Recap Lecture 1

- Standard subspace: closed real subspace $H \subset \mathcal{H}$ s.t. $H \cap iH = \{0\}$ and $H + iH \subset \mathcal{H}$ dense.
- Standard subspaces H are in natural bijections with closed antilinear densely defined involutions S_H (Tomita operators)

$$S_H : H + iH \to H + iH, \qquad h_1 + ih_2 \mapsto h_1 - ih_2.$$

- Polar decomposition $S_H = J_H \Delta_H^{1/2}$ yields one-parameter group Δ_H^{it}, $t \in \mathbb{R}$ (the modular group), and antiunitary involution J_H, s.t.

$$J_H \Delta_H^{1/2} = \Delta_H^{-1/2} J_H, \qquad \Delta_H^{it} H = H, \quad t \in \mathbb{R}, \qquad J_H H = H'.$$

- Here H' is the symplectic complement of H. We have $S_H^* = S_{H'}$.
- The modular group is characterized by an analytic extension and symmetry property (KMS condition.)

3) Inclusions of standard subspaces

So far we looked at a single standard subspace $H \in \mathrm{Std}(\mathcal{H})$.

- Standard subspaces behave like little brothers of von Neumann algebras:
  - center $\longleftrightarrow$ $H \cap H'$
  - maximal abelian vNA $\longleftrightarrow$ maximal abelian standard subspace $H = H'$
  - factor $\longleftrightarrow$ factorial standard subspace $H \cap H' = \{0\}$
  - commutant $\longleftrightarrow$ symplectic complement
  - double commutant $\longleftrightarrow$ $H = H''$

  We can classify (factorial) standard subspaces quite easily.
- Inspired by subfactors, one may look at inclusions of standard subspaces:

$$K \subset H, \qquad K, H \in \mathrm{Std}(\mathcal{H})$$

- We have already seen: If $K \subset H$ and $\|S_H\| < \infty$, then $K = H$. Let us prove the converse.

Lemma (Figliolini-Guido '00)

If $K \in \mathrm{Std}(\mathcal{H})$ has unbounded Tomita operator S_K, there exists $H \in \mathrm{Std}(\mathcal{H})$ such that $K \subsetneq H$.

Proof idea: Find $v \in \mathcal{H}$, $v \notin K + iK = \mathrm{dom}\,\Delta_K^{1/2}$ (possible because $\Delta_H^{1/2}$ is unbounded selfadjoint). Define $H := K + \mathbb{R}v \supsetneq K$. cyclic: ✔ separating: **(Exercise)**

$$k_1 + r_1 v = i(k_2 + r_2 v)$$

$$\Leftrightarrow \quad k_1 - i k_2 = \underbrace{(i r_2 - r_1)}_{=0} v \qquad \square$$

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- Any extension $K \subset H$ defines an extension

$$S_K^* S_H \supset S_H^* S_H = \Delta_H,$$

with

$$\mathrm{Fix}(S_K^* S_H) = (K' \cap H) + i(K' \cap H).$$

The extension $S_K^* S_H$ detects the relative "commutant". [Correa da Silva-L '25]

- Problem: $S_K^* S_H$ is not closable. This gets subtle and needs more tools ... ask Andreas!

Inclusions and one-parameter groups

Here we rather consider inclusions with additional symmetries.

- Inspired by quantum physics, we consider a unitary one-parameter group $T(x), x \in \mathbb{R}$, with positive generator, e.g. $T(x) = e^{iPx}$, $P \geq 0$.

Lemma

Let $H \in \mathrm{Std}(\mathcal{H})$ and T a positively generated one-parameter group. Then

$$T(x)H \subset H \quad \forall x \in \mathbb{R} \implies P = 0.$$

Proof: The assumption implies $T(x)H = H$ for all x. Hence

$$e^{iPx} = T(x) = J_H T(x) J_H = e^{-iJ_H P J_H x} \implies 0 \leq P = -J_H P J_H \leq 0.$$

The second most natural setting is half-sided invariance,

$$T(x)H \subset H, \qquad x \geq 0.$$

Theorem (Borchers '92, Florig '98)

