),$$

  and their adjoints $a_L(\xi), a_R(\xi)$.
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- Field operators $\phi_{L/R}(\xi) := a^*_{L/R}(\xi) + a_{L/R}(\xi)$ (real linear in $\xi \in \mathcal{H}$!).
- generated von Neumann algebra $\mathcal{L}_0(H) := \{\phi_L(h) \,:\, h \in H\}''$

$$\mathcal{L}_0(H) = \begin{cases} B(\mathcal{F}_0(\mathcal{H})) & H = \mathcal{H} \\ L(\mathbb{F}_{\dim \mathcal{H}}) & H \in \mathrm{Std}(\mathcal{H}), \quad \Delta_H = 1 \quad \text{[Voiculescu '85]} \\ \text{type III deformation of } L(\mathbb{F}_{\dim \mathcal{H}}) & H \in \mathrm{Std}(\mathcal{H}), \quad \Delta_H \neq 1 \quad \text{[Shlyakhtenko '97]} \end{cases}$$

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**Example:**

- $P_{F,n}(\psi_1 \otimes \ldots \otimes \psi_n) = \sum_{\pi \in S_n} \psi_{\pi(1)} \otimes \ldots \otimes \psi_{\pi(n)}$ (total symmetrization times $n!$)
- total antisymmetrization works in the same way

Total symmetrization/antisymmetrization derive from the (negative) flip

$$F(v \otimes w) = w \otimes v, \qquad v, w \in \mathcal{H}$$

according to

$$P_{\pm F,n} = \sum_{\pi \in S_n} (\pm F)_{i_1} \cdots (\pm F)_{i_{l(\pi)}}, \qquad F_k = \mathrm{id}_{\mathcal{H}}^{\otimes(k-1)} \otimes F \otimes \mathrm{id}_{\mathcal{H}}^{\otimes(n-k-1)}, \qquad \pi = \sigma_{i_1} \cdots \sigma_{i_{l(\pi)}}.$$

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- All of this can also be done for general $T \in B(\mathcal{H} \otimes \mathcal{H})$, $\|T\| \leq 1$, provided $P_{T,n} \geq 0$ for all $n$. **"Twist"**.
- It turns out that the Fock space $\mathcal{F}_T(\mathcal{H})$ carries natural left annihilation/creation operators $a^*_{T,L}(\xi), a_{T,L}(\xi), \xi \in \mathcal{H}$.
- For $T = F$ (flip): CCR algebra
- For $T = -F$: CAR algebra
- For $T = qF$, $-1 < q < 1$: $q$-deformed CCR algebra
- For general $T$: quite general quadratic algebra

**Definition (Correa da Silva / L '23)**

Let $H \in \mathrm{Std}(\mathcal{H})$ and $T \in B(\mathcal{H} \otimes \mathcal{H})$ a twist. The left twisted Araki-Woods algebra is

$$\mathcal{L}_T(H) := \{a_{T,L}^*(h) + a_{T,L}(h) \ : \ h \in H\}''.$$

- incorporates many constructions known in the literature.
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*If $[T, \Delta_H^{it} \otimes \Delta_H^{it}] = 0$ for all $t$, then $\Omega$ is cyclic and separating for $\mathcal{L}_T(H)$ if and only if*

- *$T$ solves the **Yang-Baxter Equation**.*
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- *T solves the* ***Yang-Baxter Equation****. (braid group equation)*
- *T is* ***crossing-symmetric*** *w.r.t. H. (an analytic continuation and symmetry condition for $t \mapsto (\Delta_H^{it} \otimes 1)T(1 \otimes \Delta_H^{-it})$, resembling KMS)*

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*Under the assumptions of the previous thm, the modular data* $(\Delta, J)$ *of* $(\mathcal{L}_T(H), \Omega)$ *are*

$$\Delta^{it}[\psi_1 \otimes \ldots \otimes \psi_n] = [\Delta_H^{it}\psi_1 \otimes \ldots \otimes \Delta_H^{it}\psi_n], \qquad J[\psi_1 \otimes \ldots \psi_n] = [J_H\psi_n \otimes \ldots \otimes J_H\psi_1].$$

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- For $T = F$, something special happens: $\mathcal{L}_T(H) = \mathcal{R}_T(H)$ (Bosonic second quantization, free local field)
- Generalizes many results known in special cases [Eckmann/Osterwalder '73, Baumgärtel/Jurke/Lledó '02, Leylands/Roberts/Testard '78, Shlyakhtenko '97, Buchholz/L/Summers '11, L '12]
- Internal structure of $\mathcal{L}_T(H)$ well understood for $T = qF$, $-1 < q < 1$, and general $H$ [Kumar/Skalski/Wasilewski '23], building on [Nelson '15, Miyagawa/Speicher '23]. They are non-injective factors of type $\mathrm{III}_1$, $\mathrm{III}_\lambda$, $\mathrm{II}_1$ depending on the spectrum of $\Delta_H$.

- standard subspaces can be used to construct interesting von Neumann algebras.
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Now we combine these two aspects. Fixing a twist $T$, we have an inclusion-preserving map

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- If you take the net of standard subspaces of the free current, then

$$I \mapsto \mathcal{L}_F(H(I)), \qquad [a,b] \mapsto \mathcal{L}_F(T(a)H) \cap \mathcal{L}_F(T(b)H)'$$

  is a net of von Neumann algebras satisfying "the same" properties as the net $I \mapsto H(I)$.
- This is the simplest example of a **quantum field theory (QFT)**.

- Main disadvantage of this QFT: It is “free” (describes no interaction). One wants to build richer examples of QFTs.

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A possible strategy is the following:

- Pick a twist $T$ s.t. $[T, \Delta_H^{it} \otimes \Delta_H^{it}] = 0$ and $[T, T(x) \otimes T(x)] = 0$ for all $t, x$, and consider the net

$$\begin{aligned} I \mapsto \mathcal{L}_T(H([a,b])) &:= \mathcal{L}_T(T(a)H) \cap \mathcal{L}_T(T(b)H)' \\ &= \mathcal{L}_T(T(a)H) \cap \mathcal{R}_T(T(b)H'). \end{aligned}$$

- This leads to much more complicated von Neumann algebras (intersection of left and right twisted algebras, relative commutants of twisted Araki-Woods subfactors)
- **If** these are non-trivial, they satisfy all axioms of QFT and produce examples of (mildly) interacting QFTs. Examples exist, but many questions are still open.

# Summary

- Standard subspaces are the essence of modular theory.
- An accessible yet rich theory exists, covering also inclusions and nets. Natural links with Lie group symmetries appear.
- Standard subspaces capture only partial information about von Neumann algebras (their modular theory).
- There exist methods of constructing von Neumann algebras on the basis of standard subspaces and twists. Leads to new subfactors, and has applications in QFT.

For slides and the upcoming book:

https://www.math.fau.de/mathematische-physik/qft-group/qft-research/#standard-subspaces

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