Standard Subspaces

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Motivation and overview of minicourse

standard subspaces:

- inspired by von Neumann algebras in standard form, but of independent interest.
- natural setting of Tomita-Taksesaki modular theory.
- lots of applications: KMS states, quantum field theory, geometric modular action, split inclusions, relative entropies, ...
- single standard subspaces are quite simple, but inclusions/nets are very rich!

contents (rough plan):

1) From *-algebras to standard subspaces
2) Some standard subspace theory
3) Standard subspaces and von Neumann algebras
4) Nets of standard subspaces and quantum field theory (QFT)

Standard Subspaces and Modular Theory

general theory and applications to QFT

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Abstract: This is a preview of an upcoming monograph on standard subspaces and modular theory. Currently it consists of the first chapter on fundamentals of standard subspaces, and an appendix on Hardy spaces.

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cations in quantum physics, which are collected in Section 1.8. These applications include the local standard subspaces of a free quantum Klein-Gordon field, and the standard subspace entropy operato[...] via the cutting projection. We also mention the KMS states of a type I factor, which [...] of the operator algebraic structure of Chapter 2.

Contents of Chapter 1

1) From *-algebras to standard subspaces

abstract $(C)^*$-algebra $\mathcal{A}$

*-operation $\mathcal{A} \to \mathcal{A}, \quad A \mapsto A^*$

- well-defined by definition of *-algebra
- antilinear involution
- real fixed point space $\mathcal{A}_{sa} = \{A = A^*\}$,

$\mathcal{A}_{sa} + i\mathcal{A}_{sa} = \mathcal{A}, \qquad \mathcal{A}_{sa} \cap i\mathcal{A}_{sa} = \{0\}$

- involution on $\mathcal{A}_{sa} + i\mathcal{A}_{sa}$:

$$A + iB \overset{*}{\longmapsto} A - iB$$

Representation on Hilbert space $\mathcal{H}$

Pick state ω on $\mathcal{A}$, or vector $\Omega \in \mathcal{H}$ (GNS)

$S^0_{\mathcal{A},\Omega} : \mathcal{A}\Omega \to \mathcal{A}\Omega, \qquad A\Omega \mapsto A^*\Omega$

- well-defined if Ω separating (ω faithful)
- antilinear involution
- real fixed point space $H_0 := \mathcal{A}_{sa}\Omega$,

$H_0 + iH_0 = \mathcal{A}\Omega, \qquad H_0 \cap iH_0 = \{0\}$

- involution on $H_0 + iH_0$:

$$h_1 + ih_2 \longmapsto h_1 - ih_2$$

abstract C^*-algebra $\mathcal{A}$

*-operation $\mathcal{A} \to \mathcal{A}, \quad A \mapsto A^*$

- isometric operation $\|A^*\| = \|A\|$

Does not need $\mathcal{A}$ or Ω,
just H!

Representation on Hilbert space $\mathcal{H}$

$S^0_{\mathcal{A},\Omega} : \mathcal{A}\Omega \to \mathcal{A}\Omega, \qquad A\Omega \mapsto A^*\Omega$

- isometric iff ω tracial:

$$\|A\Omega\|^2 - \|A^*\Omega\|^2 = \omega(A^*A - AA^*)$$

- typically unbounded and not closed.
- $S^0_{\mathcal{A},\Omega}$ closable $\iff$ $H := \overline{\mathcal{A}_{\mathrm{sa}}\Omega}$ satisfies

$$H \cap iH = \{0\},$$

- this is true if Ω separating for vNA $\mathcal{A}''$
- Closure of $S^0_{\mathcal{A},\Omega}$ is

$$S_{\mathcal{A},\Omega} : H + iH \to H + iH, \quad h_1 + ih_2 \mapsto h_1 - ih_2$$

Definition and first examples

$\mathcal{H}$: complex Hilbert space (always).

Definition

A **standard subspace** is a real subspace $H \subset \mathcal{H}$ such that

- H is **closed**,
- H is **separating:** $H \cap iH = \{0\}$,
- H is **cyclic:** $H + iH \subset \mathcal{H}$ is dense.

Some examples:

(1) For a von Neumann algebra $\mathcal{M} \subset B(\mathcal{H})$ and a cyclic+separating vector Ω,

$$H := \overline{\mathcal{M}_{\mathrm{sa}}\Omega}$$

is standard. Not all standard subspaces are like that.

Definition and first examples

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(2) $e^{i\alpha}\mathbb{R} \subset \mathbb{C}$ is standard ($\alpha \in \mathbb{R}$), and any 1d standard subspace looks like that.

Definition and first examples

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(3) Let $0 < \lambda \leq 1$. Then

$$H_\lambda := \left\{ \begin{pmatrix} \lambda z, & \bar{z} \end{pmatrix} : z \in \mathbb{C} \right\} \subset \mathbb{C}^2$$

is standard. Any 2d standard subspace is unitarily equivalent to one of those **(Exercise)**.