Let $H \in \mathrm{Std}(\mathcal{H})$, $T(x)$ positively generated one-parameter group, $T(x)H \subset H$, $x \geq 0$. Then

$$\Delta_H^{it} T(x) \Delta_H^{-it} = T(e^{-2\pi t} x),$$

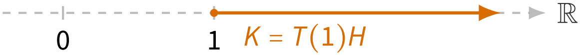
$$J_H T(x) J_H = T(-x).$$

- The modular group of H acts on the translation group $T(x)$ by scaling.
- We obtain an (anti-)unitary representation of the affine group $\mathrm{Aff}(\mathbb{R})$: group elements $(\pm e^t, x) : a \mapsto \pm e^t a + x$, representation

$$U(e^t, x) = T(x) \Delta_H^{-it/2\pi}, \qquad U(-1, 0) = J_H.$$

In the situation of Borchers' Theorem ($T(x)$ positively generated, $T(x)H \subset H$ for $x \geq 0$), consider

$$K := T(1)H \subset H$$



We have

$$\begin{aligned}
\Delta_H^{it} K &= \Delta_H^{it} T(1) H \\
&= T(e^{-2\pi t}) \Delta_H^{it} H \\
&= T(e^{-2\pi t}) H \\
&= T(1) T(e^{-2\pi t} - 1) H \\
&\subset T(1) H \qquad \text{for } t \leq 0 \\
&= K
\end{aligned}$$

Definition

A half-sided modular inclusion of two standard subspaces K, H is an inclusion $K \subset H$ s.t.

$$\Delta_H^{it} K \subset K, \qquad t \leq 0.$$

- Recall that $K \subset H$ and $\Delta_H^{it} K \subset K$ for **all** $t \in \mathbb{R}$ implies $K = H$. But **half-sided** modular inclusions exist.

In the situation of Borchers' Theorem, the translational symmetry $T(x)$ was put in by hand. Can we recover it from modular data? We have

$$\Delta_{T(1)H}^{it}\Delta_H^{-it} = T(1)\Delta_H^{it}T(-1)\Delta_H^{-it} = T(1-e^{-2\pi t}).$$

Theorem (Wiesbrock 93, Araki-Zsido 05)

Let $K \subset H$ be half-sided modular. Then there exists a positively generated one-parameter group T such that

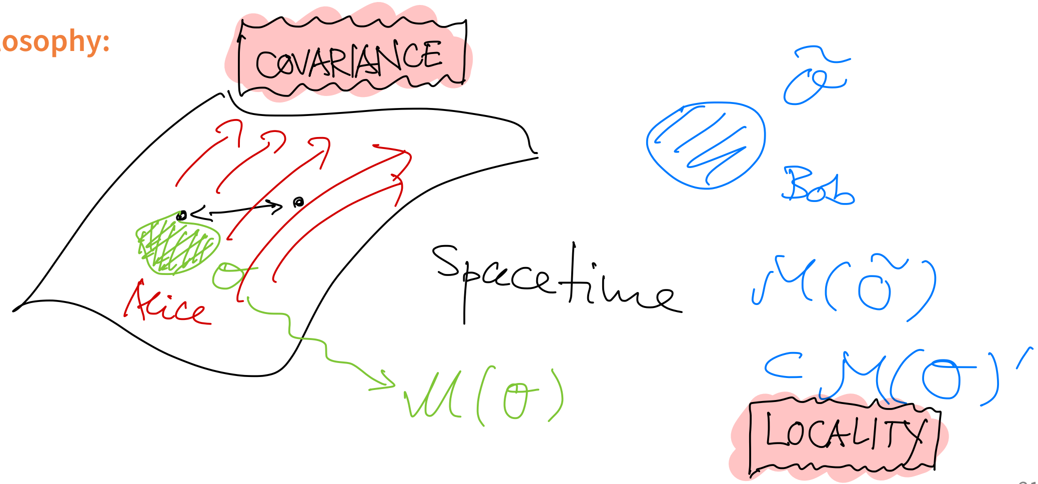
$$T(1-e^{-2\pi t}) = \Delta_K^{it}\Delta_H^{-it}, \qquad t \in \mathbb{R}.$$

- This means that half-sided modular inclusions are the same as standard subspaces with half-sided positively generated translation symmetry.
- The modular groups generate the affine group $\mathrm{Aff}(\mathbb{R})$! No symmetry has to be imposed by hand.

4) The free current as an example of a net of standard subspaces

This will be an example of a system of standard subspaces showcasing half-sided modular inclusions and much more (all main properties of a quantum field theory).

Philosophy:



Let us start with a family of real subspaces indexed by intervals $I \subset \mathbb{R}$:

$$L(I) := C_c^\infty(I \to \mathbb{R}) \subset L^2(\mathbb{R}, dx).$$

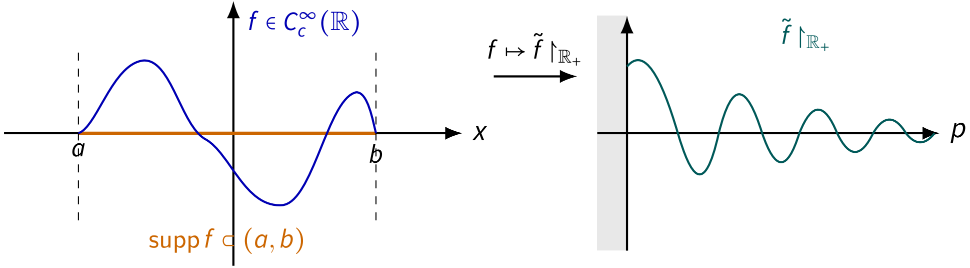
$$\mathbb{R} \ni f(x) = \underbrace{i g(x)}_{\in \mathbb{R}}$$

- separating but not cyclic for $I \neq \mathbb{R}$.

$$\Longrightarrow f = 0$$

Change representation by Fourier transform:

$$H(I) := \overline{\{\tilde{f}|_{\mathbb{R}_+} \ : \ f \in L(I)\}} \subset L^2(\mathbb{R}_+, p\, dp)$$



Note: $f \mapsto \tilde{f}|_{\mathbb{R}_+}$ is injective because reality of f implies $\tilde{f}(-p) = \overline{\tilde{f}(p)}$.

Some symmetry considerations:

- May shift functions $f \mapsto f(\,\cdot\, - x)$ in $C_c^\infty(\mathbb{R} \to \mathbb{R})$
- In $L^2(\mathbb{R}, dx)$, this is a unitary one-parameter group with non-positive generator.
- In $\mathcal{H} = L^2(\mathbb{R}_+, p\,dp)$, this is transported to the one-parameter group

$$(T(x)\psi)(p) = e^{ipx}\psi(p),$$

  with positive generator $(P\psi)(p) = p\psi(p)$.
- $\mathcal{H} = L^2(\mathbb{R}_+, p\,dp)$ carries an (anti-)unitary **irreducible** representation of $\mathrm{Aff}(\mathbb{R})$, namely

$$\begin{aligned}(U(e^t, x)\psi)(p) &= e^{ipx}\, e^t\psi(e^t p),\\ (U(j)\psi)(p) &= -\overline{\psi(p)}, \qquad j := (-1, 0) \quad \text{reflection}\end{aligned}$$

- Up to unitary equivalence, this is the unique irreducible positive energy representation (translation group has a positive generator) of $\mathrm{Aff}(\mathbb{R})$.

In the following, $I \subset \mathbb{R}$ denotes intervals (possibly unbounded), $I \neq \varnothing$, $I \neq \mathbb{R}$.

Theorem

The real subspaces $H(I) := \overline{\{\tilde{f}|_{\mathbb{R}_+} \,:\, \mathrm{supp} f \subset I\}} \subset \mathcal{H}$ *have the following properties:*

a) *The map* $I \mapsto H(I)$ *is inclusion preserving (a “net”, or “isotonous”)* ✓

b) *The net is* $\mathrm{Aff}(\mathbb{R})$*-covariant:*

$$U(\gamma)H(I) = H(\gamma I), \qquad \gamma \in \mathrm{Aff}(\mathbb{R}).$$ ✓

c) *The net is local:*

$$I_1 \cap I_2 = \varnothing \implies H(I_1) \subset H(I_2)'.$$

d) $H(I)$ *is standard.*

e) $H([1,\infty)) \subset H([0,\infty)) =: H_+$ *is a half-sided modular inclusion, with*