Definition and first examples

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(4) Let $I \subset \mathbb{R}$ be an interval and $f \mapsto \tilde{f}$ Fourier transform. Then

$$H(I) := \left\{\tilde{f}|_{\mathbb{R}_+} \,:\, f \in C_c^\infty(\mathbb{R} \to \mathbb{R}),\ \mathrm{supp}(f) \subset I\right\}^- \subset L^2(\mathbb{R}_+, p\,dp)$$

is standard **(Exercise)**. These standard subspaces generate a QFT (later).

2) Some standard subspace theory

Standard subspaces are perfect for modular theory.

- $\mathrm{Inv}(\mathcal{H}) := \{$closed densely defined antilinear involutions $S : \mathrm{dom}(S) \to \mathrm{dom}(S)$ in $\mathcal{H}\}$
- Given $H \subset \mathcal{H}$ standard, the **Tomita operator of H**

$$S_H : H + iH \to H + iH, \qquad h_1 + ih_2 \mapsto h_1 - ih_2,$$

$h = ik \qquad \underline{Sh} = h = ik = iSk = \underline{-Sh}$

lies in $\mathrm{Inv}(\mathcal{H})$.
- The fixed point space of $S \in \mathrm{Inv}(\mathcal{H})$, i.e. $H := \mathrm{Fix}(S) = \ker(S - 1)$, is standard.

This gives a bijection $\mathrm{Std}(\mathcal{H}) \longleftrightarrow \mathrm{Inv}(\mathcal{H})$.

Properties of this bijection:

1) equivariant under $\mathrm{AU}(\mathcal{H})$ (unitary and antiunitary operators), $S_{UH} = US_HU^*$.
2) $K \subset H \iff S_K \subset S_H$.

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1) equivariant under $\mathrm{AU}(\mathcal{H})$ (unitary and antiunitary operators), $S_{UH} = US_HU^*$.
2) $K \subset H \iff S_K \subset S_H$. **Remark:** $\|S_H\| < \infty \implies$ no non-trivial inclusions in H!

$\mathrm{Inv}(\mathcal{H})$ has a natural symmetry: $S \mapsto S^*$. What does this look like on standard subspaces?

Proposition

Let $H \in \mathrm{Std}(\mathcal{H})$. Then $S_H^* = S_{H'}$, with the ***symplectic complement***

$$H' = iH^{\perp_{\mathbb{R}}} = \{\xi \in \mathcal{H} : \mathrm{Im}\langle \xi, h\rangle = 0 \ \forall h \in H\}.$$

- *$H' \in \mathrm{Std}(\mathcal{H})$, and $H'' = H$.*

- The symplectic complement behaves like the “commutant” of a standard subspace.
- ⚠ The symplectic complement is not a complement, and very different from the real orthogonal complement.
- May have $H \cap H' \neq \{0\} \iff \overline{H + H'} \neq \mathcal{H}$, or $K \subsetneq H$ with $K' \cap H = \{0\}$.
- Call $Z(H) := H \cap H'$ the **center of H**, call H **factorial** if $H \cap H' = \{0\}$ and **maximal abelian** if $H = H'$.

Maximal abelian standard subspaces are very simple: $H = \overline{\mathbb{R}\text{-span}\{e_n\}}$, with (e_n) ONB of $\mathcal{H}$.

As any closed operator, Tomita operators have a polar decomposition,

$$\Delta_H := S_H^* S_H, \qquad S_H = J_H \Delta_H^{1/2}.$$

Some consequences of S_H being an antilinear involution:

- $\ker \Delta_H = \{0\}$, $\Delta_H \geq 0$, Δ_H linear (modular operator).
- $J_H^* = J_H^{-1} = J_H$ antiunitary involution (modular conjugation)
- the **modular relation**

$$J_H \Delta_H^{1/2} = \Delta_H^{-1/2} J_H.$$

Proof: $$J\Delta^{1/2} = S = S^{-1} = \Delta^{-1/2} J^{-1}$$

$$= J^{-1} \cdot J \Delta^{-1/2} J^{-1}$$ □

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Proof: □

Further consequences:

- The unitary one parameter group $\Delta_H^{it} = e^{it \log \Delta_H}$, $t \in \mathbb{R}$, exists (modular group)
- and commutes with the conjugation, $J_H \Delta_H^{it} = \Delta_H^{it} J_H$ (attention with the sign).
- Modular data of symplectic complement: $J_{H'} = J_H$, $\Delta_{H'} = \Delta_H^{-1}$ because $S_{H'} = (J_H \Delta_H^{1/2})^* = \Delta_H^{1/2} J_H = J_H \Delta_H^{-1/2}$.