$$\Delta_{H_+}^{it} = U(e^{-2\pi t}, 0), \qquad J_{H_+} = U(j).$$

Let us prove the locality c):

$$I_1 \cap I_2 = \varnothing \implies H(I_1) \subset H(I_2)'.$$

This follows by pulling back the symplectic form $\operatorname{Im}\langle\cdot,\cdot\rangle$ to $C_c^\infty(\mathbb{R} \to \mathbb{R})$:

$$\operatorname{Im}\langle \tilde{f}_1|_{\mathbb{R}_+}, \tilde{f}_2|_{\mathbb{R}_+} \rangle$$

$$= \frac{1}{2i}\left(\int_0^\infty \overline{\tilde{f}_1(p)}\, \tilde{f}_2(p)\, p\, dp - \int_0^\infty \tilde{f}_1(p)\, \overline{\tilde{f}_2(p)}\, p\, dp \right)$$

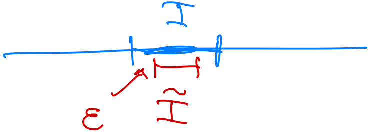
$$= \pm\frac{1}{2} \int_{\mathbb{R}} f_1(x)\, f_2'(x)\, dx = 0 \quad \left(\text{by disjoint supports}\right)$$

$\square$

Let us prove the standardness d). This is done by what is called a "Reeh-Schlieder argument" which shows the importance of positive energy.

Proof that $H(I)$ is cyclic. We have to show that $H(I) + iH(I)$ is dense, i.e. that only $\xi = 0$ satisfies $0 = \langle \xi, h \rangle, h \in H(I)$.

$\to \; 0 = \langle \xi, T(x) h \rangle \qquad h \in H(\tilde{I}) \qquad |x| < \varepsilon$



$$= \int_0^\infty \overline{\tilde{\xi}(p)}\, e^{ipx}\, \tilde{f}(p)\, p\, dp \qquad a \in \tilde{I}, \; |x| < \varepsilon$$

$$a + x \in I$$

Proof that $H(I)$ is separating.

$H(I)$ cyclic $\iff \underbrace{H(I)'}_{H(\hat{I})}$ separating

$\hat{I} \cap I = \emptyset$

$\square$

Let us prove the half-sided modular inclusion and $\Delta_+^{it}) = U(e^{-2\pi t}, 0)$ in e).

This is the simplest example of "geometric modular action" and the "Bisognano-Wichmann Theorem".

- $H_+ := H([0,\infty))$ is standard, $T(x)$ is positively generated, and $T(x)H_+ = H([x,\infty)) \subset H_+$ for $x \geq 0$. $\Longrightarrow$ half-sided modular inclusion.
- Borchers' Thm $\Longrightarrow$ $\Delta_+^{it}T(x)\Delta_+^{-it} = T(e^{-2\pi t}x)$
- We also have $U(e^{-2\pi t}, 0)T(x)U(e^{2\pi t}, 0) = T(e^{-2\pi t}x)$ (representation of $\mathrm{Aff}(\mathbb{R})$).

$$\Longrightarrow \quad (\Delta_+^{it}U(e^{2\pi t}, 0))T(x)(\Delta_+^{it}U(e^{2\pi t}, 0))^{-1} = T(x)$$

- Furthermore, $U(e^{2\pi t}, 0)$ and Δ_+^{is} commute for all t, s because $[0,\infty)$ is dilation invariant.
- irreducibility of U $\Longrightarrow$ $\Delta_+^{it} = e^{i\alpha t}U(e^{-2\pi t}, 0)$ for some $\alpha \in \mathbb{R}$
- Standardness of H_+ and $e^{i\alpha t}H_+ = H_+$ force $\alpha = 0$.
- Argument for $J_+ = U(j)$ is similar.

In this example, the whole net $I \mapsto H(I)$ is generated from the data

$$H_+ := H([0,\infty)), \qquad T(x) = U(1,x)$$

because

$$\begin{aligned} H([a,b]) &= H([a,\infty)) \cap H((-\infty,b]) \\ &= T(a)H_+ \cap T(b)U(j)H_+ \\ &= T(a)H_+ \cap T(b)H'_+. \end{aligned}$$