Theorem (Tomita-Takesaki Theorem for standard subspaces)

Let $H \in \mathrm{Std}(\mathcal{H})$. Then

$$\Delta_H^{it} H = H, \quad t \in \mathbb{R}, \qquad J_H H = H'.$$

Proof:

$$\Delta_H^{it} S_H \Delta_H^{-it} = S_H$$

$$= S_{\Delta_H^{it} H}$$

$$H \times H \longrightarrow \cancel{\mathbb{C}} \ \mathbb{R}$$

$$h_1, h_2 \longmapsto \langle h_1, J_H h_2 \rangle$$

$$\langle h, J_H h \rangle = \langle h, \Delta^{1/2} h \rangle$$

$$J\Delta^{1/2}$$

□

Any standard subspace H has an intrinsic unitary one-parameter group of automorphisms, and a conjugation mapping it to its symplectic complement H'.

$$\mathbb{R} \ni \langle h_1, J_H h_2 \rangle \Rightarrow J_H H \subset H'$$

So far: $H \to \Delta_H^{it}, J_H$. Also the converse works:

Lemma

Let $U(t) = e^{itP}$ a strongly continuous unitary one-parameter group and J an antiunitary involution with $JU(t) = U(t)J$. Then

$$H := \ker(Je^{P/2} - 1)$$

is a standard subspace, with $J_H = J$ and $\Delta_H = e^P$.

Example:

- $\mathcal{H} = L^2(\mathbb{R}, d\theta)$. Group $(U(t)f)(\theta) = f(\theta - t)$, conjugation $(Jf)(\theta) = \overline{f(\theta)}$.
  $\rightsquigarrow H \in \mathrm{Std}(\mathcal{H})$.

Exercise: Write down H explicitly. Hint: Fourier transform.

How to **characterize** the modular group Δ_H^{it} and the modular conjugation J_H?

Theorem (The KMS condition characterizes the modular group)

Let $H \in \mathrm{Std}(\mathcal{H})$ and $U(t)$ a one-par. group with $U(t)H = H$. Then $U(t) = \Delta_H^{it}$ iff for all $h_1, h_2 \in H$,

$$f(t) := \langle h_1, U(t)h_2 \rangle$$

has a continuous extension to $\{t \in \mathbb{C} : -1 \leq \mathrm{Im}(t) \leq 0\}$, analytic in $-1 < \mathrm{Im}(t) < 0$, and

$$f(t-i) = \overline{f(t)}, \qquad t \in \mathbb{R}.$$

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Corollary

Let $K \subset H$ be standard subspaces such that $\Delta_H^{it}K \subset K$, $t \in \mathbb{R}$. Then $K = H$.

Proof: KMS $\Rightarrow$ $\Delta_K = \Delta_H$ $\Rightarrow$ $\mathrm{dom}\, S_K = \mathrm{dom}\, S_H$ $\overset{K \subseteq H}{\Longrightarrow}$ $K = H$. $\square$

How to characterize the modular conjugation?

Proposition (Reflection Positivity characterizes the modular conjugation)

Let $H \in \mathrm{Std}(\mathcal{H})$ and J an anti-unitary involution. Then $J = J_H$ iff $JH \supset H'$ and

$$\langle h, Jh \rangle \geq 0, \qquad h \in H.$$

Classification of standard subspaces

$$\begin{aligned} \mathrm{Std}(\mathcal{H}) &\longleftrightarrow \mathrm{Inv}(\mathcal{H}) \\ &\longleftrightarrow (U(t), J : [U(t), J] = 0) \\ &\overset{\sim}{\longleftrightarrow} (\cancel{A > 0}, A \cong -A) \end{aligned}$$

$A = A^*$

Idea: $A = \log \Delta_H = A^*$

$J_H A J_H = \log \Delta_H^{-1} = -A.$

Classification of standard subspaces

$$
\begin{aligned}
\mathrm{Std}(\mathcal{H}) &\longleftrightarrow \mathrm{Inv}(\mathcal{H}) \\
&\longleftrightarrow (U(t), J \,:\, [U(t), J] = 0) \\
&\overset{\sim}{\longleftrightarrow} (\cancel{A > 0}, A \cong -A) \\
&\qquad\; A = A^*
\end{aligned}
$$

Proposition

$H, K \in \mathrm{Std}(\mathcal{H})$ unitarily equivalent $\iff$ $\Delta_H = U \Delta_K U^{-1}$ for some $U \in \mathcal{U}(\mathcal{H})$.

Theorem

Let V be a real Hilbert space, and $U(t)$ a strongly continuous one-parameter group of orthogonal transformations. Then there exists uniquely (up to unitary equivalence) a complex Hilbert space $\mathcal{H}$, a standard subspace $H \in \mathrm{Std}(\mathcal{H})$, and a real orthogonal map $\Phi : V \to \mathcal{H}$ such that

$$\Phi(V) = H, \qquad \Phi \circ U(t) \circ \Phi^{-1} = \Delta_H^{it}.$$